Concatenating bipartite graphs

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Abstract

Let $x, y \in (0, 1]$; and let A, B, C be disjoint nonempty subsets of a graph G, where every vertex in A has at least $x|B|$ neighbours in B, and every vertex in B has at least $y|C|$ neighbours in C. We denote by $\phi(x, y)$ the maximum z such that, in all such graphs G, there is a vertex $v \in C$ that is joined to at least $z|A|$ vertices in A by two-edge paths. The function ϕ is interesting, and we investigate some of its properties. For instance, we show that

- $\phi(x, y) = \phi(y, x)$ for all x, y; and
- for each integer $k > 1$, there is a discontinuity in $\phi(x, x)$ when $x = 1/k$: $\phi(x, x) \leq 1/k$ when $x \leq 1/k$, and $\phi(x, x) \geq \frac{2k-1}{2k(k-1)}$ when $x > 1/k$.

We raise several questions and conjectures.

1 Introduction

All graphs in this paper are finite, and have no loops or multiple edges. We denote the semi-open interval $\{x : 0 < x \leq 1\}$ of real numbers by $(0, 1]$. Let $x, y \in (0, 1]$; and let A, B, C be disjoint nonempty subsets of a graph G, where every vertex in A has at least $x|B|$ neighbours in B, and every vertex in B has at least $y|C|$ neighbours in C. If we ask for a real number z such that we can guarantee that some vertex in A can reach at least $z|C|$ vertices in C by two-edge paths, then z must be at most y, since perhaps all the vertices in B have the same neighbours in C. But in the reverse direction the question becomes much more interesting; that is, we ask for z such that some vertex in C can reach at least $z|A|$ vertices in A by two-edge paths. Then there might well be values of $z > \max(x, y)$ with this property.

Let us say this more precisely. A *tripartition* of a graph G is a partition (A, B, C) of $V(G)$ where A, B, C are all nonempty stable sets. For $x, y \in (0, 1]$, we say a graph G is (x, y)*-constrained, via a tripartition* (A, B, C), if

- every vertex in A has at least $x|B|$ neighbours in B;
- every vertex in B has at least $y|C|$ neighbours in C; and
- there are no edges between A and C.

For $v \in V(G)$, $N(v)$ denotes its set of neighbours, and $N^2(v)$ is the set of vertices with distance exactly two from v. We write $N^2_A(v)$ for $N^2(v) \cap A$, and so on. A first observation:

1.1 *Let $x, y \in (0, 1]$, and let Z be the set of all $z \in (0, 1]$ such that, for every graph G, if G is (x, y)-constrained via (A, B, C) then $|N^2_A(v)| \geq z|A|$ for some $v \in C$. Then $\sup\{z \in Z\}$ belongs to Z.*

Proof. Let $z' = \sup\{z \in Z\}$, and let G be an (x, y)-constrained graph, via (A, B, C). We must show that $|N^2_A(v)| \geq z'|A|$ for some $v \in C$. We may assume that $z' > 0$; so there exists z with $0 < z < z'$, such that $\lceil z|A| \rceil = \lceil z'|A| \rceil$. Since $z' = \sup\{z \in Z\}$ and $z < z'$, and Z is an initial interval of $(0, 1]$, it follows that $z \in Z$, and so $|N^2_A(v)| \geq z|A|$ for some $v \in C$. Consequently $|N^2_A(v)| \geq \lceil z|A| \rceil \geq z'|A|$, as required. This proves 1.1. ∎

We define $\phi(x, y)$ to be $\sup\{z \in Z\}$, as defined in 1.1. The objective of this paper is to study the properties of the function ϕ. We have a trivial lower bound:

1.2 *$\phi(x, y) \geq \max(x, y)$ for all $x, y > 0$.*

Proof. Let G be (x, y)-constrained, via (A, B, C). Since every vertex in A has at least $x|B|$ neighbours in B, and $B \neq \emptyset$, there exists $u \in B$ with at least $x|A|$ neighbours in A; let $v \in C$ be adjacent to u (this is possible since $y > 0$), and then $|N^2_A(v)| \geq x|A|$. Consequently $\phi(x, y) \geq x$. Now every vertex in A can reach at least $y|C|$ vertices in C by two-edge paths (since $x > 0$); and so by averaging, some vertex in C can reach at least $y|A|$ vertices in A by two-edge paths. Hence $\phi(x, y) \geq y$. This proves 1.2. ∎

And a trivial upper bound:

1.3 *For all $x, y \in (0,1]$,*

$$\phi(x,y) \le \frac{\lceil kx \rceil + \lceil ky \rceil - 1}{k}$$

for every integer $k \ge 1$.

Proof. Let $x, y \in (0,1]$, and let $k \ge 1$ be an integer. Let A, B, C be three sets disjoint sets each of cardinality k, where $A = \{a_1, \dots, a_k\}$, $B = \{b_1, \dots, b_k\}$ and $C = \{c_1, \dots, c_k\}$. Make a graph G with vertex set $A \cup B \cup C$ as follows. Let $g = \lceil kx \rceil$, and for $1 \le i \le k$ make a_i adjacent to $b_i, b_{i+1}, \dots, b_{i+g-1}$ (reading subscripts modulo k). Now let $h = \lceil ky \rceil$, and for $1 \le i \le k$ make b_i adjacent to $c_i, c_{i+1}, \dots, c_{i+h-1}$ (reading subscripts modulo k). Then G is (x,y)-constrained via (A, B, C); and for $1 \le i \le k$, $N_A^2(c_i) = \{a_i, a_{i-1}, \dots, a_{i-g-h+2}\}$ (again, reading subscripts modulo k). Consequently $\phi(x,y) \le (g+h-1)/k$. This proves 1.3. ∎

In particular, we have:

1.4 *For every integer $k \ge 1$, if $x, y > 0$ and $\max(x,y) = 1/k$ then $\phi(x,y) = 1/k$.*

Proof. From 1.2, $\phi(x,y) \ge 1/k$; and the graph consisting of k disjoint three-vertex paths shows that $\phi(x,y) \le 1/k$. (This also follows from 1.3, since $\lceil kx \rceil, \lceil ky \rceil = 1$.) This proves 1.4. ∎

What makes the function ϕ interesting is that for some values of x, y, 1.2 is far from best possible, and indeed 1.3 seems closer to the truth. We were originally motivated by the hope of extending Kneser's theorem from additive group theory [3] to a general graph-theoretic setting, and a corresponding wild conjecture that the bound in 1.3 is always best possible, that is, that for all $x, y \in (0,1]$, there is an integer $k > 0$ with $\phi(x,y) = \frac{\lceil kx \rceil + \lceil ky \rceil - 1}{k}$. This turns out to be false, but perhaps not ridiculously false; maybe something like it is true.

There are two other related problems:

- Let us say G is (x,y)-*biconstrained* (*via* (A,B,C)) if G is (x,y)-constrained via (A,B,C), and in addition
  - every vertex in B has at least $x|A|$ neighbours in A, and
  - every vertex in C has at least $y|B|$ neighbours in B.
- Say G is (x,y)-*exact* (*via* (A,B,C)) if G is (x,y)-constrained via (A,B,C), and in addition there exist $x' \ge x$ and $y' \ge y$ such that
  - every vertex in A has exactly $x'|B|$ neighbours in B;
  - every vertex in B has exactly $x'|A|$ neighbours in A;
  - every vertex in B has exactly $y'|C|$ neighbours in C; and
  - every vertex in C has exactly $y'|B|$ neighbours in B.

We shall sometime use "mono-constrained" to clarify that we mean the (x,y)-constrained case and not the (x,y)-biconstrained case. Let $\psi(x,y)$ be the analogue of $\phi(x,y)$ for biconstrained graphs; that is, the maximum z such that for all G, if G is (x,y)-biconstrained via (A,B,C), then $|N_A^2(v)| \ge z|A|$ for some $v \in C$. (As before, this maximum exists.) Similarly, let $\xi(x,y)$ be the analogue of ϕ and ψ for the exact case. Then we have

1.5 *For all $x, y \in (0,1]$,*

$$\max(x,y) \le \phi(x,y) \le \psi(x,y) \le \xi(x,y) \le \frac{\lceil kx \rceil + \lceil ky \rceil - 1}{k}$$

for every integer $k \ge 1$.

The proof of the non-trivial part of this is the same as the proof of 1.3. One might hope that ψ (and even more ξ) are better-behaved than ϕ.

Let us see an example. Start with the graph of figure 1. Each vertex has a number written next to it in the figure; replace each vertex v by a set X_v of new vertices of the specified cardinality, and for each edge uv of the figure make every vertex in X_u adjacent to every vertex in X_v. This results in a graph with 81 vertices, divided into three sets of 27 corresponding to the three rows of the figure; call these A, B, C. The graph produced is $(13/27, 1/9)$-biconstrained via (A, B, C), and yet $|N_A^2(v)| = 13$ for every vertex in C; so this proves that $\psi(13/27, 1/9) \le 13/27$ (and therefore equality holds, by 1.2). This shows that there need not exist an integer k with $\psi(x,y) = \frac{\lceil kx \rceil + \lceil ky \rceil - 1}{k}$. The same graph, used from bottom to top, shows that $\psi(1/9, 13/27) = 13/27$.

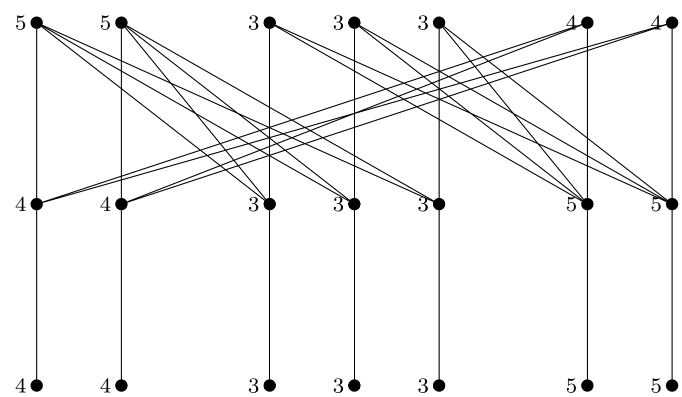

Figure 1: $\psi(13/27, 1/9) = 13/27$

The example is not yet $(13/27, 1/9)$-exact, because some vertices in B have three, four or five neighbours in C, and vice versa. We can make it exact as follows. For each edge uv of the figure with u in the second row and v in the third, the two sets X_u, X_v have the same cardinality, one of three, four, five. Delete some edges between X_u and X_v such that every vertex in X_u has exactly three neighbours in X_v and vice versa. Then the modified graph is $(13/27, 1/9)$-exact, and shows that $\xi(13/27, 1/9) = 13/27$. Consequently, even for the supposedly nicest function ξ of our three functions, there is not always an integer k with $\xi(x,y) = \frac{\lceil kx \rceil + \lceil ky \rceil - 1}{k}$.

So what can we prove about the functions ϕ and ψ? For which x, y, z is $\phi(x,y) \ge z$, or $\psi(x,y) \ge z$? In order to make the question a little more manageable, we focus on four special cases, $x = y$, $z = 1/2$, $z = 2/3$ and $z = 1/3$, but in each case the results for ϕ and for ψ are quite different. The paper is organized as follows:

- We begin with a proof that $\phi(x,y) = \phi(y,x)$ for all x, y.
- Then we give some general upper bounds on $\phi(x,y)$ and $\psi(x,y)$, particularly focussing on the case when $x = y$.

- Next we ask when $\phi(x, y) \geq 1/2$, or $\psi(x, y) \geq 1/2$. There are several theorems that this is true for certain pairs (x, y), and their union fills a good part of the (x, y)-square. We also give a number of constructions that shows the statement is *not* true for certain pairs (x, y). Ideally this would fill the complementary part of the square, but there is an "undecided" band of varying width down the middle.
- Then we do the same for 2/3 instead of 1/2; and then for 1/3.
- Finally, we discuss some other questions and approaches.

2 Weighted graphs and some linear programming

In this section we prove that $\phi(x, y) = \phi(y, x)$ for all x, y. The argument uses linear programming, and we need some preparation. We denote the set of real numbers by $\mathbb{R}$, and the non-negative reals numbers by $\mathbb{R}_+$. A *weighted graph* (G, w) consists of a graph G together with a function $w : V(G) \to \mathbb{R}_+$. If $X \subseteq V(G)$, we denote $\sum_{v \in X} w(v)$ by $w(X)$. Let (G, w) be a weighted graph, and (A, B, C) a tripartition of G. If $x, y \in (0, 1]$, a weighted graph (G, w) is (x, y)-*constrained* via (A, B, C), if:

- $\sum_{v \in A} w(v) = \sum_{v \in B} w(v) = \sum_{v \in C} w(v) = 1$;
- for each $v \in A$, $w(N(v) \cap B) \geq x$; and
- for each $v \in B$, $w(N(v) \cap C) \geq y$.

Similarly, we say (G, w) is (x, y)-*biconstrained* via (A, B, C), if in addition:

- for each $v \in B$, $w(N(v) \cap A) \geq x$; and
- for each $v \in C$, $w(N(v) \cap B) \geq y$.

To make the graph of figure 1 into an appropriate weighted graph, divide all the numbers by 27.

2.1 *For $x, y, z \in (0, 1]$, the following are equivalent:*

- *$\phi(x, y) \geq z$;*
- *$w(N_A^2(v)) \geq z$ for some $v \in C$, for every weighted graph (G, w) that is (x, y)-constrained via a tripartition (A, B, C).*

Similarly, the following are equivalent:

- *$\psi(x, y) \geq z$;*
- *$w(N_A^2(v)) \geq z$ for some $v \in C$, for every weighted graph (G, w) that is (x, y)-biconstrained via a tripartition (A, B, C).*

Proof. To prove the "if" direction of the first statement, let G be (x,y)-constrained via (A,B,C). Define $w(v)=1/|A|$, for each $v\in A$, and $w(v)=1/|B|$ for $v\in B$ and similarly for $v\in C$. Then (G,w) is an (x,y)-constrained weighted graph, and the claim follows. The "if" direction of the second statement is proved similarly.

For the "only if" direction, let (G,w) be a weighted graph, (x,y)-constrained via (A,B,C), and suppose such a weighted graph can be chosen with $w(N_A^2(v))<z$ for each $v\in C$. Consequently we may choose (G,w) such that in addition, w is rational-valued. Choose an integer $N>0$ such that $Nw(v)$ is an integer for each $v\in G$. For each $v\in V(G)$, take a set X_v of $Nw(v)$ new vertices; and make a graph G' with vertex set $\bigcup_{v\in V(G)}X_v$, by making every vertex of X_u adjacent to every vertex of X_v for all adjacent $u,v\in V(G)$. Let $A'=\bigcup_{v\in A}X_v$, and define B',C' similarly; then (A',B',C') is a tripartition of G', and G' is (x,y)-constrained via (A',B',C'). Since in G, $w(N_A^2(v))<z$ for each $v\in C$, it follows that in G', $|N_{A'}^2(v')|<z|A'|$ for each $v'\in C'$, a contradiction. The "only if" direction of the second statement is similar. This proves 2.1. ∎

Let G be a graph with a bipartition (A,B), and let $w:B\to\mathbb{R}_+$ be some function. We define $w(A\to B)$ to mean the minimum, over all $u\in A$, of $w(N(u))$ (taking $w(A\to B)=0$ if $A=\emptyset$).

2.2 *Let G be a graph with a bipartition (A,B), and let $w:B\to\mathbb{R}_+$ be some function such that $w(B)=1$. Then either*

- *there is a function $w':B\to\mathbb{R}_+$, such that $w'(B)=1$, $w'(A\to B)\geq w(A\to B)$, and $w'(v)=0$ for some $v\in B$; or*
- *there is a function $f:A\to\mathbb{R}_+$, such that $f(A)=1$ and $f(B\to A)\geq w(A\to B)$.*

Proof. We may assume that $A\neq\emptyset$. If some vertex in A has no neighbour in B, then $w(A\to B)=0$ and the second bullet holds; so we assume that each vertex in A has a neighbour in B.

Let $x=w(A\to B)$. The function w', defined by $w'(v)=1/|B|$ for each $v\in B$, satisfies $w'(A\to B)>0$, since every vertex in A has a neighbour in B. Thus we may assume that $x>0$, replacing w by w' if necessary.

Let M be the 0/1-matrix $(a_{uv}:u\in A,v\in B)$, where $a_{uv}=1$ if and only if u,v are adjacent. Let $\mathbf{1}_A\in\mathbb{R}^A$ be the vector of all 1's, and define $\mathbf{1}_B$ similarly. Then $w\in\mathbb{R}_+^B$ satisfies:

- $\mathbf{1}_B^Tw=1$; and
- $Mw\geq x\mathbf{1}_A$.

Consequently $b=w/x$ satisfies $b\in\mathbb{R}_+^B$, and

- $\mathbf{1}_B^Tb=1/x$; and
- $Mb\geq\mathbf{1}_A$.

Choose $q\in\mathbb{R}_+^B$ with $Mq\geq\mathbf{1}_A$, with $\mathbf{1}_B^Tq$ minimum. (This is possible by compactness.) Thus $\mathbf{1}_B^Tq\leq 1/x$. Since $Mq\geq\mathbf{1}_A$ and G has an edge, it follows that $\mathbf{1}_B^Tq>0$; let $1/y=\mathbf{1}_B^Tq$, and define $w'=yq$. Then $y\geq x$, and $\mathbf{1}_B^Tw'=1$ and $Mw'\geq y\mathbf{1}_A$, and so we may assume that $w'(v)>0$ for each $v\in B$, because otherwise the first bullet holds.

Now q minimizes $\mathbf{1}_B^T q$ subject to the linear programme $q \in \mathbb{R}_+^B$ and $Mq \geq \mathbf{1}_A$. From the linear programming duality theorem, there exists $p \in \mathbb{R}_+^A$ such that $p^T M \leq \mathbf{1}_B$, and $p^T \mathbf{1}_A = \mathbf{1}_B^T q = 1/y$. Define $f = yp$. Then $f : A \to \mathbb{R}_+$ satisfies $f(A) = 1$, and $f(N(v)) \leq y$ for each $v \in B$.

Let $v' \in B$; we claim that $f(N(v')) = y$. This follows from the "complementary slackness" principle, but we give the argument in full, as follows. Let $s = w'(v')(y - f(N(v')))$. Thus $s \geq 0$, and we will show $s = 0$. We have

$$x = \sum_{v \in B} y w'(v) \geq s + \sum_{v \in B} \sum_{u \in N(v)} w'(v) f(u) = s + \sum_{u \in A} \sum_{v \in N(u)} f(u) w'(v) \geq s + \sum_{u \in A} y f(u) = s + y.$$

Consequently $s = 0$, as claimed. Hence f satisfies the second bullet. This proves 2.2. ∎

From 2.2 we deduce a very useful result.

2.3 *If $x, y \in (0, 1]$ then $\phi(x, y) = \phi(y, x)$.*

Proof. Let $z = \phi(x, y)$, and choose a weighted graph (G, w) that is (x, y)-constrained via (A, B, C), such that $w(N_A^2(v)) \leq z$ for each $z \in C$. Moreover, choose G with $|V(G)|$ minimum. If there is a function $w' : B \to \mathbb{R}_+$, such that $w'(B) = 1$ and $w'(A \to B) \geq w(A \to B)$, and such that $w'(v) = 0$ for some $v \in B$, then we may replace w by a new weight function, changing w to w' on B and otherwise keeping w unchanged, and then we may delete the vertex $v \in B$ with $w'(v) = 0$, contrary to the minimality of $|V(G)|$. Thus there is no such w', and so by 2.2, there is a function $f : A \to \mathbb{R}_+$, such that $f(A) = 1$ and $f(B \to A) \geq w(A \to B) \geq x$. Similarly, there is a function $g : B \to \mathbb{R}_+$, such that $g(B) = 1$ and $g(C \to B) \geq y$. Let H be the graph with bipartition (A, C) in which $u \in A$ and $v \in C$ are adjacent if $u \notin N_A^2(v)$ in G. Thus, in H, $w(C \to A) \geq 1 - z$; and so from 2.2 and the minimality of $|V(G)|$, there is a function $h : C \to \mathbb{R}_+$, such that $h(C) = 1$ and (in H) $h(A \to C) \geq 1 - z$. Let w' be defined by the union of f, g and h in the natural sense; then (G, w') is a weighted graph and is (y, x)-constrained via (C, B, A), and $w'(N_C^2(v)) \leq z$ for each $v \in A$. This proves that $\phi(y, z) \leq z$, and so proves 2.3. ∎

We remark that we have not been able to prove an analogue of 2.3 for the biconstrained case, or for the exact case, although we have no counterexample for either one.

There is another useful application of 2.2, the following:

2.4 *Let (G, w) be an (x, y)-constrained weighted graph, via (A, B, C), such that $w(N_A^2(v)) \leq z$ for each $v \in C$. Suppose that there exists $X \subseteq A$ with $|X| < z^{-1}$ such that $\bigcup_{v \in X} N_C^2(v) = C$. Then there exists $u \in A$ and a weighted graph (G', w') such that*

- *G' is obtained from G by deleting u;*
- *(G', w') is (x, y)-constrained via (A', B, C), where $A' = A \setminus \{u\}$;*
- *in G', $w'(N_{A'}^2(v)) \leq z$ for all $v \in C$; and*
- *$w'(u) = w(u)$ for all $u \in B \cup C$.*

Proof. Suppose not. Let H be the graph with bipartition (A, C), in which $u \in A$ and $v \in C$ are adjacent if $u \notin N_A^2(v)$ in G. Then by 2.2, applied to H, there is a function $h : C \to \mathbb{R}_+$, such that $h(C) = 1$ and (in H) $h(A \to C) \geq 1 - z$. Consequently, in G, $h(N_C^2(v)) \leq z$ for each $v \in A$. In particular, $h(N_C^2(v)) \leq z$ for each $v \in X$, and so $h(C) \leq z|A| < 1$, a contradiction. This proves 2.4. ∎

3 Constructions

In this section we construct some graphs to prove upper bounds on $\phi(x,y)$ or $\psi(x,y)$ for certain values of x,y. We begin with:

3.1 *Let $x,y \in (0,1]$, and let $z \in (0,1]$ such that $z/(1-z) = \phi(x/(1-x), y/(1-y))$; then $\phi(x,y) \le z$.*

Proof. Let (G',w') be a weighted graph that is $(x/(1-x), y/(1-y))$-constrained via some tripartition (A',B',C'), such that $w'(N^2_{A'}(v)) \le z/(1-z)$ for each $v \in C'$. Add three new vertices a,b,c to G', and two edges ab and bc, forming G. Define w by

$$\begin{aligned} w(a) &= z \\ w(v) &= (1-z)w'(v) \text{ for each } v \in A' \\ w(b) &= x \\ w(v) &= (1-x)w'(v) \text{ for each } v \in B' \\ w(c) &= y \\ w(v) &= (1-y)w'(v) \text{ for each } v \in C'. \end{aligned}$$

Then G is (x,y)-constrained via $(A' \cup \{a\}, B' \cup \{b\}, C' \cup \{c\})$ and shows that $\phi(x,y) \le z$. This proves 3.1. ∎

3.2 *Let $k \ge 0$ be an integer, and let $x,y \in (0,1]$ with $\frac{x}{1-kx} + \frac{y}{1-ky} \le 1$, with strict inequality if x or y is irrational; then $\phi(x,y) < \frac{1}{k+1}$.*

Proof. By increasing x and y if necessary, we may assume that x,y are rational. Suppose first that $k=0$; then we may assume that $x+y=1$. Choose an integer $k \ge 1$ such that kx (and hence ky) is an integer. By 1.3,

$$\phi(x,y) \le \frac{\lceil kx \rceil + \lceil ky \rceil - 1}{k} = x + y - 1/k < 1.$$

This completes the proof for $k=0$. For general k we proceed by induction on k. We may assume that $k>0$; let $x,y \in (0,1]$ with $\frac{x}{1-kx} + \frac{y}{1-ky} \le 1$, with strict inequality if x or y is irrational. Let $x' = x/(1-x)$, and $y' = y/(1-y)$. Thus $x',y' \in (0,1]$ with

$$\frac{x'}{1-(k-1)x'} + \frac{y'}{1-(k-1)y'} = \frac{x}{1-kx} + \frac{y}{1-ky} \le 1,$$

with strict inequality if x' or y' is irrational. From the inductive hypothesis, $\phi(x',y') < 1/k$. Let z satisfy $z/(1-z) = \phi(x',y')$; then $z/(1-z) < 1/k$, and so $z < 1/(k+1)$. From 3.1, $\phi(x,y) \le z < 1/(k+1)$. This proves 3.2. ∎

3.3 *Let $k > 0$ be an integer, and let $x,y \in (0,1]$ with $x + (k+1)y \le 1$ and $(k+1)x + y \le 1$, with strict inequality in both if x or y is irrational; then $\psi(x,y) < \frac{1}{k+1}$.*

Proof. Again, we may assume that x, y are rational. Let $s = \max(x, y)$; thus, $s < 1/(k+1)$. Choose an integer $N \geq 1$ such that $p = xN/(1-ks)$ and $q = yN/(1-ks)$ are integers. (It follows that $p + q \leq N$, from the hypothesis.) Let G be a graph with vertex set partitioned into three sets A, B, C, with $|A| = N + k + 1$ and B, C of cardinality $N + k$; let

$$\begin{aligned} A &= \{a_1, a_2, \ldots, a_N, a'_1, \ldots, a'_k, a^*\}, \\ B &= \{b_1, b_2, \ldots, b_N, b'_1, \ldots, b'_k\}, \\ C &= \{c_1, c_2, \ldots, c_N, c'_1, \ldots, c'_k\}. \end{aligned}$$

Let G have the following edges:

- for $1 \leq i \leq N$, a_i is adjacent to $b_i, b_{i+1}, \ldots, b_{i+p-1}$ reading subscripts modulo N;
- for $1 \leq i \leq N$, b_i is adjacent to $c_i, c_{i+1}, \ldots, c_{i+q-1}$ reading subscripts modulo N;
- for $1 \leq i \leq k$, a'_i is adjacent to b'_i, and b'_i is adjacent to c'_i.
- a^* is adjacent to b_i for $1 \leq i \leq N$.

(Thus, this is the same as in the proof of 3.2, except for the extra vertex a^*.) Let r satisfy $(k+1)rN = 1/(k+1) - x$. Thus $r > 0$. For each $v \in V(G)$, define $w(v)$ as follows:

- $w(v) = kr(N+1)/N$ for $v \in \{a_1, \ldots, a_N\}$; $w(v) = 1/(k+1) - r$ for $v \in \{a'_1, \ldots, a'_k\}$;
- $w(a^*) = 1/(k+1) - Nkr$;
- $w(v) = (1-ks)/N$ for $v \in \{b_1, \ldots, b_N\}$; $w(v) = s$ for $v \in \{b'_1, \ldots, b'_k\}$; and
- $w(v) = (1-ky)/N$ for $v \in \{c_1, \ldots, c_N\}$; $w(v) = y$ for $v \in \{c'_1, \ldots, c'_k\}$.

Then (G, w) is a weighted graph. We claim it is (x, y)-biconstrained via (A, B, C), and $w(N_A^2(v)) < 1/(k+1)$ for each $v \in C$. To see this we must verify:

$$\begin{aligned} x &\leq p(1-ks)/N \\ y &\leq q(1-ky)/N \\ y &\leq q(1-ks)/N \\ x &\leq 1/(k+1) - r \\ x &\leq pkr(N+1)/N + 1/(k+1) - Nkr, \text{ and} \\ 1/(k+1) &> 1/(k+1) - Nkr + (p+q-1)kr(N+1)/N. \end{aligned}$$

The first and third hold with equality from the definitions of p, q, and the second follows since $y \leq s$. The fourth follows from the definition of r. For the fifth, on substituting for p and simplifying, we need to show that $rk(N - x(N+1)/(1-ks)) \leq 1/(k+1) - x$, and this follows from the definition of r. Finally, the sixth simplifies to $(p+q-1)(N+1)/N < N$, and this is true since $p + q \leq N$. Consequently $\psi(x, y) < \frac{1}{k+1}$, by 2.1. This proves 3.3. ∎

4 Biconstrained graphs

In this section we prove some lower bounds on $\psi(x,y)$. On the diagonal $x = y$, $\psi(x,y)$ behaves perfectly; it turns out that for all x, $\psi(x,x) = 1/k$, where k is the largest integer with $1/k \ge x$. That follows from:

4.1 *For all integers $k \ge 1$, if $x, y \in (0,1]$ with $x + ky > 1$ and $kx + \frac{x}{1-(k-1)y} \ge 1$, then $\psi(x,y) \ge 1/k$.*

Proof. By 1.2 we may assume that $x, y < 1/k$. Let G be (x,y)-biconstrained, via (A,B,C). We must show that $|N_A^2(v)| \ge |A|/k$ for some $v \in C$. Suppose not. Choose $K \subseteq C$ with $|K| \le k$, and subject to that with $|K|$ maximum such that the sets $N(v)$ $(v \in K)$ are pairwise disjoint. Let $I \subseteq A$ be the union of the sets $N_A^2(v)$ $(v \in K)$, and let $J \subseteq B$ be the union of the sets $N(v)$ $(v \in K)$. It follows that

(1) $|A \setminus I| > (1 - |K|/k)|A|$, *and* $|B \setminus J| \le (1 - |K|y)|B|$.

If $|K| = k$, then by (1), $|B \setminus J| \le (1-ky)|B| < x|B|$, and since every vertex in A has $x|B|$ neighbours in B, it follows that every vertex in A has a neighbour in J, that is, $I = A$, contrary to (1). Thus $|K| < k$.

Since each vertex in $A \setminus I$ has at least $x|B|$ neighbours in B, and they all belong to $B \setminus J$, some vertex $t \in B \setminus J$ has at least

$$x\frac{|A \setminus I|}{|B \setminus J|} \ge x\frac{1 - |K|/k}{1 - |K|y}|A|$$

neighbours in $A \setminus I$ by (1). Since $|K| \le k-1$ and so $|K|y < 1$, it follows that

$$\frac{1 - |K|/k}{1 - |K|y} \ge \frac{1-(k-1)/k}{1-(k-1)y} = \frac{1}{k(1-(k-1)y)},$$

and so t has at least $\frac{x|A|}{k(1-(k-1)y)}$ neighbours in $A \setminus I$. Let $u \in C$ be adjacent to t. From the maximality of K, u has a neighbour $w \in N(v)$ for some $v \in K$. Since w has at least $x|A|$ neighbours in I, it follows that

$$|N_A^2(u)| \ge x|A| + \frac{x|A|}{k(1-(k-1)y)} \ge |A|/k,$$

a contradiction. This proves 4.1. ∎

We deduce:

4.2 *For all integers $k \ge 1$, if $x, y \in (0,1]$ with $x + ky > 1$ and $kx + y \ge 1$, then $\psi(x,y) \ge 1/k$.*

Proof. If $k = 1$ the result is easy (and follows from 5.2 below), so we assume that $k \ge 2$; and hence we may assume that $x, y < 1/k \le 1/2$ by 1.2. By 4.1 we may assume (for a contradiction) that $kx + \frac{x}{1-(k-1)y} < 1$. Consequently $kx + \frac{x}{1-(k-1)(1-kx)} < 1$. Let $t = 1 - kx$. Then $\frac{1-t}{1-(k-1)t} < kt$, and so $k(k-1)t^2 - (k+1)t + 1 < 0$. This is quadratic in t, with discriminant $(k+1)^2 - 4k(k-1)$, and the latter is negative if $k > 2$; so we may assume that $k = 2$. Then $2t^2 - 3t + 1 < 0$, so $(2t-1)(t-1) < 0$, that is, $1/2 < t < 1$. But $t = 1 - 2x$, so $1/2 < 1 - 2x < 1$, that is, $x < 1/4$. But $2x + y \ge 1$ and $y < 1/2$, a contradiction. This proves 4.2. ∎

Consequently we have:

4.3 *For all $x \geq 0$, $\psi(x,x) = 1/k$, where k is the largest integer with $1/k \geq x$.*

Proof. Certainly $\psi(x,x) \leq 1/k$, since by 1.5,

$$\psi(x,x) \leq \frac{\lceil kx \rceil + \lceil kx \rceil - 1}{k} = 1/k.$$

Equality holds by 4.2. This proves 4.3. ∎

Next we need a lemma:

4.4 *Let $k \geq 1$ be an integer, let $(k-1)/k^2 \leq y \leq 1$, and let (A,B,C) be a tripartition of a graph G, such that:*

- *every vertex in B has at least $y|C|$ neighbours in C; and*
- *$|N_A^2(v)| < |A|/k$ for each $v \in C$.*

Then there exist $v_1, \ldots, v_k \in A$ such that $N(v_i) \cap N(v_j) = \emptyset$ for $1 \leq i < j \leq k$.

Proof. If some vertex v in A has degree zero, then we may take $v_1 = \cdots = v_k = v$. So we assume that every vertex in A has a neighbour in B. For each $v \in A$, let $c(v) = |N_C^2(v)|$, and let $A(v) \subseteq A$ be the set of vertices in A that have a neighbour in $N(v)$. Let $|A(v)| = a(v)$.

(1) *For each $v \in A$, $c(v) > kya(v)|C|/|A|$.*

If we choose $u \in N_C^2(v)$ independently at random, then since every vertex in $A(v)$ has at least $y|C|$ second neighbours in $N_C^2(v)$, the probability that a given vertex $w \in A(v)$ belongs to $N_A^2(u)$ is at least $y|C|/c(v)$, and so the expectation of $|N_A^2(u)|$ is at least $(y|C|/c(v))a(v)$. On the other hand, the expectation of $|N_A^2(u)|$ is less than $|A|/k$. This proves (1).

Let H be the graph with vertex set A, in which distinct u,v are adjacent if (in G) u,v have a common neighbour in B. Thus every vertex v has degree $a(v)-1$ in H. So $2|E(H)| = \sum_{v \in A}(a(v)-1)$; but

$$(ky|C|/|A|)\sum_{v \in A} a(v) \leq \sum_{v \in A} c(v) = \sum_{v \in A} |N_C^2(v)| = \sum_{u \in C} |N_A^2(u)| < |A| \cdot |C|/k.$$

Consequently

$$2|E(H)| < (|A| \cdot |C|/k)/(ky|C|/|A|) - |A| = |A|^2/(k^2 y) - |A| \leq |A|^2/(k-1) - |A|.$$

By Turán's theorem, H has a stable set of cardinality k. This proves 4.4. ∎

4.5 *Let $k \geq 1$ be an integer, and let $x,y \in (0,1]$ where $y \geq (k-1)/k^2$ and $kx + y > 1$. Let G be (x,y)-constrained via (A,B,C), such that every vertex in C has at least $y|B|$ neighbours in B. Then $|N_A^2(v)| \geq |A|/k$ for some $v \in C$. Consequently $\psi(x,y) \geq 1/k$.*

Proof. Suppose not; then there is a weighted graph (G', w), (x, y)-constrained via some tripartition (A', B', C'), such that

- for each $v \in C'$, $w(N(v)) \geq y|B'|$; and
- for each $v \in C'$, $w(N^2_{A'}(v)) < 1/k$.

Choose such a weighted graph (G', w) with $|V(G')|$ minimum, and let $z < 1/k$ such that $w(N^2_{A'}(v)) \leq z$ for each $v \in A'$. Then by 4.4, there exist $v_1, \ldots, v_k \in A'$ such that $N(v_1), \ldots, N(v_k)$ are pairwise disjoint. Consequently $w(N(v_1), \ldots, N(v_k)) \geq kx$; and since $w(N(u)) \geq y > 1 - kx$ for each $u \in C$, it follows that $\bigcup_{v \in X} N^2_{C'}(v) = C'$ where $X = \{v_1, \ldots, v_k\}$. But $|X| < z^{-1}$, contrary to 2.4 and the minimality of $|V(G')|$. This proves the first claim, and the second follows. This proves 4.5. ∎

5 The mono-constrained case

In this section we are mostly concerned with $\phi(x, y)$ when $x = y$. We know that ψ behaves well on the diagonal $x = y$, because of 4.3, so what about ϕ? More generally, what about an analogue of 4.1 or 4.2 with ψ replaced by ϕ?

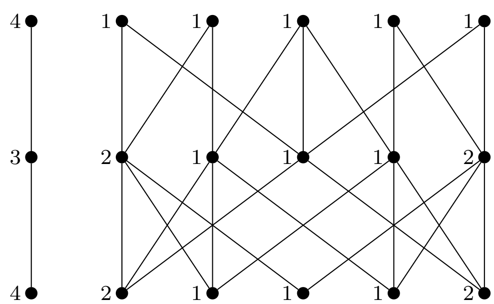

Figure 2: $\phi(3/10, 4/11) \leq 4/9$.

If we replace ψ by ϕ in 4.1, it becomes false, even with $k = 2$, because $\phi(3/10, 4/11) \leq 4/9$, as the graph of figure 2 shows (the sets A, B, C are the rows, and the numbers on the vertices are used as in figure 1). But as far as we know, 4.2 might hold with ψ replaced by ϕ. Let us state this as a conjecture:

5.1 Conjecture: *For all integers $k \geq 1$, if $x, y \in (0, 1]$ with $x + ky > 1$ and $kx + y \geq 1$, then $\phi(x, y) \geq 1/k$.*

On the other hand, we have not even been able to prove what is presumably the simplest nontrivial case of this, namely that $\phi(x, y) \geq 1/2$ for all x, y with $x, y > 1/3$. But we do have several results approaching 5.1. First, it is true with $k = 1$; we have the trivial:

5.2 *For $x, y \in (0, 1]$, if $x + y > 1$, or $x + y = 1$ and x is irrational, then $\phi(x, y) = 1$.*

Proof. Let G be (x, y)-constrained via (A, B, C). Then some vertex $v \in C$ has at least $y|B|$ neighbours in B, and strictly more if y is irrational; and so $N^2_A(v) = A$, as every vertex in A has at least $x|B|$ neighbours in B. This proves 5.2. ∎

5.1 implies that $\phi(x,x) \geq 1/2$ if $x > 1/3$. We have not been able to prove this, but we can show that $\phi(x,x) > 3/7$ if $x > 1/3$. That is implied by the following:

5.3 *Let $k \geq 2$ be an integer; then for $x, y \in (0,1]$, if $y > 1/k$ then*

$$\phi(x,y) \geq \frac{x(2-3x)}{kx(1-x)+x^2-3x+1}.$$

Moreover, if $k = 2$, then $\phi(x,y) \geq 2x - x^2$ (which is larger).

Proof. Let G be (x,y)-constrained via (A,B,C). If x is irrational then G is (x,y)-constrained via (A,B,C), for some rational $x' > x$; so we may assume that x is rational, by increasing x if necessary. Suppose that $k = 2$, and choose $v_1, v_2 \in B$ independently and uniformly at random. For each $u \in A$, the probability that u is adjacent to at least one of v_1, v_2 is at least $2x - x^2$, since u has at least $x|B|$ neighbours in B; and so we may choose v_1, v_2 such that at least $(2x - x^2)|A|$ vertices in A are adjacent to at least one of them. But v_1, v_2 have a common neighbour in C, since $y > 1/2$, and the claim follows.

Thus we may assume that $k \geq 3$. By 1.2, $\phi(x,y) \geq x$, and so we may assume that

$$\frac{x(2-3x)}{kx(1-x)+x^2-3x+1} > x,$$

that is, $x < 1/(k-1)$. Consequently $x \leq (k-2)/(k-1)$ since $k \geq 3$. Define

$$\begin{aligned} p &= \frac{x(1-x)}{kx(1-x)+x^2-3x+1}, \\ s &= \frac{x}{(k-2)(1-x)}, \text{ and} \\ m &= \frac{x(2-3x)}{kx(1-x)+x^2-3x+1}. \end{aligned}$$

These are all non-negative, and p is rational with denominator T say; and by replacing each vertex by T copies, we may assume that $p|A|$ is an integer. Since $x \leq (k-2)/(k-1)$ it follows that $s \leq 1$.

For $1 \leq i \leq k-1$, we define $v_i \in B$, and a subset P_i of $N_A(v_i)$ with $|P_i| = p|A|$, inductively, as follows. Let $Q = P_1 \cup \cdots \cup P_{i-1}$.

(1) *There exists $v_i \in B$ such that $sa + b \geq x(s|Q| + |A| - |Q|)$, where $a = |N_A(v_i) \cap Q|$ and $b = |N_A(v_i) \setminus Q|$.*

Suppose not; then summing over all $v \in B$, we deduce that

$$\sum_{v \in B} s|N_A(v) \cap Q| + \sum_{v \in B} |N_A(v) \setminus Q| < x(s|Q| + |A| - |Q|)|B|.$$

But the first sum is s times the number of edges between Q and B, and so at least $xs|Q|$; and the second is similarly at least $x(|A| - |Q|)$, a contradiction. This proves (1).

Let v_i be as in (1). Thus $sa + b \ge x(s|Q| + |A| - |Q|) \ge x(1 - (1-s)(k-2)p)|A|$. In particular, since

$$a + b \ge sa + b \ge x(1 - (1-s)(k-2)p)|A| = p|A|,$$

there exists $P_i \subseteq N_A(v_2)$ of cardinality $p|A|$. Also, since $a \le (k-2)p|A|$, and so

$$s(k-2)p|A| + b \ge sa + b \ge x(1 - (1-s)(k-2)p)|A|,$$

it follows that

$$b \ge x(1 - (1-s)(k-2)p)|A| - s(k-2)p|A| = (m-p)|A|,$$

and so

$$|N_A(v_h) \cup N_A(v_i)| \ge m|A|,$$

for $1 \le h < i$. This completes the inductive definition of $v_1, \dots, v_{k-1}$ and $P_1, \dots, P_{k-1}$.

Let $P = P_1 \cup \dots \cup P_{k-1}$. Then $|P| \le (k-1)p|A|$. Since every vertex in $A \setminus P$ has at least $x|B|$ neighbours in B, there exists $v_k \in B$ with at least $x(|A| - |P|) \ge x(1 - (k-1)p)|A|$ neighbours in $A \setminus P$. Let P_k be its set of neighbours in $A \setminus P$. Then for all i with $1 \le i \le k-1$,

$$|P_i| + |P_k| \ge (x(1 - (k-1)p) + p)|A| = m|A|.$$

Consequently, for all distinct $v, v' \in \{v_1, \dots, v_k\}$, $|N_A(v) \cup N_A(v')| \ge m|A|$. But since $y > 1/k$, some two of $v_1, \dots, v_k$ have a common neighbour $u \in C$, and so $|N_A^2(u)| \ge m$. This proves 5.3. ∎

We deduce from 5.3 a version of 4.3 for the mono-constrained case:

5.4 *For $y \in (0,1]$, if $y > 1/k$ where $k \ge 2$ is an integer, then $\phi(1/k, y) \ge \frac{2k-3}{2k^2-4k+1}$.*

Consequently $\phi(1/k, y) \le 1/k + 1/(2k^2) + O(k^{-3})$.

5.4 tells us in particular that $\phi(x,x) \ge \frac{2k-3}{2k^2-4k+1} > 1/k$ when $x > 1/k$ (if $k \ge 2$ is an integer), and since $\phi(1/k, 1/k) = 1/k$, there is a discontinuity in $\phi(x,x)$ when $x = 1/k$, and the limit of $\phi(x,x)$ as $x \to 1/k$ from above is different from $\phi(1/k, 1/k)$. What happens when $x \to 1/k$ from below? The next results investigate this. We will show that if x is sufficiently close to $1/k$ from below, then $\phi(x,x) = 1/k$.

5.5 *If $k > 0$ is an integer and $x \in (0,1]$ satisfies $(1-x)^k < x$, then $\phi(x,x) \ge 1/k$. In particular, if $x > 0.382$ then $\phi(x,x) \ge 1/2$, and if $x > 0.318$ then $\phi(x,x) > 1/3$.*

Proof. Let G be (x,x)-constrained via (A,B,C). If we choosing k vertices from C uniformly at random, the number of vertices in B nonadjacent to all of them is at most $(1-x)^k|B|$ in expectation; and so there exist $v_1, \dots, v_k \in C$ such that at most $(1-x)^k|B|$ vertices in B are nonadjacent to all of them. Since $(1-x)^k|B| < x|B|$, it follows that the sets $N_A^2(v_i)$ $(1 \le i \le k)$ have union A, and so one of them has cardinality at least $|A|/k$. This proves 5.5. ∎

The result 5.5 is of no use when $k \ge 4$ because then $(1-x)^k < x$ implies $x > 1/k$; and we will see in 6.5 a more complicated argument that prove a result stronger than 5.5 when $k = 2$. Here is another approach to the same question, more successful for larger values of k.

5.6 *Let $k \ge 1$ be an integer, and let $x \ge 1/k - \varepsilon$ where $\varepsilon = 1/(13k^3)$. Then $\phi(x,x) \ge 1/k$.*

Proof. We may assume that $x = 1/k - \varepsilon$. By 5.2 we may assume that $k \geq 2$. We leave the reader to check that

- $1/(2k) - \varepsilon > 6k^2\varepsilon$;
- $x > 1/(k+1)$; and
- $(2kx - 1)/(2k - 1) > (k\varepsilon)/(x + k\varepsilon)$.

(These are inequalities we will need later.) Let G be (x, x)-constrained via (A, B, C), and suppose that $|N_A^2(v)| < |A|/k$ for each $v \in C$. Let P be the set of vertices in B that have at most $(1/k-2k\varepsilon)|A|$ neighbours in A.

(1) *$|P| \leq |B|/(2k)$.*

Every vertex in B has fewer than $|A|/k$ neighbours in A, and so the number of edges between A and B is at most $|P|(1/k - 2k\varepsilon)|A| + (|B| - |P|)|A|/k$. On the other hand, the number of such edges is at least $(1/k - \varepsilon)|A||B|$; and so

$$|P|(1/k - 2k\varepsilon)|A| + (|B| - |P|)|A|/k \geq (1/k - \varepsilon)|A||B|,$$

which simplifies to $2k|P| \leq |B|$. This proves (1).

(2) *If $u, v \in B \setminus P$ have a common neighbour in C, then $|N_A(u) \setminus N_A(v)| \leq 2k\varepsilon|A|$.*

Since $u, v \in B \setminus P$ have a common neighbour in C, it follows that $|N_A(u) \cup N_A(v)| \leq |A|/k$. But $|N_A(u)| \geq (1/k - 2k\varepsilon)|A|$ since $u \in B \setminus P$, and so $|N_A(u) \setminus N_A(v)| \leq 2k\varepsilon|A|$. This proves (2).

(3) *There exist $v_1, \ldots, v_k \in B \setminus P$ such that for $1 \leq i < j \leq k$, there are at least $(1/(2k) - \varepsilon)|A|/k$ vertices in A that are adjacent to v_j and not to v_i.*

Choose $v_1, \ldots, v_k \in B \setminus P$ as follows. Choose $v_1 \in B \setminus P$ arbitrarily. Inductively, suppose we have defined $v_1, \ldots, v_i$ where $i < k$. Each has at most $|A|/k$ neighbours in A, and so the set of vertices in A adjacent to one of $v_1, \ldots, v_i$ has cardinality at most $(i/k)|A| \leq (1 - 1/k)|A|$. Let D be the set of vertices in A nonadjacent to each of $v_1, \ldots, v_i$; then $|D| \geq |A|/k$. Since, by (1), each vertex in D has at least $x|B| - |P| \geq (1/(2k) - \varepsilon)|B|$ neighbours in $B \setminus P$, there exists $v_{i+1} \in B \setminus P$ with at least $(1/(2k) - \varepsilon)|A|/k$ neighbours in D. This completes the inductive definition. We see that for $1 \leq i < j \leq k$, there are at least $(1/(2k) - \varepsilon)|A|/k$ vertices in A that are adjacent to v_j and not to v_i. This proves (3).

Let H be the bipartite graph $G[(B \setminus P) \cup C]$.

(4) *For $1 \leq i < j \leq k$, v_i and v_j belong to distinct components of H.*

From (2), the sets $N_C(v_1), \ldots, N_C(v_k)$ are pairwise disjoint, because $(1/(2k) - \varepsilon)|A|/k > 2k\varepsilon|A|$. Suppose that there is a path of H joining some two of $v_1, \ldots, v_k$, and take the shortest such path Q; between v_i and v_j say, where $j > i$. Let Q have m vertices in B, say $u_1, \ldots, u_m$ in order where

$u_1 = v_i$. We claim that $m \le 4$. For suppose that $m \ge 5$. From the minimality of the length of Q, u_3 has no common neighbour in C with any of $v_1, \dots, v_k$, and so the sets $N_C(v), N_C(v_1), \dots, N_C(v_k)$ are pairwise disjoint, which is impossible since $x > 1/(k+1)$. Thus $m \le 4$. By applying (3) to each pair of consecutive members of $V(Q) \cap B$, we deduce that

$$|N_A(v_j) \setminus N_A(v_i)| \le (m-1)2k\varepsilon|A| \le 6k\varepsilon|A|.$$

But $|N_A(v_j) \setminus N_A(v_i)| \ge (1/(2k) - \varepsilon)|A|/k$, and so $(1/(2k) - \varepsilon)|A|/k \le 6k\varepsilon|A|$, a contradiction. This proves (4).

For $1 \le i \le k$, let H_i be the component of H containing v_i, and let $V(H_i) \cap B = B_i$ and $V(H_i) \cap C = C_i$. If there exists $v \in B \setminus P$ that does not belong to any of $B_1, \dots, B_k$, then the sets $N_C(v), N_C(v_1), \dots, N_C(v_k)$ are pairwise disjoint, which is impossible since they all have cardinality at least $x|C|$, and $(k+1)x > 1$. Consequently the sets $B_1, \dots, B_k$ and P form a partition of B.

(5) *For $1 \le i \le k$ there exists $u_i \in C_i$ adjacent to at least $\frac{(1-k\varepsilon)}{1+k(k-1)\varepsilon}|B_i|$ vertices in B_i.*

For $1 \le i \le k$, since v_i has at least $x|C|$ neighbours in C, it follows that $|C_i| \ge x|C|$. Let $1 \le i \le k$. Since $C_1, \dots, C_k$ are pairwise disjoint, and the union of the sets C_j ($j \in \{1, \dots, k\} \setminus \{i\}$) has cardinality at least $(k-1)x|C|$, it follows that

$$|C_i| \le |C| - (k-1)x|C| = x|C| + k\varepsilon|C|.$$

There are at least $x|B_i| \cdot |C|$ edges between B_i and C_i, and so some vertex in C_i has at least

$$x(|C|/|C_i|)|B_i| \ge x(|C|/(x|C| + k\varepsilon|C|))|B_i| = (x/(x + k\varepsilon))|B_i|$$

neighbours in B_i. By substituting $x = 1/k - \varepsilon$, this proves (5).

For $1 \le i \le k$, let $A_i = N_A^2(u_i)$. Since $|A_i| < |A|/k$ for $1 \le i \le k$, there exists $v \in A$ that is in none of $A_1, \dots, A_k$. Now v has at least $x|B|$ neighbours in B, and they all belong to $B_1 \cup \dots \cup B_k$ except for at most $|P|$ of them. Consequently there exists $i \in \{1, \dots, k\}$ such that v has at least $(x|B| - |P|)|B_i|/|B \setminus P|$ neighbours in B_i. Since $v \notin A_i$, it follows that

$$(x|B| - |P|)|B_i|/|B \setminus P| + (x/(x + k\varepsilon))|B_i| \le |B_i|.$$

Since $x|B| \le |B|$ and $|P| \le |B|/(2k)$ by (1), it follows that

$$(x|B| - |P|)|B_i|/|B \setminus P| \ge (x - 1/(2k))|B_i|/(1 - 1/(2k)) = (2kx - 1)|B_i|/(2k - 1),$$

and so $(2kx - 1)/(2k - 1) \le k\varepsilon/(x + k\varepsilon)$, a contradiction. This proves 5.6. ∎

For comparison, in figure 3 we give graphs of the function $\psi(x, x)$ (which we know completely, because of 4.3), and the function $\phi(x, x)$ (which we only know partially, from 5.6 and 5.4.)

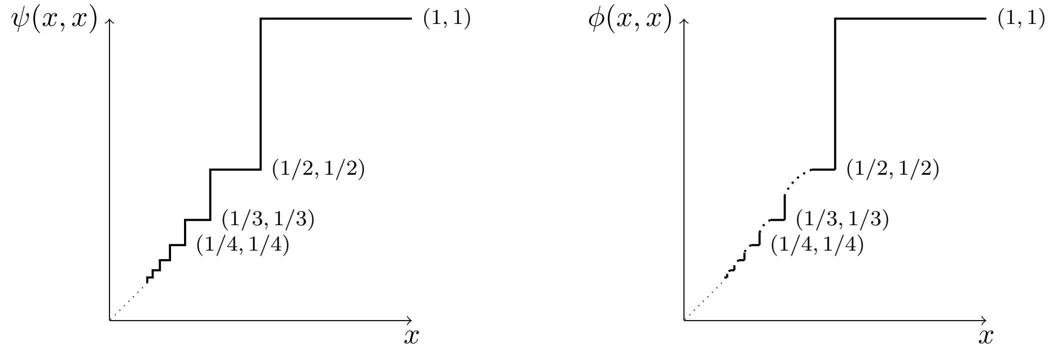

Figure 3: Graphs of $\psi(x,x)$ and $\phi(x,x)$

6 When is $\phi(x,y)$ or $\psi(x,y) \geq 1/2$?

Another way to approach the problem is to ask, given some value z, for which $x, y \in (0,1]$ is $\phi(x,y) \geq z$? Or we could ask the same question for ψ, or ask when $\phi(x,y) > z$. For instance:

6.1 *If $k \geq 1$ is an integer, then for $x, y \in (0,1]$, $\phi(x,y) > 1/k$ if and only if $\max(x,y) > 1/k$.*

This follows trivially from 1.3 and 1.2. And the same holds with ϕ replaced by ψ. But deciding when $\psi(x,y)$ or $\phi(x,y) \geq 1/k$ seems to be much less obvious. In this section we discuss when $\psi(x,y)$ or $\phi(x,y)$ is at least 1/2; and in later sections we look at when they are at least 2/3, and at least 1/3.

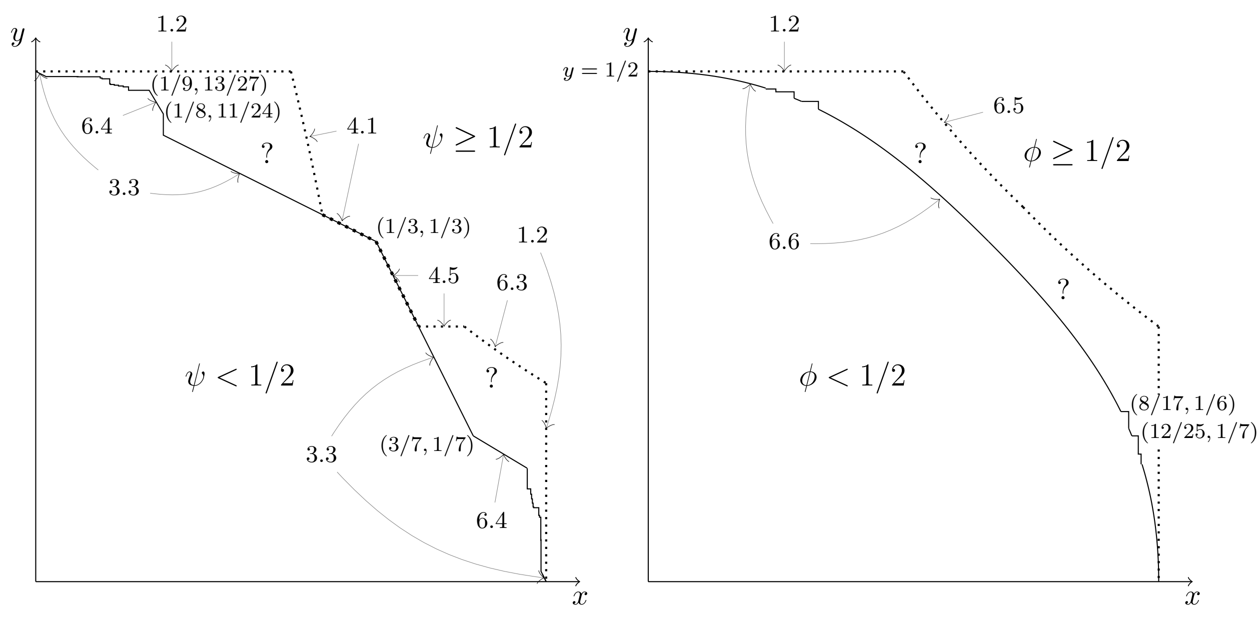

Figure 4: In the left-hand figure, $\psi(x,y) < 1/2$ for pairs (x,y) below the solid line, and $\psi(x,y) \geq 1/2$ above the dotted one; between we don't know. The right-hand figure does the same for ϕ.

For $x, y \in (0,1]$, we say (temporarily) that (x,y) is *good* if $\psi(x,y) \geq 1/2$, and *bad* otherwise. The "map" of good and bad points is shown in the left half of figure 4. The solid black curve borders the known bad points, and the dotted curve borders the good points; between them is undecided. The borders are complicated, and we have indicated in the figure which theorem is responsible for each stretch of border.

Let us explain some of the details. First, if $\max(x,y) \geq 1/2$, then (x,y) is good; and all pairs (x,y) with $x+2y, 2x+y \leq 1$ are bad, by 3.3. We searched by computer to find other examples of bad pairs (x,y), and found about 12 maximal such pairs of rationals, with numerator and denominator at most 100. In fact we only searched for pairs (x,y) where the corresponding (x,y)-biconstrained graph is similar to the graph obtained from figure 1, that is, it is obtained by "blowing up" the vertices of another graph in which the graph between two of the three parts is a matching. All these examples not only show that $\psi(x,y) < 1/2$, but also that $\psi(y,x) < 1/2$, and $\xi(x,y) < 1/2$. In particular, for every bad pair (x,y) we found by computer search, (y,x) is another. This is just an artifact of our method of search, and is not evidence that the set of all bad pairs is closed under switching x and y (though it might be; it is for ϕ, by 2.3). Anyway, for each bad pair the computer found, all pairs it dominates are also bad, and that gave us a step function bordering the area of the known bad points. We improved on this; we were able to fill out some of the steps of the step function, by means of 3.3 and 6.4, so the step function the computer found now only survives towards the ends of the solid black curve in the figure. (These "fills" are not invariant under switching x and y.) We give the coordinates of some bad pairs that we find particularly interesting. The apparent asymmetry between x and y in the left half of the figure is just asymmetry among what we have been able to prove; we have no proof of asymmetry. The right half of figure 4 does the same for ϕ. Here there is symmetry exchanging x and y, by 2.3, and so we only "explain" half of the border.

A graph has *radius* at most r if there is a vertex u such that every vertex has distance at most r from u. We will need the following theorem of Erdős, Saks and Sós [1]:

6.2 *Let G be a connected graph with radius at least r, where $r \geq 1$ is an integer. Then G has an induced path with $2r-1$ vertices, and consequently has a stable set of cardinality at least r.*

When we have more than one graph defined using the same vertices, we speak of "H-distance" to mean distance in the graph H, and so on.

6.3 *Let $x, y \in (0,1]$, such that*

$$x^2(1+3y) + x(4y^2 - y - 2) + 1 - 2y + 2y^3 < 0.$$

Then $\psi(x,y) \geq 1/2$.

Proof. Let G be (x,y)-biconstrained via (A,B,C), and suppose that $|N_A^2(v)| < |A|/2$ for each $v \in C$. Then 1.2 implies that $x, y < 1/2$. Suppose that $y > 1/4$. The given inequality implies that $56x^2 - 64x + 17 < 0$, and so $x > .41$. Since $2x+y > 1$, 4.5 implies that $y \leq 1/4$, a contradiction. Thus $y \leq 1/4$. We leave the reader to verify that the following are consequences of the given inequality:

- $\frac{x}{3-3y} > 1-2x$; and in particular, $\frac{x}{1-y} > 1-2x$, so from 4.1 it follows that $x+2y \leq 1$;
- $\frac{y}{3-6y} > 1-2x$; and so $\frac{y}{2-2y} > 1-2x$, since $y \leq 1/4$; and

- $x + 3y > 1$.

(We found the easiest way to check these is to have a computer plot the various curves.) Let H be the bipartite graph $G[B \cup C]$.

(1) *If $v, v' \in C$ have H-distance at most $2t$ where $t > 0$ is an integer, then*

$$|N_A^2(v') \setminus N_A^2(v)| < t(1/2 - x)|A|.$$

Take a path P of H joining v and v', of length at most $2t$. Let the vertices of P in C be

$$v = v_0, \dots, v_t = v',$$

in order. For $1 \le i \le t$ let $u_i \in B$ be adjacent to v_{i-1} and v_i. Then for $1 \le i \le t$, $N_A^2(v_{i-1}) \cap N_A^2(v_i)$ includes $N_A(u_i)$ and hence has cardinality at least $x|A|$; and since $|N_A^2(v_i)| < |A|/2$, it follows that $|N_A^2(v_i) \setminus N_A^2(v_{i-1})| < (1/2 - x)|A|$. But the union of the t sets $N_A^2(v_i) \setminus N_A^2(v_{i-1})$ includes $N_A^2(v') \setminus N_A^2(v)$, and so the latter has cardinality less than $t(1/2 - x)|A|$. This proves (1).

(2) *There do not exist $v_1, \dots, v_4 \in C$, pairwise with no common neighbour in B.*

Suppose that such $v_1, \dots, v_4$ exist. Then every three of $N(v_1), \dots, N(v_4)$ have union of cardinality at least $3y$; and since $3y > 1 - x$, every vertex in A has a neighbour in at least two of $N(v_1), \dots, N(v_4)$. Consequently every vertex in A belongs to at least two of $N_A^2(v_1), \dots, N_A^2(v_4)$, and so one of $N_A^2(v_1), \dots, N_A^2(v_4)$ has cardinality at least $|A|/2$, a contradiction. This proves (2).

(3) *H has at least two components.*

Suppose not, and let H' be the graph with vertex set C in which v, v' are adjacent if they have a common neighbour in H. By (2), it follows that H' has no stable set of cardinality four, and so H' has radius at most three by 6.2. Choose $v \in C$ such that every vertex in C has H'-distance at most three from v. Let $B_1 = N_B(v)$ and $A_1 = N_A^2(v)$. Every vertex in $A \setminus A_1$ has at least $x|B|$ neighbours in $B \setminus B_1$, and so some vertex $u \in B \setminus B_1$ has at least $x(|B|/|B \setminus B_1|)|A \setminus A_1|$ neighbours in $A \setminus A_1$. Let A_2 be the set of neighbours of u in $A \setminus A_1$. Since $|B \setminus B_1| \le (1 - y)|B|$ and $|A \setminus A_1| > |A|/2$, it follows that

$$|A_2| \ge (x/(1 - y))|A|/2 \ge 3(1/2 - x)|A|.$$

Let $v' \in C$ be adjacent to u. Since the H'-distance from v to v' is at most three, the H-distance from v to v' is at most six. By (1), $|N_A^2(v') \setminus N_A^2(v)| < 3(1/2 - x)|A|$, a contradiction. This proves (3).

(4) *If H' is a component of H then $|V(H') \cap B| \le (1 - x)|B|$.*

Suppose that $|V(H') \cap B| > (1 - x)|B|$; then every vertex in A has a neighbour in $V(H')$. By (2), there do not exist three vertices in $C \cap V(H')$ pairwise with no common neighbour, and so by 6.2, it follows that there is a vertex $v \in C \cap V(H')$ with H'-distance at most four from every vertex in $C \cap V(H')$. Let $A' = N_A^2(v)$; then $|A'| < |A|/2$. Since every vertex in $A \setminus A'$ has at least $y|C|$ second neighbours in $C \cap V(H')$, and $|C \cap V(H')| \le (1 - y)|C|$, some vertex v' in $V \cap V(H')$ has at least $(y/(1 - y))|A \setminus A'|$ second neighbours in $A \setminus A'$. By (1), $(y/(1 - y))|A \setminus A'| < 2(1/2 - x)|A|$.

But $|A'| < |A|/2$, so $y/(4(1-y)) \le 1/2 - x$, a contradiction. This proves (4).

(5) *Some component H' of H satisfies* $(1-x)|B| \ge |V(H') \cap B| \ge x|B|$.

By (2) and (3), H has either two or three components. If H has only two components, then they both satisfy (5), by (4); so we assume there are three. Let the components of H be H_1, H_2, H_3, and for $1 \le i \le 3$, let $V(H_i) \cap B = B_i$ and $V(H_i) \cap C = C_i$; and let $|B_i|/|B| = b_i$ and $|C_i|/|C| = c_i$. Suppose that $b_1, b_2, b_3 < x$. Consequently every vertex in A has neighbours in at least two of B_1, B_2, B_3. For $1 \le i \le 3$, let A_i be the set of vertices in A with a neighbour in B_i. Thus every vertex in A belongs to at least two of A_1, A_2, A_3, so from the symmetry we may assume that $|A_1| \ge 2|A|/3$. By (2), every two vertices in C_1 have a common neighbour in B. Choose $v \in C_1$, and let $A' = N_A^2(v)$; then $|A'| \le |A|/2$. Since every vertex in A_1 has at least $y|C|$ second neighbours in C_1, some vertex v' in C_1 has at least $(y/c_1)|A_1 \setminus A'|$ second neighbours in $A_1 \setminus A'$. By (1), $(y/c_1)|A_1 \setminus A'| < (1/2 - x)|A|$. But $|A'| < |A|/2$, so $|A_1 \setminus A'| \ge |A|/6$; and $c_1 \le 1 - 2y$, so $y/(6(1-2y)) \le 1/2 - x$, a contradiction. This proves (5).

(6) *Every vertex in C has at most $(1-x-y)|B|$ neighbours in B. Consequently*

$$|V(H') \cap B| \le (1-x-y)|V(H') \cap C|/y$$

for each component H' of H.

Suppose that $v \in C$ has more than $(1-x-y)|B|$ neighbours in B. Choose $v' \in C$ in a different component of H; so v, v' have no common neighbour in B. Consequently

$$|N(v) \cup N(v')| > ((1-x-y) + y))|B|,$$

and so every vertex in A has a neighbour in $N(v) \cup N(v')$. But then one of $|N_A^2(v)|, |N_A^2(v')| \ge |A|/2$, a contradiction. This proves the first assertion. Let H' be a component of H. Then H' has at least $y|B| \cdot |V(H') \cap B|$ edges, and at most $(1-x-y)|B| \cdot |V(H') \cap C|$ edges, so the second claim follows. This proves (6).

Let H' be as in (5), and take the union of the other (one or two) components of H'. We obtain nonnull subgraphs H_1, H_2 of H, pairwise vertex-disjoint and with union H, such that $|V(H_i) \cap B| \ge x|B|$ for $i = 1, 2$. For $i = 1, 2$, let $V(H_i) \cap B = B_i$ and $V(H_i) \cap C = C_i$; and let $|B_i|/|B| = b_i$ and $|C_i|/|C| = c_i$. Thus $b_1, b_2 \ge x$. From (6), $b_i \le (1-x-y)c_i/y$ for $i = 1, 2$; and $c_1, c_2 \ge y$, since every vertex in B_i has at least $y|C|$ neighbours in C_i. Also $b_1 + b_2 = c_1 + c_2 = 1$.

For $i = 1, 2$ let A_i be the set of vertices $v \in A$ that have more than $(b_i - y)|B|$ neighbours in B_i. Let $A_0 = A \setminus (A_1 \cup A_2)$. Hence if $u \in A_0$, then since u has at least $x|B|$ neighbours in B, u has at least $(x + y - b_2)|B|$ neighbours in B_1, and at least $(x + y - b_1)|B|$ neighbours in B_2.

Since A_1, A_2 and A_0 have union A, we may assume that $|A_1| + |A_0|/2 \ge |A|/2$. Now $A_1 \subseteq N_A^2(v)$ for each $v \in C_1$, since if $u \in A_1$, then u has more than $(b_1 - y)|B|$ neighbours in B_1, and v has at least $y|B|$ neighbours in B. Consequently $|N_A^2(v) \cap A_0| < |A_0|/2$ for each $v \in C_1$.

Let us choose $v \in C_1$ uniformly at random; then the expected number of second neighbours of v in A_0 is less than $|A_0|/2$, and so for some vertex $u \in A_0$, the probability that $u \in N_A^2(v)$ is less than $1/2$. Let D be the set of neighbours of u in B_1. Then $|D| \ge (x + y - b_2)|B|$, and the probability that

v has a neighbour in D is less than 1/2. Thus more than $|C_1|/2$ vertices in C_1 have no neighbour in D. On the other hand, the expectation of the number of neighbours of v in D is at least $|D|y/c_1$; and so there exists $v \in C_1$ with more than $2|D|y/c_1$ neighbours in D. Also there exists $v' \in C_1$ with no neighbours in D. It follows that

$$|N_B(v) \cup N_B(v')| \geq y|B| + 2|D|y/c_1 > (y + 2(x + y - b_2)y/c_1)|B|.$$

Some vertex in A_0 is not a second neighbour of either of v, v', and so

$$|N_B(v) \cup N_B(v')| < (b_1 - (x + y - b_2))|B|.$$

Consequently $y + 2(x + y + b_1 - 1)y/c_1 \leq 1 - x - y$. Now $c_1 \leq (1 - x - 2y + yb_1)/(1 - x - y)$ since

$$1 - b_1 = b_2 \leq (1 - x - y)c_2/y = (1 - x - y)(1 - c_1)/y.$$

So

$$2y(x + y + b_1 - 1)(1 - x - y)/(1 - x - 2y + yb_1) \leq 1 - x - 2y,$$

that is,

$$b_1 y(1 - x) \leq x^2(1 + 2y) + x(-2 + 4y^2) + 1 - 2y + 2y^3.$$

But $b_1 \geq x$, contrary to the hypothesis. This proves 6.3. ∎

6.4 *For $x, y \in (0, 1]$, if $x \leq 13/27$ and $y \leq 1/7$ and $3x + 5y \leq 2$, then $\psi(x, y) \leq 13/27$. If in addition $y < 1/8$, then $\psi(y, x) < 1/2$.*

Proof. For this, we return to the graph of figure 1. Let A, B, C be the three rows of vertices, in order where A is the top row. We need to adjust the vertex weights. Define $q = \max(3x/2 + y - 1/2, 8x/5 - 2/5, 27)$; $r = \max(2y, x - p/2)$; and $p = 1 - q - r$. Now, with the vertices in the same order as the figure, take vertex weights as follows:

$$\begin{array}{rrr} 10/27, 10/27, & 1/3, 1/3, 1/3, & 4/27, 4/27 \\ p/2, p/2, & r/2, r/2, r/2, & q/3, q/3 \\ 1/7, 1/7, & 1/7, 1/7, 1/7, & 1/7, 1/7 \end{array}$$

One can check (it takes some time and we omit the details) that this defines an (x, y)-constrained weighted graph showing that $\psi(x, y) \leq 13/27$. For the second statement, take the same graph and same vertex weighting, except replace the third row (of all one-sevenths) in the table above, by

$$p'/2, p'/2, \quad r'/3, r'/3, r'/3, \quad q'/2, q'/2$$

where $p' = 5/16 - y/2$, $q' = 11/31 + 3y/8$, and $r' = 21/64 + y/8$. This weighted graph is (y, x)-biconstrained via (C, B, A), and shows that $\psi(y, x) < 1/2$. (Again, we leave the reader to check that this works.) This proves 6.4. ∎

Now the mono-constrained case: for which pairs (x, y) is $\phi(x, y) \geq 1/2$? Now we have symmetry between x and y, and we found some examples of pairs (x, y) with $\phi(x, y) < 1/2$ on a computer searching randomly. (Conjecture 5.1 says that all points above both the lines $x + 2y = 1$ and $2x + y = 1$ should be good, and indeed, all the maximal examples the computer found lie in the wedges between the lines.)

The next result strengthens 5.5 when $k = 2$:

6.5 *Let $x, y \in (0, 1]$ satisfy $y(1 + \sqrt{2x})^2/2 + x > 1$; then $\phi(x, y) \geq 1/2$.*

Proof. Let G be (x, y)-constrained, via (A, B, C). We may assume that every vertex in B has exactly $y|C|$ neighbours in C. For each vertex $v \in C$, let $w(v)$ be the degree of v divided by $y|B| \cdot |C|$, which is the total number of edges between B and C. Then $w(C) = 1$. Choose $v \in C$ independently with probability $w(v)$. We may assume that the expectation of $|N_A^2(v)|$ is less than $|A|/2$, and so there exists $u \in A$ such that the probability that $u \in N_A^2(v)$ is less than $1/2$. Let B_1 be the set of neighbours of u in B, let C_1 be the set of vertices in C with a neighbour in B_1, let $|C_1| = c|C|$, and let $C_2 = C \setminus C_1$. Since each vertex in B_1 has $y|C|$ neighbours in C_1, there exists $v_1 \in C_1$ with at least $y|B_1|/c$ neighbours in B_1. Now $w(C_1)w(v) < 1/2$, and so there are fewer than $y|B| \cdot |C|/2$ edges with an end in C_1. Consequently there are at least $y|B| \cdot |C|/2$ edges with an end in C_2, and so there exists $v_2 \in C_2$ with at least $y|B|/(2(1-c))$ neighbours in B. All these neighbours belong to $B \setminus B_1$, and so

$$|N_B(v_1) \cup N_B(v_2)| \geq \frac{y|B_1|}{c} + \frac{y|B|}{2(1-c)} \geq \left(\frac{xy}{c} + \frac{y}{2(1-c)}\right)|B|.$$

The quantity $xy/c + y/(2(1-c))$ is minimized for $0 \leq c \leq 1$ when $c = \sqrt{2x}/(1 + \sqrt{2x})$, and its value is then $xy/c^2 = y(1 + \sqrt{2x})^2/2$. By hypothesis this is more than $1 - x$, and so every vertex in A has a neighbour in one of $N_B(v_1), N_B(v_2)$. This proves 6.5. ∎

6.6 *Let $x, y \in (0, 1]$ with $x \leq 1/3$ and $y < 1/2$ and $y < (1-x)^2/(2 - 4x + 6x^2)$; then $\phi(x, y) < 1/2$.*

Proof. If $y \leq 1/3$ the result follows from 1.3, so we assume that $y > 1/3$. If $x + 2y \leq 1$ then $2x + y < 1$ and the result follows from 3.3, so we assume that $x + 2y > 1$. Let $m = 1/y - 2$. It follows that $m > 0$, and the quadratic polynomial (in r) $xr^2 - m(1-x)r + mx = 0$ has distinct real roots $r_1 < r_2$ say, where $0 \leq r_1 < 1$, and $x/(1-x) \leq r_2$. Consequently we may choose a rational $s_1 < 1$ with

$$0, r_1, x/(1-x) \leq s_1 \leq r_2.$$

Thus $xs_1^2 - m(1-x)s_1 + mx \leq 0$. Choose a rational $s_2 < 1$ with

$$0, xs_1/(s_1 - x(1 + s_1)) \leq s_2 \leq m/s_1.$$

Choose an integer $N > 1$ such that s_1N, s_2N are integers. For $i = 1, 2$, choose a graph G_i that is $(s_i, 1 - s_i)$-constrained via a tripartition (A_i, B_i, C_i), such that $|A_i| = |B_i| = |C_i| = N$ and $N_{A_i}^2(v) \neq A_i$ for each $v \in C_i$. (It is easy to see that such a graph exists, for instance, one of the graphs used in 1.3.) Take the disjoint union of G_1 and G_2, and add edges to make every vertex in B_1 adjacent to every vertex in C_2. Add three more vertices a, b, c, where a is adjacent to b, b is adjacent to every vertex in C_1, and c is adjacent to every vertex in B_2, forming G. We define a weighting w of G as follows. Choose p with $(2 - 2(N-1)/N + 2(N-1)^2/N^2)p > 1/N$ and $p < 1/2$. Choose q with $q + (N-1)p/N < 1/2$ and $1 - p - q + (N-1)p/N < 1/2$. Choose f with

$$(y - 1 + s_1 + y - ys_1)/(1 - (1 - s_1)s_2) \leq f \leq (1 - 2y)/s_2$$

and $0 \leq f \leq 1$. Choose e with $e \geq y, (y - f)/(1 - s_1)$ and $e \leq 1 - s_2 f - y$.

Define w by:

$$\begin{aligned}
w(a) &= 1 - p - q;\\
w(v) &= p/N \text{ for each } v \in A_1\\
w(v) &= q/N \text{ for each } v \in A_2\\
w(b) &= x\\
w(v) &= (1 - s_2/x - x)/N \text{ for each } v \in B_1\\
w(v) &= s_2/(Nx) \text{ for each } v \in B_2\\
w(c) &= y - (1 - s_2)f\\
w(v) &= (1 - s_2 f - y)/N \text{ for each } v \in C_1\\
w(v) &= f/N \text{ for each } v \in C_2
\end{aligned}$$

Define $A = A_1 \cup A_2 \cup \{a\}$ and define B, C similarly. Then the weighted graph (G, w) is (x, y)-constrained via (A, B, C), and proves that $\phi(x, y) < 1/2$. This proves 6.6. ∎

7 The 2/3 level

When is $\phi(x, y) \geq 2/3$; or the same question for ψ? In this section we say what we know about these.

7.1 *If $x > 1/2$ then $\psi(x, 1/3) \geq 2/3$.*

Proof. Let G be $(x, 1/3)$-biconstrained via (A, B, C), and suppose for a contradiction that $|N_A^2(v)| < 2|A|/3$ for all $v \in C$. By averaging, there exists $v_0 \in A$ such that $|N_C^2(v_0)| < 2|C|/3$. Let $B_0 = N(v_0)$ and $C_0 = N_C^2(v_0)$. Hence $|B_0| \geq x|B|$, and $|C_0| < 2|C|/3$, and there are no edges between B_0 and $C \setminus C_0$, and every vertex in C_0 has a neighbour in B_0.

Choose $v_1 \in C_0$. Thus $N(v_1) \cap B_0 \neq \emptyset$. Let $B_1 = N(v_1)$ and $A_1 = N_A^2(v_1)$. So $|A_1| \geq x|A|$, and $|A_1| < 2|A|/3$. Every vertex $v \in A \setminus A_1$ has a neighbour in B_0, since $|B \setminus B_0| < |B|/2 < |N(v)|$. Consequently every vertex in $A \setminus A_1$ has at least $|C|/3 \geq |C_0|/2$ second neighbours in C_0, and by averaging it follows that some vertex $v_2 \in C_0$ has at least $|A \setminus A_1|/2$ second neighbours in $A \setminus A_1$. Let $B_2 = N(v_2)$ and $A_2 = N_A^2(v_2)$. Then $|A_2 \setminus A_1| \geq |A \setminus A_1|/2 \geq |A|/6$. If there exists $u \in B_1 \cap B_2$, then since u has at least $x|A|$ neighbours in A_1, and they all belong to A_2, it follows that

$$|A_2| = |A_2 \cap A_1| + |A_2 \setminus A_1| \geq x|A| + |A|/6 \geq 2|A|/3,$$

a contradiction. Consequently $B_1 \cap B_2 = \emptyset$.

In particular, $|B_1 \cup B_2| \geq 2|B|/3$, and so every vertex in A has a neighbour in $B_1 \cup B_2$; and so $A_1 \cup A_2 = A$. Since $|A_1|, |A_2| < 2|A|/3$, it follows that $|A_1 \cap A_2| < |A|/3$. For $i = 1, 2$,

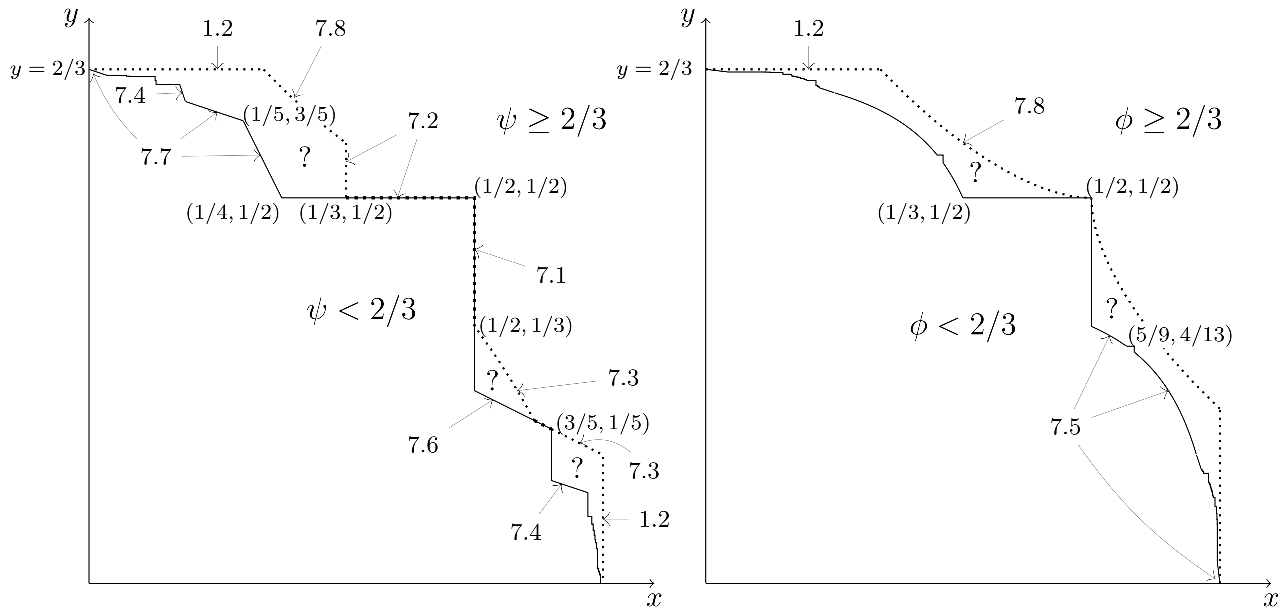

Figure 5: When $\psi(x, y) < 2/3$ and when $\phi(x, y) < 2/3$.

choose $b_i \in B_i \cap B_0$. Then $N(b_i) \cap A \subseteq A_i$ for $i = 1, 2$, and so $|N(b_1) \cap N(b_2) \cap A| < |A|/3$. Consequently $|(N(b_1) \cup N(b_2)) \cap A| > 2|A|/3$. But since $b_1, b_2 \in B_0$ and they each have at least $|C|/3$ neighbours in C_0, and $|C_0| < 2|C|/3$, it follows that they have a common neighbour $v \in C_0$. But then $N(b_1) \cup N(b_2) \cap A \subseteq N_A^2(v)$, and so $|N_A^2(v)| \geq 2|A|/3$, a contradiction. This proves 7.1. ∎

7.2 *If* $y > 1/2$ *then* $\psi(1/3, y) \geq 2/3$.

Proof. Let G be $(1/3, y)$-biconstrained via (A, B, C). There exists $v_1 \in C$ with at least $y|B|$ neighbours in B (in fact, every vertex in C has this property). Let $B_1 = N(v_1)$ and $A_1 = N_A^2(v_1)$. Thus $|B_1| \geq y|B|$. Since every vertex in $B \setminus B_1$ has at least $y|C|$ neighbours in C, some vertex $v_2 \in C$ has at least $y|B \setminus B_1|$ neighbours in $B \setminus B_1$. Let $B_2 = N(v_2)$ and $A_2 = N_A^2(v_2)$. Thus $|B_2 \setminus B_1| \geq y|B \setminus B_1|$, and so

$$|B_1 \cup B_2| \geq |B_1| + y|B \setminus B_1| = y|B| + (1 - y)|B_1| \geq y|B| + y(1 - y)|B| = (2 - y)y|B| > 3|B|/4.$$

In particular, since every vertex in A has at least $|B|/3$ neighbours in B, it follows that $A_1 \cup A_2 = A$. But $B_1 \cap B_2 \neq \emptyset$, since $|B_1|, |B_2| \geq y|B| > |B|/2$, and so there exists $b \in B_1 \cap B_2$; and since b has at least $|A|/3$ neighbours in A, and they all belong to $A_1 \cap A_2$, it follows that $|A_1 \cap A_2| \geq |A|/3$. Since $|A_1 \cup A_3| = |A|$, it follows that $|A_1| + |A_2| \geq 4|A|/3$, and so one of $|A_1|, |A_2|$ is at least $2|A|/3$. This proves 7.2. ∎

Since $\phi(4/7, 2/7) \leq 5/8$ (this is proved in section 9), we studied $\psi(4/7, 2/7)$, and proved the following.

7.3 *If* $x, y \in (0, 1]$ *such that* $\max(x, y) > 1/2$, $x \geq 1/3$, $x + 2y > 1$, *and* $3x + y/(1 - y) > 2$, *then* $\psi(x, y) \geq 2/3$.

Proof. Let G be (x, y)-biconstrained, via (A, B, C). Suppose for a contradiction that $|N^2_A(v)| < 2|A|/3$ for each $v \in C$. By 7.2, $y \leq 1/2$ since $x \geq 1/3$; and so $x > 1/2$ since $\max(x, y) > 1/2$. Hence $y < 1/3$ by 7.1. Also $x < 2/3$, by 1.2.

(1) *For all $v_1, v_2 \in C$, if $|N(v_1) \cup N(v_2)| > (1-x)|B|$ then $N(v_1) \cap N(v_2) = \emptyset$, $N^2_A(v_1) \cup N^2_A(v_2) = A$, and $|N^2_A(v_1) \cap N^2_A(v_2)| < |A|/3$.*

Every vertex in A has a neighbour in $N(u) \cup N(v)$, and so $N^2_A(u) \cup N^2_A(v) = A$. Since $|N^2_A(u)| < 2|A|/3$ and $|N^2_A(v)| < 2|A|/3$ it follows that $|N^2_A(u) \cap N^2_A(v)| < |A|/3$, and so there is no vertex in $N(u) \cap N(v)$ (since any such vertex would have at least $x|A|$ neighbours in A, all belonging to $N^2_A(u) \cap N^2_A(v)$). This proves (1).

(2) *There exist $v_1, v_2 \in C$ with $N(v_1) \cap N(v_2) = \emptyset$.*

Choose $v_1 \in C$. Since every vertex in $A \setminus N^2_A(v_1)$ has at least $x|B|$ second neighbours in $B \setminus N(v_1)$, some vertex $u_2 \in B \setminus N(v_1)$ has at least $(x/(3(1-y)))|A|$ neighbours in $A \setminus N^2_A(v_1)$. Let $v_2 \in C$ be adjacent to u_2. If v_1, v_2 have a common neighbour u_1, then since $N_A(u_1) \subseteq N^2_A(v_2)$, it follows that $|N^2_A(v_2)| \geq (x/(3(1-y)) + x)|A|$, and so $x/(3(1-y)) + x < 2/3$, that is, $(4-3y)x < 2-2y < (4-3y)/2$, and so $x < 1/2$, a contradiction. This proves (2).

(3) *If $v_1, v_2, v_3 \in C$ and $N(v_1) \cap N(v_2) = \emptyset$ then $N(v_3)$ is disjoint from exactly one of $N(v_1), N(v_2)$.*

If $N(v_3)$ is disjoint from both $N(v_1), N(v_2)$, then every two of $N(v_1), N(v_2), N(v_3)$ have union of cardinality more than $(1-x)|B|$, and so every vertex in A belongs to at least two of $N^2_A(v_i)$ $(i = 1, 2, 3)$. Consequently one of $N^2_A(v_i)$ $(i = 1, 2, 3)$ has cardinality at least $a|A|/3$, a contradiction. Now suppose that $N(u_3)$ has nonempty intersection with both $N(v_1), N(v_2)$. Thus $|N^2_A(v_i) \cap N^2_A(v_3)| \geq x|A|$ for $i = 1, 2$, and since $|N^2_A(v_1) \cap N^2_A(v_2)| < |A|/3$, it follows that $|N^2_A(v_3)| \geq (2x - 1/3)|A| \geq 2|A|/3$ since $x \geq 1/2$, a contradiction. This proves (3).

Let H be the bipartite graph $G[B \cup C]$. From (2) and (3), H has exactly two components H_1 and H_2 say. Let $C_i = V(H_i) \cup C$ and $B_i = V(H_i) \cap B$ for $i = 1, 2$. Then from (3), every two vertices in C_i have a common neighbour in B_i, for $i = 1, 2$. Let $c_i = |C_i|/|C|$, for $i = 1, 2$. Thus $c_1 + c_2 = 1$. We may assume that $b_1 \geq 1/2$. Choose $v_1 \in C_1$. Since $|A \setminus N^2_A(v_1)| > |A|/3$, and every vertex in $A \setminus N^2_A(v_1)$ has at least $y|C|$ second neighbours in C_1, some vertex $v_2 \in C_1$ has at least $(y/(3c_1))|A|$ second neighbours in $A \setminus N^2_A(v_1)$. But since $v_1, v_2 \in C_1$, they have a common neighbour in B_1; therefore $|N^2_A(v_2)| \geq (y/(3c_1) + x)|A|$, and so $y/(3c_1) + x < 2/3$. Now $c_1 \leq 1 - y$, so $3x + y/(1-y) < 2$, contrary to the hypothesis. This proves 7.3. ∎

7.4 *Let $x, y \in (0, 1]$. If either*

- $4/7 \leq x \leq 11/17$ *and* $x + 3y \leq 1$; *or*
- $5/8 < y \leq 11/17$ *and* $3x + y \leq 1$

then $\psi(x, y) < 2/3$.

Proof. Take the graph consisting of seven disjoint copies of a three-vertex path, numbered a_i, b_i, c_i in order $(1 \le i \le 7)$. For $1 \le i \le 3$ and $4 \le j \le 7$, make a_i adjacent to b_j and make a_j adjacent to b_i, forming G. Let $A = \{a_i\ 1 \le i \le 7\}$ and define B, C similarly. Choose p such that $1/6 < p < 5/27$ and $(4x-1)/9 \le p \le (1-x)/2$. For $1 \le i \le 3$, let $w(a_i) = p$, $w(b_i) = (4x-1)/9$, and $w(c_i) = 1/7$. For $4 \le i \le 7$, let $w(a_i) = (1-3p)/4$, $w(b_i) = (1-x)/3$ and $w(c_i) = 1/7$. Then this weighted graph is (x, y)-biconstrained via (A, B, C) and shows that $\psi(x, y) < 2/3$. This proves the first statement.

For the second statement, let us take the same graph and redefine w, as follows. Choose p with $1/6 < p < 5/27$ and $x \le p \le (1-4x)/3$. For $1 \le i \le 3$, let $w(a_i) = p$, $w(b_i) = (4y-1)/9$, and $w(c_i) = (1-y)/2$. For $4 \le i \le 7$, let $w(a_i) = (1-3p)/4$, $w(b_i) = (1-y)/3$ and $w(c_i) = (3y-1)/8$. Then this weighted graph is (x, y)-biconstrained via (C, B, A) and shows that $\psi(x, y) < 2/3$. This proves the second statement, and hence proves 7.4. ∎

7.5 *For all $x, y \in (0, 1]$ with $y \le 1/2$, if $\frac{x}{1-x} + \frac{y}{1-2y} \le 2$, then $\phi(x, y) < 2/3$.*

Proof. We may assume that $x > 1/2$, or else the result is true since $\psi(1/2, 1/2) = 1/2$. For the first statement, define $x' = (2x-1)/x$ and $y' = y/(1-y)$. It follows that $x'/(1-x') + y'/(1-y') \le 1$, from the hypothesis; and so from 3.2, $\phi(x', y') < 1/2$. Let $\phi(x', y') = z'$ say. By 1.1, there is a graph G', (x', y')-constrained via some tripartition (A', B', C'), such that for each $v \in C'$, $|N^2_{A'}(v)| \le z'|A'|$. Moreover, by multiplying vertices, we may assume that $|A'| = |B'| = |C'| = N$ say.

Let us add three more vertices to G', say a, b, c, and edges as follows: a is adjacent to every vertex in B'; b is adjacent to every vertex in A'; and c is adjacent to b. Let this graph be G. Let $A = A' \cup \{a\}$, $B = B' \cup \{b\}$, and $C = C' \cup \{c\}$. Then (A, B, C) is a tripartition of G. Choose $r > 0$ such that $r < (1-2z')/(3-3z')$. Define a function w by:

$$\begin{aligned} w(a) &= 1/3 + r \\ w(v) &= (2/3 - r)/N \text{ for each } v \in A' \\ w(b) &= 1 - x \\ w(v) &= x/N \text{ for each } v \in B' \\ w(c) &= y \\ w(v) &= (1-y)/N \text{ for each } v \in C'. \end{aligned}$$

Then the weighted graph (G, w) is (x, y)-biconstrained via (A, B, C). Moreover, $w(N^2_A(c)) = 2/3 - r < 2/3$, and for $v \in C'$,

$$w(N^2_A(v)) \le (z'N)(2/3 - r)/N + (1/3 + r) = (2z' + 1)/3 + r(1 - z') < 2/3$$

since $r < (1-2z')/(3-3z')$. By 2.1, $\phi(x, y) < 2/3$. This proves 7.5. ∎

7.6 *Let $x', y', z' \in (0, 1]$ such that $\psi(x', y') \le z' < 1/2$; and let $x, y \in (0, 1]$ satisfy $x \le 1/(2 - x')$, $x < 1 - (1-x')/(3(1-z'))$, $y \le y'/(1+y')$ and $x + (1-x')y/y' \le 1$. Then $\psi(x, y) < 2/3$. Consequently:*

- *$\psi(x, y) < 2/3$ if $1/2 \le x \le 3/5$ and $x + 2y \le 1$;*
- *$\psi(x, y) < 2/3$ if $3/5 \le x \le 5/8$ and $x + 3y \le 1$.*

Proof. Choose G', (x', y')-constrained via some tripartition (A', B', C'), such that for each $v \in C'$, $|N^2_{A'}(v)| \le z'|A'|$. By multiplying vertices, we may assume that $|A'| = |B'| = |C'| = N$ say.

Let us add three more vertices to G', say a, b, c, and edges as follows: a is adjacent to every vertex in B'; b is adjacent to every vertex in A'; and c is adjacent to b. Let this graph be G. Let $A = A' \cup \{a\}$, $B = B' \cup \{b\}$, and $C = C' \cup \{c\}$. Then (A, B, C) is a tripartition of G.

Since $x' \le \psi(x', y') \le z' < 1/2$ and $x \le 1/(2 - x')$ it follows that $x < 2/3$; and since $x' + y' < 1$ and $x + (1 - x')y/y' \le 1$ it follows that $x + y < 1$. Choose q with

$$\max\left(y, \frac{x - x'}{1 - x'}\right) \le q \le \min\left(1 - x, 1 - \frac{y}{y'}\right),$$

and choose p with $(x - x')/(1 - x') \le p \le 1 - x$, and $1/3 < p < (2/3 - z')/(1 - z')$. Define w by:

$$\begin{aligned}
w(a) &= p \\
w(v) &= (1 - p)/N \text{ for each } v \in A' \\
w(b) &= q \\
w(v) &= (1 - q)/N \text{ for each } v \in B' \\
w(c) &= y \\
w(v) &= (1 - y)/N \text{ for each } v \in C'.
\end{aligned}$$

Then the weighted graph (G, w) is (x, y)-biconstrained via (A, B, C), and proves that $\psi(x, y) < 2/3$. This proves the first statement of the theorem. The two statements in bullets follow by setting $x' = y' = z' = 1/3$, and then $x' = z' = 2/5$ and $y' = 1/5$. This proves 7.6. ∎

7.7 *Let $x', y', z' \in (0, 1]$ such that $\psi(x', y') \le z' < 1/2$; and let $x, y \in (0, 1]$ satisfy $y \le 1/(2 - y')$, $x < 2x'/3$, and $(1 - y')x/x' + y \le 1$. Then $\psi(x, y) < 2/3$. Consequently:*

- *$\psi(x, y) < 2/3$ if $1/2 \le y \le 3/5$ and $2x + y \le 1$; and*
- *$\psi(x, y) < 2/3$ if $1/2 \le y$ and $x + 3y \le 2$.*

Proof. Choose G', (x', y')-constrained via some tripartition (A', B', C'), such that for each $v \in C'$, $|N^2_{A'}(v)| \le z'|A'|$, and with $|A'| = |B'| = |C'| = N$ say.

Add three more vertices a, b, c to G', forming G, where a is adjacent to b, b is adjacent to every vertex in C', and c is adjacent to every vertex in B'. Let $A = A' \cup \{a\}$, $B = B' \cup \{b\}$, and $C = C' \cup \{c\}$.

Since $x' \le \psi(x', y') \le z' < 1/2$ and $x \le 1/(2 - x')$ it follows that $x < 2/3$; and since $x' + y' < 1$ and $x + (1 - x')y/y' \le 1$ it follows that $x + y < 1$. Choose p with $x \le p \le 1 - x/x'$, and $1/3 < p < (2/3 - z')/(1 - z')$; and choose q with

$$\max\left(x, \frac{y - y'}{1 - y'}\right) \le q \le \min\left(1 - y, 1 - \frac{x}{x'}\right).$$

Define w by:

$$\begin{aligned} w(a) &= p \\ w(v) &= (1-p)/N \text{ for each } v \in A' \\ w(b) &= q \\ w(v) &= (1-q)/N \text{ for each } v \in B' \\ w(c) &= 1-y \\ w(v) &= y/N \text{ for each } v \in C'. \end{aligned}$$

Then the weighted graph (G, w) is (x, y)-biconstrained via (A, B, C), and proves that $\psi(x, y) < 2/3$. This proves the first statement of the theorem. To prove the first bullet, let $2x + y \leq 1$ with $y > 1/2$. We claim there is an integer $k \geq 1$ with

$$\frac{1}{3y-1} - \frac{1}{2} \leq k \leq \frac{1}{8y-4} - \frac{1}{4}.$$

To see this, if $y \geq 5/9$ we can take $k = 1$, and if $y < 5/9$, then

$$\left(\frac{1}{8y-4} - \frac{1}{4}\right) - \left(\frac{1}{3y-1} - \frac{1}{2}\right) \geq 1,$$

and so again k exists. Let $x' = z' = k/(2k+1)$, and $y' = 1/(2k+1)$. Then the claim follows from the first statement.

For the second bullet, let $x + 3y \leq 2$ with $y \geq 1/2$. Let $x' = 2/y - 3$, and $y' = 2 - 1/y$; it follows that $x' + 2y' \leq 1$, and so $\psi(x', y') < 1/2$ by 3.2. The result follows from the first statement. This proves 7.6. ∎

For the mono-constrained question, we have:

7.8 *For $x, y \in (0, 1]$, if $y \leq 1/2$ and $x > (1-y)^2/(1-2y^2)$ then $\phi(x, y) \geq 2/3$.*

Proof. Let G be (x, y)-constrained via (A, B, C). If $x + y > 1$ the result follows from 5.2, so we may assume that $x + y \leq 1$. Since $x > (1-y)^2/(1-2y^2)$, we may also assume that

- x, y are rational; and
- every vertex in A has strictly more than $x|B|$ neighbours in B

by reducing x and y a little if necessary while retaining the property that $x > (1-y)^2/(1-2y^2)$.

Let $p = (1-x-y)/(1-2y)$. Thus p is rational, so we may assume (by multiplying vertices) that $p|B|$ is an integer. Since $x > (1-y)^2/(1-2y^2)$ and $x + y \leq 1$, it follows that there exists s such that

$$0 \leq (x - (1-y)^2)/(y - y^2) \leq s \leq \min(1, (p - y(1-y))/y^2).$$

Choose $v_1 \in C$ with at least $y|B|$ neighbours in B, and let $B_1 \subseteq N(v_1)$ with $|B_1| = y|B|$. Choose $v_2 \in C$ such that $sb_0 + b_2 \geq y(sy + (1-y))$, where $b_0|B| = |N(v_2) \cap B_1|$ and $b_2|B| = |N(v_2) \setminus B_1|$. (Such a vertex exists by averaging.) We claim that $b_0 + b_2 \geq p$; for

$$p \leq y(sy + 1 - y) \leq sb_0 + b_2 \leq b_0 + b_2.$$

Also we claim that $b_2 \geq 1 - x - y$; for

$$sy + 1 - x - y \leq y(sy + 1 - y) \leq sb_0 + b_2 \leq sy + b_2.$$

Consequently $|N(v_1) \cup N(v_2)| \geq (1 - x)|B|$, and there exist $P_1 \cup N(v_1)$ and $P_2 \subseteq N(v_2)$, both of cardinality $p|B|$. Choose $v_3 \in C$ with at least $y(1 - 2p)|B|$ neighbours in $B \setminus (P_1 \cup P_2)$. Then for $i = 1, 2$,

$$|P_i \cup N(v_3)| \geq (y(1 - 2p) + p)|B| \geq (1 - x)|B|.$$

Since every vertex in A has strictly more than $x|B|$ neighbours in B, it follows that every vertex in A belongs to at least two of the sets $N_A^2(v_i)$ $(i = 1, 2, 3)$; and so one of these sets has cardinality at least $2|A|/3$. This proves 7.8. ∎

8 The $1/3$ level

Next we do the same for $\psi(x, y) \geq 1/3$ and $\phi(x, y) \geq 1/3$. The figure summarizes our results.

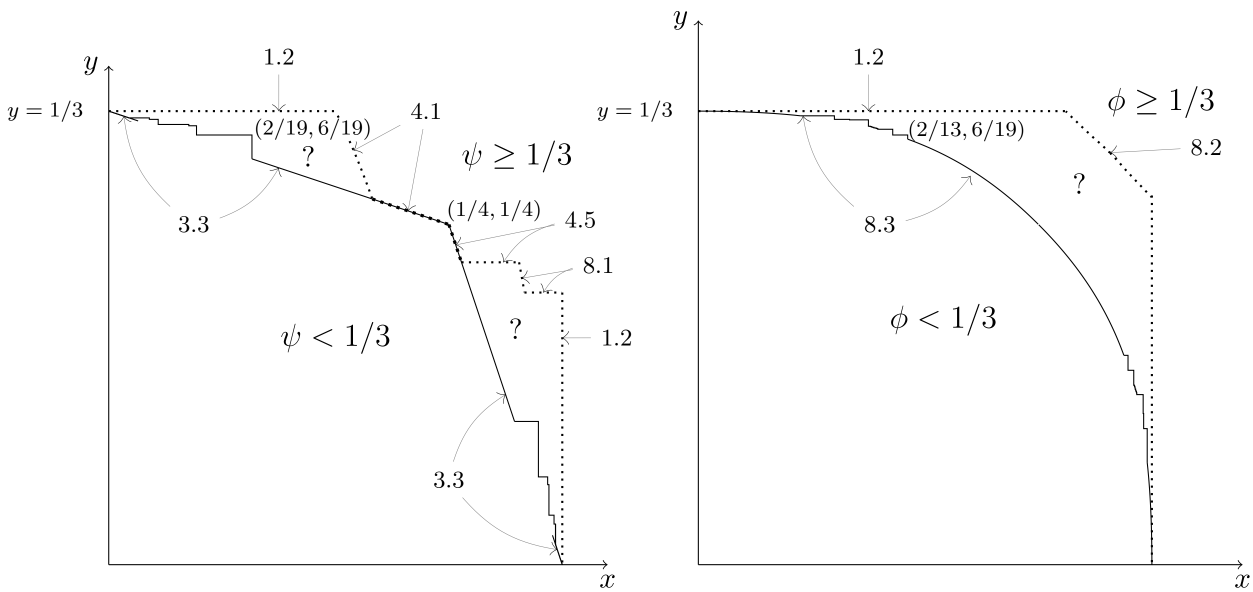

Figure 6: When $\psi(x, y) < 1/3$ and when $\phi(x, y) < 1/3$.

8.1 *Let $x, y \in (0, 1]$ with $y > \frac{1}{5}$ and $3x + \frac{y}{3(1-y)} \geq 1$. Then $\psi(x, y) \geq \frac{1}{3}$.*

Proof. We may assume that $y \leq 1/3$, by 1.2, and so $y/(3(1 - y)) \leq 1/6$. Consequently $x \geq 5/18$, and in particular $x > 2y/3$ (we will need this later). Also, since $1/5 \leq y \leq 1/3$, it follows that $3y - y/(3(1 - y)) > 1/2$; and so

$$\left(3x + \frac{y}{3(1 - y)}\right) + \left(3y - \frac{y}{3(1 - y)}\right) > \frac{3}{2},$$

and consequently $x + y > 1/2$. Let G be (x, y)-biconstrained via (A, B, C), and suppose that $|N_A^2(v)| < |A|/3$ for each $v \in C$. Let H be the subgraph induced on $B \cup C$, and let $H_1, \ldots, H_k$ be its components. Let $B_i = V(H_i) \cap B$ and $C_i = V(H_i) \cap C$, and $b_i = |B_i|/|B|$, $c_i = |C_i|/|C|$, for $1 \leq i \leq k$. Since $y > 0$, B_i, C_i are both nonempty and so $b_i, c_i \geq y$ for $1 \leq i \leq k$. For $1 \leq i \leq k$, let A_i be the set of vertices in A with a neighbour in B_i, and let A_i^* be the set of vertices in A such that $N(v) \subseteq B_i$.

(1) $k \geq 2$.

Suppose that $k = 1$, and let H' be the graph with vertex set B in which u, u' are adjacent if u, u' have a common neighbour in H. Then every stable set of H' has cardinality at most 4. By 6.2 there is a vertex $u_1 \in B$ with H'-distance at most four to every other vertex in B; and so the H-distance from u_1 to each vertex in B is at most eight. Let $v_1 \in C$ be adjacent to u_1. Let $A' = A \setminus N_A^2(v_1)$ and $B' = B \setminus N(v_1)$. Hence $|A'| > 2|A|/3$. Since every vertex in A' has at least $x|B|$ neighbours in B', and $|B'| \leq (1 - y)|B|$, some vertex $u \in B'$ has at least

$$\frac{x|A'|}{1-y} \geq \frac{2x|A|}{3(1-y)}$$

neighbours in A'. Choose a path of H between u_1 and u of length at most eight, and let its vertices be u_1-v_2-u_2-$\cdots$-v_t-$u_t = u$ say, in order. Thus $t \leq 5$, and so there exists i with $1 \leq i \leq t - 1$ such that there are at least $|N_{A'}(u)|/4$ vertices that belong to $N_A(u_{i+1}) \setminus N_A(u_i)$. Since $|N_A(u_i)| \geq x|A|$, it follows that

$$|N_A^2(v_{i+1})| \geq x|A| + |N_{A'}(u)|/4 \geq \frac{x + 2x}{12(1-y)}|A| \geq |A|/3,$$

a contradiction, since $x \geq 2y/3$ and so $x + x/(6(1 - y)) \geq x + y/(9(1 - y)) \geq 1/3$. This proves (1).

(2) *$b_i \leq 1 - x - y < 1/2$ for $1 \leq i \leq k$, and so $k \geq 3$.*

Suppose that $b_1 > 1 - x - y$ say. Thus, if $u \in A \setminus A_1$, then $u \in N_A^2(v)$ for every $v \in C \setminus C_1$; and so $|A \setminus A_1| < |A|/3$, and therefore $|A_1| > 2|A|/3$. Let H' be the graph with vertex set C_1 in which v, v' are adjacent if they have a common H_1-neighbour in B_1. Thus H' has stability number at most three (by (1)) and so H' has radius at most three, by 6.2. Choose $v_1 \in C_1$ such that every vertex in C_1 has H_1-distance at most six from v_1. Let $A' = A_1 \setminus N_A^2(v_1)$; thus $|A'| > |A|/3$. Since every vertex in A' has a neighbour in B_1 and hence has at least $y|C|$ second neighbours in C_1, there exists $v \in C_1$ such that

$$|N_{A'}^2(v)| \geq \frac{y}{|C_1|}|A'| \geq \frac{y}{3(1-y)}|A|,$$

since $|C_1| \leq (1 - y)|C|$. Choose a path of H_1 between v_1, v of length at most six, with vertices v_1-u_1-v_2-$\cdots$-u_{t-1}-$v_t = v$ say where $t \leq 4$. Then for some i with $1 \leq i \leq t - 1$,

$$|N_{A'}^2(v_{i+1}) \setminus N_{A'}^2(v_i)| \geq \frac{y}{9(1-y)}|A|,$$

and hence

$$|N_{A'}^2(v_{i+1})| \geq \left(x + \frac{y}{9(1-y)}\right)|A|$$

since all vertices of $N_A(u_i)$ belong to $N^2_A(v_{i+1})$ and do not belong to $N^2_{A'}(v_{i+1}) \setminus N^2_{A'}(v_i)$. But $3x + y/(3(1-y)) \geq 1$, a contradiction. This proves (2).

By (2), $k \geq 3$; and $k \leq 4$ since $y > 1/5$. We may assume that $|B_1|, |B_2| \geq |B_i|$ for $i \geq 3$; let $B_0 = \bigcup_{3 \leq i \leq k} B_i$, and $C_0 = \bigcup_{3 \leq i \leq k} C_i$. Hence $|B_0| \leq |B|/2$ since $k \leq 4$. Let $b_0 = |B_0|/|B|$ and $c_0 = |C_0|/|C|$; let A_0 be the set of vertices in A with a neighbour in B_0, and let A^*_0 be the set of vertices in A such that $N(v) \subseteq B_0$. For $0 \leq i \leq 2$ let $a_i = |A_i|/|A|$ and $a^*_i = |A^*_i|/|A|$. Since $b_1, b_2 \leq 1 - x - y < x + y$ and $b_0 \leq 1/2 < x + y$, we have $b_i < x + y$ for $i = 0, 1, 2$. Let $0 \leq i \leq 2$, and choose $v \in C_i$ uniformly at random. Then $A^*_i \subseteq N^2_A(v)$ because $b_i < x + y$, and the expected value of $|N^2_A(v) \cap (A_i \setminus A^*_i)|$ is at least $(y/c_i)|A_i \setminus A^*_i|$; so the expected value of $|N^2_A(v)|$ is at least

$$|A^*_i| + \frac{y}{c_i}|A_i| \setminus |A^*_i| = \left(a^*_i + \frac{y}{c_i}(a_i - a^*_i)\right)|A|.$$

Since $|N^2_A(v)| < |A|/3$, it follows that $a^*_i + (y/c_i)(a_i - a^*_i) < 1/3$ for $i = 0, 1, 2$. For $0 \leq i, j \leq 2$ with $i \neq j$ let $a_{ij} = |A_i \cap A_j|/|A|$. Hence a_i is at most the sum of the numbers a_{ij}, over all $j \in \{0, 1, 2\} \setminus \{i\}$; and so substituting for a_i, summing the three inequalities for $i = 0, 1, 2$, and subtracting the inequality $a^*_1 + a^*_2 + a^*_3 + a_{12} + a_{13} + a_{23} \geq 1$, we obtain

$$a_{12}\left(\frac{y}{c_1} + \frac{y}{c_2} - 1\right) + a_{13}\left(\frac{y}{c_1} + \frac{y}{c_3} - 1\right) + a_{23}\left(\frac{y}{c_2} + \frac{y}{c_3} - 1\right) < 0.$$

Consequently there exist distinct $i, j \in \{0, 1, 2\}$ with $y/c_i + y/c_j - 1 < 0$. But $1/c_i + 1/c_j \geq 4/(c_i + c_j)$, and $c_i + c_j \leq 1 - y$, and so $4y/(1-y) < 1$, a contradiction. This proves 8.1. ∎

For ϕ, we have the following (this result is not very sharp if figure 6 is a guide, but it is the best we have so far):

8.2 *For $x, y \in (0, 1]$, if $xy\left(1 + \sqrt{\frac{2}{3x}}\right)^2 \geq \frac{1-x-y}{1-y}$ then $\phi(x, y) \geq 1/3$.*

Let G be (x, y)-constrained via (A, B, C). We may assume that every vertex in B has exactly $y|C|$ neighbours in C, and every vertex in A has exactly $x|B|$ neighbours in B. Choose an edge uv at random between B and C, all edges being equally probable, with $v \in C$. Then there is a vertex $w \in A$ such that the probability that $w \in N^2_A(v)$ is less than 1/3; let $P = N(w)$ and $Q = N^2_C(w)$, and let $|Q| = q|C|$. It follows that there are more than $(2y/3)|B| \cdot |C|$ edges between $B \setminus P$ and $C \setminus Q$; choose $v_1 \in C \setminus Q$ with more than $2y/(3(1-q))|B|$ neighbours in $B \setminus P$. Now $|P| = x|B|$, and so there are $xy|B| \cdot |C|$ edges between P and Q; choose $v_2 \in Q$ with at least $(xy/q)|B|$ neighbours in B. Now $N(v_1) \cup N(v_2)$ has cardinality more than $y(2/(3(1-q)) + x/q)|B|$. Let $|N(v_1) \cup N(v_2)| = t|B|$; then there exists $v_3 \in C$ with at least $y(1-t)|B|$ neighbours in $B \setminus (N(v_1) \cup N(v_2))$, and so

$$|N(v_1) \cup N(v_2) \cup N(v_3)| = (t + y(1-t))|B|.$$

If this is larger than $(1-x)|B|$, then every vertex in A has a neighbour in $N(v_1) \cup N(v_2) \cup N(v_3)$ and the result follows; so we assume not, that is, $t + y(1-t) \leq 1 - x$. Hence $t \leq (1-x-y)/(1-y)$. But $t > y((2/(3(1-q)) + x/q)$, and so

$$y\left(\frac{2}{3(1-q)} + \frac{x}{q}\right) < \frac{1-x-y}{1-y}.$$

For $0 < q < 1$ the function $2/(3(1-q)) + x/q$ is minimized when $q = 1/(1+\sqrt{2/(3x)})$, and its value then is $x(1+\sqrt{2/(3x)})^2$; and so $xy(1+\sqrt{2/(3x)})^2 < (1-x-y)/(1-y)$, contrary to the hypothesis. This proves 8.2. ∎

8.3 *Let $x, y \in (0,1]$ with $x \le \frac{1}{4}$ and $y < \frac{1}{3}$ and $y < \frac{(1-2x)^2}{3-12x+16x^2}$; then $\phi(x,y) < \frac{1}{3}$.*

Proof. Apply 3.1 to 6.6. This proves 8.3. ∎

9 Triangular triples

Let $x, y, z \in (0,1]$. We say that (x,y,z) is *triangular* if no triangle-free graph G admits a tripartition A, B, C of $V(G)$ with the following properties:

- A, B, C are nonempty stable sets;
- every vertex in A has at least $x|B|$ neighbours in B;
- every vertex in B has at least $y|C|$ neighbours in C; and
- every vertex in C has at least $z|A|$ neighbours in A.

It is possible to reformulate results about $\phi(x,y)$ in terms of triangular triples, because we have:

9.1 *For $x, y, z \in (0,1]$, $\phi(x,y) > 1-z$ if and only if (x,y,z) is triangular.*

Proof. Suppose that (x,y,z) is not triangular. Then there is a triangle-free graph G with a tripartition (A,B,C), satisfying the three bullets in the definition of "triangular". Let H be the subgraph of G with $V(H) = V(G)$, obtained by deleting all edges between A and C. If $v \in C$, then $N_A^2(v)$ (defined with respect to H) contains only vertices in A that are nonadjacent to v in G, since G is triangle-free; and so $|N_A^2(v)| \le |A| - z|A|$, since in G, v has at least $z|A|$ neighbours in A. Consequently $\phi(x,y) \le 1-z$.

For the reverse implication, suppose that $\phi(x,y) \le 1-z$, and let H be (x,y)-constrained via (A,B,C), such that $|N_A^2(v)| \le |A| - z|A|$ for each $v \in C$. Make a graph G by adding certain edges to H, namely for each $v \in C$ and $u \in A$, add an edge uv if $u \notin N_A^2(v)$. Then G is triangle-free, and every vertex $v \in C$ is adjacent in G to at least $|A| - (1-z)|A| = z|A|$ vertices in A; and so (x,y,z) is not triangular. This proves 9.1. ∎

In particular, (x,y,z) is triangular if and only if (z,x,y) is triangular; so from 9.1 it follows that $\phi(x,y) \le 1-z$ if and only if $\phi(z,x) \le 1-y$, and similarly if and only if $\phi(y,z) \le 1-x$. (We call this "rotating".)

It is more awkward to cast the biconstrained problem in triangular language, but we can do so as follows. For $x, y, z \in (0,1]$ we say that (x^*, y, z) is triangular if no triangle-free graph G admits a tripartition (A,B,C) that satisfies the three bullets of the previous definition, and in addition satisfies

- every vertex in B has at least $x|A|$ neighbours in A.

Similarly, we say (x^*, y^*, z) is triangular if no triangle-free graph G admits a tripartition (A, B, C) that satisfies the three bullets of the previous definition, and in addition satisfies

- every vertex in B has at least $x|A|$ neighbours in A; and
- every vertex in C has at least $y|B|$ neighbours in B;

and so on. Then we have:

9.2 *For $x, y, z \in (0, 1]$, $\psi(x, y) > 1 - z$ if and only if (x^*, y^*, z) is triangular.*

This is not the same as saying that (x^*, y^*, z^*) is triangular, so we need to keep track of the asterisks if we rotate; but still it can be useful, as we shall see.

There will be results of the form "if $x' > x$ then (x', y, z) is triangular", and we would like some shorthand for this; let us say "(x^+, y, z) is triangular" to mean "(x', y, z) is triangular for all $x' > x$", and treat the other two coordinates similarly. We will mix these two systems of notation, in expressions such as "(x^{+*}, y^+, z) is triangular", meaning "(x'^*, y^+, z) is triangular for all $x' > x$".

Thus, in triangular language, we have the following.

- $(1/2^{+*}, 1/3^*, 1/3^+)$ is triangular: because 7.1 says that $(x^*, 1/3^*, 1/3^+)$ is triangular when $x > 1/2$.
- $(1/2^+, 1/3^+, 1/3^*)$ is triangular: because the proof of 7.2 did not use that every vertex in C has at least $x|B|$ neighbours in B, and so it proves that $(1/3^*, 1/2^+, 1/3^+)$ is triangular, and rotating gives that $(1/2^+, 1/3^+, 1/3^*)$ is triangular.
- $(1/2^+, 1/3^{+*}, 1/3^*)$ is triangular; this follows from 4.1 with $k = 2$ and rotating.
- $(1/2^+, 1/3^*, 1/3^{+*})$ is triangular; this also follows from 4.1 with $k = 2$ and rotating.

These four statements are similar, but no two are equivalent, and it would be good to find a common strengthening. Note, however, that $(1/2^{+*}, 1/3^*, 1/3^*)$ is not triangular, and indeed $(2/3^*, 1/3^*, 1/3^*)$ is not triangular. We have not been able to decide whether $(1/2^+, 1/3^+, 1/3)$ and $(1/2^+, 1/3, 1/3^+)$ are triangular, or indeed whether $(1/2^{+*}, 1/3^+, 1/3^+)$ is triangular.

Pursuing this further, what about $(1/2^+, 1/3^+, x)$ when $x < 1/3$ (perhaps with some sprinkling of asterisks)? How small can x be such that the triple remains triangular? We have examples that show that $(5/9, 5/14, 4/13)$ and $(4/7, 3/8, 2/7)$ are not triangular, and $(3/5^*, 2/5, 1/4)$ is not triangular, but as far as we know, $(1/2^{+*}, 1/3^{+*}, 1/5)$ might be triangular.

This extends to weighted graphs in the natural way. For instance, the weighted graph of figure 7 (identify the vertices on the left with those on the right, in order) shows that $(4/7, 2/7, 3/8)$ is not triangular.

10 Peaceful coexistence

We have not been able to evaluate $\phi(x, y)$ in general, but here is an easier question (that we also cannot do, but it seems to be less far out of reach). It is always true that $\phi(x, y) \geq x$, by 1.2, but if y is sufficiently small then equality may hold. For fixed x, what is the largest y such that $\phi(x, y) = x$?

Let (G, w) be a weighted graph. We say it is *x-regular* via a bipartition (A, B) if

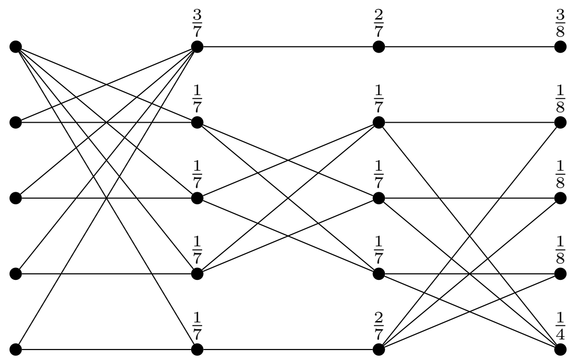

Figure 7: $(4/7, 2/7, 3/8)$ is not triangular

- $|A| = |B|$, and $w(v) > 0$ for each $v \in V(G)$;
- the $0,1$-adjacent matrix between A and B is nonsingular;
- $\sum_{u \in A} w(u) = \sum_{v \in B} w(v) = 1$; and
- for each $u \in V(G)$, $\sum_{v \in N(u)} w(v) = x$.

(Note that the fourth bullet is required to hold both for $u \in A$ and for $u \in B$.) Its *order* is $|A|$, and its *min-weight* is $\min_{v \in B} w(v)$. We will show:

10.1 *For $x, y \in (0, 1]$, $\phi(x, y) = x$ if and only if there is an x-regular bipartite weighted graph with order at most $1/y$.*

Proof. If there is a such a weighted graph (G, w), via (A, B), where $|A| = |B| = n$ say, let C be a set of n new vertices, and add a perfect matching between B and C. Extend w to C by defining $w(v) = 1/n$ for each $v \in C$. The weighted graph just made is $(x, 1/n)$-constrained, and shows that $\phi(x, 1/n) \le x$, and consequently $\phi(x, y) \le x$ (and so $\phi(x, y) = x$).

For the converse, suppose that G is (x, y)-constrained via (A, B, C), and $|N_A^2(v)| \le x|A|$ for each $v \in C$.

(1) *Each vertex in A has exactly $x|B|$ neighbours in B, and each vertex in B has exactly $x|A|$ neighbours in A.*

Each vertex $u \in B$ has at most $x|A|$ neighbours in A, since u has a neighbour $v \in C$ and $|N_A^2(v)| \le x|A|$. Since each vertex in A has at least $x|B|$ neighbours in B, averaging shows that equality holds throughout. That proves (1).

Say two vertices in A are *twins* if they have the same neighbour set in B, and two vertices in B are *twins* if they have the same neighbour set in A. This defines equivalence relations of A and B, and we call the equivalence classes *twin classes*.

(2) *For each vertex $v \in C$, all its neighbours in B are twins, and so $N(v)$ is a subset of a twin*

class of B.

By (1) each vertex in $N(v)$ has $x|A|$ neighbours in A, and all these vertices belong to $N^2_A(v)$; and since $|N^2_A(v)| = x|A|$, equality holds, and in particular, all vertices in $N(v)$ are twins. This proves (2).

Let $\mathcal{T}$ be the set of all twins classes of B. For each $T \in \mathcal{T}$, let $C(T)$ be the set of all $v \in C$ with $N(v) \subseteq T$. Thus the sets $C(T)$ $(T \in \mathcal{T})$ are nonempty, pairwise disjoint and have union C. There is one of cardinality at most $|C|/|\mathcal{T}|$, say $C(T)$; and then each vertex in T has only at most $|C|/|\mathcal{T}|$ neighbours in C, and so $y \leq 1/|\mathcal{T}|$.

Choose one vertex from each twin class of A and of B, and let H be the subgraph induced on this set. For each vertex v of H, let $w(v) = |T|/|B|$ if $v \in T$ for some twin class T of B, and $w(v) = |T|/|A|$ if $v \in T$ for some twin class T of A. Then we have:

- (H, w) is a bipartite graph, with bipartition (A_0, B_0) say;
- $\sum_{u \in A_0} w(u) = \sum_{v \in B_0} w(v) = 1$;
- for each $u \in V(H)$, $\sum_{v \in N(u)} w(v) = x$; and
- $|B_0| \leq 1/y$.

Let us choose a weighted graph (H, w) and bipartition with these properties, with $|V(H)|$ minimum. If there is a function $f : A \to \mathbb{R}$ such that $\sum_{u \in N(v)} f(u) = 0$ for each $v \in B$, not identically zero, then by adding a suitable multiple of f to the restriction of w to A, we can arrange that $w(u) = 0$ for some $u \in A$, and then u can be deleted, contrary to the minimality of $|V(H)|$. Thus there is no such f, and similarly there is no $f : B \to \mathbb{R}$ such that $\sum_{v \in N(u)} f(v) = 0$ for each $u \in A$, not identically zero. Consequently $|A_0| = |B_0| = n$ say, and the adjacency matrix between A_0 and B_0 is nonsingular. Moreover $w(v) > 0$ for each $v \in V(H)$, from the minimality of $V(H)$. This proves 10.1. ∎

By 2.3, $\phi(x, y) = x$ if and only $\phi(y, x) = x$, so this also answers the analogous question for $\phi(y, x)$. If x is irrational, there is no x-regular bipartite weighted graph, and so $\phi(x, y) > x$ for all $y > 0$. If $x \in (0, 1]$ is rational, let us define the *order* of $x \in (0, 1]$ to be the minimum order of x-regular bipartite weighted graphs. If $x = p/q$ say where $p, q > 0$ are integers, then the order of x is at most q, because one can construct an appropriate cyclic shift graph. But the order of x can be strictly less than q. For instance, the top part of the graph of figure 1 is 13/27-regular (take as vertex-weights the numbers given, divided by 27), and so the order of 13/27 is at most seven. Figure 8 gives a smaller example, showing that the order of 2/5 is at most four.

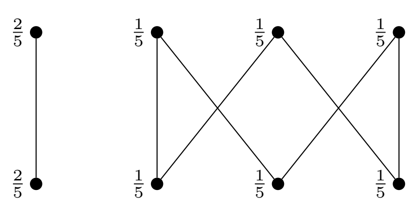

Figure 8: A 2/5-regular weighted bipartite graph of order four.

We can prove that the order is also bounded below by a function of q that goes to infinity with q. More exactly, if G is p/q-regular (in lowest terms) and has order n, then q is at most $(n+1)^{(n+1)/2}$. This follows from a theorem of Hadamard [2], that every $n \times n$ 0,1-matrix has determinant at most $(n+1)^{(n+1)/2}2^{-n}$. We do not know whether there are weighted bipartite graphs with order n that are p/q-regular (in lowest terms), where q is exponentially large in n. (Hadamard $n \times n$ 0,1-matrices have determinant that achieve Hadamard's bound, and they exist when $n+1$ is a power of two, but they give weighted bipartite graphs that are vertex-transitive, and which therefore are p/q-regular with $q = n$.)

One could ask the same question for the biconstrained problem: given x, for which values of y is it true that $\psi(x, y) = x$? A similar analysis (we omit the details) shows:

10.2 *For $x, y \in (0, 1]$, the following are equivalent:*

- $\psi(x, y) = x$;
- $\psi(y, x) = x$; *and*
- *there is an x-regular bipartite weighted graph with min-weight at least y.*

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