TREE INDEPENDENCE NUMBER V. WALLS AND CLAWS

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Abstract. Given a family $\mathcal{H}$ of graphs, we say that a graph G is $\mathcal{H}$-free if no induced subgraph of G is isomorphic to a member of $\mathcal{H}$. Let $S_{t,t,t}$ be the graph obtained from $K_{1,3}$ by subdividing each edge $t-1$ times, and let $W_{t\times t}$ be the t-by-t hexagonal grid. Let $\mathcal{L}_t$ be the family of all graphs G such that G is the line graph of some subdivision of $W_{t\times t}$. We prove that for every positive integer t there exists $c(t)$ such that every $\mathcal{L}_t \cup \{S_{t,t,t}, K_{t,t}\}$-free n-vertex graph admits a tree decomposition in which the maximum size of an independent set in each bag is at most $c(t)\log^4 n$. This is a variant of a conjecture of Dallard, Krnc, Kwon, Milanič, Munaro, Štorgel, and Wiederrecht from 2024. This implies that the Maximum Weight Independent Set problem, as well as many other natural algorithmic problems, that are known to be NP-hard in general, can be solved in quasi-polynomial time if the input graph is $\mathcal{L}_t \cup \{S_{t,t,t}, K_{t,t}\}$-free. As part of our proof, we show that for every positive integer t there exists an integer d such that every $\mathcal{L}_t \cup \{S_{t,t,t}\}$-free graph admits a balanced separator that is contained in the neighborhood of at most d vertices.

1. Introduction

All graphs in this paper are finite and simple and all logarithms are base 2. Let $G = (V(G), E(G))$ be a graph. For a set $X \subseteq V(G)$ we denote by $G[X]$ the subgraph of G induced by X, and by $G \setminus X$ the subgraph of G induced by $V(G) \setminus X$. In this paper, we use induced subgraphs and their vertex sets interchangeably. For graphs G, H we say that G *contains* H if H is isomorphic to $G[X]$ for some $X \subseteq V(G)$. In this case, we say that X *is an* H *in* G. We say that G is H*-free* if G does not contain H. For a family $\mathcal{H}$ of graphs, we say that G is $\mathcal{H}$*-free* if G is H-free for every $H \in \mathcal{H}$.

Let $v \in V(G)$. The *open neighborhood of* v, denoted by $N_G(v)$, is the set of all vertices in $V(G)$ adjacent to v. The *closed neighborhood of* v, denoted by $N_G[v]$, is $N(v) \cup \{v\}$. Let $X \subseteq V(G)$. The *open neighborhood of* X, denoted by $N_G(X)$, is the set of all vertices in $V(G) \setminus X$ with at least one neighbor in X. The *closed neighborhood of* X, denoted by $N_G[X]$, is $N_G(X) \cup X$. When there is no danger of confusion, we omit the subscript G. Let $Y \subseteq V(G)$ be disjoint from X. We say X is *complete* to Y if all edges with an end in X and an end in Y are present in G, and X is *anticomplete* to Y if there are no edges between X and Y.

For a graph G, a *tree decomposition* (T, χ) of G consists of a tree T and a map $\chi\colon V(T) \to 2^{V(G)}$ with the following properties:

(1) For every $v \in V(G)$, there exists $t \in V(T)$ such that $v \in \chi(t)$.
(2) For every $v_1v_2 \in E(G)$, there exists $t \in V(T)$ such that $v_1, v_2 \in \chi(t)$.
(3) For every $v \in V(G)$, the subgraph of T induced by $\{t \in V(T) \mid v \in \chi(t)\}$ is connected.

[†] Princeton University, Princeton, NJ, USA. Supported by NSF Grant DMS-2348219 and by AFOSR grant FA9550-22-1-0083.

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For each $t \in V(T)$, we refer to $\chi(t)$ as a *bag of* (T, χ). The *width* of a tree decomposition (T, χ), denoted by $width(T, \chi)$, is $\max_{t \in V(T)} |\chi(t)| - 1$. The *treewidth* of G, denoted by $tw(G)$, is the minimum width of a tree decomposition of G. Graphs of bounded treewidth are well understood both structurally [23] and algorithmically [3].

A *stable (or independent) set* in a graph G is a set of pairwise non-adjacent vertices of G. The *stability (or independence) number* $\alpha(G)$ of G is the maximum size of a stable set in G. The Maximum Weight Independent Set (MWIS) problem is the problem whose input is a graph G with weights on its vertices, and whose output is a stable set of maximum total weight in G. MWIS is known to be NP-hard [14], but it can be solved efficiently (in polynomial time) in graphs of bounded treewidth. Motivated by this fact, Dallard, Milanič, and Štorgel [12] defined a related graph width parameter, specifically targeting the complexity of the MWIS problem. The *independence number* of a tree decomposition (T, χ) of G is $\max_{t \in V(T)} \alpha(G[\chi(t)])$. The *tree independence number* of G, denoted tree-$\alpha(G)$, is the minimum independence number of a tree decomposition of G. Results of Yolov [24] (or of Dallard et al. [10, 12]) imply that MWIS can be solved in polynomial time on graphs of bounded tree independence number. These results further imply that MWIS can be solved in quasi-polynomial time in graph classes with tree independence number polylogarithmic in the number of vertices. Lima et al. [21] observed that the algorithm of Dallard et al [12] can be extended to a much more general class of problems. We refer the reader to [6] for a detailed discussion of the algorithmic applications of polylogarithmic bounds on the tree independence number. Tree independence number also has connections to coarse geometry [2, 18].

Graph classes admitting useful bounds on their tree independence number were studied further in [11] and [13], where [11] focused on excluding complete bipartite graphs. In particular, the following was conjectured in [11]. Let $\mathcal{S}$ be the set of forests every component of which has at most three leaves, and for a graph G let $L(G)$ denote the line graph of G.

Conjecture 1.1. *For every positive integer t and for all $S, T \in \mathcal{S}$, there exists $c = c(t, S, T)$ such that every $\{K_{t,t}, S, L(T)\}$-free graph has tree independence number at most c.*

In this paper, we study a variant of this conjecture. Our main result is the following. Let t be a positive integer. We denote by $S_{t,t,t}$ the graph obtained from the complete bipartite graph $K_{1,3}$ by subdividing each edge $t-1$ times (so each edge is replaced by a t-edge path, and $K_{1,3}$ is $S_{1,1,1}$). We call the unique degree-three vertex of $S_{t,t,t,}$ *the center* of $S_{t,t,t,}$. We denote by $W_{t \times t}$ the t-by-t hexagonal grid (also known as the $t \times t$-wall). Let $\mathcal{L}_t$ be the family of graphs G for which there exists a subdivision H of $W_{t \times t}$ such that $G = L(H)$. Let $\mathcal{M}_t$ be the class of all $\mathcal{L}_t \cup \{S_{t,t,t}, K_{t,t}\}$-free graphs. We prove:

Theorem 1.2. *For every positive integer t there exists $c = c(t)$ such that every n-vertex graph in $\mathcal{M}_t$ with $n \geq 2$ has tree independence number at most $c \log^4 n$.*

For a graph G, a function $w \colon V(G) \to [0, 1]$ is a *weight function* if $\sum_{v \in V(G)} w(v) \leq 1$. For $S \subseteq V(G)$, we write $w(S) \coloneqq \sum_{v \in S} w(v)$. A weight function w is a *normal weight function* on G if $w(V(G)) = 1$. If $0 < w(V(G)) < 1$, we call the function $w' \colon V(G) \to [0, 1]$ given by $w'(v) = \frac{w(v)}{\sum_{u \in V(G)} w(u)}$ the *normalized weight function of* w. Let $c \in [0, 1]$ and let w be a weight function on G. A set $X \subseteq V(G)$ is a (w, c)-*balanced separator* if $w(D) \leq c$ for every component D of $G \setminus X$. The set X is a *w-balanced separator* if X is a $(w, \frac{1}{2})$-balanced separator. Given two sets of vertices X and Y of G, we say that X is a *core* for Y if $Y \subseteq N[X]$. A graph G is said to be *k-breakable* if for every weight function $w \colon V(G) \to [0, 1]$, there exists a w-balanced separator with a core of size strictly less than k. When the weight function w is clear from the context, we might omit it from the notation. Let $\mathcal{M}^*_t$ be the class of all $\mathcal{L}_t \cup \{S_{t,t,t}\}$-free graphs. As part of the proof of our main result, we show the following.

Theorem 1.3. *For every positive integer t, there is an integer $d = d(t)$ such that every graph $G \in \mathcal{M}^*_t$ is d-breakable.*

This result is of independent interest. It is a significant step in the program of understanding induced subgraph (or induced minor) obstructions to small tree independence number (here by "small" we mean polylogarithmic in the size of the graph). It provides support for the following conjecture that was posed in [15] and seems to be gaining popularity in the community:

Conjecture 1.4. *For every positive integer t, there is an integer $d = d(t)$ such that every $\mathcal{L}_t$-free graph G with no induced subgraph isomorphic to a subdivision of the $t \times t$-wall is d-breakable.*

In turn, Conjecture 1.4, together with Theorem 5.1 and the methods of Section 6, are promising steps toward the following:

Conjecture 1.5. *For every positive integer t, there is an integer $d = d(t)$ such that for every $n \geq 2$, every n-vertex graph with no induced minor isomorphic to $K_{t,t}$ or to $W_{t\times t}$ has tree independence number at most $\log^d n$.*

Also, together with Lemma 7.1 of [6], Theorem 1.3 provides an alternative proof of Theorem 1.4 of [11], namely:

Theorem 1.6. *For every positive integer t and every pair of graphs $S, T \in \mathcal{S}$, there is an integer $d = d(t, S, T)$ such that the tree independence number of every $\{K_{1,t}, S, L(T)\}$-free graph is at most d.*

We remark that the majority of the proofs in this paper work in a slightly more general setup than excluding $S_{t,t,t}$, but we chose to present them in what we consider to be the most natural context. A version of Theorem 1.3 was recently proved independently in [4] by a somewhat different method.

1.1. **Proof outline and organization.** We start with Lemma 7.1 of [6]:

Lemma 1.7. *Let G be a graph, let $c \in [\frac{1}{2}, 1)$, and let d be a positive integer. If for every normal weight function w on G, there is a (w, c)-balanced separator X_w with $\alpha(X_w) \leq d$, then the tree independence number of G is at most $\frac{3-c}{1-c}d$.*

In view of Lemma 1.7, in order to prove Theorem 1.2 it is enough to show:

Theorem 1.8. *For every positive integer t, there exists $c = c(t)$ such that for every $n \geq 2$, every n-vertex graph G in $\mathcal{M}_t$, and every normal weight function w on G, there is a $(w, \frac{1}{2})$-balanced separator X_w in G with $\alpha(X_w) \leq c \log^4 n$.*

We now outline the proof of Theorem 1.8.

Our first goal is to prove Theorem 1.3. An important tool in that proof is "extended strip decompositions" from [8]. They are introduced in Section 2, and in Section 3 we prove several results about the behavior of extended strip decompositions in $\mathcal{L}_t$-free graphs.

The actual proof of Theorem 1.3 is presented in Section 4; it proceeds as follows. Let $G \in \mathcal{M}_t^*$. We may assume that G is connected. From now on we fix a weight function w and assume that G does not have a w-balanced separator with a small core. By using the normalized weight function of w, we may assume that w is normal. By Lemma 5.3 of [7] there is an induced path $P = p_1\text{-}\dots\text{-}p_k$ in G such that $N[P]$ is a w-balanced separator in G. Choosing P with k minimum, we may assume that there is a component B of $G \setminus N[P \setminus \{p_k\}]$ with $w(B) > \frac{1}{2}$. We now analyze the structure of the set $N = N(B) \subseteq N(P \setminus \{p_k\})$. We say that $v \in N$ is a *hat* if v has exactly two neighbors in P, and they are adjacent. First, we show that the set of all vertices in N that are not hats has a small core. Now we focus on one hat h, and use it to show that G (with a subset with a small core deleted) admits an extended strip decomposition. This allows us to produce a separator $S(h)$ with a small core that is not yet balanced but exhibits several useful properties. More explicitly, the component of $G \setminus S(h)$ with maximum w-weight only meets P on one side of h. So h either "points left" or "points right". Then we show that the hat with the earliest neighbors in P points right, and the hat with the latest neighbors in P points left. Now we focus on two consecutive hats h, h' where the change first occurs, and conclude that $S(h) \cup S(h')$ is a w-balanced separator in G. This completes the proof of Theorem 1.3.

Let us now continue with steps toward the proof of Theorem 1.8, and so assume that $G \in \mathcal{M}_t$. Our next goal is to show that we can choose sets $Y_1, \dots, Y_{\lceil \log n \rceil}$ such that for every j, $|Y_j| \leq d$, Y_j is a core of a w-balanced separator for G, and, for an appropriately chosen integer D, no vertex of G belongs to more than $\frac{\log n}{D}$ of the sets $N[Y_j]$. To do so, we continue selecting sets Y_j as above, keeping track of the so-called "layers" L^i_j, which are sets of vertices that belong to at least i out of the j separators chosen so far. We maintain the property that $\alpha(L^i_j)$ is bounded from above by a value that decreases geometrically with i and increases geometrically with j, but at a much slower rate (see (21) for details). The main result of Section 5 ensures that we are able to maintain this property by deleting a set of small stability number. As a consequence, the sets $L^i_{\lceil \log n \rceil}$ are empty for large enough i, and that is what we needed to achieve. We call this technique "the layered set" argument. This is done in Section 6 (in fact, the result there is more general, to accommodate the proofs in Section 7).

Next, in Subsection 7.1 we strengthen the conclusion of Theorem 1.3 and establish the existence of a more refined type of separator, that we call a *boosted separator*. Let (S, C) be a pair of subsets of $V(G)$ and let B be a component of $G \setminus C$ with maximum weight. For $\varepsilon \in (0, \frac{1}{2}]$, the pair (S, C) is said to be a (w, ε)-*boosted separator* if $w(B) \leq 1/2$ or if S is an ε-balanced separator of B. We call C the *boosting set* of (S, C). A set $X \subseteq V(G)$ is said to be a *core* of (S, C) if $S \subseteq N_G[X]$. The existence of boosted separators with small cores, where, in addition, the boosting set has a small stability number, is established in Theorem 7.6 by an application of the layered set argument (to a carefully chosen weight function different from w).

Now, another application of the layered set argument allows us to construct sets $Y_1, \dots, Y_{\lceil \log n \rceil}$ and a set C such that $|Y_i| \leq d$ (where d is a fixed integer), C has small stability number, each $(N[Y_i], C)$ is a (w, ε)-boosted separator, and no vertex of G belongs to more than $\frac{\log n}{D}$ of the sets $N[Y_i]$ (for appropriately chosen ε and D). In Subsection 7.2 we use this collection of sets to select a subcollection $Y_1, \dots, Y_{8t^2d}$ such that no vertex of G belongs to more than t of the sets $N[Y_i]$. This is done in Theorem 7.9.

We are now ready to put everything together. This is done in Section 8, where we use the results described above to complete the proof of Theorem 1.8. By a first moment argument, we produce a large set X of vertices such that for at least $4t^2d$ of the sets $Y_1, \dots, Y_{8t^2d}$ above, no two vertices of X belong to the same component of $G \setminus (C \cup N[Y_i])$; say these are sets $Y_1, \dots, Y_{4t^2d}$. Then we use a result of [5] to describe the structure of a minimal connected subgraph H of G containing X. We deduce that H contains a large set $\mathcal{P}$ of pairwise disjoint paths, each of which meets at least t of the sets $N[Y_1], \dots N[Y_{4t^2d}]$ above. We consider minimal subpaths of elements of $\mathcal{P}$ with this property. Using the fact that no vertex of G belong to t of the sets $N[Y_i]$, together with several Ramsey-type arguments, we can construct paths $P_1, \dots, P_{3d}$ of length at least t, that are subpaths of distinct elements of $\mathcal{P}$, such that the first vertex of each P_j belongs to $N[Y_1]$ (say), and the rest of P_j is disjoint from $N[Y_1]$ (and therefore anticomplete to Y_1). Since $|Y_1| \leq d$, it follows that there exist $y \in Y_1$ and three paths $P, Q, R \in \{P_1, \dots, P_{3d}\}$ such that $P \cup Q \cup R \cup \{v\}$ is an $S_{t,t,t}$ in G, a contradiction. This completes the proof of 1.8.

2. Constricted sets and extended strip decompositions

An important tool in the proof of Theorem 1.3 is "extended strip decompositions" of [8]. We explain this now. A set $C \subseteq G$ is a *hole in* G if $G[C]$ is cycle of length at least four. Similarly, a set $P \subseteq G$ is a *path in* G if $G[P]$ is a path. Let $P = \{p_1, \dots, p_k\}$ be a path in G where $p_ip_j \in E(G)$ if and only if $|j - i| = 1$. We say that p_1 and p_k are the *ends* of P. The *interior* of P, denoted by P^*, is the set $P \setminus \{p_1, p_k\}$. For $i, j \in \{1, \dots k\}$ we denote by p_i-P-p_j the subpath of P with ends p_i, p_j. Let G, H be graphs, and let $Z \subseteq V(G)$. Let W be the set of vertices of degree one in H. Let $T(H)$ be the set of all triangles of H. Let η be a map with domain the union of $E(H)$, $V(H)$, $T(H)$, and the set of all pairs (e, v) where $e \in E(H)$, $v \in V(H)$, and e incident with v, and range $2^{V(G)}$, satisfying the following conditions:

- For every $v \in V(G)$ there exists $x \in E(H) \cup V(H) \cup T(H)$ such that $v \in \eta(x)$.
- If $x, y \in E(H) \cup V(H) \cup T(H)$ and $x \neq y$, then $\eta(x) \cap \eta(y) = \emptyset$.
- For every $e \in E(H)$ and $v \in V(H)$ such that e is incident with v, $\eta(e, v) \subseteq \eta(e)$.

- Let $e, f \in E(H)$ with $e \neq f$, and $x \in \eta(e)$ and $y \in \eta(f)$. Then $xy \in E(G)$ if and only if e, f share an end-vertex v in H, and $x \in \eta(e, v)$ and $y \in \eta(f, v)$.
- If $v \in V(H)$, $x \in \eta(v)$, $y \in V(G) \setminus \eta(v)$, and $xy \in E(G)$, then $y \in \eta(e, v)$ for some $e \in E(H)$ incident with v.
- If $D \in T(H)$, $x \in \eta(D)$, $y \in V(G) \setminus \eta(D)$ and $xy \in E(G)$, then $y \in \eta(e, u) \cap \eta(e, v)$ for some distinct $u, v \in D$, where e is the edge uv of H.
- $|Z| = |W|$, and for each $z \in Z$ there is a vertex $w \in W$ such that $\eta(e, w) = \{z\}$, where e is the (unique) edge of H incident with w.

Under these circumstances, we say that η is an *extended strip decomposition of* (G, Z) *with pattern* H (see Figure 1). Let e be an edge of H with ends u, v. An *e-rung* in η is a path p_1-$\dots$-p_k (possibly $k = 1$) in $\eta(e)$, with $p_1 \in \eta(e, v)$, $p_k \in \eta(e, u)$ and $\{p_2, \dots, p_{k-1}\} \subseteq \eta(e) \setminus (\eta(e, v) \cup \eta(e, u))$. We say that η is *faithful* if for every $e \in E(H)$, there is an e-rung in η.

A set $A \subseteq V(G)$ is an *atom* of η if one of the following holds:

- $A = \eta(x)$ for some $x \in V(H) \cup T(H)$.
- $A = \eta(e) \setminus (\eta(e, u) \cup \eta(e, v))$ for some edge e of H with ends u, v.

For an atom A of η, the *boundary* $\delta(A)$ of A is defined as follows:

- If $v \in V(H)$ and $A = \eta(v)$, then $\delta(A) = \bigcup_{e \in E(H)\,:\, e \text{ is incident with } v} \eta(e, v)$.
- If $A = \eta(D)$, and $D \in T(H)$ with $D = v_1v_2v_3$, then $\delta(A) = \bigcup_{i \neq j \in \{1,2,3\}} \eta(v_iv_j, v_i) \cap \eta(v_iv_j, v_j)$.
- If $A = \eta(e) \setminus (\eta(e, u) \cup \eta(e, v))$ for some edge e of H with ends u, v, then $\delta(A) = \eta(e, u) \cup \eta(e, v)$.

A set $Z \subseteq V(G)$ is *constricted* for every $T \subseteq G$ such that T is a tree, $|Z \cap V(T)| \leq 2$.

The main result of [8] is the following.

Theorem 2.1. *Let G be a connected graph and let $Z \subseteq V(G)$ with $|Z| \geq 2$. Then Z is constricted if and only if for some graph H, (G, Z) admits a faithful extended strip decomposition with pattern H.*

We also need the following, which is an immediate corollary of Lemma 6.8 of [7]:

Lemma 2.2. *Let G, H be graphs, $Z \subseteq V(G)$ with $|Z| \geq 3$, and let η be an extended strip decomposition of (G, Z) with pattern H. Let Q_1, Q_2, Q_3 be paths in G, pairwise anticomplete to each other, and each with an end in Z. Then for every atom A of η, at least one of the sets $N[A] \cap Q_1$, $N[A] \cap Q_2$ and $N[A] \cap Q_3$ is empty.*

We finish this section with another lemma.

Lemma 2.3. *Let G, H be graphs, $Z \subseteq V(G)$ with $|Z| \geq 2$, and let η be a faithful extended strip decomposition of (G, Z) with pattern H. Let A be an atom of η. Then $\delta(A)$ has a core of size at most* 3*.*

Proof. Suppose first that $A = \eta(v)$ for some $v \in V(H)$. If v has degree one in H, then by the definition of an extended strip decomposition, $|\eta(v, e)| = 1$ (where e is the unique edge incident with v), and so $|\delta(A)| = 1$. Thus we may assume that v is incident with at least two edges, say e and f, in H. Since η is faithful, there exist $x \in \eta(e, v)$ and $y \in \eta(f, v)$. But now $\delta(A) \subseteq N[\{x, y\}]$, as required.

Next assume that $A = \eta(D)$ and $D \in T(H)$ with $D = v_1v_2v_3$. Since η is faithful, there exist $x \in \eta(v_1v_2, v_1)$ and $y \in \eta(v_1v_3, v_1)$ and $z \in \eta(v_1v_2, v_2)$. Now $\delta(A) \subseteq N[\{x, y, z\}]$ as required.

Thus we may assume that $A = \eta(e) \setminus (\eta(e, u) \cup \eta(e, v))$ for some edge e of H with ends u, v. We may assume that the degree of u in H is at least 2; let $f \neq e$ be an edge of H incident with u. Since η is faithful, there exists $x \in \eta(f, u)$, and $\eta(e, u) \subseteq N(x)$. If v has degree one in H, then $|\eta(e, v)| = 1$, and $\delta(A) \subseteq N[\eta(e, v) \cup \{x\}]$, as required. Thus we may assume that there is an edge $f' \neq e$ such that v is incident with f'. Since η is faithful, there exists $y \in \eta(f', v)$. Now $\delta(A) \subseteq N(\{x, y\})$, and the conclusion of the theorem holds. ■

3. Extended strip decompositions in graphs in $\mathcal{M}_t^*$

In this section we prove several results about the behavior of extended strip decompositions in $\mathcal{L}_t$-free graphs, that we will use in the proof of Theorem 1.3.

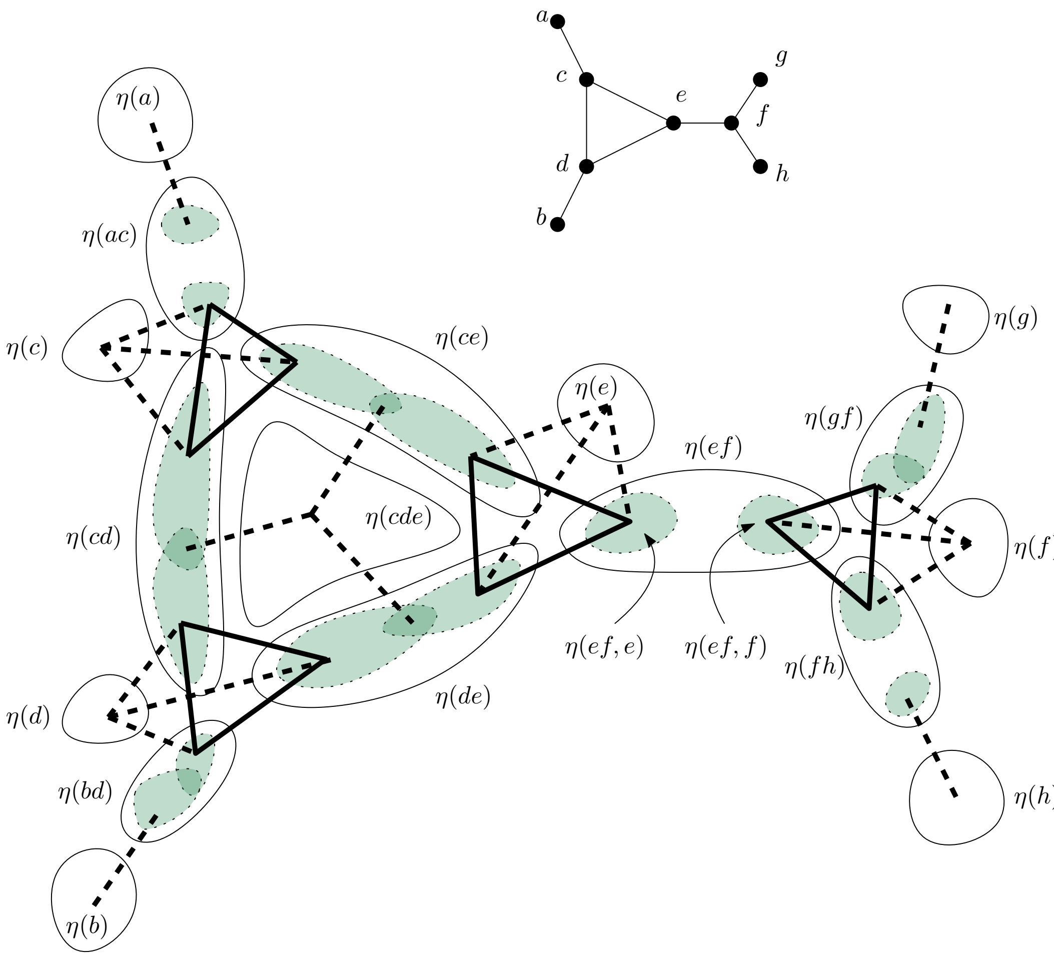

Figure 1. Example of an extended strip decomposition with its pattern (here dash lines represent potential edges). This figure was created by Paweł Rzążewski and we use it with his permission.

We need a result of [9]:

Theorem 3.1 ([9])**.** *There exist positive integers c_1 and c_2 such that for every positive integer t, every graph with no subgraph isomorphic to a subdivision of the $(t \times t)$-wall has treewidth at most $c_1 t^9 \log^{c_2} t$.*

Our first goal is to prove the following:

Theorem 3.2. *Let t be an integer, let G be an $\mathcal{L}_t$-free graph, and let $Z \subseteq V(G)$. Let η be a faithful extended strip decomposition of G with pattern H. Then $\mathsf{tw}(H) \leq c_1 t^9 \log^{c_2} t$, where c_1, c_2 are as in Theorem 3.1.*

Proof. By Theorem 3.1 it is enough to show that no subgraph of H is isomorphic to a subdivision of the $(t \times t)$-wall. Since η is faithful, for every $e \in E(H)$, we can choose an e-rung R_e in η. Let $G' = \bigcup_{e \in E(H)} R_e$. Then there exists a graph H', obtained from H by subdividing edges, such that $G' = L(H')$. Now, let F be a subgraph of H isomorphic to a subdivision of the $(t \times t)$-wall. Let $G'' = \bigcup_{e \in E(F)} R_e$. Then G'' is the line graph of a graph F'', where F'' is a graph obtained from F by subdividing edges, contrary to the fact that G is $\mathcal{L}_t$-free. ■

Before we embark on the proof of Theorem 1.3, we need one final lemma:

Lemma 3.3. *Let $t \geq 2$ be an integer. Let G be an $\mathcal{L}_t$-free graph, and let w be a weight function on G. Let D be a component of G with $w(D) > \frac{1}{2}$. Let $Z \subseteq D$, and let η be a faithful extended strip decomposition of (D, Z) with pattern H. Assume that $w(A) \leq \frac{1}{2}$ for every atom A of η. Let c_1, c_2 be as in Theorem 3.1. Then there exists $Y \subseteq V(G)$ with $|Y| \leq 3c_1 t^9 \log^{c_2} t$, such that $N[Y]$ is a w-balanced separator in G.*

Proof. By working with the normalized weight function of w, we may assume that w is normal. Let H' be obtained from H as follows. Subdivide every edge e of H once; call the new vertex v_e. For every $v \in V(H)$, add a new vertex v_v adjacent to v and with no other neighbors. For every triangle $T = uvw$ of H, add a vertex v_T adjacent to u, v, w with no other neighbors (see Figure 2). Observe that vertices of $V(H') \setminus V(H)$ correspond to atoms of η. For every component D' of $G \setminus D$, add a new isolated vertex $v_{D'}$.

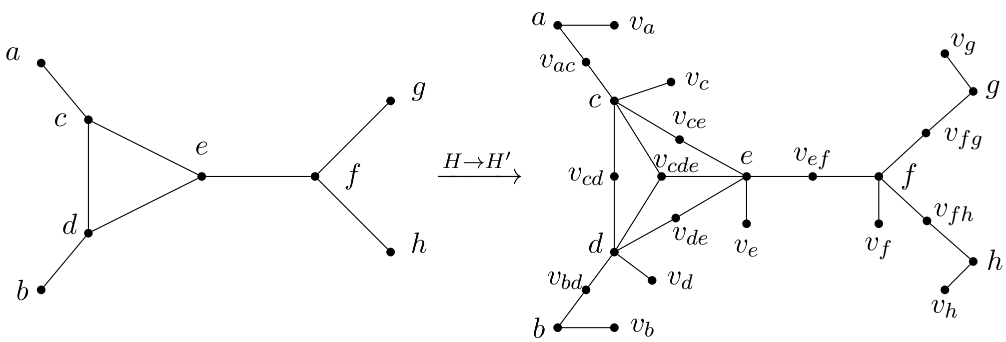

FIGURE 2. Example of H' given an H.

Now define a weight function w' on H'. For every $v \in V(H)$, let $w'(v) = w(\delta(\eta(v)))$, that is, $w'(v) = \sum_{e \in E(H) \text{ incident with } v} w(\eta(e, v))$, and let $w'(v_v) = w(\eta(v))$. For every $e \in E(H)$ with ends u, v, let $w'(v_e) = w(\eta(e) \setminus (\eta(e, v) \cup \eta(e, u)))$. For every triangle T of H, let $w'(v_T) = w(\eta(T))$. For every component D' of $G \setminus D$, let $w'(v_{D'}) = w(D')$. Now w' is a normal function on H'.

By Theorem 3.2 $\mathsf{tw}(H) \leq c_1 t^9 \log^{c_2} t$. Since H' is obtained from H by subdividing edges and adding vertices whose neighborhood is a clique of size at most three, it is easy to see that $\mathsf{tw}(H') \leq \max\{\mathsf{tw}(H), 3\}$, and so $\mathsf{tw}(H') \leq c_1 t^9 \log^{c_2} t$. By a result from [19], it follows that there exists a w'-balanced separator $X' \subseteq V(H')$ with $|X'| \leq c_1 t^9 \log^{c_2} t$. We may assume that for every component D' of $G \setminus D$, $v_{D'} \notin X'$. Next, we use X' to obtain a w-balanced separator X in G. First, for every $v \in X' \cap V(H)$, add to X the set $\bigcup_{e \in E(H) \text{ incident with } v} \eta(e, v)$. Second, for every $e \in E(H)$ with ends u, v such that $v_e \in X'$, add to X the boundary of the atom $\eta(e) \setminus (\eta(e, u) \cup \eta(e, v))$. Third, for every $v \in V(H)$ such that $v_v \in X'$, add to X the boundary of the atom $\eta(v)$. Finally, for every triangle $T = uvw$ of H with $v_T \in X'$, add to X the boundary of the atom $\eta(T)$. This completes the construction of X.

Since by Lemma 2.3 for every atom A of η, $\delta(A)$ has a core of size at most three, and since for every $v \in V(H)$, the set $\bigcup_{e \in E(H) \text{ incident with } v} \eta(e, v)$ has a core of size at most two, it follows that there exists $Y \subseteq V(H)$ with $|Y| \leq 3c_1 t^9 \log^{c_2} t$ such that $X \subseteq N[Y]$.

It remains to show that X is a w-balanced separator in G. Suppose not, and let C be a component of $G \setminus X$ with $w(C) > \frac{1}{2}$. Let $U \subseteq V(H) \cup E(H) \cup T(H)$ be such that $u \in U$ if and only if one of the following holds:

- $\eta(u) \cap C \neq \emptyset$, or
- $u \in V(H)$ and there exists $e \in E(H)$ incident with u such that $\eta(e, u) \cap C \neq \emptyset$.

Let $f : V(H) \cup E(H) \cup T(H) \to V(H') \setminus V(H)$, where $f(x) = v_x$, and let $U' = (U \cap V(H)) \cup f(U)$. Then $U' \subseteq V(H')$.

(1) *$U' \cap X' = \emptyset$.*

Suppose that there exists $u' \in U' \cap X'$. Define $u \in U$ as follows. If $u' \in U \cap V(H)$, let $u = u'$. If $u' \in f(U)$, let $u \in U$ be such that $u' = v_u$. Assume that $\eta(u) \cap C = \emptyset$. Since $u' \in U'$, it follows that $u = u'$, and there exists $e \in E(H)$ incident with u such that $\eta(e, u) \cap C \neq \emptyset$. However, $\eta(e, u) \subseteq X$ since $u' \in X'$, contrary to the fact that $C \cap X = \emptyset$. This proves that $\eta(u) \cap C \neq \emptyset$ for every $u' \in U' \cap X'$. Assume first that $u' = v_e$ for some edge $e \in E(H)$ with ends x, y. Then $u = e$ and $\eta(e, x) \cup \eta(e, y) \subseteq X$, and so, since C is connected and $C \cap \eta(e) \neq \emptyset$, it follows that $C \subseteq \eta(e) \setminus (\eta(e, x) \cup \eta(e, y))$. But then C is a subset of an atom of η, and so $w(C) \leq \frac{1}{2}$, a contradiction. It follows that $u' = v_x$ for some $x \in V(H) \cup T(H)$. Then $\delta(\eta(x)) \subseteq X'$, and so, since C is connected, it follows that $C \subseteq \eta(x)$. Again C is a subset of an atom of η, and so $w(C) \leq \frac{1}{2}$, a contradiction. This proves (1).

Since C is connected, we deduce that U' is connected, and therefore, by (1), U' is contained in a component of $H' \setminus X'$. But $w(C) \leq \sum_{u \in U'} w'(u)$, contrary to the fact that X' is a w-balanced separator of H'. ■

4. Bounded core separators in graphs in $\mathcal{M}_t^*$

We are now ready to complete the first step in the proof of Theorem 1.8, that is, Theorem 1.3, which we restate.

Theorem 1.3. *For every positive integer t, there is an integer $d = d(t)$ such that every graph $G \in \mathcal{M}_t^*$ is d-breakable.*

Proof. We may assume that $t \geq 2$. Let $G \in \mathcal{M}_t^*$ and let w be a weight function on G. By working with the normalized function of w, we may assume that w is normal. Let c_1, c_2 be as in Theorem 3.1. Let $d = 3c_1 t^9 \log^{c_2} t + 22t$. We will show that there is a set $Y \subseteq G$ with $|Y| < d$ such that $N[Y]$ is a $(w, \frac{1}{2})$-balanced separator in G. Suppose no such Y exists.

By the proof of Lemma 5.3 of [7] there is a path P in G such that $N[P]$ is a w-balanced separator in G. Let $P = p_1\text{-}\dots\text{-}p_k$, and assume that P was chosen with k minimum. It follows that there exists a component B of $G \setminus N[P \setminus \{p_k\}]$ such that $w(B) > \frac{1}{2}$. Let $N = N(B)$. Then $N \subseteq N(P \setminus \{p_k\})$. First, we show

(2) *There is no $Y \subseteq G$ with $|Y| < d$ such that $N \cup N[p_k] \subseteq N[Y]$.*

Suppose such Y exists. We will show that $N[Y]$ is a w-balanced separator in G. We may assume that there is a component D of $G \setminus N[Y]$ with $w(D) > \frac{1}{2}$. Since $w(B) > \frac{1}{2}$, we deduce that $D \cap B \neq \emptyset$. Since $N \subseteq N[Y]$, it follows that $D \subseteq B$, and so $D \cap N[P] \subseteq N[p_k]$. Since $N[p_k] \subseteq N[Y]$, we deduce that D is contained in a component of $G \setminus N[P]$, and therefore $w(D) < \frac{1}{2}$, a contradiction. This proves that $N[Y]$ is a w-balanced separator in G, contrary to our assumption, and (2) follows.

Let $a, b \in \mathbb{Z}_{\geq 0}$ such that $k = 2at + b$ and $b < 2t$. For $i \in \{1, \dots, a-1\}$ let $P_i = p_{2(i-1)t+1}\text{-}\dots\text{-}p_{2it}$, and let $P_a = p_{2(a-1)t+1}\text{-}\dots\text{-}p_k$. Let $Y_1 = P_1 \cup P_a$. Then $|Y_1| < 6t$. Let $N_1 = N \setminus N[Y_1]$. We deduce from (2):

(3) *There is no $Y \subseteq G$ with $|Y| \leq d - 6t$ such that $N_1 \subseteq N[Y]$.*

In the next several arguments, we will treat the two ends of P symmetrically since we will only use the property that $N(B) \subseteq N(P \setminus \{p_k\})$. We will state explicitly when additional properties of P come into play, and stop using this symmetry. We call $v \in N_1$ a *hat* if v has exactly two neighbors in P, and these neighbors are consecutive in P. Let $H_1 \subseteq N_1$ be the set of all hats. Our next goal is to reduce the problem to the case when $N_1 = H_1$.

(4) *For every $v \in N_1 \setminus H_1$, $\alpha(N(v) \cap P) \geq 3$.*

Suppose that there is $v \in N_1 \setminus H_1$ with $\alpha(N(v) \cap P) \leq 2$. Let r be minimum and s be maximum such that v is adjacent to p_r, p_s. Since $v \in N_1$, it follows that $r > 2t$ and $s \leq k - 2t$. Since $\alpha(N(v) \cap P) \leq 2$, we deduce that $N(v) \cap P \subseteq \{p_r, p_{r+1}, p_{s-1}, p_s\}$. Let $R = p_{r-2t+1}$-P-p_{r+2t+1} and let $S = p_{s-2t}$-P-p_{s+2t}. Let $Z = R \cup S$. Then $|Z| \leq 2(4t + 2) \leq d - 6t$, and so there exists $w \in N_1 \setminus N(Z)$.

Since v, w are both in N_1, there is a path Q from v to w with $Q^* \subseteq B$. Let i be minimum and j be maximum such that w is adjacent to p_i, p_j. Since $w \in N_1$, it follows that $i > 2t$ and $j \leq k - 2t$.

Suppose first that $r = s$. Since w is anticomplete to R, it follows that $p_i \notin R$. Now we get an $S_{t,t,t}$ with center p_r two of whose paths are subpaths of R^* and the third is a subpath of p_r-v-Q-w-p_i-P-p_{i-t+3}, a contradiction.

This proves that $r \neq s$, and since $v \notin H_1$, it follows that $s > r + 1$. Now we get an $S_{t,t,t}$ with center v whose paths are v-p_r-R-p_{r-t+1}, v-p_s-S-p_{s+t-1} and a subpath of v-Q-w-p_i-P-p_{i-t+2}, again a contradiction. This proves (4).

(5) *There do not exist $1 < i < j < \ell < a$ and $v \in N_1$ such that v has a neighbor in P_i and a neighbor in P_ℓ, and v is anticomplete to P_j.*

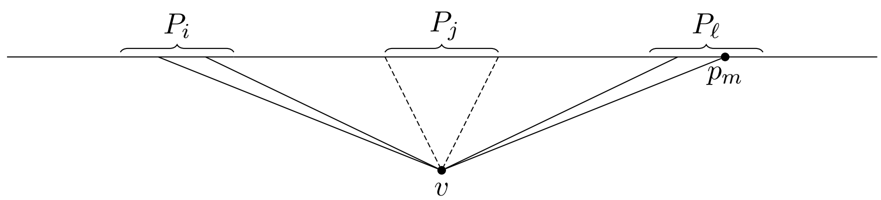

FIGURE 3. Visualization for (5).

Suppose such i, j, ℓ, v exist (see Figure 3). Then $v \notin H_1$. By (4) we may assume that v has two non-adjacent neighbors in p_1-P-$p_{2t(j-1)}$. It follows that there exist subpaths Q, R with $|Q| = |R| = t$ of p_1-P-$p_{2t(j-1)+t}$ and anticomplete to each other such that v is adjacent to exactly one end of Q and has no other neighbors in Q, and v is adjacent to exactly one end of R and has no other neighbors in R. Let m be maximum such that v is adjacent to p_m. Since $v \in N_1$, it follows that $m \leq k - 2t$. Now we get an $S_{t,t,t}$ with center v and path v-Q, v-R and v-p_m-P-p_{m+t-1}, a contradiction. This proves (5).

(6) *There exist $i, j \in \{2, \ldots, a-1\}$ such that $N_1 \setminus H_1 \subseteq N(P_i) \cup N(P_j)$.*

Let $\mathcal{I} \subseteq \{2, \ldots, a-1\}$ be such that $N_1 \setminus H_1 \subseteq \bigcup_{i \in \mathcal{I}} N(P_i)$ and with $|\mathcal{I}|$ minimum. We may assume that $|\mathcal{I}| \geq 3$; let $i, j, \ell \in \mathcal{I}$ with $i < j < \ell$. By the minimality of $|\mathcal{I}|$, there exist $v_i, v_\ell \in N_1 \setminus H_1$ such that $v_i \subseteq N(P_i) \setminus N(P_j)$ and $v_\ell \subseteq N(P_\ell) \setminus N(P_j)$. By (5), we have that v_i is anticomplete to $\bigcup_{m \geq j} P_m$ and v_ℓ is anticomplete to $\bigcup_{m \leq j} P_m$. Since both $v_i, v_\ell \in N_1$, there is a path Q from v_i to v_ℓ with $Q^* \subseteq B$. Let r be minimum and s be maximum such that v_i is adjacent to p_r, p_s. Then $s \leq 2t(j-1)$ and, by (4), $r + 1 < s$. Since $v_i \in N_1$, it follows that $r > 2t$. Let q be minimum such that v_ℓ is adjacent to p_q. Then $q > 2tj$. Now we get an $S_{t,t,t}$ with center v_i and whose paths are v_i-p_r-P-p_{r-t+1}, v_i-p_s-P-p_{s+t-1}, and a subpath of v_i-Q-v_ℓ-p_q-P-p_{q-t+2}, a contradiction. This proves (6).

By (6), there exist $i', j' \in \{2, \ldots, a-2\}$ (possibly $i' = j'$) such that $N_1 \setminus H_1 \subseteq N(P_{i'}) \cup N(P_{j'})$.

Let $Y_2 = P_{i'} \cup P_{j'}$. Then $|Y_1 \cup Y_2| < 10t$ and $N_1 \setminus H_1 \subseteq N(Y_1 \cup Y_2)$. Let $H_2 = H_1 \setminus N(Y_1 \cup Y_2)$. By (3), $H_2 \neq \emptyset$.

From now on we will use additional properties of P, so we no longer use symmetry between its two ends. Let x be minimum such that there exists $h_0 \in H_2$ with $N(h_0) \cap P = \{p_x, p_{x+1}\}$. Let y be maximum such that there exists $h_1 \in H_2$ with $N(h_1) \cap P = \{p_y, p_{y+1}\}$; we refer to this later as the *maximality of* h_1. (See Figure 4.)

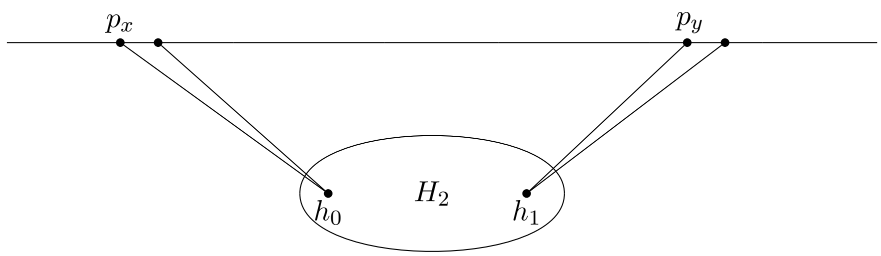

FIGURE 4. Visualization of h_0 and h_1.

For all $h \in H_2$, we define a subpath $P(h)$ of P as follows: $P(h) = p_{i-2t-1}\text{-}P\text{-}p_{i+2t+2}$, where $N(h) \cap P = \{p_i, p_{i+1}\}$. Let $H_3 = H_2 \setminus N(P(h_0))$.

(7) $H_3 \neq \emptyset$.

Suppose $H_3 = \emptyset$. Let $Y = Y_1 \cup Y_2 \cup P(h_0)$. Now $|Y| < 16t < d$ and $N \cup N[p_k] \subseteq N[Y]$, contrary to (2). This proves (7).

Our next goal is to define, for every $h \in H_3$, a graph G_h, a triple $(z_1(h), z_2(h), z_3(h))$ of vertices of G_h, and an extended strip decomposition η_h of $(G_h, \{z_1(h), z_2(h), z_3(h)\})$. So let $h \in H_3$ and let $N(h) \cap P = \{p_i, p_{i+1}\}$. Let $z_1(h) = p_{i-2t-1}$, $z_2(h) = p_{i+2t+2}$ and $z_3(h) = h$. Write $P_L(h) = p_1\text{-}P\text{-}p_{i-2t-2}$ and $P_R(h) = p_{i+2t+3}\text{-}P\text{-}p_k$. We define

$$G'_h = (G \setminus N[Y_1 \cup Y_2 \cup P(h_0) \cup P(h)]) \cup (P_L(h) \cup P_R(h) \cup (N(p_k) \cap B) \cup \{z_1(h), z_2(h), z_3(h)\}).$$

For $i \in \{1, 2, 3\}$ write $z_i = z_i(h)$. Then $z_1, z_2, z_3 \in G'_h$, and $B \subseteq G'_h$. Let G_h be the component of G'_h containing B. Then $z_3 \in G_h$. Since $p_k \in G'_h$ and p_k has a neighbor in B, it follows that $p_k \in G_h$, and consequently $z_2\text{-}P\text{-}p_k \subseteq G_h$. Since h_0 has a neighbor in B, it follows that $h_0 \in G_h$; and, since $h \in H_3$, we deduce that $p_1\text{-}P\text{-}z_1 \subseteq G_h$. (See Figure 5.)

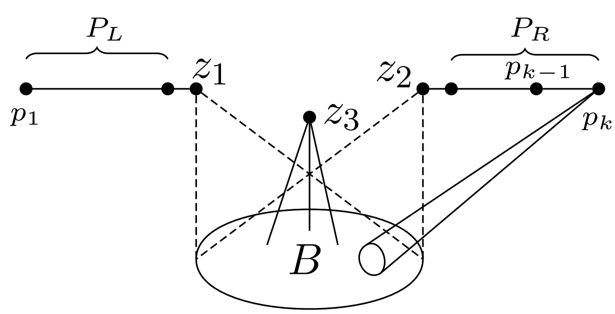

FIGURE 5. Visualization of G_h (here dashed lines represent non-edges).

(8) $\{z_1, z_2, z_3\}$ *is constricted in* G_h.

Suppose there is a tree T in G_h such that $z_1, z_2, z_3 \in V(T)$. We may choose T minimal with this property; then either T is a subdivision of $K_{1,3}$ and z_1, z_2, z_3 are leaves on T, or T is a path with ends in $\{z_1, z_2, z_3\}$. Since $G_h \setminus \{z_1, z_2, z_3\}$ is anticomplete to $P(h)$, in both cases $T \cup P(h)$ contains $S_{t,t,t}$, a contradiction. This proves (8).

By Theorem 2.1, there is a graph $H(h)$ such that $(G_h, \{z_1, z_2, z_3\})$ admits a faithful extended strip decomposition η_h with pattern $H(h)$.

(9) *There exists an atom $A(h)$ of η_h such that $w(A(h)) > \frac{1}{2}$.*

Suppose not. Then by Lemma 3.3 applied to G_h and w (with $D = G_h$), we deduce that there exists $Y_h \subseteq G_h$ with $|Y_h| \leq d - 22t$, such that $N[Y_h]$ is a w-balanced separator in G_h. It follows from the definition of G_h that $N[Y_h \cup Y_1 \cup Y_2 \cup P(h_0) \cup P(h)]$ is a w-balanced separator in G, which is a contradiction since $|Y_1 \cup Y_2 \cup P(h_0) \cup P(h)| < 22t$. This proves (9).

Let $A(h)$ be as in (9).

(10) *At least one of the sets $P_L(h) \cap A(h)$, $P_R(h) \cap A(h)$ is empty.*

Suppose not. Let $Q_1 = z_1\text{-}p_{i-2t-2}\text{-}P_L(h)$. Since $N[P]$ is a w-balanced separator, there exists $m \in N(p_k) \cap B$; let Q be a path from h to m with $Q^* \subseteq B$. Then $F = z_2\text{-}p_{i+2t+3}\text{-}P_R(h)\text{-}p_k\text{-}m\text{-}Q\text{-}h$ is a path. Since $P_R(h) \cap A(h) \neq \emptyset$, there exists $f \in A(h) \cap F$. It follows from the last bullet of the definition of an extended strip decomposition that z_2, z_3 do not belong to any atoms, and so $f \notin \{z_2, z_3\}$. Let Q_2' be the subpath of F from z_2 to f, and let Q_3' be the subpath of F from z_3 to f (see Figure 6). Let $Q_2 = Q_2' \setminus f$ and $Q_3 = Q_3' \setminus f$. Then Q_1, Q_2, Q_3 are pairwise disjoint and anticomplete to each other; z_i is an end of Q_i, and $Q_i \cap N[A] \neq \emptyset$ for every $i \in \{1, 2, 3\}$, contrary to Lemma 2.2. This proves (10).

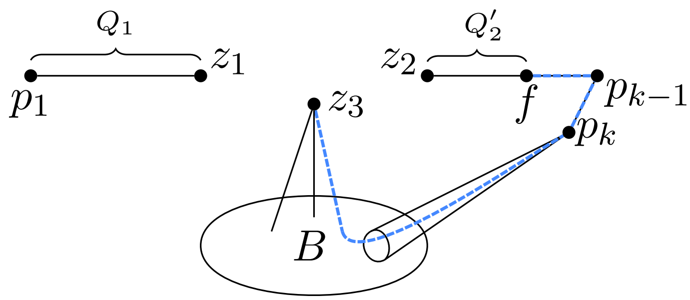

FIGURE 6. Visualization of Q_1, Q_2' and Q_3' (in blue).

Let $\delta(h)$ be the boundary of $A(h)$ in η_h and let $\gamma(h) = \delta(h) \cap P$.

(11) $|\gamma(h)| \leq 9$.

By Lemma 2.3 there exists $\Delta \subseteq G_h$ with $|\Delta| \leq 3$ such that $\delta(h) \subseteq N[\Delta]$. Since $N(P) \cap G_h \subseteq H_3$, for every $v \in \Delta$, $|N[v] \cap P| \leq 3$. Consequently, $|\delta(h) \cap P| \leq |N[\Delta] \cap P| \leq 9$, and (11) follows.

(12) *Let $Z \subseteq V(G)$ with $Y_1 \cup Y_2 \cup P(h_0) \cup P(h) \subseteq Z$ and such that $\delta(h) \subseteq N[Z]$. Let $D \subseteq G \setminus N[Z]$ be connected with $w(D) > \frac{1}{2}$. Then $D \subseteq A(h)$ and there exists $v \in D \cap N(B)$.*

Since $w(B) > \frac{1}{2}$, it follows that $B \cap D \neq \emptyset$. Similarly, since $w(A(h)) > \frac{1}{2}$, $A(h) \cap D \neq \emptyset$. Since $Y_1 \cup Y_2 \cup P(h_0) \cup P(h) \subseteq Z$ and since $\delta(h) \subseteq N[Z]$, it follows that $N_G(A(h)) \subseteq N[Z]$. We deduce that $D \subseteq A(h)$. Since $p_k \in Y_1$, and since $N[P]$ is a balanced separator in G, it follows that $D \setminus B \neq \emptyset$. Since D is connected, there exists $v \in D \setminus B$ with a neighbor in B. This proves (12).

In view of (7), let x' be minimum such that there exists $h'_0 \in H_3$ with $N(h'_0) \cap P = \{p_{x'}, p_{x'+1}\}$; we refer to this as the *minimality of* h'_0.

(13) $P_R(h'_0) \cap A(h'_0) \neq \emptyset$.

Suppose that $P_R(h'_0) \cap A(h'_0) = \emptyset$. Let Y_3 be a core of $\delta(h'_0)$; choose Y_3 with $|Y_3|$ minimum. By Lemma 2.3 $|Y_3| \leq 3$. Let $Y_4 = \gamma(h'_0)$; by (11) $|Y_4| \leq 9$.

Let $Z = Y_1 \cup Y_2 \cup Y_3 \cup Y_4 \cup P(h_0) \cup P(h'_0)$. We claim that $N[Z]$ is a balanced separator in G. Suppose not, and let D be a component of $G \setminus N[Z]$ with $w(D) > \frac{1}{2}$. By (12), $D \subseteq A(h'_0)$ and there exists $v \in D \setminus B$ with a neighbor in B. Then $v \in N[P]$; let $v' \in P$ be a neighbor of v. Since $D \subseteq A(h'_0)$, we deduce that $v \in A(h'_0)$, and so $v' \in N[A(h'_0)] \cap P$. Since $Y_1 \cup Y_2 \cup P(h'_0) \subseteq Z$, and by the minimality of h'_0, we deduce that $v' \in P_R(h'_0)$. Since $P_R(h'_0) \cap A(h'_0) = \emptyset$, we conclude that $v' \in N_{G_{h'_0}}(A(h'_0)) \cap P \subseteq \delta(h'_0) \cap P = \gamma(h'_0)$. But then $v' \in Y_4$, and so $v \in N[Z]$, contrary to the fact that $v \in D$. This proves (13).

(14) $P_L(h_1) \cap A(h_1) \neq \emptyset$.

The proof is similar to the proof of (13). Suppose that $P_L(h_1) \cap A(h_1) = \emptyset$. Let Y_3 be a core of $\delta(h_1)$; choose Y_3 with $|Y_3|$ minimum. By Lemma 2.3 $|Y_3| \leq 3$. Let $Y_4 = \gamma(h_1)$; by (11) $|Y_4| \leq 9$.

Let $Z = Y_1 \cup Y_2 \cup P(h_0) \cup Y_3 \cup Y_4 \cup P(h_1)$. We claim that $N[Z]$ is a balanced separator in G. Suppose not, and let D be a component of $G \setminus N[Z]$ with $w(D) > \frac{1}{2}$. By (12), $D \subseteq A(h_1)$ and there exists $v \in D \setminus B$ with a neighbor in B. Then $v \in N[P]$; let $v' \in P$ be a neighbor of v. Since $D \subseteq A(h_1)$, we deduce that $v \in A(h_1)$, and so $v' \in N[A(h_1)] \cap P$. Since $Y_1 \cup Y_2 \cup P(h_0) \cup P(h_1) \subseteq Z$, and by the maximality of h_1, we deduce that $v' \in P_L(h_1)$. Since $P_L(h_1) \cap A(h_1) = \emptyset$, we conclude that $v' \in N_{G_{h_1}}(A(h_1)) \cap P \subseteq \delta(h_1) \cap P = \gamma(h_1)$. But then $v' \in Y_4$, and so $v \in N[Z]$, contrary to the fact that $v \in D$. This proves (14).

By (10) and (13) $P_L(h'_0) \cap A(h'_0) = \emptyset$. In view of this, let i be maximum such that there exist $h_2 \in H_3$ with $P_L(h_2) \cap A(h_2) = \emptyset$ and $N(h_2) \cap P = \{p_i, p_{i+1}\}$. By Lemma 2.3 there is a core Z_2 for $\delta(h_2)$ with $|Z_2| \leq 3$. By (14), $i < y$, and therefore, since $y \in H_3$, there exists $j > i$ such that there exists $h_3 \in H_3$ with $N(h_3) \cap P = \{p_j, p_{j+1}\}$; we may assume that j is chosen minimum with this property. Then $P_L(h_3) \cap A(h_3) \neq \emptyset$, and therefore by (10) $P_R(h_3) \cap A(h_3) = \emptyset$. By Lemma 2.3 there is a core Z_3 for $\delta(h_3)$ with $|Z_3| \leq 3$. Let $Z = Y_1 \cup Y_2 \cup P(h_0) \cup Z_2 \cup Z_3 \cup P(h_2) \cup P(h_3) \cup \gamma(h_2) \cup \gamma(h_3)$. Then $|Z| < d$. To complete the proof, it remains to show that $N[Z]$ is a w-balanced separator in G.

Suppose not, and let D be a component of $G \setminus N[Z]$ with $w(D) > \frac{1}{2}$. By (12), $D \subseteq A(h_2) \cap A(h_3)$ and there exists $v \in D \setminus B$ with a neighbor in B. Then $v \in N[P]$, and therefore $v \in H_3$; let $v' \in P$ be a neighbor of v. Since $D \subseteq A(h_2) \cap A(h_3)$, we deduce that $v \in A(h_2) \cap A(h_3)$, and so $v' \in N[A(h_2)] \cap N[A(h_3)] \cap P$.

It follows from the choice of h_3 that $N(D) \cap P \subseteq P_L(h_2) \cup P_R(h_3) \cup P(h_2) \cup P(h_3)$, and therefore $v' \in P_L(h_2) \cup P_R(h_3) \cup P(h_2) \cup P(h_3)$. Since $P_L(h_2) \cap A(h_2) = \emptyset$ and $P_R(h_3) \cap A(h_3) = \emptyset$, and $Y_1 \cup Y_2 \cup P(h_0) \cup P(h_2) \cup P(h_3) \subseteq Z$, we deduce that $v' \notin A(h_2) \cap A(h_3)$, and therefore either $v' \in N_{G_{h_2}}(A(h_2)) \cap P \subseteq \delta(h_2) \cap P = \gamma(h_2)$ or $v' \in N_{G_{h_3}}(A(h_3)) \cap P \subseteq \delta(h_3) \cap P = \gamma(h_3)$. But then $v' \in \gamma(h_2) \cup \gamma(h_3)$, and so $v \in N[Z]$, contrary to the fact that $v \in D$. ■

5. Large stable sets in neighborhoods

For positive integers a, b let $R(a, b)$ be the smallest integer R such that every graph on R vertices contains either a stable set of size a or a clique of size b. The 2-subdivision of a graph H, denoted by $H^{(2)}$, is the graph obtained by subdividing each edge in H twice (so each edge of H is replaced by a three-edge path). In particular, $K_\gamma^{(2)}$ is the 2-subdivision of the complete graph on γ vertices.

Consider a graph G that is $K_\gamma^{(2)}$-free and $K_{t,t}$-free. We will show that for every subset $Y \subseteq V(G)$ with large independence number, the set of vertices Z with reasonably large independent sets in their neighborhoods in Y has small independence number, as follows:

Theorem 5.1. *Let $C, \gamma, t \in \mathbb{N}$ such that $C, \gamma \geq 2$, and let G be a $\{K_\gamma^{(2)}, K_{t,t}\}$-free graph. Let $Y \subseteq V(G)$. Define*

$$Z = \left\{ z \in V(G) \colon \alpha(N(z) \cap Y) \geq \frac{\alpha(Y)}{C} \right\}.$$

Then

$$\min(\alpha(Y), \alpha(Z)) \leq (512C)^{\gamma^{2t}}.$$

A pair (A, B) is a $K_{s,t}$ *in* G if $|A| = s$, $|B| = t$ and A, B are disjoint stable sets in G that are complete to each other. (So $A \cup B$ is a $K_{s,t}$ in G.) Let $Y \subseteq V(G)$. We say that (A, B) is a $K_{s,t}$ *in G with respect to Y* if (A, B) is a $K_{s,t}$ in G and $B \subseteq Y$. We say that G is *$K_{s,t}$-free with respect to Y* if there is no $K_{s,t}$ with respect to Y. We start with a lemma.

Lemma 5.2. *Let G be a graph and let $C, \gamma, s, t \in \mathbb{N}$ such that $1 \leq s \leq t$ and $C, \gamma \geq 2$. For $i \in \{1, \dots, s\}$, define*

$$c_i = (8^{s-i} \cdot C)^{\gamma^{s-i}},$$
$$f(i) = f_{s,t,\gamma,C}(i) = t(C \cdot 8^s)^{2i\gamma^s}.$$

Let $Y \subseteq V(G)$ and let

$$Z = \left\{ z \in V(G) \colon \alpha(N(z) \cap Y) \geq \frac{\alpha(Y)}{c_i} \right\}.$$

Assume that the following two conditions hold:

(a) G is $K_\gamma^{(2)}$-free and $K_{i,t}$-free with respect to Y.
(b) $\alpha(Y) > f(i)$.

Then $\alpha(Z) < f(i)$.

Proof. We proceed by induction on i. Let $i = 1$. Suppose $Y \subseteq G$ satisfies conditions (a) and (b) above. So $\alpha(Y) > f(1) > c_1 t$. Since G is $K_{1,t}$-free with respect to Y, it follows that for every $z \in V(G)$, $\alpha(N(z) \cap Y) < t < \frac{\alpha(Y)}{c_1}$. We deduce that Z is empty, so $\alpha(Z) < 1 < f(1)$, as required.

Now let $i \in \{2, \dots, s\}$ and assume inductively that the result holds for $i - 1$. Assume that Y satisfies conditions (a) and (b) above. We need to show that $\alpha(Z) < f(i)$.

Suppose not. By taking a subset of Z if necessary, we may assume that $\alpha(Z) = |Z| = f(i)$ (and in particular Z is an independent set). For each $z \in Z$, let $J''(z)$ be an independent set of size $\lceil \frac{\alpha(Y)}{c_i} \rceil$ in $N(z) \cap Y$. Write

$$d = \frac{1}{c_{i-1}} \left\lceil \frac{\alpha(Y)}{c_i} \right\rceil.$$

We now perform some cleaning steps, modifying Z and $J''(z)$.

(15) *Let $z \in Z$, and let $F(z)$ be the set of all vertices $z' \in V(G) \setminus N[z]$ such that $|N(z') \cap J''(z)| \geq d$. Then $\alpha(F(z)) < f(i-1)$.*

Let $G' = G \setminus (N[z] \setminus J''(z))$ and let $Y' = J''(z)$. If (A, B) is a $K_{i-1,t}$ with respect to Y' in G', then $(A \cup \{z\}, B)$ is a $K_{i,t}$ with respect to Y in G, and therefore, Y' satisfies condition (a) with $i - 1$ in G'. Further, for every $i \geq 2$, we have that

$$|Y'| = \left\lceil \frac{\alpha(Y)}{c_i} \right\rceil > \frac{f(i)}{c_i} \geq f(i-1),$$

and so Y' satisfies condition (b) with $i - 1$ in G'. Let $Z' = \{z \in V(G') \colon \alpha(N(z) \cap Y') \geq d\}$. It follows inductively that $\alpha(Z') < f(i-1)$. Since $F(z) \subseteq Z'$, (15) follows.

(16) *There exists $I \subseteq Z$ with $|I| = (8c_i)^{\gamma-1}$ such that for all distinct $z, z' \in I$, $|N(z) \cap J''(z')| < d$.*

Let K be a directed graph with vertex set Z where $(z_1, z_2) \in E(K)$ if and only if $|N(z_1) \cap J''(z_2)| \geq d$. Since Z is a stable set, it follows from (15) that for all $z \in Z$, the indegree of z in K is less than $f(i-1)$. By an easy degeneracy argument (see, for example, Lemma 5.2 of [1]) K contains an independent set I of size at least

$$\frac{f(i)}{2f(i-1)} = \frac{1}{2}(C \cdot 8^s)^{2\gamma^s} = \frac{1}{2}(C \cdot 8^{s-i})^{\gamma^{s-i} \cdot 2\gamma^i} 8^{2i\gamma^s} = \frac{1}{2}c_i^{2\gamma^i} 8^{2i\gamma^s} \geq \frac{1}{2}(8c_i)^{\gamma} \geq (8c_i)^{\gamma-1}.$$

This proves (16).

Let I be as in (16).

(17) *For every $z \in I$ there exists $J'(z) \subseteq J''(z)$ with $|J'(z)| \geq |J''(z)| - |I|d$ such that for all distinct $z, z' \in I$, $J'(z) \cap J'(z') = \emptyset$.*

Let $z \in I$. It follows from the definition of I that for every $z' \in I$ with $z' \neq z$, we have that $|N(z') \cap J''(z)| < d$. Define

$$J'(z) = J''(z) \setminus \bigcup_{z' \in I \setminus \{z\}} N(z').$$

It then follows that $|J'(z)| \geq |J''(z)| - |I|d$ for all $z \in I$, and the sets $J'(z)$ are pairwise disjoint. This proves (17).

(18) *For every $z \in I$ there exists $J(z) \subseteq J'(z)$ with $|J(z)| \geq |J''(z)| - |I| \cdot (f(i-1) + d)$ such that for every vertex $v \in \bigcup_{z' \in I} J(z')$ we have $|N(v) \cap J(z)| < d$.*

Let $z \in I$. Let $F'(z) = \bigcup_{z' \in I \setminus \{z\}} (J'(z) \cap F(z'))$. By (15) $\alpha(F'(z)) < |I| f(i-1)$. Define

$$J(z) = J'(z) \setminus F'(z).$$

Now $|J(z)| \geq |J''(z)| - |I| \cdot (f(i-1) + d)$ as required. This proves (18).

By known upper bounds on Ramsey numbers [22], $|I| \geq R(4c_i, \gamma)$.

Let Γ be a graph with vertex set I, where two distinct vertices $z, z' \in I$ are adjacent if and only if there is a matching of size at least $m = 2\gamma^2 d$ between $J(z)$ and $J(z')$. Since by (16) and Ramsey theorem [22] $|I| \geq R(4c_i, \gamma)$, it follows that Γ contains a stable set of size $4c_i$ or a clique of size γ. We will finish the proof by showing that both cases lead to a contradiction.

(19) Γ *has no stable set of size $4c_i$.*

Suppose there is a stable set S in Γ of size $4c_i$. Let $z, z' \in S$ be distinct. Since there is no matching of size m between $J(z)$ and $J(z')$, by König's Theorem [20] there exists $X_{zz'} \subseteq J(z) \cup J(z')$ with $|X_{zz'}| < m$ such that $J(z) \setminus X_{zz'}$ is anticomplete to $J(z') \setminus X_{zz'}$. Let $X = \bigcup_{z \neq z' \in S} X_{zz'}$. Then $J = \bigcup_{z \in S} J(z) \setminus X$ is a stable set and

$$|X| \leq (4c_i)^2 m = 16c_i^2 \frac{2\gamma^2}{c_{i-1}} \left\lceil \frac{\alpha(Y)}{c_i} \right\rceil.$$

By (18)

$$|J| \geq 4c_i(\left\lceil \frac{\alpha(Y)}{c_i} \right\rceil - |I| \left(f(i-1) + \frac{1}{c_{i-1}} \left\lceil \frac{\alpha(Y)}{c_i} \right\rceil \right)) - |X| \geq$$

$$4c_i \left(\left\lceil \frac{\alpha(Y)}{c_i} \right\rceil - |I| \left(f(i-1) + \frac{1}{c_{i-1}} \left\lceil \frac{\alpha(Y)}{c_i} \right\rceil \right) - \frac{8\gamma^2 c_i}{c_{i-1}} \left\lceil \frac{\alpha(Y)}{c_i} \right\rceil \right).$$

Since $J \subseteq Y$, it follows that $|J| \leq \alpha(Y)$. Simplifying,

$$\left\lceil \frac{\alpha(Y)}{c_i} \right\rceil \left(4c_i - \frac{4c_i|I|}{c_{i-1}} - \frac{8\gamma^2 c_i}{c_{i-1}}\right) \leq 4c_i|I|f(i-1) + \alpha(Y)\,.$$

Note that

$$\frac{c_{i-1}}{4} = \frac{(8^{s-i+1}C)^{\gamma^{s-i+1}}}{4} = \frac{(8^{s-i}C)^{\gamma^{s-i+1}} 8^{\gamma^{s-i+1}}}{4} \geq \frac{(8c_i)^\gamma}{4} \geq (8c_i)^{\gamma-1} = |I|$$

and

$$c_{i-1} = (8^{s-i+1}C)^{\gamma^{s-i+1}} \geq (8C)^\gamma \geq 16^\gamma \geq 8\gamma^2\,.$$

Therefore, $\frac{4|I|}{c_{i-1}} \leq 1$ and $\frac{8\gamma^2}{c_{i-1}} \leq 1$. It follows that

$$\left\lceil \frac{\alpha(Y)}{c_i} \right\rceil (4c_i - c_i - c_i) \leq 4c_i|I|f(i-1) + \alpha(Y)\,.$$

We may drop the ceiling signs and rearrange to obtain

$$\alpha(Y) \leq 4c_i|I|f(i-1) = \frac{1}{2}(8c_i)^\gamma f(i-1) \leq f(i)\,.$$

This is a contradiction as $\alpha(Y) > f(i)$, which proves (19).

(20) Γ *has no clique of size* γ.

Suppose there exists clique K of size γ in Γ. We will obtain a contradiction by constructing a $K_\gamma^{(2)}$ in G. Write $K = \{z_1, \ldots, z_\gamma\}$. For each $i, j \in [\gamma]$ with $i < j$, we will find a vertex $u \in J(z_i)$ and a vertex $v \in J(z_j)$ such that u is adjacent to v, so we may connect z_i to z_j via the path z_i-u-v-z_j. We will ensure that the choice of u and v for each i and j is such that the resulting graph is a $K_\gamma^{(2)}$ in G.

Loop through the pairs $(i,j) \in [\gamma]^2$ with $i < j$. Fix some i, j. We have previously connected fewer than $\binom{\gamma}{2}$ pairs (i', j'). For each such previously connected pair (i', j'), we chose $u' \in J(z_{i'})$ and $v' \in J(z_{j'})$. It follows that so far we have used fewer than $\gamma(\gamma-1)$ vertices from $\bigcup_{i' \in K} J(z_{i'})$; denote the set of these previously used vertices by X. By (18), for each $v \in X$, $|N(v) \cap J(z_i)| < d$ and $|N(v) \cap J(z_j)| < d$. Since $|X| < \gamma(\gamma-1)$, and since there is a matching of size m between $J(z_i)$ and $J(z_j)$, there exist $u \in J(z_i) \setminus X$ and $v \in J(z_j) \setminus X$ such that u is adjacent to v. Connect z_i to z_j using u and v via the path z_i-u-v-z_j. This gives us a $K_\gamma^{(2)}$, a contradiction that proves (20). ■

Proof of Theorem 5.1. Let $i = s = t$, $Z = \left\{z \in V(G) \colon \alpha(N(z) \cap Y) \geq \frac{\alpha(Y)}{C}\right\}$ and let $f(i)$ be defined as in Lemma 5.2. Note that

$$\begin{aligned} t(C \cdot 8^t)^{2t\gamma^t} &\leq (2C \cdot 8^t)^{2t\gamma^t} \\ &\leq (2C)^{2t\gamma^t} \cdot 2^{6t^2\gamma^t} \\ &\leq (2C)^{\gamma^{2t}} \cdot 2^{8\gamma^{2t}} \\ &= (512C)^{\gamma^{2t}}\,. \end{aligned}$$

We may assume that $\alpha(Y) > (512C)^{\gamma^{2t}} \geq t(C \cdot 8^t)^{2t\gamma^t} = f(t)$ as otherwise, there is nothing to show. Since G is $\{K_\gamma^{(2)}, K_{t,t}\}$-free and $\alpha(Y) > f(t)$, Lemma 5.2 applies. It follows that $\alpha(Z) < f(t) \leq (512C)^{\gamma^{2t}}$. ■

6. The layered sets argument

We start with a few more definitions. Let $\mathcal{C}$ be a class of graphs and let k be a positive integer. We say that $\mathcal{C}$ is *k-breakable* if $\mathcal{C}$ is closed under taking induced subgraphs and every graph in $\mathcal{C}$ is k-breakable. Next we define a more refined version of balanced separators. Let $\varepsilon \in (0,1]$, G be a graph and w be a weight function on G. Let (S,C) be a pair of subsets of $V(G)$ and let B be a component of $G \setminus C$ with maximum weight. The pair (S,C) is said to be a *(w,ε)-boosted separator* of G if $w(B) \leq 1/2$ or if $S \cap B$ is a (w,ε)-balanced separator of B. We call C the *boosting set* of (S,C). A set $X \subseteq V(G)$ is said to be a *core* of (S,C) if $S \subseteq N[X]$. Let $k \in \mathbb{N}$, $f\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$. A graph G is said to be *(k,f,ε)-breakable* if for every weight function w, there exists a (w,ε)-boosted separator (S,C) with a core of size at most k and $\alpha(C) \leq f(|V(G)|)$. A class $\mathcal{C}$ of graphs is (k,f,ε)-breakable if $\mathcal{C}$ is closed under taking induced subgraphs and every graph in $\mathcal{C}$ is (k,f,ε)-breakable. If $f(n)$ is a constant function with $f(n)=c$, we say that G (or $\mathcal{C}$) is *(k,c,ε)-breakable*.

The goal of this section is to prove the following:

Theorem 6.1. *For all positive integers k,γ,t,λ there exists an integer $c = c(k,\gamma,t,\lambda)$ with the following properties. Let $f\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ and $\varepsilon \in (0,1]$. Let $\mathcal{C}$ be a (k,f,ε)-breakable class of graphs, and let $G \in \mathcal{C}$ be a $\{K_\gamma^{(2)}, K_{t,t}\}$-free graph on $n \geq 2$ vertices. Let w be a weight function on G. Then there exist $C, S_1, \ldots, S_{\lceil \log n \rceil} \subseteq V(G)$ where*

- *(I) $\alpha(C) \leq c(f(n)\log n + \log^2 n)$,*
- *(II) for all i, (S_i, C) is a (w,ε)-boosted separator of G with a core X_i of size less than k,*
- *(III) if there is a component G' of $G \setminus C$ with $w(G') > \frac{1}{2}$, then no vertex of G' is contained in more than $\frac{3\log n}{k\lambda}$ of the sets $S_1, \ldots, S_{\lceil \log n \rceil}$, and*
- *(IV) for every $j \in \{1, \ldots, \lceil \log n \rceil\}$, no vertex of X_j is contained in more than $\frac{3\log n}{k\lambda}$ of the sets S_i with $i < j$.*

To prove this theorem, we give an algorithm that outputs a set of (w,ε)-boosted separators and prove that they have the required proprieties.

The following definition is key in the proof. Let G be a graph and $\mathcal{F}$ be a family of sets with ground set $V(G)$. Then the i^{th} *layer* of $\mathcal{F}$ in G is

$$L^i(G,\mathcal{F}) = \{v \in V(G) \colon |\{F \colon F \in \mathcal{F}, v \in F\}| \geq i\}.$$

Proof of Theorem 6.1. We may assume that $n \geq 2^{2k\lambda}$ as otherwise the theorem holds trivially with $c = 2^{2k\lambda}$, $C = G$ and $S_1 = \cdots = S_{\lceil \log n \rceil} = \emptyset$.

Let $d = k2^{\lambda k}$ and $T = (512d)^{\gamma^{2t}}$. Consider Algorithm 1.

Since for all $j \in \{1, \ldots, \lceil \log n \rceil\}$, (S_j, Y_j) is a (w,ε)-boosted separator of G_{j-1}, and G_j is a component of $G_{j-1} \setminus C_j$, and $Y_j \subseteq C_j \subseteq C$, it follows that each (S_j, C) is a (w,ε)-boosted separator of G, and so (II) holds.

(21) *For all $1 \leq i \leq j \leq \lceil \log n \rceil$,*

$$\alpha(L^i_j) \leq \frac{2^{j-1}}{2^{\lambda k(i-1)}} n.$$

We proceed by induction on j. Whenever $i = 1$, the bound is trivial. When $j = 1$, we must also have that $i = 1$. So the base case holds. So now assume $2 \leq i \leq j$. If $i = j$, then $L^i_{j-1} = \emptyset$. If $i < j$, by the induction hypothesis, $\alpha(L^i_{j-1}) \leq \frac{2^{j-2}}{2^{\lambda k(i-1)}} n$. Since $L^i_j \subseteq L^i_{j-1} \cup (L^{i-1}_{j-1} \cap S_j)$, it suffices to show that $\alpha(L^{i-1}_{j-1} \cap S_j) \leq \frac{2^{j-2}}{2^{\lambda k(i-1)}} n$ as well. There are two possible cases.

- *Case 1:* $\alpha(L^{i-1}_{j-1}) \leq T$. In this case $C^{i-1}_{j-1} = L^{i-1}_{j-1} \subseteq C_{j-1}$ is removed from G_{j-2} when forming G_{j-1}, and so none of the vertices in L^{i-1}_{j-1} belong to G_{j-1}. Therefore, $L^{i-1}_{j-1} \cap S_j = \emptyset$, so we are done.

Algorithm 1 Layered Sets Algorithm

$G_0 = G$
for $j = 1, \ldots, \lceil \log n \rceil$ **do**
    $(S_j, Y_j) \coloneqq$ a (w, ε)-boosted separator of G_{j-1} where $S_j = N[X_j]$, $|X_j| < k$, and $\alpha(Y_j) \leq f(n)$
    ▷ *Such a pair* (S_j, Y_j) *exists since* $\mathcal{C}$ *is* (k, f, ε)*-breakable.*
    for $i = 1, \ldots, j+1$ **do**
        $L_j^i \coloneqq L^i(G_{j-1}, \{S_1, \ldots, S_j\})$
        $Z_j^i \coloneqq \left\{ v \in V(G_{j-1}) \colon \alpha(N(v) \cap L_j^i) \geq \frac{\alpha(L_i^j)}{d} \right\}$
        if $\alpha(L_j^i) \leq T$ **then**
            $C_j^i \coloneqq L_j^i$
        else
            $C_j^i \coloneqq Z_j^i$
        end if
    end for
    $C_j = Y_j \cup \bigcup_{i=1}^{j} C_j^i$
    $G_j \coloneqq$ a maximum weight component of $G_{j-1} \setminus C_j$
end for
$C = \bigcup_{j=1}^{\log n} C_j$
return $(S_1, C), \ldots, (S_{\lceil \log n \rceil}, C)$

- *Case 2:* $\alpha(L_{j-1}^{i-1}) > T$. In this case we have that $C_{j-1}^{i-1} = Z_{j-1}^{i-1} \subseteq C_{j-1}$ is removed from G_{j-2} when forming G_{j-1}. It follows that for every vertex $v \in G_{j-1}$,
$$\alpha(N(v) \cap L_{j-1}^{i-1}) < \frac{\alpha(L_{j-1}^{i-1})}{d} \leq \frac{2^{j-2}}{d 2^{\lambda k(i-2)}} n.$$
Since $L_{j-1}^{i-1} \cap S_j \subseteq (X_j \cap L_{j-1}^{i-1}) \cup (\bigcup_{v \in X_j} N(v) \cap L_{j-1}^{i-1})$, we deduce
$$\alpha(L_{j-1}^{i-1} \cap S_j) \leq \sum_{v \in X_j} \max(\alpha(N(v) \cap L_{j-1}^{i-1}), 1)\,.$$
Therefore
$$\alpha(L_{j-1}^{i-1} \cap S_j) \leq k \frac{\alpha(L_{j-1}^{i-1})}{d} \leq \frac{2^{j-2}}{2^{\lambda k(i-1)}} n\,.$$

This completes the induction and proves (21). To prove that (III) holds, let $i > 1 + \frac{1 + 2\log n}{k\lambda}$ and $j = \lceil \log n \rceil$. By (21), $\alpha(L_j^i) \leq \frac{2^{\log n}}{2^{\lambda k(i-1)}} n < 1$, and therefore $|L_j^i| = 0$. Since, $\frac{3 \log n}{k\lambda} \geq 1 + \frac{1 + 2\log n}{k\lambda}$, this proves that (III) holds.

An immediate consequence of (21) and the same calculation as the proof of (III) is that for every j no vertex of G_j is in more than $\min\left\{j, \frac{3\log n}{k\lambda}\right\}$ of the sets $S_1, \ldots, S_j$, which proves (IV).

Finally, we have
$$\begin{aligned}
\alpha(C) = \alpha\left(\bigcup_{j=1}^{\log n} C_j\right) &= \alpha\left(\bigcup_{j=1}^{\log n} Y_j \cup \bigcup_{j=1}^{\log n} \bigcup_{i=1}^{j} C_j^i\right) \\
&\leq \sum_{j=1}^{\log n} \alpha(Y_j) + \sum_{j=1}^{\log n} \sum_{i=1}^{j} \alpha\left(C_j^i\right) \\
&\leq f(n) \log n + \log(n)^2 T \\
&= f(n) \log n + \log(n)^2 (512 k 2^{\lambda k})^{\gamma^{2t}}\,,
\end{aligned}$$

where the second inequality follows from Theorem 5.1. Setting $c = (512k2^{\lambda k})^{\gamma^{2t}}$ proves that (I) holds. ■

7. Improving the separators

7.1. **Improving the Separating Power.** The goal of this section is to "boost" the separators given by Theorem 1.3 to (w, ε)-boosted separators without changing the size of the core and while introducing a boosting set with a somewhat small stability number. This, however, will be at the cost of forbidding $K_{t,t}$. The idea of boosting the separators without increasing the size of the core is inspired by an argument of Gartland et al. ([17], Section 4).

The existence of 1/2-balanced separators with small cores allows us to obtain $1/2^i$-balanced separators with relatively small cores.

Lemma 7.1 (Analogue of Lemma 2 of [16])**.** *Let $k \geq 2$ be an integer and let $\mathcal{C}$ be a k-breakable class of graphs. Let $G \in \mathcal{C}$ and let w be a weight function on G. Then, for every positive integer i there exists $X \subseteq V(G)$ with $|X| < 2^{i+1}(k-1)$, such that $N[X]$ is a $(w, 1/2^i)$-balanced separator of G.*

Proof. We proceed by induction on i. If $i = 1$, the result follows from the fact that G is k-breakable. If $i > 1$, the induction hypothesis gives us a set X_0 with $|X_0| < 2^i(k-1)$ such that $N[X_0]$ is a $(w, 1/2^{i-1})$-balanced separator in G. Let $\mathcal{D}$ be the set of all components D of $G \setminus N[X_0]$ such that $w(D) \geq 1/2^i$. Then $|\mathcal{D}| \leq 2^i$. Let $D \in \mathcal{D}$ and let $w'(x) = 2^{i-1}w(x)$ for all $x \in D$. Then $w'(D) \leq 1$. Since $\mathcal{C}$ is k-breakable, it follows that D is k-breakable, and so there exists $X(D) \subseteq D$ such that $|X(D)| \leq k-1$ and $N[X(D)]$ is a $(w', 1/2)$-balanced separator in D. Now let $X = X_0 \cup \bigcup_{D \in \mathcal{D}} X(D)$. Then

$$|X| \leq |X_0| + \sum_{D \in \mathcal{D}} |X(D)| < 2^i(k-1) + 2^i(k-1) = 2^{i+1}(k-1)$$

and $N[X]$ is a $(w, 1/2^i)$-separator of G. ■

Corollary 7.2. *Let $k \geq 2$ be an integer. Then, for every positive integer i, every k-breakable class of graphs is $(2^{i+1}(k-1), 0, 1/2^i)$-breakable.*

Proof. Let $G \in \mathcal{C}$ and let w be a weight function on G. Let $X \subseteq G$ with $|X| \leq 2^{i+1}(k-1)$ be a core of a $(w, 1/2^i)$-separator guaranteed by Lemma 7.1. Then $(N[X], \emptyset)$ is a $(w, 1/2^i)$-boosted separator of G. ■

Let G be a graph and $\varepsilon \in (0, 1]$. We now introduce new terminology that will be useful in the next two lemmas and in which we will suppress the dependencies on ε to make the notation more concise. Let w be a weight function on G. Let $X \subseteq G$ be such that $|X| < k$ and $N[X]$ is a $(w, \frac{1}{2})$-balanced separator in G. We will say that a connected component B of G is *big* if $w(B) > \varepsilon$. We denote by $\mathcal{B}(G)$ the set of big components of G. Let $\beta \geq 0$, let B be a big component of $G \setminus N[X]$, and let $x \in X$. We call the pair (x, B) (X, β)-*problematic* if $\alpha(N(x) \cap N(B)) \geq \beta$. Let $\mathcal{P}(G, X, \beta)$ be the set of $(X, \frac{3\beta}{4})$-problematic pairs of G. Let

$$W(G, X, \beta) = \sum_{(x,B) \in \mathcal{P}(G,X,\beta)} w(B)\,.$$

Let $\beta(G, X)$ be the maximum value of β for which an (X, β)-problematic pair of G exists (with $\beta(G, X) = 0$ if no big component of $G \setminus N[X]$ exists or $X = \emptyset$).

Lemma 7.3. *Let $k, t \in \mathbb{N}$ with $k \geq 2$ and $\varepsilon \in (0, 1]$. Let $\eta = c(16(k-1), 3t+1, t, 6t)$ be as in Theorem 6.1. Let $\mathcal{C}$ be a k-breakable class of graphs, and let $G \in \mathcal{C}$ be an $\{S_{t,t,t}, K_{t,t}\}$-free graph on n vertices, where $n \geq 2$. Let w be a weight function on G. Let $X \subseteq G$ be such that $|X| < k$ and $N[X]$ is a $(w, \frac{1}{2})$-balanced separator for G. Let $\beta = \beta(G, X) > 4\eta \log^2 n$ and let $\beta' \in [\beta, \frac{4}{3}\beta]$. Then, there exists $C \subseteq V(G) \setminus X$, such that $\alpha(C) \leq \eta \log^2 n$ and $W(G \setminus C, X, \beta') \leq W(G, X, \beta') - \varepsilon$.*

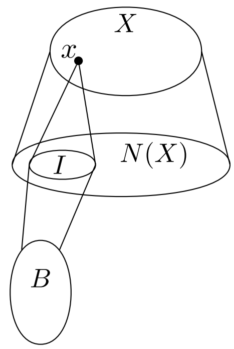

FIGURE 7. Drawing to keep in mind for the proof of Lemma 7.3.

Proof. Let (x, B) be an (X, β)-problematic pair. Let $I \subseteq N(x) \cap N(B)$ be a stable set of size β (see Figure 7).

Define a new weight function $w' \colon V(G) \to [0, 1]$ where $w'(v) = \frac{1}{|I|}$ if $v \in I$ and $w'(v) = 0$ otherwise. By Corollary 7.2, G is $(16(k-1), 0, 1/8)$-breakable. Note that since G is $S_{t,t,t}$-free, G is also $K^{(2)}_{3t+1}$-free. Applying Theorem 6.1 to $G \setminus X$ with $\gamma = 3t + 1$ and $\lambda = 6t$, we obtain sets $C, S_1, \ldots, S_{\lceil \log n \rceil} \subseteq V(G)$ with $\alpha(C) \leq \eta \log^2 n$ where (S_i, C) is a $(w', 1/8)$-boosted separator of $G \setminus X$ with a core of size less than $16(k-1)$ for every i. Moreover, if there is a component G' of $G \setminus (X \cup C)$ with $w'(G') > \frac{1}{2}$, then no vertex of G' is in more than $\frac{\log n}{32t(k-1)}$ of the sets S_i.

(22) *C is a $(w', 1/2)$-balanced separator in $G \setminus X$.*

Suppose not, and let G' be the unique connected component of $G \setminus (X \cup C)$ with $w'(G') > \frac{1}{2}$. Let $a_1, a_2, a_3 \in V(G')$. We say that $a_1a_2a_3$ is *separated by* S_i if for every component D of $G' \setminus S_i$, $|D \cap \{a_1, a_2, a_3\}| \leq 1$. Randomly choose three vertices a_1, a_2, a_3 of G' using the normalized function of w' on G' as a probability distribution. Let $i \in \{1, \ldots, \lceil \log n \rceil\}$. Since $\frac{w'(D)}{w'(G')} \leq \frac{1}{8w'(G')} \leq \frac{1}{4}$ for every component D of $G' \setminus S_i$, the probability that $a_1a_2a_3$ is separated by S_i is at least $\frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$. Therefore, by the linearity of expectation, there exist $a_1, a_2, a_3 \in G'$ and $J \subseteq \{1, \ldots, \lceil \log n \rceil\}$ with $|J| \geq \frac{3 \log n}{8}$ such that $a_1a_2a_3$ is separated by S_j for every $j \in J$.

For $1 \leq i, j \leq 3$, $i \neq j$, let P_{ij} be a shortest path from a_i to a_j in B. Since no vertex of G' belongs to more than $\frac{3 \log n}{32t}$ of the sets S_i, and since $|J| \geq \frac{3 \log n}{8}$, it follows that $|P_{ij}| \geq 4t$ for every $i, j \in \{1, 2, 3\}$.

Let Q_{ij} be the t vertex subpath of P_{ij} including a_i. Since $Q_{ij} \setminus B = a_i$, it follows that $N_G[x] \cap Q_{ij} = \{a_i\}$. Since $|P_{ij}| \geq 4t$ and by minimality of P_{ij}, it follows that Q_{12}, Q_{23}, and Q_{31} are pairwise anticomplete. Therefore, $x \cup Q_{12} \cup Q_{23} \cup Q_{31}$ is an $S_{t,t,t}$ with center x in G (see Figure 8), a contradiction. This proves (22).

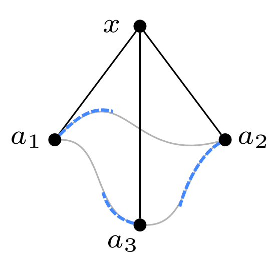

FIGURE 8. Visualization of the $S_{t,t,t}$ obtained in the proof of (22).

(23) *There is no* $(y, B') \in \mathcal{P}(G \setminus C, X, \beta')$ *such that* $y = x$ *and* $B' \subseteq B$.

Let B' be a component of $B \setminus C$ and let D be the component of $G \setminus (X \cup C)$ containing B'. By (22), $|D \cap I| \leq |I|/2$. Therefore, $I' = I \setminus (D \cup C)$ is an independent set of size at least $|I| - |I|/2 - \eta \log^2 n > \beta/4$, and I' is anticomplete to D. Since $\left(I' \cup N_{G\setminus C}(B')\right) \cap N_{G\setminus C}(x) \subseteq N_G(B) \cap N_G(x)$ and by the maximality of β, we have that

$$\beta \geq \alpha\left(\left(I' \cup N_{G\setminus C}(B')\right) \cap N_{G\setminus C}(x)\right) > \beta/4 + \alpha\left(N_{G\setminus C}(B') \cap N_{G\setminus C}(x)\right),$$

where the second inequality follows from the fact that the sets $I' \cap N_{G\setminus C}(x) = I'$ and $N_{G\setminus C}(B') \cap N_{G\setminus C}(x) \subseteq D$ are disjoint and anticomplete to each other. Therefore, $\alpha\left(N_{G\setminus C}(B') \cap N_{G\setminus C}(x)\right) \leq \frac{3}{4}\beta$ and so (x, B') is not a $(X, \frac{3}{4}\beta)$-problematic pair in $G \setminus C$. This proves (23).

Note that for every pair $(y, D) \in \mathcal{P}(G \setminus C, X, \beta')$, the pair (y, D') where D' is the unique component of G containing D, belongs to $\mathcal{P}(G, X, \beta')$. Therefore,

$$\begin{aligned} W(G \setminus C, X, \beta') &= \sum_{(y,D)\in\mathcal{P}(G\setminus C,X,\beta')} w(D) \\ &= \sum_{(y,D')\in\mathcal{P}(G,X,\beta')} \sum_{\substack{D\subseteq D' \\ (y,D)\in\mathcal{P}(G\setminus C,X,\beta')}} w(D) \\ &= \sum_{\substack{(y,D')\in\mathcal{P}(G,X,\beta') \\ (y,D')\neq(x,B)}} \sum_{\substack{D\subseteq D' \\ (y,D)\in\mathcal{P}(G\setminus C,X,\beta')}} w(D), \end{aligned}$$

where the last equality holds by (23).

Now

$$\sum_{\substack{(y,D')\in\mathcal{P}(G,X,\beta') \\ (y,D')\neq(x,B)}} \sum_{\substack{D\subseteq D' \\ (y,D)\in\mathcal{P}(G\setminus C,X,\beta')}} w(D) \leq \sum_{\substack{(y,D')\in\mathcal{P}(G,X,\beta') \\ (y,D')\neq(x,B)}} w(D').$$

We deduce

$$\sum_{\substack{(y,D')\in\mathcal{P}(G,X,\beta') \\ (y,D')\neq(x,B)}} \sum_{\substack{D\subseteq D' \\ (y,D)\in\mathcal{P}(G\setminus C,X,\beta')}} w(D) \leq \sum_{(y,D')\in\mathcal{P}(G,X,\beta')} w(D') - \varepsilon = W(G, X, \beta') - \varepsilon,$$

as required.

■

Corollary 7.4. *Let* $k, t \in \mathbb{N}$ *with* $k \geq 2$, $\varepsilon \in (0, 1]$. *Let* $\eta = c(16(k-1), 3t+1, t, 6t)$ *be as in Theorem 6.1. Let* $\mathcal{C}$ *be a* k*-breakable class of graphs, and let* $G \in \mathcal{C}$ *be an* $\{S_{t,t,t}, K_{t,t}\}$*-free graph on* n *vertices, where* $n \geq 2$. *Let* w *be a weight function on* G. *Let* $X \subseteq G$ *be such that* $|X| < k$ *and* $N[X]$ *is a* $(w, \frac{1}{2})$*-balanced separator for* G. *Let* $\beta = \beta(G, X) > 4\eta \log^2 n$. *Then, there exists* $C^* \subseteq V(G \setminus X)$ *such that* $\alpha(C^*) \leq \frac{k}{\varepsilon}\eta \log^2 n$ *and* $W(G \setminus C^*, X, \beta) = 0$.

Proof. We recursively define $C_\ell \subseteq V(G \setminus X)$ by applying Lemma 7.3 to $G \setminus \bigcup_{i<\ell} C_i$ with a weight function which is the appropriate restriction of w and with $\beta' = \beta$. Let ℓ^* be the smallest ℓ such that $W(G \setminus \bigcup_{i\leq\ell} C_i, X, \beta) = 0$. Note that $\beta(G \setminus \bigcup_{i<\ell} C_i) \leq \beta$ and, moreover, if $\beta(G \setminus \bigcup_{i<\ell} C_i) < \frac{3}{4}\beta$, then

$W(G \setminus \bigcup_{i<\ell} C_i, X, \beta) = 0$, so the conditions of Lemma 7.3 are satisfied for $\ell \in \{1, \dots, \ell^* - 1\}$. By the definition of the function W, we have that

$$W(G, X, \beta) = \sum_{x \in X} \sum_{(x,B) \in \mathcal{P}(G,X,\beta)} w(B) \leq \sum_{x \in X} 1 = |X| \leq k\,.$$

Therefore, $\ell^* \leq \frac{k}{\varepsilon}$. Setting $C^* = \bigcup_{i \leq \ell^*} C_i$ completes the proof. ■

Corollary 7.5. *Under the same conditions as in Lemma 7.3, there exist $C^* \subseteq V(G) \setminus X$ such that $\alpha(C^*) \leq \frac{k}{\varepsilon}\eta \log^3 n$ and the maximum value of γ for which an (X,γ)-problematic pair exists in $G \setminus C^*$ is at most $4\eta \log^2 n$.*

Proof. We recursively define $C_\ell \subseteq V(G \setminus X)$ by applying Corollary 7.4 to $G \setminus \bigcup_{i<\ell} C_i$ with β_i where $\beta_i = \beta(G \setminus \bigcup_{i<\ell} C_i, X)$ (so $\beta_1 = \beta(G, X)$). Since $W(G \setminus \bigcup_{j \leq i} C_j, X, \beta_i) = 0$, we have that $\mathcal{P}(G \setminus \bigcup_{j \leq i} C_j, X, \beta_i) = \emptyset$. Therefore, $\beta_{i+1} \leq \frac{3}{4}\beta_i$ and so

$$n \geq \beta_1 \geq \frac{4}{3}\beta_2 \geq \cdots \geq \left(\frac{4}{3}\right)^i \beta_i\,.$$

Let ℓ^* be the smallest ℓ such that $\beta_\ell \leq 4\eta \log^2 n$. We have that $\ell^* \leq \log n$, so setting $C^* = \bigcup_{i \leq \ell^*} C_i$ completes the proof. ■

We can now prove the main result of this subsection.

Theorem 7.6. *For all positive integers k, t with $k \geq 2$, there exist an integer $c = c(k,t)$ with the following property. Let $\mathcal{C}$ be a k-breakable class of graphs. Let $G \in \mathcal{C}$ be an $\{S_{t,t,t}, K_{t,t}\}$-free graph on n vertices, where $n \geq 2$. Let $\varepsilon \in (0,1]$ and let $f = f_{k,t} : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be the function $f(n) = \frac{c}{\varepsilon}\log^3 n$. Then G is (k, f, ε)-breakable.*

Proof. Let w be a weight function on G. Since G is k-breakable, we can find $X \subseteq G$ such that $|X| < k$ and $N[X]$ is a $(w, \frac{1}{2})$-balanced separator in G. Let $\beta = \beta(G, X)$ and let $\eta = c(16(k-1), 3t+1, t, 6t)$ be as in Theorem 6.1. If $\beta > 4\eta \log^2 n$, we apply Corollary 7.5 to obtain a set $C \subseteq V(G) \setminus X$. Otherwise, let $C = \emptyset$. In both cases, $\mathcal{P}(G \setminus C, X, 4\eta \log^2(n) + 1) = \emptyset$ which implies that none of the remaining big components of $G \setminus (C \cup N[X])$ has an independent set of size $4k\eta \log^2 n$ in its neighborhood. Let $Z = \bigcup_{B \in \mathcal{B}(G \setminus (C \cup N[X]))} N(B)$ (see Figure 9). Since there are at most $\frac{1}{\varepsilon}$ big components, $\alpha(Z) \leq \frac{1}{\varepsilon} 4k\eta \log^2 n$.

(24) *$(N[X], C \cup Z)$ is a (w, ε)-boosted separator in G.*

Let S be a component of $G \setminus (C \cup Z)$ with $w(S)$ maximum. If $w(S) \leq 1/2$, there is nothing to show. Therefore, we may assume that $w(S) > 1/2$. Since $N[X]$ is a $(w, \frac{1}{2})$-balanced separator of G, $S \cap N[X] \neq \emptyset$. Let G^* be the union of all the components of $G \setminus (C \cup Z)$ intersecting $N_{G \setminus (C \cup Z)}[X]$. It follows that $S \subseteq G^*$. Since none of the components of $G^* \setminus N[X]$ is big, $N[X]$ is a (w, ε)-balanced separator for G^*. Therefore,

$$S \setminus N[X] = (G^* \setminus (G^* \setminus S)) \setminus N[X] = (G^* \setminus N[X]) \setminus (G^* \setminus S)$$

has no component of weight greater than ε, as removing vertices cannot introduce a big component. This proves (24).

Since $\alpha(C \cup Z) \leq \frac{1}{\varepsilon} k\eta \log^3(n) + \frac{1}{\varepsilon} 4k\eta \log^2 n$, setting $c = 4k\eta$ completes the proof. ■

7.2. **Improving the Disjointedness of the Separators.** Our next goal is to combine Theorems 1.3, 6.1, and 7.6 to obtain a large set of boosted separators with pairwise anticomplete cores.

Lemma 7.7. *Let $k, N \in \mathbb{N}$ and $R \in \mathbb{R}_{\geq 0}$. Let G be a graph. Let $Y_1, \dots, Y_{\lceil kRN \rceil}$ be subsets of $V(G)$ of size less than k. Write $S_i = N[Y_i]$. Assume that for all $j \in \{1, \dots, \lceil kRN \rceil\}$, no vertex of Y_j belongs to more than R of the sets S_i with $i < j$. Then there exists $I \subseteq \{1, \dots, \lceil kRN \rceil\}$ with $|I| = N$ such that for every $i \neq j \in I$, Y_i is anticomplete to Y_j.*

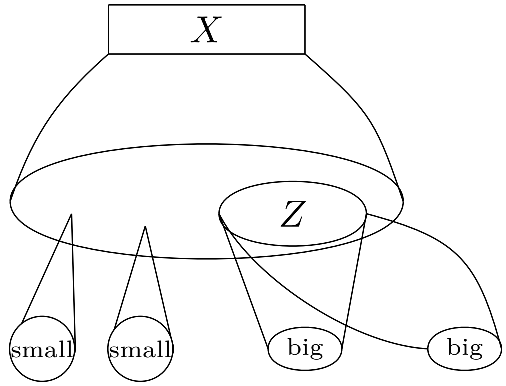

FIGURE 9. Visualization of $G \setminus C$ in Theorem 7.6.

Proof. Let H be the graph with vertex set $\{Y_1, \dots, Y_{\lceil kRN \rceil}\}$, with Y_i adjacent to Y_j for $i \neq j$ if and only if Y_i and Y_j are not anticomplete to each other. Since $|Y_j| < k$ for all j, the graph H is $\lfloor kR \rfloor$-degenerate, and therefore has a stable set of size N, as required. ∎

Lemma 7.8. *Let $t, N > 1$ be integers, and let G be a $K_{t,t}$-free graph. Let w be a weight function on G. Let $Y_1, \dots, Y_N$ be pairwise anticomplete subsets of size at most k of $V(G)$, and for every $i \in \{1, \dots, N\}$ write $S_i = N[Y_i]$. Let $Z = \{v \in V(G) \colon |\{i \colon v \in S_i\}| \geq t\}$. Then $\alpha(Z) \leq \binom{N}{t} tk^t$.*

Proof. We start with the following:

(25) *$Z \cap Y_i = \emptyset$ for every $i \in \{1, \dots, N\}$.*

Suppose there is a vertex $z \in Z \cap Y_1$. Since $t > 1$, we may assume that $z \in S_2$. But since $z \in Y_1$ and since the sets $Y_1, \dots, Y_N$ are pairwise anticomplete, it follows that z is anticomplete to Y_2, a contradiction. This proves (25).

Now let I be a largest stable set in Z. We may assume that $|I| > \binom{N}{t} tk^t$. Then I contains a subset I_1 of size tk^t such that every $v \in I_1$ is in $S_1, \dots, S_t$ (renumbering $S_1, \dots, S_N$ if necessary). Since by (25) $Z \cap Y_i = \emptyset$ for every i, it follows that every vertex in I_1 has a neighbor in each of $Y_1, \dots, Y_t$. Now there exists a subset I_2 of I_1 of size t such that for every $a \in \{1, \dots, t\}$ there exists $y_a \in Y_a$, for which every $v \in I_2$ is adjacent to y_a. But now $I_2 \cup \{y_1, \dots, y_t\}$ is a $K_{t,t}$ in G, a contradiction. ∎

We can now prove the main result of this subsection.

Theorem 7.9. *Let t be a positive integer and let $\varepsilon \in (0, 1]$. Then, there exists an integer $c = c(t, \varepsilon)$ with the following properties. Let $d = d(t)$ be as in Theorem 1.3. Let $G \in \mathcal{M}_t$ with $|G| = n \geq 2$ and let w be a weight function on G. Then, there is a set $X \subseteq V(G)$ with $\alpha(X) \leq c \log^4 n$, such that, denoting by D a component of $G \setminus X$ with $w(D)$ maximum, either*

- *$w(D) \leq \frac{1}{2}$ (and therefore X is a $(w, \frac{1}{2})$-balanced separator of G), or*
- *$w(D) > \frac{1}{2}$ and there exist pairwise anticomplete subsets $Y_1, \dots Y_{8t^2d}$ of G such that*
  - *for every i, $|Y_i| < d$, and*
  - *$N[Y_i] \cap D$ is a (w, ϵ)-balanced separator of D, and*
  - *no vertex of D is in more than t of the sets $N[Y_i]$.*

Proof. Let $N = 8t^2 d$ and $\lambda = 3N$. By Theorem 1.3, we know that $\mathcal{M}_t^*$ is d-breakable. We may assume that $d \geq 2$. It follows that Theorem 7.6 applies and there exists $c_1 = c(d, t)$ such that G is (d, f, ε)-breakable where $f(n) = \frac{c_1}{\varepsilon} \log^3 n$. We can therefore apply Theorem 6.1 to get $c_2 = c(d, 3t + 1, t, \lambda)$ and $C, Y_1, \dots, Y_{\lceil \log n \rceil} \subseteq V(G)$ such that (writing $S_i = N[Y_i]$)

- $\alpha(C) \leq c_2 \log^4 n$;

- for every i, $|Y_i| < d$;
- for every i, (S_i, C) is a (w, ε)-boosted separator of G;
- if there is a component G' of $G \setminus C$ with $w(G') > \frac{1}{2}$, then no vertex of G' is contained in more than $\frac{3 \log n}{d\lambda}$ of the sets $S_1, \dots, S_{\lceil \log n \rceil}$;
- for every $j \in \{1, \dots, \lceil \log n \rceil\}$, no vertex of Y_j is contained in more than $\frac{3 \log n}{d\lambda}$ of the sets S_i with $i < j$.

Let G' be the component of $G \setminus C$ with $w(G')$ maximum. Let $c = c_2 + \binom{N}{t} t d^t$. We may assume that $w(G') > 1/2$ as otherwise, we are done. Applying Lemma 7.7 to G' with $k = d$ and $R = \frac{3 \log n}{d\lambda}$ we obtain $I \subseteq \{1, \dots, \lceil kRN \rceil\}$ with $|I| = N$ such that for every $i \neq j \in I$, Y_i is anticomplete to Y_j. Renumbering if necessary, we may assume that $I = \{1, \dots, N\}$. Applying Lemma 7.8 to the sets $Y_1, \dots, Y_N$ gives a set Z with $\alpha(Z) \leq \binom{N}{t} t d^t = c_3$ such that every vertex of $G \setminus Z$ is in fewer than t of the sets $S_1, \dots, S_N$. Let $X = C \cup Z$. Then $\alpha(X) \leq c_3 + c_2 \log^4 n$. Let D be the component of $G \setminus X$ with $w(D)$ maximum. If $w(D) \leq 1/2$, then the first alternative in the statement of the theorem holds, and we are done. Therefore, we may assume that $w(D) > 1/2$. But now the second alternative in the statement of the theorem holds. ■

8. Bringing it all together

The final key ingredient in our proof of Theorem 1.8 is the following lemma. By a *caterpillar*, we mean a tree T with maximum degree of three such that there exists a path that contains all the vertices of degree three. For a graph H, we denote by $\mathcal{Z}(H)$ the set of vertices whose neighborhood is a clique.

Lemma 8.1 (Theorem 5.2 of [5]). *For every integer $h \geq 1$, there exists an integer $\mu = \mu(h) \geq 1$ with the following property. Let G be a connected graph. Let $Y \subseteq G$ such that $|Y| \geq \mu$, $G \setminus Y$ is connected and every vertex of Y has a neighbor in $G \setminus Y$. Then there is a set $Y' \subseteq Y$ with $|Y'| = h$ and an induced subgraph H of $G \setminus Y$ for which one of the following holds.*

- *H is a path and every vertex of Y' has a neighbor in H; or*
- *H is a caterpillar, or the line graph of a caterpillar, or a subdivided star, or the line graph of a subdivided star. Moreover, every vertex of Y' has a unique neighbor in H and $H \cap N(Y') = \mathcal{Z}(H)$.*

We prove a slightly modified version of Lemma 8.1 that will be more convenient for our use.

Lemma 8.2. *For every integer $h \geq 1$, there exists an integer $\mu = \mu(h) \geq 1$ with the following property. Let G be a connected graph. Let $S \subseteq G$ be a stable set such that $|S| \geq \mu$. Then there is an induced subgraph H of G for which one of the following conditions holds:*

- *H is a path and $|H \cap S| = h$.*
- *H is a caterpillar, or the line graph of a caterpillar, or a subdivided star, or the line graph of a subdivided star with $|H \cap S| = h$ and $H \cap S = \mathcal{Z}(H)$.*

Proof. Let $\mu = \mu(h^2)$ be as in Lemma 8.1. Define G' to be the graph obtained from G by adding, for each $v \in S$, a new vertex u_v whose unique neighbor in G' is v. Let $Y = \{u_v : v \in S\}$ (so $Y = G' \setminus G$). Then $|Y| = |S| \geq \mu$, every vertex in Y has a neighbor in $G' - Y$, and $G' - Y = G$ is connected.

Applying Lemma 8.1 to G' and Y, we obtain a set $Y' \subseteq Y$ with $|Y'| = h^2$ and an induced subgraph H of $G' - Y = G$ for which one of the following holds:

- H is a path and every vertex of Y' has a neighbor in H. Since every $v \in S$ has a unique neighbor in Y, and since Y is anticomplete to $G \setminus S$, it follows that $|H \cap S| \geq |Y'| \geq h$. Truncating H, we obtain the required conclusion.
- H is a caterpillar, or the line graph of a caterpillar, or a subdivided star, or the line graph of a subdivided star. Moreover, every vertex of Y' has a unique neighbor in H and $H \cap N(Y') = \mathcal{Z}(H)$. Again, since $|N(y)| = 1$ for every $y \in Y$, it is easy to see that $H \setminus Y$ is a caterpillar, or the line graph of a caterpillar, or a subdivided star, or the line graph of a subdivided star. We may

assume that for every path P in H, $|P \cap S| < h$, for otherwise the theorem holds. Since S is stable, the observation in the previous sentence implies that there exists an induced subgraph H' of H, and $Y'' \subseteq Y'$ with $|Y''| = h$ such that H' is a caterpillar, or the line graph of a caterpillar, or a subdivided star, or the line graph of a subdivided star, $N(Y'') = \mathcal{Z}(H')$, and every vertex of $N(Y) \cap H' = H' \cap S$ belongs to $\mathcal{Z}(H')$. It follows that $H' = \mathcal{Z}(H')$, as required. ■

We can now prove Theorem 1.8, which we restate.

Theorem 1.8. *For every positive integer t, there exists $c = c(t)$ such that for every $n \geq 2$, every n-vertex graph G in $\mathcal{M}_t$, and every normal weight function w on G, there is a $(w, \frac{1}{2})$-balanced separator X_w in G with $\alpha(X_w) \leq c \log^4 n$.*

Proof. Let G be an n-vertex graph in $\mathcal{M}_t^*$ and let $d = d(t)$ be the constant given by Theorem 1.3. Let $N = 8t^2 d, h = 10Nd$ and $\varepsilon = \frac{1}{2\mu^2}$ where $\mu = \mu(h)$ as in Lemma 8.2. Let $c = c(t, \varepsilon)$ be as in Theorem 7.9. By Theorem 7.9 we may assume that there exists $Z \subseteq V(G)$ with $\alpha(Z) \leq c \log^4 n$ such that, denoting by G'' the component of $G \setminus Z$ with $w(G'')$ maximum, $w(G'') > \frac{1}{2}$ and there exist pairwise anticomplete subsets $Y_1, \dots Y_{8t^2 d}$ of G such that

- for every i, $|Y_i| < d$, and
- $N[Y_i] \cap G''$ is a (w, ϵ)-balanced separator of G'', and
- no vertex of G'' is in more than t of the sets $N[Y_i]$.

We now randomly choose μ vertices of G'' using the normalized function of w on G'' as a probability distribution. For every i, write $S_i = N[Y_i]$. For every i and for every component D of $G'' \setminus S_i$, $\frac{w(D)}{w(G'')} \leq 2\varepsilon$, and so, applying the union bound, the probability that no two of the vertices we chose are in the same component of $G'' \setminus S_i$ is at least

$$1 - 2\varepsilon \sum_{j=1}^{\mu(h)} j \geq 1 - \varepsilon \mu(h)^2 = 1/2\,.$$

By the linearity of expectation, there exist $S \subseteq G''$ with $|S| = \mu$ and a set $I \subseteq \{1, \dots, N\}$ with $|I| = \frac{N}{2}$ such that for every $i \in I$ and every component D of $G'' \setminus S_i$, $|S \cap D| \leq 1$.

(26) *S is a stable set.*

Suppose $s, t \in S$ are adjacent. Since for every component D of $G'' \setminus S_i$, $|S \cap D| \leq 1$, it follows that for every $i \in \{1, \dots, N\}$, $|\{s, t\} \cap N[Y_i]| \geq 1$. Since no vertex of G is in more than t of the sets $N[Y_i]$, it follows that $N \leq 2t$, a contradiction. This proves (26).

In view of (26) we now apply Lemma 8.2 to G'' and S to obtain an induced subgraph H of G'' where $X = \mathcal{Z}(H) \cap S = \{x_1, \dots, x_h\}$ and H is either a path, a caterpillar, the line graph of a caterpillar, a subdivided star, or the line graph of a subdivided star. We will get a contradiction in each of the cases. We start with the following:

(27) *Let $1 \leq i < j \leq h$. Then for every path P from x_i to x_j in G'' and for every $\ell \in I$, $S_\ell \cap P \neq \emptyset$. Consequently, $|P| \geq 4t + 2$.*

Let P be a path from x_i to x_j in G''. Suppose $P \cap S_\ell = \emptyset$ for some $\ell \in I$. Then, there is a component D of $G'' \setminus S_\ell$ such that $x_i, x_j \in D$, a contradiction. This proves that $P \cap S_\ell \neq \emptyset$ for every $\ell \in I$. Since $N/2 \geq 4t^2$, and since no vertex of G'' is in more than t of the sets $N[Y_i]$, it follows that $|P \setminus \{x_i, x_j\}| \geq 4t$, as required. This completes the proof of (27).

(28) *H is not a path, a caterpillar, or the line graph of a caterpillar.*

Assume that H is a path, a caterpillar, or the line graph of a caterpillar. Without loss in generality, assume that $x_1, \dots, x_h$ appear in the natural order given by H (see Figure 10).

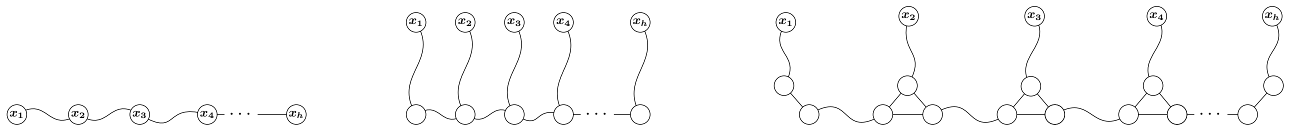

FIGURE 10. A path, a caterpillar, and the line graph of a caterpillar with X ordered.

For each $i = 1, \dots, h/2$, let P_i be a path in H from x_{2i-1} to x_{2i}.

Let $i \in \{1, \dots, \frac{h}{2}\}$. By (27), there is a minimal subpath P'_i of P_i containing x_{2i-1} and such that $P'_i \cap S_j \neq \emptyset$ for every $j \in I$. Since $N/2 > t^2$, and since no vertex of G is in more than t of the sets $N[Y_j]$, it follows that $|P'_i| > t$. Let y_i be the end of P'_i different from x_{2i-1}. By the minimality of P'_i, there exists $j \in I$ such that $y_i \in S_j$, and no other vertex of P'_i belongs to S_j; write $j = j(i)$. Since $\frac{h}{2} \geq 50Nd > 3d|I|$, there exists $j \in I$ such that $j(i) = j$ for at least $3d$ values of i. Since $|Y_j| < d$ for every $j \in I$, there is $y \in Y_j$ and distinct $i_1, i_2, i_3 \in \{1, \dots, \frac{h}{2}\}$ such that y is complete to $\{y_{i_1}, y_{i_2}, y_{i_3}\}$. Recall that y is anticomplete to $(P'_{i_1} \setminus y_{i_1}) \cup (P'_{i_2} \setminus y_{i_2}) \cup (P'_{i_3} \setminus y_{i_3})$. But now there is an $S_{t,t,t}$ in G with center y and paths contained in P'_{i_1}, P'_{i_2}, and P'_{i_3}, a contradiction. This proves (28).

(29) *H is not a subdivided star.*

Suppose that H is a subdivided star with $|H \cap X| = h$ and $H \cap X = \mathcal{Z}(H)$. Let z be the unique vertex of H with degree at least 3. Since $h \geq 4$, there exist paths P_1, P_2, P_3, P_4 of H where P_i is from z to x_i. By (27), for at least three values of i, say $i \in \{1, 2, 3\}$, $|P_i| > t$. But now $P_1 \cup P_2 \cup P_3$ contains an $S_{t,t,t}$ with center z in G (see Figure 11), a contradiction. This proves (29).

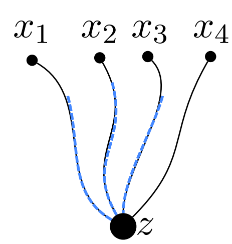

FIGURE 11. Visualization of the $S_{t,t,t}$ obtained to prove (29).

By (28) and (29), it follows that H is the line graph of a subdivided star. Then H consists of a clique $K = \{k_1, \dots, k_h\}$ and paths $P_1, \dots, P_h$, where P_i is from k_i to x_i. For every $i \in \{1, \dots, h\}$ let $I(i) \subseteq I$ be the set of all $j \in I$ such that $S_j \cap (P_i \setminus k_i) \neq \emptyset$.

(30) *$|I(i)| < \frac{N}{8}$ for at most one value of i.*

Suppose $|I(1)|, |I(2)| < \frac{N}{8}$. Since each of k_1, k_2 belongs to at most t of the sets S_i, it follows that $P_1 \cup P_2$ meets S_i for at most $2\frac{N}{8} + 2t < \frac{N}{2} = |I|$ values of i, contrary to (27). This proves (30).

By renumbering if necessary, we can assume that $|I(i)| \geq \frac{N}{8}$ for every $i \in \{1, \dots, \frac{h}{2}\}$. Let $i \in \{1, \dots, \frac{h}{2}\}$. Since $|I(i)| \geq \frac{N}{8}$, there is a minimal subpath P'_i of P_i containing x_i and such that $P'_i \cap S_j \neq \emptyset$ for every $j \in I(i)$. Since $N/8 > t^2$, and since no vertex of G is in more than t of the sets $N[Y_i]$, it follows that

$|P_i'| > t$. Let y_i be the end of P_i' different from x_i. By the minimality of P_i', there exists $j \in I$ such that $y_i \in S_j$, and no other vertex of P_i' belongs to S_j; write $j = j(i)$. Since $\frac{h}{2} \geq 50Nd > 3d|I|$, there exists $j \in I$ such that $j(i) = j$ for at least $3d$ values of i. Since $|Y_j| < d$ for every $j \in I$, there is $y \in Y_j$ and distinct $i_1, i_2, i_3 \in \{1, \dots, \frac{h}{2}\}$ such that y is complete to $\{y_{i_1}, y_{i_2}, y_{i_3}\}$. Recall that y is anticomplete to $(P_{i_1}' \setminus y_{i_1}) \cup (P_{i_2}' \setminus y_{i_2}) \cup (P_{i_3}' \setminus y_{i_3})$. But now there is an $S_{t,t,t}$ in G with center y and paths contained in P_{i_1}', P_{i_2}', P_{i_3}', a contradiction. ■

9. Acknowledgment

We are grateful to Paweł Rzążewski for allowing us to use Figure 1. Part of this research was conducted while the fourth author visited Princeton University. The final phase of the writing of this paper took place at the Dagstuhl Seminar 25041, “Solving Problems on Graphs: From Structure to Algorithms”, Jan 19–24, 2025. We are grateful to both institutions for their hospitality.

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