The Structure of Claw-Free Perfect Graphs

Maria Chudnovsky* Matthieu Plumettaz†

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Abstract

In 1988, Chvátal and Sbihi [4] proved a decomposition theorem for claw-free perfect graphs. They showed that claw-free perfect graphs either have a clique-cutset or come from two basic classes of graphs called elementary and peculiar graphs. In 1999, Maffray and Reed [6] successfully described how elementary graphs can be built using line-graphs of bipartite graphs using local augmentation. However gluing two claw-free perfect graphs on a clique does not necessarily produce claw-free graphs. In this paper we give a complete structural description of claw-free perfect graphs. We also give a construction for all perfect circular interval graphs.

1 Introduction

The class of claw-free perfect graphs was studied extensively in the past. The first structural result for this class was obtained by Chvátal and Sbihi in [4], where they proved that every claw-free Berge graph can be decomposed via clique-cutsets into two types of graphs: "elementary" and "peculiar". The structure of the peculiar graphs was determined precisely by their definition, but that was not the case for elementary graphs. Later Maffray and Reed [6] proved that an elementary graph is an augmentation of the line-graph of a bipartite multigraph, thereby giving a precise description of all elementary graphs. Their result, together with the result of Chvátal and Sbihi, gave an alternative proof of Berge's Strong Perfect Graph Conjecture for claw-free Berge graphs (the first proof was due to Parthasarathy and Ravindra [7]). However, this still does not describe the class of claw-free perfect graphs completely, as gluing two claw-free Berge graphs together via a clique-cutset may introduce a claw.

The purpose of this paper is to give a complete description of the structure of claw-free perfect graphs. Chudnovsky and Seymour proved a structure theorem for general claw-free graphs [2] and quasi-line graphs (which are a subclass of claw-free graphs) in [3]. Later we will show that every perfect claw-free graph is a quasi-line graph, however not all quasi-line graphs are perfect. The result of this paper refines the result of [3] to obtain a precise description of perfect quasi-line graphs. But before going further, we need to present some definitions. Let $G = (V, E)$ be a graph. A *clique* in G is a set $X \subseteq V$ such that every two members of X are adjacent. A set $X \subseteq V$ is a *stable set* in G if every two members of X are antiadjacent. For $X \subseteq V$, we define the subgraph $G|X$ *induced on* X as the subgraph with vertex set X and edge set all edges of G with both ends in X. The *chromatic number* of G, denoted by $\chi(G)$, is defined as the smallest number of stable sets covering the vertices of G. G is said to be *perfect* if for every induced subgraph G' of G, the chromatic number of G' is equal to the maximal clique size of G'.

In this paper we study perfect graphs, which by the strong perfect graph theorem [1], is equivalent to studying Berge graphs (the definition of Berge graphs, and more generally Berge trigraphs will be given later). Since it is easier in many cases to prove that a graph is Berge than to prove that the graph is

*Columbia University, New York, NY 10027, USA. E-mail: mchudnov@columbia.edu. Partially supported by NSF grants IIS-1117631 and DMS-1001091.

†Columbia University, New York, NY 10027, USA. E-mail: mp2761@columbia.edu. Partially supported by NSF grant DMS-1001091.

perfect, in the rest of the paper we will only deal with Berge graphs. In fact, we will work with slightly more general objects than graphs called trigraphs. A *trigraph* G consists of a finite set $V(G)$ of vertices, and a map $\theta_G : V(G)^2 \rightarrow \{-1, 0, 1\}$, satisfying:

- for all $v \in V(G)$, $\theta_G(v, v) = 0$.
- for all distinct $u, v \in V(G)$, $\theta_G(u, v) = \theta_G(v, u)$
- for all distinct $u, v, w \in V(G)$, at most one of $\theta_G(u, v), \theta_G(u, w)$=0.

For distinct $u, v \in V(G)$, we say that u, v are *strongly adjacent* if $\theta_G(u, v) = 1$, *strongly antiadjacent* if $\theta_G(u, v) = -1$, and *semiadjacent* if $\theta_G(u, v) = 0$. We say that u, v are *adjacent* if they are either strongly adjacent or semiadjacent, and *antiadjacent* if they are either strongly antiadjacent or semiadjacent. Also, we say that u is *adjacent* to v if u, v are adjacent, and that u is *antiadjacent* to v if u, v are antiadjacent. For a vertex a and a set $B \subseteq V(G)\backslash\{a\}$, we say that a is *complete* (resp. *anticomplete*) to B if a is adjacent (resp. antiadjacent) to every vertex in B. For two disjoint $A, B \subset V(G)$, we say that A is *complete* (resp. *anticomplete*) to B if every vertex in A is complete (resp. anticomplete) to B. Similarly, we say that a is strongly complete to B if a is strongly adjacent to every member of B, and so on.

A *clique* is a set $X \subseteq V(G)$ such that every two members of X are adjacent and X is a *strong clique* if every two members of X are strongly adjacent. A set $X \subseteq V(G)$ is a *stable set* if every two members of X are antiadjacent and X is a *strong stable set* if every two members of X are strongly antiadjacent. A *triangle* is a clique of size 3, and a *triad* is a stable set of size 3.

For a trigraph G and $X \subseteq V(G)$, we define the trigraph $G|X$ *induced on* X as follows. Its vertex set is X, and its adjacency function is the restriction of θ_G to X^2. We say that G *contains* H, and H is a *subtrigraph* of G if there exists $X \subseteq V(G)$ such that H is isomorphic to $G|X$.

A *claw* is a trigraph H such that $V(H) = \{x, a, b, c\}$, x is complete to $\{a, b, c\}$ and $\{a, b, c\}$ is a triad. A trigraph G is said to be *claw-free* if G does not contains a claw.

A *path* in G is a subtrigraph P with n vertices for $n \geq 1$, whose vertex set can be ordered as $\{p_1, \ldots, p_n\}$ such that p_i is adjacent to p_{i+1} for $1 \leq i < n$ and p_i is antiadjacent to p_j if $|i - j| > 1$. We generally denote P with the following sequence $p_1 - p_2 - \ldots - p_n$ and say that the path P is from p_1 to p_n. For a path $P = p_1 - \ldots - p_n$ and $i, j \in \{1, \ldots, n\}$ with $i < j$, we denote by $p_i - P - p_j$ the *subpath* P' of P defined by $P' = p_i - p_{i+1} - \ldots - p_j$.

A *cycle* (resp. *anticycle*) in G is a subtrigraph C with n vertices for some $n \geq 3$, whose vertex set can be ordered as $\{c_1, \ldots, c_n\}$ such that c_i is adjacent (resp. antiadjacent) to c_{i+1} for $1 \leq i < n$, and c_n is adjacent (resp. antiadjacent) to c_1. We say that a cycle (resp. anticycle) C is a *hole* (resp. *antihole*), if $n > 3$ and if for all $1 \leq i, j \leq n$ with $i + 2 \leq j$ and $(i, j) \neq (1, n)$, c_i is antiadjacent (resp. adjacent) to c_j. We will generally denote C with the following sequence $c_1 - c_2 - \ldots - c_n - c_1$. The *length* of C is the number of vertices of C. c_i and c_j are said to be *consecutive* if $i + 1 = j$ or $\{i, j\} = \{1, n\}$.

Now, we can finally give the definition of a Berge trigraph. A trigraph G is said to be *Berge* if G does not contain any odd holes or any odd antiholes.

A trigraph G is *cobipartite* if there exist nonempty subsets $X, Y \subseteq V(G)$ such that X and Y are strong cliques and $X \cup Y = V(G)$.

For $X, A, B, C \subseteq V(G)$, we say that $\{X|A, B, C\}$ *is a claw* if there exist $x \in X$, $a \in A$, $b \in B$ and $c \in C$ such that $G|\{x, a, b, c\}$ is a claw and x is complete to $\{a, b, c\}$. Similarly, we say that $X_1 - X_2 - \ldots - X_n - X_1$ is a *hole* (resp. *antihole*) if there exist $x_i \in X_i$ such that $x_1 - x_2 - \ldots - x_n - x_1$ is a hole (resp. antihole). To simplify notation, we will generally forget the bracket delimiting a singleton, i.e. instead of writing $\{\{x\}|A, \{y\}, B\}$ we will simply denote it by $\{x|A, y, B\}$.

Let A, B be disjoint subsets of $V(G)$. A is called a *homogeneous set* if A is a strong clique, and every vertex in $V(G)\backslash A$ is either strongly complete or strongly anticomplete to A. The pair (A, B) is called a *homogeneous pair* in G if A, B are nonempty strong cliques, and for every vertex $v \in V(G)\backslash(A \cup B)$, v is either strongly complete to A or strongly anticomplete to A, and either strongly complete to B or strongly anticomplete to B.

Let V_1, V_2 be a partition of $V(G)$ such that $V_1 \cup V_2 = V(G)$, $V_1 \cap V_2 = \emptyset$, and for $i = 1, 2$ there is a subset $A_i \subseteq V_i$ such that:

- A_i and $V_i \setminus A_i$ are not empty for $i = 1, 2$,
- $A_1 \cup A_2$ is a strong clique,
- $V_1 \setminus A_1$ is strongly anticomplete to V_2, and V_1 is strongly anticomplete to $V_2 \setminus A_2$.

The partition (V_1, V_2) is called a *1-join* and we say that G *admits a 1-join* if such a partition exists.

Let $A_1, A_2, A_3, B_1, B_2, B_3$ be disjoint subsets of $V(G)$. The 6-tuple $(A_1, A_2, A_3 | B_1, B_2, B_3)$ is called a *hex-join* if $A_1, A_2, A_3, B_1, B_2, B_3$ are strong cliques and

- A_1 is strongly complete to $B_1 \cup B_2$, and strongly anticomplete to B_3, and
- A_2 is strongly complete to $B_2 \cup B_3$, and strongly anticomplete to B_1, and
- A_3 is strongly complete to $B_1 \cup B_3$, and strongly anticomplete to B_2, and
- $\bigcup_i (A_i \cup B_i) = V(G)$.

Let G be a trigraph. For $v \in V(G)$, we define the *neighborhood* of v, denoted $N(v)$, by $N(v) = \{x : x \text{ is adjacent to } v\}$. G is said to be a *quasi-line trigraph* if for every $v \in V(G)$, $N(v)$ is the union of two strong cliques.

Here is an easy fact:

1.1. *Every claw-free Berge trigraph is a quasi-line trigraph.*

Proof. Let G be a claw-free Berge trigraph and let $v \in V(G)$. Since G is claw-free, we deduce that $G|N(v)$ does not contain a triad. Since G is Berge, we deduce that $G|N(v)$ does not contain a odd antihole. Thus $G|N(v)$ is cobipartite and it follows that $N(v)$ is the union of two strong cliques. This proves 1.1. □

A trigraph H is a *thickening* of a trigraph G if for every $v \in V(G)$ there is a nonempty subset $X_v \subseteq V(H)$, all pairwise disjoint and with union $V(H)$, satisfying the following:

- for each $v \in V(G)$, X_v is a strong clique of H
- if $u, v \in V(G)$ are strongly adjacent in G then X_u is strongly complete to X_v in H
- if $u, v \in V(G)$ are strongly antiadjacent in G then X_u is strongly anticomplete to X_v in H
- if $u, v \in V(G)$ are semiadjacent in G then X_u is neither strongly complete nor strongly anticomplete to X_v in H

A basic result about thickenings is the following.

1.2. *Let G be a trigraph and H be a thickening of G. If F is a thickening of H then F is a thickening of G.*

Proof. For $v \in V(H)$, let X_v^F be the strong clique in F as in the definition of a thickening. For $v \in V(G)$, let X_v^H be the strong clique in H as in the definition of a thickening. For $v \in V(G)$, let $Y_v \subseteq V(F)$ be defined as $Y_v = \bigcup_{y \in X_v^H} X_y^F$. Clearly, the sets Y_v are all nonempty, pairwise disjoint and their union is $V(F)$. Since X_v^H is a strong clique, we deduce that Y_v is a strong clique for all $v \in V(G)$. If $u, v \in V(G)$ are strongly adjacent (resp. antiadjacent) in G, then X_u^H is strongly complete (resp. anticomplete) to X_v^H in H and thus Y_u is strongly complete (resp. anticomplete) to Y_v in F. If $u, v \in V(G)$ are semiadjacent in G, then X_u^H is neither strongly complete nor strongly anticomplete to X_v^H in H and hence Y_u is neither strongly complete nor strongly anticomplete to Y_v in F. This proves 1.2. □

Next we present some definitions from [3].

A *stripe* is a pair (G, Z) of a trigraph G and a subset Z of $V(G)$ such that $|Z| \leq 2$, Z is a strong stable set, $N(z)$ is a strong clique for all $z \in Z$, no vertex is semiadjacent to a vertex in Z, and no vertex is adjacent to two vertices of Z.

G is said to be a *linear interval trigraph* if its vertex set can be numbered $\{v_1, \dots, v_n\}$ in such a way that for $1 \leq i < j < k \leq n$, if v_i, v_k are adjacent then v_j is strongly adjacent to both v_i, v_k. Given such a trigraph G and numbering $v_1, \dots, v_n$ with $n \geq 2$, we call $(G, \{v_1, v_n\})$ a *linear interval stripe* if no vertex is semiadjacent to v_1 or to v_n, there is no vertex adjacent to both v_1, v_n, and v_1, v_n are strongly antiadjacent. By analogy with intervals, we will use the following notation, $[v_i, v_j] = \{v_k\}_{i \leq k \leq j}$, $(v_i, v_j) = \{v_k\}_{i < k < j}$, $[v_i, v_j) = \{v_k\}_{i \leq k < j}$ and $(v_i, v_j] = \{v_k\}_{i < k \leq j}$. Moreover we will write $x_i < x_j$ if $i < j$.

Let Σ be a circle, and let $F_1, \dots, F_k \subseteq \Sigma$ be homeomorphic to the interval $[0, 1]$, such that no two of $F_1, \dots, F_k$ share an end-point. Now let $V \subseteq \Sigma$ be finite, and let G be a trigraph with vertex set V in which, for distinct $u, v \in V$,

- if $u, v \in F_i$ for some i then u, v are adjacent, and if also at least one of u, v belongs to the interior of F_i then u, v are strongly adjacent,
- if there is no i such that $u, v \in F_i$ then u, v are strongly antiadjacent.

Such a trigraph G is called a *circular interval trigraph*. We will denote by F_i^* the interior of F_i.

Let G have four vertices say w, x, y, z, such that w is strongly adjacent to x, y is strongly adjacent to z, x is semiadjacent to y, and all other pairs are strongly antiadjacent. Then the pair $(G, \{w, z\})$ is a *spring* and the pair $(G \backslash w, \{z\})$ is a *truncated spring*.

Let G have three vertices say v, z_1, z_2 such that v is strongly adjacent to z_1 and z_2, and z_1, z_2 are strongly antiadjacent. Then the pair $(G, \{z_1, z_2\})$ is a *spot*, the pair $(G, \{z_1\})$ is a *one ended spot* and the pair $(G \backslash z_2, \{z_1\})$ is a *truncated spot*.

Let G be a circular interval trigraph, and let $\Sigma, F_1, \dots, F_k$ be as in the corresponding definition. Let $z \in V(G)$ belong to at most one of $F_1, \dots, F_k$; and if $z \in F_i$ say, let no vertex be an endpoint of F_i. We call the pair $(G, \{z\})$ a *bubble*.

If H is a thickening of G, where X_v $(v \in V(G))$ are the corresponding subsets, and $Z \subseteq V(G)$ and $|X_v| = 1$ for each $v \in Z$, let Z' be the union of all X_v $(v \in Z)$; we say that (H, Z') is a *thickening* of (G, Z).

The following construction is slightly different from [3], but the resulting graphs are exactly the same. We may also assume that if (G, Z) is a stripe then $V(G) \neq Z$. Any trigraph G that can be constructed in this manner is called a *linear interval join*.

- Let $H = (V, E)$ be a graph, possibly with multiple edges and loops.
- Let $\eta : (E \times V) \cup E \to 2^{V(G)}$.
- For every edge $e = x^1x^2 \in E$ (where $x^1 = x^2$ if e is a loop)
  - Let (G_e, Y_e) be either
    - a spot or a thickening of a linear interval stripe if e is not a loop or,
    - the thickening of a bubble if e is a loop

    Moreover let ϕ_e be a bijection between Y_e and the endpoints of e.
  - Let $\eta(e, x^j) = N(\phi_e(x^j))$ for $j = 1, 2$ and $\eta(e, v) = \emptyset$ if v is not an endpoint of e.
  - Let $\eta(e) = V(G_e) \backslash Y_e$.
- Construct G with $V(G) = \bigcup_{e \in E} \eta(e)$, $G|\eta(e) = G_e$ for all $e \in E$ and such that $\eta(f, x)$ is strongly complete to $\eta(g, x)$ for all $f, g \in E$ and $x \in V$ (in particular if x is an endpoint of both f and g, then the sets $\eta(f, x)$ and $\eta(g, y)$ are nonempty and strongly complete to each other).

Moreover, we call the graph H used in the construction of a linear interval join G the *skeleton* of G, and we say that e has been *replaced* by (G_e, Y_e).

The following theorem is the main result of [3]. It is the starting point of our structure theorem for claw-free perfect graphs.

1.3. *Every connected quasi-line trigraph G is either a linear interval join or a thickening of a circular interval trigraph.*

To state our main theorem we need a few more definitions that refine the concepts used in 1.3.

Let G be a circular interval trigraph. G is a *structured circular interval trigraph* if, for some even integer $n \geq 4$, $V(G)$ can be partitioned into pairwise disjoint strong cliques $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ such that (all indices are modulo n):

(S1) $\bigcup_i (X_i \cup Y_i) = V(G)$.

(S2) $X_i \neq \emptyset \; \forall \; i$.

(S3) Y_i is strongly complete to X_i and X_{i+1} and strongly anticomplete to $V(G) \setminus (X_i \cup X_{i+1} \cup Y_i)$.

(S4) If $Y_i \neq \emptyset$ then X_i is strongly complete to X_{i+1}.

(S5) Every vertex in X_i has at least one neighbor in X_{i+1} and at least one neighbor in X_{i-1}.

(S6) X_i is strongly complete to X_{i+1} or X_{i-1} and possibly both, and strongly anticomplete to $V(G) \setminus (X_i \cup X_{i-1} \cup X_{i+1} \cup Y_i \cup Y_{i-1})$.

A bubble (G, Z) is said to be a *structured bubble* if G is a structured circular interval trigraph.

We need to define one important class of Berge circular interval trigraphs. Let G be a trigraph with vertex set the disjoint union of sets $\{a_1, a_2, a_3\}, B_1^1, B_1^2, B_1^3, B_2^1, B_2^2, B_2^3, B_3^1, B_3^2, B_3^3$ such that $|B_i^j| \leq 1$ for $1 \leq i, j \leq 3$ with adjacency as follows (all additions are modulo 3):

- For $i = 1, 2, 3$, $B_i^1 \cup B_i^2 \cup B_i^3$ is a strong clique.
- For $i = 1, 2, 3$, B_i^i is strongly complete to $\bigcup_{k=1}^3 (B_{i+1}^k \cup B_{i+2}^k)$.
- For $1 \leq i, j \leq 3$ with $i \neq j$, B_i^j is strongly complete to $\bigcup_{k=1}^3 B_j^k$.
- For $i = 1, 2, 3$, B_i^{i+1} is not strongly complete to B_{i+2}^{i+1} (In particular, $|B_i^{i+1} \cup B_{i+2}^{i+1}| \neq 1$).
- For $i = 1, 2, 3$, a_i is strongly complete to $\bigcup_{k=1}^3 (B_i^k \cup B_{i+1}^k)$ and a_i is strongly anticomplete to $\bigcup_{k=1}^3 B_{i+2}^k$.
- a_1 is either semiadjacent or strongly antiadjacent to a_3, and a_2 is strongly anticomplete to $\{a_1, a_3\}$.
- If a_1 is semiadjacent to a_3 then $B_3^1 \cup B_2^1 = \emptyset$.
- There exist $x_i \in V(G) \cap (B_i^1 \cup B_i^2 \cup B_i^3)$ for $i = 1, 2, 3$, such that $\{x_1, x_2, x_3\}$ is a triangle.

We define $\mathcal{C}$ to be the class of all such trigraphs G. Moreover we define $\mathcal{C}'$ to be the set of all pairs $(H, \{z\})$ such that there exists $i \in \{1, 2, 3\}$ with $z \in X_{a_i}$, H is a thickening of a trigraph in $\mathcal{C}$ with $B_{i+1}^{i+2} \cup B_i^{i+2} = \emptyset$ and such that $N(z) \cap (X_{a_{i+1}} \cup X_{a_{i+2}}) = \emptyset$ (with X_{a_i} as in the definition of a thickening).

A *signing* of a graph $G = (V, E)$ is a function $s : E \to \{0, 1\}$. The value $v(C)$ of a cycle C is $v(C) = \sum_{e \in C} s(e)$. A graph, possibly with multiple edges and loops, is said to be *evenly signed* by s if for all cycles C in G, C has an even value, and in that case the pair (G, s) is said to be an *evenly signed graph.*

We need to define three classes of graphs that are going to play an important role in the structure of claw-free perfect graphs.

$\mathcal{F}_1$: Let (G, s) be a pair of a graph G (possibly with multiple edges and loops) and a signing s of G such that:

- $V(G) = \{x_1, x_2, x_3\}$,
- there is an edge e_{ij} between x_i and x_j with $s(e_{ij}) = 1$ for all $\{i, j\} \subset \{1, 2, 3\}$ with $i \neq j$,
- if e and f are such that $s(e) = s(f) = 0$, then e is parallel to f.

We define $\mathcal{F}_1$ to be the class of all such pairs (G, s).

$\mathcal{F}_2$: Let (G, s) be a pair of a graph G (possibly with multiple edges and loops) and a signing s of G such that all pairs of vertices of G are adjacent and $s(e) = 1$ for all $e \in E(G)$. We define $\mathcal{F}_2$ to be the class of all such pairs (G, s).

$\mathcal{F}_3$: Let (G, s) be a pair of a graph G (possibly with multiple edges and loops) and a signing s of G such that:

- $V(G) = \{x_1, x_2, \dots, x_n\}$ with $n \geq 4$,
- there is an edge e_{12} between x_1 and x_2 with $s(e) = 1$,
- $\{x_1, x_2\}$ is complete to $\{x_3, \dots, x_n\}$,
- $\{x_3, \dots, x_n\}$ is a stable set,
- if $s(e) = 0$, then e is an edge between x_1 and x_2.

We define $\mathcal{F}_3$ to be the class of all such pairs (G, s).

An *even structure* is a pair (G, s) of a graph G and a signing s such that for all blocks A of G, $(A, s|_{V(A)})$ is either a member of $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ or an evenly signed graph.

Here is a construction; a trigraph G that can be constructed in this manner is called an *evenly structured linear interval join*.

- Let $H = (V, E)$ and the signing s be an even structure.
- Let $\eta : (E \times V) \cup E \to 2^{V(G)}$.
- For every edge $e = x^1x^2 \in E$ (where $x^1 = x^2$ if e is a loop),
  - Let (G_e, Y_e) be:
    * a spot if e is in a cycle, $x^1 \neq x^2$ and $s(e) = 1$,
    * a thickening of a spring if e is in a cycle, $x^1 \neq x^2$, and $s(e) = 0$,
    * a trigraph in $\mathcal{C}'$ if e is a loop,
    * either a spot or a thickening of a linear interval stripe if e is not in a cycle.
  - Let ϕ_e be a bijection between the endpoints of e and Y_e.
  - Let $\eta(e, x^j) = N(\phi_e(x^j))$ for $j = 1, 2$ and $\eta(e, v) = \emptyset$ if v is not an endpoint of e.
  - Let $\eta(e) = V(G_e) \backslash Y_e$.
- Construct G with $V(G) = \bigcup_{e \in E} \eta(e)$, $G|\eta(e) = G_e$ for all $e \in E$ and such that $\eta(f, x)$ is complete to $\eta(g, x)$ for all $f, g \in E$ and $x \in V$ (in particular if x is an endpoint of both f and g, then the sets $\eta(f, x)$ and $\eta(g, y)$ are nonempty and strongly complete to each other).

As for the linear interval join, we call the graph H used in the construction of an evenly structured linear interval join G the *skeleton* of G, and we say that e as been *replaced* by (G_e, Y_e) .

We can now state our main theorem:

1.4. *Every connected Berge claw-free trigraph is either*

- *an evenly structured linear interval join or*
- *a thickening of a linear interval trigraph or*
- *a thickening of a trigraph in $\mathcal{C}$.*

The goal of the paper is to prove 1.4, but first we can prove an easy result about evenly signed graphs. Here is an algorithm that will produce a signing for a graph:

Algorithm 1

- Let T be a spanning tree of G and $r \in V(G)$.
- Arbitrarily assign a value in $\{0,1\}$ to $s(e)$ for all $e \in T$.
- For every $e = xy \in E(G)\backslash T$, let $s(e) = \sum_{f\in P_x} s(f) + \sum_{f\in P_y} s(f) \pmod 2$ where P_i is the path from r to i in T.

1.5. *Algorithm 1 produces an evenly signed graph (G,s).*

Proof. Let C be a cycle in G. First, we notice that for an edge e in T, $s(e)$ can be expressed with the same formula used to calculate the signing of an edge outside of T. In fact we have that for all $e \in E(G)$, $s(e) = \sum_{f\in P_x} s(f) + \sum_{f\in P_y} s(f) \pmod 2$. Thus,

$$\sum_{e=xy\in E(C)} s(e) = \sum_{xy\in E(C)} \left(\sum_{e\in P_x} s(e) + \sum_{e\in P_y} s(e) \right) =$$

$$= 2 \cdot \sum_{x\in V(C)} \left(\sum_{e\in P_x} s(e) \right) = 0 \pmod 2$$

which concludes the proof of 1.5. □

The result of 1.5 shows that if we have a graph G, we can find all signings s such that (G,s) is an evenly signed graph by using Algorithm 1 with all possible assignments for $s(e)$ on the tree T.

The paper is organized as follow. In Section 2, we study circular interval trigraphs that contain special triangles. Section 3 examines circular interval trigraphs that contain a hole of length 4 while Section 4 covers the case when a circular interval trigraph contains a long hole. In Section 5, we analyze linear interval joins. Finally in Section 6, we gather our results and prove our main theorem 1.4.

2 Essential Triangles

In order to prove 1.4, we first prove the following:

2.1. *Let G be a Berge circular interval trigraph. Then either G is a linear interval trigraph, or a cobipartite trigraph, or a thickening of a member of $\mathcal{C}$, or G is a structured circular interval trigraph.*

Before going further, more definitions are needed. Let G be a circular interval trigraph defined by Σ and $F_1, \dots, F_k \subseteq \Sigma$. Let $T = \{c_1, c_2, c_3\}$ be a triangle. We say that T is *essential* if there exist $i_1, i_2, i_3 \in \{1, \dots, k\}$ such that $c_1, c_2 \in F_{i_1}$, $c_2, c_3 \in F_{i_2}$ and $c_3, c_1 \in F_{i_3}$, and such that $F_{i_1} \cup F_{i_2} \cup F_{i_3} = \Sigma$. Let x, y, q be three points of Σ. We denote by $\Sigma^q_{x,y}$ the subset of Σ such that there exists a homeomorphism $\phi : \Sigma^q_{x,y} \to [0,1]$ with $\phi(x) = 0$ and $\phi(y) = 1$ and such that $q \in \Sigma^q_{x,y}$. Moreover let $\overline{\Sigma}^q_{x,y} = (\Sigma \backslash \Sigma^q_{x,y}) \cup \{x, y\}$.

The following two lemmas are basic facts that will be extensively used to prove 2.1.

2.2. *Let G be a circular interval trigraph defined by Σ and $F_1, \dots, F_k$. Let $x, y, a, b \in V(G)$ such that $x \in \overline{\Sigma}^y_{a,b}$. If x is antiadjacent to a and b, then y is strongly antiadjacent to x.*

Proof. Assume not. Since x is adjacent to y, we deduce that there exists F_i such that $x, y \in F_i$. It follows that at least one of $a, b \in F_i^*$. By symmetry we may assume that $a \in F_i^*$, but it implies that a is strongly adjacent to x, a contradiction. This proves 2.2. □

2.3. *Let G be a circular interval trigraph defined by Σ and $F_1, \dots, F_k$. Let $x, y, z \in V(G)$ such that x is adjacent to y and x is antiadjacent to z. Then there exists F_i such that $\overline{\Sigma}^z_{x,y} \subseteq F_i$.*

Proof. Since x is adjacent to y there is F_i such that $x, y \in F_i$. Since z is antiadjacent to x, we deduce that $z \notin F_i^*$. Thus we conclude that $\overline{\Sigma}^z_{x,y} \subseteq F_i$. This proves 2.3. □

2.4. *Let G be a circular interval trigraph defined by Σ and $F_1, \dots, F_k$, and let $C = c_1 - c_2 - \dots - c_n - c_1$ be a hole. Then the vertices of C are in order on Σ.*

Proof. Assume not. By symmetry, we may assume that c_1, c_2, c_3, c_4 are not in order on Σ, and thus we may assume that $c_4 \in \Sigma^{c_2}_{c_1,c_3}$. Since c_3 is antiadjacent to c_1 and since c_2 is complete to $\{c_1, c_3\}$, we deduce that there exist F_i and F_j, possibly $F_i = F_j$, such that $\overline{\Sigma}^{c_3}_{c_1,c_2} \subseteq F_i$ and $\overline{\Sigma}^{c_1}_{c_2,c_3} \subseteq F_j$. If $c_4 \in \overline{\Sigma}^{c_3}_{c_1,c_2}$, then since $c_4 \in F_i^*$, we deduce that c_4 is strongly complete to $\{c_1, c_2\}$, a contradiction. If $c_4 \in \overline{\Sigma}^{c_1}_{c_3,c_2}$, then since $c_4 \in F_j^*$, we deduce that c_4 is strongly complete to $\{c_2, c_3\}$, a contradiction. This proves 2.4. □

2.5. *Let G be a circular interval trigraph defined by Σ and $F_1, \dots, F_k$. If G is not a linear interval trigraph, then there exists an essential triangle or a hole in G.*

Proof. Let F_{i_1} be such that $F_{i_1} \cap V(G)$ is maximal and let $y \notin F_{i_1}$. Let $x_0, x_1 \in F_{i_1}$ such that $\overline{\Sigma}^y_{x_0,x_1} \cap F_{i_1}$ is maximal.

Let x_2 and F_{i_2} be such that $x_2 \in F_{i_2}$, $x_2 \notin F_{i_1}$ and $\overline{\Sigma}^{x_0}_{x_1,x_2}$ is maximal.

Starting with $j = 3$ and while $x_{j-1} \notin F_{i_1}$, let x_j and F_{i_j} be such that $x_j \in F_{i_j}$, $x_j \notin F_{i_k}$, for any $k < j$ and $\overline{\Sigma}^{x_1}_{x_{j-1},x_j}$ is maximal. Since G is not a linear interval trigraph, there exists $k > 1$ such that $x_k \in F_{i_1}$.

Assume first that $k = 3$. Clearly $F_{i_1} \cup F_{i_2} \cup F_{i_3} = \Sigma$, $x_0, x_1 \in F_{i_1}$, $x_1, x_2 \in F_{i_2}$ and $x_0, x_2 \in F_{i_3}$. Hence $T = \{x_0, x_1, x_2\}$ is an essential triangle.

Assume now that $k > 3$. Clearly x_{j-1} is adjacent to x_j for $j = 1, \dots, k-1$ and x_{k-1} is adjacent to x_0. By the choice of F_{i_1} and x_0, x_1, we deduce that x_{k-1} is strongly antiadjacent to x_1. By the choice of F_{i_j}, x_{j-1} is antiadjacent to $x_{j+1 \bmod k}$ for all $j = 1, \dots, k-1$. Hence by 2.2, C is a hole. This concludes the proof of 2.5. □

2.6. *Let G be a circular interval trigraph and C a hole. Let $x \in V(G) \setminus V(C)$, then x is strongly adjacent to two consecutive vertices of C.*

Proof. Let G be defined by Σ and $F_1, \dots, F_k$ and let $C = c_1 - c_2 - \dots - c_l - c_1$. By 2.4, there exists j such that $x \in \overline{\Sigma}^{c_{j+2}}_{c_j,c_{j+1}}$. Since c_j is adjacent to c_{j+1} and antiadjacent to c_{j+2}, we deduce that there exists $i \in \{1, \dots, k\}$ such that $\overline{\Sigma}^{c_{j+2}}_{c_j,c_{j+1}} \subseteq F_i$. Hence x is strongly adjacent to c_j and c_{j+1}. This proves 2.6. □

In the remainder of this section, we focus on circular interval trigraphs that contain an essential triangle. For the rest of the section, addition is modulo 3.

2.7. *Every trigraph in $\mathcal{C}$ is a Berge circular interval trigraph.*

Proof. Let G be in $\mathcal{C}$. We let the reader check that G is indeed a circular interval trigraph, it can easily be done using the following order of the vertices on a circle: $B_1^3, B_1^1, B_1^2, a_1, B_2^1, B_2^2, B_2^3, a_2, B_3^2, B_3^3, B_3^1, a_3$.

(1) *There is no odd hole in G.*

Assume there is an odd hole $C = c_1 - c_2 - \ldots - c_n - c_1$ in G. Since B_i^i is strongly complete to $V(G)\backslash\{a_{i+1}\}$, it follows that $V(C) \cap B_i^i = \emptyset$ for all i. Since $G|(B_1^2 \cup B_1^3 \cup B_2^1 \cup B_2^3 \cup B_3^1 \cup B_3^2)$ is a cobipartite trigraph, we deduce that $|\{a_1, a_2, a_3\} \cap V(C)| \geq 1$.

Assume first that a_1, a_3 are two consecutive vertices of C. We may assume that $c_1 = a_1$ and $c_2 = a_3$. Since c_n is adjacent to c_1 and antiadjacent to c_2, we deduce that $c_n \in B_2^1 \cup B_2^3$. Symmetrically, $c_3 \in B_3^1 \cup B_3^2$. As a_1 is semiadjacent to a_3, it follows that $B_2^1 \cup B_3^1 = \emptyset$. Hence, c_3 is strongly adjacent to c_n, a contradiction.

Thus, we may assume that $c_1 = a_i$ and $\{c_2, c_n\} \cap \{a_1, a_2, a_3\} = \emptyset$. Since c_2 is antiadjacent to c_n, and c_1 is complete to $\{c_2, c_n\}$, we deduce that $\{c_2, c_n\} = B_i^{i+2} \cup B_{i+1}^{i+2}$. Without loss of generality, let $c_2 \in B_i^{i+2}$ and $c_n \in B_{i+1}^{i+2}$. Since c_{n-1} is antiadjacent to c_2, we deduce that $c_{n-1} = a_{i+1}$. Symmetrically, we deduce that $c_3 = a_{i+2}$. Since a_{i+2} is not consecutive with a_{i+1} in C, we deduce that $n > 5$. But $|\{x \in V(G) : x \text{ antiadjacent to } c_2\}| \leq 2$, a contradiction. This proves (1).

(2) *There is no odd antihole in G.*

Assume there is an odd antihole $C = c_1 - c_2 - \ldots - c_n$ in G. By (1), we may assume that C has length at least 7. Since B_i^i is strongly complete to $V(G)\backslash\{a_{i+1}\}$, it follows that $V(C) \cap B_i^i = \emptyset$ for all i.

Assume first that a_1 is adjacent to a_3. Then $B_3^1 \cup B_2^1 = \emptyset$. Since $|V(G)\backslash(B_1^1 \cup B_2^2 \cup B_3^3)| = 7$, we deduce that $V(C) = (\{a_1, a_2, a_3\} \cup B_1^2 \cup B_1^3 \cup B_3^2 \cup B_2^3)$. But a_2 has only two neighbors in $(\{a_1, a_2, a_3\} \cup B_1^2 \cup B_1^3 \cup B_3^2 \cup B_2^3)$, a contradiction. This proves that a_1 is strongly antiadjacent to a_3.

Assume now that $|V(C) \cap \{a_1, a_2, a_3\}| = 1$. We may assume that $a_1 \in V(C)$ and it follows that $V(C) = \{a_1\} \cup \bigcup_{j \neq k} B_j^k$. But $G|(\{a_i\} \bigcup_{j \neq k} B_j^k)$ is not an antihole of length 7, since the vertex of B_1^2 has 5 strong neighbors in $(\{a_i\} \bigcup_{j \neq k} B_j^k)$, a contradiction.

Hence we may assume that $|V(C) \cap \{a_1, a_2, a_3\}| \geq 2$. Since there is no triad in C, we deduce that $|C \cap \{a_1, a_2, a_3\}| = 2$ and by symmetry we may assume that $c_1 = a_1$, $c_2 = a_2$ and $a_3 \notin C$. But since $B_1^2 \cup B_1^3$ is strongly anticomplete to a_2 and $B_3^1 \cup B_3^2$ is strongly anticomplete to a_1, we deduce that $\{c_4, c_5, c_6\} \subseteq B_2^1 \cup B_2^3$, a contradiction. This proves (2).

Now by (1) and (2), we deduce 2.7. $\square$

2.8. *Let G be a Berge circular interval trigraph such that G is not cobipartite. If G has an essential triangle, then G is a thickening of a trigraph in $\mathcal{C}$.*

Proof. Let $\{x_1, x_2, x_3\}$ be an essential triangle and let F_1, F_2, F_3 be such that $x_1 \in F_1 \cap F_3$, $x_2 \in F_1 \cap F_2$, $x_3 \in F_2 \cap F_3$ and $F_1 \cup F_2 \cup F_3 = \Sigma$.

(1) *x_i is not in a triad.*

Assume x_1 is in a triad. Then there exist y, z such that $\{x_1, y, z\}$ is a triad. Since $x_1 \in F_1 \cap F_3$, we deduce that $y, z \in F_2^*$ and so y is strongly adjacent to z, a contradiction. This proves (1).

By (1) and as G is not a cobipartite trigraph, there exists a triad $\{a_1^*, a_2^*, a_3^*\}$ and we may assume that $a_i^* \in F_i\backslash(F_{i+1} \cup F_{i+2})$, $i = 1, 2, 3$. Let $\overline{a}_i \in F_i \cap \Sigma_{a_i^*, a_{i+2}^*}^{x_i}$ and $\overline{a}_i' \in F_i \cap \Sigma_{a_i^*, a_{i+1}^*}^{x_{i+1}}$ such that $\overline{a}_i, \overline{a}_i'$ are in triads and $\Sigma_{\overline{a}_i, \overline{a}_i'}^{a_i^*}$ is maximal. Let $\mathcal{A}_i = \Sigma_{\overline{a}_i, \overline{a}_i'}^{a_i^*}$, $\mathcal{B}_i = \Sigma_{a_i^*, a_{i+2}^*}^{x_i}\backslash(\mathcal{A}_i \cup \mathcal{A}_{i+2})$, $A_i = V(G) \cap \mathcal{A}_i$ and $B_i = V(G) \cap \mathcal{B}_i$. By the definition of $\overline{a}_1, \overline{a}_2, \overline{a}_3, \overline{a}_1', \overline{a}_2', \overline{a}_3'$, no vertex in $B_1 \cup B_2 \cup B_3$ is in a triad.

(2) *$\{\overline{a}_1, \overline{a}_2, \overline{a}_3\}$ and $\{\overline{a}_1', \overline{a}_2', \overline{a}_3'\}$ are triads.*

By the definition, $\overline{a}_1$ is in a triad. Let $\{\overline{a}_1, a_2, a_3\}$ be a triad, then we assume that $a_i \in A_i$, $i = 2, 3$. By 2.2, $\overline{a}_1$ is non adjacent to $\overline{a}_3$. Now, using symmetry, we deduce that $\{\overline{a}_1, \overline{a}_2, \overline{a}_3\}$ and $\{\overline{a}_1', \overline{a}_2', \overline{a}_3'\}$ are triads. This proves (2).

(3) *For all $x \in A_i$ there exist $y \in A_{i+1}, z \in A_{i+2}$ such that $\{x, y, z\}$ is a triad.*

By symmetry, we may assume that $x \in A_1$. If $|A_1| = 1$, then $x = a_1^*$ and $\{a_1^*, a_2^*, a_3^*\}$ is a triad.

Therefore, we may assume that $\overline{a}_1 \neq \overline{a}_1'$. By (2) and 2.2, x is antiadjacent to $\overline{a}_2'$ and $\overline{a}_3$. We may assume that $\{x, \overline{a}_2', \overline{a}_3\}$ is not a triad, then $\overline{a}_2'$ is strongly adjacent to $\overline{a}_3$. By (2) and 2.2, $\overline{a}_2$ is strongly antiadjacent to $\overline{a}_3'$. Since $x - \overline{a}_2 - \overline{a}_2' - \overline{a}_3 - \overline{a}_3' - x$ is not a hole of length 5, we deduce that x is not strongly complete to $\{\overline{a}_2, \overline{a}_3'\}$. But now one of $\{x, \overline{a}_2', \overline{a}_3'\}$, $\{x, \overline{a}_2, \overline{a}_3\}$ is a triad. This proves (3).

(4) *$\{x_1, x_2, x_3\}$ is a triangle such that $x_i \in B_i$ for $i = 1, 2, 3$.*

By (3), $x_i \notin A_1 \cup A_2 \cup A_3$ for $i = 1, 2, 3$. By the definition of B_i, it follows that $x_i \in B_i$ for $i = 1, 2, 3$. Moreover, $\{x_1, x_2, x_3\}$ is a essential triangle. This proves (4).

(5) *$(A_1, A_2, A_3 | B_1, B_2, B_3)$ is a hex-join.*

By the definition of $A_1, A_2, A_3, B_1, B_2, B_3$, they are clearly pairwise disjoint and $\bigcup_i (A_i \cup B_i) = V(G)$. Clearly A_i is a strong clique as $\mathcal{A}_i \subset F_i,\ i = 1, 2, 3$.

If there exist $x_i, x_i' \in B_i$ such that x_i is antiadjacent to x_i', then $\{x_i, x_i', a_{i+1}^*\}$ is a triad by 2.2, a contradiction. Thus B_i is a strong clique for $i = 1, 2, 3$.

By symmetry, it remains to show that B_1 is strongly anticomplete to A_2 and strongly complete to A_1. Since $B_1 \subset \overline{\Sigma}^{a_2^*}_{a_1^*, a_3^*}$, we deduce that B_1 is strongly anticomplete to A_2 by 2.2.

Suppose there is $a_1 \in A_1$ and $b_1 \in B_1$ such that a_1 is antiadjacent to b_1. By (3), let $a_2 \in A_2$ and $a_3 \in A_3$ be such that $\{a_1, a_2, a_3\}$ is a triad. Since a_2 is anticomplete to $\{a_1, a_3\}$, and $b_1 \in \overline{\Sigma}^{a_2}_{a_1, a_3}$, we deduce by 2.2 that b_1 is strongly antiadjacent to a_2. Thus $\{a_1, a_2, b_1\}$ is a triad, a contradiction as $b_1 \in B_1$. This concludes the proof of (5).

(6) *There is no triangle $\{a_1, a_2, a_3\}$ with $a_i \in A_i,\ i = 1, 2, 3$*

Let $a_i \in A_i,\ i = 1, 2, 3$ be such that a_1 is adjacent to $a_i,\ i = 2, 3$. By (3), let $c_i \in A_i,\ i = 2, 3$ such that $\{a_1, c_2, c_3\}$ is a triad. By 2.3, $c_2 \in \overline{\Sigma}^{a_1}_{a_2, a_3}$. By symmetry, $c_3 \in \overline{\Sigma}^{a_1}_{a_2, a_3}$. Since $\{a_2 | a_1, c_2, c_3\}$ is not a claw, we deduce that c_3 is strongly antiadjacent to a_2. By (2) and as $a_2 \in \overline{\Sigma}^{\overline{a}_3'}_{\overline{a}_2', \overline{a}_1'}$, $\overline{a}_3'$ is antiadjacent a_2. Since $a_3 \in \overline{\Sigma}^{a_2}_{c_3, \overline{a}_3'}$ and by (2), a_3 is strongly antiadjacent to a_2. This proves (6).

For the rest of the proof of 2.8, let $\{j, k, l\} = \{1, 2, 3\}$.

(7) *There is no induced 3-edge path $w - x - y - z$ such that $w \in A_j,\ x, y \in A_k,\ z \in A_l$.*

Assume that $w{-}x{-}y{-}z$ be a 3-edge path such that $w \in A_1$, $x, y \in A_2$, $z \in A_3$. Now $w{-}x{-}y{-}z{-}x_1{-}w$ is a hole of length 5, a contradiction. This proves (7).

(8) *For $i = 1, 2, 3$, let $y_i \in A_i$. Then y_k is strongly antiadjacent to at least one of y_j, y_l.*

Suppose there exist $y_i \in A_i,\ i = 1, 2, 3$ such that y_2 is adjacent to y_1 and y_3. By (6), y_1 is strongly antiadjacent to y_3. By (3), there exist $z^1, z^3 \in A_2$ such that z^1 is antiadjacent to y_1 and z^3 is antiadjacent to y_3. Since $\{y_2 | y_1, y_3, z^3\}$ and $\{y_2 | y_1, y_3, z^1\}$ are not claws, we deduce that y_1 is strongly adjacent to z^3, and y_3 is strongly adjacent to z^1. But $y_1 - z^3 - z^1 - y_3$ is a 3-edge path, contrary to (7). This proves (8).

(9) *A_j is stongly anticomplete to at least one of A_k, A_l.*

Assume not. By symmetry, we may assume there are $x \in A_1$, $y, z \in A_2$ and $w \in A_3$ such that x is adjacent to y and z is adjacent to w. By (8), x is strongly anticomplete to w, y is strongly anticomplete to w, and z is strongly anticomplete to x; and in particular $y \neq z$. But now $x - y - z - w$ is a 3-edge path, contrary to (7). This proves (9).

(10) *For $i = 1, 2, 3$, let $b_i \in B_i$ such that b_k is adjacent to b_l. Then b_j is strongly adjacent to at least one of b_k, b_l.*

By symmetry, we may assume that $j = 1$, $k = 2$ and $l = 3$. Since $b_1 - a_3^* - b_3 - b_2 - a_1^* - b_1$ is not a hole of length 5, we deduce that b_1 is strongly adjacent to at least one of b_2, b_3. This proves (10).

(11) *Let $x \in B_j$, then x is strongly complete to one of B_k, B_l.*

Assume there is $y \in B_k$ such that x is antiadjacent to y. Let $z \in B_l$. If y is antiadjacent to z, then x is strongly adjacent to z since $\{x, y, z\}$ is not a triad. By (10), if y is strongly adjacent to z, then x is strongly adjacent to z. Thus x is strongly complete to B_l. This proves (11).

By (9) and symmetry, we may assume that A_2 is strongly anticomplete to $A_1 \cup A_3$.

Let B_i^i be the set of all vertices of B_i that are strongly complete to $B_{i+1} \cup B_{i+2}$. For $j \neq i$, let B_i^j be the set of all vertices of $B_i \backslash B_i^i$ that are strongly complete to B_j. By (11), we deduce that $B_i = \bigcup_{j=1}^{3} B_i^j$.

(12) *If $B_j^k = \emptyset$, then $B_l^k = \emptyset$.*

Assume that B_j^k is empty. It implies that B_l^k is strongly complete to $B_j \cup B_k$, contrary of the definition of B_l^l and B_l^k. This proves (12).

Now, we observe that A_2, B_1^1, B_2^2, B_3^3 are homogeneous sets and $(A_1, A_3), (B_1^2, B_3^2), (B_2^3, B_1^3), (B_3^1, B_2^1)$ are homogeneous pairs. If A_1 is strongly anticomplete to A_3, then by (4) and (12), G is a thickening of a member of $\mathcal{C}$. Thus, we may assume that A_1 is not anticomplete to A_3. Since $A_1 - A_3 - B_3^1 - A_2 - B_2^1 - A_1$ is not a hole of length 5, we deduce that either $B_2^1 = \emptyset$ or $B_3^1 = \emptyset$. By (12), it follows that $B_2^1 \cup B_3^1$ is empty. Using (4) and (12), we deduce that G is a thickening of a member of $\mathcal{C}$. This concludes the proof of 2.8. □

3 Holes of Length 4

Next we examine circular interval trigraphs that contain a hole of length 4.

3.1. *Let G be a Berge circular interval trigraph. If G has a hole of length 4 and no essential triangle, then G is a structured circular interval trigraph.*

Proof. In the following proof, the addition is modulo 4. Let G be defined by Σ and $F_1, \dots, F_k$. Let $x_1^* - x_2^* - x_3^* - x_4^* - x_1^*$ be a hole of length 4. We may assume that $x_i^*, x_{i+1}^* \in F_i,\ i = 1, 2, 3, 4$.

(1) *x_i^* is strongly antiadjacent to x_{i+2}^*.*

Assume not. By symmetry we may assume that x_1^* is adjacent to x_3^*. Moreover, we may assume that there exists $i \in \{1, \dots, k\}$ such that $\Sigma_{x_1^*, x_3^*}^{x_2^*} \subseteq F_i$. If $i = 4$, it implies that $\{x_1^*, x_2^*, x_3^*, x_4^*\} \subset F_4$, and thus $x_1^* - x_2^* - x_3^* - x_4^* - x_1^*$ is not a hole, a contradiction. Symmetrically, we may assume that $i \neq 3$. But now $\{x_1^*, x_3^*, x_4^*\}$ is an essential triangle since $F_i \cup F_3 \cup F_4 = \Sigma$, a contradiction. This proves (1).

For $i = 1, 2, 3, 4$, let $\mathcal{X}_i, \mathcal{Y}_i \subset \Sigma$ and $X_i, Y_i \subset V(G)$ be such that:

(H1) each of $\mathcal{X}_i, \mathcal{Y}_i$ is homeomorphic to $[0, 1)$,

(H2) $X_i \subseteq V(G) \cap \mathcal{X}_i$, $Y_i \subseteq V(G) \cap \mathcal{Y}_i,\ i = 1, 2, 3, 4$,

(H3) $\bigcup_i (\mathcal{X}_i \cup \mathcal{Y}_i) = \Sigma$,

(H4) $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4$ are pairwise disjoint,

(H5) $\mathcal{Y}_i \subseteq \overline{\Sigma}_{x_i^*, x_{i+1}^*}^{x_{i+2}^*},\ i = 1, 2, 3, 4$, (H6) $x_i^* \in X_i,\ i = 1, 2, 3, 4,$

(H7) $X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4$ are disjoints cliques satisfying (S2)-(S6),

(H8) $\bigcup_i (X_i \cup Y_i)$ is maximal.

By (1), such a structure exists. We may assume that $V(G)\backslash \bigcup_i (X_i \cup Y_i)$ is not empty. Let $x \in V(G)\backslash \bigcup_i (X_i \cup Y_i)$. For $S \subseteq V(G)\backslash\{x\}$, we denote by S^C the subset of S that is complete to x, and by S^A the subset of S that is anticomplete to x.

For $i = 1, 2, 3, 4$, let $x_i^l, x_i^r \in X_i$ be such that $x_{i-1}^*, x_i^l, x_i^r, x_{i+1}^*$ are in this order on Σ and such that $\overline{\Sigma}_{x_i^l, x_i^r}^{x_{i+1}^*}$ is maximal.

(2) *$\{x_i^r, x_{i+1}^l\}$ is complete to $X_i \cup X_{i+1}$.*

By (S5), there exists $a \in X_i$ such that a is adjacent to x_{i+1}^r. By 2.3 and (S6), there exists F_l such that $\{a, x_i^r\} \cup X_{i+1} \subseteq F_l$ and thus x_i^r is complete to X_{i+1}. By symmetry, x_{i+1}^l is complete to X_i. This proves (2).

(3) *If X_i is not complete to X_{i+1}, then x_i^l is strongly antiadjacent to x_{i+1}^r.*

Let $a \in X_i$ and $b \in X_{i+1}$ be such that a is antiadjacent to b. By 2.2 and (S6), a is strongly antiadjacent to x_{i+1}^r. By 2.2 and (S6), x_{i+1}^r is strongly antiadjacent to x_i^l. This proves (3).

(4) *$x \notin \overline{\Sigma}_{x_i^l, x_i^r}^{x_{i+1}^l}$ for all i.*

Assume not. We may assume that $x \in \overline{\Sigma}_{x_1^l, x_1^r}^{x_2^l}$. For $i = 1, 2, 3, 4$, let $Y_i' = Y_i$, for $i = 2, 3, 4$, let $X_i' = X_i$ and let $X_1' = X_1 \cup \{x\}$. Since $Y_2 \cup Y_3 \cup X_3$ is strongly anticomplete to $\{x_1^r, x_1^l\}$ by (S3) and (S6), we deduce by 2.2 that x is strongly anticomplete to $Y_2 \cup Y_3 \cup X_3$. Since x_1^r is adjacent to x_4^r by (2), we deduce by 2.3 that x is strongly complete to Y_4 and not strongly anticomplete to X_4. By symmetry, x is strongly complete to Y_1 and not strongly anticomplete to X_2. Since x_1^l is stongly adjacent to x_1^r, we deduce that X_1' is a strong clique. If X_1 is strongly complete to X_2, it follows from 2.3 that x is strongly complete to X_2. By symmetry, if X_1 is strongly complete to X_4, then x is strongly complete to X_4. The above arguments show that $X_1', \dots, X_4', Y_1', \dots, Y_4'$ are disjoint cliques satisfying (S2)-(S6). Moreover, $\mathcal{X}_i, \mathcal{Y}_i$ $i = 1, 2, 3, 4$ clearly satisfy (H1)-(H5) with X_i', Y_i' $i = 1, 2, 3, 4$, contrary to the maximality of $\bigcup_i (X_i \cup Y_i)$. This proves (4).

By (4) and by symmetry, we may assume that $x \in \overline{\Sigma}_{x_1^r, x_2^l}^{x_3^*}$ and therefore $x \in F_1$. By 2.2 and (S3), x is strongly anticomplete to Y_3. Since $x \in F_1$, we deduce that x is strongly complete to Y_1.

(5) *X_3^C is strongly anticomplete to X_4^C.*

Assume not. We may assume there exist $x_3 \in X_3^C$ and $x_4 \in X_4^C$ such that x_3 is adjacent to x_4. By (S6), x_3 is strongly antiadjacent to x_1^* and therefore by 2.3 there exists F_i, $i \in \{1, \dots, k\}$, such that $x, x_3 \in F_i$ and $x_1^* \notin F_i$. By symmetry, there exists F_j, $j \in \{1, \dots, k\}$ such that $x, x_4 \in F_j$ and $x_2^* \notin F_j$. Moreover, as $x_2^* \in F_i$, we deduce that $F_i \neq F_j$. By (S6), x_i^* is strongly anticomplete to x_{i+2} for $i = 1, 2$. Now, since x_3 is adjacent to x_4, we deduce from 2.3 that there exists F_l such that $x_3, x_4 \in F_l$ and $l \in \{1, \dots, k\}\backslash\{i, j\}$. Since $\overline{\Sigma}_{x,x_3}^{x_4} \subseteq F_i$, $\overline{\Sigma}_{x,x_4}^{x_3} \subseteq F_j$ and $\overline{\Sigma}_{x_3,x_4}^{x} \subseteq F_k$, we deduce that $F_i \cup F_j \cup F_k = \Sigma$. Hence, $\{x, x_3, x_4\}$ is an essential triangle, a contradiction. This proves (5).

(6) *At least one of X_3^C, X_4^C is empty.*

Assume not. Let $a \in X_4^C$. By 2.3 and since a is strongly anticomplete to X_2, we deduce that there is F_i, $i \in \{1,\dots,k\}$, such that $\{a, x_4^r, x\} \in F_i$ and thus $x_4^r \in X_4^C$. Symmetrically, $x_3^l \in X_3^C$. By (5), x_4^r is strongly antiadjacent to x_3^l. By (S6), X_1 is strongly complete to X_4, and X_2 is strongly complete to X_3. By (2) and (5), x is anticomplete to $\{x_3^r, x_4^l\}$. But now by (2) and (S6), $x - x_4^l - x_2^l - x_4^r - x_3^l - x_1^r - x_3^r - x$ is an antihole of length 7, a contradiction. This proves (6).

By symmetry, we may assume that x is strongly anticomplete to X_4. By (2) and 2.3, x is strongly complete to $X_1 \cup X_2$.

(7) *x is adjacent to x_3^l.*

Assume not. By 2.2, x is strongly anticomplete to X_3. Since $x - Y_2 - x_3^r - x_4^r - X_1 - x$ and $x - Y_4 - x_4^l - x_3^l - X_2 - x$ are not holes of length 5, we deduce that x is strongly anticomplete to $Y_2 \cup Y_4$. Since $x - X_2 - X_3 - X_4 - X_1 - x$ is not a cycle of length 5, we deduce that X_1 is strongly complete to X_2. For $i = 1,2,3,4$, let $X_i' = X_i$, for $i = 2,3,4$, let $Y_i' = Y_i$, and let $Y_1' = Y_1 \cup \{x\}$. The above arguments show that $X_1', \dots, X_n', Y_1', \dots, Y_n'$ are disjoint cliques satisfying (S2)-(S6). Moreover, it is easy to find $\mathcal{X}_i', \mathcal{Y}_i'$ $i = 1,2,3,4$ satisfying (H1)-(H5), contrary to the maximality of $\bigcup_i (X_i \cup Y_i)$. This proves (7).

By 2.3 and (7), x is strongly complete to Y_2. For $i = 3,4$, let $X_i' = X_i$, for $i = 1,2,3$, let $Y_i' = Y_i$, let $Y_4' = Y_4^A$, let $X_1' = X_1 \cup Y_4^C$ and let $X_2' = X_2 \cup \{x\}$. The above arguments show that $X_1', \dots, X_n', Y_1', \dots, Y_n'$ are disjoint cliques satisfying (S2), (S3) and (S5). To get a contradiction, it remains to show that $X_1', \dots, X_n', Y_1', \dots, Y_n'$ satisfy (S4) and (S6).

First we check (S4). Since $X_3' = X_3$, $X_4' = X_4$ and $Y_3' = Y_3$, and since $X_1' \backslash X_1 \subset Y_4$ is strongly complete to X_4, it is enough to check the following:

- If $Y_2 \neq \emptyset$ then X_2' is complete to X_3'.
- If $Y_1 \neq \emptyset$ then X_1' is complete to X_2'.

For the former, we observe that if x is not strongly complete to X_3, then since $x - Y_2 - X_3 - X_4 - X_1 - x$ is not a hole of length 5, we deduce that Y_2 is empty. For the latter, since x is strongly complete to X_1, it is enought to show that if Y_1 is not empty, then Y_4^C is empty. Since X_3^C is not empty, it follows that $Y_1 \subseteq \overline{\Sigma}_{x, x_1^*}^{x_2^*}$. Now if Y_4^C is not empty, then Y_1 is empty by 2.3 and (S4).

To check (S6), we need to prove the following:

(i) If X_1' is not strongly complete to X_2' then X_2' is strongly complete to X_3'.

(ii) If X_2' is not strongly complete to X_3' then X_3' is strongly complete to X_4'.

(iii) If X_3' is not strongly complete to X_4' then X_4' is strongly complete to X_1'.

(iv) If X_4' is not strongly complete to X_1' then X_1' is strongly complete to X_2'.

For (i), first assume that x is not strongly complete to X_3. By (2.2), we deduce that x is strongly anticomplete to x_3^r. Since $x - x_2^r - x_3^r - X_4 - Y_4 - x$ and $x - x_2^r - x_3^r - X_4 - X_1 - x$ are not cycles of length 5, we deduce that Y_4^C is empty and that X_1 is strongly complete to X_2. Thus $X_1' = X_1$ and since x is strongly complete to X_1, it follow that X_1' is strongly complete to X_2'. So we may assume that x is strongly complete to X_3. By 2.3 and (S6), it follows that X_2 is strongly complete to X_3 and thus X_2' is strongly complete to X_3'. This proves (i).

For (ii), if X_3' is not strongly complete to X_4', then by (3) it follows that x_3^l is strongly antiadjacent to x_4^r. Moreover by (S4), X_2 is strongly complete to X_3. Since $x - x_3^l - x_3^r - x_4^r - X_1 - x$ is not a cycle of length 5, we deduce, using (2), that x is strongly complete to X_3 and thus X_3' is strongly complete to X_2'. This proves (ii).

For (iii) and (iv), we may assume that X'_4 is not strongly complete to X'_1. Since X_4 is stongly complete to Y_4, we deduce that X_4 is not strongly complete to X_1. But by (S6), it implies that X_4 is strongly complete to X_3 and thus X'_4 is strongly complete to X'_3, and (iii) follows. Also by (S6), we deduce that X_1 is strongly complete to X_2. Moreover by (S4), it follows that Y_4 is empty. Since x is strongly complete to X_1, we deduce that X'_1 is strongly complete to X'_2, and (iv) follows.

The above arguments show that $X'_1, \dots, X'_n, Y'_1, \dots, Y'_n$ are disjoint cliques satisfying (S2)-(S6). Moreover, it is easy to find $\mathcal{X}'_i, \mathcal{Y}'_i$ $i = 1, 2, 3, 4$ satisfying (H1)-(H5), contrary to the maximality of $\bigcup_i (X_i \cup Y_i)$. This concludes the proof of 3.1 $\square$

4 Long Holes

In this section, we study circular interval trigraphs that contain a hole of length at least 6.

A result equivalent to 4.1 has been proved independently by Kennedy and King [5]. The following was proved in joint work with Varun Jalan.

4.1. *Let G be a circular interval trigraph defined by Σ and $F_1, \dots, F_k \subseteq \Sigma$. Let $P = p_0 - p_1 - \dots - p_{p-1} - p_0$ and $Q = q_0 - q_1 - \dots - q_{q-1} - q_0$ be holes. If $p + 1 < q$ then there is a hole of length l for all $p < l < q$. In particular, if G is Berge then all holes of G have the same length.*

Proof. We start by proving the first assertion of 4.1. We may assume that the vertices of P and Q are ordered clockwise on Σ. Since P and Q are holes, it follows that $p \geq 4$ and $q > 5$. We are going to prove the following claim which directly implies the first assertion of 4.1 by induction.

(1) *There exists a hole of length $q - 1$.*

We may assume that Q and P are chosen such that $|V(Q) \cap V(P)|$ is maximal.

(2) *If there are $i \in \{0, \dots, q-1\}, j \in \{0, \dots, p-1\}$ such that*

$$q_i, q_{i+1 \mod q} \in \overline{\Sigma}_{p_j, p_{j+1 \mod p}}^{p_{j+2 \mod p}} \backslash \{p_j, p_{j+1 \mod p}\}$$

then there is a hole of length $q - 1$ in G.

We may assume that $q_1, q_2 \in \overline{\Sigma}_{p_1, p_2}^{p_3} \backslash \{p_1, p_2\}$. Since q_1 is antiadjacent to q_3, we deduce that $q_3 \notin \overline{\Sigma}_{p_1, p_2}^{p_3}$. Since $p_2 \in \overline{\Sigma}_{q_2, q_3}^{q_1}$, we deduce by 2.3 that p_2 is strongly anticomplete to $\{q_0, q_5\}$.

If p_2 is adjacent to q_4, it follows that $Q - q_1 - p_2 - q_4 - Q$ is a hole of length $q - 1$. Thus we may assume that p_2 is strongly antiadjacent to q_4. But then $Q' = Q - q_1 - p_2 - q_3 - Q$ is a hole of length q with $|V(Q') \cap V(P)| > |V(Q) \cap V(P)|$, a contradiction. This proves (2).

By (2) and since $q > p + 1$, we may assume that $|V(P) \cap V(Q)| > 1$. Let $V(P) \cap V(Q) = \{x_0, x_1, \dots, x_{n-1}\}$. We may assume that $x_0, \dots, x_{n-1}$ are in clockwise order on Σ. For $i \in \{0, \dots, n-1\}$, let $A_i = \overline{\Sigma}_{x_i, x_{i+1 \mod n}}^{x_{i+2 \mod n}}$. Since $q > p+1$, there exists $k \in \{0, \dots, n-1\}$ such that $|A_k \cap V(P)| < |A_k \cap V(Q)|$. By (2), it follows that $|A_k \cap V(P)| = |A_k \cap V(Q)| - 1$. Let P' be the subpath of P such that $V(P') = V(P) \cap A_k$. Let Q' be the subpath of Q such that $V(Q') \cap A_k = \{x_i, x_{i+1}\}$. Then $x_1 - P' - x_2 - Q' - x_1$ is a hole of length $q - 1$.

This proves (1) and the first assertion of 4.1. Since every hole in a Berge trigraph has even length, the second assertion of 4.1 follows immediately from the first. This concludes the proof of 4.1. $\square$

4.2. *Let G be a Berge circular interval trigraph. If G has a hole n with $n \geq 6$, then G is a structured circular interval trigraph.*

Proof. Let G be a Berge circular interval trigraph. Let $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ be pairwise disjoint cliques satisfying $(S2)-(S6)$ and with $|\bigcup_i (X_i \cup Y_i)|$ maximum. By (S5) and (S6), $X_1 - X_2 - \dots - X_n - X_1$ is a hole, and so n is even. We may assume that $V(G) \setminus \bigcup_i (X_i \cup Y_i)$ is not empty. Let $x \in V(G) \setminus \bigcup_i (X_i \cup Y_i)$.

For $S \subseteq V(G) \setminus \{x\}$, we denote by S^C the subset of S that is complete to x, and by S^A the subset of S that is anticomplete to x.

(1) *If $y \in X_i^C$ and $z \in X_{i+1}^C$ then y is strongly adjacent to z.*

Assume not. We may assume $y \in X_1^C$ and $z \in X_2^C$ but y is antiadjacent to z. By (S3), $Y_1 = \emptyset$. By (S6), X_2 is strongly complete to X_3 and X_n is strongly complete to X_1. Since $\{x | y, z, \cup_{i=4}^{n-1} X_i \cup_{i=3}^{n-1} Y_i\}$ is not a claw, x is strongly anticomplete to $X_4, \dots, X_{n-1}, Y_3, \dots, Y_{n-1}$. Since $x - z - X_3 - \dots - X_{n-1} - y - x$ is not a hole of length $n+1$, we deduce that x is strongly complete to at least one of X_3 or X_n. Without loss of generality, we may assume that x is strongly complete to X_3. Since $x - X_3 - X_4 - \dots - X_{n-1} - x$ is not a hole of length $n-1$, x is strongly anticomplete to X_n. Since $\{X_3 | X_4, Y_2, x\}$ and $\{X_3 | X_2, X_4, x\}$ are not claws, we deduce that x is strongly complete to $Y_2 \cup X_2$.

For $i = 3, \dots, n$, let $X'_i = X_i$, for $i = 1, \dots, n-1$, let $Y'_i = Y_i$. Let $X'_2 = X_2 \cup \{x\}$, $X'_1 = X_1 \cup Y_n^C$ and $Y'_n = Y_n^A$. Then $X'_1, \dots, X'_n, Y'_1, \dots, Y'_n$ are disjoint cliques satisfying $(S2)-(S6)$ but with $|\bigcup_i (X_i \cup Y_i)| < |\bigcup_i (X'_i \cup Y'_i)|$, a contradiction. This proves (1).

(2) *If $X_i^C \neq \emptyset$ and $X_{i+2}^C \neq \emptyset$ then $X_{i+1}^A = \emptyset$.*

Assume not. We may assume $y \in X_n^C$ and $z \in X_2^C$ and $w \in X_1^A$. Since $\{x | y, z, \cup_{i=4}^{n-2} X_i\}$ is not a claw, x is strongly anticomplete to $X_4, \dots, X_{n-2}$. Assume that $C = x - X_3 - \dots - X_{n-1} - x$ is a hole. Then C has length $n-2$. If $n = 6$, then w is strongly anticomplete to $V(C)$, contrary to 2.6. If $n \geq 8$, we get a contradiction to the last assertion of 4.1. Thus x is strongly anticomplete to at least one of X_3 or X_{n-1}. By symmetry, we may assume that x is strongly anticomplete to X_3. Since $x - X_2 - X_3 - \dots - X_{n-1} - x$ is not a hole length $n-1$, x is strongly anticomplete to X_{n-1}. By (S6) and symmetry, we may assume that X_1 is strongly complete to X_2. But now $\{z | X_3, x, w\}$ is a claw, a contradiction. This proves (2).

(3) *If $X_i^C \neq \emptyset$, then $X_{i+2}^C = \emptyset$.*

Assume not. We may assume there exist $y \in X_n^C$ and $z \in X_2^C$. By (2), x is strongly complete to X_1. Since $\{x | y, z, \cup_{i=4}^{n-2} X_i \cup_{j=3}^{n-2} Y_j\}$ is not a claw, it follows that x is strongly anticomplete to $X_4, \dots, X_{n-2}$ and $Y_3, \dots, Y_{n-2}$.

If $X_3^C \neq \emptyset$, then either $\{x | X_1, X_3, X_{n-1}\}$ is a claw or $x - X_3 - X_4 - \dots - X_n - x$ is a hole of length $n-1$ and therefore odd, hence x is strongly anticomplete to X_3. By symmetry, x is strongly anticomplete to X_{n-1}. Since $\{z | X_3, x, Y_1\}$ and $\{y | X_{n-1}, x, Y_n\}$ are not claws, x is strongly complete to $Y_1 \cup Y_n$.

For $i = 3, \dots, n-1$, let $X'_i = X_i$ and for $i = 1, 3, 4, \dots, n-2, n$, let $Y'_i = Y_i$. Let $X'_2 = X_2 \cup Y_2^C$, let $X'_1 = X_1 \cup \{x\}$, let $Y'_2 = Y_2^A$, let $X'_n = X_n \cup Y_{n-1}^C$ and let $Y'_{n-1} = Y_{n-1}^A$.

Clearly $X'_1, \dots, X'_n, Y'_1, \dots, Y'_n$ are disjoint cliques such that $|\bigcup_i (X_i \cup Y_i)| < |\bigcup_i (X'_i \cup Y'_i)|$. The above arguments show that $X'_1, \dots, X'_n, Y'_1, \dots, Y'_n$ satisfy (S2) and (S5). To get a contradiction, we need to show that $X'_1, \dots, X'_n, Y'_1, \dots, Y'_n$ satisfy (S3), (S4) and (S6).

Since $\{x | X_n, Y_1, Y_2^C\}$ is not a claw, we deduce that either $Y_1 = \emptyset$ or $Y_2^C = \emptyset$. In both cases, it implies that Y'_1 is strongly complete to X'_2. Symmetrically, Y'_n is strongly complete to X'_{n-1}. Hence, (S3) is satisfied.

It remains to prove the following.

i) If $Y_1 \neq \emptyset$, then X'_1 is strongly complete to X'_2

ii) If $Y_n \neq \emptyset$, then X'_n is strongly complete to X'_1

iii) X'_2 is strongly complete to at least one of X'_3, X'_1.

iv) X'_n is strongly complete to at least one of X'_{n-1}, X'_2.

v) X_1' is strongly complete to at least one of X_n', X_2'.

Assume that $Y_1 \neq \emptyset$. It implies by (S4), that X_1 is strongly complete to X_2. Since $\{x|Y_n, Y_1, Y_2^C\}$ is not a claw, we deduce that $Y_2^C = \emptyset$. Since $x - Y_1 - X_2^A - X_3 - \ldots - X_n - x$ is not a hole of length $n+1$, we deduce that $X_2^A = \emptyset$ and thus X_1' is strongly complete to X_2'. This proves i) and by symmetry ii) holds.

If $Y_2^C \neq \emptyset$, it follows by (S4) that X_2' is strongly complete to X_3' and iii) holds. Thus we may assume that Y_2^C is empty. If X_2^A is empty, and since by (S6), X_2 is strongly complete to at least one of X_1, X_3, it follows that X_2' is strongly complete to at least one of X_1', X_3'. Thus we may assume that $X_2^A \neq \emptyset$. Since $x - Y_1 - X_2^A - X_3 - \ldots - X_n - x$ is not a hole of length $n+1$, we deduce that $Y_1 = \emptyset$.

Assume that there exist $w \in X_2$ and $v \in X_3$ such that w is antiadjacent to v. Suppose first that $w \in X_2^C$. Since $x - w - X_2^A - v - X_4 - \ldots - X_n - x$ is not a cycle of length $n+1$, we deduce that v is strongly anticomplete to X_2^A. By (S5), there exists $a \in X_2^C$ adjacent to v. But $\{a|x, v, X_2^A\}$ is a claw, a contradiction. Thus we may assume that $w \in X_2^A$ and v is strongly complete to X_2^C. But $\{z|x, v, w\}$ is a claw, a contradiction. Hence X_2 is strongly complete to X_3. This proves iii) and by symmetry iv) holds.

We claim that x is strongly complete to at least one of X_2 or X_n. Suppose that $p \in X_n^A$ and $q \in X_2^A$. By (S5) and (S6), there is $r \in X_1$ that is adjacent to both p and q. But $\{r|p, q, x\}$ is a claw, a contradiction. This proves the claim. By symmetry we may assume that x is strongly complete to X_n. By (1), X_n is strongly complete to X_1. If $Y_{n-1}^C = \emptyset$, it follows that X_1' is strongly complete to X_n' and v) holds. Thus we may assume that $Y_{n-1}^C \neq \emptyset$. Since $\{x|X_1, Y_{n-1}^C, Y_2^C\}$ is not a claw, we deduce that $Y_2^C = \emptyset$. Since $x - Y_{n-1}^C - X_{n-1} - \ldots - X_3 - X_2^A - X_1 - x$ is not a hole of length $n+1$, we deduce that X_2^A is empty. By (1), X_1 is strongly complete to X_2 and thus X_1' is strongly complete to X_2'. This proves v). This concludes the proof of (3).

Let $C = x_1 - x_2 - \ldots - x_n - x$ be a hole of length n with $x_i \in X_i$. By 2.6, x is strongly adjacent to two consecutive vertices of C. Without loss of generality, we may assume that x is strongly complete to $\{x_1, x_2\}$. By (1), x_1 is strongly adjacent to x_2. By (3), x is strongly anticomplete to $X_3 \cup X_4 \cup X_{n-1} \cup X_n$. Since $G|(\{x\} \bigcup_i X_i)$ does not induce a cycle of length $p \neq n$ by 4.1, we deduce that x is strongly anticomplete to X_i for $i = 5, \ldots, n-2$. Similarly, x is strongly anticomplete to $Y_4 \cup \ldots \cup Y_{n-2}$ otherwise there is a hole of length $p \neq n$ in G.

Since $x - Y_2 - X_3 - \ldots - X_n - X_1 - x$ and $x - Y_n - X_n - \ldots X_2 - x$ are not holes of length $n+1$, we deduce that x is strongly anticomplete to $Y_2 \cup Y_n$.

Since $\{X_2^C|X_1^A, x, X_3\}$ and $\{X_1^C|X_2^A, x, X_n\}$ are not claws, it follows that X_1^A is strongly anticomplete to X_2^C and X_1^C is strongly anticomplete to X_2^A. Suppose there is $a \in X_1^A$. By (S4), there is $b \in X_2^A$ adjacent to a. But $G|(\{x_1, x_2, a, b\}$ is a hole of length 4 strongly anticomplete to X_4, contrary to 2.6. Thus $X_1^A = X_2^A = \emptyset$ and by (1), X_1 is strongly complete to X_2. Since $\{X_1|x, Y_1, X_n\}$ is not a claw, we deduce that x is strongly complete to Y_1.

If $Y_3^C \neq \emptyset$, it follows that $x - X_2 - X_3 - Y_3^C - x$ is a hole of length 4 strongly anticomplete to X_n, contrary to 2.6. Thus $Y_3^C = \emptyset$ and by symmetry $Y_{n-1}^C = \emptyset$.

For $i = 1, \ldots, n$, let $X_i' = X_i$, for $i = 2, \ldots, n$, let $Y_i' = Y_i$ and let $Y_1' = Y_1 \cup \{x\}$. The above arguments show that $X_1', \ldots, X_n', Y_1', \ldots, Y_n'$ are cliques satisfying $(S2) - (S6)$ but $|\bigcup_i (X_i \cup Y_i)| < |\bigcup_i (X_i' \cup Y_i')|$, a contradiction. This concludes the proof of 4.2. □

We now have all the tools to prove theorem 2.1.

Proof of 2.1. We may assume that G is not a linear interval trigraph and not a cobipartite trigraph. By 2.5, there is an essential triangle or a hole in G. Then by 2.8, 3.1 or 4.2, G is either a structured circular interval trigraph or is a thickening of a trigraph in $\mathcal{C}$. This proves 2.1. □

5 Some Facts about Linear Interval Join

In this section we prove some lemmas about paths in linear interval stripes.

5.1. *Let G be a linear interval join with skeleton H such that G is Berge. Let e be an edge of H that is in a cycle. Let $\eta(e) = V(T)\backslash Z$ where (T, Z) is a thickening of a linear interval stripe $(S, \{x_1, x_n\})$. Then the lengths of all paths from x_1 to x_n in $(S, \{x_1, x_n\})$ have the same parity.*

Proof. Assume not. Let $C = c_0 - c_1 - \ldots - c_n - c_0$ be a cycle in H such that $e = c_0c_n$. For $i = 0, \ldots, n-1$, let $c_ic_{i+1} = e_i$, $(G_{e_i}, \{x_i^1, x_i^2\})$ be such that $\eta(e_i) = V(G_{e_i})\backslash\{x_i^1, x_i^2\}$, $\phi_{e_i}(c_i) = x_i^1$ and $\phi_{e_i}(c_{i+1}) = x_i^2$ as in the definition of a linear interval join. We may assume that $\phi_e(c_n) = x_1$ and $\phi_e(c_0) = x_n$. Let $O = x_1 - o_1 - \ldots - o_{l-1} - x_n$ be an odd path from x_1 to x_n in S and $P = x_1 - p_1 - \ldots - p_{l'-1} - x_n$ be an even path from x_1 to x_n in S. For $i = 0, 1, \ldots, n-1$, let Q_i be a path in G_{e_i} from x_i^1 to x_i^2. Let Q_i' be the subpath of Q_i with $V(Q_i') = V(Q_i)\backslash\{x_i^1, x_i^2\}$.

Let $C_1 = X_{o_1} - \ldots - X_{o_{l-1}} - Q_0' - Q_1' - \ldots - Q_{n-1}' - X_{o_1}$ and $C_2 = X_{p_1} - \ldots - X_{p_{l'-1}} - Q_0' - Q_1' - \ldots - Q_{n-1}' - X_{p_1}$. Then one of C_1, C_2 is an odd hole in G, a contradiction. This proves 5.1. $\square$

Before the next lemma, we need some additional definitions. Let $(G, \{x_1, x_n\})$ be a linear interval stripe. The *right path* of G is the path $R = r_0 - \ldots - r_p$ (where $r_0 = x_1$ and $r_p = x_n$) defined inductively starting with $i = 1$ such that $r_i = x_{i^*}$ with $i^* = \max\{j | x_j \text{ is adjacent to } r_{i-1}\}$ (i.e. from r_i take a maximal edge on the right to r_{i+1}). Similarly the *left path* of G is the path $L = l_0 - \ldots - l_p$ (where $l_0 = x_1$ and $l_p = x_n$) defined inductively starting with $i = p-1$ such that $l_i = x_{i^*}$ with $i^* = \min\{j | x_j \text{ is adjacent to } l_{i+1}\}$.

5.2. *Let G be a linear interval stripe and R be the right path of G. If $x, y \in V(R)$, then $x - R - y$ is a shortest path between x and y.*

Proof. Let $P = x - p_1 - \ldots - p_{t-1} - y$ be a path between x and y of length t and let $x - r_l - \ldots - r_{s+l-2} - y = x - R - y$. By the definition of R and since G is a linear interval stripe, we deduce that $r_{l+i-1} \geq p_i$ for $i = 1, \ldots, s-1$. Hence it follows that $s \leq t$. This proves 5.2. $\square$

5.3. *Every linear interval trigraph is Berge.*

Proof. Let G be a linear interval trigraph with $V(G) = \{v_1, \ldots, v_n\}$. The proof is by induction on the number of vertices. Clearly $H = G|\{v_1, \ldots, v_{n-1}\}$ is a linear interval trigraph, so inductively H is Berge. Since G is a linear interval trigraph, it follows that $N(v_n)$ is a strong clique. But if A is an odd hole or an odd antihole in G, then for every $a \in V(A)$, it follows that $N(a) \cap V(A)$ is not a strong clique. Therefore $v_n \notin V(A)$ and consequently G is Berge. This proves 5.3. $\square$

5.4. *Let $(G, \{x_1, x_n\})$ be a linear interval stripe. Let L and Q be two paths from x_1 to x_n of length l and q such that $l < q$. Then there exists a path of length m from x_1 to x_n in G for all $l < m < q$.*

Proof. Let G' be a circular interval trigraph obtained from G by adding a new vertex x as follows:

- $V(G') = V(G) \cup \{x\}$,
- $G'|V(G) = G$,
- x is strongly anticomplete to $V(G)\backslash\{x_1, x_n\}$,
- x is strongly complete to $\{x_1, x_n\}$.

Let $l < m < q$, $C_1 = x_1 - L - x_n - x - x_1$ and $C_2 = x_1 - Q - x_n - x - x_1$. Clearly, C_1 and C_2 are holes of length $l+2$ and $q+2$ in G'. By 4.1, there exists a hole C_m of length $m+2$ in G'. Since it is easily seen from the definition of linear interval trigraph that there is no hole in G, we deduce that $x \in V(C_m)$. Let $C_m = x - c_1 - c_2 - \ldots - c_{m+1} - x$. Since $N(x) = \{x_1, x_n\}$, we may assume that $c_1 = x_1$ and $c_{m+1} = x_n$. But now $x_1 - c_2 - \ldots - c_m - x_n$ is a path of length m from x_1 to x_n in G. This proves 5.4. $\square$

We say that a linear interval stripe $(G, \{x_1, x_n\})$ has *length* p if all induced paths from x_1 to x_n have length p.

5.5. *Let $(G, \{x_1, x_n\})$ be a linear interval stripe of length p. Let $L = l_0 - \ldots - l_p$ and $R = r_0 - \ldots - r_p$ be the left and right paths. Then $r_0 < l_1 \leq r_1 < l_2 \leq r_2 < \ldots < l_{p-1} \leq r_{p-1} < l_p$.*

Proof. Since G is a linear interval trigraph and by the definition of right path, it follows that $r_0 < r_1 < r_2 < \ldots < r_p$.

We claim that if $l_i \in (r_{i-1}, r_i]$, then $l_{i-1} \in (r_{i-2}, r_{i-1}]$. Assume that $l_i \in (r_{i-1}, r_i]$. Since r_{i-1} is adjacent to r_i, we deduce that l_i is adjacent to r_{i-1}. By the definition of the left path, $l_{i-1} \leq r_{i-1}$. Since $r_{i-1} < l_i$ and by the definition of the right path, we deduce that r_{i-2} is strongly antiadjacent to l_i. Since G is a linear interval trigraph, we deduce that $l_{i-1} > r_{i-2}$. This proves the claim.

Now, since $l_p \in (r_{p-1}, r_p]$ and using the claim inductively, we deduce that $r_{i-1} < l_i \leq r_i$ for $i = 1, \ldots, p$. This proves 5.5. □

5.6. *Let $(G, \{x_1, x_n\})$ be a linear interval stripe of length p. Let $L = l_0 - \ldots - l_p$ and $R = r_0 - \ldots - r_p$ be the left and right paths. Then $[r_0, l_i)$ is strongly anticomplete to $[l_{i+1}, l_p]$ and $[r_0, r_i]$ is strongly anticomplete to $(r_{i+1}, l_p]$ for $i = 0, \ldots, p$.*

Proof. Assume not. By symmetry, we may assume that there exist i, $a \in [r_0, l_i)$ and $b \in [l_{i+1}, l_p]$ such that a is adjacent to b. Since $l_{i+1} \in (a, b]$ and since G is a linear interval trigraph, we deduce that l_{i+1} is adjacent to a. But $a < l_i$, contrary to the definition of the left path. This proves 5.6. □

5.7. *Let $(G, \{x_1, x_n\})$ be a linear interval stripe of length $p \geq 3$. Let $L = l_0 - \ldots - l_p$ and $R = r_0 - \ldots - r_p$ be the left and right paths. If l_i and r_{i+1} are strongly adjacent for some $0 < i < p$, then G admits a 1-join.*

Proof. Let i be such that l_i and r_{i+1} are strongly adjacent. Since G is a linear interval trigraph, we deduce that $[l_i, r_{i+1}]$ is a strong clique. By 5.6, it follows that $[r_0, l_i)$ is strongly anticomplete to $(r_{i+1}, r_p]$.

Suppose there exists $x \in [l_i, r_{i+1}]$ that is adjacent to a vertex $a \in [r_0, l_i)$ and $b \in (r_{i+1}, r_p]$. By 5.6, it follows that a is strongly anticomplete to $[l_{i+1}, l_p]$ and thus $x \in [l_i, l_{i+1})$. Symmetrically, $x \in (r_i, r_{i+1}]$. Hence by 5.5, we deduce that $x \in (r_i, l_{i+1})$. By the definition of the right path and since a is adjacent to x, we deduce that $a \notin [r_0, r_{i-1}]$. Hence $a \in (r_{i-1}, l_i)$. By symmetry, $b \in (r_{i+1}, l_{i+2})$.

We claim that $P = r_0 - R - r_{i-1} - a - x - b - l_{i+2} - L - l_p$ is a path. Since $r_{i-1} < a$ and by the definition of the right path, we deduce that r_{i-2} is strongly antiadjacent to a. Since $b < l_{i+2}$ and by the definition of the left path, we deduce that b is strongly antiadjacent to l_{i+3}. By 5.6 and since $a \in (r_{i-1}, l_i)$ and $b \in (r_{i+1}, l_{i+2})$, it follows that a and b are strongly antiadjacent. Moreover since $x \in (r_i, l_{i+1})$ and by the definition of the left and right path, we deduce that x is strongly anticomplete to $\{r_{i-1}, l_{i+2}\}$. This proves the claim.

But P is an induced path of length $p+1$, a contradiction. Hence for all $x \in [l_i, r_{i+1}]$, x is strongly anticomplete to at least one of $[r_0, l_i)$, $(r_{i+1}, r_p]$.

Let $V_1 = \{x \in [l_i, r_{i+1}] : x \text{ is strongly anticomplete to } (r_{i+1}, r_p]\}$ and $V_2 = [l_i, r_{i+1}] \backslash V_1$. The above arguments shows that $([r_0, l_i) \cup V_1, (r_{i+1}, r_p] \cup V_2)$ is a 1-join. This proves 5.7. □

5.8. *Let $(G, \{x_1, x_n\})$ be a linear interval stripe of length p with $p > 3$, then G admits a 1-join.*

Proof. Assume not. Let $L = l_0 - \ldots - l_p$ and $R = r_0 - \ldots - r_p$ be the left and right paths. If $r_2 = l_2$, it follows that r_2 is strongly adjacent to at least one of l_1, r_3, contrary to 5.7. Thus by 5.5, we may assume that $l_2 < r_2$.

By 5.7, we may assume that l_1 is antiadjacent to r_2 and l_2 is antiadjacent to r_3. By 5.5, it follows that $l_2 \in (r_1, r_2)$. Since G is a linear interval trigraph, we deduce that l_2 is adjacent to r_2. Hence $l_0 - l_1 - l_2 - r_2 - R - r_p$ is a path of length $p+1$, a contradiction. This proves 5.8. □

5.9. *Let $(G, \{x_1, x_n\})$ be a linear interval stripe of length three, and (H, Z) a thickening of $(G, \{x_1, x_n\})$. Then either H admits a 1-join or (H, Z) is the thickening of a spring.*

Proof. Let $L = l_0 - l_1 - l_2 - l_3$ and $R = r_0 - r_1 - r_2 - r_3$ be the left and right paths of G. If l_1 is strongly adjacent to r_2 then by 5.7, G admits a 1-join, and so does H.

Thus, we may assume that l_1 is not strongly adjacent to r_2. Suppose that there exists $a \in (r_1, l_2)$. Since $a > r_1$, we deduce that a is strongly antiadjacent to r_0. Symmetrically, a is strongly antiadjacent to l_3. By 5.5, it follows that $a \in (l_1, l_2)$. Since G is a linear interval trigraph, we deduce that a is adjacent to l_1. Symmetrically, a is adjacent to r_2. Hence $r_0 - l_1 - a - r_2 - l_3$ is a path of length 4, contrary to the fact that G has length 3. Thus $(r_1, l_2) = \emptyset$.

Since r_0 is strongly adjacent to r_1 and as G is a linear interval trigraph, we deduce that $(r_0, r_1]$ is a strong clique, and moreover, that r_0 is strongly complete to $(r_0, r_1]$. By 5.6, it follows that r_0 is strongly anticomplete to $[l_2, l_3]$. By symmetry and since $V(G) = \{r_0, l_3\} \cup (r_0, r_1] \cup [l_2, l_3)$, the above arguments show that $((r_0, r_1], [l_2, l_3))$ is a homogeneous pair. Moreover by 5.5, $l_1 \in (r_0, r_1]$ and $r_2 \in [l_2, l_3)$. Since l_1 is antiadjacent to r_2, we deduce that $(r_0, r_1]$ is not strongly complete to $[l_2, l_3)$. Since $r_2 \in [l_2, l_3)$ and by the definition of the right path, we deduce that $(r_0, r_1]$ is not strongly anticomplete to $[l_2, l_3)$.

Now setting $X_w = \{l_0\}$, $X_x = (r_0, r_1]$, $X_y = [l_2, l_3)$ and $X_z = \{r_3\}$, we observe that $(G, \{x_1, x_n\})$ is the thickening of a spring, and therefore (H, Z) is the thickening of a spring. This proves 5.9. $\square$

6 Proof of the main theorem

In this section we collect the results we have proved so far, and finish the proof of the main theorem.

6.1. *Let $(G, \{x\})$ be a connected cobipartite bubble. Then $(G, \{x\})$ is a thickening of a truncated spot, a thickening of a truncated spring or a thickening of a one ended spot.*

Proof. Let X and Y be two disjoint strong cliques such that $X \cup Y = V(G)$. We may assume that $\{x\} \subseteq X$. If $\{x\} \cup N(x) = V(G)$, it follows that $N(x)$ is a homogeneous set. Hence $(G, \{x\})$ is the thickening of a truncated spot.

Thus we may assume that $\{x\} \cup N(x) \neq V(G)$. Let $Y_1 = Y \cap N(x)$ and $Y_2 = Y \backslash Y_1$. Then x is strongly complete to Y_1 and strongly anticomplete to Y_2. Observe that $(N(x), Y_2)$ is a homogeneous pair. Since G is connected, we deduce that $|N(x)| \geq 1$ and that $N(x)$ is not strongly anticomplete to Y_2. If $N(x)$ is strongly complete to Y_2, we observe that $(G, \{x\})$ is a thickening of a one ended spot. And otherwise, we observe that $(G, \{x\})$ is a thickening of a truncated spring. This concludes the proof of 6.1. $\square$

6.2. *Let $(G, \{z\})$ be a stripe such that G a thickening of a trigraph in $\mathcal{C}$. Then $(G, \{z\})$ is in $\mathcal{C}'$.*

Proof. Let H be a trigraph in $\mathcal{C}$ such that G is a thickening of H. For $i, j = 1, 2, 3$, let $B_i^j \subseteq V(H)$ and $a_i \in V(H)$ be as in the definition of $\mathcal{C}$. For $i = 1, 2, 3$, let $X_{a_i} \subset V(G)$ be as in the definition of a thickening. For $b \in V(G) \backslash (X_{a_1} \cup X_{a_2} \cup X_{a_3})$ and since there exists i such that $X_{a_i} \cup X_{a_{i+1}} \subseteq N(b)$, and X_{a_i} is not strongly complete to $X_{a_{i+1}}$, we deduce that $b \notin \{z\}$. Thus there exists $k \in \{1, 2, 3\}$ such that $z \in X_{a_k}$. Since $\bigcup_{i=1}^3 (B_k^1 \cup B_{k+1}^i) \subseteq N(z)$ and since there exists no $c \in X_{a_{k+1}} \cup X_{a_{k+2}}$ with c strongly complete to $\bigcup_{i=1}^3 (B_k^1 \cup B_{k+1}^i)$, we deduce that $N(z) \cap (X_{a_{k+1}} \cup X_{a_{k+2}}) = \emptyset$. Since B_{k+1}^{k+2} is anticomplete to B_k^{k+2} and $B_{k+1}^{k+2} \cup B_k^{k+2} \subseteq N(z)$, we deduce from the definition of $\mathcal{C}$ that $B_{k+1}^{k+2} \cup B_k^{k+2} = \emptyset$. Hence we deduce that $(G, \{z\})$ is in $\mathcal{C}'$. This proves 6.2. $\square$

6.3. *Let G be a trigraph and let H be a thickening of G. For $v \in V(G)$, let X_v be as in the definition of thickening of a trigraph. Let $C = c_1 - c_2 - \ldots - c_n$ be an odd hole or an odd antihole. Then $|V(C) \cap X_v| \leq 1$ for all $v \in V(G)$.*

Proof. Assume not. We may assume that $|V(C) \cap X_x| \geq 2$ with $x \in V(G)$.

Assume first that C is a hole. By symmetry, we may assume that $c_1, c_2 \in X_x$. Since c_3 is antiadjacent to c_1 and adjacent to c_2, we deduce that there exists $y \in V(G)$ such that x is semiadjacent to y and

$c_3 \in X_y$. By symmetry, and since x is semiadjacent to at most on vertex in G, we deduce that $c_n \in X_y$, a contradiction since X_y is a strong clique.

Assume now that C is an antihole. By symmetry, we may assume that there exists $k \in \{3, \dots, n-1\}$ such that $c_1, c_k \in X_x$. Moreover we may assume by symmetry that k is even.

(1) *For $i \in \{1, \dots, k/2\}$, if i is odd then $c_i, c_{k-i+1} \in X_x$, and there exists $y \in V(G)$ such that if i is even then $c_i, c_{k-i+1} \in X_y$.*

By induction on i. By assumption, $c_1, c_k \in X_x$. Since c_2 is adjacent to c_k and antiadjacent to c_1, we deduce that there exists $y \in V(G)$ such that x is semiadjacent to y in G and $c_2 \in X_y$. By symmetry, and since x is semiadjacent to at most one vertex in G, we deduce that $c_{k-1} \in X_y$.

Now let $i \in \{3, \dots, k/2\}$ and assume first that i is odd. By induction, we may assume that $c_{i-1}, c_{k-i+2} \in X_y$. Since c_i is adjacent to c_{k-i+2} and antiadjacent to c_{i-1}, we deduce that $c_i \in X_x$ since y is semiadjacent only to x in G. By symmetry, we deduce that $c_{k-i+1} \in X_x$. Now if i is even, the same argument holds by symmetry. This proves (1).

By (1), there exists $z \in \{x, y\}$ such that $c_{k/2}, c_{k/2+1} \in X_z$, a contradiction. This concludes the proof of 6.3. $\square$

6.4. *Let G be a trigraph and let H be a thickening of G. Then G is Berge if and only if H is Berge.*

Proof. If $C = c_1 - c_2 - \dots - c_n - c_1$ is an odd hole (resp. antihole) in G then $C' = X_{c_1} - X_{c_2} - \dots - X_{c_n} - X_{c_1}$ is an odd hole (resp. antihole) in H.

Now assume that $C = c_1 - c_2 - \dots - c_n - c_1$ is an odd hole or an odd antihole in H. By 6.3, there is $x_i \in V(G)$ such that $c_i \in X_{x_i}$ for $i = 1, \dots, n$ and such that $x_i \neq x_j$ for all $i \neq j$. But $x_1 - x_2 - \dots - x_n - x_1$ is an odd hole or an odd antihole in G. This proves 6.4. $\square$

6.5. *Let G be a structured circular interval trigraph. Then G is Berge.*

Proof. Assume not. For $i = 1, \dots, n$, let X_i and Y_i be as in the definition of structured circular interval trigraph. Let $C = c_1 - \dots - c_n$ be an odd hole or an odd antihole in G. Since $N(y)$ is a strong clique for all $y \in \bigcup_{i=1}^n Y_i$, we deduce that $V(C) \cap \bigcup_{i=1}^n Y_i = \emptyset$. But by 6.3 and (S1)-(S6), we get a contradiction. This proves 6.5. $\square$

6.6. *Let G be a structured circular interval trigraph. Then G is a thickening of an evenly structured linear interval join.*

Proof. Let $X_1, \dots, X_n, Y_1, \dots, Y_n$ and n be as in the definition of structured circular interval trigraph. Throughout this proof, the addition is modulo n.

Let $H = (V, E)$ be a graph and s be a signing such that:

- $V \subseteq \{h_1, h_2, \dots, h_n\} \cup \{l_1^1, \dots, l_1^{|Y_1|}\} \cup \dots \cup \{l_n^1, \dots, l_n^{|Y_n|}\}$,
- if X_i is not strongly complete to X_{i+1}, then $h_{i+1} \notin V$, and there is exactly one edge e_i between h_i and h_{i+2}, and $s(e_i) = 0$,
- if X_i is strongly complete to $X_{i-1} \cup X_{i+1}$, then there are $|X_i|$ edges $e_i^1, \dots, e_i^{|X_i|}$ between h_i and h_{i+1}, and $s(e_i^k) = 1$ for $k = 1, \dots, |X_i|$,
- if $h_i \in V$, there is one edge between h_i and l_{i-1}^k with $s(h_i l_{i-1}^k) = 1$ for $k = 1, \dots, |Y_{i-1}|$.

Then G is an evenly structured linear interval join with skeleton H and such that each stripe associated with an edge e with $s(e) = 1$ is a spot. This proves 6.6. $\square$

We can now prove the following.

6.7. *Let G be a linear interval join. Then G is Berge if and only if G is an evenly structured linear interval join.*

Proof.

$\Leftarrow$ Let G be an evenly structured linear interval join. We have to show that G is Berge. By 5.3, linear interval stripes are Berge. By 2.7 and 6.4, trigraphs in $\mathcal{C}'$ are Berge. By 6.5, structured bubbles are Berge. Clearly spots, truncated spots, one ended spots and truncated springs are Berge. By 6.4 and due to the construction of evenly structured linear interval join, the only holes created are of even length due to the signing. Thus G is Berge.

$\Rightarrow$ Let G be a Berge linear interval join. Let H be a skeleton of G. We may assume that H is chosen among all skeletons of G such that $|V(H)|$ is maximal and subject to that with $|E(H)|$ maximal. Let (G_e, Y_e), $e = x^1x^2$ (with $x^1 = x^2$ if e is a loop) and $\phi_e : V(e) \to Y_e$ be associated with H as in the definition of linear interval join.

(1) *If (G_e, Y_e) is a thickening of a linear interval stripe such that e is in a cycle in H but e is not a loop, then G_e does not admit a 1-join.*

Assume not. Let $Y_e = \{y, z\}$ and $e = x^1x^2$. We may assume that $\phi_e(x^1) = y$ and $\phi_e(x^2) = z$.

Let H' be the graph obtained from H by adding a new vertex a' as follows: $V(H') = V(H) \cup \{a'\}$, $H'|V(H) = H\backslash e$ and a' is adjacent to x^1 and x^2, and to no other vertex.

Let (F_e, Z_e) be a linear interval stripe such that (G_e, Y_e) is a thickening of (F_e, Z_ei) and such that F_e admits a 1-join. Let $V_1, V_2, A_1, A_2 \subset V(F_e)$ be as in the definition of 1-join. Moreover let W_1, W_2 be the natural partition of $V(G_e)$ such that $G_e|W_k$ is a thickening of $F_e|W_k$ for $k = 1, 2$ and (W_1, W_2) is a 1-join. We may assume that $V(F_e) = \{v_1, \ldots, v_n\}$, $V_1 = \{v_1, \ldots, v_k\}$ and $V_2 = \{v_{k+1}, \ldots, v_n\}$. Let F_e^1 be such that $V(F_e^1) = \{v_1, \ldots, v_k, v'_{k+1}\}$, $F_e^1|V_1 = F_e$ and v'_{k+1} is complete to A_1 and anticomplete to $V_1\backslash A_1$. Let (G_e^1, Y_e^1) be the thickening of $(F_e^1, \{v_1, v'_{k+1}\})$ such that $G_e^1\backslash Y_e^1 = G_e|(W_1\backslash Y_e)$. Let F_i^2 be such that $V(F_e^2) = \{v'_k, v_{k+1}, \ldots, v_n\}$, $F_e^2|V_2 = F_e$ and v'_k is complete to A_2 and anticomplete to $V_2\backslash A_2$. Let (G_e^2, Y_e^2) be the thickening of $(F_e^2, \{v'_k, v_n\})$ such that $G_e^2\backslash Y_e^2 = G_e|(W_2\backslash Y_e)$.

Now G is a linear interval join with skeleton H' using the same stripes as the construction with skeleton H except for stripe (G_e^1, Y_e^1) and (G_e^2, Y_e^2) associated with the edges $a'x^1$ and $a'x^2$, contrary to the maximality of $|V(H)|$. This proves (1).

Let s be a signing such that $s(e) = 1$ if (G_e, Y_e) is a spot, and $s(e) = 0$ if (G_e, Y_e) is not a spot.

It remains to prove that:

(P1) *if e is not a loop and is in a cycle and $s(e) = 0$, then (G_e, Y_e) is a thickening of a spring, and*

(P2) *(H, s) is an even structure,*

(P3) *if e is a loop, then (G_e, Y_e) is a trigraph in $\mathcal{C}'$.*

First we prove (P1). Let $e = x^1x^2$ be in a cycle and such that $s(e) = 0$ and e is not a loop. Let (G_e, Y_e) be a thickening of a linear interval stripe such that e has been replaced by (G_e, Y_e) in the construction. Let $Y_e = \{y, z\}$. We may assume that $\phi_e(x^1) = y$ and $\phi_e(x^2) = z$. By 5.1 and 5.4, if $e \in H$ is in a cycle, then all paths from y to z have the same length. By (1), (G_e, Y_e) does not admit a 1-join, and thus by 5.8 and 5.9, (G_e, Y_e) is the thickening of a spring. This proves (P1).

Before proving (P2). We need the following claims.

(2) *Let $C = x_1 - x_2 - x_3 - x_1$ be a cycle in H with edge set $E(C) = \{e_1, e_2, e_3\}$. If $s(e_1) = s(e_2) = 0$ and $s(e_3) = 1$, then there is an odd hole in G.*

By (P1), (G_{e_1}, Y_{e_1}) and (G_{e_2}, Y_{e_2}) are springs. It follows that the springs (G_{e_1}, Y_{e_1}) and (G_{e_2}, Y_{e_2}) together with the spot (G_{e_3}, Y_{e_3}) induce a hole of length 5 in G, a contradiction. This proves (2).

(3) *Let $C = x_1 - x_2 - \ldots - x_n - x_1$ be a cycle in H such that $n > 3$ and such that $\sum_{e\in E(C)} s(e)$ is odd, then there is an odd hole in G.*

The proof of (3) is similar to the proof of (2) and is omitted.

(4) *Let $\{x_1, x_2, x_3\}$ be a triangle in H. For $i = 1, 2, 3$, let e_i be an edge between x_i and $x_{i+1 \bmod 3}$ such that $s(e_i) = 1$. If $y \in V(H)\backslash\{x_1, x_2, x_3\}$ is adjacent to at least two vertices in $\{x_1, x_2, x_3\}$, then $s(f) = 1$ for every edge f with one end y and the other end in $\{x_1, x_2, x_3\}$.*

Assume that there is an edge e_4 with one end y and the other end in $\{x_1, x_2, x_3\}$ with $s(e_4) = 0$. By symmetry, we may assume that x_1 is an end of e_4. By symmetry, we may also assume that there is an edge e_5 between y and x_2. If $s(e_5) = 0$, we deduce by (2) using $y - x_1 - x_2 - y$ that there is an odd hole in G, a contradiction. But if $s(e_5) = 1$, we deduce by (2) using $y - x_1 - x_3 - x_2 - y$ that there is an odd hole in G, a contradiction. This proves (4).

(5) *Let A be a block of H. Assume that there is a cycle $C = x_1 - x_2 - x_3 - x_1$ in H such that $s(e) = 1$ for all $e \in E(C)$. Then all maximal connected components of $A\backslash V(C)$ have size 1.*

Let B be a connected components of $A\backslash V(C)$ such that $|B| > 1$. Since $B \cup \{x_1, x_2, x_3\}$ is 2-connected, there are at least 2 vertices in B that are not anticomplete to $\{x_1, x_2, x_3\}$. Similarly, there are at least 2 vertices in $\{x_1, x_2, x_3\}$ that are not anticomplete to B. Hence, we can find $b_i, b_j \in B$ such that b_i is adjacent to x_i and b_j is adjacent to b_j with $i \neq j$. By symmetry, we may assume that $i = 1$ and $j = 2$. Since B is connected, we deduce that there is a path P from b_1 to b_2 in B. But $C_1 = x_3 - x_1 - b_1 - P - b_2 - x_2 - x_3$ and $C_2 = x_1 - b_1 - P - b_2 - x_2 - x_1$ are cycles of length greater than 3 and one of them has a odd value, thus by (3) there is an odd hole in G, a contradiction. This proves (5).

Now we prove (P2). We need to prove that every block of H is either a member of $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ or an evenly signed graph. Let A be such a block and assume that $(A, s|_A)$ is not an evenly signed graph. It follows that there exists a cycle $C = x_1 - x_2 - \ldots - x_n - x_1$ in A of odd value. By (3) and (2), we deduce that C has length 3 and $s(e) = 1$ for all edges $e \in E(C)$.

By (2), if $|V(A)| = 3$ we deduce that A is a member of $\mathcal{F}_1$. Hence we may assume that there is $x_4 \in A$. By (5) and by symmetry, we deduce that x_4 is adjacent to both x_1 and x_2. By (4), we deduce that $s(e) = 1$ for all edges e between $\{x_1, x_2, x_3\}$ and x_4.

Assume first that x_4 is adjacent to x_3. Assume that $|V(A)| > 4$. Since A is connected, there is $y \in A\backslash\{x_1, x_2, x_3, x_4\}$ such that y is not anticomplete to $\{x_1, x_2, x_3, x_4\}$. Let $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Since there is a cycle $C_{ijk} = x_i - x_j - x_k - x_i$ of length 3 with $s(e) = 1$ for all edges $e \in E(C_{ijk})$, we deduce by (5) that y is not adjacent to x_l. Hence y is anticomplete to $\{x_1, x_2, x_3, x_4\}$, a contradiction. It follows that $|V(A)| = 4$. Assume now that there is an edge e in A with $s(e) = 0$. By symmetry, we may assume that e is between x_1 and x_2. Now $x_1 - x_2 - x_3 - x_4 - x_1$, is a cycle of length 4 of odd value. By (3), it follows that G has an odd hole, a contradiction. Hence $s(e) = 1$ for all edges e in A and we deduce that A is a member of $\mathcal{F}_2$.

Assume now that x_4 is not adjacent to x_3. By (5), we deduce that $E(A\backslash\{x_1, x_2, x_3\}) = \emptyset$. Similarly by (5), it follows that $E(A\backslash\{x_1, x_2, x_4\}) = \emptyset$. Since A is 2-connected, it follows that $\{x_1, x_2\}$ is complete to $V(A)\backslash\{x_1, x_2\}$. By (4), we deduce that $s(f) = 1$ for all edges f between $\{x_1, x_2\}$ and $V(A)\backslash\{x_1, x_2\}$. Hence A is a member of $\mathcal{F}_3$. This proves (P2).

Finally we prove (P3). Let e be a loop. Let (G_e, Y_e) be a thickening of a bubble such that e has been replaced by (G_e, Y_e) in the construction. Let $Y_e = \{y\}$. Let (F, W) be a bubble such that

(G_e, Y_e) is a thickening of (F, W). By 2.1, F is a linear interval trigraph, a cobipartite trigraph, a structured circular interval trigraph or a thickening of a trigraph in $\mathcal{C}$.

Assume first that F is a linear interval trigraph. Let $\{v_1, \dots, v_n\}$ be the set of vertices of F. Let $k \in \{1, \dots, n\}$ be such that $\{v_k\} = W$. For $v_i \in V(F)$, let $X_{v_i} \subset V(G_i)$ be as in the definition of a thickening. Let $l < r$ be such that $N(v_k) = \{v_l, \dots, v_r\}$. Assume that $1 < l$ and $r < n$. Let H' be the graph obtained from H by adding two new vertices a_1, a_2 as follows: $V(H') = V(H) \cup \{a_1, a_2\}$, $H'|V(H) = H \backslash e$, a_1 and a_2 are adjacent to $\phi_e^{-1}(y)$ and to no other vertex. Let F_l be such that $V(F_l) = \{v_0, v_1, \dots, v_k\}$, $F_l \backslash v_0 = F|\{v_1, \dots, v_k\}$ and v_0 is adjacent to v_1 and to no other vertex. Let F_r be such that $V(F_r) = \{v_k, \dots, v_n, v_{n+1}\}$, $F_r \backslash v_{n+1} = F|\{v_k, \dots, v_n\}$ and v_{n+1} is adjacent to v_n and to no other vertex. Let (G_e^l, Y_e^l) be the thickening of $(F_l, \{v_0, v_k\})$ such that $G_e^l \backslash Y_e^l = G_e | \bigcup_{j=1}^{k-1} X_{v_j}$. Let (G_e^r, Y_e^r) be the thickening of $(F_r, \{v_k, v_{n+1}\})$ such that $G_e^r \backslash Y_e^r = G_e | \bigcup_{j=k+1}^{n} X_{v_j}$. Now G is a linear interval join with skeleton H' using the same stripes as the construction with skeleton H except for (G_e^l, Y_e^l) and (G_e^r, Y_e^r) instead of (G_e, Y_e), contrary to the maximality of $|V(H)|$. Hence by symmetry, we may assume that $l = 1$. Now let H' be the graph obtained from H by adding a new vertex a' as follows: $V(H') = V(H) \cup \{a'\}$, $H'|V(H) = H \backslash e$ and a' is adjacent to $\phi_e^{-1}(y)$ and to no other vertex. Let F' be such that $V(F') = \{v_1, \dots, v_n, v_{n+1}\}$, $F'|V(F) = F$ and v_{n+1} is adjacent to v_n and to no other vertex. Let (G'_e, Y'_e) be the thickening of $(F', \{v_1, v_{n+1}\})$ such that $G'_e \backslash Y'_e = G_e \backslash Y_e$. Now G is a linear interval join with skeleton H' using the same stripes as the construction with skeleton H except for (G'_e, Y'_e) instead of (G_e, Y_e), contrary to the maximality of $|V(H)|$. Hence F is not a linear interval trigraph.

Assume now that F is a structured circular interval trigraph. Using the same construction as in the proof of 6.6, it is easy to see that there exist H' with $|V(H')| > |V(H)|$ and a set of stripes $\mathcal{S}$, such that G is a linear interval join with skeleton H' using the stripes of $\mathcal{S}$, contrary to the maximality of $|V(H)|$. Hence F is not a structured circular interval trigraph.

Assume now that F is a cobipartite trigraph. Clearly any thickening of a cobipartite trigraph is a cobipartite trigraph. By 6.1, (G_e, Y_e) is a thickening of a truncated spot, a thickening of a truncated spring or a thickening of a one ended spot.

Assume that (G_e, Y_e) is a thickening of a one ended spot. Let $X_v \subset V(G_e)$ be as in the definition of a thickening. Let H' be the graph obtained from H by adding a new vertex a' as follows: $V(H') = V(H) \cup \{a'\}$, $H'|V(H) = H \backslash e$, there is $|X_v|$ edges between a' and $\phi_e^{-1}(y)$, there is a loop l on a' and a' is adjacent to no other vertex than $\phi_e^{-1}(y)$. Let the stripes associated with the edges between a' and $\phi_e^{-1}(y)$ be spots and let the stripe associated with the loop on a' be a thickening of a truncated spot. Now G is a linear interval join with skeleton H' using the same stripes as the construction with skeleton H except for additional edges, contrary to the maximality of $|V(H)|$. Hence (G_i, Y_i) is not a thickening of a one ended spot.

Assume now that (G_e, Y_e) is a thickening of a truncated spot. Let H' be the graph obtained from H by adding $|V(G_e)| - 1$ new vertices $a_1, \dots, a_{|V(G_e)|-1}$ as follows: $V(H') = V(H) \cup \{a_1, \dots, a_{|V(G_e)|-1}\}$, $H'|V(H) = H \backslash e$, and for $j \in \{1, \dots, |V(G_e)| - 1\}$, a_j is adjacent to $\phi_e^{-1}(y)$ and to no other vertex. Now G is a linear interval join with skeleton H' using the same stripes as the construction with skeleton H and such that the stripes associated with the added edges are spots, contrary to the maximality of $|V(H)|$. Hence (G_e, Y_e) is not a thickening of a truncated spot.

Assume that (G_e, Y_e) is a thickening of a truncated spring. Let H' be the graph obtained from H by adding a new vertex a' as follows: $V(H') = V(H) \cup \{a'\}$, $H'|V(H) = H \backslash e$, and a' is adjacent to $\phi_e^{-1}(y)$ and no other vertex. Now G is a linear interval join with skeleton H' using the same stripes as the construction with skeleton H and such that the stripe associated with the edge $a'\phi_e^{-1}(y)$ is a spring, contrary to the maximality of $|V(H)|$. Hence (G_e, Y_e) is not a thickening of a truncated spring.

Finally assume that G_e is a thickening of a trigraph in $\mathcal{C}$. By 6.2, it follows that (G_e, Y_e) is in $\mathcal{C}'$. This concludes the proof of (P3).

Hence G is an evenly structured linear interval join.

This concludes the proof of 6.7. □

A last lemma is needed for the proof of 1.4.

6.8. *Let G be a cobipartite trigraph. Then G is a thickening of a linear interval trigraph.*

Proof. Let Y, Z be two disjoint strong cliques such that $Y \cup Z = V(G)$. Clearly (Y, Z) is a homogeneous pair. Let H be the trigraph such that $V(H) = \{y, z\}$ and

- y is strongly adjacent to z if Y is strongly complete to Z,
- y is strongly antiadjacent to z if Y is strongly anticomplete to Z,
- y is semiadjacent to z if Y is neither strongly complete nor strongly anticomplete to Z.

Now setting $Y = X_y$ a nd $Z = X_z$, we observe that G is a thickening of H. Since H is clearly a linear interval trigraph, it follows that G is a thickening of a linear interval trigraph. This proves 6.8. □

Proof of 1.4. Let G be a Berge claw-free connected trigraph. By 1.3, G is either a linear interval join or a thickening of a circular interval trigraph. By 2.1, if G is a thickening of a circular interval trigraph, then G is a thickening of a linear interval trigraph, or a cobipartite trigraph, or a thickening of a member of $\mathcal{C}$, or G is a structured circular interval trigraph. But by 6.6, if G is a structured circular interval trigraph, then G is an evenly structured linear interval join. By 6.8, if G is a cobipartite trigraph, then G is a thickening of a linear interval trigraph. Moreover, any thickening of a linear interval trigraph is clearly an evenly structured linear interval join. Finally by 6.7, if G is a linear interval join, then G is an evenly structured linear interval join. This proves 1.4. □

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