TOURNAMENTS AND THE STRONG ERDŐS-HAJNAL PROPERTY

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Abstract. A conjecture of Alon, Pach and Solymosi, which is equivalent to the celebrated Erdős-Hajnal Conjecture, states that for every tournament S there exists $\epsilon(S) > 0$ such that if T is an n-vertex tournament that does not contains S as a subtournament, then T contains a transitive subtournament on at least $n^{\epsilon(S)}$ vertices. Let C_5 be the unique five-vertex tournament where every vertex has two inneighbors and two outneighbors. The Alon-Pach-Solymosi conjecture is known to be true for the case when $S = C_5$. Here we prove a strengthening of this result, showing that in every tournament T with no subtorunament isomorphic to C_5 there exist disjoint vertex subsets A and B, each containing a linear proportion of the vertices of T, and such that every vertex of A is adjacent to every vertex of B.

1. introduction

A *tournament* is a complete graph with directions on edges. A tournament is *transitive* if it has no directed triangles. For tournaments S, T we say that T is *S-free* if no subtourament of T is isomorphic to S. In [1] a conjecture was made concerning tournaments with a fixed forbidden subtorunament:

Conjecture 1.1. *For every tournaments S there exists $\epsilon > 0$ such that every S-free n-vertex tournament contains a transitive subtournament on at least n^ϵ vertices.*

It was shown in [1] that Conjecture 1.1 is equivalent to the Erdős-Hajnal Conjecture [7, 8]. Conjecture 1.1 is known to hold for a few types of tournaments S [2, 3], but is still wide open in general.

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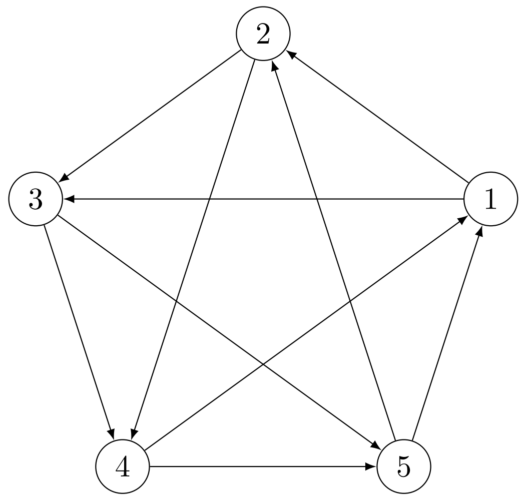

FIGURE 1. The C_5 tournament.

Let D be a directed graph, and let $A, B \subseteq V(D)$ with $A \cap B = \emptyset$. We say that A is *complete* to B if all every vertex of A is adjacent to every vertex of B, and that A is complete *from* B if every vertex of A is adjacent from every vertex of B. A class of tournaments is *hereditary* if it is closed under subtournaments. A hereditary class of tournaments $\mathcal{T}$ has *strong Erdős-Hajnal property* if there exists $\varepsilon = \varepsilon(\mathcal{T})$ such that for every $T \in \mathcal{T}$ there exist disjoint subsets A, B of $V(T)$, each of size $\varepsilon|V(T)|$ such that A is complete to B.

The following question is closely related to Conjecture 1.1.

Question 1.1. *For which tournaments S does the class of S-free tournaments have the strong Erdös-Hajnal property?*

It is easy to see [2] that if S is a tournament, and the class of S-free graphs has the strong Erdös-Hajnal property, then 1.1 is true for S. In [4] there is a list of necessary conditions for a tournament S to satisfy 1.1.

Denote by C_5 the (unique) tournament on 5 vertices in which every vertex is adjacent to exactly two other vertices. One way to construct this tournament is with vertex set $\{0, 1, 2, 3, 4\}$ and i is adjacent to $i+1 \mod 5$ and $i + 2 \mod 5$ (see Figure 1).

In [2] it was proved that C_5 satisfies the Erdős-Hajnal conjecture. Here we prove the following stronger result:

Theorem 1.2. *The class of C_5-free tournaments has the strong Erdős-Hajnal property.*

2. Regularity Tools

We recall some definitions given in [3].

Let $c > 0$, $0 < \lambda < 1$ be constants, and let w be a $\{0,1\}$-vector of length $|w|$. Let T be a tournament with $|V(T)| = n$. Denote by $tr(T)$ the largest size of the transitive subtournament of T. For $\epsilon > 0$ we call a tournament T *ϵ-critical* for $\epsilon > 0$ if $tr(T) < |T|^{\epsilon}$ but for every proper subtournament S of T we have: $tr(S) \geqslant |S|^{\epsilon}$. A sequence of disjoint subsets $(S_1, S_2, ..., S_{|w|})$ of $V(T)$ is a *(c, λ, w)-structure* if

- whenever $w_i = 0$ we have $|S_i| \geqslant cn$,
- whenever $w_i = 1$ the set $T|S_i$ is transitive and $|S_i| \geqslant c \cdot tr(T)$,
- $d^+(S_i, S_j) \geqslant 1 - \lambda$ for all $1 \leqslant i < j \leqslant |w|$.

We say that a (c, λ, w)-structure is *smooth* if the last condition of the definition of the (c, λ, w)-structure is satisfied in a stronger form, namely we have: for every $i < j$, every $v \in S_i$ has at most $\lambda|S_j|$ inneighbors in S_j, and every $v \in S_j$ has at most $\lambda|S_i|$ outneighbors in S_i.

Theorem 3.5 of [3] asserts:

Theorem 2.1. *Let S be a tournament, let w be a $\{0,1\}$-vector, and let $0 < \lambda < \frac{1}{2}$ be a constant. Then there exist $\epsilon, c > 0$ such that every S-free ϵ-critical tournament contains a smooth (c, λ, w)-structure.*

Here we need a weaker form of Theorem 2.1 for the case when w is the all-zero vector. It turns out that in that case we do need the criticality assumption. The proof consists of standard regularity lemma arguments, and can be easily reconstructed from the proof of 2.1 in [2]. Thus we have:

Theorem 2.2. *Let S be a k-vertex tournament and let w an the all-zero vector. There exists $c > 0$ such that every S-free tournament contains a smooth $(c, \frac{1}{k}, w)$-structure.*

3. A lemma on out-simplicial directed graphs

We say that a directed graph is out-simplicial if the out neighborhood of each vertex is a clique in the underlying undirected graph.

The main lemma we use for the proof of Theorem 1.2 is this:

Lemma 3.1. *If D is an out-simplicial directed graph on $n > 1$ vertices, then there exist two disjoint subsets A and B of $V(D)$, both of size $\lfloor n/6 \rfloor$, such that either*

(i) *there is no edge, in any direction, between a vertex of A and a vertex of B, or*
(ii) *there is a path from every vertex of A to every vertex of B.*

Proof. Assume this is false. Then D is not strongly connected, and, moreover, every strongly connected component C of D has size at most $n/3$, for otherwise a balanced partition of C satisfies (ii). Let $C_1, \dots, C_m$ be the strongly connected components of D. Let F be the directed graph with vertex set $C_1, \dots, C_m$, and such that C_i is adjacent to C_j if and only if there is an edge from C_i to C_j in D. For $S \in V(F)$ let $w(S) = \sum_{C_i \in S} |V(C_i)|$. Note that F is an acyclic directed graph. Let F' be the underlying undirected graph of F.

(1) F is outsimpicial.

To see this, suppose to the contrary that C_i is adjacent to C_j and to C_k but there is no edge from C_j to C_k in F'. Then in D, no vertex of C_j is adjacent to or from a vertex of C_k. Let $v_j, v_k \in C_i$, $u_j \in C_j$, $u_k \in C_k$ be such that (v_j, u_j) and (v_k, u_k) are edges in D. Since D is outsimplicial and u_j u_k are not in D, it follows that $v_j \neq v_k$. We may assume that v_j, v_k are chosen so that the directed path P in C_i from v_j to v_k is as short as possible. Let p be the outneighbor of v_j in P. It follows from the minimality of P that (p, u_k) is not an edge of D. Since C_i and C_k are distinct strongly connected components of D, it follows that (u_j, p) is not an edge. But now p and u_j are both outneighbors of v_j, and there is no edge between then in either direction, contrary to the fact that D is outsimplicial. This proves (1).

(2) F' is chordal.

Indeed, suppose C is an induced cycle of length larger than 3 in F'. Since F is out-simplicial, no vertex of C has two outneighbors in C, and therefore C is directed, a contradiction to the fact that F is acyclic.

(3) No clique of F' has weight $n/3$.

Indeed, suppose that K be a clique of weight $n/3$. Then K is a transitive subtournament of F. Let $C_{k_1}, \dots, C_{k_p}$ be the vertices of K in the transitive order. Then there is a path in D from every vertex of C_{k_i} to every vertex of C_{k_j} for $i \leqslant j$. For every $i \in [p]$, choose some order on the vertices in C_{k_i}, and consider the corresponding order $(C_{k_1}, \dots, C_{k_p})$ on $V_K = \cup_{i=1}^{p} C_{k_i}$. Let A be the first $\frac{|V_K|}{2}$ vertices in this order and let $B = V_K \setminus A$. Then A, B satisfy (ii), contradicting our assumption. This proves (3).

Since F' is chordal, F' has a tree decomposition with bags being cliques of F'. Let (T, X) be such a tree decomposition, where the bag corresponding to a vertex v of T is denoted X_v.

By Lemma 7.19 in [6] there exists a bag X_v of T such that every connected component D of $F' \setminus X_v$ has weight at most $w(V(F'))/2$. In particular, $F' \setminus X_v$ contains at least two connected components. Write $x = |V(F') \setminus X_v| \geqslant 2n/3$. Let $X_1, \ldots, X_m$ be the connected components in $F' \setminus X_v$, ordered such that $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_m$, where $x_i = |X_i|$. Then $x_i \leqslant n/2$ for all i, and thus $m \geqslant 2$, showing $x_1 \leqslant x/2$. Write $y_i = \sum_{j=1}^{i} x_j$. Let r be the index for which $y_{r-1} \leqslant x/2$ and $y_r > x/2$.

Suppose first that $r = m$. Then $a = y_{r-1}$ and $b = x_r$ are both at least of size $n/6$, since $a = x - x_r \geqslant 2n/3 - n/2 = n/6$ and $b = x - y_{r-1} \geqslant x - x/2 = x/2 \geqslant n/3$. Therefore the sets $A = \bigcup_{i=1}^{r-1} X_i$ and $B = X_r$, of sizes a and b respectively, satisfy (i), a contradiction. It follows that $r < m$; let $a = y_r$ and $b = x - y_r$. Note that $x_r \leqslant b$, and thus $x/2 + 2b \geqslant y_{r-1} + x_r + b = x$, showing $b \geqslant x/4 \geqslant n/6$, and $a \geqslant x/2 \geqslant n/3$ by the choice of r. Therefore the sets $A = \bigcup_{i=1}^{r} X_i$ and $B = \bigcup_{i=r+1}^{m} X_i$, of sizes a and b respectively, satisfy (i), again a contradiction. Thus the lemma is proved. □

4. Proof of Theorem 1.2.

Let T be a C_5-free tournament on n vertices, and let c be as in Theorem 2.2 applied with $S = C_5$. We show that there exist a constant c and disjoint subsets A, B of $V(T)$, such that $|A| = |B| = cn/6$ and A is complete to B.

Assume to the contrary that this is false. Let w be the zero vector of length 5. By Theorem 2.2 there exists $c > 0$ and a smooth $(c, \frac{1}{5}, (0, 0, 0, 0, 0))$-structure $\mathcal{S} = (V_1, ..., V_5)$. By definition, we have that $|V_i| \geqslant cn$ for all $1 \leqslant i \leqslant 5$, and for each $v_i \in V_i$, if $j > i$, then the number of inneighbors of v_i in V_j at most $\frac{1}{5}|V_j|$, and if $j < i$ then the number of outneighbors of v_i in V_j is at most $\frac{1}{5}|V_j|$. It follows that , if $j > i$, then the number of outneighbors of v_i in V_j at least $\frac{4cn}{5}$, and if $j < i$ then the number of inneighbors of v_i in V_j is at least $\frac{4cn}{5}$.

We now define a directed graph D on the set of vertices V_1 in the following way: there is an edge between two vertices u_1, v_1 of V_1 if and only if they have a common inneighbor in V_5, and in this case the direction of the edge is the same as the direction of the edge between these two vertices in T.

Claim 4.1. *If (u_1, v_1) is an edge in D then*

$$N^-(u_1) \cap V_3 \subseteq N^-(v_1) \cap V_3. \tag{1}$$

Proof. Assume $v_3 \in V_3$ is an inneighbor of u_1, but an outneighbor of v_1, and let $v_5 \in V_5$ be a common inneighbor of u_1 and v_1. Suppose first $(v_3, v_5) \in E(T)$. We claim that there exists a vertex $v_2 \in V_2$ such that $\{(v_2, v_3), (v_2, v_5), (u_1, v_2), (v_1, v_2)\} \subset E(T)$. Indeed, the number of outneighbors of each of u_1 and v_1 in V_2 is at least $\frac{cn}{5}$, and thus the number of common outneighbors of u_1 and v_1 in V_2 is at least $\frac{3cn}{5}$. Let $O_2 \in V_2$ be the set common outneighbor of u_1, v_1 in V_2. Since the number of outneighbors of v_5 in O_2 is at most $\frac{cn}{5}$, and the number of outneighbors of v_3 in O_2 is at most $\frac{cn}{5}$, there must exist a vertex $v_2 \in O_2$ that is a inneighbor of both v_3 and v_5. Now $(u_1, v_1, v_2, v_3, v_5)$ form a C_5 subtournament of T, a contradiction.

So assume $(v_5, v_3) \in E(T)$. Let $O_4 \in V_4$ be the set of common outneighbors of u_1 and v_3. Then, as before, $|O_4| \geqslant \frac{3cn}{5}$. Since the number of inneighbors v_5 in V_4 is at least $\frac{4cn}{5}$, there exists a set $N_4 \subset O_4$ of size at least $\frac{2cn}{5}$ such that every vertex $v \in N_4$ has $(u_1, v), (v_3, v), (v, v_5) \in E(T)$. Similarly, there exists a set $N_2 \subset V_2$ of size at least $\frac{2cn}{5}$, such that every vertex $v \in N_2$ has $(u_1, v), (v, v_3), (v, v_5) \in E(T)$. If there exist $v_2 \in N_2, v_4 \in N_4$ such that $(v_4, v_2) \in E(T)$ then $(v_3, u_1, v_4, v_2, v_5)$ is a C_5 subtournament of T, a contradiction. Otherwise N_2 is complete to N_4, contradicting our negation assumption. □

Claim 4.2. *D is an out-simplicial directed graph.*

Proof. In order to prove this, consider three vertices $u_1, v_1, w_1 \in V_1$ such that $(u_1, v_1), (u_1, w_1) \in E(D)$. We need to prove that there is a D edge between v_1 and w_1 in some direction, i.e., that v_1 and w_1 have a common inneighbor in V_5. Let $x_5 \in V_5$ be the common inneighbor of u_1 and v_1 and let $y_5 \in V_5$ be the common inneighbor of u_1 and w_1. Without loss of generality, $(x_5, y_5) \in E(T)$. As before, there exists a set $O_3 \subset V_3$ of size at least $\frac{3cn}{5}$ such that every vertex $v \in O_3$ is an outneighbor of both u_1 and w_1, and there exists a set $I_3 \subset V_3$ of size at least $\frac{3cn}{5}$ such that every vertex $v \in I_3$ is an inneighbor of both x_5 and y_5. Thus there exists a vertex $z_3 \in O_3 \cap I_3$, and $(u_1, w_1, z_3, x_5, y_5)$ is a C_5 subtournament in T, a contradiction. Therefore x_5 must also be an inneighbor of w_1, proving our claim. □

Now, by Lemma 3.1 there exist sets A and B of $V(D)$, each of size $|V(D)|/6 \geqslant cn/6$, satisfying either (i) or (ii). Let C be the set of vertices complete from A in T.

In case (i), there is no edge in D between A and B in any direction, implying that no two vertices $a \in A$ and $b \in B$ have a common inneighbor in V_5. Thus the set $V_5 \setminus C$ is complete from B, and either C or $V_5 \setminus C$ is of size at least $\frac{cn}{2}$, a contradiction.

In case (ii), there is a directed path from every vertex in A to every vertex of B. Let $v \in V_3 \setminus C$. Then v is an inneighbor of some $a \in A$. Let $b \in B$. There is a directed path P in D from a to b. By (1), using induction on the length of P, v is an inneighbor of every every p in P, and in particular v is an inneighbor of b. It follows that $V_3 \setminus C$ is complete to B. Since either C or $V_3 \setminus C$ is of size at least $\frac{cn}{2}$, we get a contradiction. This concludes the proof of the theorem.

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