A counterexample to a conjecture of Schwartz

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August 25, 2011; revised October 1, 2011

[1]Supported by DFG grants BR 2312/7-1 and BR 2312/10-1.
[2]Supported by NSF grants DMS-1001091 and IIS-1117631.
[3]Supported by ONR grant N00014-10-1-0680 and NSF grant DMS-0070912.

Abstract

In 1990, motivated by applications in the social sciences, Thomas Schwartz made a conjecture about tournaments which would have had numerous attractive consequences. In particular, it implied that there is no tournament with a partition A, B of its vertex set, such that every transitive subset of A is in the out-neighbour set of some vertex in B, and vice versa. But in fact there is such a tournament, as we show in this paper, and so Schwartz' conjecture is false. Our proof is non-constructive and uses the probabilistic method.

1 Introduction

The goal of this paper is to disprove a popular conjecture of Schwartz [9], but before that we need to state the conjecture. If D is a digraph, $N^-_D(v) = N^-(v)$ denotes the set of in-neighbours of a vertex v. Let G be a tournament, and suppose that ϕ is some function such that $\phi(H)$ is defined and satisfies $\phi(H) \subseteq V(H)$ for every non-null proper subtournament H of G. We say a subset $A \subseteq V(G)$ is *ϕ-retentive* if $A \neq \emptyset$ and $\phi(G|N^-(a)) \subseteq A$ for each $a \in A$.

Let $\mathcal{G}$ be the class of all non-null finite tournaments. A *tournament solution* is a function ϕ with domain $\mathcal{G}$, and with $\emptyset \neq \phi(G) \subseteq V(G)$ for each $G \in \mathcal{G}$. Let τ be the tournament solution defined inductively as follows. Assume that $\tau(G)$ is defined for all non-null proper subtournaments of G. Then $\tau(G)$ is the union of all minimal τ-retentive subsets of $V(G)$. (We see that $\tau(G)$ is nonempty, since $V(G)$ is τ-retentive.) $\tau(G)$ is called the *tournament equilibrium set*.

In 1990, motivated by applications in the social sciences, Thomas Schwartz [9] proposed the following conjecture.

1.1 (Schwartz' Conjecture.) *In every non-null tournament there is a unique minimal τ-retentive set.*

Schwartz' conjecture has been extensively studied, for instance in papers by Brandt [2], Brandt et al. [3], Dutta [4], Houy [6], and Laffond et al. [7] (we refer to Laslier [8] for further background). For instance, it is known that Schwartz' conjecture is equivalent to τ having any one of several desirable properties of tournament solutions, including monotonicity, independence of unchosen alternatives, and the "strong superset property".

In this paper, we give a counterexample to Schwartz' conjecture (with about 10^{130} vertices). Indeed, we give a series of weakenings of Schwartz' conjecture, and disprove the weakest.

2 Results

A subset X of the vertex set of a tournament G is *transitive* if it can be ordered $X = \{x_1, \dots, x_n\}$ such that $x_i x_j$ is an edge for all i, j with $1 \leq i < j \leq n$; and if so, x_1 is the *source* of X. For $G \in \mathcal{G}$, let $\beta(G)$ be the set of all vertices v of G such that v is the source of some maximal transitive subset of $V(G)$. Then β is a tournament solution. (This is called the *Banks set*, after J. S. Banks who first studied it [1].)

We need the following lemma of Schwartz [9], and we give the proof for the reader's convenience.

2.1 *$\tau(G) \subseteq \beta(G)$, and every β-retentive subset of $V(G)$ is τ-retentive, for every tournament G.*

Proof. We prove the first assertion by induction on $|V(G)|$. Let $x \in \tau(G)$; we must show that $x \in \beta(G)$. If $N^-(x) = \emptyset$, then x belongs to $\beta(G)$ as required, so we may assume that $N^-(x)$ is nonempty. Consequently $\tau(G|N^-(x))$ is nonempty; choose $w \in \tau(G|N^-(x))$. Let A be a minimal τ-retentive set containing x. It follows that $w \in A$, and so $A \setminus \{x\}$ is nonempty. From the minimality of A, it follows that $A \setminus \{x\}$ is not τ-retentive, and so there exists $y \in A \setminus \{x\}$ such that $x \in \tau(G|N^-(y))$.

From the inductive hypothesis, $\tau(G|N^-(y)) \subseteq \beta(G|N^-(y))$, and so there is a maximal transitive subset X_0 of $N^-(y)$ with source x. Thus $X_0 \cup \{y\}$ is transitive; let X be a maximal transitive subset of $V(G)$ including $X_0 \cup \{y\}$. It follows from the maximality of X_0 that no vertex of $X \setminus X_0$ belongs to $N^-(y)$, and so every vertex in $X \setminus X_0$ different from y is an out-neighbour of y and hence of x. Consequently x is the source of X, and so $x \in \beta(G)$. This proves the first assertion.

For the second assertion, let $A \subseteq V(G)$ be β-retentive, and let $a \in A$. From the first assertion, $\tau(G|N^-(a)) \subseteq \beta(G|N^-(a))$; and since A is β-retentive, $\beta(G|N^-(a)) \subseteq A$. Thus $\tau(G|N^-(a)) \subseteq A$, and so A is τ-retentive. This proves the second assertion, and so proves 2.1. ∎

Our first weakening of 1.1 is:

2.2 (First weakening.) *In every tournament G, every two β-retentive sets intersect.*

Proof that 1.1 implies 2.2. Let A_1, A_2 be β-retentive subsets of $V(G)$. By 2.1, A_1, A_2 are both τ-retentive, and hence both include a minimal τ-retentive set. Since there is only one such set by 1.1, and it is nonempty, it follows that $A_1 \cap A_2 \neq \emptyset$. This proves 2.2. ∎

If T is a subset of $V(G)$ where G is a tournament, we say that $v \in V(G) \setminus T$ *dominates* T if $vt \in E(G)$ for every $t \in T$, and if no such a vertex v exists, we say that T is *undominated in G*.

2.3 (Second weakening.) *Let (A, B) be a partition of the vertex set of a tournament G. Then one of A, B includes a transitive subset which is undominated in G.*

Proof that 2.2 implies 2.3. Assume that 2.2 holds, let G be a tournament and let (A, B) be a partition of $V(G)$. Take a second copy G' of G on a disjoint vertex set, and let (A', B') be the corresponding partition. Now make a tournament H from the disjoint union of G, G' as follows; for $v \in V(G)$ and $v' \in V(G')$, let $v'v \in E(H)$ if either $v \in A$ and $v' \in A'$, or $v \in B$ and $v' \in B'$; and otherwise let $vv' \in E(H)$.

We apply 2.2 to H, and deduce that one of $V(G), V(G')$ is not β-retentive in H, and from the symmetry we may assume that $V(G)$ is not β-retentive in H. Consequently, there exists $v \in V(G)$, and a maximal transitive subset T of $N_H^-(v)$, with source some $u \in V(G')$. From the symmetry we may assume that $v \in A$. It follows that $T \cap V(G') \subseteq A'$, since every vertex of $T \cap V(G')$ is an in-neighbour of v. In particular, $u \in A'$. Since u is the source of T, similarly every vertex of $T \cap V(G)$ belongs to A. Let $X = (T \cup \{v\}) \cap V(G)$. Suppose that some $x \in V(G) \setminus X$ dominates X. Since $T \cap V(G') \subseteq A'$, either $xy \in E(H)$ for all $y \in T \cap V(G')$, or $yx \in E(H)$ for all $y \in T \cap V(G')$, and in either case $T \cup \{x\}$ is a transitive subset of $N_H^-(v)$, contrary to the maximality of T. Thus X is undominated in G. This proves 2.3. ∎

Now, we give a counterexample to 2.3, which therefore provides a counterexample to all the previous conjectures. We need the following lemma, due to Erdős and Moser [5] (logarithms are to base two):

2.4 *For every integer $n \geq 2$ there is a tournament with n vertices and with no transitive subset of cardinality more than $3 \log n$.*

This is easily seen; a random tournament on n vertices has this property with positive probability, so such a tournament exists.

Now for the counterexample. Let $n \geq 2$ be an integer large enough that $n > (3 \log n)^3$, and let $k = 3 \log n$. By 2.4, there is a tournament G_1 with n vertices and with no transitive subset of cardinality more than k. Let $A = V(G_1)$. For each transitive subset $T \subseteq A$, let v_T be a new vertex, and let B be the set of all these new vertices. So $|B| \leq n^k$.

Let G_2 be a tournament with vertex set B and with no transitive subset of cardinality more than $3\log|B|$ (this exists by 2.4). Consequently G_2 has no transitive subset of cardinality more than $3\log(n^k) = k^2$. We construct a tournament G from the disjoint union of G_1 and G_2 as follows. For each $a \in A$ and each $b \in B$, let $ba \in E(G)$ if $a \in T$, where $T \subseteq A$ is the transitive subset of A with $b = v_T$, and let $ab \in E(G)$ otherwise. We observe:

- Every transitive subset T of A is dominated in G; because $v_T \in B$ dominates T.
- Every transitive subset Y of B is dominated in G. To see this, note first that $|Y| \leq k^2$, and since each vertex in Y has at most k out-neighbours in A, it follows that there are at most $k^3 < n$ vertices in A that are adjacent from some vertex in Y. Consequently some vertex in A dominates Y.

It follows that G, A, B do not satisfy 2.3.

In a recent paper, Brandt [2] gave a weaker version of Schwartz' conjecture. It is easy to see that Brandt's conjecture implies 2.2 and is therefore also false.

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