TREE-INDEPENDENCE NUMBER VI. THETAS AND PYRAMIDS.

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Abstract. Given a family $\mathcal{H}$ of graphs, we say that a graph G is $\mathcal{H}$-free if no induced subgraph of G is isomorphic to a member of $\mathcal{H}$. Let $W_{t\times t}$ be the t-by-t hexagonal grid and let $\mathcal{L}_t$ be the family of all graphs G such that G is the line graph of some subdivision of $W_{t\times t}$. We denote by $\omega(G)$ the size of the largest clique in G. We prove that for every integer t there exist integers $c_1(t)$, $c_2(t)$ and $d(t)$ such that every (pyramid, theta, $\mathcal{L}_t$)-free graph G satisfies:

- G has a tree decomposition where every bag has size at most $\omega(G)^{c_1(t)}\log(|V(G)|)$.
- If G has at least two vertices, then G has a tree decomposition where every bag has independence number at most $\log^{c_2(t)}(|V(G)|)$.
- For any weight function, G has a balanced separator that is contained in the union of the neighborhoods of at most $d(t)$ vertices.

These results qualitatively generalize the main theorems of [2] and [9].

Additionally, we show that there exist integers $c_3(t), c_4(t)$ such that for every (theta, pyramid)-free graph G and for every non-adjacent pair of vertices $a, b \in V(G)$,

- a can be separated from b by removing at most $w(G)^{c_3(t)}\log(|V(G)|)$ vertices.
- a can be separated from b by removing a set of vertices with independence number at most $\log^{c_4(t)}(|V(G)|)$.

1. Introduction

All graphs in this paper are finite and simple, and all logarithms are base 2. Let $G = (V(G), E(G))$ be a graph. For a set $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X, and by $G \setminus X$ the subgraph of G induced by $V(G) \setminus X$. In this paper, we use induced subgraphs and their vertex sets interchangeably. For graphs G, H we say that G *contains* H if H is isomorphic to $G[X]$ for some $X \subseteq V(G)$. In this case, we say that X *is an* H *in* G. We say that G is H*-free* if G does not contain H. For a family $\mathcal{H}$ of graphs, we say that G is $\mathcal{H}$*-free* if G is H-free for every $H \in \mathcal{H}$.

The *open neighborhood of* v, denoted by $N_G(v)$, is the set of all vertices in $V(G)$ adjacent to v. The *closed neighborhood of* v, denoted by $N_G[v]$, is $N(v) \cup \{v\}$. Let $X \subseteq V(G)$. The *open neighborhood of* X, denoted by $N_G(X)$, is the set of all vertices in $V(G) \setminus X$ with at least one neighbor in X. The *closed neighborhood of* X, denoted by $N_G[X]$, is $N_G(X) \cup X$. When there is no danger of confusion, we omit the subscript G. Let $Y \subseteq V(G)$ be disjoint from X. We say X is *complete* to Y if all possible edges with an end in X and an end in Y are present in G, and X is *anticomplete* to Y if there are no edges between X and Y.

For a graph G, a *tree decomposition* (T, χ) of G consists of a tree T and a map $\chi\colon V(T) \to 2^{V(G)}$ with the following properties:

(1) For every $v \in V(G)$, there exists $t \in V(T)$ such that $v \in \chi(t)$.

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(2) For every $v_1v_2 \in E(G)$, there exists $t \in V(T)$ such that $v_1, v_2 \in \chi(t)$.
(3) For every $v \in V(G)$, the subgraph of T induced by $\{t \in V(T) \mid v \in \chi(t)\}$ is connected.

For each $t \in V(T)$, we refer to $\chi(t)$ as a *bag of* (T, χ). The *width* of a tree decomposition (T, χ), denoted by $width(T, \chi)$, is $\max_{t \in V(T)} |\chi(t)| - 1$. The *treewidth* of G, denoted by $\mathsf{tw}(G)$, is the minimum width of a tree decomposition of G. Graphs of bounded treewidth are well-understood both structurally [29] and algorithmically [5].

A *clique* in a graph G is a set of pairwise adjacent vertices; the *clique number* of G, denoted by $\omega(G)$, is the largest size of a clique in G. A *stable (or independent) set* in a graph G is a set of pairwise non-adjacent vertices of G. The *stability (or independence) number* $\alpha(G)$ of G is the largest size of a stable set in G. Given a graph G with weights on its vertices, the Maximum Weight Independent Set (MWIS) problem is the problem of finding a stable set in G of maximum total weight. This problem is known to be NP-hard [19], but it can be solved efficiently (in polynomial time) in graphs of bounded treewidth.

The *independence number* of a tree decomposition (T, χ) of G is $\max_{t \in V(T)} \alpha(G[\chi(t)])$. The *tree independence number* of G, denoted tree-$\alpha(G)$, is the minimum independence number of a tree decomposition of G. The tree-independence number was defined by Dallard, Milanič and Štorgel [16] as a way to understand graphs whose high treewidth can be explained by the presence of a large clique, and targeting the complexity of the MWIS problem. Combining results of [16] and [14] yields an efficient algorithm for the MWIS problem for graphs of bounded tree-α. In [23], similar algorithmic results are obtained for a much more general class of problems. Recently, the study of the structure of graphs with low tree-$\alpha(G)$ has gained momentum, see for example [17, 15, 21].

A *hole* in a graph is an induced cycle of length at least four. A *path* in a graph is an induced subgraph that is a path. The *length* of a path or a hole is the number of edges in it. We denote by $P = p_1\text{-}\dots\text{-}p_k$ be a path in G where $p_ip_j \in E(G)$ if and only if $|j - i| = 1$. We say that p_1 and p_k are the *ends* of P. The *interior* of P, denoted by P^*, is the set $P \setminus \{p_1, p_k\}$. For $i, j \in \{1, \dots k\}$ we denote by $p_i\text{-}P\text{-}p_j$ the subpath of P with ends p_i, p_j.

A *theta* is a graph consisting of two distinct vertices a, b and three paths P_1, P_2, P_3 from a to b, such that $P_i \cup P_j$ is a hole for every $i, j \in \{1, 2, 3\}$. It follows that a is non-adjacent to b and the sets P_1^*, P_2^*, P_3^* are pairwise disjoint and anticomplete to each other. If a graph G contains an induced subgraph H that is a theta, and a, b are the two vertices of degree three in H, then we say that G contains a theta *with ends a and b*.

A *pyramid* is a graph consisting of a vertex a and a triangle $\{b_1, b_2, b_3\}$, and three paths P_i from a to b_i for $1 \le i \le 3$, such that $P_i \cup P_j$ is a hole for every $i, j \in \{1, 2, 3\}$. It follows that $P_1 \setminus a, P_2 \setminus a, P_3 \setminus a$ are pairwise disjoint, and the only edges between them are of the form b_ib_j. It also follows that at most one of P_1, P_2, P_3 has length exactly one. We say that a is the *apex* of the pyramid and that $b_1b_2b_3$ is the *base* of the pyramid.

(Theta, pyramid)-free graphs have received significant attention in structural graph theory. Often another family of graphs, called "prisms", is excluded. Treewidth and tree-independence number of (Theta, pyramid, prism)-free graphs were studied in [2] and [9]. In [2], a logarithmic upper bound on treewidth is obtained (where the bound depends on the clique number of the graph), while in [9] tree-independence number is bounded by a polylogarithmic function of the number of vertices.

The scope of this paper is broader in the following sense. Let $W_{t\times t}$ be the t-by-t hexagonal grid, and let $\mathcal{L}_t$ be the family of all graphs G such that G is the line graph of some subdivision of $W_{t\times t}$. Observe that $\bigcup_t \mathcal{L}_t$ contains a sequence of graphs for which the treewidth grows asymptotically as the square root of the number of vertices. It follows that $\mathcal{L}_t$ needs to be excluded to achieve sub-polynomial (in the number of vertices) bounds on treewidth. Since all graphs in $\bigcup_t \mathcal{L}_t$ are (pyramid, theta)-free and have clique number at most three, excluding $\bigcup_t \mathcal{L}_t$ is necessary even in the class of (theta, pyramid)-free graphs with bounded clique number. The situation for tree-independence number is similar.

We prove that this necessary condition is, in fact, sufficient in (theta pyramid)-free graphs. Since every $\mathcal{L}_t$ contains a prism if we choose t large enough, this is a qualitative generalization of the results of [2] and [9] (the degree of the polynomial in $\log(|V(G)|)$ in the bound on the tree-independence number is worse here). For an integer t, let $\mathcal{M}_t$ be the class of all (theta, pyramid, $\mathcal{L}_t$)-free graphs. We prove:

Theorem 1.1. *For every positive integer t, there is an integer $c = c(t)$ such that every graph $G \in \mathcal{M}_t$ has treewidth at most $\omega(G)^c \log(|V(G)|)$.*

Theorem 1.2. *For every positive integer t there exists $c = c(t)$ such that for every graph $G \in \mathcal{M}_t$ on at least 3 vertices, we have* tree-$\alpha(G) \leq \log^c n$.

We remark that Theorem 1.2 follows immediately from Theorem 1.1 using a result of [11]; we explain this in Section 7. Theorem 1.1 is tight since there exist (theta, triangle)-free graphs with logarithmic treewidth [30]. For the same reason Theorem 1.2 is tight up to the degree of the polynomial (in fact, we do not have a counterexample to $c = 1$).

As is explained in [2], by the celebrated Courcelle's theorem [13], Theorem 1.1 also implies the existence of polynomial time algorithms for a large class of NP-hard problems, such as Stable Set, Vertex Cover, Dominating Set, and Coloring, when the input restricted to graphs in $\mathcal{M}_t$ with bounded clique number. Similarly, the algorithmic implications of Theorem 1.2 using results of [23] are discussed in [9].

Finally, Theorem 1.2 is a promising step toward the following:

Conjecture 1.3 (from [7])**.** *For every positive integer t, there is an integer $d = d(t)$ such that for every $n \geq 2$, every n-vertex graph with no induced minor isomorphic to $K_{t,t}$ or to $W_{t\times t}$ has* tree-α *at most* $\log^d n$.

Let G be a graph and let $A, B \subseteq G$ be disjoint. We say that a set $X \subseteq V(G) \setminus (A \cup B)$ *separates* A from B if for every connected component D of $G \setminus X$, $D \cap A = \emptyset$ or $D \cap B = \emptyset$. Let G be a graph and let a, b be two non-adjacent vertices of G. We say that a set $X \subseteq V(G) \setminus \{a, b\}$ *separates* a from b if for every connected component D of $G \setminus X$, $|D \cap \{a, b\}| \leq 1$. A graph is said to be *k-pairwise separable* if for every pair of non-adjacent vertices of G, there exists a set X with $|X| \leq k$ that separates them from each other.

In order to prove Theorem 1.1, we need a result on pairwise-separability. We were able to obtain such a result in a more general setting (without excluding $\mathcal{L}_t$), which may be of independent interest. Here we denote by $\mathcal{H}_t$ the class of (K_t, pyramid, theta)-free graphs.

Theorem 1.4. *For every integer $t \geq 2$, there exists a positive integer c such that every n-vertex graph in $\mathcal{H}_t$ is $t^c \log n$-pairwise separable.*

Once again, using a result of [11], we also get a version of this theorem where the size of the separator is replaced by its independence number (see Section 7 for details):

Theorem 1.5. *There exists an integer c such that for every n-vertex (theta, pyramid)-free graph G and every non-adjacent pair $u, v \in V(G)$ there exists $X \subseteq V(G) \setminus \{u, v\}$ with $\alpha(X) \leq \log^c n$ such that X separates u from v.*

For a graph G, a function $w\colon V(G) \to [0,1]$ is a *weight function* if $\sum_{v\in V(G)} w(v) \leq 1$. For $S \subseteq V(G)$, we write $w(S) \coloneqq \sum_{v\in S} w(v)$. A weight function w is a *normal weight function* on G if $w(V(G)) = 1$. If $0 < w(V(G)) < 1$, we call the function $w'\colon V(G) \to [0,1]$ given by $w'(v) = \frac{w(v)}{\sum_{u\in V(G)} w(u)}$ the *normalized weight function of* w. Let $c \in [0,1]$ and let w be a weight function on G. A set $X \subseteq V(G)$ is a (w,c)*-balanced separator* if $w(D) \leq c$ for every component D of $G \setminus X$. The set X is a w*-balanced separator* if X is a $(w, \frac{1}{2})$-balanced separator. Given two sets of vertices X and Y of G, we say that X is a *core* for Y if $Y \subseteq N[X]$. A graph G is said to be k*-breakable* if for every weight function $w\colon V(G) \to [0,1]$, there exists a w-balanced separator with a core of size strictly less than k. When the weight function w is clear from the context, we may omit it from the notation. Our last result is:

Theorem 1.6. *For every positive integer t, there is an integer $d = d(t)$ such that every graph $G \in \mathcal{M}_t$ is d-breakable.*

In Section 7, we follow the outline of the proof (and use some results) of [8] to deduce Theorem 1.1 from Theorem 1.4 and Theorem 1.6.

1.1. **Proof outline and organization.** Most of the work in this paper is devoted to proving Theorem 1.4 and Theorem 1.6. Theorem 1.1, Theorem 1.2 and Theorem 1.5 are deduced from them using existing results in Section 7. An important tool for both the main proofs is "extended strip decompositions" from [12]. They are introduced in Section 2.

Let us start by outlining the proof of Theorem 1.6 that is proved in Section 3. The high-level idea of the proof is similar to [7], but the details are different because of the different properties of the graph class in question. Let $G \in \mathcal{M}_t$. We may assume that G is connected. For a contradiction, we fix a weight function w such that G does not have a w-balanced separator with a small core. By using the normalized weight function of w, we may assume that w is normal. By Lemma 5.3 of [10], there is a path $P = p_1\text{-}\dots\text{-}p_k$ in G such that $N[P]$ is a w-balanced separator in G. We choose P with k minimum; consequently we may assume that there is a component B of $G \setminus N[P \setminus \{p_k\}]$ with $w(B) > \frac{1}{2}$. We now analyze the structure of the set $N = N(B) \subseteq N(P)$. To every vertex $n \in N$ we assign a subpath $I(n)$ of P, that is the minimal subpath of P that contains $N(n) \cap P$. We define a new graph H with vertex set N where n_1 and n_2 are adjacent if and only if $I(n_1) \cap I(n_2) \neq \emptyset$. Let S be a maximum stable set in H. We first show that for every $s \in S$, we can find a small core (in G) for the set $N_H[s]$ (when viewed as a subset of G). This, in particular, allows us to assume that $|S|$ is large. Now, we focus on one vertex $n \in S$ and use it to show that G (with a subset with a small core deleted) admits an extended strip decomposition. This allows us to produce a separator $X(n)$ with a small core that is not yet balanced, but exhibits several useful properties. More explicitly, the component of $G \setminus X(n)$ with maximum w-weight only meets P on one side of $I(n)$. So n either "points left" or "points right". Then we show that the vertex of S with the earliest neighbors in P points right, and the vertex of S with the latest neighbors in P points left. Now we focus on two vertices $n, n' \in S$ such that $I(n)$ and $I(n')$ are consecutive along P where the change first occurs, and conclude that $X(n) \cap X(n')$ is a w-balanced separator in G. This completes the proof of Theorem Theorem 1.6.

We now turn to the proof of Theorem 1.4. This is the most novel part of the paper, where several new ideas are introduced. Let $G \in \mathcal{H}_t$ and let $a, b \in V(G)$ be non-adjacent. We consider carefully chosen subsets X_i of balls of radius i around a with $a \in X_i$, and iteratively construct a set C that, when the process stops, separates a from b. We will show that $|C| \leq t^c \log(|V(G)|)$ (where the same c works for all graphs in $\mathcal{H}_t$), thus proving Theorem 1.4.

Throughout the process, X_i satisfies two key properties. The first one is that X_i is "cooperative" (as defined in Section 5). The second one is that at each step of the construction, we only add to X_i vertices that have a lower value in the partition of $V(G)$ defined by Theorem 6.1 (see Section 6 for the defintion of "value").

At each step i, we examine the attachments N_i of the component D_i of $G \setminus (N[X_i] \cup C)$ with $b \in D$. Note that $N(D_i) \subseteq N(X_i)$. First, we show that $N_i \cap N(b)$ is small and add $N_i \cap N(b)$ to C. Next we define two matroids on $N_i \setminus N(b)$: $\mathcal{M}_1$ is the matching matroid into X_i, and $\mathcal{M}_2$ is the matroid whose independent sets are linkable to b by disjoint paths with interior in D_i; both matroids are defined precisely in Section 5. Suppose first that there is a large subset I of N_i that is independent in both matroids. We use the fact that I is independent in $\mathcal{M}_1$ and that X_i is cooperative to obtain a large subset Z of I that is "constricted" in $D_i \cup Z$ (see Section 2 for a precise definition). This allows us to construct an extended strip decomposition of $(D_i \cup Z, Z)$. Now, we use results from Section 4 to get a contradiction to the fact that Z is independent in $\mathcal{M}_2$.

Thus, we may assume that no such set I exists. We apply the Matroid Intersection Theorem to construct a partition (A_1, A_2) of $N_i \setminus N(b)$ such that the sum of the ranks $rk_{\mathcal{M}_i}(A_i)$ is small. By Menger's Theorem, there is a small subset Z_2 of $D_i \cup A_2$ such that Z_2 separates A_2 from b; we add Z_2 to C. By König's Theorem, there is a small subset Z_1 of $N_{X_i}(N_i) \cup N_i$ such that every edge between A_1 and X_i has an end in Z_1. We add $Z_1 \cap N_i$ to C and focus on $Z_1 \cap X_i$. By Theorem 6.1, the number of vertices with a neighbor in $Z_1 \cup X_i$ whose value is higher than the maximum value of a vertex in $Z_1 \cup X_i$ is bounded by $t^{c'}$ (where the same c' works for every graph in $\mathcal{H}_t$); we add all such vertices to C. Note that at this point $N_i \subseteq N(Z_1) \cup C$. If $N(Z_1) \not\subseteq C$, we construct $X_{i+1} = X_i \cup (N(Z_1) \setminus C)$ and continue.

By Theorem 6.1 for some $i \leq \log(|V(G)|)$ it holds that $N(Z_1) \subseteq C$. Now $N_i \subseteq C$ and consequently C separates X_i from b. Since $a \in X_i$, C has the required separation properties, and we stop the process. This completes the description of the proof of Theorem 1.4.

This paper is organized as follows. In Section 2, we define constricted sets and extended strip decompositions. In Section 3, we prove Theorem 1.6. In Section 4, we establish an important property of constricted sets in $K_{t,t}$-free graphs (a super-class of $\mathcal{H}_t$), that will be used in the proof of Theorem 1.4 to obtain a bound on the size of a set that is independent in both $\mathcal{M}_1$ and $\mathcal{M}_2$. In Section 5, we define cooperative sets and prove several lemmas about their properties; that is also where we describe the application of the Matroid Intersection Theorem. In Section 6, we prove Theorem 1.4. Finally, in Section 7, we show how to use Theorem 1.4 and Theorem 1.6 to prove Theorem 1.1, Theorem 1.2, and Theorem 1.5.

2. Constricted sets and extended strip decompositions

An important tool in the proof of Theorem 1.4 is the "extended strip decompositions" of [12]. We explain this now after introducing some definitions from [7]. Let G, H be graphs, and let $Z \subseteq V(G)$. Let W be the set of vertices of degree one in H. Let $T(H)$ be the set of all triangles of H. Let η be a map with domain the union of $E(H)$, $V(H)$, $T(H)$, and the set of all pairs (e, v) where $e \in E(H)$, $v \in V(H)$ and e incident with v, and range $2^{V(G)}$, satisfying the following conditions:

- For every $v \in V(G)$ there exists a unique $x \in E(H) \cup V(H) \cup T(H)$ such that $v \in \eta(x)$.
- For every $e \in E(H)$ and $v \in V(H)$ such that e is incident with v, $\eta(e, v) \subseteq \eta(e)$
- Let $e, f \in E(H)$ with $e \neq f$, and $x \in \eta(e)$ and $y \in \eta(f)$. Then $xy \in E(G)$ if and only if e, f share an end-vertex v in H, and $x \in \eta(e, v)$ and $y \in \eta(f, v)$.

- If $v \in V(H)$, $x \in \eta(v)$, $y \in V(G) \setminus \eta(v)$, and $xy \in E(G)$, then $y \in \eta(e, v)$ for some $e \in E(H)$ incident with v.
- If $D \in T(H)$, $x \in \eta(D)$, $y \in V(G) \setminus \eta(D)$ and $xy \in E(G)$, then $y \in \eta(e, u) \cap \eta(e, v)$ for some distinct $u, v \in D$, where e is the edge uv of H.
- $|Z| = |W|$, and for each $z \in Z$ there is a vertex $w \in W$ such that $\eta(e, w) = \{z\}$, where e is the (unique) edge of H incident with w.

Under these circumstances, we say that η is an *extended strip decomposition of* (G, Z) *with pattern* H (see Figure 1). As a slight abuse of notation, for $v \in V(G)$ we will denote by $\eta^{-1}(v)$ the unique $x \in E(H) \cup V(H) \cup T(H)$ such that $v \in \eta(x)$, as guaranteed by the first condition.

Let e be an edge of H with ends u, v. An *e-rung* in η is a path p_1-$\dots$-p_k (possibly $k = 1$) in $\eta(e)$, with $p_1 \in \eta(e, v)$, $p_k \in \eta(e, u)$ and $\{p_2, \dots, p_{k-1}\} \subseteq \eta(e) \setminus (\eta(e, v) \cup \eta(e, u))$. We say that η is *faithful* if for every $e \in E(H)$, there is an e-rung in η.

A set $A \subseteq V(G)$ is an *atom* of η if one of the following holds:

- $A = \eta(v)$ for some $v \in V(H)$.
- $A = \eta(D)$ for some $D \in T(H)$.
- $A = \eta(e) \setminus (\eta(e, u) \cup \eta(e, v))$ for some edge e of H with ends u, v.

We say that an atom is a *vertex atom*, a *triangle atom*, or an *edge atom* depending on which of the previous conditions holds. For an atom A of η, the *boundary* $\delta(A)$ of A is defined as follows:

- If $v \in V(H)$ and $A = \eta(v)$, then $\delta(A) = \bigcup_{e \in E(H)\,:\, e \text{ is incident with } v} \eta(e, v)$.
- If $A = \eta(D)$, and $D \in T(H)$ with $D = v_1v_2v_3$, then $\delta(A) = \bigcup_{i \neq j \in \{1,2,3\}} \eta(v_iv_j, v_i) \cap \eta(v_iv_j, v_j)$
- If $A = \eta(e) \setminus (\eta(e, u) \cup \eta(e, v))$ for some edge e of H with ends u, v, then $\delta(A) = \eta(e, u) \cup \eta(e, v)$.

A set $Z \subseteq V(G)$ is *constricted* if it is stable and for every $T \subseteq G$ such that T is a tree, $|Z \cap V(T)| \leq 2$.

The main result of [12] is the following.

Theorem 2.1. *Let G be a connected graph and let $Z \subseteq V(G)$ with $|Z| \geq 2$. Then Z is constricted if and only if for some graph H, (G, Z) admits a faithful extended strip decomposition with pattern H.*

3. Dominated Balanced Separators in $\mathcal{M}_t$

We need the following results from [7]:

Lemma 3.1. *There exists an integer c with the following property. Let $t \geq 2$ be an integer. Let G be an $\mathcal{L}_t$-free graph, and let w be a weight function on G. Let D be a component of G with $w(D) > \frac{1}{2}$. Let $Z \subseteq D$, and let η be a faithful extended strip decomposition of (D, Z) with pattern H. Assume that $w(A) \leq \frac{1}{2}$ for every atom A of η. Then there exists $Y \subseteq V(G)$ with $|Y| \leq ct^9 \log^c t$, such that $N[Y]$ is a w-balanced separator in G.*

Lemma 3.2. *Let G, H be graphs, $Z \subseteq V(G)$ with $|Z| \geq 2$, and let η be a faithful extended strip decomposition of (G, Z) with pattern H. Let A be an atom of η. Then $\delta(A)$ has a core of size at most 3.*

We also need the following, which is an immediate corollary of Lemma 6.8 of [10]:

Lemma 3.3. *Let G, H be graphs, $Z \subseteq V(G)$ with $|Z| \geq 3$, and let η be an extended strip decomposition of (G, Z) with pattern H. Let Q_1, Q_2, Q_3 be paths in G, pairwise anticomplete to*

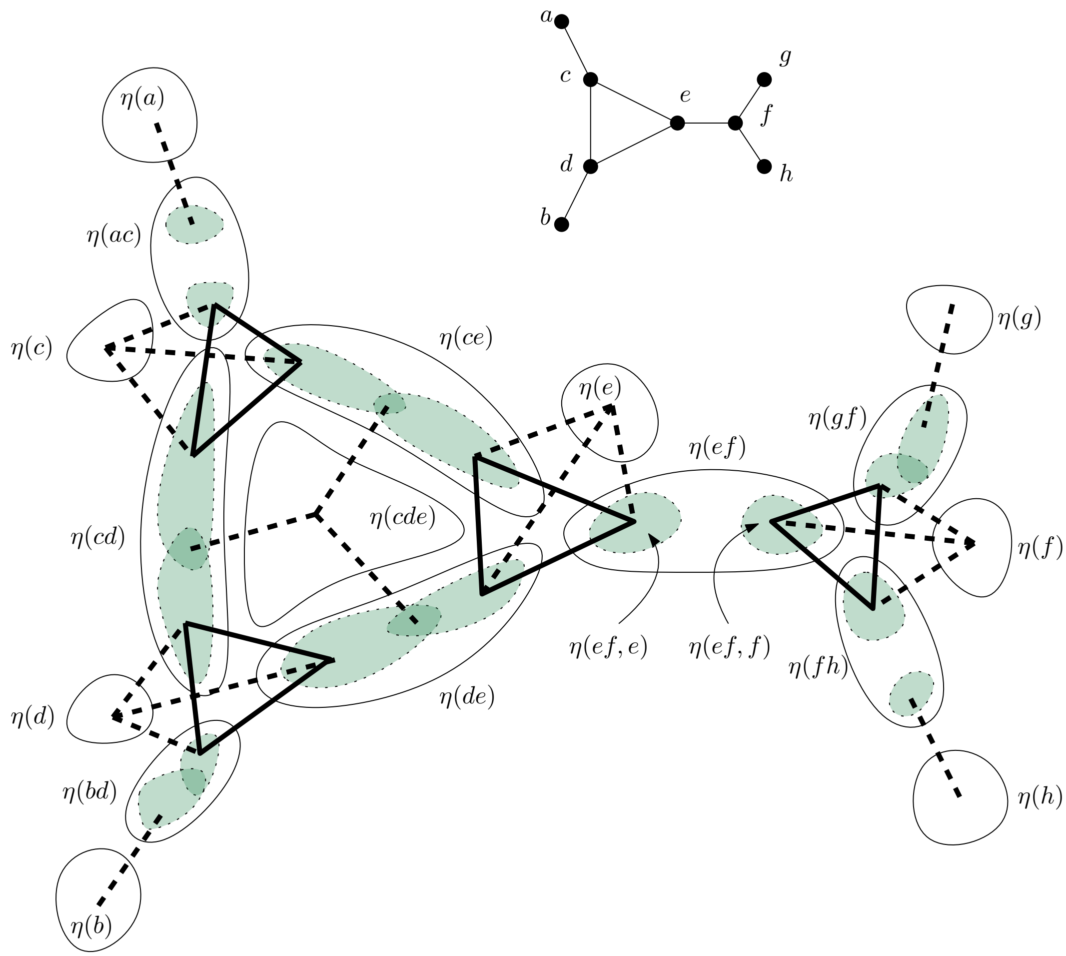

FIGURE 1. Example of an extended strip decomposition with its pattern (here dash lines represent potential edges). This figure was created by Paweł Rzążewski and we use it with his permission.

each other, and each with an end in Z. Then for every atom A of η, at least one of the sets $N[A] \cap Q_1$, $N[A] \cap Q_2$ and $N[A] \cap Q_3$ is empty.

Finally, we need the following result from [1].

Lemma 3.4. *Let x_1, x_2, x_3 be three distinct vertices of a graph G. Assume that H is a connected induced subgraph of $G \setminus \{x_1, x_2, x_3\}$ such that $V(H)$ contains at least one neighbor of each of x_1, x_2, x_3, and that $V(H)$ is minimal subject to inclusion. Then, one of the following holds:*

- *(i) For some distinct $i, j, k \in \{1, 2, 3\}$, there exists P that is either a path from x_i to x_j or a hole containing the edge $x_i x_j$ such that*
  - *$V(H) = V(P) \setminus \{x_i, x_j\}$; and*
  - *either x_k has two non-adjacent neighbors in H or x_k has exactly two neighbors in H and its neighbors in H are adjacent.*
- *(ii) There exists a vertex $a \in V(H)$ and three paths P_1, P_2, P_3, where P_i is from a to x_i, such that*
  - *$V(H) = (V(P_1) \cup V(P_2) \cup V(P_3)) \setminus \{x_1, x_2, x_3\}$;*
  - *the sets $V(P_1) \setminus \{a\}$, $V(P_2) \setminus \{a\}$ and $V(P_3) \setminus \{a\}$ are pairwise disjoint; and*

- *for distinct $i, j \in \{1, 2, 3\}$, there are no edges between $V(P_i) \setminus \{a\}$ and $V(P_j) \setminus \{a\}$, except possibly $x_i x_j$.*

(iii) There exists a triangle $a_1a_2a_3$ in H and three paths P_1, P_2, P_3, where P_i is from a_i to x_i, such that
- *$V(H) = (V(P_1) \cup V(P_2) \cup V(P_3)) \setminus \{x_1, x_2, x_3\}$;*
- *the sets $V(P_1)$, $V(P_2)$ and $V(P_3)$ are pairwise disjoint; and*
- *for distinct $i, j \in \{1, 2, 3\}$, there are no edges between $V(P_i)$ and $V(P_j)$, except $a_i a_j$ and possibly $x_i x_j$.*

We are now ready to prove Theorem 1.6.

Proof. We may assume that $t \geq 2$. Let $G \in \mathcal{M}_t$ and let w be a weight function on G. By working with the normalized function of w, we may assume that w is normal. Let c be as in Lemma 3.1. Let $d = ct^9 \log^c t + 100$. We will show that there is a set $Y \subseteq G$ with $|Y| < d$ such that $N[Y]$ is a $(w, \frac{1}{2})$-balanced separator in G. Suppose no such Y exists.

By the proof of Lemma 5.3 of [10], there is a path P in G such that $N[P]$ is a w-balanced separator in G. Let $P = p_1\text{-} \dots \text{-}p_k$, and assume that P was chosen with k minimum. It follows that there exists a component B of $G \setminus N[P \setminus \{p_k\}]$ such that $w(B) > \frac{1}{2}$. Let $N = N(B)$. Then $N \subseteq N(P \setminus \{p_k\})$.

(1) *There is no $Y \subseteq G$ with $|Y| < d$ such that $N \cup N[p_k] \subseteq N[Y]$.*

Suppose such Y exists. We will show that $N[Y]$ is a w-balanced separator in G. We may assume that there is a component D of $G \setminus N[Y]$ with $w(D) > \frac{1}{2}$. Since $w(B) > \frac{1}{2}$, we deduce that $D \cap B \neq \emptyset$. Since $N \subseteq N[Y]$, it follows that $D \subseteq B$, and so $D \cap N[P] \subseteq N[p_k]$. Since $N[p_k] \subseteq N[Y]$, we deduce that D is contained in a component of $G \setminus N[P]$, and therefore $w(D) < \frac{1}{2}$, a contradiction. This proves that $N[Y]$ is a w-balanced separator in G, contrary to our assumption, and (1) follows.

For every vertex $n \in N$ let $l(n)$ be the minimum $i \in \{1, \dots, k-1\}$ such that n is adjacent to p_i and let $r(n)$ be the maximum $i \in \{1, \dots, k-1\}$ such that n is adjacent to p_i. Let $I(n) = l(n)\text{-}P\text{-}r(n)$. Let H be the graph with vertex set N and such that $n_1n_2 \in E(H)$ if and only if $I(n_1) \cap I(n_2) \neq \emptyset$. Let N_0 be a stable set of size $\alpha(H)$ in H. Write $N_0 = \{k_1, \dots, k_m\}$ where $r(k_i) < l(k_{i+1})$ for every $i \in \{1, \dots, m-1\}$. Observe that H is an interval graph, and therefore the complement of H is perfect [4]. It follows that there exists a partition $K_1, \dots, K_m$ of $V(H)$ such that for every i, K_i is a clique of H and $k_i \in K_i$. Since K_i is a clique of H, it follows from the Helly property of the line that there exists $j(i) \in \{1, \dots m\}$ such that $p_{j(i)} \in I(n)$ for every $n \in K_i$.

(2) *Let $N' \subseteq N$ be such that the sets $I(n_1) \cap I(n_2) \neq \emptyset$ for all $n_1, n_2 \in N'$. Then there exists $X' \subseteq P$ with $|X'| \leq 3$ and $n' \in N'$ such that $N' \subseteq N[X' \cup \{n'\}]$.*

It follows from the Helly property of the line that there exists $p_j \in \bigcap_{n \in N'} I(n)$. Let $X' = \{p_s : s \in \{j-1, j, j+1\} \cap \{1, \dots, k-1\}\}$. If $N' \in N[X']$, then (2) holds, setting n' to be an arbitrary element of N'. Thus, we may assume that there exists $n' \in N'$ such that n' is anticomplete to X'. It is now enough to show that every $n \in N' \setminus (N[X'] \cup \{n'\})$ is adjacent to n'. Suppose not, and let $n \in N' \setminus N[X' \cup \{n'\}]$. Since $p_j \in I(n) \cap I(n')$ it follows that both n' and n have neighbors in $p_1\text{-}P\text{-}p_{j-2}$ and in $p_{j+2}\text{-}P\text{-}p_{k-1}$. It follows that there is a path P_1 from n' to n with interior in $p_1\text{-}P\text{-}p_{j-2}$ and a path P_2 from n' to n with interior in $p_{j+2}\text{-}P\text{-}p_{k-1}$. Since $n', n \in N$, there is a path from n' to n with interior in B. But now $P_1 \cup P_2 \cup P_3$ is a theta with ends n, n', a contradiction. (See Fig. 2) This proves (2).

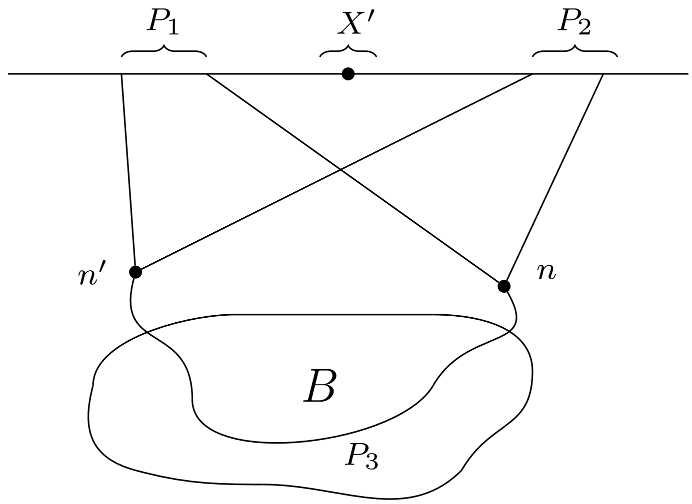

FIGURE 2. Visualisation for (2)

In the remainder of the proof, we will consider, for each $i \in \{1, \dots, m\}$, and "interval" of N_0 starting at i, but we will need to choose its end in a particular way. We will explain this next. Let $i \in \{2, \dots, m\}$ and let $J(i) = p_{l(k_i)-1}$-P-$p_{r(k_i)+1}$. Let $P_L(i)$ be the component of $P \setminus J(i)^*$ containing p_1, and let $P_R(i)$ be the component of $P \setminus J(i)^*$ containing p_k.

The following is immediate from (2):

(3) *There exists $k_i' \in K_i$ such that every $k \in K_i$ has a neighbor in $J(i) \cup \{k_i'\}$.*

(4) *There exists $Y_i' \subseteq P \cup N$ with $|Y_i'| \leq 18$ with $p_{l(k_i)-1}, p_{l(k_i)}, p_{l(k_i)+1}, p_{r(k_i)-1}, p_{r(k_i)}, p_{r(k_i)+1} \in Y_i'$ such that $N(J(i)) \cap N \subseteq N[Y_i']$.*

Let $Z_1 = \{p_{l(k_i)-1}, p_{l(k_i)}, p_{l(k_i)+1}\}$ and let $Z_2 = \{p_{r(k_i)-1}, p_{r(k_i)}, p_{r(k_i)+1}\}$. Let N_1 be the set of vertices $n \in N$ with $I(n) \subseteq I(k_i)$, N_2 the set of vertices in $n \in N$ such that $p_{l(k_i)-1} \in I(n)$ and N_3 the set of vertices in $n \in N$ such that $p_{r(k_i)+1} \in I(n)$. Then $N(J(i)) \cap N = N_1 \cup N_2 \cup N_3$. By the maximality of N_0 it follows that the sets $I(n)$ pairwise meet for all $n \in N_1$. Now (2) implies that for every $i \in \{1, 2, 3\}$ there exist $n_i' \in N_i$ and $X_i' \subseteq P$ with $|X_i'| \leq 3$ such that $N_i \subseteq N(X_i' \cup n_i')$. Let

$$Y_i' = X_1' \cup X_2' \cup X_3' \cup \{n_1', n_2', n_3'\} \cup Z_1 \cup Z_2.$$

Now $N(J_i) \cap N \subseteq N[Y_i']$. This proves (4).

Let Y_i' be as in (4).

(5) *There exists $X_i' \subseteq P \cap N(k_i)$ with $|X_i'| \leq 4$ with the following proprety. For every path $Q = q_1$-$\dots$-q_s in $G \setminus N[Y_i']$ where q_1 has a neighbor in $J(i)$ and q_s has a neighbor in B, we have that Q meets $N[X_i' \cup \{k_i\}]$.*

We may assume that $Q \setminus q_s$ is anticomplete to B, and that $Q \setminus q_1$ is anticomplete to $J(i)$. Since $q_1 \notin N[Y_i']$, we deduce that $s > 1$. If $|N(k_i) \cap V(P)| \leq 4$, let $X_i' = N(k_i) \cap P$. If $|N(k_i) \cap V(P)| > 4$, let J be the set of consisitng of the two minimum values of j, and the two maximum values of j such that $p_j \in N(k_i) \cap J(i)$, and let $X'_i = \{p_j : j \in J\}$. Assume that $Q \cap N[X'_i \cup \{k_i\}] = \emptyset$. It follows that k_i is anticomplete to Q. Let R be a path from k_i to q_s with interior in B.

Suppose first that q_1 has a neighbor v in $J(i) \setminus N(k_i)$. Since $q_1 \notin N[Y'_i]$, it follows that there exists a subpath $P' = p_j\text{-}P\text{-}p_l$ of $J(i)$ with $j < l$ such that p_j, p_l are adjacent to k_i, k_i is anticomplete to $\{p_{j+1}, \dots, p_{l-1}\}$ and $v \in \{p_{j+1}, \dots, p_{l-1}\}$ (see Fig. 3).

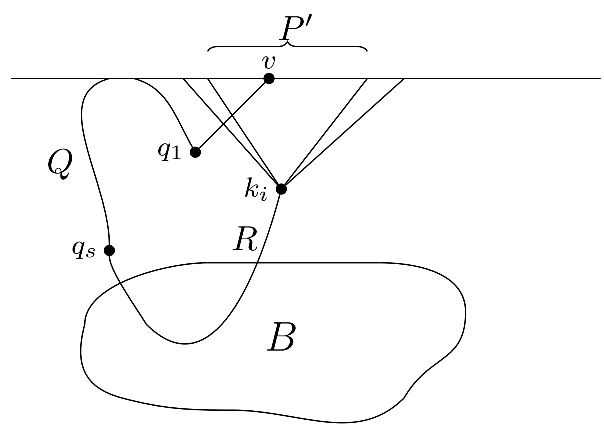

FIGURE 3. Visualisation for (5)

If q_1 has two non-adjacent neighbors in P', then $P' \cup Q \cup R$ contains a theta with ends q_1, k_i; if v is the unique neighbor of q_1 is P', then $P' \cup Q \cup R$ is a theta with ends v, k_i; and if q_1 has exactly two neighbors u, v in P' and u is adjacent to v, then $P' \cup Q \cup R$ is a pyramid with apex k_i and base uvq_1. This proves that $N(q_1) \cap J(i) \subseteq N(k_i) \cap J(i)$.

Since q_s has a neighbor in B and q_{s-1} is anticomplete to B, it follows that $q_s \in N$, and therefore q_s has a neighbor in P. Since $s > 1$, q_s is anticomplete to $J(i)$ and has a neighbor in $P \setminus J(i)$. Assume that q_s has a neighbor in $P_R(i)$ (the argument for the other case is analogous). Let $l > r(k_i)$ be minimum such that p_l has a neighbor in Q. Let $j < r(k_i)$ be maximum such that q_1 is adjacent to p_j. Then $j \in N(k_i) \cap J(i)$. Let $S = k_i\text{-}p_{r(k_i)}\text{-}P\text{-}p_l$. Since $p_j \notin X'_i$, it follows that p_j is anticomplete to $S \setminus k_i$.Now if p_l has two non-adjacent neighbors in $Q \cup R$, then $Q \cup R \cup S$ contains a theta with ends p_l, k_i; if p_l has a unique neighbor v in $Q \cup R$, then $Q \cup R \cup S$ is a theta with ends v, k_i; and if p_l has exactly two neighbors $u, v \in Q \cup R$ and u is adjacent to v, then $Q \cup R \cup S$ is a pyramid with apex k_i and base $p_l uv$, in all cases a contradiction. This proves (5).

Let X'_i be as in (5) and let $Y_i = Y'_i \cup X'_i \cup \{k_i, k'_i\}$. Then $k_i, k'_i, p_{l(k_i)-1}, p_{l(k_i)}, p_{r(k_i)}, p_{r(k_i)+1} \in Y_i$. Let $Z_i = (N[Y_i] \cap N(P)) \cup J(i)^*$. Let $U_i = (V(G) \setminus Z_i) \cup \{k_i, k_1, p_{l(k_i)-1}, p_{r(k_i)+1}\}$. Then $B \subseteq U_i$. Let D_i be the component of U_i containing B. Then $k_1, p_k \in D_i$, and therefore $P \setminus J(i)^* \subseteq D_i$. Also, $k_i \in D_i$. Write $(z_1, z_2, z_3) = (p_{l(k_i)-1}, k_i, p_{r(k_i)+1})$. Let $G'_i = G[D_i]$.

From now on, we assume that $i > 1$.

(6) *One of the following holds.*

(i) There is a component D'_i of $G'_i \setminus N[z_2]$ with $z_1, z_3 \in D'_i$, and $n_i \in N(z_2) \cap N(D'_i)$ such that the set $\{z_1, z_2, z_3\}$ is constricted in the graph $G_i = G[D'_i \cup \{n_i, z_2\}]$, or

(ii) There is a path T_i from z_1 to z_3 in G'_i such that either z_2 has a unique neighbor in T_i, or z_i has at least two non-adjacent neighbors in T_i , or

(iii) no component of $G'_i \setminus N[z_2]$ contains both z_1 and z_3 .

We may assume that there is a component D'_i of $G'_i \setminus N[z_2]$ such that $z_1, z_3 \in D'_i$. Then there is a neighbor n_i of z_2 such that n_i has a neighbor in D'_i. Let $G_i = G[D'_i \cup \{n_i, z_2\}]$. We may also assume that no path T_i as in outcome (ii) exists.

Suppose that that $\{z_1, z_2, z_3\}$ is not constricted in G_i. Then there is a tree T containing z_1, z_2, z_3. Since z_1, z_2, z_3 have degree one in G_i, it follows that T is a subdivision of $K_{1,3}$ and z_1, z_2, z_3 are the leaves of T. Let t be the unique vertex of T of degree three, and for $j \in \{1, 2, 3\}$ let P_j be the path of T from t to z_j. Since $P_1 \cup P_3$ does not satisfy outcome (ii), it follows that t is non-adjacent to z_2. By (5), $D_i \setminus \{z_1, z_2, z_3\}$ is anticomplete to $J(i)^*$ in G, and in particular $T \setminus \{z_1, z_2, z_3\}$ is anticomplete to $J(i)^*$ in G. Now, if $l(z_2) = r(z_2)$, we get that $T \cup J(i)^*$ is a theta with ends $t, p_{l(z_2)}$; if $r(z_2) = l(z_2) + 1$, we get that $T \cup J(i)^*$ is a pyramid with base $z_2 p_{l(z_2)} p_{r(z_2)}$ and apex t; and if $r(z_2) \geq l(z_2) + 2$, we get that $T \cup \{p_{l(z_2)}, p_{r(z_2)}\}$ is theta with ends t, z_2; in all cases, a contradiction. This proves (6).

Our next goal is to define a special connected set A_i.

(7) *Suppose that outcome (ii) or outcome (iii) of (6) holds. Then there exists $\Delta_i \subseteq G \setminus N[P]$ with $|\Delta_i| \leq 2$ and a component A_i of $G'_i \setminus N[\{k_i\} \cup \Delta_i]$ such that*

(1) $w(A_i) > \frac{1}{2}$, and

(2) at least one of the sets $P_L(i) \cap N_G[A_i]$ and $P_R(i) \cap N_G[A_i]$ is empty.

Suppose first that outcome (iii) of (6) holds, and so no component of $G'_i \setminus N[z_2]$ contains both z_1 and z_3. Let $\Delta_i = \emptyset$. Since $(N[Y_i] \cap N(P)) \cup N[\{z_2\}]$ is not a balanced separator in G, there is a component A_i of $G'_i \setminus N[z_2]$ with $w(A_i) > \frac{1}{2}$. Then $|A_i \cap \{z_1, z_3\}| \leq 1$. Since $(N[Y_i] \cap N(P)) \cup N[\{z_2\}]$ is disjoint from $P_L(i) \cup P_R(i)$, it follows that one of the sets $N[A_i] \cap P_L(i)$ and $N[A_i] \cap P_R(i)$ is empty, as required.

Thus, we may assume that outcome (ii) of (6) holds. Let $T_i = t_1 \text{-} \dots \text{-} t_m$ be a path as in outcome (ii) of (6) where $t_1 = z_1$ and $t_m = z_3$. Let q be minimum and r be maximum such that z_2 is adjacent to t_q, t_r. Then $r \neq q+1$. Let $\Delta_i = \{t_q, t_r\}$. Since $k_i \in Y_i$, it follows that $\Delta_i \subseteq G \setminus N[P]$. Since $(N[Y_i] \cap N(P)) \cup N[\{k_i, t_q, t_r\}]$ is not a w-balanced separator in G, it follows that there is a component A_i of $G'_i \setminus N[\{k_i, t_q, t_r\}]$ with $w(A_i) > \frac{1}{2}$.

Suppose $P_L(i) \cap N[A_i] \neq \emptyset$ and $P_R(i) \cap N[A_i] \neq \emptyset$. Since $k_i \in Y_i$, it follows that $t_q, t_r \notin N(P)$, and consequently $N[\{k_i, t_q, t_r\}] \cap (P_L(i) \cup P_R(i)) = \emptyset$. We deduce that $N_G(A_i) \cap (P_L(i) \cup P_R(i)) = \emptyset$; therefore $A_i \cap P_L(i) \neq \emptyset$ and $A_i \cap P_R(i) \neq \emptyset$, and so $P_L(i) \cup P_R(i) \subseteq A_i$. Consequently, there is a path R from z_1 to z_3 with $R^* \subseteq A_i$. Recall that $R^* \cap N[\{k_i, t_q, t_r\}] = \emptyset$, and $R^* \cap N[Y_i] \cap N(P) = \emptyset$. Let R' be a minimal subpath of R from a vertex r with a neighbor in $t_1 \text{-} T_i \text{-} t_{q-1}$ to a vertex r' with a neighbor in $t_{r+1} \text{-} T_i \text{-} t_m$. Since z_1 has a unique neighbor in G'_i, it follows that r is non-adjacent to

z_1, and similarly r' is non-adjacent to z_3. Since $J(i) \cup T_i \cup \{k_i\}$ is not a theta or a pyramid, we deduce that either $p_r(k_i) > p_l(k_i) + 1$, or $r > q + 1$. Now exactly of the following holds.

(i) There is a path S from k_i to r' with $S^* \subseteq t_r\text{-}T_i\text{-}t_m$ and such that S is anticomplete to $t_1\text{-}T_i\text{-}t_q$.
(ii) $r = q$, and t_{q+1} is the unique neighbor of r' in $t_{q+1}\text{-}T_i\text{-}t_m$.

Also, switching the roles of z_1 and z_3, exactly one of the following holds:

(i) There is a path S' from k_i to r with $S^* \subseteq t_1\text{-}T_i\text{-}t_q$ and such that S' is anticomplete to $t_r\text{-}T_i\text{-}t_m$.
(ii) $r = q$, and t_{q-1} is the unique neighbor of r in $t_1\text{-}T_i\text{-}t_{q-1}$.

We claim that at least one of the following holds:

(i) There is a path S from k_i to r' with $S^* \subseteq t_r\text{-}T_i\text{-}t_m$ and such that S is anticomplete to $t_1\text{-}T_i\text{-}t_q$.
(ii) There is a path S' from k_i to r with $S^* \subseteq t_1\text{-}T_i\text{-}t_q$ and such that S' is anticomplete to $t_r\text{-}T_i\text{-}t_m$.

Suppose not. Then $q = r$, t_{q-1} is the unique neighbor of r in $t_1\text{-}T_i\text{-}t_{q-1}$, and t_{q+1} is the unique neighbor of r' in $t_{q+1}\text{-}T_i\text{-}t_m$. From the minimality of R' we have that $R' \setminus \{r, r'\}$ is anticomplete to T_i. By (5), $G'_i \setminus \{z_1, z_2, z_3\}$ is anticomplete to $J(i)^*$. But now $J(i) \cup T_i \cup R'$ is a theta with ends t_{q-1}, t_{q+1}, a contradiction. This proves the claim.

Exchanging the roles of z_1, z_3 if necessary, we may assume that

- There is a path S from k_i to r' with $S^* \subseteq t_r\text{-}T_i\text{-}t_m$ and such that S is anticomplete to $t_1\text{-}T_i\text{-}t_q$.

Let $1 \leq x \leq y \leq q - 1$ such that x is minimum and y is maximum with t_x, t_y adjacent to r. Recall that r is non-adjacent to $p_l(k_i)$, and r is non-adjacent to t_q. Now if $x = y$, then $t_1\text{-}T_i\text{-}t_q \cup S \cup R' \cup \{p_l(k_i)\}$ is a theta with ends t_x, k_i; if $y = x+1$, then $t_1\text{-}T_i\text{-}t_q \cup S \cup R' \cup \{p_l(k_i)\}$ is a pyramid with apex k_i and base rt_xt_y; and if $y > x + 1$, then $t_1\text{-}T_i\text{-}t_x \cup t_y\text{-}T_i\text{-}t_q \cup S \cup R' \cup \{p_l(k_i)\}$ is a theta with ends r, k_i; in all cases a contradiction. This proves (7).

We have defined a set A_i in the case when outcome (ii) or outcome (iii) of (6) holds. Now we define A_i in the remaining case. Thus, assume that outcome (i) of (6) holds and $\{z_1, z_2, z_3\}$ is constricted in G_i. By Theorem 2.1, there is a graph H_i such that $(G_i, \{p_{l(k_i)-1}, k_i, p_{l(k_i)+1}\})$ admits a faithful extended strip decomposition η_i with pattern H_i.

(8) *There exists an atom A of η_i, and a component A_i of A such that $w(A_i) > \frac{1}{2}$.*

Suppose not. Then by Lemma 3.1 applied to G_i, we deduce that there exists $W_i \subseteq G_i$ with $|W_i| \leq d - 100$, such that $N[W_i]$ is a w-balanced separator in G_i. It follows from the definition of G_i that $N[Y_i \cup W_i \cup \{p_k, k_1, p_{l(k_i)-1}, p_{r(k_i)+1}, n_i\}]$ is a w-balanced separator in G, which is a contradiction since $|Y_i| < 80$. This proves (8).

If outcome (i) of (6) holds, let A_i be as in (8). This completes the definition of A_i.

(9) *If $P_L(i) \cap N_G[A_i] \neq \emptyset$, then $|P_R(i) \cap N_G[A_i]| \leq 2$.*

If outcome (ii) or outcome (iii) of (6) holds, this follows immediately from (7). Thus we may assume that outcome (i) of (6) holds and $\{z_1, z_2, z_3\}$ is constricted in G_i. Suppose that both

$P_L(i) \cap N_G[A_i] \neq \emptyset$ and $|P_R(i) \cap N_G[A_i]| > 2$. It follows from the definition of G_i that $P_L(i) \cap N_{G_i}[A_i] \neq \emptyset$ and $|P_R(i) \cap N_{G_i}[A_i]| > 2$.

Let $Q_1 = z_1$-$P_L(i)$. Since $N[P]$ is a w-balanced separator, we have that $N(p_k) \cap B \neq \emptyset$; let Q be a path from z_2 to p_k with $Q^* \subseteq B$. Then $F = z_3$-$P_R(i)$-p_k-Q-z_2 is a path. Let Q_3 be the minimal subpath of F from z_3 to a vertex of $N_{G_i}[A_i]$, and let Q_2 be the minimal subpath of F from z_2 to a vertex of $N_{G_i}[A_i]$. Since $|P_R(i) \cap N_{G_i}[A_i]| > 2$, it follows that Q_2 is anticomplete to Q_3. But now Q_1, Q_2, Q_3 are pairwise disjoint and anticomplete to each other; z_i is an end of Q_i, and $Q_i \cap N_{G_i}[A_i] \neq \emptyset$ for every $i \in \{1, 2, 3\}$, contrary to Lemma 3.3. This proves (9).

If outcome (i) of (6) holds let δ_i be the boundary of A_i in η_i, let Δ'_i be a core for δ_i with $|\Delta'_i| \leq 3$ (such Δ'_i exists by Lemma 3.2), and let $\Delta_i = \Delta'_i \cup \{n_i\}$, where n_i is as in (6)(i). If outcome (ii) or outcome (iii) of (6) holds, let Δ_i be as in (7) and let $\delta_i = N[\Delta_i]$. In both cases let $\gamma_i = N_G[A_i] \cap P$.

(10) *Let $Z \subseteq V(G)$ with $Y_i \cup \{p_k, k_i, p_{l(k_i)-1}, p_{r(k_i)+1}\} \subseteq Z$ and such that $\delta_i \subseteq N[Z]$. Let $D \subseteq G \setminus N[Z]$ be connected with $w(D) > \frac{1}{2}$. Then $D \subseteq A_i$, $N[D] \cap J(i) = \emptyset$, and there exists $v \in D \cap N(B)$.*

Since $w(B) > \frac{1}{2}$, it follows that $B \cap D \neq \emptyset$. Similarly, since $w(A_i) > \frac{1}{2}$, $A_i \cap D \neq \emptyset$. Since $Y_i \subseteq Z$, it follows from (4) and (5) that $J(i) \cap N[D] = \emptyset$. Since $Y_i \cup \{p_k, k_i, p_{l(k_1)-1}, p_{r(k_i)+1}\} \subseteq Z$ and since $\delta_i \subseteq N[Z]$, it follows that $N_{G \setminus J(i)}(A_i) \subseteq N[Z]$. We deduce that $D \subseteq A_i$. Since $p_k \in Z$, and since $N[P]$ is a balanced separator in G, it follows that $D \setminus B \neq \emptyset$. Since D is connected, there exists $v \in D \setminus B$ with a neighbor in B. This proves (10).

(11) $|P_R(2) \cap N_G[A_2]| > 2$.

Suppose that $P_R(2) \cap N_G[A(2)] \leq 2$. Let

$$Z = Y_1 \cup Y_2 \cup \{p_k\} \cup \Delta_i \cup (\gamma_2 \cap P_R(2)).$$

We claim that $N[Z]$ is a balanced separator in G. Suppose not, and let D be a component of $G \setminus N[Z]$ with $w(D) > \frac{1}{2}$. By (10) $D \subseteq A_2$, $N[D] \cap J(2) = \emptyset$ and there exists $v \in D \setminus B$ with a neighbor in B. Then $v \in N \cap A_2$. Let $v' \in P$ be a neighbor of v, choosing $v' \in P_R(2)$ if possible. Since $N[D] \cap J(2) = \emptyset$ and since $v \in A_2$, it follows that $v' \in N[A_2] \cap (P \setminus J(2))$. Since $Y_1 \subseteq Z$, it follows from (4) that v is anticomplete to $J(1)$. Since $k'_1 \in Y_1$ and $k'_2 \in Y_2$, (3) implies that $v \notin K_1 \cup K_2$. Since $v \in N$, it follows from the choice of v' that $v' \in \gamma_2 \cap P_R(2)$. But now $v' \in Z$ and so $v \in N[Z]$, contrary to the fact that $v \in D$. This proves that $N[Z]$ is a balanced separator in G, contrary to the fact that $|Z| < d$, and (11) follows.

(12) $P_L(m) \cap N_G[A_m] \neq \emptyset$.

Suppose that $P_L(m) \cap N_G[A_m] = \emptyset$. Let

$$Z = Y_m \cup \{p_k\} \cup \Delta_i.$$

We claim that $N[Z]$ is a balanced separator in G. Suppose not, and let D be a component of $G \setminus N[Z]$ with $w(D) > \frac{1}{2}$. By (10) $D \subseteq A_m$, $N[D] \cap J(m) = \emptyset$, and there exists $v \in D \setminus B$ with a neighbor in B. Then $v \in N \cap A_m$; let $v' \in P$ be a neighbor of v, choosing $v' \in P_L(m)$ if possible. Since $N[D] \cap J(m) = \emptyset$ and since $v \in A_m$, we deduce that $v' \in N[A_m] \cap (P \setminus J(m))$. Since $k'_m \in Y_m$, it follows from (3) that $v \notin K_m$. By the choice of v', we deduce that $v' \in P_L(m)$, a contradiction. This proves that $N[Z]$ is a balanced separator in G, contrary to the fact that $|Z| < d$, and (12) follows.

By (11) $|P_R(2) \cap N_G[A(2)]| > 2$, and therefore by (9) $P_L(2) \cap N_G[A(2)] = \emptyset$. In view of this, let i be maximum such that $P_L(i) \cap N_G[A_i] = \emptyset$. By (12), $i < m$. By the maximality of i, $P_L(i+1) \cap N_G[A_{i+1}] \neq \emptyset$, and therefore by (9) $|P_R(i+1) \cap N_G[A_{i+1}]| \leq 2$. Let

$$Z = Y_1 \cup Y_i \cup Y_{i+1} \cup \Delta_i \cup \Delta_{i+1} \cup (\gamma_{i+1} \cap P_R(i+1)) \cup \{p_k\}.$$

Then $|Z| < d$. To complete the proof, we obtain a contradiction by showing that $N[Z]$ is a w-balanced separator in G.

Suppose not, and let D be a component of $G \setminus N[Z]$ with $w(D) > \frac{1}{2}$. By (10), $D \subseteq A_i \cap A_{i+1}$, $N[D] \cap (J(i) \cup J(i+1)) = \emptyset$, and there exists $v \in D \setminus B$ with a neighbor in B. Then $v \in N \cap A_i \cap A_{i+1}$; let $v' \in P$ be a neighbor of v. Then $v' \in N_G[A_i] \cap N_G[A_{i+1}] \cap (P \setminus (J(i) \cup J(i+1)))$, and therefore $v' \in P_L(i+1) \cup P_R(i+1)$. Since $P_L(i) \cap N[A_i] = \emptyset$ and $P_R(i+1) \cap N[A_{i+1}] \subseteq Z$, it follows that $v' \in Z$. But now $v \in N[Z]$, contrary to the fact that $v \in D$.

■

4. Constricted sets in $K_{t,t}$-free graphs

Although we only use the results of this section on graphs in $\mathcal{H}_t$, they hold for a much larger class, namely $K_{t,t}$-free graphs. We, therefore, prove them in their full generality. Let G and H be graphs, $Z \subseteq V(G)$ with $|Z| \geq 2$ and let η be an extended strip decomposition of (G, Z) with pattern H. We say that a path $P = p_1\text{-}\dots\text{-}p_k$ in G is a *leaf path starting at p_1 and ending at p_k* if $p_1 \in Z$.

The main result of this section is the following:

Theorem 4.1. *For every $t \in \mathbb{N}$ there exists $c = c(t) \in \mathbb{N}$ with the following property. Let G be a $K_{t,t}$-free graph, $Z \subseteq V(G)$ with $|Z| \geq 2$ and H be a graph. Let η be an extended strip decomposition of (G, Z) with pattern H. Then for every vertex $v \in V(G) \setminus Z$, there exists $X \subseteq V(G) \setminus v$ such that X intersects all the leaf paths ending at v and $\alpha(X) \leq c(t)$.*

In the remainder of the proof, we will need the following consequence of Theorem 4.1:

Theorem 4.2. *For every $t \in \mathbb{N}$ there exists $c = c(t) \in \mathbb{N}$ with the following property. Let G be a $K_{t,t}$-free graph, let $\omega = \omega(G)$, and let $Z \subseteq V(G)$ be constricted. Then for every vertex $v \in V(G) \setminus Z$, there exist at most ω^c pairwise vertex-disjoint (except at v) paths starting in Z and ending at v.*

We first prove Theorem 4.2 assuming Theorem 4.1. For positive integers a, b let $R(a, b)$ be the smallest integer R, often called the Ramsey number, such that every graph on R vertices contains either a stable set of size a or a clique of size b.

Theorem 4.3 (Ramsey [28]). *For all $c, s \in \mathbb{N}$, $R(c, s) \leq c^{s-1}$*

Lemma 4.4. *For every $t \in \mathbb{N}$ there exists $c = c(t) \in \mathbb{N}$ with the following property. Let G be a $K_{t,t}$-free graph, $\omega = \omega(G)$, $Z \subseteq V(G)$ with $|Z| \geq 2$. Let H be a graph, and let η be an extended strip decomposition of (G, Z) with pattern H. Then, for every vertex $v \in V(G) \setminus Z$, there are at most ω^c pairwise vertex-disjoint (except at v) leaf paths ending at v.*

Proof. Let $c(t)$ be the constant given by Theorem 4.1. By Lemma 4.1, there exists a set X that intersects all the leaf paths ending in v and $\alpha(X) \leq c(t)$. Since G is $K_{\omega+1}$-free, we have that $|X| \leq R(c(t), \omega+1) \leq \omega^{2c(t)}$, and the result follows. ■

Proof of Theorem 4.2. By Theorem 2.1, there exists an extended strip decomposition of (G, Z). Now, the theorem follows from Lemma 4.4. ■

We now turn to the proof of Theorem 4.1. For the rest of this section, fix an integer $t > 1$ and let G be a $K_{t,t}$-free graph, $Z \subseteq V(G)$ with $|Z| \geq 2$, and H be a graph. Let η be an extended strip decomposition of (G, Z) with pattern H. Let $x, y \in V(H)$, we say that y is a *special neighbor* of x if $xy \in E(H)$ and $\alpha(\eta(xy, x)) \geq t$. In that case, we call xy a *special edge* of x. We say that a vertex $v \in V(G)$ is *safe* if either

- there is no edge xy of H such that $v \in \eta(xy, x)$, or
- there exists an edge xy of H such that y is a special neighbor of x, and $v \in \eta(xy, x)$.

Let $Safe(G)$ denote the set of all safe vertices of G. We observe:

Lemma 4.5. *Let G be a $K_{t,t}$-free graph, and let $X_1, \dots, X_k \subseteq V(G)$ be disjoint and pairwise complete, then either*

- $\alpha\left(\bigcup_{i=1}^{n} X_i\right) < t$*, or*
- *there exists i^* such that $\alpha(X_{i^*}) \geq t$ and $\alpha\left(\bigcup_{i \neq i^*} X_i\right) < t$*

Consequently, every vertex of H has at most one special neighbor.

Proof. Notice that any stable set in $\bigcup_{i \neq i} X_i$ is a subset of one of the X_i. Therefore, the first case holds if there is no i^* such that $\alpha(X_i) \geq t$. So we can assume that such an i^* exists. Since G is $K_{t,t}$-free, i^* is unique. Moreover, since X_{i^*} is complete to $\bigcup_{i \neq i^*} X_i$, we have that $\alpha\left(\bigcup_{i \neq i^*} X_i\right) < t$. ■

For a vertex $x \in V(H)$, let $Em(x) = \bigcup \eta(xy, x)$ where the union is taken over all neighbors y of x that are not special. We call $Em(x)$ *the set emulating* x in G. Observe that $Em(x) \cap Safe(G) = \emptyset$. It follows immediately from Lemma 4.5 that $\alpha(Em(x)) < t$ for every $x \in V(H)$.

We are now ready to prove Theorem 4.1.

Proof. Let $c = 4t + 6$ and suppose that $v \in V(G)$ violates the conclusion of Theorem 4.1.

Let A be an atom of η. If $A = \eta(x)$ is a vertex atom, we say that A *points to an edge e of H* if $e = xy$ and y is a special neighbor of x. If $A = \eta(x_1x_2x_3)$ is a triangle atom, we say that A *points to an edge e of H* if $e = x_ix_j$ for some $i, j \in \{1, 2, 3\}$ and x_i is a special neighbor of x_j, and x_j is a special neighbor of x_i.

(13) *Every vertex atom and every triangle atom points to at most one edge.*

Suppose first that $A = \eta(x)$ is a vertex atom. Then A only points to edges incident with x. Let $y \in V(H)$ be such that A points to the edge xy. Then y is a special neighbor of x, and therefore xy is unique by Lemma 4.5.

Thus, we may assume that $A = \eta(x_1x_2x_3)$ is a triangle atom. We may assume that A points to the edge x_1x_2. Then x_2 is a special neighbor of x_1, and x_1 is a special neighbor of x_2. By Lemma 4.5 x_3 is not a special neighbor of x_1, and x_3 is not a special neighbor of x_2. Consequently, A does not point at x_1x_3, and A does not point at x_2x_3. This proves (13).

Let $A = \eta(x)$ be a vertex atom of η. If A does not point to any edge, let $\overline{A} = A$ and let $\Delta(A) =$

$Em(x)$. If A points to an edge xy (which is unique by (13)), let $\overline{A}$ be the union of $\eta(xy) \cap Safe(G)$ with all the vertex and triangle atoms that point to xy, and let $\Delta(A) = Em(x) \cup Em(y)$.

(14) *If A is a vertex atom, then $\alpha(\Delta(A)) < 2t$, and $\Delta(A)$ separates $\overline{A}$ from $V(G) \setminus (\overline{A} \cup \Delta(A))$.*

Since $\Delta(A)$ is contained in the union of at most two sets emulating a vertex of H, it follows that $\alpha(\Delta(A)) < 2t$. The second statement of (14) follows from the definition of a strip structure. This proves (14).

(15) *There is no $x \in V(H)$ such that $v \in \eta(x)$.*

Suppose $v \in A = \eta(x)$ for some $x \in V(H)$. Let $X = \Delta(A) \cup (\overline{A} \cap Z)$. Since there is at most one edge e of H such that $\overline{A}$ meets the set $\eta(e)$, it follows that $|\overline{A} \cap Z| \leq 2$, and so $\alpha(X) < 2t + 2$. Since by (14) X meets every leaf path ending at v that does not start in $\overline{A}$, (15) follows.

Let $A = \eta(x_1x_2x_3)$ be a triangle atom of η. If A does not point to any edge, let $\overline{A} = A$ and let $\Delta(A) = Em(x_1) \cup Em(x_2) \cup Em(x_3)$. If A points to an edge x_ix_j (which is unique by (13)), let $\overline{A}$ be the union of $\eta(x_ix_j) \cap Safe(G)$ with all the vertex and triangle atoms that point to x_ix_j, and let $\Delta(A) = Em(x_i) \cup Em(x_j)$.

(16) *If A is a triangle atom, then $\alpha(\Delta(A)) < 3t$, and $\Delta(A)$ separates $\overline{A}$ from $V(G) \setminus (\overline{A} \cup \Delta(A))$.*

Since $\Delta(A)$ is contained in the union of at most three sets emulating a vertex of H, it follows that $\alpha(\Delta(A)) < 3t$. The second statement of (16) follows from the definition of a strip structure. This proves (16).

(17) *There is no triangle $x_1x_2x_3$ of H such that $v \in \eta(x_1x_2x_3)$.*

Suppose $v \in A = \eta(x_1x_2x_3)$ for some triangle $x_1x_2x_3$ of H. Let $X = \Delta(A) \cup (\overline{A} \cap Z)$. Since there is at most one edge e of H such that $\overline{A}$ meets the set $\eta(e)$, it follows that $|\overline{A} \cap Z| \leq 2$, and so $\alpha(X) < 2t + 2$. Since by (16) X meets every leaf path ending at v that does not start in $\overline{A}$, (17) follows.

(18) *$v \notin Safe(G)$.*

Suppose that $v \in Safe(G)$. By (15) and (17), $v \in \eta(xy) \cap Safe(G)$ for some edge xy of H. Let $X = Em(x) \cup Em(y) \cup (\eta(xy) \cap Z)$. Then $v \notin X$. Since $|\eta(xy) \cap Z| \leq 2$, we deduce $\alpha(X) < 2t + 2$. It follows from the defintion of a strip structure that $Em(x) \cup Em(y)$ meets every leaf path ending at v that does not start in $\eta(xy) \cap Safe(G)$, and (18) follows.

By (18), there exists an edge xy of H such that y is not a special neighbor of x and $v \in \eta(xy, x)$. Moreover, if $v \in \eta(xy, y)$, then x is not a special neighbor of y. Let x' be the special neighbor of x (if one exists), and let y' be the special neighbor of y (if one exists). Note that $x' \neq y$, but possibly $y' = x$. Suppose first that either

- $x = y'$, or
- $x \neq y'$ and $v \in \eta(xy, y)$.

Let $\mathcal{A}$ be the union of the following sets

- $\{v\}$,

- $\eta(xy) \cap Safe(G)$,
- $\eta(xx') \cap Safe(G)$ (if x' is defined),
- $\eta(yy') \cap Safe(G)$ (if y' is defined),
- all vertex and triangle atoms that point to xx' (if x' is defined),
- all vertex and triangle atoms that point to yy' (if y' is defined),
- $\eta(x)$,
- all triangle atoms $\eta(xyw)$ with $w \in V(H) \setminus \{x, y\}$.

Let Δ be the union of the following sets

- $Em(x) \setminus \{v\}$,
- $Em(y) \setminus \{v\}$,
- $Em(x')$ (if x' is defined),
- $Em(y')$ (if y' is defined).

Now assume that $x \neq y'$ and $v \notin \eta(xy, y)$. Let $\mathcal{A}$ be the union of the following sets

- $\{v\}$,
- $\eta(xy) \cap Safe(G)$,
- $\eta(xx') \cap Safe(G)$ (if x' is defined),
- all vertex and triangle atoms that point to xx' (if x' is defined),
- $\eta(x)$,
- all triangle atoms $\eta(xyw)$ with $w \in V(H) \setminus \{x, y\}$.

Let Δ be the union of the following sets

- $Em(x) \setminus \{v\}$,
- $Em(y) \setminus \{v\}$,
- $Em(x')$ (if x' is defined).

Since Δ is contained in the union of at most four sets emulating a vertex of H, it follows that $\alpha(\Delta) < 4t$. It follows from the definition of a strip structure that Δ separates $\mathcal{A}$ from $V(G) \setminus (\mathcal{A} \cup \Delta)$, and therefore Δ meets every leaf path ending at v that does not start in $\mathcal{A}$. Let $X = \Delta \cup (\mathcal{A} \cap Z)$. Since there are at most three edges e of H such that $\overline{A}$ meets the set $\eta(e)$, it follows that $|\overline{A} \cap Z| \leq 6$. We deduce that $\alpha(X) < 4t + 6$, and X meets every leaf path that ends at v, a contradiction. ∎

5. Cooperative sets

In this section, we introduce the notion of *cooperative sets*, which will be central to the rest of the proof. Cooperative sets are used to prove Theorem 1.4: instead of trying to separate two vertices directly, we will construct a "cooperative set" starting with one of them and use its properties to separate it from the other vertex. In this section, we develop the necessary properties of cooperative sets, culminating with Theorem 5.9. The main tools are Corollary 5.2 (Theorem 4.4 from [6]) and Theorem 5.8 (The Matroid Intersection Theorem [18]). The use of cooperative sets is explained in Section 6.

We start with two results from the literature.

Lemma 5.1 (Lemma 2 from [24])**.** *For all positive integers a and b, there is a positive integer $C = C(a, b)$ such that if a graph G contains a collection of C pairwise disjoint subsets of vertices, each of size at most a and with at least one edge between every two of them, then G contains a $K_{b,b}$-subgraph*

Corollary 5.2 (immediate corollary of Theorem 4.4 from [6])**.** *There exists a polynomial q, such that the average degree of every theta-free graph G is at most $q(\omega(G))$.*

Let G be a graph. A *matching* $M \subseteq E(G)$ is a set of disjoint edges. We denote by $V(M)$ the set of all endpoints of the edges of M. We say that a vertex x is *matched* by M (or that M *matches* x) if there exists an edge in M such that v is one of its endpoints. Let $X, Y \subseteq V(G)$. A matching *from X to Y* is a matching each of whose edges has one end in X and the other end in Y. We say that X *matches into* Y if there exists a matching from X to Y and that matches every vertex in X. A matching is said to be induced if $E(G[V(M)]) = M$. We denote $V(M)$ the set of all vertices matched by the matching M.

In the following, by a polynomial in two variables x and y, we mean a finite sum of terms of the form $a_{ij}x^iy^j$, where a_{ij} are real non-zero coefficients, and i, j are non-negative integers. We show:

Lemma 5.3. *There exists a polynomial $p(x, y)$ for which the following holds. Let $k \in \mathbb{N}$, G be a theta-free graph, and $M = \{(x_i, y_i)\}_{i=1}^{p(\omega(G),k)}$ a matching (not necessarily induced) in G then there exist $M' \subseteq M$ such that $|M'| \geq k$ and M' is an induced matching.*

Proof. Let $C(a, b)$ be defined as in Lemma 5.1, let q be the polynomial from Corollary 5.2 and let $p(x, k) = k^{C(2,3)}(q(x) + 1)^2$. Let

$$X = \left\{x_i | i \in 1, \dots, k^{C(2,3)}(q(x) + 1)^2\right\}$$

and

$$Y = \left\{y_i | i \in 1, \dots, k^{C(2,3)}(q(x) + 1)^2\right\}.$$

By Corollary 5.2, G is $q(\omega(G))$ degenerate and therefore $q(\omega(G)) + 1$ colorable. It follows that we can find a subset $I \subseteq \{1, \dots, p(\omega(G))\}$ such that such that $|I| = k^{C(2,3)}$ and $X' = \{x_i\}_{i \in I}$ and $Y' = \{y_i\}_{i \in I}$ are stable sets. Let H be the graph with the vertex set $\{(x_i, y_i) | i \in I\}$ and where (a, b) is adjacent to (b, d) if $\{a, b\}$ and $\{c, d\}$ are not anticomplete. Since, by Theorem 4.3, $|I| \geq R(k, C(2, 3))$, H contains either a clique of size $C(2, 3)$ or a stable set of size k. If H contains a stable set of size k, we are done since this corresponds to an induced matching of size k. Therefore, we can assume that H contains a clique of size $C(2, 3)$. This implies that we have a set of $C(2, 3)$ sets of two vertices (the edges) with an edge between every two of them. Therefore, by Lemma 5.1, $G[Y' \cup X']$ contains $K_{3,3}$ as a (non necessarily induced) subgraph. Finally, since both X' and Y' are independent sets, this $K_{3,3}$ is actually induced, which is a contradiction as G is theta-free. ■

Let G be a graph, and $X \subseteq V(G)$ be connected. We define the *boundary* of X in G, which we denote by $\delta^G(X) = \delta_1^G(X)$, to be the set of vertices in X having at least a neighbor in $G \setminus X$. Let $\delta_i^G(X) = \delta^G(X \setminus \bigcup_{k<i} \delta_k^G(X))$.

We say that X is *cooperative* in G if the following three conditions hold

- every vertex in $\delta_1^G(X)$ has a neighbor in $X \setminus \delta_1^G(X)$
- every vertex in $\delta_2^G(X)$ has a neighbor in $X \setminus (\delta_1^G(X) \cup \delta_2^G(X))$
- $X \setminus (\delta_1^G(X) \cup \delta_2^G(X))$ is connected

See Fig. 4.

When the graph G is clear from the context, we may omit it from the notation.

Lemma 5.4. *Let G be a graph and X be a cooperative set in G. Let $D_1, \dots, D_k$ be the connected components of $G \setminus N[X]$. Then for all $1 \leq i \leq k$, $X \cup N_G(D_i)$ is a cooperative set in $X \cup N[D_i]$.*

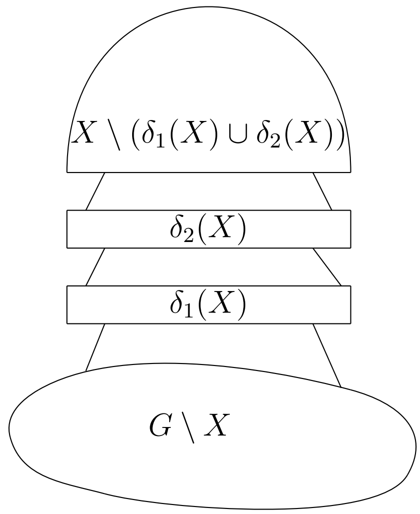

FIGURE 4. Visualization of a cooperative set

Proof. Let us fix an i arbitrarily and let $G' = X \cup N_G[D_i]$ and $X' = X \cup N_G(D_i) = N_{G'}[X]$. We have that $\delta_1^{G'}(X') = N_G(D_i) = N_{G'}(D_i)$ and $\delta_2^{G'}(X') = N_{G'}(N_{G'}(D_i)) \cap X'$. See Fig. 5.

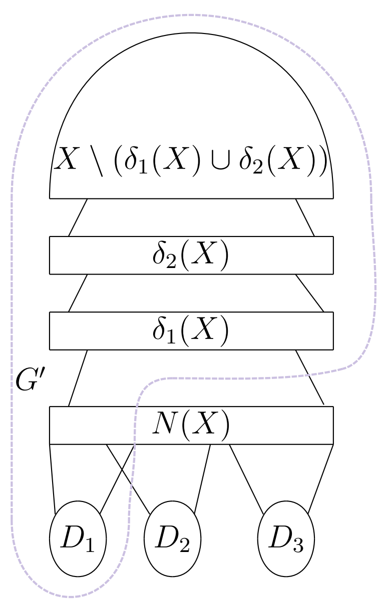

FIGURE 5. Visualization for Lemma 5.4

Let us now verify that the three conditions to be a cooperative set hold.

(19) *The first condition holds*

We have that

$$N_{G'}(X' \setminus \delta_1^{G'}(X')) = N_{G'}(X) = N_{G'}(D_i) = \delta_1^{G'}(X').$$

This proves (19).

(20) *The second condition holds*

Moreover, we have that

$$\begin{aligned} N_{G'}(X' \setminus (\delta_1^{G'}(X') \cup \delta_2^{G'}(X'))) &= N_{G'}\left((X \cup N_{G'}(D_i)) \setminus (N_{G'}(D_i) \cup (N_{G'}(N_{G'}(D_i)) \cap X'))\right) \\ &= N_{G'}(X \setminus N_{G'}(N_{G'}(D_i))) \\ &= \delta_2^{G'}(X') \end{aligned}$$

This proves (20).

(21) *The third condition holds*

Notice that $\delta_2^{G'}(X') \subseteq \delta_1^G(X)$. Therefore, $X' \setminus (\delta_1^{G'}(X') \cup \delta_2^{G'}(X')) = (X \setminus \delta_1^G(X)) \cup Y$ for some $Y \subseteq \delta_1^G(X)$. Since X is cooperative in G, $X \setminus \delta_1^G(X)$ is connected and every vertex in $\delta_1^G(X)$ has a neighbor in $\delta_2^G(X)$ and thus, every vertex in Y has a neighbor in $X \setminus \delta_1^G(X)$. This proves (21). ■

Lemma 5.5. *Let G be a graph, X be a cooperative set in G, and $C \subseteq (G \setminus X) \cup \delta_1^G(X)$. Let D be a connected component of $G \setminus (X \cup C)$ which is not anticomplete to $X \setminus C$. Then $X' = \big((X \setminus \delta_1^G(X)) \cup N(D)\big) \cap (X \setminus C)$ is cooperative in $G' = (N[D] \setminus C) \cup X'$.*

Proof. We have that $\delta_1^{G'}(X') = N_{G'}(D)$. Since $X' \setminus \delta_1^{G'}(X') = X \setminus \delta_1^G(X)$ the first condition of being cooperative holds. We also have that $\delta_2^{G'}(X') \subseteq \delta_2^G(X)$ and that $X' \setminus (\delta_1^{G'}(X') \cup \delta_2^{G'}(X')) \supseteq X \setminus (\delta_1^G(X) \cup \delta_2^G(X))$ so the second condition holds. Finally, we have that $X' \setminus (\delta_1^{G'}(X') \cup \delta_2^{G'}(X')) = (X \setminus (\delta_1^G(X)) \cup \delta_2^G(X)) \cup Y$ for some $Y \subseteq \delta_2^G(X)$. Since $X \setminus (\delta_1^G(X) \cup \delta_2^G(X))$ is connected and every vertex in Y has a neighbor in this set, $X' \setminus (\delta_1^{G'}(X') \cup \delta_2^{G'}(X'))$ is connected, and the third condition of being cooperative holds. ■

In the remainder of this section, we will assume familiarity with basic matroid theory (see [27] for an introduction to the subject). Let $X \subseteq V(G)$ be a cooperative set of G and let b be a vertex anticomplete to X. Let $\mathcal{M}^1_{X,b}$ on G be the pair $(N(X), \mathcal{I}_1)$ where $\mathcal{I}_1$ is the set of subsets of $N(X)$ that are matched into $\delta(X)$.

Let $\mathcal{M}^2_{X,b}$ on G be the pair $(N(X), \mathcal{I}_2)$ where $\mathcal{I}_2$ is the set of subsets Y of $N(X)$ for which they are $|Y|$ vertex-disjoint paths (except at b) from Y to b that are internally disjoint from $N[X]$.

Lemma 5.6. *Both $\mathcal{M}^1_{X,b}$ and $\mathcal{M}^2_{X,b}$ on G are matroids.*

Proof. For $\mathcal{M}^1_{X,b}$, consider the bipartite graph G_1 that is built from $G[\delta(X) \cup N(X)]$ by removing the internal edges in both $\delta(X)$ and $N(X)$. Then $\mathcal{M}^1_{X,b}$ is the transversal matroid on G_1 (see Theorem 1.61 of [27]). For $\mathcal{M}^2_{X,b}$, consider the digraph G_2 with vertex set $V(G) \setminus X$ and where $(x, y) \in E(G_2)$ if $\{x, y\} \in E(G)$ and $y \notin N(X)$. Let G_2' be the digraph made from G_2 by adding $|V(G) \setminus N(X)|$ copies of b along with its incident edges. Then, $\mathcal{M}^2_{X,b}$ is the gammoid on G_2' made with the pairwise vertex disjoint paths from $N_G(X)$ to copies of b in G_2' (see [25]). ■

Lemma 5.7. *Let G be a (theta, pyramid)-free graph, X be a cooperative set in G, and $b \in V(G)$ be anticomplete to X. There exists a polynomial p for which the size of the largest common independent set of $\mathcal{M}^1_{X,b}$ and $\mathcal{M}^2_{X,b}$ on G is strictly smaller than $p(\omega(G))$.*

Proof. Since G is theta-free, it follows that G is $K_{3,3}$-free. Let $c = c(3)$ given by Theorem 4.2 and let $p(\omega(G)) = q(\omega(G), 2q(\omega(G), \omega(G)^c + 1))$ where q is given by Lemma 5.3. Suppose the statement is false for this choice of p for some G, X, and b. Let I be an independent set of both $\mathcal{M}^1_{X,b}$ and $\mathcal{M}^2_{X,b}$ of size $p(\omega(G))$. By Lemma 5.3, there exists $I' \subseteq I$ such that $|I'| = 2q(\omega(G), \omega^c + 1)$ and there is an induced matching M_1 that matches I' into $\delta_1(X)$. Let Y be the vertices of $\delta_1(X)$ matched by M_1.

(22) *No vertex in $\delta_2(X)$ is adjacent to more than 2 vertices of Y*

Suppose not and let $x_1, x_2, x_3 \in Y$ and v in $\delta_2(X)$ such that $\{x_1, x_2, x_3\} \subseteq N(v)$. Let m_1, m_2, m_3 be the vertices in $N(X)$ matched to respectively x_1, x_2, x_3 by M_1. Let us now consider $G' = G \setminus N[X] \cup \{x_1, x_2, x_3, m_1, m_2, m_3\}$. Since the set $\{m_1, m_2, m_3\}$ is independent in $\mathcal{M}^2_{X,b}$, x_1, x_2, x_3 are in the same connected component of G'. Let H be a minimal connected subgraph of G' such that $\{x_1, x_2, x_3\} \subseteq N(H)$. We apply Lemma 3.4 to analyze the structure of H. Since x_1, x_2, x_3 each have a unique neighbor in H and $\{m_1, m_2, m_3\}$ is a stable set, the first outcome of Lemma 3.4 cannot happen. If the second outcome happens, then $\{v\} \cup \bigcup_{i \leq 3} P_i$ forms a theta, which is a contradiction. Therefore, the last outcome of Lemma 3.4 happens, but then $\{v\} \cup \bigcup_{i \leq 3} P_i$ forms a pyramid, which is also a contradiction. This proves (22).

Since every vertex of $\delta_1(X)$ has a neighbor in $\delta_2(X)$ and by (22), we can find a (non-induced) matching of size $q(\omega(G), \omega(G)^c + 1)$ that matches Y into $\delta_2(X)$. Therefore, by Lemma 5.3, there exists $Y' \subseteq Y$ with $|Y'| = \omega(G)^c + 1$ and an induced matching M_2 that matches Y' into to $\delta_2(X)$. Using the edges incident with Y' in both M_1 and M_2, we get a set of $\omega(G)^c + 1$ pairwise anticomplete paths $P^1, \dots, P^{\omega(G)^c+1}$ of length 3 from $N(X)$ to $\delta_2(X)$. Let $Z \subseteq I'$ be the set of the ends of these paths in $N(X)$.

(23) *Z is a constricted set in $(G \setminus N[X]) \cup Z$.*

Suppose not; let $x_1, x_2, x_3 \in Z$ and let T be an induced subgraph of G such that T is a tree and $x_1, x_2, x_3 \in T$. We may assume that T is chosen to be minimal with these properties. Then either T is a path with ends in $\{x_1, x_2, x_3\}$, or T is a subdivision of the bipartite graph $K_{1,3}$ and x_1, x_2, x_3 are the leaves of T. We may assume that x_i is the end of the path P^i for every $i \in \{1, 2, 3\}$.

Let $G' = X \setminus (\delta_1(X) \cup \delta_2(X)) \cup P^1 \cup P^2 \cup P^3$, and let H be a minimal connected subgraph of G' such that $\{x_1, x_2, x_3\} \subseteq N(H)$. As before, we apply Lemma 3.4 to analyze the structure of H. Since x_1, x_2, x_3 each have a unique neighbor in H and since these neighbors are distinct and form a stable set, the first outcome of Lemma 3.4 cannot happen. If the second outcome happens, then $T \cup \bigcup_{i \leq 3} P_i$ forms a theta, which is a contradiction. Therefore, the last outcome of Lemma 3.4 happens, but then $T \cup \bigcup_{i \leq 3} P_i$ forms a pyramid, which is also a contradiction. This proves (23).

By Theorem 4.2, there are at most $\omega(G)^c$ vertex disjoint paths from Z to b in $G \setminus X$, which contradicts the fact that I' is an independent set of $\mathcal{M}^2_{X,b}$. ∎

We remind the reader of the celebrated Matroid Intersection Theorem [18].

Theorem 5.8 (Matroid Intersection Theorem [18])**.** *Let $M_1 = (U, \mathcal{I}_1)$ and $M_2 = (U, \mathcal{I}_2)$ be two matroids with the same ground set U. Then*

$$\max\{\, |I| \colon I \in \mathcal{I}_1 \cap \mathcal{I}_2 \,\} = \min\{\, rank_{M_1}(A) + rank_{M_2}(U \setminus A) \Big| A \subseteq U \,\}.$$

From Lemma 5.7 and Theorem 5.8, we deduce:

Theorem 5.9. *There exists a polynomial q such that the following holds. Let G be a (theta, pyramid)-free graph, X be a cooperative set in G, and $b \in V(G)$ anticomplete to X. Then there exists a partition $(A, N(X) \setminus A)$ of $N(X)$ such that:*

- *the maximum matching from A to $\delta(X_1)$ is of size at most $q(\omega(G))$ and*
- *there are a most $q(\omega(G))$ vertex-disjoint paths (except at b) from $N(X) \setminus A$ to b that are internally disjoint from $N[X]$.*

6. Cooperative pairs and degenerate partitions

In this section, we prove Theorem 6.6 and deduce from it our first main result Theorem 1.4.

Let G be a graph, and let $X \subseteq G$ be connected and $C \subseteq G$ be such that $C \cap X = \emptyset$. We call a pair (X, C) a *cooperative pair in G* if X is cooperative in the connected component of $G \setminus C$ that contains X. Let $G' \subseteq G$ and let a and b be distinct vertices of G. We say that a cooperative pair (X, C) in G' *separates* a from b in G if the following conditions hold:

- $a \in X \setminus \delta^{G'}(X)$, and
- $b \in G \setminus (X \cup C)$, and
- $\delta^{G'}(X) \cup C$ separates a from b in G.

In order to prove Theorem 6.6 we start by defining a cooperative set $X_0 = N^2_G(a)$, and create an improving sequence of pairs (X_i, C_i) and subgraphs G_i such that (X_i, C_i) is a cooperative pair in G_i that separates a from b in G. To make the notion of improvement precise, we define the "value" of a cooperative pair. To do so, we make use of a vertex partition for theta-free graphs first introduced in [3]. This allows us to bound the number of improvement steps, and as a result, bound the size of the separator. We now explain this in detail.

Following the proof of Theorem 7.1 of [3] and using Theorem 4.4 of [6] we deduce:

Theorem 6.1. *There exists a polynomial p such that the following holds: let G be theta-free with $|V(G)| = n$ and clique number ω. Then, there exists a partition $(S_1, \dots, S_k)$ of $V(G)$ with the following properties:*

(1) $k \leq p(\omega) \log n$.
(2) S_i is a stable set for every $i \in \{1, \dots, k\}$.
(3) For every $i \in \{1, \dots, k\}$ and $v \in S_i$,
we have $\deg_{G \setminus \bigcup_{j<i} S_j}(v) \leq p(\omega)$.

Let $\Pi = (S_1, \dots, S_k)$ be a partition of $V(G)$ as in Theorem 6.1 and let and $v \in V(G)$. We define the *value* of v in G denoted $val^G_\Pi(v)$ as the index i such that $v \in S_i$. For a set $X \subseteq V(G)$, we define its *value* in G, denoted $val^G_\Pi(X)$ as follows. If $X \neq \emptyset$, then $val^G_\Pi(X) = \max\left\{ val^G_\Pi(v) \Big| v \in \delta^G(X) \right\}$ and if $X = \emptyset$, then $val^G_\Pi(X) = 0$. Similarly, we define the *value* of the cooperative pair in G as $val^G_\Pi(X, C) = val^{G \setminus C}_\Pi(X)$.

Lemma 6.2. *There exists a polynomial p for which the following holds. Let G be a (theta, pyramid)-free graph, let X be a cooperative set in G, and let $b \in N(X)$. Then $|N(b) \cap X| \leq p(\omega(G))$.*

Proof. Suppose not and let $p(x) = x^3 p(x,3) > R(3, x+1)p(x,3)$ (by Theorem 4.3) where $p(x,y)$ is defined as in Lemma 5.3. Since no vertex in $\delta_2(X)$ has more than $R(3,\omega(G)+1)$ common neighbour with b (as it would create a theta), and since every vertex in $\delta_1(X)$ has a neighbour in $\delta_2(X)$, there exist a matching from $N(b) \cap X$ to $\delta_2(X)$ of size $p(\omega(G), 3)$. By Lemma 5.3, there exists an induced matching M from $N(b)$ to $\delta_2(X)$ such that $|M| = 3$. Write $M = \{x_1y_1, x_2y_2, x_3y_3\}$ where $x_1, x_2, x_3 \in \delta_1(X)$ and $y_1, y_2, y_3 \in \delta_2(X)$. Let $G' = X \setminus (\delta_1(X) \cup \delta_2(X)) \cup \{x_1, x_2, x_3, y_1, y_2, y_3\}$ and let H be a minimal connected subgraph of G' such that $\{x_1, x_2, x_3\} \subseteq N(H)$. We apply Lemma 3.4 to analyze the structure of H. Since each of x_1, x_2, x_3 has a unique neighbor in H, and since $\{y_1, y_2, y_3\}$ forms a stable set, the first outcome of Lemma 3.4 cannot happen. If the second outcome happens, then $\{b\} \cup \bigcup_{i\leq 3} P_i$ forms a theta, which is a contradiction. Therefore, the last outcome of Lemma 3.4 happens, but then $\{b\} \cup \bigcup_{i\leq 3} P_i$ forms a pyramid, which is also a contradiction. ■

We will need two classical results in graph theory:

Theorem 6.3 (Menger's Theorem (Vertex Version)[26])**.** *For any two nonadjacent sets of vertices X and Y in a finite graph G, the size of the smallest set of vertices whose removal separates X from Y is equal to the maximum number of pairwise internally vertex-disjoint paths between X and Y.*

Theorem 6.4 (Kőnig's Theorem [22])**.** *In any bipartite graph $G = (U, V, E)$, the size of a maximum matching is equal to the size of a minimum vertex cover.*

The next lemma explains how to make one improvement step in the construction of a sequence of cooperative pairs.

Lemma 6.5. *There exists a polynomial p for which the following holds. Let G be a (theta, pyramid)-free graph, and let $a, b \in V(G)$. Let Π be a partition of $V(G)$ with the properties given by Theorem 6.1. Let (X, C) be a cooperative pair separating a and b in G such that $val_\Pi(X, C) > 1$. Then, there exists either*

- *$G' \subseteq G$ and a cooperative pair (X', C') in G' separating a and b in G such that $|C'| \leq |C| + p(\omega(G))$ and $val_\Pi^{G'}(X', C') < val_\Pi^{G}(X, C)$, or*
- *$C' \subseteq V(G)$ such that C' separates a from b in G and $|C'| \leq |C| + \ p(\omega(G))$.*

Proof. Let p_1, p_2 and p_3 be the polynomials given by Lemma 6.2, Theorem 5.9 and Theorem 6.1, respectively, and let $p(x) = p_1(x) + 2p_2(x) + p_2(x)p_3(x)$. Let $C_1 = C \cup (N(b) \cap X)$. If C_1 separates a from b in G we are done, so we may assume that there is a connected component J of $G \setminus (X \cup C_1)$ such that J is not anticomplete to $X \setminus C_1$ and such that $b \in J$. Let $X_1 = (X \setminus \delta_1^{G\setminus C}(X)) \cup N_{G\setminus C}(J) \cap (X \setminus C_1)$ and let $G_1 = N[J] \setminus C_1 \cup X_1$. It follows from Lemma 5.5 that (X_1, C_1) is a cooperative pair in G_1.

Let $\mathcal{M}^1_{X,b}$ and $\mathcal{M}^2_{X,b}$ on $G_1 \setminus C_1$ be defined as before. By Theorem 5.9, there exists a partition (A, B) of $N_{G_1}(X_1)$ for which

- the maximum matching from A to $\delta(X_1)$ is of size at most $p_2(\omega(G))$ and
- there are a most $p_2(\omega(G))$ vertex-disjoint paths (except at b) from B to b that are internally disjoint from $N[X]$.

By Theorem 6.3, there exists a set $K \subseteq G_1 \setminus (X_1 \cup C_1)$ such that $|K| \leq p_2(\omega(G))$ and every path from B to b in $G_1 \setminus (X_1 \cup C_1 \cup K)$ intersects A. Let $C_2 = C_1 \cup K$. By Theorem 6.4, there exists a set $F \subseteq N_{G_1}(X_1) \cup \delta^{G_1}(X_1)$ such that $|F| \leq q_2(\omega(G))$ and $A \setminus F$ is anticomplete to $\delta^{G_1}(X_1) \setminus F$.

Let $F_1 = F \cap \delta^{G_1}(X_1)$ and $F_2 = F \cap A$. Let $C_3 = C_2 \cup F_2$. Let

$$H = \bigcup_{u \in F_1} \{v | v \in N(u) \text{ such that } val_\Pi(u) < val_\Pi(v)\}.$$

Let $C' = C_3 \cup (H \cap N(X))$. By the third property of the partition Π, we have that

$$|C'| \leq |C_3| + |F_1| p_3(\omega(G)) \leq |C| + p_1(\omega(G)) + 2p_2(\omega(G)) + p_2(\omega(G)) p_3(\omega(G)) = |C| + p(\omega(G)).$$

If C' separates a from b in G, we are done. Therefore, we may assume that the connected component D in $G \setminus (C' \cup X_1)$ containing b is not anticomplete to $\delta^{G_1}(X_1) \setminus C'$. Let

$$X'' = X_1 \setminus \delta^{G_1}(X_1) \cup (N_{G \setminus (C' \cup X_1)}(D) \cap X_1 \setminus C')$$

and $G'' = X'' \cup N_{G \setminus (C' \cup X_1)}[D] \setminus C'$. It follows from Lemma 5.5 that (X'', C') is a cooperative pair in G''.

Let Γ be the connected component of $G'' \setminus N_{G''}[X'']$ containing b and let $X' = X'' \cup N_{G''}(\Gamma)$. By Lemma 5.4, we have that (X', C') is a cooperative pair in $G' = X'' \cup N_{G''}[\Gamma]$. Moreover, we have that $\delta_1^{G'}(X') \cup C'$ separates a from b in G since $N_{G \setminus C'}(\Gamma) = \delta_1^{G'}(X')$. To conclude that (X', C') and the subgraph G' satisfy the first bullet in the statement of the theorem, it remains to show that $val^{G'}(X', C') < val^G(X, C)$. This follows since $\delta^{G'}(X') \subseteq N_G(X) \setminus H$ and so every vertex in $\delta^{G'}(X')$ has a lower value than $val^G(X, C)$. ■

We can now prove the main result of this section.

Theorem 6.6. *There exists a polynomial p for which the following holds. Let G be an n-vertex (theta, pyramid)-free graph, and a, b be non-adjacent vertices in G. Then, there exist and (a, b)-separator $C \subseteq V(G)$ such that $|C| \leq p(\omega(G)) \log n$.*

Proof. Let p_1 and p_2 be defined as in Theorem 6.1 and Lemma 6.5, respectively. Let $p(x) = x^6 + (p_1(x) + 1)p_2(x)$. Let $C_0 = N(a) \cap N(b)$, $G_0 = G \setminus C_0$, and $X_0 = N^2_{G_0}(a)$. Then (X_0, C_0) is a cooperative pair separating a from b in G. Now let us recursively define a sequence $(X_i, C_i)_{i=0}^k$ where (X_i, C_i) are the sets obtained by applying Lemma 6.5 to (X_{i-1}, C_{i-1}) and G as long as the first outcome of that lemma happens. Since the value of the cooperative pairs in this sequence is strictly decreasing, Theorem 6.1 implies that $k \leq p_1(\omega(G)) \log(n)$. Applying Lemma 6.5 one more time gives us C such that C separates a from b in G and $|C| \leq |C_0| + (k + 1)\ p_2(\omega(G))$.

(24) $|C_0| \leq R(3, \omega(G) + 1)$.

Suppose not, then $N(a) \cap N(b)$ contains an independent set of size 3, which together with a and b forms a theta. This proves (24).

Therefore, using Theorem 4.3, we have that $|C| \leq \omega(G)^6 + (k + 1)\ p_2(\omega(G)) \leq p(\omega(G))$ as required. ■

From Theorem 6.6, we deduce Theorem 1.4, which we now restate.

Theorem 1.4. *For every integer $t \geq 2$, there exists a positive integer c such that every n-vertex graph in $\mathcal{H}_t$ is $t^c \log n$-pairwise separable.*

Proof. Let $G \in \mathcal{H}_t$ and let p be the polynomial from Theorem 6.6. Then, by Theorem 6.6, G is $p(\omega(G)) \log n$-pairwise separable. Since $p(\omega(G)) \leq p(t) \leq t^c$ for a large enough c, the theorem follows. ■

7. Proofs of the main results

We have already proved Theorem 1.4; we prove Theorem 1.5 next. We make use of the following result from [11]:

Theorem 7.1 (Theorem 1.4 from [11])**.** *For every positive integer c there exists an integer $d = d(c)$ with the following property. If $\mathcal{C}$ is a hereditary graph class that is $(\omega(G)\log|V(G)|)^c$-pairwise separable, then for every $G \in \mathcal{C}$ on at least 3 vertices and for every two non-adjacent vertices $u, v \in V(G)$, there exists a set $X \subseteq V(G)$ disjoint from $\{u, v\}$, with $\alpha(X) \leq \log^d(|V(G)|)$, that separates u from v.*

Proof of Theorem 1.5. It follows from Theorem 7.1 and Theorem 6.6. ■

Next, we prove Theorem 1.1. The structure of the proof is similar to [8]. We use the following:

Lemma 7.2 (Lemma 8 of [20])**.** *Let G be a graph. If for every weight function w, there exists a $w-$balanced separator X such that $|X| \leq d$, then $\mathsf{tw}(G) \leq 2d$.*

Theorem 7.3 (Corollary of Theorem 9.2 from [8])**.** *Let $d, L \in \mathbb{N}$. Let G be $L-$pairwise separable and let w be a weight function on G. If G is d-breakable then there exists a w-balanced separator X in G such that $|X| \leq 3Ld$.*

We are now ready to prove Theorem 1.1.

Proof. Every graph G is $K_{\omega(G)+1}-$free. Let c' be such that $p(x+1) \leq x^{c'}$ where p is defined as in Theorem 6.6, and let $d = d(t)$ be defined as in Theorem 1.6. By Theorem 1.4, G is $\omega(G)^{c'}\log n$-pairwise separable. By Theorem 1.6, G is d-breakable. Therefore, by Theorem 7.3, for every weight function μ there exist a mu-balanced separator X of G such that $|X| \leq 3d\omega(G)^{c'}\log n$. By Lemma 7.2 this implies that $\mathsf{tw}(G) \leq 6d\omega(G)^{c'}\log n$. Taking c large completes the proof (in fact, $c = c' + \log(6d)$ is enough). ■

Finally, we prove Theorem 1.2. We use the following result from [11]:

Theorem 7.4 (Theorem 1.1 from [11])**.** *Let $\mathcal{C}$ be a hereditary graph class. The following are equivalent:*

(i) There exists an integer $c_1 > 0$ such that for every $G \in \mathcal{C}$ on at least 3 vertices we have $\text{tree-}\alpha(G) \leq (\log|V(G)|)^{c_1}$
(ii) There exists an integer $c_3 > 0$ such that for every $G \in \mathcal{C}$ on at least 3 vertices we have $\mathsf{tw}(G) \leq (\omega(G)\log|V(G)|)^{c_3}$

Proof of Theorem 1.2. It follows from Theorem 1.1 and Theorem 7.4. ■

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