The structure of bull-free graphs II and III — a summary

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Abstract

The *bull* is a graph consisting of a triangle and two pendant edges. A graph is called *bull-free* if no induced subgraph of it is a bull. This is a summary of the last two papers [2, 3] in a series [1, 2, 3]. The goal of the series is to give a complete description of all bull-free graphs. We call a bull-free graph *elementary* if it does not contain an induced three-edge-path P such that some vertex $c \notin V(P)$ is complete to $V(P)$, and some vertex $a \notin V(P)$ is anticomplete to $V(P)$. Here we prove that every elementary graph either belongs to one a few basic classes, or admits a certain decomposition, and then use this result together with the results of [1] to give an explicit description of the structure of all bull-free graphs.

1 Introduction

All graphs in this paper are finite and simple, unless stated otherwise. The *bull* is a graph with vertex set $\{x_1, x_2, x_3, y, z\}$ and edge set

$$\{x_1x_2, x_2x_3, x_1x_3, x_1y, x_2z\}.$$

Let G be a graph. We say that G is *bull-free* if no induced subgraph of G is isomorphic to the bull.

This is a summary of the last two papers [2, 3] in a series [1, 2, 3]. The goal of the series is to give a complete description of all bull-free graphs. In this paper we give all the necessary definitions and state the main result of the series, which is an explicit description of the structure of all bull-free graphs. Some of the more complicated proofs from [2, 3] have been either

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completely omitted, or replaced by a very general "proof outline". Thus a reader who is interested to learn about the result, without spending too much time and effort on studying the proofs may find this summary useful.

The complement of G is the graph $\overline{G}$, on the same vertex set as G, and such that two vertices are adjacent in G if and only if they are non-adjacent in $\overline{G}$. A *clique* in G is a set of vertices, all pairwise adjacent. A *stable set* in G is a clique in the complement of G. A clique of size three is called a *triangle* and a stable set of size three is a *triad*. For a subset A of $V(G)$ and a vertex $b \in V(G) \setminus A$, we say that b is *complete* to A if b is adjacent to every vertex of A, and that b is *anticomplete* to A if b is not adjacent to any vertex of A. For two disjoint subsets A and B of $V(G)$, A is *complete* to B if every vertex of A is complete to B, and A is *anticomplete* to B every vertex of A is anticomplete to B. For a subset X of $V(G)$, we denote by $G|X$ the subgraph induced by G on X, and by $G \setminus X$ the subgraph induced by G on $V(G) \setminus X$.

An obvious example of a bull-free graph is a graph with no triangle, or a graph with no triad; but there are others. Let us call a graph G an *ordered split graph* if there exists an integer n such that the vertex set of G is the union of a clique $\{k_1, \dots, k_n\}$ and a stable set $\{s_1, \dots, s_n\}$, and s_i is adjacent to k_j if and only if $i + j \leq n + 1$. It is easy to see that every ordered split graph is bull-free. A large ordered split graph contains a large clique and a large stable set, and therefore the three classes (triangle-free, triad-free and ordered split graphs) are significantly different. Another way to make a bull-free graph that has both a large clique and a large stable set is by using the operation of substitution (this is a well known operation, but, for completeness, we define it in Section 6). It turns out, however, that we can give and explicit description of the structure of all bull-free graphs that are not obtained from smaller bull-free graphs by substitution. To do so, we first define "bull-free trigraphs", which are objects generalizing bull-free graphs: while in a graph every two vertices are either adjacent or nonadjacent, in a trigraph every pair of vertices is either adjacent, or antiadjacent or semi-adjacent (this is done in Section 2).

Let us call a bull-free graph G *elementary* it does not contain an induced three-edge-path P such that some vertex $c \notin V(P)$ is complete to $V(P)$ and some vertex $a \notin V(P)$ is anticomplete to $V(P)$. Our first goal in this paper is to prove that every elementary graph either belongs to a one of a few basic classes, or admits a decomposition (this is the main result of [2], and theorem 3.2 here.) Sections 3 and 4 are devoted to the proof of 3.2. In Section 3 we describe the class $\mathcal{T}_1$ of bull-free trigraphs and the decompositions needed to state the theorem, and state 3.2. We also define the class of "unfriendly trigraphs", which is the subject of most of the theorems in Section 4. In Section 4 we first discuss unfriendly trigraphs, that contain a "prism" (an induced subtrigraph consisting of two disjoint cliques and a matching between them, for a precise definition see Section 4). We prove that every such trigraph satisfies one of the outcomes of 3.2 . Then we study the behavior of an unfriendly trigraph relative to an induced triangle-free subtrigraph (again, see Section 4 for the definitions). We prove that one of the outcomes of 3.2 holds for every unfriendly trigraph that contains an induced three-edge path. We finish Section 4 with a proof of 3.2, using the main result of [1].

In the remainder of the paper, we use 3.2 to describe the structure of all bull-free trigraphs. To do that, we need to restrict the list of decompositions we use. In Section 5, we describe the class $\mathcal{T}_2$ of trigraphs, and state a theorem that says that, up to taking complements, every elementary bull-free trigraph either belongs to one of the classes $\mathcal{T}_1, \mathcal{T}_2$, or admits a decomposition from the "restricted list" (this is 5.7). At this point, we recall a result of [1], that says that every non-elementary bull-free trigraph either belongs to the class $\mathcal{T}_0$ (defined in [1]) or admits a decomposition from the "restricted list" (this is 5.6). In Section 6, we turn 5.6 and 5.7 into a "composition theorem", which is our main result, 6.2. Roughly, 6.2 says that every bull-free trigraph that is not obtained from smaller bull-free trigraphs by substitution is an "expansion" of a trigraph in $\mathcal{T}_0 \cup \mathcal{T}_1 \cup \mathcal{T}_2$ (we postpone the definition of an "expansion" to Section 6). The rest of the paper is devoted to proving 5.7.

2 Trigraphs

In order to prove our main result, we consider objects, slightly more general than bull-free graphs, that we call "bull-free trigraphs". A *trigraph* G consists of a finite set $V(G)$, called the *vertex set* of G, and a map $\theta : V(G)^2 \rightarrow \{-1, 0, 1\}$, called the *adjacency function*, satisfying:

- for all $v \in V(G)$, $\theta_G(v, v) = 0$
- for all distinct $u, v \in V(G)$, $\theta_G(u, v) = \theta_G(v, u)$
- for all distinct $u, v, w \in V(G)$, at most one of $\theta_G(u, v), \theta_G(u, w) = 0$.

Two distinct vertices of G are said to be *strongly adjacent* if $\theta(u, v) = 1$, *strongly antiadjacent* if $\theta(u, v) = -1$, and *semi-adjacent* if $\theta(u, v) = 0$. We say that u and v are *adjacent* if they are either strongly adjacent, or semi-adjacent; and *antiadjacent* if they are either strongly antiadjacent, or semi-adjacent. If u and v are adjacent (antiadjacent), we also say that u is *adjacent (antiadjacent) to* v, or that u is a *neighbor (antineighbor)* of v. Similarly, if u and v are strongly adjacent (strongly antiadjacent), then u is a *strong neighbor (strong antineighbor)* of v. Let $\eta(G)$ be the set of all strongly adjacent pairs of G, $\nu(G)$ the set of all strongly antiadjacent pairs of G, and $\sigma(G)$ the set of all pairs $\{u, v\}$ of vertices of G, such that u and v are distinct and semi-adjacent. Thus, a trigraph G is a graph if $\sigma(G)$ empty.

Let G be a trigraph. The complement $\overline{G}$ of G is a trigraph with the same vertex set as G, and adjacency function $\overline{\theta} = -\theta$. Let $A \subset V(G)$ and $b \in V(G) \setminus A$. For $v \in V(G)$ let $N(v)$ denote the set of all vertices in $V(G) \setminus \{v\}$ that are adjacent to v, and let $S(v)$ denote the set of all vertices in $V(G) \setminus \{v\}$ that are strongly adjacent to v. We say that b is *strongly complete* to A if b is strongly adjacent to every vertex of A, b is *strongly anticomplete* to A if b is strongly antiadjacent to every vertex of A, b is *complete* to A if b is adjacent to every vertex of A, and b is *anticomplete* to A if b is antiadjacent to every vertex of A. For two disjoint subsets A, B of $V(G)$, B is *strongly complete (strongly anticomplete, complete, anticomplete)* to A if every vertex of B is strongly complete (strongly anticomplete, complete, anticomplete, respectively) to A. We say that b is *mixed* on A, if b is not strongly complete and not strongly anticomplete to A. A *clique* in G is a set of vertices all pairwise adjacent, and a *strong clique* is a set of vertices all pairwise strongly adjacent. A *stable set* is a set of vertices all pairwise antiadjacent, and a *strongly stable set* is a set of vertices all pairwise strongly antiadjacent. A (strong) clique of size three is a *(strong) triangle* and a (strong) stable set of size three is a *(strong) triad*. For $X \subset V(G)$, the trigraph *induced by G on X* (denoted by $G|X$) has vertex set X, and adjacency function that is the restriction of θ to X^2. Isomorphism between trigraphs is defined in the natural way, and for two trigraphs G and H we say that H is an *induced subtrigraph* of G (or G *contains H as an induced subtrigraph*) if H is isomorphic to $G|X$ for some $X \subseteq V(G)$. We denote by $G \setminus X$ the trigraph $G|(V(G) \setminus X)$.

A *bull* is a trigraph with vertex set $\{x_1, x_2, x_3, v_1, v_2\}$ such that $\{x_1, x_2, x_3\}$ is a triangle, v_1 is adjacent to x_1 and antiadjacent to x_2, x_3, v_2, and v_2 is adjacent to x_2 and antiadjacent to x_1, x_3. For a trigraph G, a subset X of $V(G)$ is said to be a *bull* if $G|X$ is a bull. We say that a trigraph is *bull-free* if no induced subtrigraph of it is a bull, or, equivalently, no subset of its vertex set is a bull.

Let G be a trigraph. An induced subtrigraph P of G with vertices $\{p_1, \dots, p_k\}$ is a *path* in G if either $k = 1$, or for $i, j \in \{1, \dots, k\}$, p_i is adjacent to p_j if $|i - j| = 1$ and p_i is antiadjacent to p_j if $|i - j| > 1$. Under these circumstances we say that P is a path *from p_1 to p_k*, its *interior* is the set $P^* = V(P) \setminus \{p_1, p_k\}$, and the *length* of P is $k - 1$. We also say that P is a $(k-1)$-edge-path. Sometimes we denote P by p_1-$\dots$-p_k. An induced subtrigraph H of G with vertices $h_1, \dots, h_k$ is a *hole* if $k \geq 4$, and for $i, j \in \{1, \dots, k\}$, h_i is adjacent to h_j if $|i - j| = 1$ or $|i - j| = k - 1$; and h_i is antiadjacent to h_j if $1 < |i - j| < k - 1$. The *length* of a hole is the number of vertices in it. Sometimes we denote H by h_1-$\dots$-h_k-h_1. An *antipath* (*antihole*) in G is an induced subtrigraph of G whose complement is a path (hole) in $\overline{G}$.

Let G be a trigraph, and let $X \subseteq V(G)$. Let G_c be the graph with vertex set X, and such that two vertices of X are adjacent in G_c if and only if they are adjacent in G, and let G_a be be the graph with vertex set X, and such that two vertices of X are adjacent in G_a if and only if they are strongly adjacent in G. We say that X (and $G|X$) is *connected* if the graph G_c is connected, and that X (and $G|X$) is *anticonnected* if $\overline{G_a}$ is connected. A *connected component* of X is a maximal connected subset of X, and an *anticonnected component* of X is a maximal anticonnected subset of X. For a trigraph G, if X is a component of $V(G)$, then $G|X$ is a component of G.

We finish this section by two easy observations from [1].

2.1 *If G be a bull-free trigraph, then so is $\overline{G}$.*

2.2 *Let G be a trigraph, let $X \subseteq V(G)$ and $v \in V(G) \setminus X$. Assume that $|X| > 1$ and v is mixed on X. Then there exist vertices $x_1, x_2 \in X$ such that v is adjacent to x_1 and antiadjacent to x_2. Moreover, if X is connected, then x_1 and x_2 can be chosen adjacent.*

3 Elementary bull-free trigraphs

In this section we state a decomposition theorem for elementary bull-free trigraphs. We start by describing a few special types of trigraphs.

Clique connectors. Let G be a trigraph. Let $K = \{k_1, \dots, k_t\}$ be a strong clique in G, and let A, B, C, D be strongly stable sets, such that the sets K, A, B, C, D are pairwise disjoint and $A \cup B \cup C \cup D \cup K = V(G)$. Let $A_1, \dots, A_t$ be disjoint subsets of A with $\bigcup_{i=1}^{t} A_i = A$, and let $B_1, \dots, B_t, C_1, \dots, C_t, D_1, \dots, D_t$ be defined similarly. Let us now describe the adjacencies in G:

- For $i \in \{1, \dots, t\}$
  A_i is strongly complete to $\{k_1, \dots, k_{i-1}\}$,
  A_i is complete to $\{k_i\}$,
  A_i is strongly anticomplete to $\{k_{i+1}, \dots, k_t\}$,
  B_i is strongly complete to $\{k_{t-i+2}, \dots, k_t\}$,
  B_i is complete to $\{k_{t-i+1}\}$, and
  B_i is strongly anticomplete to $\{k_1, \dots, k_{t-i}\}$.

Let A'_i be the set of vertices of A_i that are semi-adjacent to k_i, and let B'_{t-i+1} be the set of vertices of B_{t-i+1} that are semi-adjacent to k_i. (Thus $|A'_i| \leq 1$ and $|B'_{t-i+1}| \leq 1$.)

- For $i, j \in \{1, \dots, t\}$, if $i + j \neq t$ and A_i is not strongly complete to B_j, then $|A| = |B| = |K| = 1$ and A is complete to B.

- A'_i is strongly complete to B_{t-i}, B'_{t-i} is strongly complete to A_i, and the adjacency between $A_i \setminus A'_i$ and $B_{t-i} \setminus B'_{t-i}$ is arbitrary.

- $A \cup K$ is strongly anticomplete to D, and $B \cup K$ is strongly anticomplete to C.
- For $i \in \{1, \dots, t\}$, C_i is strongly complete to $\bigcup_{j<i} A_j$, and C_i is strongly anticomplete to $\bigcup_{j>i} A_j$.
- For $i \in \{1, \dots, t\}$, C_i is strongly complete to A'_i, every vertex of C_i has a neighbor in A_i, and otherwise the adjacency between C_i and $A_i \setminus A'_i$ is arbitrary.
- For $i \in \{1, \dots, t\}$, D_i is strongly complete to $\bigcup_{j<i} B_j$, and D_i is strongly anticomplete to $\bigcup_{j>i} B_j$.
- For $i \in \{1, \dots, t\}$, D_i is strongly complete to B'_i, every vertex of D_i has a neighbor in B_i, and otherwise the adjacency between D_i and $B_i \setminus B'_i$ is arbitrary.
- For $i, j \in \{1, \dots, t\}$, if $i + j > t$, then C_i is strongly complete to D_j, and otherwise the adjacency between C_i and D_j is arbitrary.

If $A_t \neq \emptyset$ and $B_t \neq \emptyset$, then G is a (K, A, B, C, D)-*clique connector*.

Melts. Let G be a trigraph, such that $V(G)$ is the disjoint union of four sets K, M, A, B, where A and B are strongly stable sets, and K and M are strong cliques. Assume that $|A| > 1$ and $|B| > 1$. Let $K = \{k_1, \dots, k_m\}$ and $M = \{m_1, \dots, m_n\}$. Let A be the union of pairwise disjoint subsets $A_{i,j}$ where $i \in \{0, \dots, m\}$ and $j \in \{0, \dots, n\}$, and let B be the disjoint union of subsets $B_{i,j}$ where $i \in \{0, \dots, m\}$ and $j \in \{0, \dots, n\}$. Let $A_{0,0} = B_{0,0} = \emptyset$. Assume also that

- K is strongly anticomplete to M
- for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ $A_{i,j}$ is
  strongly complete to $\{k_1, \dots, k_{i-1}\} \cup \{m_{n-j+2}, \dots, m_n\}$,
  complete to $\{k_i\} \cup \{m_{n-j+1}\}$,
  strongly anticomplete to $\{k_{i+1}, \dots, k_m\} \cup \{m_1, \dots, m_{n-j}\}$,
  and the set $B_{i,j}$ is
  strongly complete to $\{k_{m-i+2}, \dots, k_m\} \cup \{m_1, \dots, m_{j-1}\}$,
  complete to $\{k_{m-i+1}\} \cup \{m_j\}$,
  strongly anticomplete to $\{k_1, \dots, k_{m-i}\} \cup \{m_{j+1}, \dots, m_n\}$.
- for $i \in \{1, \dots, m\}$, $A_{i,0}$ is
  strongly complete to $\{k_1, \dots, k_{i-1}\}$,
  complete to $\{k_i\}$,
  strongly anticomplete to $\{k_{i+1}, \dots, k_m\} \cup M$
- for $j \in \{1, \dots, n\}$, $A_{0,j}$ is
  strongly complete to $\{m_{n-j+2}, \dots, m_n\}$,
  complete to $\{m_{n-j+1}\}$,
  strongly anticomplete to $K \cup \{m_1, \dots, m_{n-j}\}$

- for $i \in \{1, \dots, m\}$, $B_{i,0}$ is
strongly complete to $\{k_{m-i+2}, \dots, k_m\}$,
complete to $\{k_{m-i+1}\}$,
strongly anticomplete to $\{k_1, \dots, k_{m-i}\} \cup M$

- for $j \in \{1, \dots, n\}$, $B_{0,j}$ is
strongly complete to $\{m_1, \dots, m_{j-1}\}$,
complete to $\{m_j\}$,
strongly anticomplete to $K \cup \{m_{j+1}, \dots, m_n\}$

- the sets $\bigcup_{0 \le j \le n} A_{m,j}$, $\bigcup_{0 \le j \le n} B_{m,j}$, $\bigcup_{0 \le i \le m} A_{i,n}$ and $\bigcup_{0 \le i \le m} B_{i,n}$ are all non-empty

- Let $i, i' \in \{0, \dots, m\}$ and $j, j' \in \{0, \dots, n\}$, and suppose that $i' > i$ and $j' > j$. Then at least one of the sets $A_{i,j}$ and $A_{i',j'}$ is empty, and at least one of the sets $B_{i,j}$ and $B_{i',j'}$ is empty

- For $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$, $A_{i,j}$ is strongly complete to B, and $B_{i,j}$ is strongly complete to A

- For $i, i' \in \{1, \dots, m\}$ and $j, j' \in \{1, \dots, n\}$, $A_{i,0}$ is strongly complete to $B_{i',0}$, and $A_{0,j}$ is strongly complete to $B_{0,j'}$

- for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots n\}$, $A_{i,0}$ is the disjoint union of sets $A^k_{i,0}$ with $k \in \{0, \dots, n\}$, and $A_{0,j}$ is the disjoint union of sets $A^k_{0,j}$ with $k \in \{0, \dots, m\}$,

- for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots n\}$, $B_{i,0}$ is the disjoint union of sets $B^k_{i,0}$ with $k \in \{0, \dots, n\}$, and $B_{0,j}$ is the disjoint union of sets $B^k_{0,j}$ with $k \in \{0, \dots, m\}$.

- for $i \in \{1, \dots, m\}$, every vertex of $A^0_{i,0}$ is strongly anticomplete to $\bigcup_{1 \le j \le n} B_{0,j}$, and has a neighbor in $\bigcup_{1 \le j \le m} \bigcup_{1 \le k \le n} B_{j,k}$

- for $j \in \{1, \dots, n\}$, every vertex of $A^0_{0,j}$ is strongly anticomplete to $\bigcup_{1 \le i \le m} B_{i,0}$, and has a neighbor in $\bigcup_{1 \le i \le m} \bigcup_{1 \le k \le n} B_{i,k}$

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- for $j \in \{1, \dots, n\}$, every vertex of $B^0_{0,j}$ is strongly anticomplete to $\bigcup_{1 \le i \le m} A_{i,0}$, and has a neighbor in $\bigcup_{1 \le i \le m} \bigcup_{1 \le k \le n} A_{i,k}$

- for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$,
every vertex of $A^i_{0,j}$ has a neighbor in $B_{i,0}$,
every vertex of $B^j_{i,0}$ has a neighbor in $A_{0,j}$,
every vertex of $A^j_{i,0}$ has a neighbor in $B_{0,j}$,
every vertex of $B^i_{0,j}$ has a neighbor in $A_{i,0}$,

$A^i_{0,j}$ is strongly complete to $\bigcup_{1\leq s<i} B_{s,0}$
$A^i_{0,j}$ is strongly anticomplete to $\bigcup_{i<s\leq m} B_{s,0}$
$A^j_{i,0}$ is strongly complete to $\bigcup_{1\leq s<j} B_{0,s}$
$A^j_{i,0}$ is strongly anticomplete to $\bigcup_{j<s\leq n} B_{0,s}$
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$B^i_{0,j}$ is strongly complete to $\bigcup_{1\leq s<i} A_{s,0}$
$B^i_{0,j}$ is strongly anticomplete to $\bigcup_{i<s\leq m} A_{s,0}$

- for $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$ let
  $A'_{i,0}$ be the set of vertices of $A_{i,0}$ that are semi-adjacent to k_i
  $A'_{0,j}$ be the set of vertices of $A_{0,j}$ that are semi-adjacent to m_{n-j+1},
  $B'_{i,0}$ be the set of vertices of $B_{i,0}$ that are semi-adjacent to k_{m-i+1},
  $B'_{0,j}$ be the set of vertices of $B_{0,j}$ that are semi-adjacent to m_j.
  Then
  $A'_{i,0}$ is strongly complete to $\bigcup_{1\leq s\leq n} B^i_{0,s}$,
  $A'_{0,j}$ is strongly complete to $\bigcup_{1\leq s\leq m} B^j_{s,0}$,
  $B'_{i,0}$ is strongly complete to $\bigcup_{1\leq s\leq n} A^i_{0,s}$,
  $B'_{0,j}$ is strongly complete to $\bigcup_{1\leq s\leq m} A^j_{s,0}$.
- there exist $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$ such that either $A_{i,j} \neq \emptyset$, or $B_{i,j} \neq \emptyset$.
- Let $i, s, s' \in \{1,\dots,m\}$ and $j, t, t' \in \{1,\dots,n\}$ such that $t' \geq j \geq n+1-t$ and $s \geq i \geq m+1-s'$. Then at least one of $A_{s,t}$ and $B_{s',t'}$ is empty.

Under these circumstances we say that G is a *melt*. We say that a melt is an *A-melt* if $B_{i,j} = \emptyset$ for every $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$. We say that a melt is a *B-melt* if $A_{i,j} = \emptyset$ for every $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$. We say that a melt is a *double melt* if there exist $i, i' \in \{1,\dots,m\}$ and $j, j' \in \{1,\dots,n\}$ such that $A_{i,j} \neq \emptyset$, and $B_{i',j'} \neq \emptyset$.

Let H be a graph. For a vertex $v \in V(H)$, the *degree* of v in H, denoted by $deg(v)$, is the number of edges of H incident with v. If H is the empty graph let $maxdeg(H) = 0$, and otherwise we define $maxdeg(H) = \max_{v\in V(H)} deg(v)$.

The class $\mathcal{T}_1$. Before giving a precise definition of the class $\mathcal{T}_1$, let us describe roughly what a trigraph in this class looks like. The idea is the following. Every trigraph in $\mathcal{T}_1$ consists of a triangle-free part X (in what follows $V(X)$ is the union of L, the sets $h(e)$, and the sets $h(e,v) \cap B$), and a collection of pairwise disjoint and pairwise anticomplete strong cliques Y_v (in what follows Y_v is the union of $h(v)$ and the sets $h(e,v) \setminus B$ for all edges e incident with v). Every vertex of X has neighbors in at most two cliques

Y_v. Each Y_v, together with vertices of X at distance at most two from Y_v, induces a clique connector. If every vertex of X has neighbors in at most one Y_v, this describes the graph completely. Describing the adjacency rules for vertices of X that have neighbors in two different cliques, Y_u and Y_v is more complicated (we need to explain how the clique connectors for Y_u and Y_v overlap). Without going into details, the structure there is locally a melt.

Let us now turn to the precise definition of $\mathcal{T}_1$. Let H be a loopless triangle-free graph with $maxdeg(H) \leq 2$ (H may be empty, and may have parallel edges). We say that a trigraph G admits an *H-structure* if there exist a subset L of $V(G)$ and a map

$$h : V(H) \cup E(H) \cup (E(H) \times V(H)) \to 2^{V(G) \setminus L}$$

such that

- every vertex of $V(G) \setminus L$ is in $h(x)$ for exactly one element x of $V(H) \cup E(H) \cup (E(H) \times V(H))$, and
- $h(v) \neq \emptyset$ for every $v \in V(H)$ of degree zero, and
- $h(e) \neq \emptyset$ for every $e \in E(H)$, and
- $h(e, v) \neq \emptyset$ if e is incident with v, and
- $h(e, v) = \emptyset$ if e is not incident with v, and
- for $u, v \in V(H)$, $h(u)$ is strongly anticomplete to $h(v)$, and
- $h(v)$ is a strong clique for every $v \in V(H)$, and
- every vertex of L has a neighbor in at most one of the sets $h(v)$ where $v \in V(H)$, and
- $G|(L \cup (\bigcup_{e \in E(H)} h(e)))$ has no triangle, and
- for every $e \in E(H)$, every vertex of L is either strongly complete or strongly anticomplete to $h(e)$, and
- $h(e)$ is either strongly complete or strongly anticomplete to $h(f)$ for every $e, f \in E(H)$; if e and f share an endpoint, then $h(e)$ is strongly complete to $h(f)$, and
- for every $e \in E(H)$ and $v \in V(H)$, $h(e)$ is strongly anticomplete to $h(v)$, and
- for $v \in V(H)$, let S_v be the vertices of L with a neighbor in $h(v)$, and let T_v be the vertices of $(L \cup (\bigcup_{e \in E(H)} h(e))) \setminus S_v$ with a neighbor in S_v. Then there is a partition of S_v into two sets A_v, B_v, and a partition of T_v into two sets C_v, D_v such that $G|(h(v) \cup S_v \cup T_v)$ is an $(h(v), A_v, B_v, C_v, D_v)$-clique connector, and

- for $v \in V(H)$, if there exist $a \in A_v$ and $b \in B_v$ antiadjacent with a common neighbor in $h(v)$, then v has degree zero in H.

Moreover, let e be an edge of H with ends u, v. Then

- if $f \in E(H) \setminus \{e\}$ is incident with v, then $h(e, v)$ is strongly complete to $h(f, v)$, and
- $G|(h(e) \cup h(e, v) \cup h(e, u))$ is an $h(e)$-melt, such that if (K, M, A, B) are as in the definition of a melt, then $K \subseteq h(e, v)$, $M \subseteq h(e, u)$, $A = h(e)$, $B \subseteq h(e, v) \cup h(e, u)$, every vertex of $h(e, v) \cap B$ has a neighbor in K, and every vertex of $h(e, u) \cap B$ has a neighbor in M (and, in particular, $h(e, v)$ is strongly anticomplete to $h(e, u)$); and
- $h(e, v)$ is strongly complete to $h(v)$, and $h(e, v)$ is strongly anticomplete to $h(w)$ for every $w \in V(H) \setminus \{v\}$, and
- $h(e, v)$ is strongly anticomplete to $h(f, w)$ for every $f \in E(H) \setminus \{e\}$, and $w \in V(H) \setminus \{v\}$, and
- $h(e, v)$ is strongly anticomplete to $h(f)$ for every $f \in E(H) \setminus \{e\}$.

Furthermore, either the following statements all hold, or they all hold with the roles of $A_u \cup A_v$ and $B_u \cup B_v$ switched:

- $h(e)$ is strongly complete to $B_u \cup B_v$, and
- $h(e, v)$ is strongly complete to A_v and strongly anticomplete to $L \setminus A_v$, and, and
- every vertex of $(L \cup (\bigcup_{f \in E(H)} h(f))) \setminus (A_u \cup A_v)$ with a neighbor in $A_u \cup A_v$ is strongly complete to $h(e)$.

Let us say that G belongs to $\mathcal{T}_1$ if either G is a double melt, or G admits an H structure for some loopless triangle-free graph H with maximum degree at most two.

In the definition of an H-structure, we did not specify the adjacencies between the sets $h(e)$ for disjoint edges e of H, except that

- $h(e)$ is either strongly complete or strongly anticomplete to $h(f)$ for every $e, f \in E(H)$.

In fact, the only constraints on these adjacencies come from the condition that

- $G|(L \cup (\bigcup_{e \in E(H)} h(e)))$ has no triangle.

To tighten the structure, one might want to add another ingredient, which is a triangle-free supergraph F of the line graph of H, that would "record" for which pairs of disjoint edges e, f of H, the sets $h(e)$ and $h(f)$ are strongly complete to each other. We did not do that here, since such a graph F can be easily reconstructed from the H-structure. The situation concerning the adjacencies between the vertices of L and the sets $h(e)$ is similar.

We observe the following:

3.1 *Every clique connector, every melt and every trigraph in $\mathcal{T}_1$ is bull-free.*

For the proof of 3.1 see [2].

Next let us describe some decompositions (some of these definitions appear in [1], but we repeat them for completeness). Let G be a trigraph. A proper subset $X$of $V(G)$ is a *homogeneous set* in G if every vertex of $V(G) \setminus X$ is either strongly complete or strongly anticomplete to X. We say that G admits *a homogeneous set decomposition*, if there is a homogeneous set in G of size at least two.

For two disjoint subsets A and B of $V(G)$, the pair (A, B) is a *homogeneous pair* in G if A is a homogeneous set in $G \setminus B$ and B is a homogeneous set in $G \setminus A$. We say that the pair (A, B) is *tame* if

- $|V(G)| - 2 > |A| + |B| > 2$, and
- A is not strongly complete and not strongly anticomplete to B.

The graph G admits a *homogeneous pair decomposition* if there is a tame homogeneous pair in G.

Let $S \subseteq V(G)$. A *center* for S is a vertex of $V(G) \setminus S$ that is complete to S, and an *anticenter* for S is a vertex of $V(G) \setminus S$ that is anticomplete to S. A vertex of G is a *center (anticenter)* for an induced subgraph H of G if it is a center (anticenter) for $V(H)$.

We say that a trigraph G is *elementary* if there does not exist a path P of length three in G, such that some vertex c of $V(G) \setminus V(P)$ is a center for P, and some vertex a of $V(G) \setminus V(P)$ is an anticenter for P. The main result of [2] is the following:

3.2 *Let G be an elementary bull-free trigraph. Then either*

- *one of $G, \overline{G}$ belongs to $\mathcal{T}_1$, or*
- *G admits a homogeneous set decomposition, or*
- *G admits a homogeneous pair decomposition.*

In the next section we describe the proof of 3.2. Let us call a bull-free trigraph that does not admit a homogeneous set decomposition, or a homogeneous pair decomposition, and does not contain a path of length three with a center *unfriendly.* In view of the main result of [1], in the next few sections of this paper we deal mainly with unfriendly graphs (for a precise explanation, see the end of Section 4).

4 The proof of 3.2

Let G be a trigraph. A *k-prism* in G is a trigraph whose vertex set is the disjoint union of two cliques $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_k\}$; and such that for every $i, j \in \{1, \dots, k\}$, a_i is adjacent to b_j if $i = j$ and a_i is antiadjacent to b_j if $i \neq j$. A *prism* is a 3-prism.

The first step in the proof of 3.2 is the following:

4.1 *Let G be an unfriendly trigraph. Assume that for some integer $n \geq 3$, G contains an induced subtrigraph that is an n-prism. Then G is a prism.*

To prove 4.1 we start with a maximal structure W in G that we call a *hyperprism.* A hyperprism consists of pairwise disjoint non-empty subsets $A_1, \dots A_k, B_1, \dots, B_k$, where $k \geq 3$ is an integer, such that for $i, j \in \{1, \dots, k\}$

- A_i is complete to A_j and B_i is complete to B_j
- if $i \neq j$, then A_i is anticomplete to B_j
- every vertex of A_i has a neighbor in B_i
- every vertex in B_i has a neighbor in A_i
- $k \geq 3$.

Since G contains an n-prism, there is a hyperprism in G. Next we analyze how the vertices of $V(G) \setminus W$ attach to W. It turns out the the structure there is pretty tight, forcing either a homogeneous set, or a homogeneous pair decomposition, contrary to the fact that G is unfriendly. Thus $V(G) = W$. But now, again since G is unfriendly, it follows that G is a prism. For details, please see [2].

Next we prove (or state without proof) a few lemmas about unfriendly trigraphs, all from [2].

4.2 *Let G be an unfriendly graph, let $m > 2$ be an integer, and let $Y_1, \dots, Y_m$ be pairwise disjoint anticonnected sets, such that for $i, j \in \{1, \dots, m\}$, Y_i is complete to Y_j. Let $v \in V(G) \setminus (\bigcup_{i=1}^{m} Y_i)$, assume that $|Y_1| > 1$ and v has a neighbor and an antineighbor in $\bigcup_{i=2}^{m} Y_i$. Then v is either strongly complete, or strongly anticomplete to Y_1.*

Proof. Suppose not. Then v has a neighbor a and an antineighbor a' in Y_1, and by 2.2 we may assume that a and a' are distinct and antiadjacent. From the symmetry, we may assume that v has a neighbor $x \in Y_2$ and an antineighbor $h \in Y_3$. But now v-a-h-a' is a path, and x is a center for it, contrary to the fact that G is unfriendly. This proves 4.2. ∎

4.3 *Let G be an unfriendly trigraph such that there is no prism in G, and let a_1-a_2-a_3-a_4-a_1 be a hole of length four. Let K be the set of vertices that are complete to $\{a_1, a_2\}$ and anticomplete to $\{a_3, a_4\}$. Then K is a strong clique.*

Proof. Suppose some two vertices of K are not strongly adjacent, and let C be an anti-component of K with $|C| > 1$. Since G is unfriendly, it follows that C is not a homogeneous set in G, and so, by 2.2 applied in $\overline{G}$, there exist vertices c, c', v such that $c, c' \in C$, $v \notin C$, v is adjacent to c' and antiadjacent to c, and c' is antiadjacent to c. Since $\{a_4, a_1, c', a_2, c\}$ is not a bull, it follows that $v \neq a_1$, and from the symmetry $v \neq a_2$. Since a_4-c'-a_2-c is not a path with center a_1, it follows that $v \neq a_4$, and from the symmetry $v \neq a_3$.

Suppose first that v is anticomplete to $\{a_1, a_2\}$. Since $\{v, c', a_2, a_1, a_4\}$ is not a bull, it follows that v is strongly adjacent to a_4, and, similarly, v is strongly adjacent to a_3. But now $G|\{a_1, a_2, c', a_3, a_3, v\}$ is a prism, a contradiction. So we may assume that v is strongly adjacent to a_1, and by 4.2, v is strongly adjacent to a_2. Since $\{c, a_2, c', v, a_4\}$ is not a bull, it follows that v is strongly antiadjacent to a_4, and similarly to a_3. But now $v \in C$, a contradiction. This proves 4.3. ∎

4.4 *Let G be an unfriendly trigraph such that there is no prism in G, let a_1-a_2-a_3-a_4-a_1 be a hole in G, and let c be a center and a an anticenter for $\{a_1, a_2, a_3, a_4\}$. Then c is strongly antiadjacent to a.*

Proof. Suppose c is adjacent to a.

(1) Let $i \in \{1, \ldots, 4\}$. Then a_i is strongly adjacent to a_{i+1} (here the addition is performed mod 4*), c is strongly adjacent to a_i, and a is strongly antiadjacent to a_i.*

Since a_i-a_{i+3}-a_{i+2}-a_{i+1} is not a path with a center c, it follows that a_i is strongly adjacent to a_{i+1}. Since $\{a_i, a_{i+1}, a_{i+2}, c, a\}$ is not a bull, it follows that a_i is strongly adjacent to c. Finally, since a-a_i-a_{i+1}-a_{i+2} is not a path with center c, we deduce that a is strongly antiadjacent to a_i. This proves (1).

Let A_1, A_2, A_3, A_4 be connected subsets of $V(G)$, where $a_i \in A_i$ for $i \in \{1, \ldots, 4\}$, such that

- for $i \in \{1, \ldots, 4\}$, A_i is strongly complete to A_{i+1} (with addition *mod* 4),
- for $i = 1, 2$, A_i is anticomplete to A_{i+2},
- c is strongly complete to $A_1 \cup A_2 \cup A_3 \cup A_4$

- a is strongly anticomplete to $A_1 \cup A_2 \cup A_3 \cup A_4$.

Let $W = A_1 \cup A_2 \cup A_3 \cup A_4$, and assume that A_1, A_2, A_3, A_4 are chosen with W maximal. Since G is unfriendly, it follows that $A_1 \cup A_3$ is not a homogeneous set in G, and so some vertex v of $V(G) \setminus (A_1 \cup A_3)$ is mixed on $A_1 \cup A_3$. Then $v \notin A_2 \cup A_3 \cup \{a, c\}$. We may assume that v has a neighbor $v_1 \in A_1$, and antineighbor $v_3 \in A_3$. Since $A_1 \cup A_3, A_2 \cup A_4$ and $\{c\}$ are three anticonnected sets complete to each other, 4.2 implies that v is either strongly complete or strongly anticomplete to $A_2 \cup A_4 \cup \{c\}$.

Suppose first that v is strongly anticomplete to $A_2 \cup A_4 \cup \{c\}$. Since $\{v, v_1, a_2, c, a\}$ is not a bull, it follows that v is adjacent to a. But now v-a-c-v_1-v is a hole of length four, and a_2, a_4 are two antiadjacent vertices, each complete to $\{v_1, c\}$ and anticomplete to $\{v, a\}$, contrary to 4.3. This proves that v is strongly complete to $A_2 \cup A_4 \cup \{c\}$. Since a-v-a_2-v_3 is not a path with center c, it follows that v is strongly antiadjacent to a. If v is anticomplete to A_3, then replacing A_1 by $A_1 \cup \{v\}$ contradicts the maximality of W, so v has a strong neighbor in A_3, and therefore $A_3 \neq \{v_3\}$. Since A_3 is connected, 2.2 implies that there exist vertices $x, y \in A_3$, such that v is adjacent to x and antiadjacent to y, and x is adjacent to y. But now y-x-v-v_1 is a path, and c is a center for it, contrary to the fact that G is unfriendly. This proves 4.4. ∎

4.5 *Let H be a trigraph such that no induced subtrigraph of H is a path of length three. Then either*

1. *H is not connected, or*
2. *H is not anticonnected, or*
3. *there exist two vertices $v_1, v_2 \in V(H)$ such that v_1 is semi-adjacent to v_2, and $V(H) \setminus \{v_1, v_2\}$ is strongly complete to v_1 and strongly anticomplete to v_2.*

The proof is similar to the proof of the analogous result for graphs, and we omit it (see [2] for details).

4.6 *Let G be an unfriendly trigraph with no prism, and let $u, v \in V(G)$ be adjacent. Let A, B be subsets of $V(G)$ such that*

- *u is strongly complete to A and strongly anticomplete to B,*
- *v is strongly complete to B and strongly anticomplete to A,*
- *No vertex of $V(G) \setminus (A \cup B)$ is mixed on A, and*
- *if $x, y \in B$ are adjacent, then no vertex of $V(G) \setminus (A \cup B)$ is mixed on $\{x, y\}$.*

Then $A = K \cup S$, where K is a strong clique and S is a strongly stable set.

The proof of this lemma is too long to include here, and we refer the reader to [2].

4.7 *Let G be an unfriendly bull-free trigraph with no prism. Then there do not exist six vertices $a, b, c, d, x, y \in V(G)$ such that*

- *the pairs ab, cd, xy are adjacent,*
- *$\{a, b\}$ is anticomplete to $\{c, d\}$, and*
- *$\{x, y\}$ is complete to $\{a, b, c, d\}$.*

Proof. Since b-a-y-c is not a path with center x, it follows that y is strongly adjacent to b, and from the symmetry, $\{x, y\}$ is strongly adjacent to $\{a, b, c, d\}$.

Let $k \geq 2$ be an integer, and let $Y_0, \dots, Y_k$ be pairwise disjoint anticonnected sets, such that

- Y_0 is strongly complete to $\bigcup_{i=1}^{k} Y_i$,
- for $i, j \in \{1, \dots, k\}$, Y_i is complete to Y_j, and
- $\{a, b, c, d\} \subseteq Y_0$.

We may assume that $Y_0, \dots, Y_k$ are chosen with $W = \bigcup_{i=0}^{k} Y_i$ maximal.

(1) Let $v \in V(G) \setminus W$ and assume that v has a neighbor in Y_0. Then v is strongly anticomplete to $W \setminus Y_0$.

We may assume that v has a neighbor in $W \setminus Y_0$. Suppose first that v is mixed on Y_0. By 4.2, it follows that v strongly complete to $W \setminus Y_0$, and therefore $Y_0 \cup \{v\}, Y_1, \dots, Y_k$ contradict the maximality of W. This proves that v is strongly complete to Y_0.

Next suppose that v has a neighbor in Y_1, and v is not complete to Y_1. Then $|Y_1| > 1$, and 4.2 implies that v is strongly complete to $W \setminus Y_1$. But then replacing Y_1 with $Y_1 \cup \{v\}$ contradicts the maximality of W. Using the symmetry, this proves that if v has a neighbor in Y_i with $1 \leq i \leq k$, then v is complete to Y_i.

Let I be the set of all $i \in \{1, \dots, k\}$, such that v is complete to Y_i, and let $J = \{1, \dots, k\} \setminus I$. Then v is strongly anticomplete to $\bigcup_{j \in J} Y_j$. From the symmetry we may assume that $I = \{1, \dots, t\}$ for some $t \in \{1, \dots, k\}$. Let $Z_{t+1} = \{v\} \cup \bigcup_{j \in J} Y_j$. Then $Y_0, Y_1, \dots, Y_t, Z_{t+1}$ contradict the maximality of W. This proves (1).

Since $W \setminus Y_0$ is strongly complete to Y_0, and since Y_0 is not a homogeneous set in G, it follows that some vertex of $V(G) \setminus Y_0$ has a neighbor in

Y_0. Let Z_0 be the set of all vertices of $V(G) \setminus W$ with a neighbor in Y_0. Then $Z_0 \neq \emptyset$, and by (1), Z_0 is strongly anticomplete to $W \setminus Y_0$. Moreover, no vertex of $V(G) \setminus (Y_0 \cup Z_0)$ is mixed on Y_0.

Since Y_0 is strongly complete to $W \setminus Y_0$, and Z_0 is strongly anticomplete to $W \setminus Y_0$, and since $W \setminus Y_0$ is not a homogeneous set in G, it follows that some vertex $z_1 \in V(G) \setminus (W \cup Z_0)$ is mixed on $W \setminus Y_0$. Since Z_0 is strongly anticomplete to $W \setminus Y_0$, it follows that $z_1 \notin Z_0$, and therefore z_1 is strongly anticomplete to Y_0. We may assume that z_1 has a neighbor $y_1 \in Y_1$ and antineighbor $y_2 \in Y_2$.

(2) z_1 is strongly complete to Z_0.

Suppose $z_0 \in Z_0$ is antiadjacent to z_1. Let $y_0 \in Y_0$ be a neighbor of z_0. Then $\{z_0, y_0, y_2, y_1, z_1\}$ is a bull, a contradiction. This proves (2).

(3) Let $s, t \in Z_0$ be adjacent, and let $v \in V(G) \setminus (Y_0 \cup Z_0)$. Then v is not mixed on $\{s, t\}$.

Suppose that v is adjacent to s and antiadjacent to t. Let $y_s \in Y_0$ be adjacent to s, and y_t to t, choosing $y_s = y_t$ if possible. Since v is mixed on Z_0, it follows that $v \notin (W \setminus Y_0)$. Since $v \notin Z_0$, it follows that v is strongly antiadjacent to y_s, y_t.

Assume first that $y_s = y_t$. Since $\{v, s, t, y_t, w\}$ is not a bull for any $w \in W \setminus Y_0$, it follows that v is strongly complete to $W \setminus Y_0$. But now $Y_0 \cup \{v\}, Y_1, \ldots, Y_k$ contradict the maximality of W. This proves that $y_s \neq y_t$, and therefore s is antiadjacent to y_t, and t to y_s. Since $\{y_s, s, z_1, t, y_t\}$ is not a bull, it follows that y_s is strongly adjacent to y_t. But now $G|\{s, t, z_1, y_s, y_t, y_1\}$ is a prism, a contradiction. This proves (3).

Now y_1, z_1 are adjacent, and Y_0, Z_0 are subsets of $V(G)$ such that

- y_1 is strongly complete to Y_0 and strongly anticomplete to Z_0,
- z_1 is strongly complete to Z_0 and strongly anticomplete to Y_0,
- No vertex of $V(G) \setminus (Y_0 \cup Z_0)$ is mixed on Y_0, and
- if $s, t \in Z_0$ are adjacent, then no vertex of $V(G) \setminus (Y_0 \cup Z_0)$ is mixed on $\{s, t\}$.

By 4.6, we deduce that $Y_0 = K \cup S$, where K is a strong clique and S is a strongly stable set. But then at least one of a, b is in K, and at least one of c, d is in K, contrary to the fact that $\{a, b\}$ is strongly anticomplete to $\{c, d\}$. This proves 4.7. ∎

We have now reached the heart of the proof of 3.2, which is understanding unfriendly trigraphs that contain a three edge path and do not contain a prism. Let G be such a trigraph. We choose a maximal subtrigraph H of G such that there is no triangle in H, and analyze how the vertices of $V(G)\setminus V(H)$ attach to H. It turns out that each component of $V(G)\setminus V(H)$ is a strong clique, no vertex of H has neighbors in more than two components of $V(G)\setminus V(H)$, and we can describe how each of the cliques "connects" to H, thus proving that $G\in\mathcal{T}_1$.

We start with a lemma.

4.8 *Let G be an unfriendly trigraph with no prism, and let h_1-h_2-h_3-h_4-h_5-h_1 be a hole of length five in G, say H. Then no vertex of $V(G)\setminus V(H)$ is adjacent to h_1, h_2, h_5.*

Proof. Suppose some $v\in V(G)\setminus V(H)$ is adjacent to h_1, h_2, h_5. Since $\{h_2, v, h_1, h_5, h_4\}$ and $\{h_2, h_1, v, h_5, h_4\}$ are not bulls, it follows that h_2 is strongly complete to $\{v, h_1\}$, and from the symmetry, h_5 is strongly complete to $\{v, h_1\}$. Since h_5-v-h_2-h_3 is not a path with center h_1, it follows that h_3 is strongly antiadjacent to h_1, and therefore h_3 is strongly anticomplete to $\{v, h_1\}$. From the symmetry h_4 is strongly anticomplete to $\{v, h_1\}$.

Let X the set of vertices of $V(G)\setminus\{h_2, h_3, h_4, h_5\}$ that are strongly complete to $\{h_2, h_5\}$ and strongly anticomplete to $\{h_3, h_4\}$ and let C be a component of X such that $v, h_1\in C$. Since G is unfriendly, it follows that C is not a homogeneous set in G, and therefore some vertex $w\in V(G)\setminus C$ is mixed on C. Then $w\notin V(H)$. By 2.2, there exists $c, c'\in C$ such that c is adjacent to c', and w is adjacent to c and antiadjacent to c'.

Assume first that w is antiadjacent to h_5. Since $\{w, c, c', h_5, h_4\}$ is not a bull, it follows that w is strongly adjacent to h_4. If w is antiadjacent to h_2, then, form the symmetry, w is strongly adjacent to h_3, and $\{h_2, h_3, w, h_4, h_5\}$ is a bull, a contradiction; thus w is strongly adjacent to h_2. Since c-h_2-h_3-h_4 is not a path with center w, it follows that w is strongly antiadjacent to h_3. But now, $\{h_5, c, w, h_2, h_3\}$ is a bull, a contradiction. This proves that w is strongly adjacent to h_5, and so, from the symmetry, w is strongly adjacent to h_2. Since h_5-c-h_2-h_3 is not a path with center w, it follows that w is strongly antiadjacent to h_3, and from the symmetry, w is strongly antiadjacent to h_4. But then $w\in C$, a contradiction. This proves 4.8. ∎

A *frame* is a trigraph T such that

- T is connected, and
- there is no triangle in T, and
- T has an induced subtrigraph which is a path of length three.

A trigraph is called *framed* if some induced subtrigraph of it is a frame. We prove the following:

4.9 *Every unfriendly framed trigraph with no prism is in $\mathcal{T}_1$.*

Proof. Let G be an unfriendly framed trigraph, and let F be an induced subtrigraph of G that is a frame. We may assume that there is a triangle in G, for otherwise G admits an H-structure where H is the empty graph. Since G is unfriendly, it follows that G is connected. Assume that F is chosen with $|V(F)|$ maximum, subject to that with $|\eta(F)| + |\sigma(F)|$ maximum.

(1) Every vertex of $V(G) \setminus V(F)$ has a neighbor in $V(F)$.

Suppose some vertex of $V(G) \setminus V(F)$ is strongly anticomplete to $V(F)$. Since G is connected, there exist adjacent vertices $u, v \in V(G) \setminus V(F)$ such that u has a neighbor in $V(F)$, and v is strongly anticomplete to $V(F)$. Let N be the set of neighbors of u in $V(F)$, and let $M = V(F) \setminus N$. By the maximality of $|V(F)|$, there are two adjacent vertices in N. Let C be a component of N with $|C| > 1$. Since G is unfriendly, F contains a path of length three and u is complete to C, it follows that $C \neq V(F)$. Since F is connected, some vertex $f \in V(F) \setminus C$ has a neighbor in C, and since C is a component of N, it follows that f belongs to M. Let $c \in C$ be adjacent to f. Since C is connected, it follows that c has a neighbor, say c', in C. Since F is triangle-free, we deduce that f is strongly antiadjacent to c'. But now $\{v, u, c', c, f\}$ is a bull, a contradiction. This proves (1).

For a vertex $v \in V(G) \setminus V(F)$, let $N_F(v)$ be the set of neighbors of v in $V(F)$, and let $M(v) = V(F) \setminus N_F(v)$.

(2) Let H be a triangle free trigraph, no induced subtrigraph of which is a path of length three, and assume that H is connected. Then $V(H) = S_1 \cup S_2$, where S_1 and S_2 are disjoint strongly stable sets, complete to each other. Moreover, if both $|S_2| > 1$ and $|S_2| > 1$, then S_1 is strongly complete to S_2.

By 4.5, and since H is connected, one of the following holds:

- H is not anticonnected, or
- there exist two vertices $v_1, v_2 \in V(H)$ such that v_1 is semi-adjacent to v_2, and $V(H) \setminus \{v_1, v_2\}$ is strongly complete to v_1 and strongly anticomplete to v_2.

Assume first that H is not anticonnected. Since H is triangle free, H has exactly two anti-components, and each of them is a strongly stable set, and (2) holds.

Next assume that there exist two vertices $v_1, v_2 \in V(H)$ such that v_1 is semi-adjacent to v_2, and $V(H) \setminus \{v_1, v_2\}$ is strongly complete to v_1 and strongly anticomplete to v_2. Since H is triangle free, it follows that $V(H) \setminus \{v_1\}$ is strongly stable, and again (2) holds. This proves (2).

(3) Let $v \in V(G) \setminus V(F)$. Then there exist non-empty strongly stable sets

$S_1(v)$ and $S_2(v)$ in F, such that $N_F(v) = S_1(v) \cup S_2(v)$, $S_1(v)$ is complete to $S_2(v)$, and if both $|S_1(v)| > 1$ and $|S_2(v)| > 1$, then $S_1(v)$ is strongly complete to $S_2(v)$.

Let $H = F|N_F(v)$. Since G is unfriendly, it follows that no induced subtrigraph of H is a path of length tree. If H is connected, (3) follows from (2), so we may assume not. It follows from the maximality of $|V(F)|$ that some two vertices of $N_F(v)$ are adjacent. Let C be component of $N_F(v)$ with $|C| > 1$. Since H is not connected, it follows that $N_F(v) \neq C$. Since F is connected, some vertex $m \in V(F) \setminus C$ has a neighbor in C, and since C is a component of $N_F(v)$, we deduce that $m \in M(v)$. Let $c \in C$ be a neighbor of m. Since C is connected and F is triangle free, there exists $c' \in C$ such that c' is adjacent to c and antiadjacent to m. Since $\{m, c, c', v, n\}$ is not a bull for any $n \in N_F(v) \setminus C$, it follows that m is strongly complete to $N_F(v) \setminus C$. Since F is triangle-free, it follows that the set $N_F(v) \setminus C$ is strongly stable.

By (2), $C = C_1 \cup C_2$, such that C_1 and C_2 are disjoint non-empty strongly stable sets, and C_1 is complete to C_2. Let $n \in N_F(v) \setminus C$. If both $|C_1| > 1$ and $|C_2| > 1$, then $G|C$ contains a hole of length four, with center v and anticenter n, contrary to 4.4. So we may assume that $|C_1| = 1$, say $C_1 = \{c_1\}$. Let $F' = G|((V(F) \setminus \{c_1\}) \cup \{v\})$. By the choice of F, $|\eta(F')| + |\sigma(F')| \leq |\eta(F)| + |\sigma(F)|$, and therefore some vertex $m_1 \in M(v)$ is adjacent to c_1. By the argument in the previous paragraph with m replaced by m_1, we deduce that m_1 is strongly complete to $N_F(v) \setminus C$. Now c_1-m_1-n-v-c_1 is a hole of length four, and, since F is triangle-free, it follows that every vertex of C_2 is complete to $\{c_1, v\}$ and anticomplete to $\{m_1, n\}$. By 4.3, it follows that C_2 is a strong clique, and therefore $|C_2| = 1$, say $C_2 = \{c_2\}$. Exchanging the roles of c_1 and c_2, we deduce that some vertex $m_2 \in M(v)$ is adjacent to c_2 and to n. Since F is triangle-free, it follows that $m_1 \neq m_2$, and since $\{m_1, c_1, v, c_2, m_2\}$ is not a bull, it follows that m_2 is strongly adjacent to m_1. But now $\{m_1, m_2, n\}$ is a triangle in F, a contradiction. This proves (3).

(4) Let $u, v \in V(G) \setminus V(F)$ be adjacent. Then there exist $s_1, s_2 \in N_F(u) \cap N_F(v)$ such that s_1 is adjacent to s_2.

Let $S_1(u), S_1(v), S_2(u), S_2(v)$ be as in (3). Since $S_1(u), S_1(v), S_2(u), S_2(v)$ are non-empty strongly stable sets, and since $S_1(u)$ is complete to $S_2(u)$, and $S_1(v)$ to $S_2(v)$, we may assume that $S_1(u) \cap S_2(v) = S_2(u) \cap S_1(v) = \emptyset$.

If both $S_1(u) \cap S_1(v)$ and $S_2(u) \cap S_2(v)$ are non-empty then (3) holds, so we may assume that $S_2(u) \cap S_2(v) = \emptyset$. From the maximality of $|V(F)|$, there exist $t_u \in S_2(u)$ and $t_v \in S_2(v)$.

Suppose $S_1(u) \cap S_1(v) \neq \emptyset$, and choose $s \in S_1(u) \cap S_1(v)$. Since F is triangle free and s is adjacent to both t_u and t_v, it follows that t_u is antiadjacent to t_v. But now t_u-u-v-t_v is a path, and s is a center for it, contrary to the fact that G is unfriendly. This proves that $S_1(u) \cap S_1(v) = \emptyset$.

If $|S_1(u)| > 1$ and $|S_2(u)| > 1$, then $G|(S_1(u) \cup S_2(u))$ contains a hole of length four, say H; and u is a center for H and v is an anticenter for H, contrary to 4.4, since u is adjacent to v. So we may assume that $S_1(u) = \{s_u\}$, say. Similarly, we may assume that $S_1(v) = \{s_v\}$.

Suppose s_u is strongly antiadjacent to s_v. Let $F' = (F \setminus \{s_u, s_v\}) + \{u, v\}$. Then F' is triangle-free, and therefore $|\eta(F')| + |\sigma(F')| \leq |\eta(F)| + |\sigma(F)|$. Consequently, we may assume from the symmetry, that s_u has a neighbor $m \in M(u)$. Then m is strongly anticomplete to $S_2(u)$. Since $\{m, s_u, t_u, u, v\}$ is not a bull, it follows that $m \in N_F(v)$; and since s_u is strongly antiadjacent to s_v, we deduce that $m \in S_2(v)$. Now u-s_u-m-v-u is a hole of length four, and, since F is triangle free, $S_2(u)$ is complete to $\{u, s_u\}$ and anticomplete to $\{m, v\}$. Therefore, 4.3 implies that $S_2(u)$ is a strong clique, and therefore $|S_2(u)| = 1$, namely $S_2(u) = \{t_u\}$.

Since F is triangle free, it follows that t_u is strongly antiadjacent to m. Since $G|\{u, s_u, t_u, v, m, s_v\}$ is not a prism, it follows that s_v is strongly antiadjacent to t_u. Let $F'' = (F \setminus \{t_u, s_v\}) + \{u, v\}$. Then F'' is triangle-free, and therefore $|\eta(F'')| + |\sigma(F'')| \leq |\eta(F)| + |\sigma(F)|$. Consequently, either t_u has a neighbor in $M(u)$, or s_v has a neighbor in $M(v)$. If s_v has a neighbor $x \in M(v)$, then $x \neq s_u, t_u$, and so $\{x, s_v, m, v, u\}$ is a bull, a contradiction. Thus t_u has a neighbor $y \in M(u)$. Since $\{y, t_u, s_u, u, v\}$ is not a bull, it follows that $y \in S_2(v)$. Then $y \neq m$, and since F is triangle free, we deduce that y is strongly antiadjacent to s_u. But then $\{m, s_u, u, t_u, y\}$ is a bull, a contradiction. This proves that s_u is adjacent to s_v.

Now u-s_u-s_v-v-u is a hole of length four, $S_2(u)$ is complete to $\{u, s_u\}$ and anticomplete to $\{v, s_v\}$, and $S_2(v)$ complete to $\{v, s_v\}$ and anticomplete to $\{u, s_u\}$. Thus, 4.3 implies that $|S_2(u)| = |S_2(v)| = 1$, and therefore $S_2(u) = \{t_u\}$, and $S_2(v) = \{t_v\}$. Now, reversing the roles of $S_1(u)$ and $S_2(u)$, and of $S_1(v)$ and $S_2(v)$, we deduce that t_u is adjacent to t_v. But then, since F is triangle free, it follows that $G|\{u, s_u, t_u, v, s_v, t_v\}$ is a prism, a contradiction. This proves (4).

(5) Let $u, v \in V(G) \setminus V(F)$ be antiadjacent. Then $N_F(u) \cap N_F(v)$ is a strongly stable set.

Let $S_1(u), S_2(u), S_1(v), S_2(v)$ be as in (3). Suppose $s_1, s_2 \in N_F(u) \cap N_F(v)$ are adjacent. We may assume that $s_1 \in S_1(u) \cap S_1(v)$, and $s_2 \in S_2(u) \cap S_2(v)$. Then $S_2(u) \cap S_1(v) = S_1(u) \cap S_2(v) = \emptyset$.

First we claim that $N_F(u) = N_F(v)$. Suppose $S_2(u) \setminus S_2(v) \neq \emptyset$, and let $t \in S_2(u) \setminus S_2(v)$. Then t-u-s_2-v is a path, and s_1 is a center for it, contrary to the fact that G is unfriendly. Therefore, $S_2(u) \setminus S_2(v) = \emptyset$, and, form the symmetry, this implies that $N_F(u) = N_F(v)$, and the claim follows. Let $S_1(u) = S_1(v) = S_1$, and $S_2(u) = S_2(v) = S_2$.

Let C_0 be the set of all vertices of $V(G) \setminus V(F)$ that are complete to $S_1 \cup S_2$

and strongly anticomplete to $V(F) \setminus (S_1 \cup S_2)$. Let C be an anticomponent of C_0 with $u, v \in C$. Since C is not a homogeneous set in G, some vertex $x \in V(G) \setminus C$ is mixed on C. By 2.2, there exist $c_1, c_2 \in C$ such that c_1 is antiadjacent to c_2, and x is adjacent to c_1 and antiadjacent to c_2.

Suppose first that $x \notin S_1 \cup S_2$. By 4.2, it follows that x is either strongly complete or strongly anticomplete to $S_1 \cup S_2$. If x is strongly complete to $S_1 \cup S_2$, then, $x \in V(G) \setminus V(F)$, and since x is antiadjacent to c_2, the claim above implies that $N_F(x) = N_F(c_2) = S_1 \cup S_2$, contrary to the fact that $x \notin C$. Therefore x is strongly anticomplete to $S_1 \cup S_2$. Since $x \notin S_1 \cup S_2$, and since x is adjacent to c_1, it follows that $x \in V(G) \setminus V(F)$. But now (4) implies that $N_F(x) \cap N_F(c_1) \neq \emptyset$, contrary to the fact that x is strongly anticomplete to $S_1 \cup S_2$. This proves that $x \in S_1 \cup S_2$, and, since x was chosen arbitrarily, that every vertex of $V(G) \setminus C$ that is mixed on C belongs to $S_1 \cup S_2$. We may assume that $x \in S_1$. Since for any $s \in S_1 \setminus \{x\}$, x-c_1-s-c_2 is not a path with center s_2, it follows that $S_1 = \{x\}$. Since $(C, \{x\})$ is not a homogeneous pair in G, it follows that some vertex y is mixed on C. Since every vertex that is mixed on C belongs to $S_1 \cup S_2$, it follows that $y \in S_2$, and therefore, from the symmetry between x and y, $S_2 = \{y\}$ and y is semi-adjacent to some vertex $c_3 \in C$. Since x is semi-adjacent to c_2, it follows that $c_2 \neq c_3$. Suppose that there exist $x', y' \in V(F) \setminus \{x, y\}$ such that x' is adjacent to x, and y' to y. Since F is triangle free, it follows that x' is strongly antiadjacent to y, and y' to x. Since $\{x', x, u, y, y'\}$ is not a bull, we deduce that x' is adjacent to y'. But now x-y-y'-x'-x is a hole of length four, and $\{u, v\}$ is complete to $\{x, y\}$ and anticomplete to $\{x', y'\}$, contrary to 4.3. So we may assume from the symmetry that y is strongly anticomplete to $V(F) \setminus \{x, y\}$. Since F is connected and since there is a three-edge path in F, it follows that there exists a vertex $x' \in V(F) \setminus \{x, y\}$ adjacent to x. Since $\{x', x, c_3, y, c_2\}$ is not a bull, it follows that c_2 is strongly adjacent to c_3. Since C is anticonnected, there is an antipath Q from c_2 to c_3 with $V(Q) \subseteq C$. Since x is complete to C and G is unfriendly, it follows that Q has a unique internal vertex, say q. Then q is complete to $\{x, y\}$ and strongly antiadjacent to x'. But now $\{x', x, q, y, c_2\}$ is a bull, a contradiction. This proves (5).

(6) Let C be a component of $V(G) \setminus V(F)$. Then C is a strong clique.

Suppose C is not a strong clique. Then, since C is connected, there exist vertices $x, y, z \in C$, such that y is adjacent to both x and z; and x is antiadjacent to z. By (4), there exist $a, b, c, d \in V(F)$ such that a is adjacent to b, c is adjacent to d, $\{x, y\}$ is complete to $\{a, b\}$ and $\{y, z\}$ is complete to $\{c, d\}$. By (5), z is not complete to $\{a, b\}$, and x is not complete to $\{c, d\}$; and therefore $\{a, b\} \neq \{c, d\}$. Suppose b is complete to $\{z, d\}$. Since F is triangle-free, it follows that a is strongly antiadjcaent to d. Then, by (5), x is strongly antiadjacent to d, and z to a. But now $\{x, a\}$ is anticomplete to

$\{z,d\}$, and $\{y,b\}$ is complete to $\{x,a,z,d\}$, contrary to 4.7. This proves that b is not complete to $\{z,d\}$, and, in particular, $b \neq c$. From the symmetry, this implies that a is not complete to $\{z,c\}$, and that $\{a,b\} \cap \{c,d\} = \emptyset$. Since $a,b,c,d, \in N_F(y)$, by (3) and the symmetry we may assume that a is adjacent to c and b to d. Since F is triangle-free, it follows that b is strongly antiadjacent to c. Since b is adjacent to d, it follows that b is antiadjacent to z, and, since a is adjacent to c, it follows that a is antiadjacent to z. But now z-c-a-b is a path, and y is a center for it, contrary to the fact that G is unfriendly. This proves (6).

Let C be a component of $V(G) \setminus V(F)$, and let $f \in V(F)$. We denote by $C(f)$ the set of vertices of C that are adjacent to f, and by $N_F(C)$ the set of vertices of F with a neighbor in C.

(7) Let C be a component of $V(G) \setminus V(F)$, and let $c \in C$. For $i = 1,2$ let $S_i(c)$ be defined as in (3). Then, for $i = 1,2$ there exists $s_i \in S_i(c)$ such that s_i is complete to C.

Choose $s_1 \in S_1(c)$ with $C(s_1)$ maximal. We may assume that $C(s_1) \neq C$, for otherwise (7) holds. Let $c' \in C \setminus C(s_1)$. By (4), c' has a neighbor $s_1' \in S_1(c)$. It follows from the maximality of $C(s_1)$ that there exists $c_1 \in C(s_1)$ such that s_1' is strongly antiadjacent to c_1. But now s_1-c_1-c'-s_1' is a path with center c, a contradiction. This proves (7).

(8) Let C be a component of $V(G) \setminus V(F)$. Then $N_F(C) = S_1(C) \cup S_2(C)$ where each of $S_1(C), S_2(C)$ is a non-empty strongly stable set.

Let $c \in C$, and let $S_1(c), S_2(c)$ be as in (3). By (7), for $i = 1,2$ there exists $s_i \in S_i(c)$ such that C is complete to s_i. Now, by (3), we may assume that for every $c' \in C$, $S_1(c')$ is complete to s_2, and $S_2(c')$ is complete to s_1. For $i = 1,2$, let $S_i(C) = \bigcup_{c' \in C} S_i(c')$. Then $N_F(C) = S_1(C) \cup S_2(C)$. But $S_1(C)$ is complete to s_2, and $S_2(C)$ is complete to s_1, and therefore, since F is triangle free, it follows that each of $S_1(C)$ and $S_2(C)$ is strongly stable. This proves (8).

For a component C of $V(G) \setminus V(F)$ we call the sets $S_1(C), S_2(C)$ defined in (8) the *anchors* of C.

(9) Let C be a component of $V(G) \setminus V(F)$. Let $S_1(C), S_2(C)$ be the anchors of C, for $i = 1,2$ let $T_i(C)$ be the set of vertices of $V(F) \setminus (S_1(C) \cup S_2(C))$ with a neighbor in $S_i(C)$; and for $s_i \in S_i(C)$, let $T_i(s_i)$ be the set of neighbors of s_i in $V(F) \setminus (S_1(C) \cup S_2(C))$. Then

- *for every $s, s' \in S_1(C)$ either s is strongly complete to $C(s')$, or s' is strongly complete to $C(s)$,*

- *Let $s_1 \in S_1(C)$ be antiadjacent to $s_2 \in S_2(C)$. Then every vertex of C is strongly adjacent to one of s_1, s_2. If some $c \in C$ is adjacent to both s_1 and s_2, then $C = \{c\}$, $N_F(C) = \{s_1, s_2\}$ and s_1 is semi-adjacent to s_2.*

- *for every $s, s' \in S_1(C)$, if some vertex of $C(s')$ is antiadjacent to s, then s is strongly complete to $T(s')$.*

- *$T_1(s_1)$ is disjoint from and strongly complete to $T_2(s_2)$ for every $s_1 \in S_1(c)$, $s_2 \in S_2(c)$ and $c \in C$.*

- *let $c \in C$, $s_1 \in S_1(C)$ and $s_2 \in S_2(C)$ such that c is adjacent to both s_1 and s_2. Then every vertex of C is strongly adjacent to at least one of s_1, s_2.*

Let $s, s' \in S_1(C)$, and suppose there exist $c \in C$ adjacent to s and antiadjacent to s', and $c' \in C$ adjacent to s' and antiadjacent to s. By (4), there is $s_2 \in S_2(C)$ adjacent to both c, c'. By (3), s_2 is adjacent to both s and s'. But now s-c-c'-s' is a path, and s_2 is a center for it, contrary to the fact that G is unfriendly. This proves the first assertion of (9).

Next assume that $s_1 \in S_1(C)$ is antiadjacent to $s_2 \in S_2(C)$. Suppose first that some $c \in C$ is adjacent to both s_1 and s_2. By (3), it follows that $S_1(c) = \{s_1\}$, $S_2(c) = \{s_2\}$, and s_1 is semi-adjacent to s_2. Suppose there exists $c' \in C \setminus \{c\}$. By (4), c' is complete to $\{s_1, s_2\}$. Suppose c' has a neighbor $f \in V(F) \setminus \{s_1, s_2\}$. By (3), we may assume that f is adjacent to s_1 and antiadjacent to s_2. But now f-s_1-c-s_2 is a path, and c' is a center for it, a contradiction. Therefore, $N_F(C) = \{s_1, s_2\}$. Since s_1 is semi-adjacent to s_2, it follows that C is strongly complete to $N_F(C)$, and C is a homogeneous set in G, contrary to the fact that G is unfriendly. Thus $C = \{c\}$, and the second assertion of (9) holds. So we may assume that $C(s_1) \cap C(s_2) = \emptyset$. Suppose there exists a vertex $c \in C$ anticomplete to $\{s_1, s_2\}$. For $i = 1, 2$, let $c_i \in C$ be adjacent to s_i. If c, c_1, c_2 are all distinct, then $\{s_1, c_1, c, c_2, s_2\}$ is a bull, a contradiction. Thus we may assume that $c = c_1$. By (7), there exists a vertex $s \in S_2(C)$ adjacent to both c_1 and c_2. Since c_1 is semi-adjacent to s_1, it follows that c_1 is strongly antiadjacent to s_2, and so $s \neq s_2$. By (3), s is adjacent to s_1. But now $\{s_1, s, c_1, c_2, s_2\}$ is a bull, a contradiction. This proves the second assertion of (9).

Next let $s, s' \in S_1(C)$, and assume that some vertex $c' \in C(s')$ is antiadjacent to s, and some vertex $t' \in T_1(s')$ is antiadjacent to s. Let $s_2 \in S_2(C)$ be complete to C (such a vertex s_2 exists by (7)). By the second assertion of (9), and since both s, s' have neighbors in C, it follows that s_2 is adjacent to both s, s'. But now, since F is triangle-free, $\{t', s', c', s_2, s\}$ is a bull, a contradiction. This proves the third assertion of (9).

Next, let $c \in C$, and for $i = 1, 2$, let $s_i \in S_i(c)$, and let $t_i \in T_i(s_i)$. By (3), s_1 is adjacent to s_2. Since F is triangle-free, s_1 is strongly antiadjacent to t_2, and s_2 to t_1, and therefore $t_1 \neq t_2$. Now since $\{t_1, s_1, c, s_2, t_2\}$ is not a bull, it follows that t_1 is strongly adjacent to t_2, and the fourth assertion of (9) follows.

Finally, suppose that there exist $c, c' \in C$, $s_1 \in S_1(C)$ and $s_2 \in S_2(C)$ such that c is adjacent to both s_1 and s_2, and c' is antiadjacent to both s_1, s_2. Since c is semi-adjacent to at most one of s_1, s_2, it follows that c is strongly adjacent to at least one of s_1, s_2, and so $c \neq c'$. By the second assertion of (9), s_1 is adjacent to s_2. Since c' is semi-adjacent to at most one of s_1, s_2, we may assume that s_1 is strongly antiadjacent to c'. By (7), there exists $s \in S_1(C)$ complete to C. Then $s \neq s_1$. By the second assertion of (9), since s_2 has a neighbor in C, it follows that s is adjacent to s_2. But now s_1-s_2-s-c' is a path, and c is a center for it, contrary to the fact that G is unfriendly. This proves the fifth assertion of (9), and completes the proof of (9).

(10) Let X be a component of $V(G) \setminus (F)$, with anchors S_1, S_2. For $i = 1, 2$, let T_i be the set of vertices of $V(F) \setminus (S_1 \cup S_2)$ with a neighbor in S_i. Then $G|(X \cup S_1 \cup S_2 \cup T_1 \cup T_2)$ is a (X, S_1, S_2, T_1, T_2)-clique connector.

Let $|X| = t$. By (9), we can number the vertices of X as $\{x_1, \dots, x_t\}$ such that for every $s \in S_1$, $N(s) \cap C = \{x_1, \dots, x_i\}$ for some $i \in \{1, \dots, t\}$, and s is strongly complete to $\{x_1, \dots, x_{i-1}\}$, and for every $s \in S_2$, $N(s) \cap C = \{x_{t-i+1}, \dots, x_t\}$ for some $i \in \{1, \dots, t\}$, and s is strongly complete to $\{x_{t-i+2}, \dots, x_t\}$. Let $i \in \{1, \dots, t\}$. Let A_i be the set of vertices of S_1 that are strongly complete to $\{x_1, \dots, x_{i-1}\}$, adjacent to x_i and strongly anticomplete to $\{x_{i+1}, \dots, x_t\}$. Let A'_i be the set of vertices of A_i that are semi-adjacent to x_i. Let B_i be the set of vertices of S_2 that are strongly complete to $\{x_{t-i+2}, \dots, x_t\}$, adjacent to x_{t-i+1} and strongly anticomplete to $\{x_1, \dots, x_{t-i}\}$. Let B'_i be the set of vertices of B_i that are semi-adjacent to x_{t-i+1}. Then $S_1 = \bigcup_{i=1}^{t} A_i$, and $S_2 = \bigcup_{i=1}^{t} B_i$. Let $i \in \{1, \dots, t\}$. Let C_i be the set of vertices of T_1 with a neighbor in A_i, and that are strongly anticomplete to $\bigcup_{j>i} A_j$, and let D_i be the set of vertices of T_2 with a neighbor in B_i, and that are strongly anticomplete to $\bigcup_{j>i} B_j$. Then $T_1 = \bigcup_{i=1}^{t} C_i$, and $T_2 = \bigcup_{i=1}^{t} D_i$. We show that the sets $X, A_1, \dots, A_t, B_1, \dots, B_t, C_1, \dots, C_t, D_1 \dots, D_t$ satisfy the axioms of a clique connector.

If $i + j \neq t$, then either some vertex of X is complete to $A_i \cup B_j$, or some vertex of C is anticomplete to $A_i \cup B_j$. Therefore, (9) implies, that if $i + j \neq t$, and A_i is not strongly complete to A_j, then $|X| = |S_1| = |S_2| = 1$, and S_1 is complete to S_2. Since for every i, x_i is anticomplete to $A'_i \cup B_{t-i}$, it follows from (9) that A'_i is strongly complete to B_{t-i}, and from the symmetry B'_{t-i} is strongly complete to A_i.

Next we show that S_1 is strongly anticomplete to T_2. Suppose $s_1 \in S_1$ has a neighbor $u \in T_2$. Let $s_2 \in S_2$ be a neighbor of u. Then, since F is triangle-free, it follows that s is strongly antiadjacent to u, and so $s_1 \in A_i \setminus A'_i$ and $s_2 \in B_{t-i} \setminus B'_{t-i}$ for some $i \in \{1, \dots, t\}$. Now x_i-x_{i+1}-s_2-u-s_1-x_i is a hole of length five. By (7), there exists $s'_1 \in S_1$ complete to X. Then $s'_1 \neq s_1$, and s_1' is adjacent to x_i, x_{i+1}, and, by (9), to s_2, contrary to 4.8. This proves that S_1 is strongly anticomplete to T_2. Similarly, S_2 is strongly anticomplete to T_1.

By (9), for $i \in \{1, \ldots, t\}$, C_i is strongly complete to $\bigcup_{j<i} A_j$, and D_i is strongly complete to $\bigcup_{j<i} B_j$.

We claim that for $i \in \{1, \ldots, t\}$, C_i is strongly complete to A_i'. Suppose $c \in C_i$ is antiadjacent to $a' \in A_i'$. Since a' is semi-adjacent to x_i, it follows that a' is strongly antiadjacent to c. Since $c \in C_i$, there is a vertex $a \in A_i \setminus \{a'\}$ that is adjacent to c. But then a is adjacent to both x_i and c, and a' is antiadjacent to both c_i and c, contrary to (9). This proves that C_i is strongly complete to A_i'. Similarly, for $i \in \{1, \ldots, t\}$, D_i is strongly complete to B_i'.

Finally, let $i, j \in \{1, \ldots, t\}$, such that $i + j > t$. We claim that C_i is strongly complete to D_j. Suppose $c \in C_i$ is antiadjacent to $d \in D_j$. Let $a_i \in A_i$ be adjacent to c, and let $b_j \in B_j$ be adjacent to d. Since $j > t - i$, it follows that b_j is adjacent to c_i. But now $\{c, a_i, x_i, b_j, d\}$ is a bull, a contradiction.

Finally, by (7), $A_t \neq \emptyset$ and $B_t \neq \emptyset$. Thus, all the axioms of a clique connector are satisfied. This proves (10).

Now, if $N_F(C_1) \cap N_F(C_2) = \emptyset$ for every two components C_1, C_2 of $V(G) \setminus V(F)$, then taking H to be the graph whose vertices are the components of $V(G) \setminus V(F)$, and with $E(H) = \emptyset$, we observe, using (10), that G admits an H-structure and thus $G \in \mathcal{T}_1$.

Let us now sketch the general case (see [2] for details). Let C_1, C_2 be components of $V(G) \setminus V(F)$. Renumbering the anchors if necessary, we may assume that $S_1(C_1) \cap S_2(C_2) = S_2(C_1) \cap S_1(C_2) = \emptyset$. Let

$$i(C_1, C_2) = \begin{cases} 0 & if \quad N_F(C_1) \cap N_F(C_2) = \emptyset \\ 1 & if \quad S_1(C_1) \cap S_1(C_2) \neq \emptyset \text{ and } S_2(C_1) \cap S_2(C_2) = \emptyset \\ 1 & if \quad S_1(C_1) \cap S_1(C_2) = \emptyset \text{ and } S_2(C_1) \cap S_2(C_2) \neq \emptyset \\ 2 & if \quad S_1(C_1) \cap S_1(C_2) \neq \emptyset \text{ and } S_2(C_1) \cap S_1(C_2) \neq \emptyset \end{cases}$$

Let H be the graph whose vertices are the components of $V(G) \setminus V(F)$, and such that if $C_1, C_2 \in V(H)$, then there are $i(C_1, C_2)$ edges with ends C_1, C_2. Then H is a loopless graph, and one can show that H is triangle-free, $maxdeg(H) \leq 2$, and G admits an H-structure. Thus $G \in \mathcal{T}_1$. This completes the proof of 4.9. ▮

We can now prove 3.2, which we restate.

4.10 *Let G be an elementary bull-free trigraph. Then either*

- *one of $G, \overline{G}$ belongs to $\mathcal{T}_1$, or*
- *G admits a homogeneous set decomposition, or*

- *G admits a homogeneous pair decomposition.*

Let us first remind the reader the main result of [1]. First we repeat the definition of the **class** $\mathcal{T}_0$. Let G be the trigraph with vertex set

$$\{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2\}$$

and adjacency as follows: $\{b_1, b_2, c_1, c_2\}$ is a strong clique; a_1 is strongly adjacent to b_1, b_2 and semi-adjacent to c_1; a_2 is strongly adjacent to c_1, c_2 and semi-adjacent to b_1; d_1 is strongly adjacent to a_1, a_2; d_2 is either strongly adjacent or semi-adjacent to d_1; and all the remaining pairs are strongly antiadjacent. Let X be a subset of $\{b_1, b_2, c_1, c_2\}$ such that $|X| \leq 1$. Then $G \setminus X \in \mathcal{T}_0$. We observe that since $|X| \leq 1$, every trigraph in $\mathcal{T}_0$ contains a three-edge path with a center and an anticenter, and therefore no trigraph in $\mathcal{T}_0$ is elementary.

The main result of [1] is the following:

4.11 *Let G be a bull-free trigraph. Let P and Q be paths of length three, and assume that there is a center for P and an anticenter for Q in G. Then either*

- *G admits a homogeneous set decomposition, or*
- *G admits a homogeneous pair decomposition, or*
- *G or $\overline{G}$ belongs to $\mathcal{T}_0$.*

Proof of 4.10. We may assume that G does not admit a homogeneous set decomposition or a homogeneous pair decomposition. Assume first that there are paths P and Q, each of length three, in G, and that there is a center for P and an anticenter for Q in G. By 4.11, one of $G, \overline{G}$ belongs to $\mathcal{T}_0$, contrary to the fact that G is elementary. Consequently, no such paths P, Q exist in G, and therefore we may assume that either G or $\overline{G}$ is unfriendly. Since one of the outcomes of 4.10 holds for G if and only if one of the outcomes of 4.10 holds for $\overline{G}$, we may assume that G is unfriendly. Since if G is a prism, then $\overline{G}$ has no triangle, and therefore admits and H-structure with H being the empty graph, 4.1 implies that no induced subtrigraph of G is a prism.

If G is framed, then $G \in \mathcal{T}_1$ by 4.9, and if G is not framed, we use 4.5 to show that $G \in \mathcal{T}_1$. This proves 4.10. ▮

5 The decomposition theorem for trigraphs

In this section we state a decomposition theorem for bull-free trigraphs. We start by describing a special type of trigraphs.

1-thin trigraphs. Let G be a trigraph. Let $a, b \in V(G)$ be distinct vertices, and let $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_m\}$ be disjoint subsets of $V(G)$ such that $A \cup B = V(G) \setminus \{a, b\}$. Let us now describe the adjacency in G.

- a is strongly complete to A and strongly anticomplete to B.
- b is strongly complete to B and strongly anticomplete to A.
- a is semi-adjacent to b.
- If $i, j \in \{1, \dots, n\}$, and $i < j$, and a_i is adjacent to a_j, then a_i is strongly complete to $\{a_{i+1}, \dots, a_{j-1}\}$, and a_j is strongly complete to $\{a_1, \dots, a_{i-1}\}$.
- If $i, j \in \{1, \dots, m\}$, and $i < j$, and b_i is adjacent to b_j, then b_i is strongly complete to $\{b_{i+1}, \dots, b_{j-1}\}$, and b_j is strongly complete to $\{b_1, \dots, b_{i-1}\}$.
- If $p \in \{1, \dots, n\}$ and $q \in \{1, \dots, m\}$, and a_p is adjacent to b_q, then a_p is strongly complete to $\{b_{q+1}, \dots, b_m\}$, and b_q is strongly complete to $\{a_{p+1}, \dots, a_n\}$.

Under these circumstances we say that G is 1-*thin.* We call the pair (a, b) the *base* of G.

5.1 *Every* 1-*thin trigraph is bull-free.*

We omit the proof, it can be found in [3].

2-thin trigraphs. Let G be a trigraph. Let $x_{AK}, x_{AM}, x_{BK}, x_{BM}$ be pairwise distinct vertices of G, and let A, B, K, M be pairwise disjoint subsets of $V(G)$, such that K, M are strong cliques, A, B are strongly stable sets and

$$A \cup B \cup K \cup M \cup \{x_{AK}, x_{AM}, x_{BK}, x_{BM}\} = V(G).$$

Let $t, s \geq 0$ be integers and let $K = \{k_1, \dots, k_t\}$ and $M = \{m_1, \dots, m_s\}$ (so if $t = 0$ then $K = \emptyset$, and if $s = 0$ then $M = \emptyset$). Let A be the disjoint union of sets $A_{i,j}$, and B the disjoint union of sets $B_{i,j}$, where $i \in \{0, \dots, t\}$ and $j \in \{0, \dots, s\}$.

Assume that :

- A is strongly complete to B
- K is strongly anticomplete to M
- A is strongly complete to $\{x_{AK}, x_{AM}\}$ and strongly anticomplete to $\{x_{BK}, x_{BM}\}$
- B is strongly complete to $\{x_{BK}, x_{BM}\}$ and strongly anticomplete to $\{x_{AK}, x_{AM}\}$

- K is strongly complete to $\{x_{AK}, x_{BK}\}$ and strongly anticomplete to $\{x_{AM}, x_{BM}\}$
- M is strongly complete to $\{x_{AM}, x_{BM}\}$ and strongly anticomplete to $\{x_{AK}, x_{BK}\}$
- x_{AK} is semi-adjacent to x_{BM}
- x_{AM} is semi-adjacent to x_{BK}
- the pairs $x_{AK}x_{BK}$ and $x_{AM}x_{BM}$ are strongly adjacent, and the pairs $x_{AK}x_{AM}$ and $x_{BK}x_{BM}$ are strongly antiadjacent.

Let $i \in \{0, \dots, t\}$ and $j \in \{0, \dots, s\}$. Then

- if $i' \in \{0, \dots, t\}$ and $j' \in \{0, \dots, s\}$ such that $i > i'$ and $j > j'$, then at least one of the sets $A_{i,j}$, $A_{i',j'}$ is empty, and at least one of the sets $B_{i,j}$, $B_{i',j'}$ is empty.
- $A_{i,j}$ is strongly complete to $\{k_1, \dots, k_{i-1}\} \cup \{m_{s-j+2}, \dots, m_s\}$,
  $A_{i,j}$ is complete to $\{k_i, m_{s-j+1}\}$,
  $A_{i,j}$ is strongly anticomplete to $\{k_{i+1}, \dots, k_t\} \cup \{m_1, \dots, m_{s-j}\}$,
- $B_{i,j}$ is strongly complete to $\{k_{t-i+2}, \dots, k_t\} \cup \{m_1, \dots, m_{j-1}\}$,
  $B_{i,j}$ is complete to $\{k_{t-i+1}, m_j\}$,
  $B_{i,j}$ is strongly anticomplete to $\{k_1, \dots, k_{t-i}\} \cup \{m_{j+1}, \dots, m_s\}$.

Then G is *2-thin with base* $(x_{AK}, x_{BM}, x_{BK}, x_{AM})$. We call (A, B, K, M) the *partition of G with respect to the base* $(x_{AK}, x_{BM}, x_{BK}, x_{AM})$.

5.2 *Every 2-thin trigraph is bull-free.*

Proof. Let G be 2-thin. We observe that G is 1-thin with base (x_{AK}, x_{BM}), and the result follows from 5.1. This proves 5.2. ∎

For a semi-adjacent pair a_0b_0 in a trigraph G, we say that a_0b_0 is *doubly dominating* if every vertex of $V(G) \setminus \{a_0, b_0\}$ is strongly adjacent to one of a_0, b_0 and strongly anti-adjacent to the other. 2-thin trigraphs are a subclass of 1-thin trigraphs, they arise in the following way (see [3] for the proof):

5.3 *Let G be a 1-thin trigraph with base (a_0, b_0) and let $x, y \in V(G) \setminus \{a_0, b_0\}$ be such that (x, y) is a doubly dominating semi-adjacent pair. Then (possibly exchanging the roles of x and y)*

- *$x \in A$, $y \in B$, and*
- *G is 1-thin with base (x, y), and*
- *G is 2-thin with base (a_0, b_0, x, y).*

Moreover, there exist 1-thin trigraphs that are not 2-thin, for a example all 1-thin trigraphs with a unique semi-adjacent pair.

We will need the following three classes of trigraphs in order to state our main theorem. The class $\mathcal{T}_0$ was defined in [1] and in Section 4, and the class $\mathcal{T}_1$ was defined in Section 3.

The class $\mathcal{T}_2$. Let G be a bull-free trigraph and let (A, B) be a homogeneous pair in G. Let C be the set of vertices of G that are strongly complete to A and strongly anticomplete to B, and let D be the set of vertices of G that are strongly complete to B and strongly anticomplete to A. We say that (A, B) is *doubly dominating* if $V(G) = A \cup B \cup C \cup D$, and both C and D are non-empty.

Let G_1, G_2 be bull-free trigraphs, and for $i = 1, 2$ let (a_i, b_i) be a doubly dominating semi-adjacent pair in G_i, let A_i be the set of vertices of G_i that are strongly complete to a_i, and let B_i be the set of vertices of G_i that are strongly complete to b_i. We say that G is obtained from G_1 and G_2 by *composing along* (a_1, b_1, a_2, b_2) if $V(G) = A_1 \cup A_2 \cup B_1 \cup B_2$, for $i = 1, 2$ $G|(A_i \cup B_i) = G_i|(A_i \cup B_i)$, A_1 is strongly complete to A_2 and strongly anticomplete to B_2, and B_1 is strongly complete to B_2 and strongly anticomplete to A_2. We observe that if (x, y) is a doubly dominating semi-adjacent pair in G_i and $\{x, y\} \neq \{a_i, b_i\}$, then (x, y) is a doubly dominating semi-adjacent pair in G; and these are all the doubly dominating semi-adjacent pairs in G.

Let H be either the complete graph on two vertices, or the complete graph on three vertices, or the graph on three vertices with no edges. We say that a trigraph G is an *H-pattern* if the vertex set of G consist of two distinct copies a_v, b_v of every vertex v of H, and such that

- for every $v \in V(H)$, a_v is semi-adjacent to b_v, and
- if $u, v \in V(H)$ are adjacent, then a_u is strongly adjacent to a_v and strongly antiadjacent to b_v, and b_u is strongly adjacent to b_v and strongly antiadjacent to a_v, and
- if $u, v \in V(H)$ are non-adjacent, then a_u is strongly adjacent to b_v and strongly antiadjacent to a_v, and b_u is strongly adjacent to a_v and strongly antiadjacent to b_v.

Thus for every $v \in V(H)$, (a_v, b_v) is a doubly dominating semi-adjacent pair in G, and there are no other semi-adjacent pairs in G. We say that G is a *triangle pattern* if H is the complete graph on three vertices, an *edge pattern* if H is the complete graph on two vertices, and a *triad pattern* if H is the graph on three vertices with no edges. We remark that edge patterns are 2-thin graphs, however, it is convenient to have a special name for them.

Let $k \geq 1$ be an integer, and let $G'_1, \ldots, G'_k$ be trigraphs, such that for $i \in \{1, \ldots, k\}$, G'_k is either a triangle pattern, or a triad pattern, or a 2-thin trigraph (possibly an edge pattern). For $i \in \{2, \dots, k\}$, let (c_i, d_i) be a doubly dominating semi-adjacent pair in G'_i. For $j \in \{1, \dots, k-1\}$, let (x_j, y_j) be a doubly dominating semi-adjacent pair in G'_q for some $q \in \{1, \dots, j\}$, and such that the pairs $\{c_2, d_2\}, \dots \{c_k, d_k\}, \{x_1, y_1\}, \dots, \{x_{k-1}, y_{k-1}\}$ are all distinct (and therefore pairwise disjoint).

Let $G_1 = G'_1$. Then (x_1, y_1) is a doubly dominating semi-adjacent pair in G_1. For $i \in \{1, \dots, k-1\}$, let G_{i+1} be the trigraph obtained by composing G_i and G'_{i+1} along $(x_i, y_i, c_{i+1}, d_{i+1})$. Let $G = G_k$. We call such a trigraph G a *skeleton*. Every skeleton is in $\mathcal{T}_2$.

We observe that a semi-adjacent pair $\{u, v\}$ is doubly dominating in G if and only if (u, v) is a doubly dominating semi-adjacent pair in some G'_i with $i \in \{1, \dots, k\}$, and $\{u, v\}$ is not one of $\{c_2, d_2\}, \dots, \{c_k, d_k\}, \{x_1, y_1\}, \dots, \{x_{k-1}, y_{k-1}\}$.

Let G'_0 be a skeleton, and for $i \in \{1, \dots, n\}$ let (a_i, b_i) be a doubly dominating semi-adjacent pair in G'_0, such that the pairs $\{a_1, b_1\}, \dots, \{a_n, b_n\}$ are all distinct (and therefore pairwise disjoint). For $i = \{1, \dots, n\}$, let G'_i be a trigraph such that

- $V(G'_i) = A_i \cup B_i \cup \{a'_i, b'_i\}$, and
- the sets $A_i, B_i, \{a'_i, b'_i\}$ are all non-empty and pairwise disjoint, and
- a'_i is strongly complete to A_i and strongly anticomplete to B_i, and
- b'_i is strongly complete to B_i and strongly anticomplete to A_i, and
- a'_i is semi-adjacent to b'_i, and either
  - both A_i, B_i are strong cliques, and there do not exist $a \in A_i$ and $b \in B_i$, such that a is strongly anticomplete to $B_i \setminus \{b\}$, b is strongly anticomplete to $A_i \setminus \{a\}$, and a is semi-adjacent to b, or
  - both A_i, B_i are strongly stable sets, and there do not exist $a \in A_i$ and $b \in B_i$, such that a is strongly complete to $B_i \setminus \{b\}$, b is strongly complete to $A_i \setminus \{a\}$, and a is semi-adjacent to b, or
  - one of $G'_i, \overline{G'_i}$ is a 1-thin trigraph with base (a'_i, b'_i), and G'_i is not a 2-thin trigraph.

Let $G_0 = G'_0$, and for $i \in \{1, \dots, n\}$, let G_i be obtained by composing G_{i-1} and G'_i along (a_i, b_i, a'_i, b'_i). Let $G = G_n$. Then $G \in \mathcal{T}_2$.

By [1] and 3.1, every trigraph in $\mathcal{T}_0 \cup \mathcal{T}_1$ is bull-free. The following is a theorem from [3] that we state here without a proof.

5.4 *Every trigraph in $\mathcal{T}_2$ is bull-free.*

We also observe that

5.5 *$\overline{G} \in \mathcal{T}_2$ for every trigraph $G \in \mathcal{T}_2$.*

The proof of 5.5 is easy and we omit it.

Next let us describe some more decompositions, in addition to the ones from Section 3. Let G be a trigraph. We say that G admits a *1-join*, if $V(G)$ is the disjoint union of four non-empty sets A, B, C, D such that

- B is strongly complete to C, A is strongly anticomplete to $C \cup D$, and B is strongly anticomplete to D;
- $|A \cup B| > 2$ and $|C \cup D| > 2$, and
- A is not strongly complete and not strongly anticomplete to B, and
- C is not strongly complete and not strongly anticomplete to D.

We need three special kinds of homogeneous pairs. Let (A, B) be a homogeneous pair in G. Let C be the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to A and strongly anticomplete to B, D the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to B and strongly anticomplete to A, E the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to $A \cup B$, and F the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly anticomplete to $A \cup B$. A *homogeneous pair of type zero in G* was defined in [1].

We say that (A, B) is a *homogeneous pair of type one in G* if

- at least one member of C is adjacent to at least one member of F, and
- at least one member of D is adjacent to at least one member of F, and
- $E = \emptyset$, and
- $|A| + |B| > 2$, and A is not strongly complete and not strongly anticomplete to B, and
- both A and B are strongly stable sets.

A trigraph T is a forest if there are no holes and no triangles in T. Thus, for every two vertices of T, there is at most one path between them. A forest T is a *tree* if T is connected. A *rooted forest* is a $(k+1)$-tuple $(T, r_1, \ldots, r_k)$, where T is a forest with components $T_1, \ldots, T_k$, and $r_i \in V(T_i)$ for $i \in \{1, \ldots, k\}$. Let $u, v \in V(F)$ be distinct. We say that u is a *child* of v, if for some $i \in \{1, \ldots, k\}$, both $u, v \in V(T_i)$, and u is adjacent to v, and if P is the unique path of T_i from r_i to u, then $v \in V(P)$. We say that u is a *descendant* of v if for some $i \in \{1, \ldots, k\}$, both $u, v \in V(T_i)$, and if P is the unique path of T_i from r_i to u, then $v \in V(P)$.

Let $(T, r_1, \ldots, r_k)$ be a rooted forest. We say that the trigraph T' is *the closure* of $(T, r_1, \ldots, r_k)$, if $V(T') = V(T)$, $\sigma(T) = \sigma(T')$, and u is adjacent to v in T' if and only if one of u, v is a descendant of the other.

Finally, we say that (A, B) is a *homogeneous pair of type two in G* if

- at least one member of C is adjacent to at least one member of F, and
- $D \neq \emptyset$, and
- D strongly anticomplete to F, and
- $E = \emptyset$, and
- $|A| + |B| > 2$, and A is not strongly complete and not strongly anticomplete to B, and
- A is strongly stable, and
- there exists a rooted forest $(T, r_1, \ldots, r_k)$ such that $G|B$ is the closure of $(T, r_1, \ldots, r_k)$, and
- if $b, b' \in B$ are semi-adjacent, then, possibly with the roles of b and b' exchanged, b is a leaf of T and a child of b', and
- if $a \in A$ is adjacent to $b \in B$, then a is strongly adjacent to every descendant of b in T, and
- let $u, v \in B$ and assume that u is a child of v. Let $i \in \{1, \ldots, k\}$ and let T_i be the component of T such that $u, v \in V(T_i)$. Let P be the unique path of T_i from v to r_i, and let X be the component of $T_i \setminus (V(P) \setminus \{v\})$ containing v (and therefore u). Let Y be the set of vertices of X that are semi-adjacent to v. Let $a \in A$ be adjacent to u and antiadjacent to v. Then a is strongly complete Y and to $B \setminus (V(X) \cup V(P))$, and a is strongly anticomplete to $V(P) \setminus \{v\}$.

Please note that every homogeneous pair of type zero, one, or two is tame in both G and $\overline{G}$, and therefore if there is a homogeneous pair of type zero, one or two in either G or $\overline{G}$, then G admits a homogeneous pair decomposition.

We remind the reader a result from [1]

5.6 *Let G be a bull-free trigraph that is not elementary. Then either*

- *one of $G, \overline{G}$ belongs to $\mathcal{T}_0$, or*
- *one of $G, \overline{G}$ contains a homogeneous pair of type zero,or*
- *G admits a homogeneous set decomposition.*

The goal of most of the remainder of this paper is to modify 3.2 to obtain the following:

5.7 *Let G be an elementary bull-free trigraph. Then either*

- *one of $G, \overline{G}$ belongs to $\mathcal{T}_1 \cup \mathcal{T}_2$, or*

- *one of $G, \overline{G}$ contains a homogeneous pair of type one or two, or*
- *G admits a homogeneous set decomposition.*

Then we use 5.6 and 5.7 to prove our main theorem, which we state in the next section.

6 The main theorem

Let G be a bull-free trigraph, and let $a, b \in V(G)$ be semi-adjacent. Let C be the set of vertices of $V(G) \setminus \{a, b\}$ that are strongly adjacent to a and strongly antiadjacent to b, D the set of vertices of $V(G) \setminus \{a, b\}$ that are strongly adjacent to b and strongly antiadjacent to a, E the set of vertices of $V(G) \setminus \{a, b\}$ that are strongly complete to $\{a, b\}$, and F the set of vertices of $V(G) \setminus \{a, b\}$ that are strongly anticomplete to $\{a, b\}$. Then $V(G) = \{a, b\} \cup C \cup D \cup E \cup F$. We say that ab is a semi-adjacent pair of *type zero* if

- $D = \emptyset$, and
- some member of C is antiadjacent to some member of E, and
- $|C \cup E \cup F| > 2$.

We say that ab is a semi-adjacent pair of *type one* if

- at least one member of C is adjacent to at least one member of F, and
- at least one member of D is adjacent to at least one member of F, and
- $E = \emptyset$.

Finally, we say that ab is a semi-adjacent pair of *type two* if

- at least one member of C is adjacent to at least one member of F, and
- $D \neq \emptyset$, and
- D strongly anticomplete to F, and
- $E = \emptyset$.

We say that ab is of *complement type zero, one* or *two* if ab is of type zero, one or two in $\overline{G}$, respectively. We remark that the type of a semi-adjacent pair is well defined with one exception—a pair ab may be of both type zero, and complement type zero. Also, not every semi-adjacent pair in a bull-free trigraph needs to be of one of the types above, but it turns out that these are the only types of semi-adjacent pairs that are needed to describe the structure of bull-free trigraphs.

We say that H is an *elementary expansion* of G if for every vertex v of G there exists a non-empty subset X_v of $V(H)$, all pairwise disjoint and with union $V(H)$, such that

- for $u, v \in V(G)$, if u is strongly adjacent to v, then X_u is strongly complete to X_v, and if u is strongly antiadjacent to v, then X_u is strongly anticomplete to X_v,
- if $v \in V(G)$ does not belong to any semi-adjacent pair of type 1 or 2 or of complement type 1 or 2, then $|X_v| = 1$
- if u is semi-adjacent to v, and neither of uv, vu is a semi-adjacent pair of type 1 or 2 or of complement type 1 or 2, then the unique vertex of X_u is semi-adjacent to the unique vertex of X_v
- if uv is a semi-adjacent pair of type 1 or 2 in G, then either $|X_v| = |X_u| = 1$ and the unique vertex of X_u is semi-adjacent to the unique vertex of X_v, or (X_u, X_v) is a homogeneous pair of type 1 or 2, respectively, in H
- if uv is a semi-adjacent pair of complement type 1 or 2 in G, then either $|X_v| = |X_u| = 1$ and the unique vertex of X_u is semi-adjacent to the unique vertex of X_v, or (X_u, X_v) is a homogeneous pair of type 1 or 2, respectively, in $\overline{H}$.

We say that H is a *non-elementary expansion* of G if for every vertex v of G there exists a non-empty subset X_v of $V(H)$, all pairwise disjoint and with union $V(H)$, such that

- for $u, v \in V(G)$, if u is strongly adjacent to v, then X_u is strongly complete to X_v, and if u is strongly antiadjacent to v, then X_u is strongly anticomplete to X_v,
- if $v \in V(G)$ does not belong to any semi-adjacent pair of type 0 or of complement type 0, then $|X_v| = 1$
- if u is semi-adjacent to v, and neither of uv, vu is a semi-adjacent pair of type 0 or of complement type 0, then the unique vertex of X_u is semi-adjacent to the unique vertex of X_v
- if uv is a semi-adjacent pair that is both of type 0 and of complement type zero, then either $|X_v| = |X_u| = 1$ and the unique vertex of X_u is semi-adjacent to the unique vertex of X_v, or (X_u, X_v) is a homogeneous pair of type 0 either in H or in $\overline{H}$
- if uv is a semi-adjacent pair of type 0 in G and not in $\overline{G}$, then either $|X_v| = |X_u| = 1$ and the unique vertex of X_u is semi-adjacent to the unique vertex of X_v, or (X_u, X_v) is a homogeneous pair of type 0 in H
- if uv is a semi-adjacent pair of type 0 in $\overline{G}$ and not in G, then either $|X_v| = |X_u| = 1$ and the unique vertex of X_u is semi-adjacent to the unique vertex of X_v, or (X_u, X_v) is a homogeneous pair of type 0 in $\overline{H}$.

We leave it to the reader to verify that an elementary expansion of an elementary bull-free trigraph is another elementary bull-free trigraph, and that a non-elementary expansion of a bull-free trigraph is another bull-free trigraph.

Before we can state our main theorem, we need to define an operation. Let G_1, G_2 be bull-free trigraphs with disjoint vertex sets. We say that G is obtained from G_1, G_2 by *substitution* if

- there exist a vertex $v \in V(G_1)$ such that no vertex of $V(G_1) \setminus \{v\}$ is semi-adjacent to v, and
- $V(G) = (V(G_1) \cup V(G_2)) \setminus \{v\}$, and
- $G|(V(G_1) \setminus \{v\}) = G_1 \setminus \{v\}$, and
- $G|V(G_2) = G_2$, and
- for $x \in V(G_1)$ and $y \in V(G_2)$, x is strongly adjacent to y if x is strongly adjacent to v, and x is strongly antiadjacent to y otherwise.

It is easy to check that a trigraph obtained from two bull-free trigraphs by substitution is another bull-free trigraph.

We can now describe the structure of all bull-free trigraphs (and therefore of all bull-free graphs). First let us state a theorem that describes the structure of elementary bull-free trigraphs that are not obtained from smaller bull-free trigraphs by substitutions.

6.1 *Let G be an elementary bull-free trigraph that is not obtained from smaller bull-free trigraphs by substitution. Then one of G, $\overline{G}$ is an elementary expansion of a member of $\mathcal{T}_1 \cup \mathcal{T}_2$; and every elementary expansion of a trigraph H such that either H or $\overline{H}$ is member of $\mathcal{T}_1 \cup \mathcal{T}_2$ is elementary.*

Finally, we describe the structure of all bull-free trigraphs.

6.2 *Let G be a bull-free trigraph. Then either*

- *G is obtained by substitution from smaller bull-free trigraphs, or*
- *G is a non-elementary expansion of an elementary bull-free trigraph, or*
- *one of $G, \overline{G}$ belongs to $\mathcal{T}_0$, or*
- *one of G, $\overline{G}$ is an elementary expansion of a member of $\mathcal{T}_1 \cup \mathcal{T}_2$,*

and every trigraph obtained this way is bull-free.

We remark that in view of 5.5 and the definition of an elementary expansion, 6.2 may be restated as follows:

6.3 *Let G be a bull-free trigraph. Then either*

- *G is obtained by substitution from smaller bull-free trigraphs, or*
- *G is a non-elementary expansion of an elementary bull-free trigraph, or*
- *one of $G, \overline{G}$ belongs to $\mathcal{T}_0$, or*
- *one of G, $\overline{G}$ is an elementary expansion of a member of $\mathcal{T}_1$, or*
- *G is an elementary expansion of a member of $\mathcal{T}_2$,*

and every trigraph obtained this way is bull-free.

In the remainder of this section we prove 6.1 and 6.2 assuming 5.6 and 5.7, and some lemmas from Section 7.

Proof of 6.1 assuming 5.7. Let G be an elementary bull-free trigraph that is not obtained from smaller bull-free trigraphs by substitution. The proof is by induction on $|V(G)|$. By 5.7 either

- one of $G, \overline{G}$ belongs to $\mathcal{T}_1 \cup \mathcal{T}_2$, or
- one of $G, \overline{G}$ contains a homogeneous pair of type one or two, or
- G admits a homogeneous set decomposition.

We may assume that neither of $G, \overline{G}$ belongs to $\mathcal{T}_1 \cup \mathcal{T}_2$, for then 6.1 holds. If G admits a homogeneous set decomposition, then G is obtained from smaller bull-free trigraphs by substitution, a contradiction. Consequently, there exists a homogeneous pair (A, B) in G, such that (A, B) is of type 1 or 2 in one of $G, \overline{G}$. Since the conclusion of 6.1 is invariant under taking complements, we may assume that (A, B) is a homogeneous pair of type 1 or 2 in G. Let C be the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to A and strongly anticomplete to B, D the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to B and strongly anticomplete to A, E the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to $A \cup B$, and F the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly anticomplete to $A \cup B$. Let G' be the trigraph obtained from $G|(C \cup D \cup E \cup F)$ by adding two new vertices a and b, such that a is strongly complete to $C \cup E$ and strongly anticomplete to $D \cup F$, b is strongly complete to $D \cup E$ and strongly anticomplete to $C \cup F$, and a is semi-adjacent to b. We observe that for $i = 1, 2$, if (A, B) is a homogeneous pair of type i in G, then ab is a semi-adjacent pair of type i in G'. Since $|V(G')| < |V(G)|$, it follows from the inductive hypothesis, that either G' is obtained by substitution from smaller bull-free trigraphs, or one of G', $\overline{G'}$ is an elementary expansion of a member of $\mathcal{T}_1 \cup \mathcal{T}_2$. It is easy to check that if G' is obtained by substitution from smaller elementary trigraphs then so is G, and so we may assume that one of G', $\overline{G'}$ is an elementary expansion of a member of $\mathcal{T}_1 \cup \mathcal{T}_2$. We observe that if G' is an elementary expansion of a trigraph K, then $\overline{G'}$ is an elementary expansion of $\overline{K}$. Thus there exists a trigraph K such that one of $K, \overline{K}$ belongs to $\mathcal{T}_1 \cup \mathcal{T}_2$, and for every vertex v of K there exists a non-empty subset X_v of $V(G')$, all pairwise disjoint and with union $V(G')$, such that

- for $u, v \in V(K)$, if u is strongly adjacent to v, then X_u is strongly complete to X_v, and if u is strongly antiadjacent to v, then X_u is strongly anticomplete to X_v,
- if $v \in V(K)$ does not belong to any semi-adjacent pair of type 1 or 2 or of complement type 1 or 2, then $|X_v| = 1$
- if u is semi-adjacent to v, and neither of uv, vu is a semi-adjacent pair of type 1 or 2 or of complement type 1 or 2, then the unique vertex of X_u is semi-adjacent to the unique vertex of X_v
- if uv is a semi-adjacent pair of type 1 or 2 in K, then either $|X_v| = |X_u| = 1$ and the unique vertex of X_u is semi-adjacent to the unique vertex of X_v, or (X_u, X_v) is a homogeneous pair of type 1 or 2, respectively, in G'
- if uv is a semi-adjacent pair of complement type 1 or 2 in K, then either $|X_v| = |X_u| = 1$ and the unique vertex of X_u is semi-adjacent to the unique vertex of X_v, or (X_u, X_v) is a homogeneous pair of type 1 or 2, respectively, in $\overline{G'}$.

Suppose first that $a, b \in X_v$ for some $v \in V(K)$. Then, since $|X_v| > 2$, there exist $u \in V(K)$ such that uv is a semi-adjacent pair of type 1 or 2, or of complement type 1 or 2, and, consequently, some vertex $V(K) \setminus \{u, v\}$ is strongly adjacent to v. But then some vertex of $V(G')$ is strongly adjacent to both a and b, contrary to the fact that ab is a semi-adjacent pair of type 1 or 2 in G'. Thus there exist distinct $u, v \in V(K)$ such that $a \in X_u$ and $b \in X_v$. Since a is semi-adjacent to b, it follows that u is semi-adjacent to v in K.

We claim that uv is a semi-adjacent pair of type 1 or 2 in K. Since ab is of type 1 or 2 in G', it follows that no vertex of G' is adjacent to both a and b, and, consequently, no vertex of K is adjacent to both u and v, which implies that uv is not of complement type 1 or 2. Since uv is the only semi-adjacent pair of K involving u or v, if $|X_u| > 1$ or $|X_v| > 1$, then it follows from the definition of an elementary expansion that uv is of type 1 or 2 in K, and the claim holds. So we may assume that $X_u = \{a\}$ and $X_v = \{b\}$. But now uv has the same type in K as ab is in G', and therefore uv is of type 1 or 2 in K. This proves the claim.

Now, if uv is of type one in K, then 7.4 implies that $((X_u \setminus \{a\}) \cup A, (X_v \setminus \{b\}) \cup B)$ is a homogeneous pair of type one in G; and if uv is of type two in K, then 7.5 implies that $((X_u \setminus \{a\}) \cup A, (X_v \setminus \{b\}) \cup B)$ is a homogeneous pair of type two in G. In both cases, replacing X_u by $(X_u \setminus \{a\}) \cup A$ and X_v by $(X_v \setminus \{b\}) \cup B$, we observe that G is an elementary expansion of K. This proves 6.1. ∎

Proof of 6.2 assuming 5.6. Let G be a bull-free trigraph. The proof is by induction on $|V(G)|$. We may assume that G is not obtained from smaller trigraphs by substitutions. If G is elementary, then, by 6.1, one of $G, \overline{G}$ is an elementary expansion of a member of $\mathcal{T}_1 \cup \mathcal{T}_2$, and 6.2 holds. So we may assume that G is not elementary. So, by 5.6 either

- one of $G, \overline{G}$ belongs to $\mathcal{T}_0$, or
- one of $G, \overline{G}$ contains a homogeneous pair of type zero, or
- G admits a homogeneous set decomposition.

We may assume that neither of $G, \overline{G}$ belongs to $\mathcal{T}_0$, for then 6.2 holds. If G admits a homogeneous set decomposition, then G is obtained from smaller bull-free trigraphs by substitution, a contradiction. Consequently, there exists a homogeneous pair (A, B) in G, such that (A, B) is of type zero in one of $G, \overline{G}$. Since the conclusion of 6.2 is invariant under taking complements, we may assume that (A, B) is a homogeneous pair of type zero in G. Let C be the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to A and strongly anticomplete to B, D the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to B and strongly anticomplete to A, E the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to $A \cup B$, and F the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly anticomplete to $A \cup B$. Since (A, B) is of type zero in G, it follows that $D = \emptyset$, and some vertex of C is antiadjacent to some vertex of E. Let G' be the trigraph obtained from $G|(C \cup D \cup E \cup F)$ by adding two new vertices a and b such that a is strongly complete to $C \cup E$ and strongly anticomplete to $D \cup F$, b is strongly complete to $D \cup E$ and strongly anticomplete to $C \cup F$, and a is semi-adjacent to b. Then ab is a semi-adjacent pair of type zero in G'. Since $|V(G')| < |V(G)|$, by the inductive hypothesis, one of the outcomes of 6.2 holds for G'. Therefore, either

- G' is obtained by substitution from smaller bull-free trigraphs, or
- one of $G', \overline{G'}$ is an elementary expansion of a member of $\mathcal{T}_1 \cup \mathcal{T}_2$, or
- one of $G', \overline{G'}$ belongs to $\mathcal{T}_0$, or
- G' is a non-elementary expansion of an elementary bull-free trigraph.

If G' is obtained by substitution from smaller bull-free trigraphs, then so is G, so we may assume not. If one of $G', \overline{G'}$ is an elementary expansion of a member of $\mathcal{T}_1 \cup \mathcal{T}_2$, then G' is an elementary trigraph, and so setting $X_v = \{v\}$, for $v \in V(G') \setminus \{a, b\}$, and setting $X_a = A$ and $X_b = B$, we observe that G is a non-elementary expansion of G'. So we may assume that neither of $G', \overline{G'}$ is an elementary expansion of a member of $\mathcal{T}_1 \cup \mathcal{T}_2$. We observe that if H is a trigraph such that either H or $\overline{H}$ belongs to $\mathcal{T}_0$, then for every semi-adjacent pair xy of H, there is a vertex of $V(H) \setminus \{x, y\}$ that is strongly adjacent to x and strongly antiadjacent to y, and a vertex of $V(H) \setminus \{x, y\}$ that is strongly adjacent to y and strongly antiadjacent to x, and hence there is no semi-adjacent pair of type zero in H. Consequently neither of $G', \overline{G'}$ belongs to $\mathcal{T}_0$. This implies that G' is a non-elementary expansion of an elementary bull-free trigraph. This means that there is an elementary trigraph K such that for every vertex v of K there exists a non-empty subset X_v of $V(G')$, all pairwise disjoint and with union $V(G')$, such that

- for $u, v \in V(K)$, if u is strongly adjacent to v, then X_u is strongly complete to X_v, and if u is strongly antiadjacent to v, then X_u is strongly anticomplete to X_v,
- if $v \in V(K)$ does not belong to any semi-adjacent pair of type 0 or of complement type 0, then $|X_v| = 1$
- if u is semi-adjacent to v, and neither of uv, vu is a semi-adjacent pair of type 0 or of complement type 0, then the unique vertex of X_u is semi-adjacent to the unique vertex of X_v
- if uv is a semi-adjacent pair that is both of type 0 and of complement type zero, then either $|X_v| = |X_u| = 1$ and the unique vertex of X_u is semi-adjacent to the unique vertex of X_v, or (X_u, X_v) is a homogeneous pair of type 0 either in G' or in $\overline{G'}$
- if uv is a semi-adjacent pair of type 0 in K and not in $\overline{K}$, then either $|X_v| = |X_u| = 1$ and the unique vertex of X_u is semi-adjacent to the unique vertex of X_v, or (X_u, X_v) is a homogeneous pair of type 0 in G'
- if uv is a semi-adjacent pair of type 0 in $\overline{K}$ and not in K, then either $|X_v| = |X_u| = 1$ and the unique vertex of X_u is semi-adjacent to the unique vertex of X_v, or (X_u, X_v) is a homogeneous pair of type 0 in $\overline{G'}$.

Since for every $v \in V(K)$, X_v is either a strongly stable set or a strong clique, it follows that there exist distinct $u, v \in V(K)$ such that $a \in X_u$ and $b \in X_v$. Suppose that either $|X_u| > 1$ or $|X_v| > 1$. Then (X_u, X_v) is a homogeneous pair of type zero in either G' or $\overline{G'}$, and so (from the definition of a homogeneous pair of type zero) some vertex of G' is strongly adjacent to b and strongly antiadjacent to a, contrary to the fact that $D = \emptyset$. This proves that $|X_u| = |X_v| = 1$, and so $X_u = \{a\}$ and $X_v = \{b\}$. Since ab is a semi-adjacent pair of type zero in G', it follows that that uv is a semi-adjacent pair of type zero in K. But now, replacing X_u by A and X_v by

B, we observe that G is a non-elementary expansion of K. This proves 6.2. ∎

From now on we will turn our efforts to proving 5.7. The difference between 3.2, that we have already proved, and 5.7 is that while we have no control over the homogeneous pairs that come up in 3.2, in 5.7 only special kinds of homogeneous pair decomposition are permitted (at the expense of introducing a new basic class $\mathcal{T}_2$.) Thus, most of the work in the remainder of this paper will be devoted to understanding what kind of homogeneous pairs can occur in an elementary bull-free trigraph.

We start Section 7 by understanding elementary bull-free trigraphs that admit only very special tame homogeneous pairs, called "doubly dominating". We show that every such trigraph is obtained by repeated substitutions from trigraphs in $\mathcal{T}_2$ and their complements. Then we classify other tame homogeneous pairs in elementary bull-free trigraphs, proving that (up to taking complements) every elementary bull-free trigraph either belongs to $\mathcal{T}_1 \cup \mathcal{T}_2$, or admits a homogeneous set decomposition, or a 1-join, or a homogeneous pair of type one, two or three (7.2). Section 8 shows that homogeneous pairs of type three are in fact unnecessary (8.1). Finally, in Section 9 we prove that no minimum size counterexample to 5.7 admits a 1-join, thus proving 5.7.

7 Understanding homogeneous pairs

Let G be a bull-free trigraph, and let (A, B) be a homogeneous pair in G. We remind the reader that (A, B) is *doubly dominating* if every vertex of $V(G) \setminus (A \cup B)$ is either strongly complete to A and strongly anticomplete to B, or strongly complete to B and strongly anticomplete to A. As it turns out, elementary bull-free trigraphs that admit tame doubly dominating homogeneous pairs but no other homogeneous pairs are very restricted. In [3] we prove the following:

7.1 *Let G be an elementary bull-free trigraph. Assume that there is a doubly dominating tame homogeneous pair in G, and that every tame homogeneous pair in G is doubly dominating. Then either G admits a homogeneous set decomposition, or one of $G, \overline{G}$ belongs to $\mathcal{T}_2$.*

The proof of 7.1 is quite involved, and we omit it here.

We now turn our attention to other homogeneous pairs in elementary bull-free trigraphs. We remind the reader that homogeneous pairs of types zero, one and two are defined in Section 3. Let (A, B) be a tame homogeneous pair in G. Let C be the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to A and strongly anticomplete to B, D the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to B and strongly anticomplete to A, E the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to

$A \cup B$, and F the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly anticomplete to $A \cup B$. We say that (A, B) is *a homogeneous pair of type three in* G if

- A is a strongly stable set, and
- B is a strong clique, and
- C is not strongly anticomplete to F, and
- C is not strongly complete to E.

We observe that the pair (A, B) is a of type three in G if and only if (B, A) is of type three in $\overline{G}$.

Our goal is to prove the following:

7.2 *Let G be an elementary bull-free trigraph. Assume that G does not admit a homogeneous set decomposition. Let (A, B) be a tame homogeneous pair in G that is not doubly dominating. Then one of $G, \overline{G}$ admits a 1-join, or a homogeneous pair of type one, two or three.*

First, given a tame homogeneous pair (A, B), we study the behavior of $G \setminus (A \cup B)$.

7.3 *Let G be an elementary bull-free trigraph, and let (A, B) be a tame homogeneous pair in G. Let C be the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to A and strongly anticomplete to B, D the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to B and strongly anticomplete to A, E the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to $A \cup B$, and F the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly anticomplete to $A \cup B$. Assume that $E \cup F \neq \emptyset$. Then either*

1. *G admits a homogeneous set decomposition, or*
2. *one of $G, \overline{G}$ admits a 1-join, or*
3. *(possibly with the roles of C and D switched) each of the sets C, D, F is non-empty, $E = \emptyset$, D is strongly anticomplete to F, and C is not strongly anticomplete to F, or*
4. *(possibly with the roles of C and D switched) each of the sets C, D, E is non-empty, $F = \emptyset$, D is strongly complete to E, and C is not strongly complete to E, or*
5. *both of the following two statements hold:*
    - *D is not strongly complete to E, or C is not strongly anticomplete to F, and*
    - *C is not strongly complete to E, or D is not strongly anticomplete to F.*

Proof. First we observe that G satisfies the hypotheses of 7.3 if and only if $\overline{G}$ does, and G satisfies the conclusions of 7.3 if and only if $\overline{G}$ does. Moreover, passing to $\overline{G}$ exchanges the roles of C and D, and the roles of E and F we may assume that neither of $G, \overline{G}$ admits 1-join, and that G (and therefore $\overline{G}$) does not admit a homogeneous set decomposition.

(1) If $F \neq \emptyset$, then F is not strongly anticomplete to $C \cup D$.

Suppose $F \neq \emptyset$, and F is strongly anticomplete to $C \cup D$. Since G does not admit a homogeneous set decomposition, it follows that $E \neq \emptyset$, and there exist vertices $e \in E$ and $f \in F$ such that e is adjacent to f. Choose $a \in A$ and $b \in B$ adjacent. Since $\{f, e, b, a, c\}$ is not a bull for any $c \in C$, it follows that e is strongly complete to C, and similarly e is strongly complete to D. Let E_0 be the set of vertices of E with a neighbor in F. Then E_0 is strongly complete to $C \cup D$. Let E' be the union of anticomponents X of E such that $X \cap E_0 \neq \emptyset$. We claim that E' is strongly complete to $C \cup D$. First we observe that if e_1-e_2-e_3 is an antipath with $e_1 \in E_0$, $e_2 \in E \setminus E_0$ and $e_3 \in C \cup D \cup (E \setminus E_0)$, then, choosing $f_1 \in F$ adjacent to e_1, we get that one of $\{f_1, e_1, e_3, b, e_2\}$ and $\{f_1, e_1, e_3, a, e_2\}$ is a bull, a contradiction. So no such antipath e_1-e_2-e_3 exists. This implies that every vertex of $E' \setminus E_0$ has an antineighbor in E_0, and, consequently, that E' is strongly complete to $C \cup D$. But now, since $E \setminus E'$ is strongly complete to E' and strongly anticomplete to F, it follows that $X = A \cup B \cup C \cup D \cup (E \setminus E')$ is a homogeneous set in G, and $e, f \in V(G) \setminus X$, contrary to the fact that G does not admit a homogeneous set decomposition. This proves (1).

Passing to the complement if necessary, we may assume that $F \neq \emptyset$. By (1), we may assume that some vertex $c \in C$ is adjacent to some vertex $f \in F$. Now we may assume that C is strongly complete to E, and that D is strongly anticomplete to F, for otherwise the fifth outcome of 7.3 holds.

(2) If $E \neq \emptyset$, then 7.3 holds.

Suppose $E \neq \emptyset$. Since C is strongly complete to E, (1) applied in $\overline{G}$ implies that there exists a vertex $d \in D$ antiadjacent to a vertex $e \in E$. Passing to $\overline{G}$ if necessary, we may assume that f is antiadjacent to e. But now, choosing $a \in A$ and $b \in B$ antiadjacent, we observe that $\{f, c, a, e, b\}$ is a bull, a contradiction. This proves (2).

In view of (2) we may assume that $E = \emptyset$. Now, since G does not admit a 1-join, it follows that $D \neq \emptyset$, and the fourth outcome of 7.3 holds. This proves 7.3. ∎

Next we prove two useful lemmas about the structure of the sets A and B of a homogeneous pair (A, B).

7.4 *Let G be a bull-free trigraph, and let (A, B) be a homogeneous pair in G. Let C be the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to A and strongly anticomplete to B, D the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to A and strongly anticomplete to B, E the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to $A \cup B$, and F the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly anticomplete to $A \cup B$. Assume that G does not admit a homogeneous set decomposition. Then:*

1. *If some vertex of C is adjacent to some vertex of F, then A is strongly stable.*

2. *If some vertex of D is antiadjacent to some vertex of E, then A is a strong clique.*

Proof. Since the second assertion of 7.4 follows from the first by passing to $\overline{G}$, it is enough to prove the first assertion. Let $c \in C$ be adjacent to $f \in F$. Suppose A is not strongly stable, and let X be a component of A with $|X| > 1$. Since G does not admit a homogeneous set decomposition, it follows that some vertex $v \in V(G) \setminus X$ is mixed on X. Since (A, B) is a homogeneous pair in G, and X is a component of A, it follows that $v \in B$. By 2.2, there exist vertices $x, y \in X$ such that x is adjacent to y, and v is adjacent to x and antiadjacent to y. But now $\{v, x, y, c, f\}$ is a bull, a contradiction. This proves 7.4. ∎

7.5 *Let G be a bull-free trigraph, and let (A, B) be a homogeneous pair in G. Let C be the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to A and strongly anticomplete to B, D the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to B and strongly anticomplete to A, and F the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly anticomplete to $A \cup B$. Assume that $V(G) = A \cup B \cup C \cup D \cup F$, and that G does not admit a homogeneous set decomposition. Suppose that each of the sets C, D, F is non-empty, D is strongly anticomplete to F, and C is not strongly anticomplete to F. Then (A, B) is a homogeneous pair of type two in G.*

We omit the proof and refer the reader to [3]. We can now prove 7.2

Proof of 7.2. Let C be the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to A and strongly anticomplete to B, D the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to B and strongly anticomplete to A, E the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to $A \cup B$, and F the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly anticomplete to $A \cup B$. We may assume that neither of $G, \overline{G}$ admits a 1-join. Since G does not admit a homogeneous set decomposition, it follows that one of the last three outcomes of 7.3 holds. Passing to $\overline{G}$ if necessary, we may assume that $F \neq \emptyset$ and C is not strongly anticomplete to F. Since $F \neq \emptyset$, we deduce that either the third, or the fifth outcome of 7.3 holds. If the third outcome of 7.3 holds, then by 7.5 G admits a homogeneous pair of type two, so we may assume that the fifth outcome of 7.3 holds. Since C is not strongly anticomplete to F, 7.3 implies that either C is not strongly complete to E, or D is not strongly anticomplete to F.

Since C is not strongly anticomplete to F, 7.4 implies that A is a strongly stable set. If C is not strongly complete to E, then, by 7.4 applied in $\overline{G}$, we deduce that B is a strong clique and (A, B) is a homogeneous pair of type three in G. So we may assume that D is not strongly anticomplete to F. But then, again by 7.4, B is a strongly stable set, and (A, B) is a homogeneous pair of type one in G. This proves 7.2. ∎

8 Dealing with homogeneous pairs of type three

Let us first summarize what we know about the structure of elementary bull-free trigraphs so far:

8.1 *Let G be an elementary bull-free trigraph. Then either*

- *one of $G, \overline{G}$ belongs to $\mathcal{T}_1 \cup \mathcal{T}_2$, or*
- *G admits a homogeneous set decomposition, or*
- *one of $G, \overline{G}$ admits a 1-join, or*
- *one of $G, \overline{G}$ admits a homogeneous pair decomposition of type one, two or three.*

Proof. By 3.2, one of the following holds:

- one of $G, \overline{G}$ belongs to $\mathcal{T}_1$, or
- G admits a homogeneous set decomposition, or
- G admits a homogeneous pair decomposition.

We may assume that G admits a homogeneous pair decomposition, for otherwise one of the outcomes of 8.1 holds. Thus there is a tame homogeneous pair in G. If every tame homogeneous pair in G is doubly dominating, then by 7.1, either G admits a homogeneous set decomposition, or one of $G, \overline{G}$ belongs to $\mathcal{T}_2$, and again 8.1 holds. Thus we may assume that there is a homogeneous pair in G which is not doubly dominating. Now, by 7.2, one of $G, \overline{G}$ admits a 1-join, or a homogeneous pair of type one, two or three. This proves 8.1. ∎

In fact, 8.1 can be strengthened further, omitting one of the outcomes, namely a homogeneous pair decomposition of type three. We prove the following:

8.2 *Let G be an elementary bull-free trigraph. Then either*

- *one of $G, \overline{G}$ belongs to $\mathcal{T}_1 \cup \mathcal{T}_2$, or*
- *G admits a homogeneous set decomposition, or*
- *one of $G, \overline{G}$ admits a 1-join, or*
- *one of $G, \overline{G}$ admits a homogeneous pair decomposition of type one or two.*

Proof. Suppose 8.2 is false, and let G be a counterexample to 8.2 with $|V(G)|$ minimum. It follows from 8.1 that one of $G, \overline{G}$ admits a homogeneous pair decomposition of type three, and therefore both G and $\overline{G}$ admit a homogeneous pair decomposition of type three. Let (P, Q) be a homogeneous pair of type three in G (and so (Q, P) is a homogeneous pair of type three in $\overline{G}$). Let C be the set of vertices of $V(G) \setminus (P \cup Q)$ that are strongly complete to P and strongly anticomplete to Q, D the set of vertices of $V(G) \setminus (P \cup Q)$ that are strongly complete to Q and strongly anticomplete to P, E the set of vertices of $V(G) \setminus (P \cup Q)$ that are strongly complete to $P \cup Q$, and F the set of vertices of $V(G) \setminus (P \cup Q)$ that are strongly anticomplete to $P \cup Q$. Let G' be the trigraph obtained from $G \setminus (P \cup Q)$ by adding two new vertices a and b such that a is strongly complete to $C \cup E$ and strongly anticomplete to $D \cup F$, b is strongly complete to $D \cup E$ and strongly anticomplete to $C \cup F$, and a is semi-adjacent to b. Then G' is an elementary bull-free trigraph. From the minimality of $|V(G)|$, it follows that one of the outcomes of 8.2 holds for G'. Since so far we have preserved the symmetry between G and $\overline{G}$, we may assume that either:

- $G' \in \mathcal{T}_1 \cup \mathcal{T}_2$, or
- G' admits a homogeneous set decomposition, or
- G' admits a 1-join, or
- G' admits a homogeneous pair decomposition of type one or two.

Now one can show (see [3] for details) that G satisfies the same outcome of 8.2 as G. This proves 8.2. ∎

9 The proof of 5.7

In this section we finish the proof of 5.7, which we restate.

9.1 *Let G be an elementary bull-free trigraph. Then either*

- *one of $G, \overline{G}$ belongs to $\mathcal{T}_1 \cup \mathcal{T}_2$, or*
- *one of $G, \overline{G}$ contains a homogeneous pair of type one or two, or*
- *G admits a homogeneous set decomposition.*

Proof. Suppose 9.1 is false, and let G be a counterexample of 9.1 with $|V(G)|$ minimum. Then $\overline{G}$ is also a counterexample to 9.1, and $|V(G)| = |V(\overline{G})|$. By 8.2, and since both G and $\overline{G}$ are counterexamples to 9.1, we may assume that G admits a 1-join. Therefore, $V(G)$ is the disjoint union of four non-empty sets A, B, C, D such that

- B is strongly complete to C, A is strongly anticomplete to $C \cup D$, and B is strongly anticomplete to D;
- $|A \cup B| > 2$ and $|C \cup D| > 2$, and
- A is not strongly complete and not strongly anticomplete to B, and
- C is not strongly complete and not strongly anticomplete to D.

Let G_1 be the trigraph obtained from $G|(A \cup B)$ by adding two new vertices c and d, such that c is strongly complete to B and strongly anticomplete to A, and d is semi-adjacent to c and strongly anticomplete to $A \cup B$. Let G_2 be the trigraph obtained from $G|(C \cup D)$ by adding two new vertices a and b, such that b is strongly complete to C and strongly anticomplete to D, and a is semi-adjacent to b and strongly anticomplete to $C \cup D$.

Since for $i = 1, 2$, $|V(G_i)| < |V(G)|$, it follows that one of the outcomes of 9.1 holds for G_i, and one can show (see [3]) that $G_1, G_2 \in \mathcal{T}_1$. Since every vertex in a double melt has a strong neighbor in the melt, it follows that G_1, G_2 are not double melts. Therefore, there exist graphs H_1, H_2 each with maximum degree at most two, such that for $i = 1, 2$ G_i admits an H_i-structure. Let $L_i \subseteq V(G_i)$ and

$$h_i : V(H_i) \cup E(H_i) \cup (E(H_i) \times V(H_i)) \rightarrow 2^{V(G_i) \setminus L_i}$$

be as in the definition of an H_i-structure. Since for every $e \in E(H_i)$ with ends u, v, $G_i|(h(e) \cup h(e, v) \cup h(e, u))$ is an $h(e)$-melt, and since every vertex of a melt has a strong neighbor in the melt, it follows that $d \notin h_1(e) \cup h_1(e, v)$ for any $e \in E(H_1)$, $v \in V(H_1)$. Similarly, $a \notin h_2(e) \cup h_2(e, v)$ for any $e \in E(H_2)$, $v \in V(H_2)$. Since every vertex of $h_i(v)$ has a strong neighbor in $V(G_i)$ it follows that $d \notin h_1(v)$ for any $v \in V(H_1)$, and $a \notin h_2(v)$ for any $v \in V(H_2)$. Consequently, $d \in L_1$ and $a \in L_2$. Since d has no strong neighbors in $V(G_1) \setminus \{d\}$, and d is semi-adjacent to c, it follows that $c \in L_1$ and similarly $b \in L_2$.

By 7.4, B and C are strongly stable sets, and one can show that $B \subseteq L_1 \cup (\bigcup_{e \in E(H_1)} h_1(e))$, and similarly, $C \subseteq L_2 \cup (\bigcup_{e \in E(H_2)} h_2(e))$.

Let $L = (L_1 \cup L_2) \setminus \{a, b, c, d\}$, let H be the disjoint union of H_1 and H_2. We observe that $G|L$ has no triangle. Now, defining

$$h : V(H) \cup E(H) \cup (E(H) \times V(H)) \rightarrow 2^{V(G) \setminus L}$$

as $h(x) = h_i(x)$ for $x \in V(H_i) \cup E(H_i) \cup (E(H_i) \times V(H_i))$, we observe that G admits an H-structure, and therefore $G \in \mathcal{T}_1$, contrary to the fact that G is a counterexample to 9.1. This proves 9.1. ∎

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