Decomposing and Clique-Colouring (Diamond, Odd-Hole)-Free Graphs

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Abstract

A *diamond* is a graph on 4 vertices with exactly one pair of non-adjacent vertices, and an *odd hole* is an induced cycle of odd length. If G, H are graphs, G is H*-free* if no induced subgraph of G is isomorphic to H. A *clique-colouring* of G is an assignment of colours to the vertices of G such that no inclusion-wise maximal clique of size at least 2 is monochromatic. We show that every (diamond, odd-hole)-free graph is 3-clique-colourable, answering a question of Bacsó [1].

1 Introduction

All graphs in this paper are finite and simple. For a graph G, we let $V(G)$ denote the vertex set, $E(G)$ the edge set and $\chi(G)$ the chromatic number. We let $\overline{G}$ denote the *complement* of the graph G, which is the graph on the same vertex set with uv an edge in $\overline{G}$ if and only if it is not an edge in G.

A *clique* in G is a subset X of $V(G)$ such that every two members of X are adjacent. A k*-vertex-colouring* of G is a labeling of $V(G)$ using at most k colours, such that no pair of adjacent vertices is monochromatic. G is k*-colourable* if it admits a k-vertex-colouring. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest number k such that G is k-colourable. A k*-clique-colouring* of G is a labeling of $V(G)$ using at most k colours, such that no inclusion-wise maximal clique of size at least 2 is monochromatic. G is k*-clique-colourable* if it admits a k-clique-colouring. The *clique-chromatic number* of G, denoted by $\chi_C(G)$, is the smallest number k such that G is k-clique-colourable. A graph G is *perfect* if, for every induced subgraph H of G, $\chi(H)$ is equal to the size of the largest clique of H. A *diamond* is a graph on 4 vertices with exactly one pair of non-adjacent vertices. A graph is *diamond-free* if it contains no diamond as an induced subgraph. In general, a graph is H-free if it has no graph isomorphic to H as an induced subgraph, and if a class $\mathcal{G}$ of graphs is H-free we say that H is a *forbidden induced subgraph* in $\mathcal{G}$. A graph G is *Berge* if both G and $\overline{G}$ are odd-hole-free.

A forbidden induced subgraph characterization of perfect graphs is given by the Strong Perfect Graph Theorem [2], which states that a graph is perfect if and only if it is Berge. However, much is still unknown about the structure and behaviour of perfect graphs. Duffus et al. [3] conjectured that all perfect graph are k-clique-colourable for some constant k. This problem is still open, moreover Kratochvíl and Tuza [4] proved that the problem of deciding whether a K_4-free perfect graph is 2-clique-colourable is NP-complete. However, there are also many partial results in the positive direction. Bacsó et al. [1] proved that all (claw, odd-hole)-free graphs are 2-clique-colourable, Défossez [5] proved that all (bull, odd-hole)-free graphs are 2-clique-colourable, and Penev [6] proved that all perfect graphs with no balanced skew partition are 2-clique-colourable. In addition, Bacsó [1] et al. showed that almost all perfect graphs are 3-clique-colourable, and conjectured that all perfect graphs are 3-clique colourable. The subclass of (diamond, odd-hole)-free graphs has been of particular interest. Bacsó et al. [1] conjectured that all (diamond, odd-hole)-free graphs

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are 3-clique-colourable, and gave a simple example of a diamond-free graph that is not 2-clique-colourable. Défossez [5] has demonstrated that all (diamond, odd-hole)-free graphs are 4-clique-colourable. In this paper, we prove that all (diamond, odd-hole)-free graphs are 3-clique-colourable. The class of (diamond, odd-hole)-free graphs is in fact equivalent to the class of diamond-free perfect graphs [7], which is also equivalent to the class of diamond-free Berge graph. Hence we refer to these classes interchangeably throughout the paper and make use of all the associated properties.

The proof of the theorem relies on examining the structural properties of diamond-free perfect graphs, and follows similar steps to the proof of the Strong Perfect Graph Theorem. In Section 2, we state the main results of this paper. In Section 3, we prove a decomposition theorem for diamond-free perfect graphs. Finally, in Section 4, we use the decomposition theorem to show that all diamond-free perfect graphs are 3-clique-colourable.

1.1 Preliminaries

In this section, we present some definitions and set the notation that will be used throughout the paper. Additional graph theoretic objects used in specific sections of the proof will be defined later, as required.

A *clique* is a set of pairwise adjacent vertices. An *anticlique* is a set of pairwise non-adjacent vertices. For a vertex $v \in V(G)$, we let $N(v)$ denote the set of vertices adjacent to x. The *complete graph on n vertices K_n* is a graph such that $V(K_n)$ is a clique, and $|V(K_n)| = n$. Let $X \subseteq V(G)$ and $Y \subseteq V(G) \backslash X$. We say that X *is complete to* Y, or that X is Y*-complete*, if for every $x \in X$ and $y \in Y$, the vertices x, y are adjacent. We say that X *is anticomplete to* Y if X is complete to Y in $\overline{G}$. If $Y = \{y\}$ we will simply say that X is complete to y, or that X is y-complete. If $x \in V(G)$ and $e = uv \in E(G)$, we say that e is an *x-complete edge* if $\{u, v\}$ is x-complete.

If $X \subseteq V(G)$, the subgraph of G *induced* by X, denoted by $G|X$, is given by deleting all the vertices not in X, so that $V(G|X) = X, E(G|X) = \{uv \in E(G) \ : \ u, v, \in X\}$. For graphs G and H, we say that G *contains H (as an induced subgraph)* if there exists $X \subseteq V(G)$ so that $G|X$ is isomorphic to H, and otherwise we say that G *is H-free*. We remark that almost all subgraphs in this paper will be induced, and we will specify when subgraphs are not induced.

Let $n \geq 0$ be an integer. A *track Q of length n* in G is a non-null subgraph on a set of vertices $\{v_1, v_2, \ldots, v_{n+1}\}$, not necessarily induced, with edge set $E(Q) = \{v_i v_{i+1} : 1 \leq i \leq n\}$. A *path P of length n*, denoted by P_{n+1}, is an induced track of length n. We let $P^* = \{v_2, \ldots, v_n\}$ denote the *interior* of P, and call v_1 and v_{n+1} the *ends* of the path. For clarity, we often describe P_{n+1} by the sequence $v_1 - v_2 - \cdots - v_{n+1}$. (We note that paths and tracks are frequently termed induced paths and paths respectively in conventional literature, but our choice of terminology saves us from repeatedly specifying that paths are induced.) An *antipath of length n* is a subgraph that forms a path of length n in the complement.A *cycle C of length n* in G is a non-null subgraph on a set of vertices $\{v_1, v_2, \ldots, v_n\}$ with edge set $E(C) = \{v_i v_{i+1} : 1 \leq i \leq n\} \cup \{v_n v_1\}$. We will often describe a cycle C by the sequence $v_1 - v_2 - \cdots - v_n - v_1$. If $X, Y \subseteq V(G)$ are disjoint, we say that P is an *$X - Y$ path* if the ends of P are in X and Y respectively, and $P^* \subseteq V(G) \backslash (X \cup Y)$. If $X = \{x\}$ and $Y = \{y\}$, we say that P is an $x - y$ path.

Let G_1, G_2 be graphs. Their disjoint union, $G_1 + G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. A graph G is *connected* if there does not exist a partition (A, B) of the vertices such that G is the disjoint union of $G|A$ and $G|B$. A graph G is *anticonnected* if $\overline{G}$ is connected. We say that a set $X \subseteq V(G)$ is *connected* if $G|X$ is connected, and *anticonnected* if $\overline{G}|X$ is connected. Throughout the paper, we will assume that all ambient graphs G are connected, unless otherwise specified.

Let G be a graph, $A, B \subseteq V(G)$ disjoint subsets and $G|A, G|B$ the subgraphs induced by A and B respectively. Then the set of *attachments* of A in $G|B$ is the set of vertices in B with neighbours in A. We will interchangeably refer to the set of attachments of the set A in $G|B$, and the set of attachments of the graph $G|A$ in $G|B$.

A *diamond* is a graph on 4 vertices with exactly one pair of non-adjacent vertices. We say that a set D *is a diamond (in G)* if the graph $G|D$ is a diamond, and that a set T *is a triangle (in G)* if the graph $G|T$ is the complete graph on 3 vertices. A *hole* of G is an induced subgraph of G that is a cycle of length at least

4. An *antihole* of G is an induced subgraph of G that is a hole in $\overline{G}$. A hole is *odd* if it has odd length, and *even* if it has even length. A graph G is *Berge* if both G and $\overline{G}$ are odd-hole-free.

Throughout the proof, we will be interested in certain classes of 'basic' graphs, which we now define. A *bipartite* graph is a graph whose vertex set can be partitioned into two disjoint anticliques. The *line graph* $L(G)$ of a graph G is the graph with vertex set $V(L(G) = E(G)$, and edge set such that two vertices $e, f \in V(L(G))$ are adjacent if and only if e and f share an end in G.

A *double split graph* is a graph G whose vertices can be partitioned into four sets $\{a_1, \ldots, a_m\}$, $\{b_1, \ldots, b_m\}$, $\{c_1, \ldots, c_n\}$, $\{d_1, \ldots, d_n\}$ for some $m, n \geq 2$ such that the following properties hold:

- a_i is adjacent to b_i for $1 \leq i \leq m$, and c_j is non-adjacent to d_j for $1 \leq j \leq n$;
- $\{a_i, b_i\}$ is anticomplete to $\{a_{i'}, b_{i'}\}$ for $1 \leq i < i' \leq m$, and $\{c_j, d_j\}$ is complete to $\{c_{j'}, d_{j'}\}$ for $1 \leq j < j' \leq n$; and
- there are exactly two edges between $\{a_i, b_i\}$ and $\{c_j, d_j\}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, and these two edges do not share an end.

Definition 1.1.1. *A graph G is* basic *if either G or $\overline{G}$ is bipartite, or either G or $\overline{G}$ is the line graph of a bipartite graph, or G is a double split graph.*

2 Main Results

The main results of this paper are a decomposition theorem for diamond-free perfect graphs, and the 3-clique-colourability of diamond-free perfect graphs. Our decomposition theorem is based on the decomposition theorem for perfect graphs in [2] that was the main ingredient in the proof of the Strong Perfect Graph Theorem. To state these two decomposition theorems, we first need to define some decompositions.

A 2*-join of* G is a partition (A, B) of $V(G)\backslash X$ and disjoint nonempty sets A_1, A_2 and B_1, B_2 such that, for $i = 1, 2$, $A_i \subseteq A$ and $B_i \subseteq B$, A_i is complete to B_i, and there are no other edges between A and B. We say that G *admits* a 2-join, and that A and B are the *blocks* of the 2-join. The decomposition theorem for perfect graphs in [2] is as follows.

Theorem 2.1 (Strong Perfect Graph Theorem [2])**.** *Let G be a perfect graph. Then either G is basic, or one of $G, \overline{G}$ admits a proper 2-join, or G admits a balanced skew partition.*

(We remark that that since proper 2-joins, skew partitions and balanced skew partitions are not of interest in the analysis of diamond-free graphs, we do not define these objects in full here. A proper 2-join is a 2-join with additional properties, and skew partitions and balanced skew partitions are decompositions of the graph into vertex sets that are either complete or anticomplete to each other.)

Initially, we had hoped to directly modify Theorem 2.1 to obtain a decomposition theorem for diamond-free graphs that would be amenable to clique-colouring algorithms. However, this proved difficult, as not all proper 2-joins yielded suitable decompositions, and balanced skew partitions gave little to no insight to the clique-colourability of the graph. In order to derive any results of interest, we needed to retrace the steps of the proof using slightly different decompositions, which we now define.

Let G be a graph, and let $X \subset V(G)$ be a set of vertices. If X induces a clique in G and $G\backslash X$ is not connected, we say that X is a *clique cutset* of G, and that G *admits* a clique cutset. If $V(G) = A \cup B$, where $A \cap B = X$, we say that A and B are the *blocks* of the decomposition. If $X = \{x\}$ for some vertex $x \in V(G)$, then we say that x is a *cut vertex*.

A *complete* 2*-join of* G is a 2-join such that: A_i, B_i are cliques for $i = 1, 2$; and $|A_i \cup B_i| \geq 3$ for $i = 1, 2$. We say that G *admits* a complete 2-join if G admits a 2-join satisfying these properties.

A *vertex-complete multijoin of* G is given by a vertex $x \in V(G)$, a partition (A, B) of $V(G)\backslash\{x\}$ and disjoint nonempty cliques $A_1, \ldots, A_k$, $B_1, \ldots, B_k$ such that the following hold:

- $A_i \subseteq A$ and $B_i \subseteq B$ for all $1 \leq i \leq k$;

- A_i and B_i are complete to x for all $1 \leq i \leq k$;
- A_i is complete to B_i for all $1 \leq i \leq k$, and there are no other edges between A and B.

We say that G *admits* an x-complete multijoin. If $k = 2$, we say that G *admits a vertex-complete* 2*-join*, or more specifically an x*-complete* 2*-join*. We remark that if $k = 1$, then $A_1 \cup B_1 \cup \{x\}$ is a clique cutset.

Our decomposition theorem for diamond-free perfect graphs, and the resultant characterization of the clique-colourability of diamond-free perfect graphs, are as follows.

Theorem 2.2. *Let G be a diamond-free perfect graph. Then either G is basic, or G admits a clique cutset, a complete* 2*-join or a vertex-complete multijoin.*

Theorem 2.3. *Let G be a diamond-free perfect graph. Then G is* 3*-clique-colourable.*

The remainder of this paper is set out as follows. The entirety of Section 3 will be devoted to proving Theorem 2.2. In Section 3.2, we define our classes of basic graphs. In Sections 3.3 and 3.4 we consider graphs containing a sufficiently large line graph or the complement of a sufficiently large line graph. In Sections 3.5 and 3.6, we consider graphs containing a prism or the complement of a prism and not in the previous classes. Finally, in Section 3.7, we consider graphs containing wheels and not in the previous graph classes. In each case, we show that the graph is either a basic graph or admits one of the required graph decompositions. In Section 3.8, we show that this covers all diamond-free perfect graphs. Finally, in Section 4, we prove Theorem 2.3.

3 A Decomposition Theorem for Diamond-free Perfect Graphs

This section of the paper will be devoted to proving Theorem 2.2, which we restate here.

Theorem 3.1. *Let G be a diamond-free perfect graph. Then at least one of the following holds:*

- *G is a basic graph;*
- *G admits a clique cutset;*
- *G admits a complete* 2*-join;*
- *there is a vertex $x \in V(G)$ such that G admits an x-complete multijoin.*

3.1 Preliminary Lemmas

We begin with a few easy lemmas that will be used frequently throughout the paper.

Lemma 3.1.1. *Let G be a diamond-free graph, and let $A, B \subseteq V(G)$ be disjoint nonempty subsets of vertices such that A is complete to B in G. Then either: $|A| = 1$ and B is a union of cliques; or $|B| = 1$ and A is a union of cliques; or A and B are both cliques; or A and B are both anticliques.*

Proof. Suppose B is not a union of cliques. Then it contains an induced P_3 given by $v_1 - v_2 - v_3$. Let $v_0 \in A$. Then v_0 is complete to $\{v_1, v_2, v_3\}$, so $\{v_0, v_1, v_2, v_3\}$ is a diamond, contradiction. Hence A and B are both unions of cliques. Suppose now that A contains a pair of non-adjacent vertices a_1, a_2, and B contains a pair of adjacent vertices b_1, b_2. Then $\{a_1, b_1, a_2, b_2\}$ is a diamond, contradiction. Hence, by symmetry, if $|A|, |B| \geq 2$, then either both A and B are cliques, or both A and B are anticliques. □

Lemma 3.1.2. *Let G be a diamond-free graph, let $K \subseteq V(G)$ induce a clique in G, and let $v \in V(G) \backslash K$. Then either v has at most one neighbour in K, or v is complete to K.*

Proof. Suppose that v has at least two neighbours $k_1, k_2 \in K$ and a nonneighbour $k_3 \in K$. Then $\{v, k_1, k_3, k_2\}$ is a diamond in G. □

Let $x \in V(G)$ be a vertex, let P be a path in $G \setminus \{x\}$ given by $v_1 - v_2 - \cdots - v_{n+1}$, and let C be a cycle in $G \setminus \{x\}$ given by $w_1 - w_2 - \cdots - w_n - w_1$. Let the *parity* of P (or $V(P)$) denote the parity of the length of P, and let the *x-parity* of P (or $V(P)$) denote the parity of the number of x-complete edges in $E(P)$. Similarly, let the *parity* of C denote the parity of the length of C, and let the *x-parity* of C denote the parity of the number of x-complete edges in $E(C)$.

Lemma 3.1.3. *Let G be a perfect graph. Let $x \in V(G)$, let P be a path in $G \backslash \{x\}$ whose ends are adjacent to x, and let Q be a hole in $G \backslash \{x\}$ with an x-complete edge. Then the x-parity of P is the same as the parity of P, and either C has even x-parity, or there are exactly two x-complete vertices in $V(C)$ and they are adjacent.*

Proof. We first prove the statement for P by induction on the number of neighbours of x in P^*. If x has no neighbours in P^*, then $P \cup \{x\}$ induces a hole in G, so either P is an edge and there is one x-complete edge in P, or P has even length and there are no x-complete edges in P. If x has a neighbour in P^*, the result follows by induction. For C, let $u, v \in V(C)$ be a pair of non-adjacent vertices adjacent to x. If such u, v do not exist then the theorem holds. Let the two distinct $u - v$ paths with interiors in $V(C)$ be given by P_1, P_2. Then the number of x-complete edges in P is the sum of the number in P_1 and the number in P_2, and as the P_i are paths this has the same parity as the sum of the lengths of P_1 and P_2. This proves the lemma. $\square$

Recall that a *triangle* in G is a set of three vertices, mutually adjacent.

Definition 3.1.4. *We say a vertex v can be* linked *onto a triangle $\{a_1, a_2, a_3\}$ (via paths P_1, P_2, P_3) if:*

- *the three paths P_1, P_2, P_3 are mutually vertex-disjoint,*
- *for $i = 1, 2, 3$, the ends of P_i are v and a_i; and*
- *for $1 \leq i < j \leq 3$, $a_i a_j$ is the unique edge of G between $V(P_i)$ and $V(P_j)$.*

Lemma 3.1.5 ([2] 2.4)**.** *Let G be perfect, and suppose v can be linked onto a triangle $\{a_1, a_2, a_3\}$. Then v is adjacent to at least two of a_1, a_2, a_3.*

Proof. Let v be linked onto triangle $\{a_1, a_2, a_3\}$ via paths P_1, P_2, P_3. Then at least two of the P_i have the same parity, so we may assume that P_1, P_2 have the same parity. Since $C = v - P_1 - a_1 - a_2 - P_2 - v$ is not an odd hole in G, it follows that C is a triangle and v is adjacent to a_1 and a_2. $\square$

3.2 Basic Diamond-Free Perfect Graphs

In this section, we characterize all basic diamond-free perfect graphs. Recall that a graph G is *basic* if either G or $\overline{G}$ is bipartite, or either G or $\overline{G}$ is the line graph of a bipartite graph, or G is a double split graph.

A *claw* is a graph on 4 vertices with one vertex adjacent to all the other three, and no other edges. In this section, we use the following forbidden induced subgraph characterization of line graphs of bipartite graphs.

Lemma 3.2.1 ([8])**.** *A graph G is the line graph of a bipartite graph if and only if G is (claw, diamond, odd-hole)-free.*

This allows us to characterize diamond-free perfect graphs that are complements of the line graph of a bipartite graph.

Lemma 3.2.2. *Let G be a diamond-free perfect graph. Suppose that G is the complement of the line graph of a bipartite graph, and that G is not bipartite and does not contain a clique cutset. Then G is either a complete graph, or an induced subgraph of $L(K_{3,3})$.*

Proof. We have assumed that G is diamond-free and odd-hole-free, and it follows from Lemma 3.2.1 that $\overline{G}$ is claw-free and diamond-free. Let K be a maximal clique in G. Suppose that $|K| \geq 4$, and let $v \in V(G) \setminus K$. Then, as $\overline{G}$ is claw-free, v has at least 2 neighbours in K, and so Lemma 3.1.2 implies that v is complete to K. Hence $V(G) = K$ is a clique and the theorem holds. So we may assume that G is K_4-free.

As G is not bipartite, it contains a triangle. Let $T = \{v_1, v_2, v_3\}$ be a triangle in G, and let $v \in V(G)\backslash T$. Since $\overline{G}$ is claw-free, v has a neighbour in T, and since G is diamond-free and K_4-free, it follows that v has exactly one neighbour in T. Hence $V(G)\backslash T = A_1 \cup A_2 \cup A_3$, where A_i is complete to v_i and anticomplete to v_j for $j \neq i$. If A_1 contains a pair of non-adjacent vertices a, a', then $\{v_2, a, v_3, a'\}$ is a diamond in $\overline{G}$, contradiction. Hence, by symmetry, A_i is a clique for every i, and as G is K_4-free, $|A_i| \leq 2$ for each i. Also, as no subset of T is a clique cutset, $|A_i| \geq 1$ for each i.

Consider now the graph H with $V(H) = A_1 \cup A_2 \cup A_3$ and $E(H) = \{uv \in E(G) : u \in A_i, v \in A_j, i \neq j\}$. Suppose that H contains an induced 2-edge-path $u-v-w$. If $\{u, v, w\} \subseteq A_1 \cup A_2$, then $\{v_3, u, v, w\}$ induces a claw in $\overline{G}$, which is a contradiction. Thus we may assume that $u \in A_1, v \in A_2, w \in A_3$, and $v_1-u-v-w-v_3$ induces a C_5 in G, contradiction. Hence H does not contain an induced 2-edge-path. Moreover, if there exist adjacent vertices $a, a' \in A_1$ and a vertex $v \in A_2 \cup A_3$ that is anticomplete to $\{a, a'\}$, then $\{a, a', v_1, v\}$ induces a claw in $\overline{G}$, contradiction. Finally, if there exist $a_i \in A_i$ for $i = 1, 2, 3$ such that $\{a_1, a_2, a_3\}$ is an anticlique, then $\{v_1, a_1, a_2, a_3\}$ induces a diamond in $\overline{G}$, contradiction. It follows that G is an induced subgraph of $L(K_{3,3})$. □

A *birdcage* is a graph G whose vertices can be partitioned into two sets A and B such that: A and B are cliques; every $a \in A$ has at most one neighbour in B; and every $b \in B$ has at most one neighbour in A.

Proposition 3.2.3. *Let G be a diamond-free perfect graph. Suppose that G is a basic graph, and that G does not contain a cut vertex or a clique cutset. Then G is one of the following:*

- *a clique;*
- *a bipartite graph;*
- *a birdcage; or*
- *the line graph of a bipartite graph.*

Proof. By Lemma 3.2.2, as every induced subgraph of a line graph is a line graph, it suffices to consider when G is either the complement of a bipartite graph or a double split graph.

Suppose that G is the complement of a bipartite graph. Then $V(G) = A \cup B$ where A and B induce cliques in G. For every vertex $a \in A$, by Lemma 3.1.2, either a is complete to B, or a has at most one neighbour in B. A similar statement holds for vertices in B. Let K_A, K_B be the maximal cliques containing A and B respectively. Since $K = K_A \cap K_B$ is a clique and is complete to $G\backslash K$, by Lemma 3.1.1, either $K = V(G)$ or $|K| \leq 1$. If $K = V(G)$, then G is a clique. If $|K| = 1$, then Lemma 3.1.1 implies that $K_A\backslash K$ is anticomplete to $K_B\backslash K$ and K is a cut vertex. If $|K| = 0$, then $K_A = A, K_B = B$ and G is a birdcage.

Hence we may assume that G is a double split graph, with $V(G) = A \cup B \cup C \cup D$, where $A = \{a_1, \ldots, a_m\}$, $B = \{b_1, \ldots, b_m\}$, $C = \{c_1, \ldots, c_n\}$ and $D = \{d_1, \ldots, d_n\}$, $m, n \geq 2$. As C is a clique in G and d_1 is non-adjacent to c_1 and complete to $C\backslash\{c_1\}$, by Lemma 3.1.2 it holds that $n = |C| = 2$. Moreover, after possibly exchanging labels a_i and b_i for some values of i, we may assume that a_i is adjacent to c_1 for every i. Let A_1 be the set of $a \in A$ that are adjacent to c_2, and $A_2 = A\backslash A_1$ be the set of $a \in A$ that are adjacent to d_2. Then A_1 is complete to the clique $\{c_1, c_2\}$, so Lemma 3.1.1 implies that A_1 is a clique. As $A_1 \subseteq A$ is an independent set, $|A_1| \leq 1$. Similarly, $|A_2| \leq 1$, so in fact $|A_1| = |A_2| = 1$, $|A| = |B| = 2$ and G is the unique induced subgraph of $L(K_{3,3})$ on 8 vertices. Hence G is the line graph of a bipartite graph, and the proposition holds. □

3.3 Generalized Line Graphs

We begin the decomposition theorem proper with the case when either G or $\overline{G}$ contains a line graph of a sufficiently large bipartite graph as an induced subgraph, and follow the structural decomposition for such graphs found in [2]. First, we consider the case where G contains a nondegenerate line graph $L(H)$ as an induced subgraph, grow $L(H)$ into a generalized line graph (S, N), and consider the attachments of the components of the rest of the graph $G \setminus V(S, N)$ in the generalized line graph (S, N). We then consider the case when G contains a degenerate line graph $L(H)$ as an induced subgraph, but no nondegenerate line graph as an induced subgraph. We now make these notions precise.

A *branch-vertex* of a graph H is a vertex with degree at least 3; and a *branch* of H is a maximal path P in H such that no vertex in P^* is a branch-vertex. *Subdividing* an edge uv means deleting the edge uv, adding a new vertex w, and adding two new edges uw and wv. A *subdivision* of a graph J is obtained by repeatedly subdividing edges, effectively replacing each edge uv of J with a $u-v$ path.

If G is a graph and J is a 3-connected graph, we say that J *appears in* G if there is a bipartite subdivision H of J such that $L(H)$ is isomorphic to an induced subgraph G' of G. We call $L(H)$ an appearance of J in G. If J is 3-connected, we say a graph J' is a J*-enlargement* if J' is 3-connected, and has a proper subgraph which is isomorphic to a subdivision of J. If v is a vertex of H, the set of edges of H incident with v is denoted by $\delta_H(v)$, and is in correspondence with a set of vertices in G. If $uv \in E(J)$, we let B_{uv} denote the unique $u-v$ branch in $L(H)$, with endpoints in $\delta_H(u)$ and $\delta_H(v)$ respectively.

If $J = K_4$, we say that an appearance $L(H)$ is *degenerate* if there is a cycle of H of length four containing the four vertices of H that have degree three in H, and *nondegenerate* otherwise. If $J \neq K_4$, we say that an appearance $L(H)$ of J in G is *degenerate* if $J = H = K_{3,3}$, and *nondegenerate* otherwise.

(Note that, if $L(H)$ is isomorphic to some induced subgraph K of G, there is another subdivision H' isomorphic to H, made from H by replacing each edge of H by the corresponding vertex of K; and now $L(H') = K$, rather than just being isomorphic to it. So whenever it is convenient we shall assume that the isomorphism between $L(H)$ and K is just equality, without further explanation. Note in particular that $E(H) = V(K)$, and so some vertices of G are edges of H.)

We first show that the case when $\overline{G}$ contains the line graph of a bipartite graph reduces to the case when G contains the line graph of a bipartite graph.

Lemma 3.3.1. *Let G be a diamond-free perfect graph, let J be 3-connected, and let $L(H)$ be an appearance of J in $\overline{G}$. Then either $J = H = K_{3,3}$, or $J = K_4$ and every branch B of H has length at most 2. Moreover, G also contains an appearance of J.*

Proof. As G is diamond-free, $\overline{L(H)}$ is diamond-free, so H does not contain $P_3 + P_2 + P_2$ as a (possibly non-induced) subgraph. In particular, H does not contain a track on 7 vertices as a subgraph.

Let u, v be two branch vertices of J, chosen to be adjacent in H, if possible. As J is 3-connected, there exist three vertex-disjoint $u-v$ paths in J, and hence also in H. Let the paths in H be $u-Q_1-v$, $u-Q_2-v$ and $u-Q_3-v$, where u, v are the endpoints of Q_i for each i. Let l_i be the length of Q_i, and assume without loss of generality that $l_1 \leq l_2 \leq l_3$. Let $u^* \in Q_2^*$ be adjacent to u, and let $v^* \in Q_3^*$ be adjacent to v. Then $Q = u^* - Q_2^* - v - Q_1 - u - Q_3^* - v^*$ is a track on $l_1 + l_2 + l_3 - 1$ vertices, and hence $l_1 + l_2 + l_3 \leq 7$.

As H is bipartite, all the Q_i have the same parity, and as H is simple, $l_2, l_3 \geq 2$. Hence, if all the Q_i are odd, then $l_1 = 1$ and $l_2 = l_3 = 3$, and if all the Q_i are even, then $l_1 = l_2 = l_3 = 2$. In both cases, Q contains $P_3 + P_2$ as an induced subgraph, so $V(H) \setminus V(Q)$ is an anticlique. Moreover, since H is a subdivision of a 3-connected graph, if $v \in V(H) \setminus V(Q)$, then v has at least 2 neighbours in $V(Q)$.

Consider first the case when all the Q_i are odd. If there exists $w \in V(H) \setminus V(Q)$, then w has neighbours outside of $\{u, v\}$, so by symmetry we may assume that w is adjacent to u^*. Hence $w - Q - v^*$ is a track on 7 vertices, contradiction. Hence $V(H) = V(Q)$, and since H is a bipartite subdivision of a 3-connected graph, either $J = H = K_{3,3}$, or $J = K_4$ and H is K_4 with a pair of non-adjacent edges subdivided and $L(H)$ is the unique induced subgraph of $L(K_{3,3})$ on 8 vertices. In both outcomes, G also contains $L(H)$ as an induced subgraph, and the result holds.

Thus we may assume that all the Q_i are even, and so no pair of branch vertices in J are adjacent in H, since u, v are chosen to be adjacent if possible. It follows that $l_i = 2$, and $Q_i^* = \{w_i\}$ for all i. Let $u', v' \notin \{u, v\}$ be another two branch vertices of J, which exist since J is 3-connected. Then $u', v' \notin V(Q)$, and all their neighbours in H are in $V(Q) \setminus \{u, v\}$. Hence $\{u', v'\}$ is complete to $\{w_1, w_2, w_3\}$ and $u' - w_1 - v - w_2 - u - w_3 - v'$ is a track on 7 vertices in H, contradiction. This proves the lemma. □

As a consequence of Lemma 3.3.1, in this section we examine the structure of G only when it contains the line graph of a bipartite graph. The structure of the proof has a similar flavour to the proof of the strong perfect graph theorem in [2]. Roughly speaking, we grow the line graph into a generalized line graph, replacing each edge adjacent to a branch-vertex with a set of vertices, and each branch with multiple interchangeable 'rungs'. The definitions are as follows.

Let A, B, C be disjoint subsets of $V(G)$. We call $S = (A, C, B)$ a *strip* if A, B are nonempty, and every vertex of $A \cup B \cup C$ belongs to an $A - B$ path with interior in C. We call such a path a *rung* of the strip S. When $S = (A, C, B)$ is a strip, we let $V(S)$ denote $A \cup B \cup C$. The *reverse* of a strip (A, C, B) is the strip (B, C, A). An *antistrip* is a triple that is a strip in $\overline{G}$, and the corresponding induced antipaths are called *antirungs*. If P is a rung with ends $a \in A$ and $b \in B$, we speak of the rung $a - P - b$ for brevity. We will always use the uppercase letter for names of sets, and the lowercase letter for vertices contained in that set.

Definition 3.3.2. *Let J be 3-connected, and let G be Berge. A J-*strip system *(S, N) in G is a family of strips $S_{uv} = (N_{uv}, T_{uv}, N_{vu}) \subseteq V(G)$ for each edge uv of J, satisfying the following conditions:*

- *the sets $V(S_{uv}), uv \in E(J)$ are pairwise disjoint;*
- *if $uv, wx \in E(J)$ with u, v, w, x all distinct, then there are no edges between S_{uv} and S_{wx};*
- *if $uv, uw \in E(J)$ with $v \neq w$, then N_{uv} is complete to N_{uw}, and there are no other edges between $V(S_{uv})$ and $V(S_{uw})$;*
- *for every cycle C of J and every choice of uv-rungs for edges $uv \in C$, the sum of the lengths of the uv-rungs has the same parity as $|V(C)|$;*

*where for $uv \in E(J)$, a uv-*rung *is an $N_{uv} - N_{vu}$ path in the strip S_{uv}.*

For ease of notation, we define $V(S, N) = \cup_{uv \in E(J)} V(S_{uv})$, and $N_u = \cup_{uv \in E(J)} N_{uv}$ $\forall u \in V(J)$.

For each edge uv of J, choose a uv-rung R_{uv}. It was shown in [2] that the subgraph of G induced on the union of the vertex sets of these rungs is a line graph of a bipartite subdivision H of J. We follow the notation in [2] and say that this choice of rungs *forms* $L(H)$.

A J-strip system (S, N) in G is *maximal* if there is no J-strip system (S', N') in G such that $V(S, N) \subset V(S', N')$, and $S'_{uv}|V(S, N) = S_{uv}$ for every $uv \in E(J)$, and $N_v \subseteq N'_v$ for every $v \in V(J)$. A J-strip system (S, N) is *nondegenerate* if there is some choice of rungs such that the line graph $L(H)$ formed is a nondegenerate appearance of J.

The following lemmas considerably simplify our analysis of J-strip systems in diamond-free graphs.

Lemma 3.3.3. *Let G be a diamond-free graph, and let (V, S) be a J-strip system in G. Then for each $u \in V(J)$, N_u is a clique.*

Proof. We note that $N_u = \cup_{v: uv \in E(J)} N_{uv}$, where for $v_1 \neq v_2$, N_{uv_1} is complete to N_{uv_2}. Since J is 3-connected, u has at least 3 neighbours in J, so by Lemma 3.1.1, N_{uv} is a clique for every v. $\square$

We now move to classifying the attachments of components of $V(G) \backslash V(S, N)$ in (S, N). It turns out that all vertices either have many attachments in (S, N), or are part of a component that has very few attachments in (S, N). The following definitions and results make this precise.

Let (S, N) be a J-strip system in G, and let some choice of uv-rungs form $L(H)$. A vertex $x \in V(G) \backslash L(H)$ is *major* with respect to $L(H)$ if for every branch-vertex v of H, the vertex x has at most neighbour in $\delta_H(v)$. Similarly, a vertex $x \in V(G) \backslash V(S, N)$ is *major* with respect to (S, N) if for every $u \in V(J)$, there is at most one neighbour of u in J such that N_{uv} is not complete to x.

An appearance $L(H)$ of J in G is *overshadowed* if there is a branch B of H with odd length ≥ 3, with ends b_1, b_2, such that some vertex v of G is non-adjacent in G to at most one vertex in $\delta_H(b_1)$ and at most one vertex in $\delta_H(b_2)$. We say that the vertex v is *overshadowing* with respect to $L(H)$ and (S, N), and that it overshadows the branch B. We let $O(G, L(H))$ denote the set of vertices that are overshadowing and not major with respect to $L(H)$, and similarly let $O(G,(S,N))$ denote the set of vertices that are overshadowing and not major with respect to (S,N). In both cases, we denote this set by $O(G)$ when the choice line graph or strip system is evident.

A vertex $x \in V(G) \backslash L(H)$ is *minor* with respect to $L(H)$ if it is neither major nor overshadowing with respect to $L(H)$. Similarly, a vertex $x \in V(G) \backslash V(S,N)$ is *minor* with respect to (S,N) if it is neither major nor overshadowing with respect to (S,N).

Let $X \subseteq V(L(H))$. We say that X is *local* with respect to $L(H)$ if either $X \subseteq \delta_H(v)$ for some $v \in V(J)$, or X is a subset of the edge-set of some branch of H. We say that X is *semi-local* with respect to $L(H)$ if $X \subseteq \delta_H(u) \cup \delta_H(v) \cup B_{uv}$ for some branch uv of H. Similarly, let $X \subseteq V(S,N)$. We say that X is *local* with respect to (S,N) if either $X \subseteq N_v$ for some $v \in V(J)$, or $X \subseteq V(S_{uv})$ for some $uv \in E(J)$. We say that X is *semi-local* with respect to (S,N) if $X \subseteq N_u \cup N_v \cup T_{uv}$ for some $uv \in E(J)$.

The following lemmas about major and overshadowing vertices are easy consequences of these definitions.

Lemma 3.3.4. *Let G be a diamond-free graph. Then a vertex x is major with respect to $L(H)$ (or (S,N)) if and only if it is complete to $\delta_H(v)$ (or N_v) for every branch-vertex v of H; and it is overshadowing with respect to (S,N) if and only if there are branch-vertices b_1, b_2 with a rung in $S_{b_1b_2}$ of even length at least 2 such that x is complete to N_{b_1} and N_{b_2}.*

Proof. This follows from Lemma 3.1.2 and the fact that for every branch vertex $v \in V(J)$, the sets $\delta_H(v), N_v$ have size at least 3. ☐

Lemma 3.3.5. *Let (S,N) be a maximal J-strip system in G, let some choice of uv-rungs form $L(H)$, and let $x \in V(G) \backslash V(S,N)$. Then x is major with respect to (S,N) if and only if it is major with respect to $L(H)$.*

Proof. Clearly if x is major with respect to (S,N), then it is major with respect to $L(H)$. Suppose x is major with respect to $L(H)$. Then, for every branch-vertex v of H, x is complete to $\delta_H(v)$, which is a subset of N_v of size at least 3. Since G is diamond-free, by Lemma 3.3.5, N_v is a clique, so x is complete to N_v. Since this is true for all branch-vertices v of H, x is major with respect to (S,N). ☐

As a result of this lemma, in this section, we speak of 'major' vertices without specifying whether they are major with respect to $L(H)$ or (S,N).

Lemma 3.3.6. *Let G be a diamond-free graph, let J be a 3-connected graph, let $L(H)$ be an appearance of J in G, and let $u, v \in V(J)$. Then there is at most one vertex that is complete to $\delta_H(u) \cup \delta_H(v)$.*

Proof. This follows from Lemma 3.1.1, since $\delta_H(u) \cup \delta_H(v)$ is neither a clique nor an anticlique. ☐

Lemma 3.3.7. *Let G be a diamond-free graph, let (S,N) be a maximal J-strip system in G, and let $L(H)$ be formed by some choice of rungs. Then for every $uv \in E(J)$, there is at most one vertex that overshadows B_{uv}. In particular, if there a vertex that is major with respect to (S,N) (or $L(H)$), then there is exactly one vertex that is major with respect to (S,N) (or $L(H)$), and $O(G) = \emptyset$.*

Proof. The first statement follows directly from Lemma 3.3.6. To show the second statement, we note that Lemmas 3.3.4 and 3.3.5 imply that a major vertex is complete to $\delta_H(u)$ (and N_u) for every branch vertex u. Hence Lemma 3.3.6 implies that there is at most one major vertex and $O(G) = \emptyset$. ☐

The bulk of the proof relies on the following structural theorem proven in [2], which says that all components of $G \setminus V(S,N)$ that are composed of minor vertices have local attachments.

Theorem 3.3.8 ([2] 8.5)**.** *Let G be perfect, let J be a 3-connected graph, and let (S,N) be a maximal J-strip system in G. Assume that there is no J-enlargement with a nondegenerate appearance in G. Assume moreover that if $J = K_4$ then (S,N) is nondegenerate and there is no overshadowed appearance of J in G. Let $F \subseteq V(G) \backslash V(S,N)$ be connected, such that no member of F is major with respect to (S,N). Then the set of attachments of F in (S,N) is local.*

To deal with overshadowing vertices, we will need the following lemma.

Lemma 3.3.9. *Let G be a diamond-free perfect graph, let J be a 3-connected graph, and let (S, N) be a maximal J-strip system in G. Assume that there is no J-enlargement with a nondegenerate appearance in G. Assume moreover that if $J = K_4$ then (S, N) is nondegenerate. Let $x \in O(G)$, and let $F \subseteq G\backslash(O(G) \cup V(S, N))$ be chosen such that some member of F is a neighbour of x, and no member of F is major with respect to (S, N). Then the set attachments of $F \cup \{x\}$ in (S, N) is semi-local.*

Proof. Suppose that the set X of attachments of $F \cup \{x\}$ in (S, N) is not semi-local. We may assume that $|F|$ is minimal such that this holds, and note that $F = \emptyset$ if the attachments of x are not semilocal. As x is overshadowing, there is some edge uv of J such that x is complete to both N_u and N_v. As x is not major and X is not semi-local, there exists an edge $u'v'$ of J such that, without loss of generality, x is not complete to $N_{v'}$ (and hence has at most one neighbour in $N_{v'}$), $u' \neq v$, and $(X \cap S_{u'v'})\backslash N_u \neq \emptyset$. Note that, by Theorem 3.3.8, the set X_F of attachments of F in (S, N) is local, and hence by the minimality of $|F|$, $X_F \subseteq S_{u'v'} \cup N_{u'} \cup N_{v'}$.

Let $K_{v'}$ be the maximal clique containing $N_{v'}$. We show that x can be linked to a triangle $T \subseteq K_{v'}$ via paths P'_0, P'_1, P'_2. Since x has at most one neighbour in $N_{v'}$, it has at most one neighbour in $K_{v'}$ and hence also in T, which gives a contradiction.

Let P_0 be a $x - K_{v'}$ path with interior in $(F \cup S_{u'v'})\backslash N_u$ such that $|V(P_0)|$ is minimal. As J is 3-connected, $J\backslash\{u'\}$ and $J\backslash\{e = u'v'\}$ are both 2-connected. If $V(P_0)$ meets $N_{u'}$, let $J' = J\backslash\{u'\}$, and otherwise let $J' = J\backslash\{u'v'\}$. Note that in the first case $u' \neq u$, since $(X \cap S_{u'v'})\backslash N_u \neq \emptyset$, and so $u \in J'$ in both cases. Also, it is clear that $v, v' \in J'$ in both cases. Hence by 2-connectivity, there exist vertex-disjoint paths P^J_1, P^J_2 from $\{u, v\}$ to v' in J'. Let P^J_i be given by $p^i_0 - p^i_1 - \cdots - p^i_{m_i}$ for $i = 1, 2$, where $p^1_0 = u$, $p^2_0 = v$, and $p^i_{m_i} = v'$ for $i = 1, 2$. For each edge $p^i_j p^i_{j+1}$ in P^J_i, let $R^i_j \subseteq V(S_{p^i_j p^i_{j+1}})$ be a $p^i_j p^i_{j+1}$-rung in (S, N), and let P_i be given by $R^i_0 - R^i_1 - \cdots - R^i_{m_i-1}$. Let the final vertex of P_i be v_i for $i = 1, 2, 3$, and let $T = \{v_0 v_1 v_2\}$.

By the definition of rungs, P_i is a path in G for $i = 0, 1, 2$, and by construction, $V(P_i) \cap K_{v'} = \{v_i\}$ for $i = 0, 1, 2$, and the first end of P_i is adjacent to x for $i = 1, 2$. Moreover, the P^J_i are vertex disjoint, and by construction if $V(P_0)$ meet $N_{u'}$ then $V(P^j_i)$ does not contain u' for $i = 1, 2$. Hence for each $w \in V(J)\backslash\{v'\}$, there is at most one $i \in \{0, 1, 2\}$ such that P_i meets N_w, and for each $e \in E(J)$, there is at most one $i \in \{0, 1, 2\}$ such that $V(P_i)$ meets S_e. Finally, $V(P_0) \cup V(P_1) \cup V(P_2) \subseteq V(S, N) \cup V(F) \cup \{x\}$, and only $V(P_0)$ meets $\{x\} \cup F \cup X_F$, so the only edges between $V(P_0), V(P_1)$ and $V(P_2)$ are those from x to $V(P_1) \cup V(P_2)$ and those in $T = \{v_0, v_1, v_2\}$. For $i = 1, 2$, let P'_i be the minimal $x - v_i$ path with interior $P^*_I \subseteq V(P_i)$, and let $P'_0 = P_0$. Then x can be linked onto the triangle T via paths P'_0, P'_1, P'_2, giving the required contradiction. ☐

We are now ready to prove the following structural decomposition theorem for graphs containing nondegenerate line graphs.

Proposition 3.3.10. *Let G be a diamond-free perfect graph, let J be a 3-connected graph, and let (S, N) be a maximal J-strip system in G. Assume that there is no J-enlargement with a nondegenerate appearance in G. Assume moreover that if $J = K_4$ then (S, N) is nondegenerate. Then G admits a clique cutset, or a complete 2-join, or a vertex-complete 2-join.*

Proof. Let M be the set of vertices that are major with respect to the strip system. Suppose first that there is a component F of $G\backslash(O(G) \cup M \cup V(S, N))$ whose set of attachments in (S, N) is contained in N_u for some branch-vertex u. We show that there is a clique containing N_u that is a clique cutset. Suppose $x \in O(G)$ is an overshadowing vertex with a neighbour in $F \cup N_u$. Then by Lemma 3.3.9, N_u is contained in the branch overshadowed by x, so x is complete to N_u. Hence $M \cup O(G) = O_1 \cup O_2$, where O_1 is complete to N_u and O_2 is anticomplete to $N_u \cup F$. As G is diamond-free, $O_1 \cup N_u$ is a clique by Lemma 3.1.1. As every path from F to $(V(S, N) \setminus N_u) \cup O_2$ has a vertex in $N_u \cup O_1$, $O_1 \cup N_u$ is a clique cutset.

Hence, for every $u \in V(J)$, no component of $G\backslash(M \cup O(G) \cup V(S, N))$ attaches only in N_u. Suppose now that $|M| \geq 0$. Then, by Lemma 3.3.7, $M = \{x\}$ for some vertex x and $O(G) = \emptyset$. Also, by Theorem 3.3.8, if F is a component of $G \setminus (V(S,N) \cup \{x\})$, then F has local attachments in (S,N), and so attaches in S_{uv} for some $uv \in E(J)$. Let $A_1 = N_{uv}, A_2 = N_{vu}, B_1 = N_u \setminus N_{uv}$ and $B_2 = N_v \setminus N_{vu}$. Let A be the union of $V(S_{uv})$ with all the components of $G \setminus (V(S,N) \cup \{x\})$ that attach in S_{uv}, and let $B = V(G) \setminus (A \cup \{x\})$. Then (A,B) is an x-complete multijoin. So we may assume that there are no major vertices.

Let $uv \in E(J)$, and let X be the set of vertices that overshadow the branch B_{uv}, which by Lemma 3.3.7 has size at most 1. Let K_u and K_v be the maximal cliques of $G \setminus X$ containing N_u and N_v respectively. It follows from Lemma 3.1.2 that X is complete to $K_u \cup K_v$. Let $A_1 = N_{uv}, A_2 = N_{vu}$, and let $B_1 = K_u \setminus N_{uv}$ and $B_2 = K_v \setminus N_{vu}$. Finally, let A be the union of $V(S_{uv})$ with all the components of $G \setminus (X \cup V(S,N) \cup K_u \cup K_v)$ whose set of attachments in (S,N) is contained in $V(S_{uv})$, and let $B = V(G) \setminus (X \cup A)$. Note that A des not contain any overshadowing vertices.

Suppose there is an overshadowing vertex $x \in O(G)$ with a neighbour $a \in A \setminus V(S,N)$. Let F be the component of $A \setminus V(S,N)$ containing a. Then by Lemma 3.3.9, the attachments of $F \cup \{x\}$ in (S,N) are semilocal, so $x \in X$. Hence, by Theorem 3.3.8, every path in $G \setminus X$ from A to B contains an edge with ends in A_i and B_i for some $i = 1,2$. Finally, $A_1 \cup B_1 = K_u$ and so $|A_1 \cup B_1| \geq 3$, and similarly $|A_2 \cup B_2| \geq 3$. Hence if $X = \emptyset$, then (A,B) with $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$ defines a complete 2-join. Otherwise, if $X = \{x\}$, then (A,B) is an x-complete 2-join. □

3.4 Degenerate Line Graphs

In this section, we prove Theorem 3.1 for graphs containing degenerate appearances of K_4. We follow the methods of [2], which view such graphs as maximal sets of strips and antistrips satisfying certain properties. We make this precise below.

Let $S = (A,C,B)$ be a strip and $T = (X,Z,Y)$ an antistrip, with $V(S) \cap V(T) = \emptyset$. We say S,T are *parallel* if: A is complete to $X \cup Z$, and B is complete to $Y \cup Z$; and X is anticomplete to $B \cup C$, and Y is anticomplete to $A \cup C$. We say S,T are *co-parallel* if S, T' are parallel, where T' is the reverse of T.

Now let S_1, S_2 be strips and T an antistrip, where S_1, S_2, T are pairwise disjoint. We say that S_1, S_2 *agree* on T if either the pairs S_1, T and S_2, T are both parallel, or both co-parallel; and they *disagree* if one pair is parallel and the other pair is co-parallel. If S is a strip and T_1, T_2 are antistrips, pairwise disjoint, we similarly define whether T_1, T_2 agree or disagree on S.

Now let S_1, S_2 be strips, and let T_1, T_2 be antistrips, all pairwise disjoint. We call the quadruple (S_1, S_2, T_1, T_2) a *twist* if S_1, S_2 agree on one of T_1, T_2 and disagree on the other (equivalently, if T_1, T_2 agree on one of S_1, S_2, and disagree on the other). Note that if (S_1, S_2, T_1, T_2) is a twist, then so is (S_1', S_2, T_1, T_2), where S_1' is the reverse of S_1.

Definition 3.4.1. *A* striation *in a graph G is a family of strips $\{S_i = (A_i, C_i, B_i)\}_{1 \leq i \leq m}$ together with a family of antistrips $\{T_j = (X_j, Z_j, Y_j)\}_{1 \leq j \leq n}$, satisfying the following conditions:*

- *all the strips and antistrips are pairwise disjoint, and all their rungs and antirungs have odd length;*
- *the integers m, n are at least 2;*
- *for $1 \leq i < i' \leq m$, S_i is anticomplete to $S_{i'}$, and for $1 \leq j < j' \leq n$, T_j is complete to $T_{j'}$;*
- *for $1 \leq i \leq m$ and $1 \leq j \leq n$, S_i and T_j are either parallel or co-parallel;*
- *for $1 \leq i < i' \leq m$ there exist distinct j, j' with $1 \leq j, j' \leq n$ such that $(S_i, S_{i'}, T_j, T_{j'})$ is a twist; and*
- *for $1 \leq j < j' \leq n$ there exist distinct i, i' with $1 \leq i, i' \leq m$ such that $(S_i, S_{i'}, T_j, T_{j'})$ is a twist.*

(Note that if we replace some (A_i, C_i, B_i) by its reverse (B_i, C_i, A_i), we obtain another striation.) We denote the striation by L, and the union of the vertex sets of all its strips and antistrips by $V(L)$.

If $L(H)$ is a degenerate appearance of K_4 in G, it can be viewed as a striation. For let $m = n = 2$, S_i, T_j be paths For suppose the degenerate appearance of K_4 has vertices $\{1,2,3,4\}$ and branches $B_{1,3}, B_{1,4}, B_{2,3}, B_{2,4}$ of length 0, where the branch $B_{i,j}$ has endpoints $r_{i,j}$ and $r_{j,i}$ respectively. Let $m = n = 2$, $S_1 = B_{1,2}, S_2 =$

$B_{3,4}$, let T_1 be the antipath $r_{1,3} - r_{2,4}$, and T_2 the antipath $r_{1,4} - r_{2,3}$. Then it is easy to check that this is a striation.

We say that a subset $X \subseteq V(L)$ is *local* with respect to L if:

- at most one of $X \cap V(S_1), \ldots, X \cap V(S_m)$ is nonempty;
- for $1 \leq j \leq n$, every T_j-antirung has a vertex not in X; and
- $X \cap (V(S_1) \cup \cdots \cup V(S_m))$ is complete to $X \cap (V(T_1) \cup \cdots \cup V(T_n))$.

We say X *resolves* L if $V(L) \backslash X$ is local with respect to the striation in $\overline{G}$ obtained from L by exchanging the strips and antistrips. Equivalently, X resolves L if:

- there is at most one of $T_1, \ldots, T_n$ that is not a subset of X;
- for $1 \leq i \leq m$, every S_i-rung meets X; and
- X contains at least one end of every edge between $V(S_1) \cup \cdots \cup V(S_m)$ and $V(T_1) \cup \cdots \cup V(T_n)$.

A striation L in G is *maximal* if there is no striation L' in G with $V(L) \subset V(L')$.

Let $L(H)$ be a degenerate appearance of K_4, which may be viewed either as a strip system (S, N) or as a striation L. We note that it is proven in [2] (result 9.2) that if a set X is local with respect to L if and only if it is local with respect to (S, N), and the neighbour set X of a vertex x resolves L if x is major with respect to (S, N) and X meets both $V(S_1)$ and $V(S_2)$.

We use the following structural theorems from [2].

Theorem 3.4.2 ([2] 9.4)**.** *Let G be perfect, such that there is no appearance in G or in $\overline{G}$ of any K_4-enlargement, and there is no overshadowed appearance of K_4 in G or in $\overline{G}$. Let L be a maximal striation in G. Let $x \in V(G) \backslash V(L)$, and let X be the set of neighbours of x in $V(L)$. Then either X is local with respect to L, or X resolves L.*

Theorem 3.4.3 ([2] 9.5)**.** *Let G be perfect, such that there is no appearance in G or in $\overline{G}$ of any K_4-enlargement, and there is no overshadowed appearance of K_4 in G or in $\overline{G}$. Let L be a maximal striation in G. Let $F \subseteq V(G) \backslash V(L)$ be such that $G|F$ is connected, and for each $f \in F$, the set of its neighbours in $V(L)$ is local with respect to L. Then the set of attachments of F in $V(L)$ is local with respect to L.*

These theorems essentially say that if G is perfect, does not contain an overshadowed appearance of K_4, and contains a maximal striation L, then we may view the rest of the vertices either as 'major' vertices whose neighbour sets resolve L, or 'minor' vertices, and that all components of 'minor' vertices have local attachments. If we impose the condition that G is diamond-free, the following lemmas show that all vertices not in $V(L)$ are 'minor', and that there are no appearances of K_4 that are both overshadowed and degenerate, allowing us to directly applying Theorems 3.4.2 and 3.4.3.

Lemma 3.4.4. *Let G be a diamond-free perfect graph, and let $L = (\cup_{i=1}^{m}(S_i = (A_i, C_i, B_i)), \cup_{j=1}^{n}(T_j = (X_j, Z_j, Y_j)))$ be a maximal striation in G. Let $x \in V(G) \backslash V(L)$ and let X be the neighbours of x in $V(L)$. Then X does not resolve L.*

Proof. Suppose that X resolves L. Let (S_1, S_2, T_1, T_2) be a twist and assume without loss of generality that S_1 is parallel to T_j for $j = 1, 2$, S_2 is parallel to T_1 and co-parallel to T_2, and $T_1 \subseteq X$.

We show first that $T_2 \cap X = \emptyset$. For suppose that $t \in T_2 \cap X$. Let $x_1 \in X_1, z_1 \in Z_1$ be non-adjacent vertices in T_1, and notice that by definition t is complete to $\{x, x_1, z_1\}$ and by assumption x is complete to $\{x_1, z_1\}$. Hence $\{t, x, x_1, z_1\}$ is a diamond in G, contradiction.

Let $x_i \in X_i$ for $i = 1, 2$ and let $a \in A_1$. Then by definition $\{a, x_1, x_2\}$ induces a triangle. In particular, ax_2 is an edge between $V(S_1) \cup \cdots V(S_m)$ and $V(T_1) \cup \cdots V(T_n)$ and $x_2 \notin X$, so since X resolves L, $a \in X$. But then $\{x, x_1, x_2, a\}$ is a diamond in G, contradiction. ◻

Lemma 3.4.5. *Let G be perfect, let $G^* \in \{G, \overline{G}\}$, and let $L(H)$ be a degenerate appearance of K_4 in G^*. Then $L(H)$ is not an overshadowed appearance of K_4 in G^*.*

Proof. Let v_1, v_2, v_3, v_4 be the branch-vertices of H, and let $v_1 - v_2 - v_3 - v_4 - v_1$ be a cycle in H. Suppose that there is a vertex $x \in V(G) \setminus L(H)$ that is overshadowing with respect to $L(H)$ in G^*. We may assume that x overshadows the branch with ends v_1 and v_3. But since H is bipartite, the branch with ends v_1, v_3 has even length, contradiction. □

The following lemma further details the structure of strips and antistrips in diamond-free graphs.

Lemma 3.4.6. *Let G be a diamond-free perfect graph, and let L be a maximal striation in G. Let $S_i = (A_i, C_i, B_i)$ be a strip, and let $T_j = (X_j, Z_j, Y_j)$ be an antistrip. Then A_i, B_i are cliques, $|X_j| = |Y_j| = 1$, and $Z_j = \emptyset$.*

Proof. Let (S_1, S_2, T_1, T_2) be a twist and assume without loss of generality that S_1 is parallel to T_1 and T_2, and S_2 is parallel to T_1 and co-parallel to T_2, and $T_1 \subseteq X$. It suffices to consider the case $i = j = 1$.

We show first that A_1, B_1 are cliques. Since A_1 is nonempty and complete to $X_1 \cup X_2$, which contains an edge, it follows from Lemma 3.1.1 that A_1 is a clique. By symmetry, B_1 is a clique.

We now show that $|X_1| = |Y_1| = 1$ and $|Z_1| = \emptyset$. As $X_1 \cup Z_1$ is nonempty and complete to $A_1 \cup A_2$, which contains a pair of non-adjacent vertices, it follows from Lemma 3.1.1 that $X_1 \cup Z_1$ is an anticlique. In addition, $X_1 \cup Z_1$ is complete to $X_2 \cup A_2$, which contains an edge, so it follows from Lemma 3.1.1 that $X_1 \cup Z_1$ is a clique. Hence $X_1 \cup Z_1$ is a single vertex, and as X_1 is nonempty, $|X_1| = 1$ and Z_1 is empty. By symmetry, $|Y_1| = 1$. □

We now prove the main result of this section.

Proposition 3.4.7. *Let G be a diamond-free perfect graph, such that every appearance of K_4 in G is degenerate. Then G admits a clique cutset, or G admits a complete 2-join.*

Proof. Let $L = (\cup_{i=1}^{m}(S_i = (A_i, C_i, B_i)), \cup_{j=1}^{n}(T_j = (X_j, Z_j, Y_j)))$ be a maximal striation in G. Let us assume that G does not admit a clique cutset or a complete multijoin. Note that, by Proposition 3.3.10, we may assume that there is no appearance in G of any K_4-enlargement. In addition, since every appearance of K_4 in G is degenerate, Lemma 3.4.5 implies that there is no overshadowed appearance of K_4 in G. Moreover, by Lemma 3.3.1, if $\overline{G}$ contains an appearance of J then G also contains an appearance of J, so we may assume that there is no appearance in $\overline{G}$ of any K_4-enlargement. Finally, Lemma 3.3.1 also implies that if there is an appearance $L(H)$ in $\overline{G}$ of K_4, then every branch B of H has length at most 2, and hence $L(H)$ is not an overshadowed appearance of K_4.

Let $F \subseteq V(G) \backslash V(L)$ induce a connected component of $G \backslash V(L)$. Then by Theorems 3.4.2, 3.4.3 and Lemma 3.4.4, the set X of attachments of F in $V(L)$ is local with respect to L. We may assume without loss of generality that $X \cap V(S_i) = \emptyset$ for all $i > 1$.

Suppose first that X meets $\cup_j V(T_j)$. We show that X induces a clique in G, and hence is a clique cutset. Let $X_S = X \cap V(S_1)$, and let $X_T = X \cap (\cup_j V(T_j))$. As X is local, every T_j-antirung has a vertex not in X. Moreover, by Lemma 3.4.6, $|T_j| = 2$ for all j, so since T_j is complete to $T_{j'}$ for all $j \neq j'$, X_T is a clique. Moreover, X_T is complete to X_S, so either $X_S \subseteq A_1$ or $X_S \subseteq B_1$. In either case, by Lemma 3.4.6, X_S is contained in a clique, so X is a clique cutset, contradiction.

Hence we may assume that for any component of $G \backslash V(L)$ induced by $F \subseteq V(G) \backslash V(L)$, there exists an i such that the attachments of F are contained in S_i. We show that G admits a complete 2-join (M, N). Let M be the union of S_1 with all the components of $G \backslash V(L)$ with attachments in S_1, and let $N = G \backslash M$. By assumption, the only edges between M and N are the edges between S_1 and $V(L) \backslash S_1$. But these are precisely the edges between S_1 and $\cup_j T_j$, and by Lemma 3.4.6 there exists a labeling $\{x_j, y_j\}_{j=1}^{n}$ such that $T_j = \{x_j, y_j\}$, A_1 is complete to $\{x_j\}_{j=1}^{m}$, B_1 is complete to $\{y_j\}_{j=1}^{n}$, and there are no other edges beteween S_1 and $V(L) \backslash S_1$. Let $M_1 = A_1, M_2 = B_1, N_1 = \{x_j\}_{j=1}^{n}$ and $N_2 = \{y_j\}_{j=1}^{n}$. Since, by Lemma 3.4.6, M_1, M_2 are disjoint cliques in M and $m, n \geq 2$, it follows that (M, N) is a complete 2-join of G. □

This proposition, together with Proposition 3.3.10, shows the following.

Proposition 3.4.8. *Let G be a diamond-free perfect graph, and let J be a 3-connected graph. Suppose that there is an appearance of J in either G or $\overline{G}$. Then G admits a clique cutset, or a complete 2-join, or a vertex-complete 2-join.*

Proof. We show the result concurrently for all 3-connected J'. If there is an appearance of J' in $\overline{G}$, then Lemma 3.3.1 implies that there is also an appearance of J' in G, so it suffices to consider appearances in G.

Let $L(H)$ be an appearance in G of some 3-connected J', where $L(H), J'$ are chosen so that the appearance is nondegenerate, if possible. We may assume that there is no J-enlargement with a nondegenerate appearance in G. If the appearance of J' is nondegenerate or $J \neq K_4$, then by Proposition 3.3.10 the result holds. Hence we may assume that if J' appears in G, then $J' = K_4$ and the appearance is degenerate, and so the result holds by Proposition 3.4.7. This completes the proof. □

3.5 Prisms

We move now to the structure of graphs that do not contain an appearance of K_4, but do contain a prism. The pertinent definitions are as follows.

A *prism* is a graph consisting of two vertex-disjoint triangles $\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}$, and three paths R_1, R_2, R_3, where each R_i has ends a_i, b_i, and for $1 \leq i < j \leq 3$ the only edges between $V(R_i)$ and $V(R_j)$ are a_ia_j and b_ib_j. The three paths R_1, R_2, R_3 are said to *form* the prism. The prism is *long* if at least one of the three paths has length greater than 1, and it is *short* otherwise. It is clear that if R_1, R_2, R_3 form a prism in a perfect graph, then they all have the same parity. If R_1 is odd, we say that the prism is *odd*, and otherwise we say that the prism is *even*.

A subset $X \subseteq V(G)$ *saturates* the prism if at least two vertices of each triangle belong to X; and a vertex is *major* with respect to the prism if its neighbour set saturates it. A subset $X \subseteq V(K)$ is *local* with respect to the prism if either $X \subseteq V(R_i)$ for some i, or X is a subset of one of the triangles.

We will consider three types of prisms: short odd prisms; even prisms; and long odd prisms. The first two types are relatively easy to handle, and we deal with them in this section. We analyze long odd prisms in the next section.

Proposition 3.5.1. *Let G be a diamond-free perfect graph containing a short odd prism. Then G is the line graph of a bipartite graph, or the complement of a bipartite graph, or G contains a clique cutset, or G contains a long odd prism or a degenerate appearance of K_4.*

Proof. Let $A = \{a_1, \dots, a_m\}$, $B = \{b_1, \dots, b_m\}$ be cliques in G, chosen such that a_i is adjacent to b_j if and only if $i = j$, and m is maximal given these constraints. As G contains a short odd prism, $m \geq 3$.

(1) Suppose there exist non-adjacent $a \in A$, $b \in B$ and an $a - b$ path with interior in $G \backslash (A \cup B)$. Then G contains a long odd prism or a degenerate appearance of K_4.

Let $a - P - b$ be a path with ends a, b and $P^* \subseteq G \backslash (A \cup B)$, and let $u, v \in P^*$ be adjacent to a and b respectively. We may assume that a, b and P are chosen so that $|V(P)|$ is minimal, and $a = a_1, b = b_2$. By the minimality of $|V(P)|$, at most one of a_2, b_1 has neighbours in $P^* \setminus \{u, v\}$, and if $i \neq 2$ then a_i does not have neighbours in $P^* \setminus \{u\}$, and if $j \neq 1$ then b_j does not have neighbours in $P^* \setminus \{v\}$. As A and B are cliques of size at least 3 and G is diamond-free, by Lemma 3.1.2, u either has exactly one neighbour in A or is complete to A, and v either has exactly one neighbour in B or is complete to B. Also, as G is diamond-free, if a vertex is complete to A, then it has no neighbours in B, and vice versa. In particular, if $u = v$, then u has exactly one neighbour in each of A and B, so $\{a_1, u, b_2, b_3, a_3\}$ is a C_5, contradiction. Hence $u \neq v$.

Suppose that u is complete to A. Then u is anticomplete to B, and v is anticomplete to $A \setminus \{a_2\}$. If v has exactly one neighbour in B, then the paths $a_1 - b_1, u - P^* - v - b_2, a_3 - b_3$ form a long odd prism with triangles $\{a_1, u, a_3\}, \{b_1, b_2, b_3\}$. Thus we may assume that v is complete to B. If uv is an edge, then u and v can be added to A and B respectively, contradicting the maximality of A and B. Hence $|P^*| \geq 3$, so the paths induced by $a_1 - b_1, u - P^* - v$ and $a_2 - b_2$ form a long odd prism with triangles $\{a_1, u, a_2\}, \{b_2, v, b_2\}$. By symmetry, we may assume that u and v each have exactly one neighbour in A and B respectively.

Suppose a_2 has no neighbours in P^*. Then b_2 can be linked onto the triangle a_1, a_2, a_3 via $b_2 - P - a_1$, $b_2 - a_2$ and $b_2 - b_3 - a_3$, contradicting Lemma 3.1.5. So a_2 has neighbours in P^*, and by symmetry, b_1 also has neighbours in P^*. Hence, since $|V(P)|$ is minimal, the only edges between $\{a_2, b_1\}$ and P^* are a_2v and b_1u. Then, since $m \geq 3$, it holds that G contains a degenerate appearance of K_4. This completes the proof of (1).

Let F be a connected component of $G\backslash(A \cup B)$ and let X be the set of its attachments in $A \cup B$.

(2) If G does not contain a long odd prism or a degenerate appearance of K_4, then $X \subseteq A$, or $X \subseteq B$, or $X = \{a_i, b_i\}$ for some i.
Suppose $X \not\subseteq A$ and $X \not\subseteq B$. By (1), if $a \in X \cup A$ and $b \in X \cup B$ then a and b are adjacent. It follows that $X = \{a_i, b_i\}$ for some i, proving (2).

Consider A, B as defined above. If $G\backslash(A \cup B)$ has some connected component F with set of attachments X in $(A \cup B)$, then by (2) X is contained in a clique, and so is a clique cutset. Otherwise $V(G) = A \cup B$ and G is the complement of a bipartite graph. This proves the proposition. □

We now move to the case of the even prism.

Definition 3.5.2. *A* hyperprism *is a collection of nine sets $\{A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3\}$ satisfying the following properties:*

- *all the A_i, B_i, C_i are nonempty and pairwise disjoint;*
- *for $1 \leq i \leq 3$, $S_i = (A_i, C_i, B_i)$ is a strip;*
- *for $1 \leq i < j \leq 3$, A_i is complete to A_j and B_i is complete to B_j, and there are no other edges between S_i and S_j; and*
- *some path between A_1 and B_1 with interior in C_1 is even.*

Let $A = A_1 \cup A_2 \cup A_3$, $B = B_1 \cup B_2 \cup B_3$, $H = \cup_i S_i$ and $V(H) = A \cup B \cup C$. A subset $X \subseteq V(H)$ is *local* with respect to the hyperprism H if X is a subset of one of the $V(S_i)$, or of A or B. We note that in a diamond-free graph, both A and B are cliques. Moreover, it is clear that in a hyperprism H, for $1 \leq i \leq 3$, all rungs in S_i have even length.

The following lemma was proven in [2], and will be useful in showing that every diamond-free perfect graph that contains an even prism is either basic, or admits either a clique cutset or a complete 2-join.

Lemma 3.5.3 ([2], 10.6 (ii))**.** *Let G be a perfect graph such that there is no nondegenerate appearance of K_4 in G. Suppose G contains a maximal hyperprism H. For any connected subset F of $V(G)\backslash V(H)$, if no vertex of F is major with respect to some prism $K \subseteq H$ with triangles in A and B respectively, then the set of attachments of F in H is local.*

Our main result for graphs containing an even prism is as follows.

Proposition 3.5.4. *Let G be a diamond-free perfect graph, such that there is no nondegenerate appearance of K_4 in G. If G contains an even prism, then G admits one of the following:*

- *a clique cutset; or*
- *a complete 2-join; or*
- *a vertex-complete 2-join.*

Proof. Let $H = \cup_{i_1}(S_i = (A_i, C_i, B_i))$ be a maximal hyperprism in G, and let Y be the set of vertices in G that are complete to $A \cup B$.

(1) If $v \notin V(H)$ is major with respect to some prism $K \subseteq H$ with triangles in A and B respectively, then $v \in Y$.

If v is major with respect to some prism $K \subseteq H$ with triangles in A and B respectively, then v has at least two neighbours in each of A and B. Since G is diamond-free, and A and B are both cliques, v is complete to $A \cup B$, which proves (1).

(2) $|Y| \leq 1$.

Since G is diamond-free, Y is complete to $A \cup B$. As $A \cup B$ contains both a pair of adjacent vertices and a pair of non-adjacent vertices, by Lemma 3.1.1, it holds that $|Y| \leq 1$, which proves (2).

Let F be a connected subset of $V(G)\backslash(V(H) \cup Y)$, and let X be its set of attachments in H. By (1), no vertex of F is major with respect to some prism $K \subseteq H$ with triangles in A and B respectively. Hence, by Lemma 3.5.3, X is local. Also if $X \subseteq A$, then $A \cup Y$ is a clique cutset, and similarly if $X \subseteq B$, then $B \cup Y$ is a clique cutset. Thus, by Lemma 3.5.3, we may assume that all components of $V(G)\backslash(V(H) \cup Y)$ have attachments in at most one of the S_i.

Let K_A and K_B be the maximal cliques of G containing A and B respectively. Let M be the union of S_1 with all the components of $V(G) \setminus (V(H) \cup Y)$ with attachments in S_1, and let $N = V(G)\backslash M$, $M_1 = A_1$, $M_2 = B_1$, $N_1 = K_A \backslash M_1$ and $N_2 = K_B \backslash M_2$. If $Y = \emptyset$, then since $|K_A|, |K_B| \geq 3$, (M, N) is a complete 2-join of G. Otherwise, $Y = \{y\}$, and (M, N) is a y-complete 2-join. □

3.6 Long Odd Prisms

In this section, we consider long odd prisms as instances of a type of graph that we call a 'staircase', and show that all diamond-free perfect graphs that do not contain appearances of K_4 and even prisms admit a clique cutset, a complete multijoin or a vertex-complete multijoin. We first give the precise definition for a 'staircase' and its various components.

Let $S = (A, C, B)$ be a strip in G. A *step* is a pair $a_1 - R_1 - b_1, a_2 - R_2 - b_2$ of rungs such that: $V(R_1) \cap V(R_2) = \emptyset$; and a_1 is adjacent to a_2, b_1 is adjacent to b_2, and there are no other edges between $V(R_1)$ and $V(R_2)$. The edges a_1a_2 and b_1b_2 such that there exists a step as above are called *stepped* edges. We say that S is *step-connected* if every vertex of $V(S)$ is in a step, and for every partition (X, Y) of A or of B into two nonempty sets, there is a step R_1, R_2 such that R_1 has an end in X and R_2 has an end in Y .

Let $S = (A, C, B)$ be a step-connected strip in a perfect graph G. A vertex $v \in V(G) \setminus V(S)$ is a *left-star* for the strip if it is complete to A and anticomplete to $B \cup C$, and it is a *right-star* if it is complete to B and anticomplete to $A \cup C$. A *banister* (with respect to the strip) is a path $a - R - b$ of $G \setminus V(S)$, such that a is a left-star, b is a right-star, and there are no edges between R^* and $V(S)$. (Here, a and b are the endpoints of R, and we distinguish between $a - R - b$ and $b - R - a$; we follow the convention that when describing a banister relative to a strip, the end which is the left-star is listed first.) A banister can have length 1.

If $S = (A, C, B)$ is a step-connected strip in G, and $a_0 - R_0 - b_0$ be a banister of length ≥ 3, we call the pair $K = (S, R_0)$ a *staircase*, and define $V(K) = V(R_0) \cup V(S)$. (For brevity we let the staircase $K = (S = (A, C, B), a_0 - R_0 - b_0)$ mean that $K = (S, R_0)$ is a staircase, $S = (A, C, B)$, and R_0 has ends a_0, b_0, where a_0 is a left-star and b_0 is a right-star.) The staircase is *maximal* if there is no staircase $(S' = (A', C', B'), a'_0 - R'_0 - b'_0)$ such that $A \subseteq A', B \subseteq B', C \subseteq C'$ and $V(S) \subsetneq V(S')$.

Let $K = (S = (A, C, B), a_0 - R_0 - b_0)$ be a staircase in G. A subset $X \subseteq V(K)$ is *local* with respect to K if X is a subset of one of $V(S), V(R_0), A \cup \{a_0\}, B \cup \{b_0\}$. A vertex $v \in V(G)\backslash V(K)$ is: *minor* with respect to K if the set of its neighbours in $V(K)$ is local; and *major* with respect to K if it has neighbours in all of A, B and $V(R_0)$. A vertex $v \in V(G)\backslash V(K)$ is: *left-diagonal* with respect to K if v is $(A \cup \{b_0\})$-complete; *right-diagonal* with respect to K if it is $(B \cup \{a_0\})$-complete; and *central* with respect to K if it is $(A \cup B)$-complete, and is non-adjacent to both a_0 and b_0.

We say a staircase $K = (S = (A, C, B), a_0 - R_0 - b_0)$ is *strongly maximal* if it is maximal, and in addition, either $C \neq \emptyset$, or there is no staircase (S', R') in $\overline{G}$ with $V(S) \subsetneq V(S')$.

Diamond-free staircases have a particular structure, which we describe in the following lemmas.

Lemma 3.6.1. *Let G be a diamond-free perfect graph, and let $K = (S = (A, B, C), a_0 - R_0 - b_0)$ be a staircase in G. Then A and B are cliques.*

Proof. We show first that A is a clique. As A is complete to $\{a_0\}$, it follows from Lemma 3.1.1 that A is a disjoint union of cliques. But S is a step-connected strip, and so A is connected. Hence A is a clique. By symmetry, B is a clique. □

Lemma 3.6.2. *Let G be a diamond-free perfect graph, let $K = (S = (A, B, C), a_0 - R_0 - b_0)$ be a staircase in G. Then there are no central vertices with respect to K.*

Proof. Suppose that $v \in V(G) \setminus V(K)$ is central with respect to K. Then $\{v, a_0\}$ is an anticlique and is complete to A, which is a clique of size at least 2, contradicting Lemma 3.1.1. □

In tracing the decomposition for long odd prisms found in [2], we will also use some of the constructions and results in [2] as black boxes. The relevant definitions and results are as follows.

Definition 3.6.3. *A triple (S, F, Q) is called a* 1-breaker *in G if it satisfies the following:*

- *$S = (A, C, B)$ is a step-connected strip in G;*
- *$F \subseteq V(G) \backslash V(S)$ is connected, such that there are no edges between F and $V(S)$, and there is a left-star and a right-star, both with neighbours in F;*
- *$Q \subseteq V(G) \backslash (V(S) \cup F)$ is anticonnected;*
- *some vertex in A has a nonneighbour in Q, and so does some vertex in B;*
- *every vertex in Q has a neighbour in F and a neighbour in $A \cup B \cup C$;*
- *some left-star with a neighbour in F is Q-complete;*
- *no vertex in Q is a left-star.*

This section will be structured as follows. We first show that if G is perfect and diamond-free, then it does not contain a 1-breaker. We then show that if G does not contain a 1-breaker, an appearance of K_4 or an even prism, then it admits a clique cutset, or complete 2-join, or a vertex-complete 2-join. To show that no diamond-free perfect graph contains a 1-breaker, we will need the following lemma from [2].

Lemma 3.6.4 ([2] 11.5.2)**.** *Let G be a diamond-free perfect graph, such that there is no appearance of K_4 in G and no even prism in G. Let (S, F, Q) be a 1-breaker in G such that $|F| + |Q|$ is maximum. Then there is no left- or right-star in Q, and every left- or right-star with a neighbour in F is Q-complete.*

We now prove our result for 1-breakers in diamond-free perfect graphs.

Lemma 3.6.5. *Let G be a diamond-free perfect graph, such that there is no appearance of K_4 in G and no even prism in G. Then there is no 1-breaker in G.*

Proof. Suppose that G contains a 1-breaker (S, F, Q). We may assume that $|F| + |Q|$ is maximum. Let l be a left-star in G with a neighbour in F, and let r be a right-star in G with a neighbour in F. Then, by Lemma 3.6.4, l and r are both Q-complete. Since $A \cup \{l\}$ and $B \cup \{r\}$ are cliques, Q is complete to $\{l, r\}$, and G is diamond-free, Lemma 3.1.1 implies that Q is an anticlique, and Lemma 3.1.2 implies that every vertex $q \in Q$ is either complete or anticomplete to A, and either complete or anticomplete to B.

We show now that there is a vertex $q \in Q$ that is anticomplete to A and has a neighbour in C. By the definition of a 1-breaker, there is a vertex in A with a nonneighbour $q \in Q$. Then q is anticomplete to A. Suppose that q does not have a neighbour in C. Since $q \in Q$, q has a neighbour in $A \cup B \cup C$, so q has a neighbour in B. Hence $q \in Q$ is complete to B and a right-star, which contradicts Lemma 3.6.4.

Let $c \in C$ be a neighbour of q, let $a_1 - R_1 - b_1, a_2 - R_2 - b_2$ be a step such that $c \in R_1$, and consider the triangle induced by $T = \{a_1, a_2, l\}$. For $i = 1, 2$, let P_i be a shortest $q - a_i$ path with interior in $V(R_i) \cup \{r\}$, and let P_3 be the edge ql. Then q can be linked to T via the paths P_1, P_2, P_3. Moreover, q is adjacent to l and anticomplete to $\{a_1, a_2\}$, which gives the required contradiction. □

We now consider the attachments of components to a staircase in a diamond-free perfect graph with no 1-breaker. The results from [2] for general perfect graphs are as follows.

Lemma 3.6.6 ([2] 12.1)**.** *Let G be a perfect graph, such that there is no appearance of K_4 in G, no even prism in G, and no 1-breaker in G. Let $K = (S = (A, C, B), a_0 - R_0 - b_0)$ be a maximal staircase in G, and let $v \in V(G) \backslash V(K)$. Then exactly one of the following holds:*

1. *v is minor, and in that case, either v is a left-star or v is not A-complete, and either v is a right-star or v is not B-complete;*
2. *v is major, and in that case, it is either left- or right-diagonal or central;*
3. *v is a left-star with a neighbour in $R_0 \backslash \{a_0\}$, or a right-star with a neighbour in $R_0 \backslash \{b_0\}$.*

Lemma 3.6.7 ([2] 12.3)**.** *Let G be a perfect graph, such that there is no appearance of K_4 in G, no even prism in G, and no 1-breaker in G. Let $K = (S = (A, C, B), a_0 - R_0 - b_0)$ be a maximal staircase in G, and let $F \subseteq V(G) \backslash V(S)$ be connected, containing a left-star and with an attachment in $B \cup C$. (Note that F may intersect $V(R_0)$.) Then F contains either a major vertex or a banister.*

Our decomposition theorem in the diamond-free case is as follows.

Proposition 3.6.8. *Let G be a diamond-free perfect graph, such that there is no appearance of K_4 in G, no even prism in G and no 1-breaker in G. Suppose that G contains a long odd prism. Then G admits a clique cutset or a complete 2-join or a vertex-complete 2-join.*

Proof. Let $K = (S = (A, C, B), a_0 - R_0 - b_0)$ be a maximal staircase in G, let A_0 and B_0 be the sets of left- and right-stars respectively, let Y be the set of major vertices with respect to K, and let $v \in V(G) \backslash (V(S) \cup A_0 \cup B_0 \cup Y)$. By Lemma 3.6.6, either $v \in R_0$, or v is minor and neither A-complete nor B-complete.

(1) Y is complete to $A \cup A_0 \cup B \cup B_0$, and $|Y| \leq 1$.

Let $y \in Y$. By Lemmas 3.6.6 and 3.6.2, y is either left- or right-diagonal. Suppose that y is left-diagonal. Then y is complete to $A \cup \{b_0\}$ where $|A| \geq 3$ and $b_0 \in B_0$, and y has a neighbour in B. Hence y has at least two neighbours in each of the cliques $A \cup A_0$ and $B \cup B_0$, so by Lemma 3.1.2 y is complete to both cliques. Hence Lemma 3.1.1 implies that $|Y| \leq 1$. This proves (1).

Let $F \subseteq V(G) \backslash (V(S) \cup A_0 \cup B_0 \cup Y)$ induce a component of $G \backslash (V(S) \cup A_0 \cup B_0 \cup Y)$, and let X be the attachments of F in $V(S) \cup A_0 \cup B_0$. If $X \subseteq A_0 \cup A$, then $A_0 \cup A \cup Y$ is a clique cutset. Hence we may assume that $X \not\subseteq A_0 \cup A$, and similarly $X \not\subseteq B_0 \cup B$. Suppose $X \not\subseteq V(S)$. We may assume without loss of generality that X meets A_0. Then, by Lemma 3.6.7, X does not meet $B \cup C$. Morever, since $X \not\subseteq A_0 \cup A$, X meets B_0, and so X does not meet $A \cup C$. Hence, either $X \subseteq V(S)$, or $X \cap V(S) = \emptyset$.

Let M be the union of $V(S)$ with all the components of $G \backslash (V(S) \cup A_0 \cup B_0 \cup Y)$ that have attachments in S, let $N = V(G) \backslash M$, let $M_1 = A$, $M_2 = B$, $N_1 = A_0$ and $N_2 = B_0$. Then $M_1, M_2 \subseteq M$ are disjoint cliques, $N_1, N_2 \subseteq N$ are disjoint cliques, $R_0 \subseteq N$, M_1 is complete to N_1 and M_2 is complete to N_2, and these are the only edges between M and N. If $Y = \emptyset$, then (M, N) is a complete 2-join of G. Otherwise, by (1), $Y = \{y\}$ is complete to $M_1 \cup M_2 \cup N_1 \cup N_2$, and so (M, N) is a y-complete 2-join. □

We close this section by noting that every long odd prism contains the complement of a diamond. In particular, if G is diamond-free, then $\overline{G}$ does not contain a long odd prism as an induced subgraph.

Lemma 3.6.9. *Let G be a graph, and let R_1, R_2, R_3 form a long odd prism or an even prism in G with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$. Then $\overline{G}$ contains a diamond as an induced subgraph.*

Proof. Let us assume that $|V(R_1)| \geq |V(R_2)| \geq |V(R_3)|$. If R_1, R_2, R_3 form a long odd prism, then $|V(R_1^*)| \geq 2$. Let $u, v \in V(R_1^*)$ be adjacent. Then $\{u, a_2, v, b_3\}$ is a diamond in $\overline{G}$. If R_1, R_2, R_3 form an even prism, then $|V(R_i^*)| \geq 1$ for all i. Let $v_i \in V(R_i^*)$ for $i = 1, 2, 3$ and let $v_0 \in V(R_1)$ be adjacent to v_1. Then $\{v_0, v_2, v_1, v_3\}$ is a diamond in $\overline{G}$. □

In conclusion, we have proven the following statement for diamond-free perfect graphs containing prisms.

Proposition 3.6.10. *Let G be a diamond-free perfect graph, such that there is no appearance of K_4 in G. Suppose that either G or $\overline{G}$ contains a long odd prism or an even prism, or G contains a short prism. Then G admits a clique cutset or a complete 2-join or a vertex-complete 2-join.*

Proof. By Lemma 3.6.9, we may assume that G contains a prism. Proposition 3.4.8 also implies that we may assume that there is no appearance of any 3-connected graph in G. Hence, by Proposition 3.5.1 we may assume that G contains either an even prism or a long odd prism, and by Proposition 3.5.4 we may assume that G does not contain an even prism. Thus Propositions 3.6.5 and 3.6.8 show that G admits a clique cutset or a complete 2-join or a vertex-complete 2-join. □

3.7 Wheels

In this section, we show that if neither G nor $\overline{G}$ contain an appearance of K_4 or a long odd prism, G does not contain an even prism, and G contains a wheel, then appropriately chosen wheels give rise to either a clique cut-set, or a vertex-complete multijoin.

A *wheel* in a graph G is a pair (C, Y), satisfying:

- C is a hole of length ≥ 6;
- Y is a nonempty anticonnected set disjoint from C; and
- There are two disjoint Y-complete edges of C.

We call C the *rim* and Y the *hub* of the wheel. We note that since there are two disjoint Y-complete edges of C, in a diamond-free graph $Y = \{y\}$ for some vertex y. Hence we will refer to wheels in diamond-free graphs as pairs (C, y) instead of $(C, \{y\})$, and speak of y-complete sets instead of Y-complete sets.

Recall that for a path $P \subseteq V(G) \backslash \{y\}$, the *$y$-parity* of P is the parity of the number of y-complete edges in P. Let us say that distinct vertices u, v of the rim of a wheel (C, y) have the *same wheel-parity* if there is a path of the rim joining them that has even y-parity, and *opposite wheel-parity* otherwise. Note that, by Lemma 3.1.3, both $u - v$ paths in C have the same y-parity.

As stated before, we are interested in the ways in which vertices can be connected to a wheel. The following lemma shows that every connected subset of $G \backslash (V(C) \cup \{y\})$ either contains a y-complete vertex, or attaches to vertices on the rim C in a specific manner.

Lemma 3.7.1. *Let G be a diamond-free perfect graph, such that there is no appearance of K_4 in G or $\overline{G}$, no even prism in G, and no long odd prism in G or $\overline{G}$. Let (C, y) be a wheel in G, and let $F \subseteq V(G) \backslash (V(C) \cup \{y\})$ be connected, such that y has no neighbours in F. Let $X \subseteq V(C)$ be the set of attachments of F in C. Then either X are the two ends of a y-complete edge, or all the vertices in X have the same wheel-parity.*

The proof of this lemma follows immediately from a result for perfect graphs proved in [2], which, in turn, requires an additional definition. We present the definition and result below.

Definition 3.7.2. *A* double diamond *is a graph with eight vertices $a_1, \ldots, a_4, b_1, \ldots, b_4$ and with the following adjacencies:*

- *a_i, a_j are adjacent if and only if $\{i, j\} \neq \{3, 4\}$;*
- *b_i, b_j are adjacent if and only if $\{i, j\} \neq \{3, 4\}$; and*
- *a_i, b_i are adjacent for all $1 \leq i \leq 4$.*

We note that a double diamond, by definition, contains two diamonds as vertex disjoint induced subgraphs.

Lemma 3.7.3 ([2] 16.2)**.** *Let G be a perfect graph, such that there is no appearance of K_4 in G or $\overline{G}$, no double diamond or even prism in G, and no long odd odd prism in G or $\overline{G}$. Let (C, Y) be a wheel in G, and let $F \subseteq V(G)\backslash(V(C) \cup Y)$ be connected, such that no vertex in F is Y-complete. Let $X \subseteq V(C)$ be the set of attachments of F in C. Suppose that there exist vertices in X with opposite wheel-parity, and there are two vertices in X that are non-adjacent. Then either:*

1. *there is a vertex $v \in F$ such that $(C, Y \cup \{v\})$ is a wheel; or*

2. *there is a vertex $v \in F$ with at least four neighbours in C, and a 3-vertex path $p_1 - p_2 - p_3$ in C, such that p_1, p_2, p_3 are all $Y \cup \{v\}$-complete, and every other neighbour of v in C has the same wheel-parity as p_1; or*

3. *the vertices of C may be numbered as $p_1, \dots, p_n$ in order, such that p_1, p_2, p_3 are all Y-complete, and there is a path $p_1 - f_1 - \dots - f_k - p_3$ with interior in F, such that there are no edges between $\{f_1, \dots, f_k\}$ and $\{p_4, \dots, p_n\}$.*

Proof of Lemma 3.7.1. Suppose G is a diamond-free perfect graph, such that there is no appearance of K_4 in G or $\overline{G}$, no even prism in G, and no long odd prism in G or $\overline{G}$. Then, as G is diamond-free, G has no double diamond, so G satisfies the assumptions of Lemma 3.7.3.

Let (C, y) be a wheel in G, and let $F \subseteq V(G)\backslash(V(C) \cup \{y\})$ be connected, such that no vertex in F is y-complete. Let $X \subseteq V(C)$ be the set of attachments of F in C. Suppose that X is not a y-complete edge of C, and two vertices in X have opposite wheel-parity. Then $(C, y), F$ and X satisfy the assumption in Lemma 3.7.3, so one of the three conclusions of the lemma holds. In the first case, there is a vertex $v \in F$ such that $(C, \{y, v\})$ is a wheel. Then $|\{y, v\}| \geq 2$ is the hub of a wheel, so G contains a diamond, contradiction. In the second and third cases, y is complete to $p_1 - p_2 - p_3$, so $\{p_1, p_2, p_3, y\}$ is a diamond, contradiction. □

In the rest of this section, we fix a vertex y that is the hub of some wheel (C, y), and consider the paths between maximal cliques containing y. If C is a hole, $uv \in E(C)$ and u, v are the only neighbours of y in $V(C)$, then we say that (C, uv) is a *y-anchored cycle* with interior $C^* = V(C) \setminus \{u, v\}$. Roughly speaking, we show that for every edge uv that is complete to y, uv is either in a wheel with hub y, or in a y-anchored cycle, but not both. In order to to prove this, we study tracks whose endpoints are the endpoints of a y-complete edge in a wheel with hub y. We will need the following definition.

Definition 3.7.4. *A quintuple $(y, K, C, P = \cup_{i \in [l]} P_i, F)$ is an* l-rack *if it satisfies the following:*

- *(C, y) is a wheel in G;*
- *$K \cup \{y\}$ is a maximal clique containing y;*
- *$(P_i, p_{i-1}p_i)$ is a y-anchored cycle for $1 \leq i \leq l$;*
- *$V(C) \cap K = \{p_0, p_l\}$*
- *if we let $P_0^* := V(C) \setminus \{p_0, p_l\}$, then $P_i^* \cap P_j^* = \emptyset$ and P_i^* is anticomplete to P_j^* for all $i \neq j$;*
- *for every $v \in V(G)\backslash(K \cup V(C) \cup V(P))$, there is at most one i such that v has neighbours in P_i^*;*
- *$F = f_1 - \dots - f_m$ is a path in $G \setminus (\{y\} \cup K \cup V(C) \cup V(P))$, possibly empty; and*
- *f_1 has neighbours in $V(P)$, f_m has neighbours in $C\backslash K$, and there are no other edges between $V(F)$ and $(\cup_i P_i^*) \cup (C\backslash K)$.*

We say that the l-rack is *secured* if $F \neq \emptyset$ and f_1 has attachments in $\cup_i P_i^*$. We call y the *hub* and C the *rim* of the wheel, K the *base* and P_i the *loops* of the rack, l the number of loops, and F the *lock*. For brevity, we let $P^* = \cup_i P_i^*$. We let $C = v_0 - v_1 - v_2 - \dots - v_k - v_{k+1} - v_0$, where $v_0 = p_0, v_{k+1} = p_l$, and let j, j' be respectively minimal and maximal such that f_m is adjacent to v_j and $v_{j'}$.

We remind the reader that if $v \notin K$ and G is diamond-free, then v has at most 1 neighbour in K. Hence, if (y, K, C, P, F) is an l-rack in a diamond-free graph, then $|P_i^*| \geq 2$ for all i.

Lemma 3.7.5. *Let G be a diamond-free perfect graph, such that there is no appearance of K_4 in G or $\overline{G}$, no even prism in G, and no long odd prism in G or $\overline{G}$. Suppose that G contains an l-rack. Then either G contains a secured l-rack, or G admits a clique cutset.*

Proof. Let (y, K, C, P, F) be an l-rack in G. Suppose that $K \cup \{y\}$ is not a clique cutset. Then there is a connected component of $V(G)\backslash(V(C) \cup K \cup \{y\} \cup P)$ with attachments in both $\cup_i P_i^*$ and $C\backslash K$. Let F' be a minimal connected subgraph of this component with attachments in both $\cup_i P_i^*$ and $C\backslash K$. Then (y, K, C, P, F') is a secured l-rack in G. □

The two technical lemmas of this section will show that if a diamond-free perfect graph without appearances of K_4 or prisms contains an l-rack, then it admits a clique cutset. It follows that tracks between vertices on a wheel of opposite wheel-parity have a certain structure. The following easy observation will be useful in the first technical lemma.

Lemma 3.7.6. *Let G be a diamond-free perfect graph, such that there is no appearance of K_4 in G or $\overline{G}$, no even prism in G, and no long odd prism in G or $\overline{G}$. Let (y, K, C, P, F) be a 1-rack. Then P^* is anticomplete to $V(C)\backslash K$.*

Proof. As (P, p_0p_1) is a y-anchored cycle, y has no neighbours in P^*. Since the set of attachments of P^* in C contains $\{p_0, p_1\}$, Lemma 3.7.1 implies that P^* has no more attachments in C. Hence P^* is anticomplete to $V(C)\backslash K$. □

Lemma 3.7.7. *Let G be a diamond-free perfect graph, such that there is no appearance of K_4 in G or $\overline{G}$, no even prism in G, and no long odd prism in G or $\overline{G}$. Then G does not contain a secured 1-rack.*

Proof. Suppose that G contains a secured 1-rack (y, K, C, P, F), with $C = p_0 - v_1 - v_2 - \cdots - v_k - p_1$. We may assume that for fixed y, p_0, p_1, the secured 1-rack is chosen such that $|K \cup V(C) \cup V(P) \cup V(F)|$ is minimal. We remark that since $F \cap K = \emptyset$, if $f \in F$ then it has at most one neighbour in K.

For $i = 0, 1$, let p_i^* be the neighbour of p_i in P^*, and let Q_i be the $f_1 - p_i$ path with $Q_i^* \subseteq P^*$. Let D_i be the $f_m - p_i$ path with $D_i^* \subseteq C \setminus \{p_0, p_1\}$, and let $D_i^- = D_i \setminus \{f_m\}$. We note that since $F \cup P^*$ has attachments in C that are non-adjacent and of opposite wheel-parity, $V(F)$ contains a neighbour of y. We now determine which vertices in $\{p_0, p_1\}$ have neighbours in $V(F)$.

(1) *At least one of $\{p_0, p_1\}$ has neighbours in $V(F)$*

Suppose that $\{p_0, p_1\}$ is anticomplete to $V(F)$. For $i = 0, 1$, $f_1 - F - f_m - D_i - p_i - Q_i - f_1$ induces a cycle C_i in G of size at least 4. Moreover, $p_i \in V(C_i)$ is adjacent to y, and its neighbours in C_i are nonneighbours of y, so by Lemma 3.1.3, C_i contains an even number of y-complete edges. Since y has no neighbours in Q_i, F and in D_i have the same y-parity. Note that this means that D_1 and D_2 have the same y-parity.

Consider the cycle C' formed by $f_1 - F - f_m - D_0 - p_0 - p_1 - Q_1 - f_1$. Recall that j and j' are respectively minimal and maximal such that f_m is adjacent to $v_j, v_{j'}$. If f_1 has a neighbour in $P^* \setminus \{p_0^*\}$ and $j \neq k$, then C' is an induced cycle in G of size at least 4 with a y-complete edge p_0p_1 and a neighbour of $y \neq p_0, p_1$ in F. Moreover, F and D_1 have the same y-parity, so C' has odd y-parity, contradiction. Symmetrically, if f_1 has a neighbour in $P^* \setminus \{p_1^*\}$ and $j' \neq 1$, we obtain a contradiction. Since $p_0^* \neq p_1^*$, we may assume that the only neighbour of f_1 in P^* is p_0^*, and that $j = j' = 1$. But then by considering the hole C, we see that the D_0^- and D_1^- have opposite y-parity, contradiction. This proves (1).

(2) *Both p_0 and p_1 have neighbours in $V(F)$.*

Suppose that only p_0 has neighbours in $V(F)$, and let these neighbours be $f_{i_1}, \ldots, f_{i_n}$ where $i_1 < i_2 < \cdots < i_n$. For $0 \leq s \leq n$, let $F_s = \{f_{i_s}, f_{i_s+1}, \ldots, f_{i_{s+1}}\}$, where $f_{i_0} = f_1$ and $f_{i_{n+1}} = v_{j'}$. We note that for every s, f_{i_s} is not a neighbour of y, otherwise $\{p_1, p_0, f_{i_s}, y\}$ would induce a diamond.

Suppose that $j' > 1$. Let $C' = p_1 - f_{i_n} - F_n - v_{j'} - D_2^- - p_2 - p_1$, and let $F' = F\backslash F_n$. Then C' is a hole, and we show that (y, K, C', P, F') is a 1-rack and that $(V(C') \cup V(P) \cup V(F')) \subseteq$

$(V(C) \cup V(P) \cup V(F)) \backslash \{w\}$, contradicting the minimality of $|K \cup V(C) \cup V(P) \cup V(F)|$. It suffices to show that (C', y) is a wheel. If C' contains a neighbour $v' \notin \{p_0, p_1\}$ of y, then Lemma 3.1.3 implies that (C', y) is a wheel. Hence we may assume that F_n is anticomplete to y. Then, since $v_{j'}$ is not adjacent to p_0, Lemma 3.7.1 implies that p_0 and $v_{j'}$ are of the same wheel-parity, so $v_{j'}$ and p_1 are of opposite wheel-parity. Hence $D_2^- \backslash \{p_2\}$ contains a neighbour of y, and (C', y) is a wheel.

Thus we may assume that $j' = 1$. Let $C' = p_1 - Q_1 - f_1 - F - f_m - D_1 - p_1$. As p_1 has no neighbours in F, (C', y) is a wheel in G and so C' has an even number of y-complete edges. Since $j' = 1$, D_1^- has odd y-parity, and Q_1 has no y-complete edges, so $V(F) \cup \{v_1\}$ has odd y-parity. Moreover, since y is anticomplete to $\{v_1, f_{i_1}, \dots, f_{i_m}\}$, this means that some F_s has odd y-parity. Then, since p_1 is adjacent to y, $F_s \cup \{p_0\}$ induces a hole with an odd number of y-complete edges and at least 3 neighbours of y, contradiction. This proves (2).

Hence p_0, p_1 both have neighbours in F. Then we may assume that there exist $i_1 < i_2$ such that p_s is adjacent to f_{i_s} for $s = 0, 1$, and that $\{p_0, p_1\}$ is anticomplete to $\{f_{i_0+1}, \dots, f_{i_1-1}\}$.

Suppose that y has no neighbours in $F_1 = \{f_{i_0}, \dots, f_{i_1}\}$. Let P' be the cycle induced by $\{p_0, p_1\} \cup F_1$ and let $F' = f_{i_1} - \dots - f_m$. Then Lemma 3.7.6 implies that (y, K, C, P', F') is also a secured 1-rack with the same y, p_1, p_2, but $(V(C) \cup V(F') \cup V(P')) \subseteq (V(C) \cup V(F))$, contradicting the minimality of $|K \cup V(C) \cup V(P) \cup V(F)|$. Hence we may assume that y has a neighbour $f \in F_1$. Then $C' = p_0 - f_{i_0} - F_1 - f_{i_1} - p_1 - p_0$ is a hole of size at least 4 in G that contains a y-complete edge p_0p_1 and a y-complete vertex $f \notin \{p_0, p_1\}$. Hence Lemma 3.1.3 implies that there are an even number of y-complete edges in C' and (C', y) is a wheel with $p_0p_1 \in E(C')$. By Lemma 3.7.1, since P^* is connected and has no y-complete vertices, it follows that P^* has no attachments in C' outside of $\{p_0, p_1\}$, so $i_1 \neq 1$. Hence if we let $F' = f_1 - \dots - f_{i_1-1}$, then by Lemma 3.7.6 (y, K, C', P, F') is a secured 1-rack with the same y, p_0, p_1, but $(V(C') \cup V(P) \cup V(F')) \subseteq (V(P) \cup V(F))$, contradicting the minimality of $|K \cup V(C) \cup V(P) \cup V(F)|$. □

Lemma 3.7.8. *Let G be a diamond-free perfect graph, such that there is no appearance of K_4 in G or $\overline{G}$, no even prism in G, and no long odd prism in G or $\overline{G}$. Suppose that G does not admit a clique cutset. Then for every $l \geq 1$, G does not contain an l-rack.*

For clarity, we first isolate some of the key lemmas used in the proof of Lemma 3.7.8.

Lemma 3.7.9. *Let G be a diamond-free perfect graph, such that there is no appearance of K_4 in G or $\overline{G}$, no even prism in G, and no long odd prism in G or $\overline{G}$. Let $(y, K, C, P = \cup_{i \in [l-1]} P_i, F)$ be an l-rack for some $l \geq 1$. Suppose that $F \neq \emptyset$, and let i' be maximal so that f_1 has a neighbour in $V(P_{i'}) \setminus \{p_{i'-1}\}$ (where $V(P_0) = p_0$). Suppose that the following conditions hold:*

(i) $i' < l$;

(ii) the set of attachments of F in $V(P) \cup V(C)$ is not $\{v_0 = p_0, v_1\}$;

(iii) $p_{i'}$ is anticomplete to $V(F) \setminus \{f_1\}$, and p_l is anticomplete to $V(F) \setminus \{f_m\}$; and

(iv) y has a neighbour in $V(F) \cup \{v_{j'}, v_{j'+1}, \dots, v_k\}$.

Then either G admits a clique cutset, or there exist $l' < l$ and induced cycles C', P' such that $V(C') \cap K = \{p_0, p_l\}$ and $(y, K, C', P', \emptyset)$ is an l'-rack in G.

Proof. Let Q be the shortest $p_{i'} - f_1$ path with interior in $P_{i'}^*$. Let $D = f_m - v_{j'} - v_{j'+1} - \dots - v_{k+1} = p_l$, and let $C' = f_1 - F - f_m - D - p_l - p_{i'} - Q - f_1$. Then it follows from conditions (i), (ii) and (iii) that C' induces a hole. If C' induces a triangle, then $f_1 = f_m$ is complete to $\{p_{i'}, p_l\}$ and hence is in K, contradiction. Hence C' has size at least 4. Moreover, C' contains a y-complete edge $p_{i'}p_l$, and by condition (iv) $V(C') \backslash \{p_{i'}, p_l\}$ also contains a y-complete vertex. Hence by Lemma 3.1.3, (C', y) is a wheel.

Let $P'_i = P_{i+i'}$, $l' = l - i'$ and $P' = \cup_{i \in [l']} P'_i$. Suppose there does not exist a path F' such that (y, K, C', P', F') is a secured l'-rack. Then Lemma 3.7.5 implies that G admits a clique cutset. This proves the lemma. □

Lemma 3.7.10. *Let G be a diamond-free perfect graph, such that there is no appearance of K_4 in G or $\overline{G}$, no even prism in G, and no long odd prism in G or $\overline{G}$. Let $(y, K, C, \cup_{i\in[l]} P_i, F)$ be an l-rack for some $l \geq 1$. Suppose further that $V(F)$ is anticomplete to y, that the only edges between $V(F)$ and $V(P)$ are between f_1 and $V(P)$, and that the set of attachments of F in $V(P) \cup V(C)$ is not $\{p_0, v_1\}$. Then either G admits a clique cutset, or there exist $l' < l$, cycles C', P' and a path F' such that $V(C') \cap K = \{p_0, p_l\}$ and G contains a secured l'-rack (y, K, C', P', F').*

Proof. Let F' be the maximal connected subgraph of $F \cup P^*$ containing F. Since $F \cup P^*$ has no y-complete vertices, by Lemma 3.7.1, all the attachments of F' in $V(C)$ have the same wheel-parity, and we may assume that they all have the same wheel-parity as v_1. We show that $(y, K, C, \cup_{i\in[l]} P_i, F)$ satisfies the conditions in Lemma 3.7.9. Condition (ii) holds by assumption. Since F' does not attach to p_l and f_1 is not adjacent to p_l, conditions (i) and (iii) hold. Recall that j' is maximal such that f_m is adjacent to $v_{j'}$. Since $v_{j'}$ and p_l have opposite wheel-parity, there is a y-complete edge in $\{v_{j'}, v_{j'+1} \dots, v_k\}$, so condition (iv) holds. This proves the lemma. □

Proof of Lemma 3.7.8. Let l be minimal such that G contains an l-rack. Then, as G does not admit a clique cutset, Lemma 3.7.5 implies that G contains a secured l-rack $(y, K, C, \cup_{i\in[l]} P_i, F)$, and hence Lemma 3.7.7 implies that $l \geq 2$. Note that if any l-rack satisfies the conditions in either Lemma 3.7.9 or Lemma 3.7.10, then either G admits a clique cutset, or l is not minimal, contradiction. We remark that all secured l-racks automatically satisfy the condition that the set of attachments of F in $V(P) \cup V(C)$ is not $\{p_0, v_1\}$. Let j'' be the unique index so that f_1 has a neighbour in $P^*_{j''}$, and let D be a $f_1 - p_l$ path with interior in $V(F) \cup C^*$.

(1) *We may assume that f_1 does not have neighbours in $P_l \setminus \{p_{l-1}\}$.*
Lemma 3.7.6 implies that G does not contain a secured 1-rack, so in particular if f_1 is not adjacent to y, then f_1 does not have neighbours in both $P_1 \setminus \{p_1\}$ and $P_l \setminus \{p_{l-1}\}$. If f_1 is adjacent to y then f_1 has no neighbours in $K \setminus \{y\}$, since f_1 has attachments in at most one of the P^*_i, it follows that f_1 also does not have neighbours in both $P_1 \setminus \{p_1\}$ and $P_l \setminus \{p_{l-1}\}$. Hence, possibly after reversing the labels of the P_i, we obtain (1).

(2) *There exists a vertex $f \in F \setminus \{f_1\}$ with neighbours in $K \cap P$.*
Suppose that $F \setminus \{f_1\}$ is anticomplete to $K \cap P$. If y has no neighbours in F, then (y, K, C, P, F) satisfies the conditions in Lemma 3.7.10, contradiction. Hence y has a neighbour in F, so (y, K, C, P, F) satisfies the conditions in Lemma 3.7.9, contradiction. This proves (2).

(3) *If there exists $l' < l$ such that $p_{l'}$ has a neighbour in $V(F)$, then $l' = 0$ and $j' = 1$.*
Let i be maximal such that f_i is adjacent to $p_{l'}$. Since G is diamond-free and $K \cap P$ is a clique and $f_i \notin K$, Lemma 3.1.2 imples that $p_{l'}$ is the unique neighbour of f_i in $(K \cap P)$, and f_i is not adjacent to y. Suppose that $(l', j') \neq (0, 1)$, and let $F' = f_i - f_{i+1} - \cdots - f_m$. If y has no neighbours in $V(F')$, then (y, K, C, P, F') satisfies the conditions in Lemma 3.7.10, contradiction. So we may assume that y has a neighbour in F'. Thus (y, K, C, P, F') satisfies the conditions in Lemma 3.7.9, contradiction. This proves (3).

(4) *y has a neighbour in D^*.*
Let us assume that y does not have a neighbour in D^*. Suppose that p_l does not have a neighbour in $V(F)$. It follows from (2) that p_1 has a neighbour in $V(F)$, so (3) implies that $D^* \supseteq C^*$, contradiction. Hence p_l has a neighbour in $V(F)$, and $D^* \subseteq V(F)$. Let Q be the $p_{j''-1} - f_1$ path with interior in $P^*_{j''}$, let and let $(P'_{j''}, p_{j''-1}p_l)$ be the y-anchored cycle with interior in $V(Q) \cup V(D)$. Then $(y, K, C, (\cup_{1\leq i<j''} P_i) \cup P'_{j''}, \emptyset)$ is a j''-rack, contradicting the minimality of l. This proves (4).

Let Q' be the $p_{j''} - f_1$ path with interior in $P^*_{j''} \setminus V(D)$, and let $C' = p_{j''} - Q' - f_1 - D - p_l$. It follows from (4) that C' contains the y-complete edge $p_{j''}p_l$ and a neighbour of y in D^*, so (C', y) is a wheel in G and $(y, K, C, \cup_{j''<i<l} P_i, \emptyset)$ is an $(l - j'')$-rack, contradicting the minimality of l. This proves the lemma. □

Lemma 3.7.11. *Let G be a diamond-free perfect graph, such that there is no appearance of K_4 in G or $\overline{G}$, no even prism in G, and no long odd prism in G or $\overline{G}$. Suppose that $y \in V(G)$, and that G does not contain a clique cutset. Then for every y-complete edge ab, at most one of the following holds:*

- *ab is an edge of a wheel with hub y; or*
- *there exists an $a-b$ track with no y-complete edges.*

Proof. Suppose that there exists a y-complete edge ab such that ab is an edge in a wheel (C, y), and there exists an $a-b$ track P in G with no y-complete edges. Let a, b, C and P be chosen so that $V(C) \cup V(P)$ is minimal. Let $K_0, K_1, K_2, \dots, K_m$ be all the maximal cliques of $G \backslash \{y\}$ that have size at least 2 and are y-complete, and let $a, b \in K_0$. Let $\{p_0, \dots, p_{n+1}\} = V(P) \cap (\cup_j K_j)$, where $p_0 = a$ and $p_{n+1} = b$. Let $P = p_0 - P_0 - p_1 - P_1 - \dots - p_n - P_n - p_{n+1}$. Then by Lemma 3.1.3, for each i it holds that either P_i is a $p_i - p_{i+1}$ path, or $(P_i, p_i p_{i+1})$ is a y-anchored cycle. The minimality of $V(C) \cup V(P)$ also implies that P_i^* is anticomplete to $P_{i'}^*$ for all $i \neq i'$.

(1) *The p_i are all in K_0.*
Suppose not. Then, as p_0, p_{n+1} are in K_0, there exist indices $i_1 < i_2 < i_3$ and j such that p_{i_1}, p_{i_3} are in K_j, and $p_{i_2} \notin K_j$. Assume that i_1, i_2, i_3, j are chosen so that $i_3 - i_1 \geq 2$ is minimal.

Let $C' = p_{i_1} - P_{i_1} - p_{i_1+1} - \dots - P_{i_3-1} - p_{i_3} - p_{i_1}$. Suppose that C' is a hole. Then it contains exactly one y-complete edge, namely $p_{i_1} p_{i_3}$, and it contains a neighbour p_{i_2} of y, contradiction. Hence there is a subsequence $i_1 < a_1 < b_1 < a_2 < b_2 < \dots < a_k < b_k < i_2$ such that the set $\{p_i : a_j \leq i \leq b_1\}$ is a clique for all i, and there are no other edges in the set of p_i. In particular, if we let $P_i' = p_{a_i} - P_{a_i} - \dots - P_{b_i-1} - p_{b_i}$ and $P_i^{'*} = V(P_i') \setminus \{p_{a_i}, p_{b_i}\}$, then $V(C') \setminus \cup_i (P_i^{'*})$ induces a hole C'' with y-complete edges $\{p_{a_i} p_{b_i}, 1 \leq i \leq k\} \cup \{p_{i_1} p_{i_3}\}$. Let K be the maximal y-complete clique containing a_1, b_1. Then $(y, K, C'', P_1', \emptyset)$ is a $(b_1 - a_1)$-rack in G, contradicting Lemma 3.7.8. This proves (1).

(2) *If $l \geq 1$ or $l \leq n-1$ then P_l^* is anticomplete to C^*.*
Suppose that there exists some index $1 \leq l \leq n$ and vertices p, v so that $p \in P_l^*$, $v \in C^*$ and p, v adjacent. Let Q be the $p_l - p$ path with interior in P_l^*. We may assume that o is chosen so that $Q^* \cap V(C) = \emptyset$. Let D^- be the $v - p_0$ path with interior in C^*, and let D^+ be the $v - p_{n+1}$ path with interior in C^*. Let C^- be the hole $v - D^- p_0 - p_l - Q - p - v$, let C^+ be the hole $v - D^+ - p_{n+1} - p_l - Q - p - v$, let $P^- = \cup_{i=1}^{l-1} P_i$, and let $P^+ = \cup_{i=l}^{n} P_i$. Then at least one of D^-, D^+ contains y-complete edges, and so at least one of $(C^-, y), (C^+, y)$ is a wheel, and so at least one of $(C^-, P^-), (C^+, P^+)$ contradicts the minimality of $|V(C) \cup V(P)|$. If there exists some index $0 \leq l \leq n-1$ so that P_l^* is not anticomplete to C^*, by symmetry there is a contradiction. This proves (2).

(3) *$P = p_0 - P_0 - p_1$, i.e. $n = 0$.*
Suppose that $n \geq 1$. Then for all l, either $l \geq 1$ or $l \leq n-1$, so $(\cup_l P_l^*)$ is anticomplete to C^*. It follows that $(\cup_l P_l^*) \cap V(C) = \emptyset$, and so $(y, K_0, C, P, \emptyset)$ is an $(n+1)$-rack, contradicting Lemma 3.7.8.

Hence $P = p_0 - P_0 - p_1$. Recall that $C = v_0 - v_1 - \dots - v_{k+1} - v_0$, where $v_0 = p_0, v_{k+1} = p_1$. Let indices $0 = j_1 < \dots < j_l = k+1$ be chosen so that $\{j_1, \dots, j_l\} = V(P) \cap V(C)$. If $l = 2$ then $(y, K_0, C, P, \emptyset)$ is a 1-rack, contradicting Lemma 3.7.7. Hence $l \geq 3$. Since v_{j_1} and v_{j_l} have opposite wheel-parity, there exists an index m so that $v_{j_m}, v_{j_{m+1}}$ have opposite wheel-parity. Moreover, as $v_{j_m}, v_{j_{m+1}} \in V(P)$ and $l \geq 3$, they are not both adjacent to y, so they are not adjacent. Let Q be the $v_{j_m} - v_{j_{m+1}}$ path with interior in P^*. Then Q^* does not contain any neighbours of y or any vertices of $V(C)$, but attaches in two non-adjacent vertices of C with different wheel-parity, contradicting Lemma 3.7.1. This proves the Lemma. $\square$

Lemmas 3.7.8 and 3.7.11 allow us to prove the following decomposition theorem for diamond-free perfect graphs containing wheels.

Theorem 3.7.12. *Let G be a diamond-free perfect graph, such that there is no appearance of K_4 in G or $\overline{G}$, no even prism in G, and no long odd prism in G or $\overline{G}$. Suppose that G contains a wheel (C, y). Then G admits a clique cutset, or G admits a vertex-complete multijoin.*

Proof. Let a_1b_1 and a_2b_2 be y-complete edges of C such that a_1, a_2 have the same wheel-parity and b_1, b_2 have the same wheel-parity and y has no neighbours in the b_1b_2 path in C not containing a_1. Let $K_1, K_2, \ldots, K_m$ be all the maximal cliques of $G \backslash \{y\}$ that have size at least 2 and are y-complete, where we may assume that $\{a_i, b_i\} \subseteq K_i$ for $i = 1, 2$.

Suppose that G does not admit a clique cutset. Then Lemma 3.7.11 implies that every $a_i - b_i$ track in G contains a y-complete edge. Consequently, every $a_1 - b_2$ track in G contains a y-complete edge, for if some $a_1 - b_2$ track did not contain a y-complete edge, it could be extended to an $b_1 - b_2$ track not containing a y-complete edge. Let $A = \{a \in V(G) : \text{some } a_1 - a \text{ track does not contain a } y\text{-complete edge}\}$, and let $B = V(G) \backslash (A \cup \{y\})$. We show that (A, B) is a y-complete multijoin of G.

For $1 \leq i \leq m' \leq m$, let $A_i = K_i \cap A$ and let $B_i = K_i \cap B$, where we may assume after relabeling that for all $i > m'$, $K_i \cap A = \emptyset$. Clearly A_i is complete to B_i for $1 \leq i \leq m'$, so it remains to show that there are no other edges between A and B.

Suppose there is some edge ab where $a \in A$, $b \in B$. Then there is an $a_1 - a$ track P that does not contain a y-complete edge, and every $a_1 - b$ track contains a y-complete edge. In particular, $b \notin P$, so since $a_1 - P - a - b$ is an $a_1 - b$ track, it follows that ab is a y-complete edge. Thus there exists some i such that $a, b \in K_i$, so $a \in A_i$ and $b \in B_i$. Hence every edge from A to B is contained in the set of $A_i - B_i$ edges, $1 \leq i \leq m'$. This completes the proof. □

3.8 Proof of the Decomposition Theorem

In this section, we prove the decomposition theorem, Theorem 3.1. We first show that every diamond-free perfect graph containing a triangle either admits a clique cutset, or contains a prism or a wheel. We then invoke the structure lemmas in the previous sections to show that diamond-free perfect graphs containing prisms or wheels are either basic, or admit clique cutsets, complete 2-joins or vertex-complete multijoins.

Lemma 3.8.1. *Let G be a diamond-free perfect graph. Suppose that G is not bipartite and does not admit a clique cutset. Then G contains a prism or a wheel.*

Proof. Let K be a maximal clique in G. As G is perfect and not bipartite, $|K| \geq 3$. Since G does not admit a clique cutset, $G \backslash K$ is connected, and every vertex in K has a neighbour in $V(G) \backslash K$. Moreover, as G is diamond-free and K is a maximal clique in G, by Lemma 3.1.2, every vertex in $V(G) \backslash K$ has at most one neighbour in K.

Let $\{v_1, v_2, v_3\} \subseteq K$, and F be a connected subgraph of $G \backslash K$ such that each v_i has neighbours in $V(F)$, and $|V(F)|$ is minimal. Note that F need not be an induced subgraph. Since $V(F) \subseteq V(G) \setminus K$, each vertex in f has at most one neighbour in $\{v_1, v_2, v_3\}$. Since F is not necessarily induced, we may assume that $E(F)$ forms a spanning tree for $G|V(F)$. Then every vertex $f \in F$ is either a cut vertex in F, or a leaf in F and the unique neighbour of v_i for some i. Hence F has at most 3 leaves.

Suppose that F has 3 leaves w_1, w_2, w_3, where $w_i \neq f$ is adjacent to v_i. Then there is exactly one vertex $f \in V(F)$ with degree 3 in F, and $F = P_1 \cup P_2 \cup P_3$, P_i has ends u_i, w_i and $f - u_i - P_i - w_i - v_i$ are $f - v_i$ paths for each i, and there are no other edges in F. Then by the minimality of $|V(F)|$, $V(P_i) \cup \{f, v_i\}$ induces a path in G for each i, and there are no edges between $V(P_i)$ and $V(P_j) \setminus \{u_j\}$ for $i \neq j$. Since G is perfect, f cannot be linked to $\{v_1, v_2, v_3\}$ and so there exist i, j so that u_iu_j is an edge. By the minimality of $|V(F)|$, $\{u_1, u_2, u_3\}$ does not induce a connected subgraph of G. Hence we may assume that the only edge in $G|\{u_1, u_2, u_3\}$ is u_1u_2, and the rungs $u_1 - P_1 - w_1 - v_1$, $u_2 - P_2 - w_2 - v_2$ and $f - u_3 - P_3 - w_3 - v_3$ form a prism with triangles $\{u_1, u_2, f\}$ and $\{v_1, v_2, v_3\}$. Hence G contains a prism.

Hence, as $|F| \neq 1$, F has 2 leaves. So $F = f_1 - f_2 - \cdots - f_n$, where we may assume that v_1 is adjacent to f_1, v_2 is adjacent to f_n and there are no other edges between $\{v_1, v_2\}$ and $V(F)$. Moreover, $n \geq 3$ and v_3 has at least one neighbour in $V(F)$. Suppose that f_1f_n is an edge in G. Then the minimality of $|V(F)|$ implies that $V(F)$ induces a cycle in G, and that $n = 3$ and v_3 is adjacent to f_2. Hence $\{v_1, v_2, v_3\}$ and $\{f_1, f_3, f_2\}$

are the triangles of a short prism in G. Hence we may assume that $f_1 f_n$ is not an edge. If $V(F)$ induces a path in G, let $C = v_1 - f_1 - F - f_n - v_2 - v_1$. Then C induces a hole in G and contains the v_3-complete edge $v_1 v_2$ and at least one other neighbour of v_3 in $V(F)$, so (C, v_3) is a wheel in G. Hence we may assume that there is an edge $f_i f_{i'}$ in G, where $i + 1 < i'$, and by the minimality of $|V(F)|$ we obtain that $i' = i + 2$ and the only neighbour of v_2 in $V(F)$ is f_{i+1}. Then the paths $v_1 - f_1 - \cdots - f_i$, $v_2 - f_n - \cdots f_{i+2}$ and $v_3 - f_{i+1}$ form a prism with triangles $\{v_1, v_2, v_3\}$ and $\{f_i, f_{i+2}, f_{i+1}\}$ and G contains a prism. □

We are now ready to prove Theorem 3.1, which we state again below.

Theorem 3.8.2. *Let G be a diamond-free perfect graph. Then at least one of the following holds:*

- *G is a basic graph;*
- *G admits a clique cutset;*
- *G admits a complete 2-join;*
- *there is a vertex $x \in V(G)$ such that G admits an x-complete multijoin.*

Proof. Let G be a diamond-free perfect graph, and suppose that G is not basic and does not admit a clique cutset, a complete 2-join, or a vertex-complete multijoin. By Lemma 3.8.1, G contains either a prism or a wheel. If G contains a wheel, then Lemma 3.6.9 and Theorem 3.7.12 imply that there is either an appearance of K_4 in G or $\overline{G}$, or an even prism or a long odd prism in G. If G contains a prism, then Proposition 3.6.10 implies that either G or $\overline{G}$ contains an appearance of K_4. Moreover, Lemma 3.3.1 implies that G contains an appearance of K_4. But then Proposition 3.4.8 implies that G admits a clique cutset or a complete 2-join or a vertex-complete 2-join, contradiction. □

4 3-Clique-Colouring Diamond-free Perfect Graphs

In this section, we prove that every diamond-free perfect graph admits a 3-clique-colouring. We will in fact prove that every diamond-free graph admits a class of slightly more involved colourings, which we now define.

Definition 4.1. *A k-*partition *of a graph G is a partition of its vertices into at most k disjoint sets $\mathcal{X} = (X_1, X_2, \dots, X_k)$ such that $X_i \cap X_j = \emptyset$ for $i \neq j$, and $\cup_i X_i = V(G)$. A k-*clique-colouring *of a graph G is a k-partition $\mathcal{X} = (X_1, X_2, \dots, X_k)$ of $V(G)$ such that $\cup_i X_i = V(G)$, and for every inclusion-wise maximal clique K in G and every index $1 \leq i \leq k$, $K \not\subseteq X_i$.*

Let H, H' be induced subgraphs of G with $V(H) \subseteq V(H') \subseteq V(G)$, let $\mathcal{X}_H = (X_{H,1}, \dots, X_{H,k})$ be a k-partition of H, and let $\mathcal{X}_{H'} = (X_1, \dots, X_k)$ be a k-partition of H'.

We say that $\mathcal{X}_{H'}$ extends *$\mathcal{X}_H$ if the following hold:*

- *$X_{H,i} \subseteq X_i$ $\forall 1 \leq i \leq k$; and*
- *for every inclusion-wise maximal clique K such that $K \subseteq H'$ and $K \not\subseteq H$, it holds that $K \not\subseteq X_i$ for every $1 \leq i \leq k$.*

We say that $\mathcal{X}_H$ is the restriction *of $\mathcal{X}_{H'}$ to H if $X_{H,i} = X_{H',i} \cap H$ $\forall$ $1 \leq i \leq k$.*

It is clear that not all partitions are clique-colourings. We remark that if $\mathcal{X}_{H'}$ extends $\mathcal{X}_H$ and K, then $\mathcal{X}_{H'}$ is, roughly speaking, as close to a clique-colouring of H' as possible, given that it agrees with $\mathcal{X}_H$ on $V(H)$. In particular, if a partition $\mathcal{X}_G$ of G extends a partition $\mathcal{X}_\emptyset$ of the empty set, or extends a clique-colouring $\mathcal{X}_H$ of $H \subseteq V(G)$, then it is also a clique-colouring of G.

Our main theorem in this section is as follows.

Theorem 4.2. *Let G be a diamond-free perfect graph, and let $K \subseteq V(G)$ induce a (possibly empty) clique in G. Let $\mathcal{X}_K$ be a 3-partition of K. Then there is a 3-partition of G that extends $\mathcal{X}_K$.*

Theorem 2.3 follows as an immediate corollary.

Corollary 4.3. *Let G be a diamond-free perfect graph. Then G is 3-clique-colourable.*

We will prove Theorem 4.2 by showing that if G is a vertex-minimal counterample to the theorem, then it cannot be basic, and it cannot admit a clique cutset, a complete multijoin or a vertex-complete multijoin.

Lemma 4.4. *Fix $k \geq 3$. Let G be a diamond-free perfect graph such that there exist a (possibly empty) clique K of G and a k-partition $\mathcal{X}_K$ of K, such that no k-partition of G extends $\mathcal{X}_K$.*

Suppose that $|V(G)|$ is minimal such that these properties hold. Then G is not a clique, and G does not admit a clique cutset.

Proof. Let G, K and $\mathcal{X}_K$ satisfy the given assumptions. Suppose that G is a clique. If K is empty, let $v \in V(G)$, $X_1 = \{v\}$, $X_2 = V(G) \setminus \{v\}$. Otherwise, let $i \in [3]$ be chosen so that $K \not\subseteq X_{K,i}$, let $X_i = (V(G) \setminus K) \cup X_{K,i}$, and for $j \in [3] \setminus \{i\}$ let $X_j = X_{K,j}$. Then $\mathcal{X} = (X_1, X_2, X_3)$ extends $\mathcal{X}_K$ to G, contradiction.

Suppose now that G admits a clique cutset L. Let $V(G) = A \cup B$, where $L = A \cap B$, $L \neq A, B$ and $A \backslash L$ and $B \backslash L$ are nonempty and anticomplete to each other. Now K is contained in A or K is contained in B, so we may assume that $K \subseteq A$. Then $G|A$ is diamond-free, perfect, and has fewer vertices than G, so by the minimality of $|V(G)|$ there exists a k-partition $\mathcal{X}_A$ of $G|A$ that extends $\mathcal{X}_K$. Let $\mathcal{X}_L$ denote the restriction of $\mathcal{X}_A$ to L. Then $G|B$ is diamond-free, perfect, and has fewer vertices than G, so by the minimality of $|V(G)|$ there exists a k-partition $\mathcal{X}_B$ of $G|B$ that extends $\mathcal{X}_L$. Let $\mathcal{X} = (X_1, \ldots, X_k)$ be defined by $X_i = X_{A,i} \cup X_{B,i}$. We show that $\mathcal{X}$ extends $\mathcal{X}_K$ to G. Clearly $X_{K,i} \subseteq X_{A,i} \subseteq X_i$ for all $1 \leq i \leq k$. Moreover, if $v \in V(G)$ then either $v \in (A \backslash B) \cup (B \backslash A)$ and there exists exactly one index i for which $v \in X_i$, or $v \in A \cap B = L$ and there is exactly one index i for which $v \in X_{A,i}$. Hence $X_i \cap X_j = \emptyset$ for all $i \neq j$. To show that $\mathcal{X}$ extends $\mathcal{X}_K$ to G, it remain to check that for every maximal clique $K' \neq K$ in G and every index i, $K' \not\subseteq X_i$.

Let K' be a maximal clique in G. Then K' is contained in A or K' is contained in B. If K' is contained in A and $K' \neq K$, then K' meets at least 2 of the $X_{A,i}$, and hence $K' \not\subseteq X_i$ for all i. If K' is contained in B and is not contained in A, then $K' \neq L$, and so a similar argument shows that $K' \not\subseteq X_i$ for all i. Hence $\mathcal{X}$ extends $\mathcal{X}_K$ to G, contradiction. This proves the lemma. ☐

Lemma 4.5. *Fix $k \geq 3$. Let G be a diamond-free perfect graph such that there exist a (possibly empty) clique K of G and a k-partition $\mathcal{X}_K$ of K, such that no k-partition of G extends $\mathcal{X}_K$.*

Suppose that $|V(G)|$ is minimal such that these properties hold. Then for every $v \in V(G) \backslash K$, v is contained in at least 3 distinct maximal cliques.

Proof. Suppose that G, K and $\mathcal{X}_K$ satisfy the given assumptions, and that there is a vertex $v \in V(G) \backslash K$ contained in at most 2 distinct maximal cliques. If v is contained only in one maximal clique K', then either G is a clique or $K' \backslash \{v\}$ is a clique cutset, contradicting Lemma 4.4. Hence we may assume that v is contained in exactly two maximal cliques, K_1 and K_2.

Consider the graph $G' = G \backslash \{v\}$. Then G' is still diamond-free and perfect, so by the minimality of $|V(G)|$ there exists a k-partition $\mathcal{X}_{G'} = (X'_1, \ldots, X'_k)$ of G' that extends $\mathcal{X}_K$. Let i be chosen so that $K_1 \setminus \{v\} \not\subseteq X'_i$ and $K_2 \backslash \{v\} \not\subseteq X'_i$. Since $k \geq 3$, such i exists. Let $X_j = X'_j \; \forall \, j \neq i$, $X_i = X'_i \cup \{v\}$, and let $\mathcal{X} = (X_1, \ldots, X_k)$. Then $\mathcal{X}$ extends $\mathcal{X}_K$ to G, contradiction. This proves the lemma. ☐

Lemma 4.6. *Let G be a diamond-free perfect graph such that there exist a (possibly empty) clique K of G and a k-partition $\mathcal{X}_K$ of K, such that no k-partitiosuch that there exist a (possibly empty) clique K of G and a k-partition $\mathcal{X}_K$ of Kn of G extends $\mathcal{X}_K$. , such that no k-partition of G extends $\mathcal{X}_K$.*

Suppose that $|V(G)|$ is minimal such that the above properties hold. Then G does not admit a 2-join.

Proof. Suppose that G, K and $\mathcal{X}_K$ satisfy the given properties, and that $|V(G)|$ is minimal subject to these conditions. We may assume without loss of generality that K is a maximal clique. Suppose that G admits a 2-join. Then G admits a partition $V(G) = A \cup B$ where $A \cap B = \emptyset$ and there are disjoint nonempty cliques $A_1, A_2 \subseteq A$, $B_1, B_2 \subseteq B$ such that A_i is complete to B_i for $i = 1, 2$, $|A_i \cup B_i| \geq 3$ for $i = 1, 2$, and there are no other edges between A and B.

Let $\mathcal{X}_{K\cap A}$ be the restriction of $\mathcal{X}_K$ to $G|(K\cap A)$. Consider the subgraph $G|A$ of G. It is diamond-free, perfect, and has fewer vertices than G, so by the minimality of $|V(G)|$ there exists a k-partition $\mathcal{X}_A$ of $G|A$ that extends $\mathcal{X}_{K\cap A}$. Similarly, there exists a k-partition $\mathcal{X}_B$ of $G|B$ that extends $\mathcal{X}_{K\cap B}$. Let $\mathcal{X} = (X_1, \dots, X_k)$ be defined by $X_i = X_{A,i} \cup X_{B,i}$. We show that $\mathcal{X}$ extends $\mathcal{X}_K$ to G. Clearly $X_{K,i} \subseteq X_{A,i} \cup X_{B,i} \subseteq X_i$ for all $1 \leq i \leq k$, and since $A \cup B = V(G)$ and $A \cap B = \emptyset$, for every $v \in V(G)$ there is exactly one index i such that $v \in X_i$.

We show now that for every maximal clique $K' \neq K$ in G and every index i, $K' \not\subseteq X_i$. Let K' be a maximal clique in G. If K' is contained in A and $K' \neq K$, then K' meets at least 2 of the $X_{A,i}$, and hence $K' \not\subseteq X_i$ for any i. Similarly, if K' is contained in B, then $K' \not\subseteq X_i$ for any i. Finally, if $K' \cap A$ and $K' \cap B$ are both nonempty, then $K' = A_j \cup B_j$ for some j (as K' is a maximal clique). Since $|A_j \cup B_j| \geq 3$, at least one of $|A_j|, |B_j| \geq 2$, so we may assume without loss of generality that $|A_j| \geq 2$. Then $A_j \not\subseteq K$, so by the definition of $\mathcal{X}_A$ it meets at least two of the $X_{A,i}$. Hence $\mathcal{X}$ extends $\mathcal{X}_K$, contradiction. This proves the lemma. □

Lemma 4.7. *Let G be a diamond-free perfect graph such that there exist a (possibly empty) clique K of G and a k-partition $\mathcal{X}_K$ of K, such that no k-partitiosuch that there exist a (possibly empty) clique K of G and a k-partition $\mathcal{X}_K$ of Kn of G extends $\mathcal{X}_K$. , such that no k-partition of G extends $\mathcal{X}_K$.*

Suppose that $|V(G)|$ is minimal such that the above properties hold. Then for every vertex $x \in V(G)$, G does not admit an x-complete multijoin.

Proof. Suppose that G, K and $\mathcal{X}_K$ satisfy the given properties, and that $|V(G)|$ is minimal subject to these conditions. We may assume without loss of generality that K is a maximal clique. Suppose that there is a vertex $x \in V(G)$ such that G admits an x-complete multijoin. Then there are sets $V(G) = A \cup B$ such that $A \cap B = \{x\}$ and disjoint cliques $A_1, \dots, A_k \subseteq A$, $B_1, \dots, B_k \subseteq B$ such that A_i and B_i are complete to $\{x\}$ and to each other for all i, and there are no other edges between $A \setminus \{x\}$ and $B \setminus \{x\}$. For brevity, let $A' = A \setminus \{x\}$, and let $B' = B \setminus \{x\}$.

We may assume that $K \cap A$ is nonempty, and let $\mathcal{X}_{K\cap A}$ be the restriction of $\mathcal{X}_K$ to $G|(K \cap A)$. Consider the subgraph $G|A$. It is diamond-free, perfect, and has fewer vertices than G, so by the minimality of $|V(G)|$ there exists a k-partition $\mathcal{X}_A$ of $G|A$ that extends $\mathcal{X}_{K\cap A}$. Now if $(K \cap B) \setminus \{x\}$ is nonempty, then $K = A_i \cup B_i \cup \{x\}$ for some i, and so $x \in K$. In this case, let $\mathcal{X}_{K\cap B}$ be the restriction of $\mathcal{X}_K$ to $G|(K \cap B)$. Otherwise, $(K \cap B) \subseteq \{x\}$, so we may let $\mathcal{X}_{K\cap B}$ be the restriction of $\mathcal{X}'_A$ to $\{x\}$. By the minimality of $|V(G)|$, there exists a k-partition $\mathcal{X}_B$ of $G|B$ that extends $\mathcal{X}_{K\cap B}$.

Let $\mathcal{X} = (X_1, \dots, X_k)$ be defined by $X_i = X_{A,i} \cup X_{B,i}$. We show that $\mathcal{X}$ extends $\mathcal{X}_K$ to G. Clearly $X_{K,i} \subseteq X_{A,i} \cup X_{B,i} \subseteq X_i$ for all $1 \leq i \leq k$, and since $A \cup B = V(G)$ and $A \cap B = \{x\}$, by the definiton of $\mathcal{X}_B$, for every $v \in V(G)$ there is exactly one index i such that $v \in X_i$. Hence it remain to check that for every maximal clique $K' \neq K$ in G and every index i, $K' \not\subseteq X_i$.

Let K' be a maximal clique in G. If $K' \neq K$ is contained in A or B, then $K' \not\subseteq X_i$ for any i so $\mathcal{X}$ extends $\mathcal{X}_K$, contradiction. Hence we may assume that $K' = A_j \cup B_j \cup \{x\}$ for some j (as K' is a maximal clique). Then $|A_j \cup \{x\}| \geq 2$, and as $K' \neq K$ it holds that $A_j \cup \{x\} \not\subseteq K$, and so $A_j \cup \{x\}$ meets at least 2 of the $X_{A,i}$. As $\mathcal{X}$ extends $\mathcal{X}_A$, this implies that $\mathcal{X}$ extends $\mathcal{X}_K$, contradiction. This proves the lemma. □

Lemma 4.8. *Let G be a diamond-free perfect graph such that there exist a clique K of G and a 3- partition $\mathcal{X}_K$ of K, such that no 3-partition $\mathcal{X}$ of G extends $\mathcal{X}_K$.*

Suppose that $|V(G)|$ is minimal such that these properties hold. Then G is not a basic graph.

Proof. Suppose that G, K and $\mathcal{X}_K$ satisfy the given assumptions, and G is basic. Without loss of generality, we may assume that K is a maximal clique, and that G is not a clique. By Proposition 3.2.3, G is one of the following: a bipartite graph; a birdcage; or the line graph of a bipartite graph.

(1) G is not a bipartite graph.

Suppose that G is bipartite, and let $V(G) = A \cup B$ where A, B are independent sets and $A \cap B = \emptyset$. If $|K| \leq 1$ or $|K| = 2$ and K is not monochromatic, we may assume that $X_{K,1} = K \cap A$, $X_{K,2} = K \cap B$ and simply let $X_1 = A, X_2 = B, \mathcal{X} = (X_1, X_2, X_3)$. Otherwise, $|K| = 2$ and we may assume that

$X_{K,1} = K$. Let $X_1 = K, X_2 = A \setminus K, X_3 = B \setminus K$. Then the 3-partition $\mathcal{X} = (X_1, X_2, X_3)$ extends $\mathcal{X}_K$, contradiction.

(2) G is not a birdcage, and not the line graph of a bipartite graph.

Suppose that G is a birdcage. Then every vertex in $V(G)$ is contained in at most 2 distinct maximal cliques. Similarly, if G is the line graph of a bipartite graph, then G is claw-free, so every vertex in $V(G)$ is contained in at most 2 distinct maximal cliques [8]. Hence, by Lemma 4.5, it suffices to show that there is a vertex in $V(G)\backslash K$. But this follows from the assumption that G is not a clique, giving the required contradiction.

Hence in all the cases, we obtain a 3-partition of G that extends $\mathcal{X}_K$, contradiction. □

Proof of Theorem 4.2. Suppose that G is a diamond-free perfect graph such that there exist a clique K of G and a 3-partition $\mathcal{X}_K$ of K, such that there is no 3-partition $\mathcal{X}$ of G that extends $\mathcal{X}_K$. We may assume that $|V(G)|$ is minimal such that this holds. By Theorem 3.1, G is a basic graph, or G admits a clique cutset or a complete 2-join or a vertex-complete multijoin. But in each of these cases, Lemmas 4.8, 4.4, 4.6 and 4.7 respectively lead to a contradiction.

□

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