

# Solution of three problems of Cornuéjols

Maria Chudnovsky<sup>1</sup>  
Columbia University, New York, NY 10027

Paul Seymour<sup>2</sup>  
Princeton University, Princeton, NJ 08544

May 4, 2006; revised June 28, 2007

<sup>1</sup>This research was conducted while the author served as a Clay Mathematics Institute Research Fellow at Princeton University.

<sup>2</sup>Supported by ONR grant N00014-01-1-0608 and NSF grant DMS-0070912.

### **Abstract**

A graph is *balanced* if it is bipartite and every induced cycle has length divisible by four. In his book [6], Gérard Cornuéjols proposed a number of open questions, offering \$5000 for the solution of any of them. Here we solve three of them, about balanced graphs.

# 1 Introduction

A graph is said to be *balanced* if it is bipartite, and every induced cycle has length divisible by four. In his excellent book [6], Gérard Cornuéjols proposed eighteen conjectures, and offered \$5000 for a proof or counterexample for any of them. Two, concerned with perfect graphs, were settled by the solution of the strong perfect graph conjecture [2]. Now we are happy to report the solution of three more, concerned with balanced graphs; conjectures 9.23, 9.28 and 9.29 of [6]. We give a counterexample to the first two, and a proof of the third.

## 2 A counterexample to conjectures 9.23 and 9.28 of [6]

Conjecture 9.23 on page 98 of [6] asserts:

**2.1 Conjecture** (*Conforti, Cornuéjols and Rao [4]*) *If  $G$  is a balanced graph that is not totally unimodular, then  $G$  is either a  $W_{pq}$  or has a biclique cutset or a 2-join.*

We need to explain these terms. A graph is *totally unimodular* if it admits a bipartition  $(A, B)$  such that every square submatrix of the matrix  $(m_{ab} : a \in A, b \in B)$  has determinant  $\pm 1$  or 0, where  $m_{ab} = 1$  if  $a, b$  are adjacent and 0 otherwise. The graphs  $W_{pq}$  are a particular class of balanced graphs that we do not need to define here (they are essentially a special case of what we call crossmatchings below). A *biclique cutset* is a pair of disjoint nonempty sets  $A, B \subseteq V(G)$ , such that every vertex in  $A$  is adjacent to every vertex in  $B$ , and  $G \setminus (A \cup B)$  is disconnected. A graph  $G$  has a *2-join* if its vertex set can be partitioned into  $V_1, V_2$  in such a way that, for each  $i = 1, 2$ , there exist disjoint nonempty subsets  $A_i, B_i \subseteq V_i$ , such that

- every vertex of  $A_1$  is adjacent to every vertex of  $A_2$ ,
- every vertex of  $B_1$  is adjacent to every vertex of  $B_2$ ,
- there are no other adjacencies between  $V_1$  and  $V_2$ ,
- for  $i = 1, 2$   $V_i$  contains at least one path from  $A_i$  to  $B_i$ , and
- for  $i = 1, 2$ , if  $|A_i| = |B_i| = 1$  then the graph induced by  $V_i$  is not a chordless path between  $A_i$  and  $B_i$ .

(Remark: the definition of a 2-join in [6] contains a minor error, and the fifth condition above has been amended to fix this error.)

Two disjoint subsets  $A, B$  of the vertex set  $V(G)$  are said to be *matched* in  $G$  if  $A, B$  are stable sets in  $G$  and each member of  $A$  has a unique neighbour in  $B$  and vice versa. Here is a class of balanced graphs. Let  $p, q \geq 1$  be integers, and let  $C$  be a cycle with vertices

$$a_1, \dots, a_{4p-3}, b_1, \dots, b_{4q-3}, c_1, \dots, c_{4p-3}, d_1, \dots, d_{4q-3}, a_1$$

in order. Take  $p + q$  new vertices  $x_1, \dots, x_p, y_1, \dots, y_q$ , and add edges as follows:

- $x_i, y_j$  are adjacent for all  $i, j$  with  $1 \leq i \leq p$  and  $1 \leq j \leq q$

- for  $1 \leq i \leq p$ ,  $x_i$  and  $c_{4i-3}$  are adjacent
- for  $1 \leq j \leq q$ ,  $y_j$  and  $d_{4j-3}$  are adjacent
- $\{a_1, a_5, a_9, \dots, a_{4p-3}\}$  and  $\{x_1, \dots, x_p\}$  are matched
- $\{b_1, b_5, b_9, \dots, b_{4q-3}\}$  and  $\{y_1, \dots, y_q\}$  are matched

and there are no other edges. Let us call such a graph a *crossmatching*. It is easy to check that every crossmatching is balanced.

In particular, let  $p = 3, q = 2$ , and take a crossmatching such that the pairs

$$a_1x_1, a_9x_2, a_5x_3, b_1y_1, b_5y_2$$

are edges. This is balanced, and does not satisfy 2.1 (we leave it to the reader to check this). The same graph is also a counterexample to conjecture 9.28 of [6] (we do not state this in full, because it is just a strengthening of 2.1, and needs several further definitions).

### 3 Conjecture 9.29 of [6]

The goal of the remainder of this paper is to prove conjecture 9.29 on page 100 of [6], which asserts the following:

**3.1 Conjecture** (*Conforti, Cornuéjols, Kapoor and Vušković [3]*) *Every balanceable bipartite graph that is not regular has a double star cutset.*

We need first to define these terms. A graph is *eulerian* if every vertex has even degree (we do not require it to be connected). If  $G$  is a graph and  $w : E(G) \rightarrow \{-1, 1\}$  is a map, and  $H$  is a subgraph of  $G$ , we denote  $\sum_{e \in E(H)} w(e)$  by  $w(H)$ . A bipartite graph  $G$  is *balanceable* if there is a map  $w : E(G) \rightarrow \{-1, 1\}$  such that  $w(C)$  is a multiple of four for every induced cycle  $C$  of  $G$ . A bipartite graph  $G$  is *regular* if there is a map  $w : E(G) \rightarrow \{-1, 1\}$  such that  $w(H)$  is a multiple of four for every induced eulerian subgraph  $H$  of  $G$ . (This definition of “regular” is more convenient for us than the definition used in [6]; they are equivalent, because of Camion’s theorem [1].) Any such map  $w$  is called a *t.u. signing* of  $G$ .

A *cutset* in  $G$  is a subset  $X \subseteq V(G)$  such that  $G \setminus X$  has at least two components. (This is not quite the definition from [6], but the difference is not significant.) A *star cutset* in  $G$  is a cutset  $X$  such that some  $u \in X$  is adjacent to all other members of  $X$ . Then  $u$  is called a *centre* of the star cutset. A *double star cutset* in  $G$  is a cutset  $X$  such for some edge  $uv$  with  $u, v \in X$ , every member of  $X$  is adjacent to one of  $u, v$ ; and then  $uv$  is called a *centre* of the double star cutset.

A remark: the definition of “double star cutset” above is the standard definition used in many of Cornuéjols’ papers, such as [3]. However, in [6] the definition is different; he requires in addition that the subgraph induced on the cutset is a tree. This is presumably a mistake in [6], because with this definition it is easy to give counterexamples to 3.1; for instance, take the graph with ten vertices

$$a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, d_1, d_2,$$

and adjacency as follows:  $a_i b_i$  is an edge for  $i = 1, 2, 3$ ;  $\{b_1, b_2, b_3\}$  is complete to  $\{c_1, c_2\}$ ;  $\{c_1, c_2\}$  is complete to  $\{d_1, d_2\}$ ; and  $\{d_1, d_2\}$  is complete to  $\{a_1, a_2, a_3\}$ . Then this graph is a counterexample

to 3.1 using the definition of “double star cutset” from [6]. Henceforth then, we use the standard definition.

If  $v \in V(G)$  we denote the union of  $\{v\}$  and the set of neighbours of  $v$  by  $N[v]$ ; and if  $uv$  is an edge of  $G$  then  $N[uv]$  denotes  $N[u] \cup N[v]$ . If  $v, w \in V(G)$  are distinct, we say that  $v$  *dominates*  $w$  if every vertex adjacent to  $w$  is also adjacent to  $v$  (and hence  $v, w$  are nonadjacent). We observe:

**3.2** *Let  $G$  be a bipartite graph with  $|V(G)| \geq 5$  and  $E(G) \neq \emptyset$  and with no double star cutset. Then*

- *$G$  is connected*
- *$G$  has no star cutset*
- *no vertex of  $G$  dominates another*
- *for every edge  $uv$ , the subgraph induced on  $V(G) \setminus N[uv]$  is nonnull and connected, and every vertex in  $N[uv] \setminus \{u, v\}$  has a neighbour in  $V(G) \setminus N[uv]$ .*

**Proof.** Suppose that  $G$  is not connected, and let  $uv$  be an edge, chosen from the component  $C$  of  $G$  that has most vertices. Then either  $G$  has at least three components, or  $|V(C)| \geq 3$ , and in either case  $\{u, v\}$  is a double star cutset, a contradiction. Thus  $G$  is connected.

Suppose that  $X$  is a star cutset with centre  $u$ . If there exists  $v \in X \setminus \{u\}$ , then  $X$  is also a double star cutset with centre  $uv$ , a contradiction; so  $X = \{u\}$ . Let  $A_1$  be a component of  $G \setminus X$ , and let  $A_2 = V(G) \setminus (A_1 \cup X)$ ; thus  $A_2 \neq \emptyset$ . Since  $G$  is connected,  $u$  has neighbours  $v_i \in A_i$  for  $i = 1, 2$ . Since  $X \cup \{v_i\}$  is not a double star cutset, it follows that  $A_i = \{v_i\}$  for  $i = 1, 2$ , and so  $|V(G)| = 3$ , a contradiction. Thus  $G$  has no star cutset.

Now suppose that  $v$  dominates  $w$ . Let  $X$  be the union of  $\{v\}$  and the set of all neighbours of  $w$ . Since  $X$  is not a star cutset with centre  $v$ , it follows that  $X \cup \{w\} = V(G)$ . Let  $u$  be adjacent to  $w$ . Since  $\{u, v, w\}$  is not a star cutset with centre  $u$ , we deduce that  $|V(G)| \leq 4$ , a contradiction. Thus no vertex dominates another.

Finally, let  $uv$  be an edge. Suppose first that  $N[uv] = V(G)$ . Then  $u$  dominates every neighbour of  $v$  different from  $u$ , so by what we just proved,  $v$  has degree one, and similarly  $u$  has degree one, a contradiction. Thus  $N[uv] \neq V(G)$ . Since  $N[uv]$  is not a double star cutset, it follows that the subgraph induced on  $V(G) \setminus N[uv]$  is connected. Now let  $w \in N[uv]$  with  $w \neq u, v$ ; say  $w$  is adjacent to  $u$ . Since  $v$  does not dominate  $w$ , it follows that  $w$  has a neighbour in  $V(G) \setminus N[uv]$ . This completes the proof of 3.2. ■

## 4 Operations preserving regularity

In this section we discuss some lemmas stating that if we piece two regular graphs together in prescribed ways, then the graph we produce is also regular.

If  $X \subseteq V(G)$ , we denote the subgraph induced on  $X$  by  $G|X$ . Let  $G$  be a connected bipartite graph that admits a 2-join, and let  $V_i, A_i, B_i$  ( $i = 1, 2$ ) be as in the definition of a 2-join. Let  $G_1$  be the graph obtained from  $G|V_1$  by adding a path  $p_1-p_2-\dots-p_k$  of new vertices, where  $p_1$  is adjacent to every vertex in  $A_1$ ,  $p_k$  is adjacent to every vertex in  $B_1$ , and there are no other edges between  $V_1$  and  $\{p_1, \dots, p_k\}$ , and  $k \geq 3$ , and  $k$  is chosen so that  $G_1$  is bipartite. Define  $G_2$  similarly, adding a path to  $G|V_2$ . We call  $G_1, G_2$  a pair of *blocks* of the 2-join. We need first:

**4.1** Let  $G$  be a connected bipartite graph that admits a 2-join, and let  $V_i, A_i, B_i$  ( $i = 1, 2$ ) be as before, and let  $G_1, G_2$  be a pair of blocks of the 2-join. If  $G_1, G_2$  are both regular then so is  $G$ .

This result is well-known, and related to the fact that 3-sums in matroid theory preserve matroid regularity. Another closely related result is that if  $G_1, G_2$  are totally unimodular then so is  $G$ , and that is proved in lemma 2.3 of [4]. The proof given there can easily be adapted to prove 4.1, and we omit the details.

Let  $G$  be a bipartite graph. A partition  $(V_1, V_2)$  of  $V(G)$  is a 6-join if  $|V_1|, |V_2| \geq 4$  and there exist disjoint nonempty subsets  $A_1, A_3, A_5 \subseteq V_1$  and  $A_2, A_4, A_6 \subseteq V_2$ , satisfying:

- for  $i = 1, \dots, 6$   $A_i$  is complete to  $A_{i+1}$ , where  $A_7$  means  $A_1$
- there are no other edges between  $V_1$  and  $V_2$ .

In this case, let  $G_1$  be obtained from  $G|V_1$  by adding three new vertices  $b_2, b_4, b_6$ , where for  $i = 2, 4, 6$ ,  $b_i$  is adjacent to every vertex of  $A_{i-1} \cup A_{i+1}$  (reading subscripts modulo 6), and there are no other new edges. Similarly, define  $G_2$  by adding four vertices  $b_1, b_3, b_5$  to  $G|V_2$ , where for  $i = 1, 3, 5$ ,  $b_i$  is adjacent to every vertex of  $A_{i-1} \cup A_{i+1}$ , and there are no other new edges. We call  $G_1, G_2$  a pair of *blocks* of the 6-join. We need a result analogous to 4.1 for 6-joins, but first a lemma:

**4.2** Let  $a_1, \dots, a_6$  be integers such that  $a_1, a_3, a_5$  are all even or all odd, and  $a_2, a_4, a_6$  are all even or all odd. Then

$$a_1a_2 - a_2a_3 + a_3a_4 - a_4a_5 + a_5a_6 - a_6a_1$$

is a multiple of four.

**Proof.** Changing the value of  $a_1$  by two does not change the value of the expression modulo four, since  $a_1$  multiplies  $a_2 - a_6$ , which is even. Thus we may assume that  $a_1 \in \{0, 1\}$ , and similarly  $a_2, \dots, a_6 \in \{0, 1\}$ . Since  $a_1, a_3, a_5$  are all even or all odd, they are all equal, and so are  $a_2, a_4, a_6$ ; and hence the expression is zero. ■

The analogue of 4.1 is:

**4.3** Let  $(V_1, V_2)$  be a 6-join in a connected bipartite graph  $G$ , and let  $G_1, G_2$  be a pair of blocks of this 6-join. If  $G_1, G_2$  are both regular then so is  $G$ .

**Proof.** Let  $A_1, \dots, A_6$  be as in the definition of a 6-join, and let  $b_1, \dots, b_6$  be the new vertices of  $G_1, G_2$  as above. (Throughout this proof we read subscripts modulo 6.) Let  $a_i \in A_i$  for  $i = 1, 3, 5$ . Let  $w_1$  be a t.u. signing of  $G_1$ . If  $Y \subseteq V(G_1)$  and we replace  $w_1(e)$  by  $-w_1(e)$  for every edge  $e$  of  $G_1$  with exactly one end in  $Y$ , we obtain another t.u. signing of  $G_1$ ; and we may therefore choose  $w_1$  such that:

- for  $j = 2, 4, 6$ ,  $w_1(e) = 1$  for every edge  $e$  incident with  $b_j$  and a vertex of  $A_{j-1}$ , and
- for  $j = 2, 4$ ,  $w_1(e) = -1$  for the edge  $e = b_j a_{j+1}$ .

Since the subgraph of  $G_1$  induced on  $\{a_1, b_2, a_3, b_4, a_5, b_6\}$  is eulerian and  $w_1$  is a t.u. signing, it follows that also  $w_1(e) = -1$  for the edge  $e = b_6 a_1$ . Also, for each choice of  $a'_1 \in A_1$ , since the subgraph induced on  $\{a'_1, b_2, a_3, b_4, a_5, b_6\}$  is eulerian, it follows that  $w_1(e) = -1$  for the edge  $e = b_6 a'_1$ . Similarly for  $j = 2, 4$  and for each  $a'_{j+1} \in A_{j+1}$ , it follows that  $w_1(e) = -1$  for the edge  $e = b_j a'_{j+1}$ . Thus in summary we have:

- for  $j = 2, 4, 6$ ,  $w_1(e) = 1$  for every edge  $e$  incident with  $b_j$  and a vertex of  $A_{j-1}$ , and
- for  $j = 2, 4, 6$ ,  $w_1(e) = -1$  for every edge  $e$  incident with  $b_j$  and a vertex of  $A_{j+1}$ .

Similarly we may choose a t.u. signing  $w_2$  of  $G_2$  such that:

- for  $j = 1, 3, 5$ ,  $w_2(e) = 1$  for every edge  $e$  incident with  $b_j$  and a vertex of  $A_{j+1}$ , and
- for  $j = 1, 3, 5$ ,  $w_2(e) = -1$  for every edge  $e$  incident with  $b_j$  and a vertex of  $A_{j-1}$ .

For each edge  $e$  of  $G$ , either  $e \in E(G|V_i)$  for some  $i \in \{1, 2\}$ , or  $e = uv$  where  $u \in A_i$  and  $v \in A_{i+1}$  for some  $i \in \{1, \dots, 6\}$ . In the first case let  $w(e) = w_i(e)$ , and in the second case let  $w(e) = 1$  if  $i$  is odd and  $-1$  if  $i$  is even. We claim that  $w$  is a t.u. signing of  $G$ . For let  $X \subseteq V(G)$  such that  $G|X$  is eulerian.

Let  $x_i = |X \cap A_i|$  for  $1 \leq i \leq 6$ . We say that for  $1 \leq i \leq 6$ ,  $x_i$  is *exceptional* if  $x_i + x_{i+2}, x_i + x_{i-2}$  are both odd (and therefore  $x_{i+2} + x_{i-2}$  is even). Thus at most one of  $x_1, x_3, x_5$  is exceptional, and at most one of  $x_2, x_4, x_6$ ; and if there is one of each, say  $x_i$  and  $x_j$ , we claim that  $j \neq i + 1, i - 1$ . To see the last assertion, suppose that  $x_1, x_2$  are exceptional, say. Thus  $x_1 + x_3, x_2 + x_4$  are odd, and  $x_4 + x_6$  is even; and so

$$(x_1 + x_3)(x_2 + x_4) + (x_1 + x_5)(x_4 + x_6)$$

is odd. But this equals

$$x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_6 + x_6x_1$$

modulo 2, and so the total number of edges between  $V_1 \cap X$  and  $V_2 \cap X$  is odd, contradicting that  $G|X$  is eulerian. This proves our assertion. Consequently, from the symmetry we may assume that  $x_1, x_5, x_2, x_4$  are not exceptional, that is,  $x_1 + x_5$  and  $x_2 + x_4$  are both even.

Let  $X_1 \subseteq V(G_1)$  be defined as follows. Let  $X_1 \cap V_1 = X \cap V_1$ , let  $b_2, b_4 \notin X_1$ , and let  $b_6 \in X_1$  if and only if  $x_6$  is exceptional. Similarly, let  $X_2 \subseteq V(G_2)$  where  $X_2 \cap V_2 = X \cap V_2$ ,  $b_1, b_5 \notin X_2$ , and  $b_3 \in X_2$  if and only if  $x_3$  is exceptional.

(1)  $G_1|X_1$  is eulerian.

For let  $v \in X_1$ ; we must check that its degree  $d_1$  say in  $G_1|X_1$  is even. If  $v = b_6$  then its degree is  $x_1 + x_5$ , which is even, so we may assume that  $v \in X$ ; let its degree in  $G|X$  be  $d$ . Thus  $d$  is even. If  $v \in V_1 \setminus (A_1 \cup A_3 \cup A_5)$  then  $d_1 = d$  and therefore is even; if  $v \in A_3$  then  $d_1 = d - (x_2 + x_4)$ , and therefore is even; if  $v \in A_1$  then  $d_1 = d - (x_2 + x_6)$  if  $b_6 \notin X_1$  (that is, if  $x_2 + x_6$  is even), and  $d_1 = d - (x_2 + x_6) + 1$  if  $b_6 \in X_1$  (that is, if  $x_2 + x_6$  is odd), and in either case  $d_1$  is even; and similarly  $d_1$  is even if  $v \in A_5$ . This proves (1).

Define  $X_2 \subseteq V(G_2)$  similarly. Then  $w_1(G_1|X_1)$  and  $w_2(G_2|X_2)$  are multiples of four, so let us examine  $w(G|X) - w_1(G_1|X_1) - w_2(G_2|X_2)$ . First,

$$w(G|X) = w(G|(X \cap V_1)) + w(G|(X \cap V_2)) + \sum_{i=1,3,5} x_i(x_{i+1} - x_{i-1}).$$

Let  $y_6 = 1$  if  $b_6 \in X_1$ , and  $y_6 = 0$  otherwise, and define  $y_3$  similarly; then

$$w_1(G_1|X_1) = w(G|(X \cap V_1)) + y_6(x_5 - x_1)$$

and

$$w_2(G_2|X_2) = w(G|(X \cap V_2)) + y_3(x_4 - x_2).$$

Thus  $w(G|X) - w_1(G_1|X_1) - w_2(G_2|X_2) = R$ , where by definition

$$R = x_1x_2 - x_2(x_3 - y_3) + (x_3 - y_3)x_4 - x_4x_5 + x_5(x_6 - y_6) - (x_6 - y_6)x_1.$$

But since  $x_1, x_5$  are not exceptional, the definition of  $y_3$  ensures that  $x_1, x_3 - y_3, x_5$  are all odd or all even; and similarly  $x_2, x_4, x_6 - y_6$  are all odd or all even. By 4.2, it follows that  $R$  is a multiple of four. Consequently  $w(G|X)$  is a multiple of four, and so  $w$  is a t.u. signing of  $G$ . This proves 4.3. ■

Third, we need the following. Let us say distinct vertices of  $G$  are *twins* if they have the same neighbour sets (and consequently are nonadjacent to each other).

**4.4** *Let  $u, v$  be twins in  $G$ , and suppose that  $G \setminus \{v\}$  is regular. Then  $G$  is regular.*

**Proof.** Let  $w$  be a t.u. signing of  $G \setminus \{v\}$ , and extend the domain of  $w$  to  $E(G)$  by defining  $w(vx) = w(ux)$  for each edge  $vx$  of  $G$ . We claim that  $w$  is a t.u. signing of  $G$ . For let  $X \subseteq V(G)$  such that  $G|X$  is eulerian. If  $v \notin X$  then  $w(G|X)$  is a multiple of four since  $w$  is a t.u. signing of  $G \setminus \{v\}$ , and if  $u \notin X$  the same conclusion follows from the symmetry between  $u, v$ . Thus we may assume that  $u, v \in X$ . Let  $X' = X \setminus \{u, v\}$ ; then  $G|X'$  is eulerian, and so  $w(G|X')$  is a multiple of four. But  $w(G|X) = w(G|X') + 2z$ , where  $z$  is the sum of  $w(ux)$  over all edges  $ux$  with  $x \in X'$ ; and since  $G|X$  is eulerian, it follows that  $z$  is even. Hence  $w(G|X)$  is a multiple of four, and so  $w$  is a t.u. signing of  $G$ , and therefore  $G$  is regular. This proves 4.4. ■

## 5 Some 6-join lemmas

A 6-join  $(V_1, V_2)$  in a bipartite graph  $G$  is said to be *skeletal* if  $|V_2| = 7$ , and  $V_2$  can be numbered as  $\{a_2, a_4, a_6, c_2, c_4, c_6, c_8\}$  such that

- $c_8$  has degree three in  $G$ , with neighbours  $c_2, c_4, c_6$
- for  $i = 2, 4, 6$ ,  $c_i$  has degree two in  $G$ , with neighbours  $a_i, c_8$
- there are disjoint nonempty subsets  $A_1, A_3, A_5 \subseteq V_1$  such that for  $i = 2, 4, 6$ ,  $a_i$  is complete to  $A_{i-1} \cup A_{i+1}$  (where  $A_7$  means  $A_1$ ) and there are no other edges between  $V_1$  and  $V_2$ .

An induced subgraph of  $G$  that is a cycle is called a *hole* in  $G$ , and a hole of length  $k$  is a *k-hole*. If  $G$  is a balanceable bipartite graph, an induced subgraph  $H$  is said to be an *irregularity* in  $G$  if  $H$  is not regular, and every induced subgraph of  $G$  with fewer vertices than  $H$  is regular. We need:

**5.1** *Let  $G$  be balanceable, and let  $(V_1, V_2)$  be a skeletal 6-join. Let  $V_2 = \{a_2, a_4, a_6, c_2, c_4, c_6, c_8\}$  as in the definition of “skeletal”. Let  $H$  be an irregularity in  $G$ ; then  $c_2, c_4, c_6, c_8 \notin V(H)$ .*

**Proof.** Let  $A_1, \dots, A_6$  be as in the definition of 6-join, where  $A_i = \{a_i\}$  for  $i = 2, 4, 6$ . Let  $w : E(G) \rightarrow \{-1, 1\}$  such that  $w(C)$  is a multiple of four for every induced cycle  $C$  of  $G$ . As usual we may assume that  $w(a_i a_{i+1}) = 1$  for  $i = 1, 3, 5$  and all  $a_i \in A_i$ , and  $w(a_i a_{i-1}) = -1$  for  $i = 1, 3, 5$  and all  $a_i \in A_i$ , where  $A_0$  means  $A_6$ . Now  $w$  induces a t.u. signing of  $J$  for every regular induced



subgraph  $J$  of  $G$ . Since  $H$  is an irregularity in  $G$ , it follows that  $H$  is eulerian; and since  $w$  induces a t.u. signing of every proper induced subgraph of  $H$  and not of  $H$  itself, we deduce that  $w(H)$  is not a multiple of four. Suppose that one of  $c_2, c_4, c_6, c_8 \in V(H)$ . Hence we may assume that  $a_2, c_2, c_8, c_4, a_4 \in V(H)$  and  $c_6 \notin V(H)$ . For  $i = 1, 3, 5$ , let  $x_i = |V(H) \cap A_i|$ .

Let  $Y = \{a_2, c_2, c_8, c_4, a_4\}$ ; then  $w(G|Y)$  is a multiple of four, since  $w(C)$  is a multiple of four where  $C$  is the hole  $c_8-c_2-a_2-a_3-a_4-c_4-c_8$  for some  $a_3 \in A_3$ . Suppose first that  $a_6 \in V(H)$ , and let  $X = V(H) \cap V_1$ . Then  $G|X$  is eulerian, and therefore regular from the minimality of  $|V(H)|$ , and so  $w(G|X)$  is a multiple of four. But  $w(H) = w(G|X) + w(G|Y)$ , and so  $w(H)$  is a multiple of four, a contradiction. Thus  $a_6 \notin V(H)$ . Let  $X = (V(H) \cap V_1) \cup \{a_6\}$ . Then again  $G|X$  is eulerian, and has fewer vertices than  $H$ , and so  $w(G|X)$  is a multiple of four. But

$$w(H) = w(G|X) + w(G|Y) - 2x_5 + 2x_1,$$

and  $x_5 - x_1$  is even since  $a_6$  has even degree in  $G|X$ . It follows again that  $w(H)$  is a multiple of four, a contradiction. This proves 5.1.  $\blacksquare$

A 6-join  $(V_1, V_2)$  in a bipartite graph  $G$  is said to be *internal* if  $|V_1|, |V_2| \geq 8$ . We need several results saying that balanceable graphs containing certain induced subgraphs admit either double star cutsets or internal 6-joins.

If  $X, Y \subseteq V(G)$ , we say that  $X$  is *anticomplete* to  $Y$  if  $X \cap Y = \emptyset$  and there is no edge  $xy$  with  $x \in X$  and  $y \in Y$ . The proof of theorem 6.3 of [3] also proves the following:

**5.2** *Let  $G$  be balanceable, and let  $a_1-b_2-a_3-b_1-a_2-b_3-a_1$  be a 6-hole  $C$  in  $G$ . Suppose that there are subsets  $A, B \subseteq V(G)$  with the following properties:*

- $A, B, V(C)$  are pairwise disjoint, and  $G|A, G|B$  are connected;
- $a_1, a_2, a_3$  have neighbours in  $A$ , and  $b_1, b_2, b_3$  do not;
- $b_1, b_2, b_3$  have neighbours in  $B$ , and  $a_1, a_2, a_3$  do not; and
- $A$  is anticomplete to  $B$ .

*Then either  $G$  admits a double star cutset, or  $G$  admits a 6-join  $(V_1, V_2)$  with  $A \cup \{a_1, a_2, a_3\} \subseteq V_1$  and  $B \cup \{b_1, b_2, b_3\} \subseteq V_2$ .*

## 6 Big dominoes

A triple  $(ab, C_1, C_2)$  is a *domino* in  $G$  if  $C_1, C_2$  are holes in  $G$ , and  $ab$  is an edge, and  $V(C_1) \cap V(C_2) = \{a, b\}$ , and  $V(C_1) \setminus \{a, b\}$  is anticomplete to  $V(C_2) \setminus \{a, b\}$ . An *odd theta* is a graph consisting of two nonadjacent vertices  $u, v$  and three odd length paths between  $u, v$ , such that the interiors of these three paths are pairwise disjoint and pairwise anticomplete. An *odd wheel* is a graph consisting of a cycle  $C$  and another vertex  $v \notin V(C)$ , such that  $v$  has an odd number, at least three, of neighbours in  $V(C)$ . We need the following easy and well-known lemma (we omit the proof).

**6.1** *If  $G$  is a balanceable bipartite graph, then no induced subgraph of  $G$  is an odd theta or an odd wheel.*

Let us say two vertices  $u, v$  in the same component of a bipartite graph  $G$  have the *same biparity* if every path between them has even length, and otherwise they have *opposite biparity* (and therefore every path between them has odd length). We begin with a lemma.

**6.2** *Let  $(a_0b_0, C_1, C_2)$  be a domino in a balanceable graph  $G$ , such that  $C_1, C_2$  both have length at least six. For  $i = 1, 2$ , let  $P_i = C_i \setminus \{a_0, b_0\}$ ; then  $P_i$  is a chordless path of length at least three with ends  $a_i, b_i$  say, where  $a_i$  is adjacent to  $b_0$  and  $b_i$  to  $a_0$ . Suppose that  $G$  does not admit a double star cutset, and does not admit a 6-join  $(V_1, V_2)$  such that  $V(C_i) \setminus \{a_0, b_0\} \subseteq V_i$  for  $i = 1, 2$ , and  $V_1, V_2$  each contain exactly one of  $a_0, b_0$ . Let  $q_1 \cdots q_n$  be a chordless path such that*

- *for  $1 \leq i \leq n$ ,  $q_i$  has a neighbour in the interior of  $P_1$  if and only if  $i = 1$ , and  $q_i$  has a neighbour in the interior of  $P_2$  if and only if  $i = n$ , and*
- *$q_1, \dots, q_n$  are all nonadjacent to both  $a_0, b_0$ .*

*Then either*

- (a)  *$a_1$  is adjacent to one of  $q_2, \dots, q_n$ , and  $a_2$  is adjacent to one of  $q_1, \dots, q_{n-1}$ , and  $b_1$  is nonadjacent to  $q_2, \dots, q_n$ , and  $b_2$  is nonadjacent to  $q_1, \dots, q_{n-1}$ , or*
- (b)  *$b_1$  is adjacent to one of  $q_2, \dots, q_n$ , and  $b_2$  is adjacent to one of  $q_1, \dots, q_{n-1}$ , and  $a_1$  is nonadjacent to  $q_2, \dots, q_n$ , and  $a_2$  is nonadjacent to  $q_1, \dots, q_{n-1}$ .*

*Moreover, if (a) holds then either*

- *$q_1$  is adjacent to both  $a_1, a_2$ , and  $a_2$  is nonadjacent to  $q_2, \dots, q_{n-1}$ , or*
- *$q_n$  is adjacent to both  $a_1, a_2$ , and  $a_1$  is nonadjacent to  $q_2, \dots, q_{n-1}$ .*

*An analogous statement holds if (b) is true.*

**Proof.** Let us say  $a_1$  or  $b_1$  is *active* if it is adjacent to one of  $q_2, \dots, q_n$ , and  $a_2$  or  $b_2$  is active if it is adjacent to one of  $q_1, \dots, q_{n-1}$ .

(1) *At least one of  $a_1, b_1, a_2, b_2$  is active, and so  $n \geq 2$ .*

For suppose not. We may assume that  $q_1, a_0$  have opposite biparity. If  $q_1$  has more than one neighbour in  $P_1$ , there are three paths between  $q_1, a_0$  forming an odd theta, namely two with interior in  $V(C_1)$  and the third with interior in  $\{q_2, \dots, q_n\} \cup (V(P_2) \setminus \{a_2\})$ , a contradiction. Thus  $q_1$  has a unique neighbour,  $p_1$  say, in  $P_1$ . Since  $q_1$  has a neighbour in the interior of  $P_1$  it follows that  $p_1 \neq a_1$ ; and there are three paths between  $p_1, b_0$  forming an odd theta, namely two with interior in  $V(C_1)$  and the third with interior in  $\{q_2, \dots, q_n\} \cup (V(P_2) \setminus \{b_2\})$ , a contradiction. This proves (1).

(2) *Not both  $a_1, b_1$  are active.*

For if they are, there are three paths between  $a_1, b_1$  forming an odd theta, namely  $P_1$ ,  $a_1-b_0-a_0-b_1$  and a path with interior in  $\{q_2, \dots, q_n\}$ , a contradiction.

(3) *Not both  $b_1, b_2$  have neighbours in  $\{q_2, \dots, q_{n-1}\}$ .*

For if they do, let  $R$  be a chordless path between  $b_1, b_2$  with interior in  $\{q_2, \dots, q_{n-1}\}$ . Then  $a_1, a_2$  are both anticomplete to  $V(R)$ , by (2), and so

$$b_1-P_1-a_1-b_0-a_2-P_2-b_2-R-b_1$$

is a hole and  $a_0$  has three neighbours in it, a contradiction. This proves (3).

In view of (1) and (2) we may assume that  $b_1$  is active and  $a_1$  is not. Let  $j \in \{2, \dots, n\}$  be maximum such that  $q_j, b_1$  are adjacent.

(4) *One of  $a_2, b_2$  is adjacent to one of  $q_1, \dots, q_{j-1}$ .*

For suppose not. Let  $R$  be a chordless path between  $b_0$  and  $q_j$  with interior in  $(V(P_1) \setminus \{b_1\}) \cup \{q_1, \dots, q_{j-1}\}$ . Let  $S, T$  be chordless paths between  $q_j, b_0$  with interior in  $\{q_{j+1}, \dots, q_n\} \cup (V(P_2) \setminus \{b_2\})$  and in  $\{q_{j+1}, \dots, q_n, a_0\} \cup (V(P_2) \setminus \{a_2\})$  respectively. Then  $b_0-R-q_j-S-b_0$  and  $b_0-R-q_j-T-b_0$  are holes, and  $b_1$  has one more neighbour in the second hole than in the first; and so by 6.1,  $b_1$  has exactly one neighbour in  $R$ , namely  $q_j$ . But then there are three paths between  $q_j$  and  $b_0$  that form an odd theta, namely  $q_j-R-b_0$ ,  $q_j-S-b_0$  and  $q_j-b_1-a_0-b_0$ , a contradiction. This proves (4).

(5)  *$a_2$  is not active.*

For suppose that  $a_2$  is active. Then  $b_2$  is not, by (2); and so by (4),  $a_2$  is adjacent to one of  $q_1, \dots, q_{j-1}$ . Let  $i \in \{1, \dots, j-1\}$  be minimum such that  $a_2, q_i$  are adjacent. If  $j > i+1$ , there are three paths between  $b_1, a_2$  forming an odd theta, namely one with interior in  $V(P_1) \cup \{q_1, \dots, q_i\}$ , one with interior in  $\{q_j, \dots, q_n\} \cup V(P_2)$ , and  $b_1-a_0-b_0-a_2$ . Thus  $j = i+1$ ; but then the 6-hole  $a_0-b_1-q_j-q_i-a_2-b_0-a_0$ , and the two subsets  $(V(P_2) \setminus \{a_2\}) \cup \{q_{j+1}, \dots, q_n\}$  and  $(V(P_1) \setminus \{b_1\}) \cup \{q_1, \dots, q_{i-1}\}$  satisfy the hypotheses of 5.2, and consequently there is either a double star cutset or a 6-join that violates the hypothesis of the theorem. This proves (5).

From (4) and (5), it follows that  $b_2$  is adjacent to one of  $q_1, \dots, q_{j-1}$ , and so there is symmetry between  $b_1$  and  $b_2$ . By (3) one of  $b_1, b_2$  is nonadjacent to  $q_2, \dots, q_{n-1}$ , so by exchanging  $C_1, C_2$  if necessary, we may assume that  $b_2$  is nonadjacent to  $q_2, \dots, q_{n-1}$ . Consequently  $b_2$  is adjacent to  $q_1$ . (Note that possibly  $b_2$  is adjacent to  $q_n$ , and possibly  $j = n$ .)

(6)  *$b_1$  is adjacent to  $q_1$ .*

For suppose not. If  $q_1$  has at least two neighbours in  $P_1$ , there are three paths between  $q_1$  and  $b_1$  forming an odd theta, namely two with interior in  $V(C_1)$  and one with interior in  $\{q_2, \dots, q_j\}$ , a contradiction. If  $q_1$  has a unique neighbour  $p_1$  in  $P_1$ , then  $p_1, a_0$  are nonadjacent and there are three paths between  $p_1, a_0$  forming an odd theta, namely two paths of  $C_1$  and a path with interior in  $\{q_1, \dots, q_j\}$ , again a contradiction. This proves (6), and completes the proof of 6.2.  $\blacksquare$

The lemma is used for the following.

**6.3** Let  $(a_0b_0, C_1, C_2)$  be a domino in a balanceable graph  $G$ , such that  $C_1, C_2$  both have length at least six. Then either  $G$  admits a double star cutset, or  $G$  admits a 6-join  $(V_1, V_2)$  such that  $V(C_i) \setminus \{a_0, b_0\} \subseteq V_i$  for  $i = 1, 2$ , and  $V_1, V_2$  each contain exactly one of  $a_0, b_0$ .

**Proof.**

For  $i = 1, 2$ , let  $P_i = C_i \setminus \{a_0, b_0\}$ ; then  $P_i$  is a chordless path of length at least three with ends  $a_i, b_i$  say, where  $a_i$  is adjacent to  $b_0$  and  $b_i$  to  $a_0$ . We assume that  $G$  does not admit a double star cutset and does not admit a 6-join satisfying the theorem. Hence there is a chordless path  $q_1 \cdots q_n$  as in 6.2, and again from 6.2 we may assume that  $q_1$  is adjacent to  $b_1, b_2$ , and  $b_1$  is adjacent to one of  $q_2, \dots, q_n$ , and  $b_2$  is nonadjacent to  $q_2, \dots, q_{n-1}$ , and  $a_1$  is nonadjacent to  $q_2, \dots, q_n$ , and  $a_2$  is nonadjacent to  $q_1, \dots, q_{n-1}$ .

Since  $b_1$  is adjacent to  $q_1$  and to one of  $q_2, \dots, q_n$ , it follows that  $n \geq 3$ . Let  $p_2$  be the neighbour of  $b_2$  in  $P_2$ . Since  $G$  does not admit a double star cutset, there is a chordless path  $R$  between  $q_2$  and some vertex  $r$  such that  $r$  has a neighbour in  $V(C_1) \cup V(C_2) \setminus \{a_0, b_1, b_2, p_2\}$ , and  $a_0, b_2$  are both nonadjacent to every vertex of  $R$ . By choosing  $R$  minimal, it follows that  $r$  is the only vertex of  $R$  with a neighbour in  $V(C_1) \cup V(C_2) \setminus \{a_0, b_1, b_2, p_2\}$ . (However,  $b_1, p_2$  may have neighbours in  $V(R) \setminus \{r\}$ .)

(1)  $r$  is adjacent to  $b_0$ .

For suppose not. Let  $\mathbf{S}_1$  be the statement that  $r$  has a neighbour in  $V(P_1) \setminus \{b_1\}$ , and  $\mathbf{S}_2$  the statement that some vertex of  $R$  has a neighbour in  $V(P_2) \setminus \{b_2\}$  (in other words, either  $r$  has a neighbour in  $V(P_2) \setminus \{b_2\}$  or some vertex of  $R$  is adjacent to  $p_2$ ). Thus at least one of  $\mathbf{S}_1, \mathbf{S}_2$  holds. We claim that if  $\mathbf{S}_1$  holds then  $r$  has a neighbour in  $V(P_1) \setminus \{a_1, b_1\}$ . For suppose that  $a_1$  is the unique neighbour of  $r$  in  $V(P_1) \setminus \{b_1\}$ . Then there are three paths between  $b_1, a_1$  that form an odd theta, namely  $b_1-a_0-b_0-a_1$ , a path with interior in  $\{q_2, \dots, q_n\} \cup V(R)$ , and  $P_1$ , a contradiction.

We claim also that if  $\mathbf{S}_2$  holds then some vertex of  $R$  has a neighbour in  $V(P_2) \setminus \{a_2, b_2\}$ . For suppose that  $a_2$  is the unique neighbour of  $r$  in  $V(P_2) \setminus \{b_2\}$ , and there are no other edges between  $V(R)$  and  $V(P_2) \setminus \{b_2\}$ . Then there are three paths between  $b_2, a_2$  that form an odd theta, namely  $b_2-a_0-b_0-a_2$ , a path with interior in  $\{q_1\} \cup V(R)$ , and  $P_2$ , a contradiction.

Now suppose that both  $\mathbf{S}_1$  and  $\mathbf{S}_2$  hold. Then there is a subpath  $T$  of  $R$  that satisfies the initial hypotheses for the path  $q_1 \cdots q_n$ . Moreover, no vertex of  $T$  is adjacent to both  $b_1, b_2$ , and no vertex of  $V(T) \setminus \{r\}$  is adjacent to  $a_1$  or to  $a_2$ , contrary to 6.2. This proves that not both  $\mathbf{S}_1, \mathbf{S}_2$  hold.

Next suppose that  $\mathbf{S}_1$  holds, and hence  $\mathbf{S}_2$  is false. Then  $V(R) \cup \{q_2, \dots, q_n\}$  includes the vertex set of a minimal path  $T$  between  $r$  and some vertex  $t$  that has a neighbour in  $V(P_2) \setminus \{a_2, b_2\}$ . But  $a_0, b_0$  have no neighbours in this path, and  $a_2, b_2$  have no neighbours in this path different from  $t$  (since  $b_2$  is nonadjacent to  $q_2, \dots, q_{n-1}$ ), contrary to 6.2.

Next suppose that  $\mathbf{S}_2$  holds, and so  $\mathbf{S}_1$  is false. Since  $r$  has a neighbour in

$$V(C_1) \cup V(C_2) \setminus \{a_0, b_1, b_2, p_2\},$$

it follows that  $r$  has a neighbour in  $V(P_2) \setminus \{b_2\}$ . Let  $T$  be a chordless path between  $q_1$  and some vertex  $t$  that has a neighbour in  $V(P_2) \setminus \{a_2, b_2\}$ , with  $V(T) \subseteq V(R) \cup \{q_1\}$ , and choose  $T$  minimal. Then no vertex of  $T$  is adjacent to  $a_0$  or to  $b_0$ , and  $a_2, b_2$  both have no neighbours in  $V(T) \setminus \{t\}$ , contrary to 6.2. This proves (1).

Let  $T$  be a chordless path between  $q_1$  and  $r$  with interior in  $V(R)$ . If  $r$  has no neighbour in  $V(P_1)$ , then there are three paths between  $q_1, b_0$  forming an odd theta, namely  $q_1-T-r-b_0$ , a path with interior in  $V(P_1)$ , and  $q_1-b_2-a_0-b_0$ , a contradiction. Thus  $r$  has a neighbour in  $V(P_1)$ . If  $r, b_1$  are nonadjacent, then there are three paths joining  $r, b_1$  forming an odd theta, namely  $r-b_0-a_0-b_1$ , a path with interior in  $V(P_1) \setminus \{a_1\}$ , and a path with interior in  $V(R) \cup \{q_2, \dots, q_n\}$ , a contradiction. Thus  $r, b_1$  are adjacent. Let  $b_1$  have  $k$  neighbours in  $T$ ; thus  $k \geq 2$ . Since we can complete  $T$  to a hole via a subpath of  $P_1$  that contains no neighbour of  $b_1$ , it follows that  $k$  is even. But we can also complete  $T$  to a hole via  $r-b_0-a_0-b_2-q_1$ , and in this hole  $b_1$  has  $k+1$  neighbours, contrary to 6.1. This completes the proof of 6.3.  $\blacksquare$

This has the following useful corollary. Let us say a domino  $(ab, C_1, C_2)$  is *big* if for  $i = 1, 2$ ,  $C_i$  has length at least six, and if  $C_i$  has length six then both the vertices of  $C_i \setminus \{a, b\}$  that are adjacent to  $a$  or  $b$  have degree at least three in  $G$ .

**6.4** *Every balanceable graph that contains a big domino admits either a double star cutset or an internal 6-join.*

**Proof.** Let  $(a_1a_2, C_1, C_2)$  be a big domino in a balanceable graph  $G$ . We may assume that  $G$  does not admit a double star cutset. By 6.3,  $G$  admits a 6-join  $(V_1, V_2)$  such that  $V(C_i) \setminus \{a_1, a_2\} \subseteq V_i$  for  $i = 1, 2$ , and  $V_1, V_2$  each contain exactly one of  $a_1, a_2$ . Let  $A_1, \dots, A_6$  be as in the definition of 6-join. We may assume that  $a_1 \in A_1$  and  $a_2 \in A_2$ , and we suppose for a contradiction that  $|V_1| \leq 7$ . Let  $a_6$  be the neighbour of  $a_1$  in  $C_2$  different from  $a_2$ , and let  $a_3$  be the neighbour of  $a_2$  in  $C_1$  different from  $a_1$ . Thus  $a_6 \in A_2 \cup A_6$ , since  $a_6 \in V_2$  and  $a_6$  is adjacent to  $a_1$ , and similarly  $a_3 \in A_1 \cup A_3$ . Since  $a_3, a_6$  are nonadjacent, it follows that  $a_3 \in A_3$  and  $a_6 \in A_6$ . Since  $a_6$  has no neighbour in  $V(C_1)$  except  $a_1$ , it follows that  $V(C_1) \cap (A_1 \cup A_5) = \{a_1\}$ . Also,  $V(C_1) \cap A_3 = \{a_3\}$  since  $a_1, a_3$  are the only neighbours of  $a_2$  in  $V(C_1)$ . Consequently all vertices of  $C_1$  except three belong to  $A_0$ , where  $A_0 = V_1 \setminus (A_1 \cup A_3 \cup A_5)$ . Since  $A_1, A_3, A_5$  are nonempty, it follows that  $|V_1| \geq |V(C_1)|$ . But  $|V_1| \leq 7$ , and  $|V(C_1)|$  is even, and so  $C_1$  is a 6-hole. Let the vertices of  $C_1$  be  $a_1-a_2-a_3-c_4-c_5-c_6-a_1$  in order. Since  $G$  is bipartite,  $c_5$  has no neighbour in  $A_1 \cup A_3 \cup A_5$ .

(1) *If  $a_5 \in A_5$ , then  $a_5$  is adjacent to both or neither of  $c_4, c_6$ .*

For suppose that  $a_5$  is adjacent to  $c_6$  and not to  $c_4$  say. Let  $a_4 \in A_4$ . Then the paths  $c_6-a_1-a_2-a_3$ ,  $c_6-a_5-a_4-a_3$  and  $c_6-c_5-c_4-a_3$  form an odd theta, contrary to 6.1. This proves (1).

Suppose first that  $|A_0| = 3$ , and so  $A_0 = \{c_4, c_5, c_6\}$ . Since we may assume that  $A_1 \cup A_3 \cup \{a_2\}$  is not a double star cutset, one of  $c_4, c_5, c_6$  (and therefore both  $c_4, c_6$ , by (1), and not  $c_5$ , since  $c_5, a_5$  would have the same biparity) has a neighbour  $a_5 \in A_5$ . But then  $a_5$  dominates  $c_5$ , contrary to 3.2.

Thus  $|A_0| > 3$ . Consequently  $|A_0| = 4$ , and  $|A_i| = 1$  for  $i = 1, 3, 5$ . Let  $A_5 = \{a_5\}$ . Suppose that  $c_4, c_6$  are adjacent to  $a_5$ . Since  $a_5$  does not dominate  $c_5$  by 3.2, some neighbour  $x$  of  $c_5$  is nonadjacent to  $c_5$  and in particular is different from  $c_4, c_6$ . Hence  $x \in A_0$ . For the same reason, some neighbour of  $x$  is nonadjacent to  $c_4$ , and so  $x, a_1$  are adjacent; and similarly  $x, a_3$  are adjacent. But then  $G|V_1$  is an odd wheel with centre  $c_5$ , contrary to 6.1.

Thus not both  $c_4, c_6$  are adjacent to  $a_5$ , and hence by (1),  $c_4, c_6$  are both nonadjacent to  $a_5$ . But  $c_6$  has degree at least three in  $G$ , since  $(a_1a_2, C_1, C_2)$  is a big domino; let  $x \neq a_1, c_5$  be adjacent to  $c_6$ . Thus  $x \in A_0$ . But none of  $c_4, c_5, c_6, x$  are adjacent to  $a_5$  (because  $c_5, x$  are nonadjacent to  $a_5$

since they have the same biparity), and therefore  $\{a_1, a_2, a_3\}$  is a star cutset, contrary to 3.2. Thus  $|V_1| \geq 8$ , and similarly  $|V_2| \geq 8$ . This proves 6.4.  $\blacksquare$

## 7 Small dominoes

A domino  $(ab, C_1, C_2)$  is *small* if  $C_1, C_2$  are both 4-holes. Our next goal is an analogue of 6.4 for small dominoes, but we first need two more lemmas. The first is theorem 6.2 of [3]. (The graph  $R_{10}$  consists of a ten-vertex cycle with edges between the five opposite pairs of vertices of the cycle.)

**7.1** *Let  $G$  be balanceable, with an induced subgraph isomorphic to  $R_{10}$ . Then either  $G$  is isomorphic to  $R_{10}$ , or  $G$  admits a double star cutset.*

Let  $(a_0b_0, C_1, C_2)$  be a small domino in a bipartite graph  $G$ . A *left ear* for  $(a_0b_0, C_1, C_2)$  is a hole  $H_1$  such that  $(a_1b_1, C_1, H_1)$  is a domino (where  $V(C_1) = \{a_0, b_0, a_1, b_1\}$ ) and  $V(C_2) \setminus \{a_0, b_0\}$  is anticomplete to  $H_1$ . A *right ear* for  $(a_0b_0, C_1, C_2)$  is a left ear for  $(a_0b_0, C_2, C_1)$ .

**7.2** *Let  $G$  be balanceable, and let  $(a_0b_0, C_1, C_2)$  be a small domino with a left ear and a right ear. Then either  $G$  is isomorphic to  $R_{10}$ , or  $G$  admits a double star cutset or an internal 6-join.*

**Proof.** We may assume that  $G$  admits no double star cutset. For  $i = 1, 2$ , let  $C_i$  have vertices  $a_0b_i-a_i-b_0-a_0$  in order. Let  $H_1$  be a left ear with vertices  $a_1-p_1-p_2-\dots-p_m-b_1-a_1$  in order, and let  $H_2$  be a right ear with vertices  $a_2-q_1-q_2-\dots-q_n-b_2-a_2$  in order. Thus  $\{p_1, \dots, p_m\}$  is anticomplete to  $V(C_2)$ , and  $\{q_1, \dots, q_n\}$  is anticomplete to  $V(C_1)$ . However, the sets  $\{p_1, \dots, p_m\}$  and  $\{q_1, \dots, q_n\}$  may not be anticomplete to each other, and may even not be disjoint. If either  $\{p_1, \dots, p_m\}$  and  $\{q_1, \dots, q_n\}$  are not disjoint, or are disjoint but not anticomplete to each other, let  $k(H_1, H_2) = 0$ . If  $\{p_1, \dots, p_m\}$  and  $\{q_1, \dots, q_n\}$  are disjoint and anticomplete, define  $k(H_1, H_2)$  to be the minimum  $k$  such that there is a path  $r_1-\dots-r_k$  with  $r_1$  adjacent to one of  $p_1, \dots, p_m$ , and  $r_k$  adjacent to one of  $q_1, \dots, q_n$ , and  $a_0, b_0$  nonadjacent to  $r_1, \dots, r_k$  (such a path exists since  $G$  does not admit a double star cutset). We proceed by induction on  $k(H_1, H_2)$ .

(1) *If  $k(H_1, H_2) = 0$  then the theorem holds.*

For suppose first that one of  $p_1, \dots, p_{m-1}$  either equals or is adjacent to one of  $q_1, \dots, q_{n-1}$ . Then there is a chordless path  $R$  between  $a_1$  and  $a_2$  with interior in  $\{p_1, \dots, p_{m-1}, q_1, \dots, q_{n-1}\}$ , and therefore  $b_1, b_2$  have no neighbours in the interior of  $R$ . But then  $b_0$  has three neighbours in the hole  $a_1-R-a_2-b_2-a_0-b_1-a_1$ , so  $G$  contains an odd wheel, contrary to 6.1. Thus  $\{p_1, \dots, p_{m-1}\}$  is disjoint from and anticomplete to  $\{q_1, \dots, q_{n-1}\}$ . Since  $p_m \notin \{q_1, \dots, q_n\}$  since  $p_m$  is adjacent to  $b_1$ , and similarly  $q_n \notin \{p_1, \dots, p_m\}$ , it follows that  $\{p_1, \dots, p_m\}$  is disjoint from  $\{q_1, \dots, q_n\}$ . Moreover, for  $1 \leq i \leq m$  and  $1 \leq j \leq n$ , if  $p_i, q_j$  are adjacent then either  $i = m$  or  $j = n$ . Similarly either  $i = 1$  or  $j = 1$ . Thus the only pairs  $p_iq_j$  that might be adjacent are  $p_1q_n$  and  $p_mq_1$ . Since  $k(H_1, H_2) = 0$  it follows that at least one of these is an edge, so from the symmetry we may assume that  $p_1, q_n$  are adjacent. If  $p_mq_1$  is not an edge then

$$p_1-\dots-p_m-b_1-a_0-b_0-a_2-q_1-\dots-q_n-p_1$$

is a hole, containing three neighbours of  $a_1$ , contrary to 6.1. Thus  $p_m q_1$  is an edge. Since the three paths  $p_1 \cdots p_m$ ,  $p_1 a_1 b_1 p_m$  and  $p_1 q_n \cdots q_1 p_m$  do not form an odd theta, it follows that  $m = 2$  and similarly  $n = 2$ ; but then  $G$  contains an induced subgraph isomorphic to  $R_{10}$  and the theorem holds by 7.1. This proves (1).

Henceforth then we assume that  $\{p_1, \dots, p_m\}$  and  $\{q_1, \dots, q_n\}$  are disjoint and anticomplete, so  $k(H_1, H_2) > 0$ . Choose a path  $r_1 \cdots r_k$  such that  $r_1$  is adjacent to one of  $p_1, \dots, p_m$ , and  $r_k$  is adjacent to one of  $q_1, \dots, q_n$ , and  $a_0, b_0$  are nonadjacent to  $r_1, \dots, r_k$ , with  $k = k(H_1, H_2)$ ; then this path is chordless. Hence  $\{r_1, \dots, r_{k-1}\}$  is anticomplete to  $\{q_1, \dots, q_n\}$ , and  $\{r_2, \dots, r_k\}$  is anticomplete to  $\{p_1, \dots, p_m\}$ . However, there may be edges between  $\{a_1, b_1, a_2, b_2\}$  and  $\{r_1, \dots, r_k\}$ .

(2) *We may assume that either  $\{a_1, b_2\}$  is anticomplete to  $\{r_1, \dots, r_k\}$ , or  $\{a_2, b_1\}$  is anticomplete to  $\{r_1, \dots, r_k\}$ .*

For suppose not. If some  $r_i$  is adjacent to two of  $a_1, b_1, a_2, b_2$ , then  $G[\{a_0, b_0, a_1, b_1, a_2, b_2, r_i\}]$  is an odd wheel, a contradiction. Thus each  $r_i$  is adjacent to at most one of  $a_1, b_1, a_2, b_2$ . Choose a chordless path  $c_2 \cdots c_{t-1}$  with  $t$  minimum such that some  $c_1 \in \{a_1, b_2\}$  is adjacent to  $c_2$  and some  $c_t \in \{a_2, b_1\}$  is adjacent to  $c_{t-1}$ , and  $c_2, \dots, c_{t-1} \in \{r_1, \dots, r_k\}$ . Thus  $t \geq 4$ . From the minimality of  $t$ , none of  $c_3, \dots, c_{t-2}$  is adjacent to any of  $a_1, b_1, a_2, b_2$ . From the symmetry we may assume that  $c_1 = a_1$ . If  $c_t = a_2$  then  $b_0$  has three neighbours in the hole  $c_1 \cdots c_t b_2 a_0 b_1 c_1$ , a contradiction. If  $c_t = b_1$ , let  $H_3$  be the hole  $c_1 c_2 \cdots c_t c_1$ ; then  $k(H_2, H_3) < k$  and the result follows from the inductive hypothesis. This proves (2).

Thus we may assume that  $\{a_2, b_1\}$  is anticomplete to  $\{r_1, \dots, r_k\}$ .

(3) *Either  $a_1$  is adjacent to one of  $r_2, \dots, r_k$ , or  $b_2$  is adjacent to one of  $r_1, \dots, r_{k-1}$ , and in particular  $k > 1$ .*

For suppose not. If there is a chordless path  $P'$  between  $a_1$  and  $r_1$  with interior in  $\{p_1, \dots, p_{m-1}\}$  and a chordless path  $Q'$  between  $a_2$  and  $r_k$  with interior in  $\{q_1, \dots, q_{n-1}\}$ , then

$$a_0 b_1 a_1 P' r_1 \cdots r_k Q' a_2 b_2 a_0$$

is a hole containing three neighbours of  $b_0$ , contrary to 6.1. So we may assume that there is no such path  $P'$  say, and therefore  $p_m$  is the only neighbour of  $r_1$  in  $\{a_1, p_1, \dots, p_m\}$ . Let  $Q'$  be a chordless path between  $r_k$  and  $b_2$  with interior in  $\{q_1, \dots, q_n\}$ ; then

$$a_0 b_0 a_1 p_1 \cdots p_m r_1 \cdots r_k Q' b_2 a_0$$

is a hole containing three neighbours of  $b_1$ , contrary to 6.1. This proves (3).

From (3) we may assume that  $a_1$  is adjacent to some  $r_j$  with  $j > 1$ . Choose  $j \leq k$  maximum with this property.

(4)  *$b_2$  is adjacent to at least one of  $r_1, \dots, r_{j-1}$ .*

For suppose not. Let  $Q', Q''$  be chordless paths from  $r_j$  to  $b_2$  and  $a_2$  respectively with interiors in

$$\{r_{j+1}, \dots, r_k, q_1, \dots, q_n\},$$

and choose  $h$  with  $1 \leq h \leq m$  maximum such that  $r_1, p_h$  are adjacent. Then

$$a_0-b_1-p_m-\dots-p_h-r_1-\dots-r_j$$

is a chordless path, and it can be completed to a hole via  $r_j-Q'-b_2-a_0$  and via  $r_j-Q''-a_2-b_0-a_0$ . The numbers of neighbours of  $a_1$  in these two holes differ by one, and yet  $b_1, r_j$  are neighbours of  $a_1$  that belong to both holes, and so  $G$  contains an odd wheel, contrary to 6.1. This proves (4).

Choose  $i$  with  $1 \leq i < j$  minimum such that  $b_2, r_i$  are adjacent.

(5)  $j = i + 1$ .

For suppose not; then  $i \leq j - 2$ , and there are three paths between  $a_1$  and  $b_2$  that form an odd theta, namely a path with interior in  $\{p_1, \dots, p_m, r_1, \dots, r_i\}$ , a path with interior in  $\{r_j, \dots, r_k, q_1, \dots, q_n\}$ , and the path  $a_1-b_0-a_0-b_2$ , contrary to 6.1. This proves (5).

Now  $a_0-b_2-r_i-r_j-a_1-b_0-a_0$  is a 6-hole. Let

$$\begin{aligned} A &= \{b_1, p_1, \dots, p_m, r_1, \dots, r_{i-1}\} \\ B &= \{a_2, q_1, \dots, q_n, r_{j+1}, \dots, r_k\}. \end{aligned}$$

Then  $G|A, G|B$  are connected, and the hypotheses of 5.2 are satisfied, and since  $G$  admits no double star cutset, it follows that  $G$  admits a 6-join  $(V_1, V_2)$  with  $A \cup \{a_0, a_1, r_i\} \subseteq V_1$  and  $B \cup \{b_0, b_2, r_j\} \subseteq V_2$ . Suppose that  $|V_1| \leq 7$ . Then

$$|\{b_1, p_1, \dots, p_m, r_1, \dots, r_{i-1}, r_i, a_0, a_1\}| \leq 7,$$

and so  $m \leq 3$ ; and since  $m$  is even it follows that  $m = 2$ . Also,  $i \leq 2$ . Now  $a_1, r_i$  have the same biparity (since  $a_1, r_{i+1}$  are adjacent). If  $r_1$  is adjacent to  $p_2$ , then it follows that  $i = 2$  (since  $a_1, r_i$  have the same biparity), and so  $V_1 = \{p_1, p_2, a_1, b_1, r_1, r_2, a_0\}$ . But then  $a_1, p_2$  are the only neighbours of  $p_1$ , and so  $\{a_1, p_2, b_1\}$  is a star cutset, contrary to 3.2. Hence  $r_1$  is adjacent to  $p_1$ . Since  $a_1, r_i$  have the same biparity it follows that  $i = 1$ , and so  $a_1, r_2$  are adjacent. Since  $a_1$  does not dominate  $p_2$  by 3.2, it follows that  $p_2$  has a neighbour  $x$  nonadjacent to  $a_1$ , and in particular  $x \neq p_1, b_1$ ; and so  $V_1 = \{p_1, p_2, a_1, b_1, r_1, x, a_0\}$ . Since  $x$  has a neighbour nonadjacent to  $p_1$  by 3.2, it follows that  $x, a_0$  are adjacent. But then  $x-p_2-p_1-a_1-b_0-a_0-x$  is a 6-hole and  $b_1$  has three neighbours in it, contrary to 6.1. This completes the proof of 7.2. ■

**7.3** *Every balanceable graph not isomorphic to  $R_{10}$  that contains a small domino admits either a double star cutset or an internal 6-join.*

**Proof.** Let  $G$  be a balanceable graph not isomorphic to  $R_{10}$ , and let  $(a_0b_0, C_1, C_2)$  be a small domino in  $G$ . By 7.2 we may assume that  $G$  does not contain a right ear for this domino. For  $i = 1, 2$  let  $C_i$  have vertices  $a_0-b_i-a_i-b_0-a_0$  in order. By 3.2 there is a hole  $H$  such that  $(a_2b_2, C_2, H)$  is a domino;



let  $H$  have vertices  $a_2-p_1-\dots-p_m-b_2-a_2$  in order. Since  $H$  is not a right ear, one of  $a_1, b_1$  is adjacent to one of  $p_1, \dots, p_m$ . From the symmetry we may assume that  $b_1$  is adjacent to one of  $p_1, \dots, p_m$ ; choose  $h, j$  with  $1 \leq h, j \leq m$  minimum and maximum respectively such that  $b_1$  is adjacent to  $p_h, p_j$ . If  $a_1$  is nonadjacent to all of  $p_{j+1}, \dots, p_m$ , then (since  $a_1, p_j$  have the same biparity and are therefore nonadjacent)

$$b_1-p_j-\dots-p_m-b_2-a_2-b_0-a_1-b_1$$

is a hole containing three neighbours of  $a_0$ , contrary to 6.1. So  $a_1$  is adjacent to one of  $p_{j+1}, \dots, p_m$ , and in particular  $j < m$ . If  $h = j$  then the three paths  $p_h-b_1-a_0-b_2$ ,  $p_h-p_{h-1}-\dots-p_1-a_2-b_2$  and  $p_h-p_{h+1}-\dots-p_m-b_2$  form an odd theta, contrary to 6.1, so  $h < j$ . Choose  $i, k$  with  $1 \leq i, k \leq m$  minimum and maximum respectively such that  $a_1$  is adjacent to  $p_i, p_k$ . Thus  $k > j$ , and from the symmetry it follows that  $h < i < k$ . If  $i \geq j$ , then the numbers of neighbours of  $b_1$  in the two holes  $H$  and

$$a_2-p_1-\dots-p_i-a_1-b_0-a_2$$

differ by one, and  $b_1$  has at least three neighbours in the second hole (since  $h < j$ ), contrary to 6.1. Thus  $i < j$ .

Let us choose the hole  $H$  described above such that  $b_1$  has as few neighbours in it as possible. Choose  $h'$  with  $h < h' \leq m$  minimum such that  $b_1, p_{h'}$  are adjacent. We may assume that there is a chordless path  $p_{h+1}-r_1-\dots-r_n-b_0$  such that  $r_1, \dots, r_n$  are nonadjacent to  $a_0, b_1$ , for otherwise  $G$  admits a double star cutset. From the choice of  $H$  it follows that every vertex of  $H$  that belongs to  $\{r_1, \dots, r_{n-1}\}$  or has a neighbour in  $\{r_1, \dots, r_{n-1}\}$  belongs to  $\{p_h, p_{h+1}, \dots, p_{h'}\}$ .

(1)  $r_n$  is adjacent to one of  $p_1, \dots, p_{h-1}$ .

For suppose not. Let  $P$  be a chordless path between  $p_h$  and  $b_0$  with interior in  $\{p_{h+1}, r_1, \dots, r_n\}$ ; then the three paths  $P$ ,  $p_h-b_1-a_0-b_0$  and  $p_h-p_{h-1}-\dots-p_1-a_2-b_0$  form an odd theta (note that  $r_n$  is nonadjacent to  $a_2, p_h$  since they have the same biparity), a contradiction. This proves (1).

(2) Every neighbour of  $r_n$  in  $H$  belongs to  $\{b_2, p_1, \dots, p_{h-1}, p_{h+1}\}$ .

For  $b_1-a_0-b_0-r_n$  is a chordless path, and by (1) there is a chordless path between  $b_1$  and  $r_n$  with interior in  $\{p_1, \dots, p_h\}$ , so if there is a chordless path between  $b_1$  and  $r_n$  with interior in  $\{p_{h+2}, p_{h+3}, \dots, p_m\}$  then these three paths would form an odd prism. This proves (2).

Let  $P$  be a chordless path between  $p_{h'}$  and  $b_2$  with interior in  $\{a_2, p_1, p_2, \dots, p_{h'-1}, r_1, \dots, r_n\} \setminus \{p_h\}$ . It can be completed to a hole via  $p_{h'}-p_{h'+1}-\dots-p_m-b_2$ . The number of neighbours of  $b_1$  in this hole is exactly one fewer than the number of neighbours of  $b_1$  in  $H$ , and so by 6.1,  $b_1$  has exactly two neighbours in  $H$ , that is,  $h' = j$ . But then  $P$  and the paths  $p_j-b_1-a_0-b_2$  and  $p_j-p_{j+1}-\dots-p_m-b_2$  form an odd theta, contrary to 6.1. This completes the proof.  $\blacksquare$

## 8 A proof of conjecture 9.29 of [6]

A bipartite graph  $G$  is *strongly balanceable* if it is balanceable and no induced subgraph is a cycle with exactly one chord. For the proof of 3.1 we also need theorem 6.1 of [3], the following:

**8.1** *Every connected balanceable bipartite graph that is not strongly balanceable either equals  $R_{10}$  or admits a 2-join, a 6-join, or a double star cutset.*

**Proof of 3.1.** We prove 3.1 by induction on  $|V(G)|$ . Suppose then that  $G$  is a nonregular balanceable graph (and consequently  $|V(G)| \geq 6$ ), and every nonregular balanceable graph with fewer vertices than  $G$  admits a double star cutset. Suppose for a contradiction that  $G$  does not admit a double star cutset. By 3.2,  $G$  is connected.

(1)  $G$  does not admit a 2-join.

For suppose it does, and let  $V_i, A_i, B_i$  ( $i = 1, 2$ ) be as in the definition of 2-join. Suppose first that there exist  $x \in A_1$  and  $y \in B_1$ , adjacent. Every path between  $V_1 \setminus \{x, y\}$  and  $V_2 \setminus (A_2 \cup B_2)$  contains a member of  $N[xy]$  in its interior, and since  $V_1 \setminus \{x, y\} \neq \emptyset$  and  $G$  does not admit a double star cutset, it follows that  $V_2 = A_2 \cup B_2$ . Hence there is an edge between  $A_2, B_2$ , and so similarly  $V_1 = A_1 \cup B_1$ . Not both  $|A_1|, |B_1| = 1$  from the definition of a 2-join, and if say  $v, w \in A_1$  are distinct then by 3.2, since neither of  $v, w$  dominates the other, it follows that  $|B_1| > 1$ . Moreover, the same argument proves that there exist  $a_1, a'_1 \in A_1$  and  $b_1, b'_1 \in B_1$  such that  $a_1b_1$  and  $a'_1b'_1$  are edges, and  $a_1, b'_1$  are nonadjacent, and  $a'_1, b_1$  are nonadjacent. Similarly there exist  $a_2, a'_2 \in A_2$  and  $b_2, b'_2 \in B_2$  such that the only edges between  $\{a_2, a'_2\}$  and  $\{b_2, b'_2\}$  are  $a_2b_2$  and  $a'_2b'_2$ . But then the subgraph induced on  $\{a_2, a_1, b_1, b'_2, b'_1, a'_1\}$  is a cycle, and  $b_2$  has exactly three neighbours in it, contrary to 6.1. Hence there are no edges between  $A_1$  and  $B_1$ , and similarly no edges between  $A_2$  and  $B_2$ .

Let  $P_2$  be a chordless path of  $G|V_2$  between  $A_2$  and  $B_2$  with no internal vertex in  $A_2 \cup B_2$ . Suppose that  $G|(V_1 \cup V(P_2))$  is not regular. Let  $P_2$  have vertices  $p_1 - \dots - p_k$  say, where  $p_1 \in A_2$  and  $p_k \in B_2$ . Every vertex in  $A_1$  has a neighbour in  $V_1 \setminus A_1$  (from the definition of a 2-join if  $|A_1| = 1$ , and by 3.2 if  $|A_1| > 1$ ); and every component of  $G|(V_1 \setminus A_1)$  contains a vertex of  $B_1$ , by 3.2 applied to  $G$  and the edge  $p_1p_2$ . It follows that  $G|((V_1 \setminus A_1) \cup \{p_k\})$  is connected, and every vertex in  $A_1$  has a neighbour in  $(V_1 \setminus A_1) \cup \{p_k\}$ ; and an analogous statement holds with  $A_1, B_1$  exchanged and  $p_1, p_k$  exchanged.

We claim that if  $k = 3$  then  $|V_2| \geq 6$ . For if  $|A_2| = |B_2| = 1$  then  $V(P)$  is a double star cutset (since  $P_2 \neq G|V_2$  from the definition of a 2-join), so from the symmetry we may assume that  $|B_2| > 1$ . Choose  $b_2 \in B_2$  with  $b_2 \neq p_3$ . From 3.2, there is a chordless path between  $b_2$  and  $A_1$  containing no neighbour of either of  $p_2, p_3$  (except possibly  $b_2$ ). In particular, this path is disjoint from  $B_1$ , and therefore contains a vertex of  $V_2 \setminus (A_2 \cup B_2)$  different from  $p_2$ , and a vertex of  $A_2$  different from  $p_1$ . Consequently  $|V_2| \geq 6$ , as claimed.

Let  $G_1 = G|(V_1 \cup V(P_2))$ . If  $k > 3$  let  $G' = G_1$  and let  $P'_2 = P_2$ . If  $k = 3$  let  $G'$  denote the graph obtained from  $G_1$  by subdividing twice some edge of  $P_2$  (that is, replacing some edge of  $P_2$  by a three-edge path), and let  $P'_2$  be the path obtained from  $P_2$  by this double subdivision. Then in either case no edge of  $P'_2$  is the centre of a double star cutset of  $G'$ . Yet  $G'$  is balanceable and not regular, and since  $|V(G')| < |V(G)|$ , the inductive hypothesis implies that there is an edge  $uv$  of  $G'$  that is the centre of a double star cutset of  $G'$ . It follows that at least one of  $u, v \in V_1$ , and  $uv$  is an edge of  $G_1$ , and therefore  $uv$  is the centre of a double star cutset of  $G_1$ .

There is a subset  $X \subseteq N[uv] \cap V(G_1)$  with  $u, v \in X$  such that  $G_1 \setminus X$  is disconnected. Since  $X \cup (N[uv] \setminus V(G_1))$  is not a double star cutset of  $G$ , there is a chordless path  $Q$  of  $G$  with ends  $x, y$  belonging to different components of  $G_1 \setminus X$ , and such that no internal vertex of  $Q$  belongs to  $V(G_1) \cup N[uv]$ . Since  $Q^* \subseteq V_2$ , there is a chordless path  $P$  of  $G_1$  between  $x, y$  with  $P^* \subseteq V(P_2)$ ; and therefore  $N[uv] \cap P^* \neq \emptyset$ . Since at least one of  $u, v \in V_1$ , it follows that  $\{u, v\} \cap (A_1 \cup B_1) \neq \emptyset$ ,

say  $u \in A_1$ . Then  $A_2 \cap Q^* = \emptyset$ , and since  $x, y$  both have neighbours in  $Q^* \subseteq V_2 \setminus A_2$ , it follows that  $x, y \in V_2 \cup B_1$ . So  $P^* \cap A_2 = \emptyset$ , and therefore  $N[uv] \cap V_2 \not\subseteq A_2$ . Hence  $v \in A_2$ , and so  $v = p_1$ , the end of  $P_2$  in  $A_2$ . Thus  $x, y \notin A_2$ , and so  $N[uv] \cap P^* = \emptyset$ , a contradiction.

This proves that  $G_1 = G|(V_1 \cup V(P_2))$  is regular. Similarly, let  $P_1$  be a chordless path of  $G|V_1$  between  $A_1$  and  $B_1$  with no internal vertex in  $A_1 \cup B_1$ ; then  $G_2 = G|(V_2 \cup V(P_1))$  is regular. But then by 4.1,  $G$  is regular, a contradiction. This proves (1).

(2) *If  $(V_1, V_2)$  is a 6-join in  $G$ , then one of  $(V_1, V_2), (V_2, V_1)$  is skeletal.*

For let  $A_1, \dots, A_6$  be as in the definition of a 6-join, and choose  $a_i \in A_i$  for  $1 \leq i \leq 6$ . By 4.3, not both blocks of the 6-join are regular; so we may assume that  $G_1$  is not regular, where  $G_1$  is the block obtained by adding three vertices  $b_2, b_4, b_6$  to  $G|V_1$ , with adjacency as before. For convenience we assume (as we may) that  $b_i = a_i$  for  $i = 2, 4, 6$ . Since  $G_1$  is therefore an induced subgraph of  $G$ , it follows that  $G_1$  is balanceable. Define  $A_7 = V_1 \setminus (A_1 \cup A_3 \cup A_5)$ . Let  $H$  be obtained from  $G_1$  by adding four vertices  $c_2, c_4, c_6, c_8$  to  $G_1$  and the edges  $c_i c_8$  and  $c_i a_i$  for  $i = 2, 4, 6$ . We shall show that  $G, H$  are isomorphic. Certainly  $H$  is balanceable (to see this, take a map  $w : E(G_1) \rightarrow \{-1, 1\}$  such that  $w(C)$  is a multiple of four for every induced cycle  $C$  of  $G_1$ ; by reversing the signs of  $w(e)$  on some edge-cutsets if necessary we may assume as usual that  $w(a'_i a_{i+1}) = 1$  and  $w(a'_i a_{i-1}) = -1$  for  $i = 1, 3, 5$  and each  $a'_i \in A_i$ , where  $a_0$  means  $a_6$ ; then extend the domain of  $w$  to  $E(H)$  by defining  $w(e) = 1$  for every edge  $e \in E(H) \setminus E(G_1)$ ; and it is easy to check that  $w(C)$  is a multiple of four for every induced cycle  $C$  of  $H$ .)

We recall that  $G$  admits no double star cutset. Suppose that  $H$  admits a double star cutset  $X$ , with centre  $uv$  say. Up to symmetry there are five possibilities for  $uv$ , namely  $c_2 c_8, c_2 a_2, a_1 a_2, a_7 a_1$  for some  $a_7 \in A_7$ , and  $a_7 a'_7$  for some  $a_7, a'_7 \in A_7$ . If  $uv = c_2 c_8$ , then  $A_1 \cup A_3 \cup A_5 \cup \{a_4, a_6\}$  is a subset of the vertex set of one component of  $H \setminus X$ , and so  $H$  and hence  $G$  is disconnected, a contradiction. If  $uv = c_2 a_2$ , then the members of  $A_5 \cup \{a_4, c_4, a_6, c_6\}$  all belong to the same component of  $H \setminus X$ , as does every vertex of  $A_1 \cup A_3$  not in  $X$ , and so there is a component  $C$  of  $H \setminus X$  with  $V(C) \subseteq A_7$ . Hence  $C$  is a component of  $G \setminus N[a_2]$ , and therefore  $G$  admits a star cutset, contrary to 3.2. If  $uv = a_1 a_2$ , then the members of  $A_5 \cup \{a_4, c_4, c_6, c_8\}$  all belong to the same component of  $H \setminus X$ , as does every member of  $A_3 \setminus X$ , and so there is a second component  $C$  say with  $V(C) \subseteq A_7 \cup A_1$ . But then  $(X \setminus \{c_2\}) \cup A_2 \cup A_6$  is a double star cutset of  $G$ , a contradiction. If  $uv = a_7 a_1$  for some  $a_7 \in A_7$ , then  $a_4, c_2, c_4, c_6, c_8$  all belong to the same component of  $H \setminus X$ , as does every member of  $(A_3 \cup A_5 \cup \{a_2, a_6\}) \setminus X$ . Hence there is a component  $C$  of  $H \setminus X$  with  $V(C) \subseteq A_7 \cup A_1$ , and so  $(X \cap V_1) \cup (A_2 \cup A_6)$  is a double star cutset of  $G$ , a contradiction. Finally, if  $u, v \in A_7$ , then  $a_2, a_4, a_6, c_2, c_4, c_6, c_8$  all belong to the same component of  $H \setminus X$ , as does every member of  $(A_1 \cup A_3 \cup A_5) \setminus X$ , and so there is a component  $C$  of  $H \setminus X$  with  $V(C) \subseteq A_7$ ; but then  $X$  is a double star cutset of  $G$ , a contradiction. It follows that  $H$  does not admit a double star cutset.

From the inductive hypothesis, we deduce that  $|V(H)| \geq |V(G)|$ , and so  $|V_2| \leq 7$ . Let  $A_8 = V_2 \setminus (A_2 \cup A_4 \cup A_6)$ . If  $A_8 = \emptyset$ , then since  $|V_2| \geq 4$  it follows that two members of  $V_2$  are twins, contradicting 3.2. Thus  $A_8 \neq \emptyset$ . Suppose that some vertex in  $A_8$  has neighbours in two of  $A_2, A_4, A_6$ , say  $a_8 \in A_8$  is adjacent to  $a_2 \in A_2$  and to  $a_4 \in A_4$ . Since  $N[a_3 a_2] \setminus \{a_8\}$  is not a double star cutset, there is a chordless path  $P$  between  $a_8$  and  $A_6$  such that  $V(P) \setminus \{a_8\}$  is anticomplete to  $a_2$ . Choose  $P$  minimal; then all its vertices belong to  $A_8$  except for its final vertex  $a_6$  say in  $A_6$ . Since the subgraph induced on  $\{a_8, a_1, \dots, a_6\}$  is not an odd wheel by 6.1, it follows that  $a_8, a_6$  are nonadjacent, and

since  $a_8, a_6$  have opposite biparity it follows that  $P$  has odd length, and length at least three. Since  $|V_2| \leq 7$  and so  $|A_8| \leq 4$ , it follows that  $P$  has length three; let its vertices be  $a_8-p_1-p_2-a_6$  in order. Now the paths  $a_8-P-a_6, a_8-a_2-a_1-a_6$  and  $a_8-a_4-a_5-a_6$  do not form an odd theta, by 6.1, and so  $a_4$  is adjacent to  $p_2$ . But then every neighbour of  $p_1$  is adjacent to  $a_4$ , contrary to 3.2. This proves that no vertex of  $A_8$  has neighbours in two of  $A_2, A_4, A_6$ .

Let  $C$  be a component of  $G|A_8$ . Since  $G$  does not admit a double star cutset, at least one member of  $A_i$  has a neighbour in  $C$  for  $i = 2, 4, 6$ , and so we may assume that for  $i = 2, 4, 6$ ,  $a_i$  is adjacent to  $c_i \in C$ . Hence  $c_2, c_4, c_6$  are all distinct, since no vertex of  $A_8$  has neighbours in two of  $A_2, A_4, A_6$ . Moreover, since  $C$  is connected, there is a vertex  $c_8 \in C$  such that  $a_2, c_8$  have the same biparity. Hence  $A_8 = \{c_2, c_4, c_6, c_8\}$ , and  $A_i = \{a_i\}$  for  $i = 2, 4, 6$ ; and since  $C$  is connected, it follows that  $c_8$  is adjacent to each of  $c_2, c_4, c_6$ . But then  $(V_1, V_2)$  is skeletal. This proves (2).

From (2) and 6.4 and 7.3, it follows that  $G$  contains no big domino or small domino.

(3) *Let  $(V_1, V_2)$  be a 6-join, and let  $A_1, \dots, A_6$  be defined as usual. If  $v \in V_1$  has a neighbour in  $A_1$  and a neighbour in  $A_3$ , then  $v$  is complete to  $A_1 \cup A_3$ .*

For let  $v$  be adjacent to  $a_1 \in A_1$  and  $a_3 \in A_3$ , and suppose it is nonadjacent to some  $a'_1 \in A_1$  say. Choose  $a_2 \in A_2$  and  $a_6 \in A_6$ ; then

$$(a_1a_2, a_1a_2a'_1a_6a_1, a_1a_2a_3va_1)$$

is a small domino, a contradiction. This proves (3).

(4) *Let  $(v_1v_2, C, D)$  be a domino, where  $|V(C)| \geq 8$  and  $|V(D)| \geq 6$ . For  $i = 1, 2$ , let  $c_i, d_i$  be the neighbours of  $v_i$  in  $C \setminus \{v_1, v_2\}, D \setminus \{v_1, v_2\}$  respectively, and let  $d_1$  have degree at least three in  $G$ . Then every vertex of  $G$  adjacent to both  $c_2$  and  $v_1$  is adjacent to every neighbour of  $v_2$  except possibly  $d_2$ , and  $d_2$  belongs to no irregularity in  $G$ .*

For by 6.3,  $G$  admits a 6-join  $(V_1, V_2)$  such that  $V(C) \setminus \{v_1, v_2\} \subseteq V_1$  and  $V(D) \setminus \{v_1, v_2\} \subseteq V_2$ , and  $V_1, V_2$  each contain exactly one of  $v_1, v_2$ . Let  $\{i, j\} = \{1, 2\}$ , where  $v_j \in V_1$  and  $v_i \in V_2$ . Since  $C$  has length at least eight, it follows that  $(V_2, V_1)$  is not skeletal, and so by (2)  $(V_1, V_2)$  is skeletal. Hence  $d_i$  has degree two, and by 5.1  $d_i$  does not belong to any irregularity in  $G$ . Since  $d_1$  has degree at least three, it follows that  $i = 2$  and  $j = 1$ . Let the sets  $A_1, \dots, A_6$  be defined as usual, where  $c_2 \in A_1, v_2 \in A_2, v_1 \in A_3$  and  $d_1 \in A_4$ . Then  $N[v_2] = A_1 \cup A_3 \cup \{v_2, d_2\}$ , and  $v_2$  is the only vertex in  $V_2$  adjacent to both  $c_2, v_1$ , and by (3) every vertex in  $V_1$  adjacent to both  $c_2, v_1$  is complete to  $A_1 \cup A_3$ . This proves (4).

Since  $R_{10}$  is regular, it follows from 8.1 that either  $G$  is strongly balanceable, or  $G$  admits a 2-join, or  $G$  admits a 6-join. The first is impossible since it is a theorem of [5] that every strongly balanceable graph is regular. So by (1) and (2), it follows that  $G$  admits a skeletal 6-join  $(V_1, V_2)$ . Let  $A_1, \dots, A_6$  be as in the definition of a 6-join, and for  $1 \leq i \leq 6$  choose  $a_i \in A_i$ . Thus  $A_i = \{a_i\}$  for  $i = 2, 4, 6$ . Since  $G$  is not regular, it follows that there is an irregularity  $H$ , and from 5.1  $V(H) \cap V_2 \subseteq \{a_2, a_4, a_6\}$ . Choose  $H$  such that  $|V(H) \cap \{a_2, a_4, a_6\}|$  is as small as possible.

(5) If  $a_2 \in V(H)$  then there is no vertex  $v \in V_1$  with a neighbour in  $A_1$  and a neighbour in  $A_3$ .

For suppose that  $a_2 \in V(H)$  and such a vertex  $v$  exists. By (3)  $v$  is complete to  $A_1 \cup A_3$ . Since  $G$  does not admit a double star cutset, there is a chordless path  $v-p_1-\dots-p_k$  of  $G$  such that  $p_1, \dots, p_{k-1} \in V_1$  and  $\{p_1, \dots, p_k\}$  is anticomplete to  $\{a_1, a_2\}$ , and  $p_k \in A_1 \cup A_3 \cup A_5$  (and therefore  $p_k \in A_5$ , since  $p_k$  is nonadjacent to  $a_2$ ). By 6.1,  $v$  does not have three neighbours in the hole induced on  $\{a_1, \dots, a_6\}$ , and so  $k > 1$ . Since the paths  $v-p_1-\dots-p_k$ ,  $v-a_3-a_4-p_k$  and  $v-a_1-a_6-p_k$  do not form an odd theta by 6.1, it follows that  $a_3$  is adjacent to one of  $p_1, \dots, p_k$ . Choose  $i$  with  $1 \leq i \leq k$  minimum such that  $a_3, p_i$  are adjacent. Since  $v, p_i$  have the same biparity, it follows that  $i < k$ , and  $i$  is even. If  $i = 2$  then

$$(va_3, v-a_1-a_2-a_3-v, v-p_1-p_2-a_3-v)$$

is a small domino, a contradiction; so  $i \geq 4$ . Since  $(V_1, V_2)$  is skeletal, there is a chordless path  $Q$  of length four between  $a_6$  and  $a_4$  with interior in  $V_2 \setminus \{a_2, a_4, a_6\}$ . Let  $C$  be the hole  $v-a_1-a_6-Q-a_4-a_3-v$ , and let  $D$  be the hole  $v-p_1-\dots-p_i-a_3-v$ . Then  $(va_3, C, D)$  is a domino, and  $C$  has length eight, and  $D$  has length at least six. Let  $v_1 = a_3$  and  $v_2 = v$ , and for  $i = 1, 2$ , let  $c_i, d_i$  be the neighbours of  $v_i$  in  $C \setminus \{v_1, v_2\}, D \setminus \{v_1, v_2\}$  respectively; then  $d_1 = p_i$ , and therefore  $d_1$  has degree at least three in  $G$ . By (4) every vertex of  $G$  adjacent to both  $c_2$  and  $v_1$  is adjacent to every neighbour of  $v_2$  except possibly  $d_2$ , and  $d_2$  belongs to no irregularity in  $G$ . Since  $c_2 = a_1$ , and  $d_2 = p_1$ , and  $a_2$  is adjacent to both  $a_1, a_3$ , it follows that  $a_2$  is adjacent to every neighbour of  $v_2$  except  $p_1$ . But the neighbour set of  $a_2$  is  $A_1 \cup A_3 \cup \{u_2\}$  for some  $u_2 \in V_2 \setminus \{a_2, a_4, a_6\}$ , where no irregularity contains  $u_2$  by 5.1; and since we have already seen that  $v$  is complete to  $A_1 \cup A_3$ , it follows that the neighbour set of  $v$  is  $A_1 \cup A_3 \cup \{p_1\}$ . If  $v \in V(H)$  then since  $p_1, u_2 \notin V(H)$  it follows that  $v, a_2$  are twins in  $H$ , which is impossible. Thus  $v \notin V(H)$ . Since  $p_1, u_2 \notin V(H)$ , it follows that the subgraph induced on  $(V(H) \setminus \{a_2\}) \cup \{v\}$  is isomorphic to  $H$ , and therefore is also an irregularity, contrary to the minimality of  $|V(H) \cap \{a_2, a_4, a_6\}|$ . This proves (5).

Let  $I$  be the set of all  $i \in \{2, 4, 6\}$  such that no vertex in  $V_1$  has a neighbour in  $A_{i-1}$  and a neighbour in  $A_{i+1}$ , where  $A_7$  means  $A_1$ . Let  $J$  be the subgraph of  $G$  induced on  $V_1 \cup \{a_i : i \in I\}$ .

(6)  $J$  does not admit a double star cutset.

For suppose there is a double star cutset  $X$  in  $J$ , with centre  $uv$  say, and let  $C_1, C_2$  be distinct components of  $J \setminus X$ . Let  $X' = X \cup (N[uv] \setminus V(J))$ . Since  $X'$  is not a double star cutset of  $G$ , it follows that  $C_1, C_2$  are both subgraphs of the same component  $C$  of  $G \setminus X'$ . In particular, for  $j = 1, 2$ , some vertex  $p_j$  of  $C_j$  has a neighbour  $q_j \in V(C) \setminus V(C_j)$  that is nonadjacent to both  $u, v$ . Thus  $q_j \notin V(J)$ , and hence  $p_j \in A_1 \cup A_3 \cup A_5 \cup \{a_i : i \in I\}$ . For  $i = 2, 4, 6$ , if  $i \in I$  let  $a'_i = a_i$ , and if  $i \notin I$  let  $a'_i$  be some vertex in  $V_1$  that is complete to  $A_{i-1} \cup A_{i+1}$  (this exists, by (3) and the definition of  $I$ ). We observe that for  $i = 2, 4, 6$ ,  $a'_i$  is anticomplete to  $A_{i+3}$ ; for this is clear if  $a'_i = a_i$ , and if  $a'_i \neq a_i$  then it follows since otherwise  $a'_i$  would have three neighbours in a 6-hole contained in  $A_1 \cup \dots \cup A_6$ . Let  $R$  be the subgraph of  $G$  induced on  $A_1 \cup A_3 \cup A_5 \cup \{a'_2, a'_4, a'_6\}$ . Then  $R$  is a subgraph of  $J$ , and is connected, and both  $p_1, p_2 \in V(R)$ .

Suppose first that  $u = a_2$  say, and therefore  $2 \in I$ . From the symmetry we may assume that  $v \in A_1$ . Hence  $X \cap (A_5 \cup \{a'_4\}) = \emptyset$ , and so we may assume that  $A_5 \cup \{a'_4\} \subseteq V(C_1)$ . Thus  $p_2 \notin A_3 \cup A_5 \cup \{a'_2, a'_4, a'_6\}$ , and so  $p_2 \in A_1$ . But then  $q_2 = a_6$ , and so  $q_2$  is adjacent to  $v$ , a

contradiction. Thus  $u \neq a_2$ , and similarly  $u, v \neq a_2, a_4, a_6$ .

Next suppose that  $u \in A_1$ . Thus  $v$  has a neighbour in  $A_1$ , and so is anticomplete to one of  $A_3, A_5$ , say  $A_5$  without loss of generality. Hence  $X \cap (A_5 \cup \{a'_4\}) = \emptyset$ , and so we may assume that  $A_5 \cup \{a'_4\} \subseteq V(C_1)$ . Consequently  $p_2 \in A_1 \cup \{a'_2\}$ . If  $p_2 = a'_2$ , then  $a'_2 = a_2$  and so  $2 \in I$ , and so  $v$  has no neighbour in  $A_3$ ; but then  $A_3 \cap X = \emptyset$ , and so  $A_3 \subseteq V(C_1)$ , a contradiction. Thus  $p_2 \in A_1$ , and so  $q_2$  is adjacent to  $u$ , a contradiction. Thus  $u \notin A_1$ , and similarly  $u, v \notin A_1 \cup A_3 \cup A_5$ .

Next suppose that  $a'_2, a'_4$  belong to the same component of  $J \setminus X$ , say  $a'_2, a'_4 \in V(C_1)$ . Then  $V(C_2) \cap (A_1 \cup A_3 \cup A_5) = \emptyset$ , and so  $p_2 = a'_6 = a_6$  and  $6 \in I$ . But then  $A_1 \cup A_5 \subseteq X$ , and so one of  $u, v$  is complete to  $A_1 \cup A_5$ , contradicting that  $6 \in I$ .

Next suppose that  $a'_2, a'_4 \in X$ . Thus we may assume that  $u$  is adjacent to  $a'_2, a'_4$ . Since  $u \notin A_1 \cup A_3 \cup A_5$ , it follows that  $a'_2, a'_4 \notin V_2$ . But then

$$(a'_2 a_3, a'_2 - u - a'_4 - a_3 - a'_2, a'_2 - a_1 - a_2 - a_3 - a'_2)$$

is a small domino in  $G$ , a contradiction. Thus at most one of  $a'_2, a'_4, a'_6$  belongs to  $X$ , and so we may assume that  $a'_2, a'_4 \notin X$ .

Since  $a'_2, a'_4$  are not both in the same component of  $J \setminus X$ , we may therefore assume that  $a'_2 \in V(C_1)$  and  $a'_4 \in V(C_2)$ . Consequently  $A_3 \subseteq X$ , and so we may assume that  $u$  is complete to  $A_3$ . Suppose that  $v$  is adjacent to  $a'_6$ , and therefore  $a'_6 \neq a_6$ . If  $u, a_1$  are adjacent then

$$(ua_1, u - a_3 - a_2 - a_1 - u, u - v - a'_6 - a_1 - u)$$

is a small domino, a contradiction. Thus  $u$  is nonadjacent to  $a_1$  and similarly to  $a_5$ . But then the subgraph induced on  $\{u, v, a_1, a_2, a_3, a_4, a_5, a'_6\}$  is an odd theta, contrary to 6.1. Thus  $v, a'_6$  are nonadjacent, and so  $a'_6 \notin X$ . Since  $u$  is complete to  $A_3$ , it follows that  $u$  is anticomplete to one of  $A_1, A_5$ , say  $A_1$  without loss of generality. It follows that  $A_1 \cap X = \emptyset$  (since  $v$  has the same biparity as the members of  $A_1$ ) and so  $A_1 \subseteq V(C_1)$ ; and since  $a'_6 \notin X$ , we deduce that  $a'_6 \in V(C_1)$ . Consequently  $A_5 \subseteq X$ , and so  $u$  is complete to  $A_5$ . It follows that  $4 \notin I$ , and therefore  $p_2 \neq a'_4$ ; but  $p_2 \in V(R) \cap V(C_2) \subseteq \{a'_4\}$ , a contradiction. This proves (6).

Now  $|V(J)| < |V(G)|$ , since  $|I| \leq 3$  and  $|V_2| \geq 4$ . From (5),  $V(H) \subseteq V(J)$  and so  $J$  is not regular. But from (6),  $J$  has no double star cutset, contrary to the inductive hypothesis. Thus our assumption that  $G$  has no double star cutset is false. This completes the proof of 3.1.  $\blacksquare$

## 9 Acknowledgement

Our sincere thanks to the referees, who noticed a major mistake in the first version of this paper.

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