Tree independence IX. A Trichotomy for Excluded Induced Minors.

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Abstract

An *independent set* in a graph G is a set of pairwise non-adjacent vertices. A *tree decomposition* of G is a pair (T, χ) where T is a tree and $\chi : V(T) \to 2^{V(G)}$ is a function satisfying the following two axioms: for every edge $uv \in E(G)$ there is an $x \in V(T)$ such that $\{u, v\} \subseteq \chi(x)$, and for every vertex $u \in V(G)$ the set $\{x \in V(T) \mid u \in \chi(x)\}$ induces a non-empty and connected subtree of T. The sets $\chi(x)$ for $x \in V(T)$ are called the *bags* of the tree decomposition. The *tree-independence* number of G is the minimum taken over all tree decompositions of G of the maximum size of an independent set of the graph induced by a bag of the tree decomposition. A graph H is an *induced minor* of a graph G if a graph isomorphic to H can be obtained from G by a sequence of vertex deletions and edge contractions (suppressing parallel edges).

We prove that for every positive integer t there exists an $\epsilon \in (0, 1)$ such that every graph G either contains the complete bipartite graph $K_{t,t}$ or the wall $W_{t\times t}$ as an induced minor, or G has tree-independence at most $O(2^{O((\log n)^{1-\epsilon})})$. This leads to algorithms with running time $2^{2^{O((\log n)^{1-\delta})}} = 2^{n^{o(1)}}$, for $\delta > 0$, for a wide range of problems, including Independent Set, Feedback Vertex Set and k-Coloring, on $\{K_{t,t}, W_{t\times t}\}$-induced minor free graphs. Our result is a substantial generalization of existing bounds for the tree-independence and tree-width on various graph classes, and a partial resolution of the conjecture of Chudnovsky, E S, and Lokshtanov [Arxiv, 2025] that $\{K_{t,t}, W_{t\times t}\}$-induced minor free graphs have poly-logarithmic tree independence number. The generality comes at the cost of a *sub-polynomial*, rather than *poly-logarithmic* upper bound.

Our result leads to a complete classification of induced-minor closed classes into ones that have sub-polynomial tree-independence, tree-independence equal to $\sqrt{n}$ up to poly-logarithmic factors, and linear tree-independence.

Keywords: tree-independence number; induced minors; tree decompositions; balanced separators; subexponential algorithms.

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1 Introduction

A *tree decomposition* of a graph G is a pair (T, χ) in which T is a tree and $\chi\colon V(T) \to 2^{V(G)}$ is a function subject to two constraints: every edge of G is contained in $\chi(x)$ for some $x \in V(T)$, and for every vertex $v \in V(G)$ the set of nodes x with $v \in \chi(x)$ induces a non-empty connected subtree of T. The sets $\chi(x)$ for $x \in V(T)$ are the *bags* of the decomposition. Tree decompositions are among the most powerful tools in modern graph theory. On the structural side they are the backbone of the Graph Minors project of Robertson and Seymour [63], and on the algorithmic side they underlie a vast body of efficient algorithms, from Courcelle's theorem [7, 10, 29], the isomorphism algorithm of Grohe and Marx [44] (see also [60]), and the bidimensionality theory of Demaine et al. [37] to the almost-linear time algorithm of Korhonen, Pilipczuk, and Stamoulis for minor testing [55] (see also [52]) and the linear time algorithm of Korhonen for finding the edge-k-connected components [53], to name just a few.

The standard way to measure the quality of a tree decomposition (T, χ) is by its *width*, which is defined as $\max_{x \in V(T)} |\chi(x)| - 1$. The *treewidth* $\mathrm{tw}(G)$ of G is the minimum width over all its tree decompositions. For many purposes, however, this measure is overly pessimistic: a tree decomposition whose bags are large but structurally simple can still be very useful algorithmically. This has prompted the introduction of several alternative width measures, among them *minor-matching hypertree width* [66], the number of bounded-radius balls, stars, or other prescribed objects needed to cover each bag [18, 39, 43, 61], and the *tree-independence number* [33, 66]. In this paper we focus on the tree-independence number. The *independence number* $\alpha(H)$ of a graph H is the maximum size of a set of pairwise non-adjacent vertices of H. The *tree-independence number* $\mathrm{tw}_\alpha(G)$ of G is the minimum, over all tree decompositions (T, χ) of G, of $\max_{x \in V(T)} \alpha\big(G[\chi(x)]\big)$. That is, rather than charging a bag its number of vertices, we charge it only the size of the largest independent set in the bag.

To streamline the discussion, it is convenient to speak of the treewidth and the tree-independence of an entire *class* of graphs. For a class $\mathcal{H}$ we define $\mathbf{tw}^{\mathcal{H}}\colon \mathbb{N} \to \mathbb{N}$ by letting $\mathbf{tw}^{\mathcal{H}}(n)$ be the maximum treewidth of a graph in $\mathcal{H}$ on at most n vertices, and we define the *tree-independence* $\mathbf{tw}^{\mathcal{H}}_\alpha(n)$ of $\mathcal{H}$ analogously, with tw_α in place of tw.

The appeal of the tree-independence number stems from two facts. First, since $\alpha(G[\chi(x)]) \leq |\chi(x)|$ for every bag $\chi(x)$, it never exceeds the treewidth by more than one, yet for many classes it is dramatically smaller: chordal graphs have treewidth $n-1$ but tree-independence number 1 [42]; even-hole-free graphs and 3-path-configuration-free graphs have treewidth $n-1$ but tree-independence number $\log^{O(1)} n$ [22, 23]; and unit disk graphs have treewidth $n-1$ but tree-independence number $O(\sqrt{n})$ [36, 59]. Second, while a small tree-independence number does not confer the full algorithmic power of a small treewidth, it still suffices to solve a rich family of problems efficiently. Concretely, for every fixed integer q and every fixed CMSO (counting monadic second-order) formula φ, a maximum-weight induced subgraph of treewidth at most q satisfying φ can be found efficiently on graphs of small tree-independence number [59] (see also [58]). This already captures many classical problems, among them Independent Set and Feedback Vertex Set, even though it falls short of the full generality of Courcelle's theorem on graphs of bounded treewidth. Still other well-studied problems that lie outside this framework are nonetheless known to be tractable on graphs of small tree-independence number; for example, k-Coloring for every fixed k [33, 66].

Treewidth is closed under taking minors: deleting a vertex, deleting an edge, or contracting an edge never increases it. This monotonicity motivates a systematic study of how excluding one or more graphs as a minor constrains the treewidth of a class, and here the picture is completely understood. A minor-closed class $\mathcal{H}$ falls into exactly one of three cases: *(i)* $\mathcal{H}$ contains all graphs, in which case $\mathbf{tw}^{\mathcal{H}}(n) = \Theta(n)$, or *(ii)* $\mathcal{H}$ contains the t by t wall for all t, but not all graphs, in which

case $\mathbf{tw}^{\mathcal{H}}(n) = \Theta(\sqrt{n})$, or *(iii)* $\mathcal{H}$ excludes the t by t wall for some t, in which case $\mathbf{tw}^{\mathcal{H}}(n) = \Theta(1)$. Here the *$t$ by t wall*, denoted by $W_{t\times t}$, is defined as the graph with vertex set

$$\{(x, y) \ : \ 0 \le x < 2t \text{ and } 0 \le y < t\} \setminus \{(0, 0), (x_t, t-1)\}$$

where $x_t = 0$ if t is even and $2t-1$ if t is odd, and every pair (x, y) and (x', y') of vertices are adjacent if $y = y'$ and $|x - x'| = 1$ or if $|y - y'| = 1$, $x = x'$ and $\max(y, y') \equiv x \mod 2$. (see Figure 1)

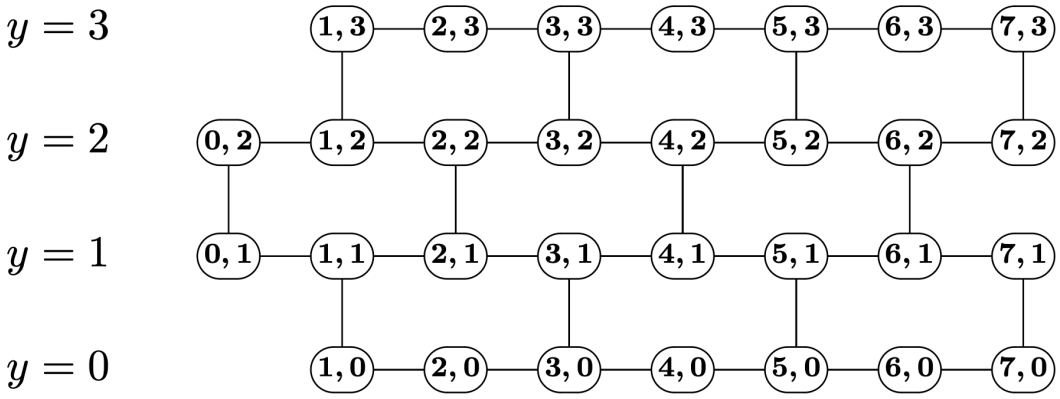


Figure 1: *The wall $W_{4\times 4}$*

The $O(\sqrt{n})$ upper bound in case *(ii)* goes back to a classical theorem of Alon, Seymour, and Thomas [6] (see also [9]), who proved that every n-vertex graph excluding K_h as a minor has a balanced separator of size $O(h^{3/2}\sqrt{n})$, and hence treewidth $O_h(\sqrt{n})$. The t by t grids have treewidth t [38], and therefore show that planar, and therefore K_5-minor free graphs, have treewidth $\Omega(\sqrt{n})$. The dividing line between case *(ii)* and *(iii)* follows from the Excluded Grid Theorem [13, 27, 62], which states that every graph that excludes $W_{t\times t}$ as a minor has treewidth at most $\tilde{O}(t^9)$. Here $\tilde{O}$ notation suppresses polylogarithmic factors.

The tree-independence number is closed under *induced minors* — deleting a vertex or contracting an edge never increases it — but, unlike treewidth, it is *not* closed under taking minors, since deleting an edge *can* increase it. This motivates studying how excluding one or more graphs as an induced minor affects the tree-independence number. Here, in stark contrast to the clean minor-closed trichotomy for treewidth, the picture is far from understood. We summarize the known results most relevant to our work.

- If $\mathcal{H}$ contains the complete bipartite graph $K_{t,t}$ for every t, then $\mathbf{tw}^{\mathcal{H}}_{\alpha}(n) = \Theta(n)$, since the tree-independence number of $K_{t,t}$ is t [33].
- If $\mathcal{H}$ contains the $t\times t$ wall $W_{t\times t}$ for every t, then $\mathbf{tw}^{\mathcal{H}}_{\alpha}(n) = \Omega(\sqrt{n})$, since the tree-independence number of $W_{t\times t}$ is at least $t/2 = \Theta(\sqrt{|V|})$. Here the $t/2$ lower bound for the tree-independence of walls follows from the fact that the treewidth of $W_{t\times t}$ is equal to t [38] and that $W_{t\times t}$ is bipartite.
- The class of $\{K_{5,5}, W_5\}$-induced-minor-free graphs has tree-independence number $\Omega(\log n)$ [26, 65], so excluding even a small complete bipartite graph and a small grid does not force a bounded tree-independence number.

- Despite considerable effort [25, 32, 34, 46, 47], no clean conjecture is known for precisely which induced-minor-closed classes have tree-independence number $O(1)$; the analogous question of which induced-subgraph-closed classes have bounded *treewidth* is known to lack a nice answer [5].

Our main theorem shows that, despite this apparent complexity, a surprisingly clean picture emerges once one is willing to tolerate sub-polynomial slack between the upper and lower bounds: up to this slack, the tree-independence number of every induced-minor-closed class is governed by just two obstructions — the complete bipartite graph $K_{t,t}$ and the wall $W_{t\times t}$ — which place the class into one of three regimes.

Theorem 1.1. *Let $\mathcal{H}$ be an induced-minor-closed class of graphs.*

1. *If there is an integer t with $K_{t,t} \notin \mathcal{H}$ and $W_{t\times t} \notin \mathcal{H}$, then there is an $\epsilon \in (0,1)$ such that $\mathbf{tw}_\alpha^{\mathcal{H}}(n) \leq 2^{O((\log n)^{1-\epsilon})}$.*
2. *If there is an integer t with $K_{t,t} \notin \mathcal{H}$, but $W_{t\times t} \in \mathcal{H}$ for every t, then $\mathbf{tw}_\alpha^{\mathcal{H}}(n) = \tilde{\Theta}(\sqrt{n})$.*
3. *If $K_{t,t} \in \mathcal{H}$ for every t, then $\mathbf{tw}_\alpha^{\mathcal{H}}(n) = \Theta(n)$.*

The three cases are exhaustive and mutually exclusive. An immediate consequence of Theorem 1.1 is that the growth rate of the tree-independence number of an induced-minor-closed class $\mathcal{H}$, $\limsup_{n\to\infty} \frac{\log \mathbf{tw}_\alpha^{\mathcal{H}}(n)}{\log n}$, is always equal to 0, $\frac{1}{2}$, or 1; no intermediate value can occur.

This trichotomy is, however, considerably coarser than its minor-closed counterpart for treewidth. There, each of the three regimes pins down $\mathbf{tw}^{\mathcal{H}}(n)$ up to a constant factor, for every class. Here the sub-polynomial regime (1) is left much wider: a class in it may have tree-independence number anywhere from $O(1)$ to $2^{(\log n)^{1-\epsilon}}$, and Theorem 1.1 does not distinguish between these. Determining the exact tree-independence number within this regime remains open.

Our main graph theoretic contribution is the tree independence upper bound underlying Theorem 1.1, point (1). Towards this, for every positive integer t, define $\mathcal{C}_t$ to be the set of all $\{K_{t,t}, W_{t\times t}\}$-induced minor free graphs. We show the following theorem.

Theorem 1.2. *For every positive integer t, there exists a positive integer $\nu'(t)$ and rational number $\epsilon(t) \in (0,1)$ such that, for every graph $G \in \mathcal{C}_t$ on n vertices,*

$$tw_\alpha(G) \quad \leq \quad 2^{\nu'(t)\log^{1-\epsilon(t)} n}.$$

Theorem 1.2 is a significant step towards a conjecture of Chudnovsky et al. [20] that for every positive integer t there exists a positive integer d such that every graph $G \in \mathcal{C}_t$ has $\mathrm{tw}_\alpha(G) \leq O((\log n)^d)$. Together with classic Ramsey bounds for graphs excluding a $K_{t,t}$ as an induced subgraph [40] our result immediately implies that for every integer t there exists an $\epsilon \in (0,1)$, $\delta > 0$ such that for every graph G in $\mathcal{C}_t$, either G contains a complete graph K_ω with $\omega \geq 2^{\delta((\log n)^{1-\epsilon})}$ as an induced subgraph or $\mathrm{tw}(G) \leq 2^{O((\log n)^{1-\epsilon})}$. This unifies and generalizes tree-independence and treewidth upper bounds shown in [1–3, 14, 15, 17, 19, 21–23, 25, 32, 33, 46, 50], at the cost of sub-polynomial rather than constant or poly-logarithmic bounds. For each of the listed papers, to see that our bound generalizes the one obtained there it is sufficient to compare the forbidden structures required for the respective bound with the forbidden structures for $\mathcal{C}_t$ from Theorem 3.3.

A few remarks are in order. First, several of the cited papers obtain asymptotically tight bounds in cases where being asymptotically tight matters. Second, our upper bounds only generalize the tree-independence or treewidth bounds proved in the listed papers, many of them also show other results that are of independent interest and are not necessarily covered (even with weaker bounds)

by Theorem 1.1. Finally, our proof of Theorem 1.1 directly or indirectly invokes the bounds shown in [15, 50], and hence, even though Theorem 1.1 is a generalization of [15, 50] it does not give an alternative or independent proof of these two results.

To complete the trichotomy we also give a new upper bound on the tree-independence number of $K_{t,t}$-induced-minor free graphs.

Theorem 1.3. *For every positive integer t there exists a positive integer d such that for every $K_{t,t}$-induced-minor free graph G, $\mathrm{tw}_\alpha(G) \leq O(\sqrt{n}(\log n)^d)$.*

Theorem 1.3 follows quite easily from a combination of known results: that every graph with large tree-independence contains an induced subgraph with large treewidth and poly-logarithmic clique number [20], that every $K_{t,t}$-induced minor free graph with poly-logarithmic clique number has poly-logarithmic average degree [11], and that every $K_{t,t}$-induced minor free graph G has a separator with $O(\sqrt{|E(G)|})$ vertices [35, 54].

Algorithmic Consequences. Lokshtanov, Pilipczuk and Rzazewski [59] (see also [58]) gave an algorithm for the following problem, called Max Weight-(ϕ, r)-Induced Subgraph. Here input is a graph G, integer r and CMSO_2-formula ϕ, and a weight function $w : V(G) \to \mathbb{N}$. The task is to find a vertex subset S such that $\mathrm{tw}(G[S]) \leq r$, $G[S]$ satisfies the property ϕ, and $w(S) = \sum_{v \in S} w(v)$ is maximized, or determines that no such set S exists. We refer to [59] and references within for a definition of CMSO_2. Here it suffices to know that Max Weight-(ϕ, r)-Induced Subgraph generalizes a host of well-known problems, including Independent Set and Feedback Vertex Set. The algorithm of Lokshtanov, Pilipczuk and Rzazewski [59] runs in time $f(r, \phi)n^{g(r,\phi)k}$ if a tree decomposition with tree-independence k is given as input. Additionally, if the tree-independence number of G is at most k, a tree-decomposition with tree-independence number at most $8k$ can be computed in time $2^{O(k^2)}n^{O(k)}$ using the algorithm of Dallard et al. [31] (see also [66]).

Dallard et al. [33] observed that a r-colorable graph with tree-independence k has treewidth at most rk. A tree decomposition of tree-width at most $2rk$ can then be computed in time $2^{O(rk)}n$, using the algorithm of Korhonen [51]. It is well-known that r-Coloring can be solved in time $O(r^{k'})n$ if a tree decomposition of tree-width k' is given as input [30]. Together with our upper bound on the tree-independence of graphs in $\mathcal{C}_t$, the algorithms above yield the following theorem.

Theorem 1.4. *For every CMSO_2 formula ϕ and integers t, r there exists an $\epsilon \in (0, 1)$ such that* Max Weight-(ϕ, r)-Induced Subgraph *and* r-Coloring *have algorithms with running time $2^{2^{O((\log n)^{1-\epsilon})}}$ on graphs in $\mathcal{C}_t$.*

The running time of the algorithms of Theorem 1.4 is strongly sub-exponential in the sense that it grows slower than 2^{n^δ} for *every* $\delta > 0$. Thus, if any of the problems covered by Theorem 1.4, including Independent Set and Feedback Vertex Set, are NP-hard on $\{K_{t,t}, W_{t\times t}\}$-induced-minor free graphs for some fixed integer t, then every problem in NP admits algorithms with running time $O(2^{n^\delta})$ for every $\delta > 0$, a complexity theoretic collapse almost as dramatic as P = NP. We remark that the existence of a *polynomial* time algorithm for Independent Set on even-hole-free graphs, a class much less general than $\mathcal{C}_t$, remains a prominent open problem [4, 12, 28, 48, 56].

Proof Overview. We give a brief outline of the proof. As the proof relies on ideas from multiple previous works, the overview is interleaved with a brief summary of the most relevant existing literature. Gartland and Lokshtanov [41] introduced the "layered sets" technique in the context of branching algorithms for the Independent Set problem. Chudnovsky et al. [23] observed that the layered sets technique can be used to prove structural bounds for the tree-independence

of certain, quite restricted, graph classes. Subsequently, Chudnovsky et al. [17] showed that the structural lemma underlying the layered set technique to bound tree-independence holds in $\mathcal{C}_t$ for all t. In other words, the layered sets technique can be used in all induced-minor closed classes where one could hope to prove sub-polynomial upper bounds on tree-independence. In a recent paper, Chudnovsky et al. [16] proved that on graphs in $\mathcal{C}_t$ that do not have large cliques (cliques of poly-logarithmic size are ok), the structural lemma underlying the layered set technique can be generalized to constant radius balls (see Proposition 4.4).

The restriction to graphs with poly-logarithmic clique number required for Proposition 4.4 to work might seem to limit its applicability for proving tree-independence bounds for graphs in $\mathcal{C}_t$ with arbitrary clique number. However, Chudnovsky et al. [20] show that in order to prove an upper bound for the tree-independence number of graphs in $\mathcal{C}_t$, it is sufficient to upper bound the tree-independence number (or treewidth) of graphs in $\mathcal{C}_t$ with poly-logarithmic clique number (see Proposition 7.3). In light of this, Chudnovsky et al. [15] gave a sub-polynomial $2^{O((\log n)^{1-\epsilon})}$ upper bounds on the tree-width of graphs in $\mathcal{C}_t$ with *constant* clique number. In an attempt to bridge the gap between constant and poly-logarithmic clique number, Chudnovsky et al. [18] show that for every integer t there exists an $\epsilon \in (0, 1]$ such that every graph in $\mathcal{C}_t$ has a tree decomposition where every bag can be covered by $2^{O((\log n)^{1-\epsilon})}$ balls of radius at most 8 (see Proposition 4.6). The proof of this result is a direct invocation of the main result of [15] coupled with a graph partitioning scheme inspired by Scott [64] and Klein et al. [49].

The main reason that the treewidth bound of Chudnovsky et al. [15] only works for graphs with *constant*, rather than poly-logarithmic, clique number is that Chudnovsky et al. [15] rely on a recursive contraction scheme pioneered by Korhonen [50] (see also [8]). The contraction scheme makes the ϵ in the $2^{O((\log n)^{1-\epsilon})}$ upper bound drop at an (at least) exponential rate as a function of the clique number, rendering the method toothless already for graphs with clique size $\log \log n$. The contraction scheme is used by Chudnovsky et al. [15] to (essentially) guarantee that the graph G under consideration has a tree decomposition in which every bag is covered by $2^{O((\log n)^{1-\delta})}$ stars (i.e. balls of radius 1). This tree decomposition is used in conjunction with the layered sets technique to give a tree decomposition of G of width at most $2^{O((\log n)^{1-\epsilon})}$, where ϵ is lower bounded in terms of t and δ.

The very short summary of the proof of this paper is that we work with graphs in $\mathcal{C}_t$ with poly-logarithmic clique number, and carry through the proof of Chudnovsky et al. [15], but avoid the recursive contraction scheme which causes the exponential decay in ϵ. The recursive contraction scheme was used to guarantee a tree decomposition in which every bag is covered by $2^{O((\log n)^{1-\delta})}$ balls of radius 1. Instead we invoke Chudnovsky et al. [18] (Proposition 4.6) to obtain a tree decomposition where every bag can be covered by $2^{O((\log n)^{1-\delta})}$ balls of radius at most 8. To turn this tree decomposition into one where every bag has $2^{O((\log n)^{1-\epsilon})}$ *vertices*, we use the layered set technique for constant radius balls (Section 4), powered by the recent result of Chudnovsky et al. [16] (Proposition 4.4).

Several technical complications occur, and in order to overcome them we prove and use two new approximate characterizations of the class $\mathcal{C}_t$. The first (see Lemma 3.5) is that for every positive integer t there exists a t' such that for every graph G in $\mathcal{C}_t$ the graph G° does not contain an induced $K_{t',t'}$. Here G° is the graph with a vertex for every connected vertex set in G, and two vertices in G° are adjacent if the corresponding sets in G have non-empty intersection or an edge between them. This characterization in terms of G° is interesting in its own right, as Dallard et al. [33] have previously observed that $tw_\alpha(G^\circ) = tw_\alpha(G)$. The second (see Lemma 3.7) is that for every t there exist integers p, q, such that for every graph G in $\mathcal{C}_t$ there does not exist a collection of p pairwise disjoint and anti-complete connected sets $S_1, \ldots, S_p$, such that for every pair S_i, S_j, there are at

least q internally disjoint and anti-complete paths from S_i to S_j.

Paper Organization. Section 2 contains notation and preliminaries. Section 3 proves the approximate characterizations of $\mathcal{C}_t$ used later: first via "complete bipartite brambles", and then via "strong slimness". Section 4 develops the bounded-radius layered-sets lemma and specializes it to balanced separators and A–B separators. Section 5 proves the existence of a combinatorial object in the class $\mathcal{C}_t$ that we need, called *strong-barriers*. Section 6 then uses strong-barriers to show that pairs of vertex subsets that are "slim" can be easily separated in $\mathcal{C}_t$. Section 7 consolidates the results from the previous sections to prove Theorem 1.2, and Section 8 contains the proof of Theorem 1.3. We conclude the paper with Section 9.

2 Preliminaries and notation

All graphs in this paper are finite and simple. If $X \subseteq V(G)$, then $G[X]$ denotes the subgraph of G induced by X, and $G \setminus X$ denotes $G[V(G) \setminus X]$. If H is an induced subgraph of G, we write $H \subseteq G$. In this paper, we use induced subgraphs and their vertex sets interchangeably. A *clique* in a graph G is a set of pairwise adjacent vertices. We use $\omega(G)$ to denote the size of the largest clique in G.

For $v \in V(G)$ and an integer $r \geq 0$, let $N^r_G[v]$ denote the set of vertices at distance at most r from v in G. For $X \subseteq V(G)$, set

$$N^r_G[X] = \bigcup_{x \in X} N^r_G[x].$$

We also write $B_r(X, G)$ for $N^r_G[X]$ and use $N_G[X]$ to denote $N^1_G[X]$. For $X, Y \subseteq V(G)$, we say that X and Y are *anticomplete* if there is no edge of G with one end in X and one end in Y. We say that X and Y *touch* if $N_G[X] \cap Y \neq \emptyset$. All logarithms are base 2. If $X, Y \subseteq V(G)$, then $\mathrm{dist}_G(X, Y)$ is the minimum length of a path in G with one end in X and one end in Y, with value ∞ if no such path exists.

For a graph class $\mathcal{H}$, we write $\mathbf{tw}^{\mathcal{H}}_\alpha(n)$ for the maximum value of $tw_\alpha(G)$ over all graphs $G \in \mathcal{H}$ with at most n vertices. For a positive integer t, we denote by $\mathcal{F}_t$ the class of $K_{t,t}$-induced-minor-free graphs. Recall that $\mathcal{C}_t$ denotes the class of graphs with no induced minor isomorphic to $K_{t,t}$ or $W_{t\times t}$. Thus $\mathcal{C}_t \subseteq \mathcal{F}_t$. Both classes are hereditary.

Let G be a graph and let $S \subseteq V(G)$. We say that $\hat{S}$ is an *r-cover* of S in G if $S \subseteq N^r_G[\hat{S}]$. The set S is *(k, r)-coverable* in G if it has an r-cover $\hat{S}$ in G with $|\hat{S}| \leq k$. We call such a set $\hat{S}$ a *(k, r)-cover* of S. We say that S is *internally (k, r)-coverable* in G if it has a (k, r)-cover contained in S.

Let $w : V(G) \to \mathbb{R}_{\geq 0}$. For $X \subseteq V(G)$, write $w(X) = \sum_{v \in X} w(v)$. We call w a *weight function* if $w(V(G)) \leq 1$. A graph G is *(w, ε)-balanced* if every connected component C of G satisfies $w(C) \leq \varepsilon$. A set $X \subseteq V(G)$ is a *(w, ε)-balanced separator* if $G \setminus X$ is (w, ε)-balanced.

As a slight abuse of notation, for a set $Y \subseteq V(G)$ we may denote by Y the weight function which gives weight $1/|Y|$ (if $Y \neq \emptyset$) to all vertices in Y and 0 elsewhere. Therefore, for $Y \subseteq V(G)$ and $\varepsilon \in (0, 1)$, a set $X \subseteq V(G)$ is a *(Y, ε)-balanced separator* if every connected component of $G \setminus X$ contains at most $\varepsilon|Y|$ vertices of Y.

We use the following standard fact (see, for example [38]):

Proposition 2.1. *Let G be a graph and w be a weight function on G. If (T, χ) is a tree decomposition of G, then some bag $\chi(x)$ is a $(w, \frac{1}{2})$-balanced separator in G.*

3 Approximate Characterizations of $\mathcal{C}_t$.

When comparing the class $\mathcal{C}_t$ against other classes, it is often more convenient to use an approximate characterization of $\mathcal{C}_t$ in terms of forbidden induced subgraphs, rather than forbidden induced minors. We begin this section by deriving such a classification.

Let s, l be positive integers. An *(s,l)-constellation* is a graph C with an independent set S_C of size s such that $C - S_C$ has exactly l components, each of which is a path that contains at least one neighbor of every vertex in S_C. Observe that an (s,l)-constellation contains a $K_{s,l}$-induced-minor. We need the following, which follows immediately from the main result in [24] by Chudnovsky, Hajebi, and Spirkl.

Proposition 3.1. *For all positive integers s, l, r, there is a positive integer $a = a(s,l,r)$ with the following property. Let G be a graph that contains a $K_{a,a}$-induced-minor. Then one of the following holds.*

- *There is an induced subgraph of G isomorphic to either $K_{r,r}$, a subdivision of $W_{r\times r}$, or the line graph of a subdivision of $W_{r\times r}$.*
- *There is an (s,l)-constellation in G.*

We also need the following theorem of Aboulker, Adler, Kim, Sintiari, and Trotignon [1], which allows us to translate wall-induced-minors into induced subgraphs.

Proposition 3.2. *For all positive integers t, there is a positive integer $b = b(t)$ with the following property. Let G be a graph that contains a $W_{b\times b}$-induced-minor. Then G contains as an induced subgraph either a subdivision of $W_{t\times t}$, or the line graph of a subdivision of $W_{t\times t}$.*

Combining Propositions 3.2 and 3.1 immediately yields the following approximate classification of $\mathcal{C}_t$.

Theorem 3.3. *For every positive integer r, there exists a positive integer t such that if a graph G contains $K_{t,t}$ or $W_{t\times t}$ as an induced minor, then G contains as an induced subgraph a subdivision of the wall $W_{r\times r}$, or the line graph of a subdivision of $W_{r\times r}$, or $K_{r,r}$ or an (r,r)-constellation.*

Proof. Given r, set $t = \max(a(r,r,r), b(r))$ where a is the function guaranteed by Proposition 3.1, while b is the function guaranteed by Proposition 3.2.

Suppose now that G contains $K_{t,t}$ or $W_{t\times t}$ as an induced minor. If G contains $K_{t,t}$ as an induced minor, Proposition 3.1 implies that G contains an induced subgraph isomorphic to $K_{r,r}$, a subdivision of $W_{r\times r}$ or the line graph of a subdivision of $W_{r\times r}$, or G contains an (r,r)-constellation. Otherwise G contains $W_{t\times t}$ as an induced minor, and Proposition 3.2 implies that G contains as an induced subgraph either a subdivision of $W_{r\times r}$, or the line graph of a subdivision of $W_{r\times r}$. □

Now we obtain two more characterizations of $\mathcal{C}_t$. Recall that two distinct vertices u, v of a graph are called *true-twins* if $N[u] = N[v]$. Let G be a graph. We define $\mathsf{twin}(G)$ to be the graph obtained from G by adding a true-twin for each vertex of the graph. That is, for each $v \in V(G)$, we add a new vertex $v^\star$, where $v^\star$ is adjacent to every $u \in N_G[v]$ and two new vertices $u^\star$ and $v^\star$ are adjacent if and only if $uv \in E(G)$. For a vertex subset S, we use $S^\star$ to denote $\{v^\star \ : \ v \in S\}$. We need the following lemma.

Lemma 3.4. *For every positive integer t, if $\mathrm{twin}(G)$ contains an induced subgraph isomorphic to $K_{t,t}$, a subdivision of $W_{t\times t}$, the line graph of a subdivision of $W_{2t\times 2t}$, or a (t,t)-constellation, then G contains $W_{t\times t}$ or $K_{t,t}$ as an induced minor.*

Proof. The case $t = 1$ is immediate since $W_{1\times 1}$ is empty, so assume $t \geq 2$. Let $G' := \text{twin}(G)$ and let H be an induced subgraph of G' as in the statement, chosen vertex minimal with respect to taking induced subgraphs. We remark that none of $K_{t,t}$, a subdivision of $W_{t\times t}$, the line graph of a subdivision of $W_{2t\times 2t}$, or a vertex minimal (t,t)-constellation contains a pair of vertices that are true-twins. Thus no two vertices of H are true-twins in G', and so H is isomorphic to an induced subgraph of G.

If H is a biclique, a subdivision of a wall, or a constellation, then it immediately gives an induced-minor model of $W_{t\times t}$ or $K_{t,t}$ in G. Otherwise, since $W_{2t\times 2t}$ contains the one-subdivision of $W_{t\times t}$ as a subgraph, and hence H contains an induced subgraph isomorphic to the line graph of a subdivision of $W_{t\times t}$ in which every edge is subdivided at least once. Here we use that, for every subgraph J' of a graph J, the line graph $L(J')$ is the induced subgraph $L(J)[E(J')]$. This again yields an induced-minor model of $W_{t\times t}$ in G, which completes the proof of the lemma. □

We remark that Theorem 3.3 is an approximate characterization of $\mathcal{C}_t$ in the sense that a graph G that contains one of the obstructions guaranteed by Theorem 3.3 with parameter $2r$ is not in $\mathcal{C}_r$ by Lemma 3.4. On the other hand, any graph not in $\mathcal{C}_t$ (where t is the integer guaranteed by the theorem for parameter $2r$) contains at least one such obstruction. Thus Theorem 3.3 shows that for any r there exists a t such that every graph G not in $\mathcal{C}_t$ contains a "nice" obstruction witnessing that G is not in $\mathcal{C}_r$.

A *complete bipartite bramble of order t* in G, denoted by $\tilde{\mathcal{K}}_{t,t}$ is a family $\{A_1, \ldots, A_t\} \cup \{B_1, \ldots, B_t\}$ of non-empty vertex subsets of G such that:

- A_i and B_i induce connected subgraphs for every $i \in [t]$;
- A_i and A_j do not touch, and B_i and B_j do not touch, for every distinct $i, j \in [t]$;
- A_i and B_j touch for every $i, j \in [t]$.

We say that a graph *has a $\tilde{\mathcal{K}}_{t,t}$* if it has a collection of vertex subsets which forms a complete bipartite bramble of order t. Now we are ready to state and prove the first main lemma of this section.

Lemma 3.5. *Corresponding to every positive integer t, there exists a positive integer $\tilde{t} = \tilde{t}(t)$ such that no graph $G \in \mathcal{C}_t$ has a $\tilde{\mathcal{K}}_{\tilde{t},\tilde{t}}$.*

Proof. Set $\tilde{t} := a(t, t, 2t)$, where a is as in Proposition 3.1. Suppose for contradiction that $G \in \mathcal{C}_t$ has a $\tilde{\mathcal{K}}_{\tilde{t},\tilde{t}}$, witnessed by $\{A_1, \ldots, A_{\tilde{t}}\} \cup \{B_1, \ldots, B_{\tilde{t}}\}$. Let $G' := \mathsf{twin}(G)$ and observe that by construction $G' - V(G)$ is isomorphic to G. So the sets $B_1^\star, \ldots, B_{\tilde{t}}^\star$ are connected and pairwise non-touching. Furthermore, since each A_i touches each B_j, we have that $N_{G'}(A_i) \cap B_j^\star \subseteq N_{G'}[A_i] \cap B_j^\star$ is non-empty for every $i, j \in [\tilde{t}]$. Consequently $\{A_1, \ldots, A_{\tilde{t}}\} \cup \{B_1^\star, \ldots, B_{\tilde{t}}^\star\}$ forms an induced minor model of a $K_{\tilde{t},\tilde{t}}$ in G'.

By Proposition 3.1, G' contains an induced subgraph H isomorphic to one of the following: $K_{t,t}$, a subdivision of $W_{2t\times 2t}$, the line graph of a subdivision of $W_{2t\times 2t}$, or a (t,t)-constellation. Now, applying Lemma 3.4, we conclude that G contains $W_{t\times t}$ or $K_{t,t}$ as an induced minor, which contradicts $G \in \mathcal{C}_t$. □

Remark. *For a positive integer t, we use $\tilde{t}^{3.5}(t)$ to denote the constant implied by Lemma 3.5 and when t is clear from context, we simply write $\tilde{t}^{3.5}$.*

For the final characterization, let p, q be positive integers and let G be a graph. Two vertex sets $A, B \subseteq V(G)$ are *q-wide* in G if G contains a set $\mathcal{P}$ of at least q A–B paths whose interiors are

pairwise vertex-disjoint and anticomplete. Otherwise they are called *q-slim* in G. We also say that the *the pair A, B is q-slim or q-wide*. A graph G is *(p, q)-slim* if for every independent set S of cardinality p, there exists distinct vertices $u, v \in S$ such that the pair $\{u\}, \{v\}$ is q-slim in G. We say that a hereditary graph class is *(p, q)-slim* if every graph in that class is. We need the following theorem by Chudnovsky, Codsi, Fischer, and Lokshtanov [15].

Proposition 3.6. *For every positive integer t, there exist positive integers $p = p(t)$ and $q = q(t)$ such that $\mathcal{C}_t$ is (p, q)-slim.*

G is said to be *strongly-(p, q)-slim* if for every family $\{S_1, S_2, \ldots, S_p\}$ of p non-empty, pairwise disjoint and anticomplete sets with the property that $G[S_i]$ is connected for each $i \in [p]$, there exists at least one distinct pair S_i, S_j that is q-slim in G. Finally, we say that a hereditary graph class is *strongly-(p, q)-slim* if every graph in that class is. We now state and prove the second main lemma of this section.

Lemma 3.7. *For every positive integer t, there exist positive integers $p = p(t)$ and $q = q(t)$ such that $\mathcal{C}_t$ is strongly-(p, q)-slim.*

Proof. For $t = 1$, the class $\mathcal{C}_1$ is empty, so we may take $p = q = 1$; hence assume $t \geq 2$. Let $a := a(t, t, 2t)$ be given by Proposition 3.1 and let $b := b(2t)$ be given by Proposition 3.2. Apply Proposition 3.6 with parameter $\rho = \max\{a, b\}$, and let p, q be the resulting integers. Let $\{S_1, \ldots, S_p\}$ be pairwise disjoint, anticomplete, connected sets in $G \in \mathcal{C}_t$. Let $G' := \mathsf{twin}(G)$ and define G'' to be the graph obtained from G' by contracting each $S_i \subseteq V(G')$ to a single vertex s_i. Since the S_i are pairwise disjoint and anticomplete in G, $\hat{S} = \{s_1, \ldots, s_p\}$ is an independent set of size p in G''. Now, suppose for contradiction that no distinct pair S_i, S_j is q-slim in G.

Claim 3.7.1. *If $s_i, s_j \in \hat{S}$ with $i \neq j$ then the pair $\{s_i\}, \{s_j\}$ is q-wide in G''.*

Proof of the Claim. Let $s_i, s_j \in \hat{S}$ be distinct vertices. Since S_i, S_j are q-wide in G, there exists a set $\mathcal{P}$ of q S_i–S_j paths in G that are internally vertex-disjoint and internally anticomplete. By shortening each path if necessary, we may assume that the internal vertices of every path in P lie outside $S_i \cup S_j$. For every $P \in \mathcal{P}$, define P' to be the subgraph of G' obtained by replacing every internal vertex $v \in V(P) \cap \bigcup_{s \in [p]} S_s$ with $v^\star$, and observe that P' is an S_i–S_j path in G'. Since the paths in $\mathcal{P}$ are internally vertex-disjoint and internally anticomplete, $\mathcal{P}' := \{P' \ : \ P \in \mathcal{P}\}$ is a set of S_i–S_j paths in G' with no internal vertices from $\bigcup_{s \in [p]} S_s$, that also have pairwise disjoint and anticomplete interiors. For every $P' \in \mathcal{P}'$, let P'' be obtained from P' by replacing its endpoints with s_i and s_j. Since P' has no internal vertex in $\bigcup_{s \in [p]} S_s$, each P'' is an s_i–s_j path in G''. Hence $\mathcal{P}'' := \{P'' \ : \ P' \in \mathcal{P}'\}$ is internally vertex-disjoint and internally anticomplete, witnessing that $\{s_i\}, \{s_j\}$ is q-wide in G''. ◁

By the contrapositive of Proposition 3.6, G'' must contain a $K_{a,a}$ or $W_{b \times b}$ as an induced minor, witnessed by some induced subgraph $H'' \subseteq G''$. But then, the induced subgraph H' of G' obtained by replacing every $s_i \in V(H'') \cap \hat{S}$ with $S_i \subseteq V(G')$ and taking the induced subgraph on the resulting vertex set witnesses a $K_{a,a}$ or $W_{b \times b}$ as an induced minor of G'. Hence, from Proposition 3.1 and Proposition 3.2 we conclude that G' contains an induced subgraph H isomorphic to one of: $K_{t,t}$, a subdivision of $W_{2t \times 2t}$, the line graph of a subdivision of $W_{2t \times 2t}$, or a (t, t)-constellation. Then by Lemma 3.4, G contains $W_{t \times t}$ or $K_{t,t}$ as an induced minor, which contradicts $G \in \mathcal{C}_t$. □

Remark. *For a positive integer t, we use $p^{3.7}(t)$ and $q^{3.7}(t)$ to denote the constants implied by Lemma 3.7. When t is clear from context, we simply write $p^{3.7}, q^{3.7}$.*

4 Layered Sets with Balls of Radius r.

Let $\mathcal{P}$ be a hereditary graph property. For a graph G and $X \subseteq V(G)$ say that G is *X-close* to $\mathcal{P}$ if $G \setminus X \in \mathcal{P}$. We say that a graph is *$(k, r, \mathcal{P})$-close* if there exists a (k, r)-coverable set X such that G is X-close to $\mathcal{P}$. We say that a graph family $\mathcal{F}$ is *$(k, r, \mathcal{P})$-close* if it is hereditary and every graph in $\mathcal{F}$ is $(k, r, \mathcal{P})$-close. A *vertex labeling* is a function l from $V(G)$ to a set of labels $\mathcal{L}$. A *vertex-labeled graph*, is a graph G together with a labeling function. An induced subgraph of a vertex-labeled graph (G, l) is a labeled graph (H, l') such that H is an induced subgraph of G and l' is l restricted to $V(H)$. We say that a vertex-labeled graph (G, l) belongs to a class of unlabeled graphs $\mathcal{F}$ if $G \in \mathcal{F}$. As a slight abuse of notation, we extend the notion of hereditary properties and $(k, r, \mathcal{P})$-close to vertex-labeled graphs.

Two important examples of hereditary properties of vertex-labeled graphs are being *ε-balanced* and being *A-B-separated*. A vertex-labeled graph (G, w) is said to be *ε-balanced* if $w : V(G) \to \mathbb{R}_{\geq 0}$ and for every connected component C of G, we have that $w(C) \leq \varepsilon$ (where $w(C)$ sums over all the vertices of C). If $\varepsilon = 1/2$, we may simply say balanced instead of 1/2-balanced. A vertex-labeled graph (G, l) is said to be *A-B-separated* if $l : V(G) \to 2^{\{A,B\}}$ and no connected component of G has both a vertex label containing A and a vertex with label containing B. Let $\mathcal{P}_{\text{sep}}$ denote the hereditary property of being A-B-separated.

In this section, we prove Lemma 4.5, which is a bounded-radius version of the layered-sets theorem from [17]. Informally, given a hereditary property $\mathcal{P}$ and a graph G in a $(k, r, \mathcal{P})$-close family, we show that there exists a bounded-size set $T \subseteq V(G)$ such that either G is already T-close to $\mathcal{P}$ or T can be enlarged in many disjoint ways to produce sets whose deletion results in a graph satisfying $\mathcal{P}$. We then apply Lemma 4.5 to two hereditary properties mentioned above: being balanced and A–B separated. These applications make use of a tree decomposition of graphs in $\mathcal{C}_t$ from [18] with desirable properties.

Proof Overview: The proof iteratively builds a sequence of easily coverable sets whose deletion gives a graph satisfying $\mathcal{P}$. At each step, an exceptional set is removed so that the "level sets", which record how often vertices are covered by previously chosen sets, shrink geometrically. This step crucially uses a recent result of [16] to ensure that the process is not too costly. We then extract a pairwise anticomplete subsequence.

For a sequence $\vec{S} = S_1, \ldots, S_k$, we denote by $\vec{S}_{\leq j}$ the subsequence $S_1, \ldots, S_j$. We define $S_{\leq 0} = \emptyset$. Given a graph G and a sequence $\vec{S} = (S_1, \ldots, S_\ell)$ of vertex sets in G and integer i, we define the *ith level set* in G to be

$$L_i(G, \vec{S}) = \{v \in V(G) : |\{j \in [\ell] : v \in S_j\}| \geq i\}$$

(note that $L_0(G, S) = V(G)$). For a graph G, a tuple $\vec{S} = (S_1, \ldots, S_\ell)$ of sets, and positive integer a and r, we say that a vertex set $C \subseteq V(G)$ is *(a, r)-clean with respect to $\vec{S}$* if for every $i \geq 1$ and vertex $v \in C$ it holds that

$$\left|L_i(N_C^r[v], \vec{S})\right| \quad \leq \quad \frac{|G|}{2^i}\frac{1}{a}.$$

A *(k, r, a)-cleaning sequence* in a graph G consists of three sequences of sets $(S_1, \ldots, S_\ell)$, $(\hat{S}_1, \ldots, \hat{S}_\ell)$, and $(G_1, \ldots, G_\ell)$ such that :

1. For every $1 \leq i \leq j \leq \ell$, $G_j \subseteq G_i \subseteq G$.
2. For every $1 \leq i \leq \ell$, $\hat{S}_i$ is a (k, r)-cover of S_i in G_i. Moreover, for every $1 \leq i < j \leq \ell$, we have $\hat{S}_i \cap V(G_j) = \emptyset$.

3. For every $1 \le i \le \ell$ the set G_i is $(2ka, r)$-clean with respect to $\vec{S}_{\le i-1}$

4. G_1 is (ka, r)-clean with respect to G (viewed as a one-element sequence)

5. $\ell \le a$,

We call ℓ the *length* of the (k, r, a)-cleaning sequence.

In the next three lemmas, we show that, from any long enough (k, r, a)-cleaning sequence, it is possible to extract a fairly long (k, r, a)-cleaning subsequence $(\vec{S}, \vec{\hat{S}}, \vec{G})$ where $S_i \cap \hat{S}_j = \emptyset$ whenever $i \neq j$.

Lemma 4.1 (Closure under subsequences)**.** *Let $(S_1, \dots, S_\ell)$, $(\hat{S}_1, \dots, \hat{S}_\ell)$, and $(G_1, \dots, G_\ell)$ be a (k, r, a)-cleaning sequence in a graph G. Let $1 \le i_1 < i_2 < \cdots < i_p \le \ell$ be an increasing sequence of indices. Let $\vec{S}' = (S_{i_1}, \dots, S_{i_p}), \vec{\hat{S}}' = (\hat{S}_{i_1}, \dots, \hat{S}_{i_p}), \vec{G}' = (G_{i_1}, \dots, G_{i_p})$. Then $\vec{S}'$, $\vec{\hat{S}}'$ $\vec{G}'$ is a (k, r, a)-cleaning sequence.*

Proof. Condition 3 holds since for every $t \le p$, $b \ge 1$, and every $v \in G_{i_t}$ we have $\vec{S}'_{\le t-1} \subseteq \vec{S}_{\le i_t - 1}$, and therefore

$$L_b\Big(N^r_{G_{i_t}}[v], \vec{S}'_{\le t-1}\Big) \quad \subseteq \quad L_b\Big(N^r_{G_{i_t}}[v], \vec{S}_{\le i_t - 1}\Big).$$

Hence,

$$\left|L_b\left(N^r_{G_{i_t}}[v], \vec{S}'_{\le t-1}\right)\right| \quad \le \quad \left|L_b\left(N^r_{G_{i_t}}[v], \vec{S}_{\le i_t - 1}\right)\right| \quad \le \quad \frac{|G|}{2^b(2ka)}.$$

Condition 4 holds since the property is monotone under taking subgraphs, so G_{i_1} is (ka, r)-clean with respect to G. All remaining conditions are inherited directly from the original sequence, so $\vec{S}', \vec{\hat{S}}', \vec{G}'$ forms a (k, r, a)-cleaning sequence. □

Lemma 4.2. *Let G be a graph and let $\vec{S}$, $\vec{\hat{S}}$, and $\vec{G}$ be a (k, r, a)-cleaning sequence in G of length ℓ. Then, for all $1 \le i, j \le \ell$,*

$$\left|L_i(G_j, \vec{S}_{\le j})\right| \quad \le \quad \frac{|G|}{2^i}\frac{j}{a}.$$

Proof. First we prove the case $i = 1$. Let $1 \le q \le \ell$. Since $\hat{S}_q$ is a (k, r)-cover of S_q in G_q, we have

$$S_q \subseteq \bigcup_{x \in \hat{S}_q} N^r_{G_q}[x].$$

Moreover, $G_q \subseteq G_1$. Hence, for every $x \in \hat{S}_q$, we have $N^r_{G_q}[x] \subseteq N^r_{G_1}[x]$. By Condition 4, G_1 is (ka, r)-clean with respect to G, and so

$$|S_q| \quad \le \quad \sum_{x \in \hat{S}_q} \left|N^r_{G_q}[x]\right| \quad \le \quad |\hat{S}_q|\frac{|G|}{2ka} \quad \le \quad \frac{|G|}{2a}.$$

Therefore, for every $1 \le j \le \ell$,

$$\left|L_1(G_j, \vec{S}_{\le j})\right| \quad \le \quad \sum_{q \le j} |S_q| \quad \le \quad \frac{|G|}{2}\frac{j}{a}.$$

We now prove the statement for $i \ge 2$ by induction on j. The case $j = 1$ is trivial, since $L_i(G_1, \vec{S}_{\le 1}) = \emptyset$ for every $i \ge 2$. Let $j \ge 1$, and suppose that the statement holds for j. We prove

it for $j+1$. If $i > j+1$, then $L_i(G_{j+1}, \vec{S}_{\leq j+1}) = \emptyset$, so there is nothing to prove. Thus, we may assume that $2 \leq i \leq j+1$. We have

$$\left|L_i(G_{j+1}, \vec{S}_{\leq j+1})\right| \quad \leq \quad \left|L_i(G_j, \vec{S}_{\leq j})\right| \; + \; \left|L_{i-1}(S_{j+1}, \vec{S}_{\leq j})\right|.$$

By the induction hypothesis, or trivially if $i > j$,

$$\left|L_i(G_j, \vec{S}_{\leq j})\right| \quad \leq \quad \frac{|G|}{2^i}\frac{j}{a}.$$

It remains to bound the second term. Since $\hat{S}_{j+1}$ is a (k,r)-cover of S_{j+1} in G_{j+1},

$$\left|L_{i-1}(S_{j+1}, \vec{S}_{\leq j})\right| \quad \leq \quad \sum_{x \in \hat{S}_{j+1}} \left|L_{i-1}(N^r_{G_{j+1}}[x], \vec{S}_{\leq j})\right|.$$

Since $i \geq 2$, we have $i-1 \geq 1$. Also, by Condition 3, G_{j+1} is $(2ka, r)$-clean with respect to $\vec{S}_{\leq j}$. Hence

$$\left|L_{i-1}(N^r_{G_{j+1}}[x], \vec{S}_{\leq j})\right| \quad \leq \quad \frac{|G|}{2^{i-1}}\frac{1}{2ka}.$$

Therefore

$$\left|L_{i-1}(S_{j+1}, \vec{S}_{\leq j})\right| \quad \leq \quad |\hat{S}_{j+1}|\frac{|G|}{2^{i-1}}\frac{1}{2ka} \quad \leq \quad k\frac{|G|}{2^{i-1}}\frac{1}{2ka} \quad = \quad \frac{|G|}{2^i a}.$$

Combining both inequalities yields

$$\left|L_i(G_{j+1}, \vec{S}_{\leq j+1})\right| \quad \leq \quad \frac{|G|}{2^i}\frac{j}{a} \; + \; \frac{|G|}{2^i a} \quad = \quad \frac{|G|}{2^i}\frac{j+1}{a},$$

as desired. □

Lemma 4.3. *Let a, k, n, p, r be positive integers such that $a > pk\lceil \log n \rceil$ and $n \geq 2$. Let G be an n-vertex graph and $\vec{S}$, $\hat{\vec{S}}$, and $\vec{G}$ be a (k, r, a)-cleaning sequence in G of length $pk\lceil \log n \rceil$. Then there exists a subsequence $i_1, \ldots, i_p$ such that $\hat{S}_{i_e} \cap S_{i_f} = \emptyset$ for every $e \neq f$.*

Proof. By induction on p. If $p = 1$ then the statement is trivially true. Let $\ell = pk\lceil \log n \rceil$. By Lemma 4.2, we have that

$$\left|L_{\lceil \log n \rceil}(G_\ell, \vec{S}_{\leq \ell})\right| \quad \leq \quad \frac{n}{2^{\lceil \log n \rceil}}\frac{\ell}{a} \quad < \quad 1$$

so $L_{\lceil \log n \rceil}(G_\ell, \vec{S}_{\leq \ell}) = \emptyset$. Therefore, there are fewer than $k\lceil \log n \rceil$ of the sets in S intersecting $\hat{S}_\ell$. Taking the subsequence not intersecting $\hat{S}_\ell$ and applying the induction hypothesis completes the proof. □

To make use of the above, we need to construct (k, r, a)-cleaning sequences with useful properties. Doing so will require a recent result of Chudnovsky, Codsi, and Kaneshiro [16]. Recall that, for a positive integer t, we denote by $\mathcal{F}_t$ the class of $K_{t,t}$-induced-minor-free graphs.

Proposition 4.4. *For all $t, r \in \mathbb{N}$, there exists a positive integer $\mu = \mu(r, t)$ such that the following holds. Let $C, s \in \mathbb{N}$ such that $C \geq 2$, let $G \in \mathcal{F}_t$ with $\omega(G) < s$, and let $Y \subseteq V(G)$. Then there is a subset $Z \subseteq V(G)$ with $|Z| \leq (Cs)^\mu$ such that for every $v \in G \setminus Z$ we have that*

$$|N^r_{G \setminus Z}[v] \cap Y| \quad \leq \quad \frac{|Y|}{C}.$$

We denote by $Z(G, Y, r, C)$ the set Z obtained by applying Proposition 4.4 to a set Y.

Lemma 4.5. *Let $r, t \in \mathbb{N}$. There exists a positive integer $\mu = \mu(r,t)$ such that the following holds. Let a, k, n, p, s be positive integers such that $n \geq 2$ and $a > pk\lceil \log n \rceil$. Let $\mathcal{P}$ be a hereditary property of (possibly vertex-labeled) graphs. Let G be an n-vertex (vertex-labeled) graph in $\mathcal{F}_t$ such that $\omega(G) < s$, and suppose that every induced (labeled) subgraph of G is $(k, r, \mathcal{P})$-close. There exists a set $T \subseteq V(G)$, with $|T| \leq (2aks)^\mu$ such that either G is T-close to $\mathcal{P}$ or there exist non-empty sets $S_1, \ldots, S_p \subseteq V(G)$, pairwise disjoint and anticomplete to each other, with the following properties:*

1. *each S_i has at most k connected components,*
2. *each S_i is internally (k, r)-coverable,*
3. *for every i, $G \setminus T$ is S_i-close to $\mathcal{P}$.*

Proof. Let $\mu = \mu'(2r+1, t) + 3$ where μ' is defined as in Proposition 4.4. Let $G_0 = G$, $\hat{S}_0 = \emptyset$ and $T_0 = Z(G, G, 2r+1, 2ka)$. Set $\ell = pk\lceil \log n \rceil$. For every $0 < j \leq \ell$, let $G_j = G_{j-1} \setminus (T_{j-1} \cup \hat{S}_{j-1})$, and let S_j be a (k, r)-coverable set in G_j with cover $\hat{S}_j$ such that G_j is S_j-close to $\mathcal{P}$. By enlarging S_j if necessary, we may assume that $S_j = N^r_{G_j}[\hat{S}_j]$. Let $S'_j = N^{2r+1}_{G_j}[\hat{S}_j]$. Let

$$T_j = \bigcup_{i \leq j} Z\left(G_j, L_i(G_j, \vec{S}'_{\leq j}), 2r+1, 2ak\right).$$

Finally, let $T = \bigcup_{0 \leq j \leq \ell} \left(T_j \cup \hat{S}_j\right)$. By Proposition 4.4, we have that $|T| \leq a^2(2aks)^{\mu'(2r+1,t)} + ka \leq (2aks)^\mu$.

If G is T-close to $\mathcal{P}$ there is nothing to show. Therefore, we may assume that $G \setminus T \notin \mathcal{P}$. We check that $\vec{S}'$, $\vec{\hat{S}}$, and $\vec{G}$ form a $(k, 2r+1, a)$-cleaning sequence. We prove this by induction on ℓ. If $\ell = 1$, we have that $G_1 = G \setminus T_0$ and so G_1 is $(ka, 2r+1)$-clean with respect to G, and the other conditions to be a $(k, 2r+1, a)$-cleaning sequence are clearly satisfied. We may assume that $\vec{S}'_{\leq \ell-1}$, $\vec{\hat{S}}_{\leq \ell-1}$ and $\vec{G}_{\leq \ell-1}$ forms a $(k, 2r+1, a)$-cleaning sequence. Since $(k, 2r+1, a)$-cleaning sequences are closed under taking prefixes, it suffices to verify that G_ℓ is $(2ka, 2r+1)$-clean with respect to $\vec{S}'_{\leq \ell-1}$. For every $m \in \mathbb{N}$ and $v \in G_\ell$, we have that

$$\begin{aligned}
\left|L_m\left(N^{2r+1}_{G_\ell}[v], \vec{S}'_{\leq \ell-1}\right)\right| &\leq \frac{\left|L_m(G_{\ell-1}, \vec{S}'_{\leq \ell-1})\right|}{2ka} \\
&\leq \frac{|G|}{2^m}\frac{\ell-1}{a}\frac{1}{2ka} \\
&\leq \frac{|G|}{2^m}\frac{1}{2ka}.
\end{aligned}$$

Therefore, $\vec{S}', \vec{\hat{S}}, \vec{G}$ form a $(k, 2r+1, a)$-cleaning sequence.

We apply Lemma 4.3 to the cleaning sequence $\vec{S}'$, $\vec{\hat{S}}$, and $\vec{G}$. Thus, there are indices $i_1 < \cdots < i_p$ such that $\hat{S}_{i_e} \cap S'_{i_f} = \emptyset$ for every $e \neq f$. For $e \in [p]$, let $R_e = S_{i_e}$. We claim that $R_1, \ldots, R_p$ satisfy the conclusion. Fix $e < f$. Since $\hat{S}_{i_f} \cap S'_{i_e} = \emptyset$ and $S'_{i_e} = N^{2r+1}_{G_{i_e}}[\hat{S}_{i_e}]$, while $\hat{S}_{i_f} \subseteq V(G_{i_f}) \subseteq V(G_{i_e})$, we have $\mathrm{dist}_{G_{i_e}}(\hat{S}_{i_e}, \hat{S}_{i_f}) > 2r+1$. Since $G_{i_f} \subseteq G_{i_e}$, if R_e and R_f were not disjoint and anticomplete, then there would be a path in G_{i_e} of length at most $2r+1$ from $\hat{S}_{i_e}$ to $\hat{S}_{i_f}$, a contradiction. Thus the sets $R_1, \ldots, R_p$ are pairwise disjoint and anticomplete.

Moreover, each R_e is internally (k,r)-coverable, with cover $\hat{S}_{i_e} \subseteq R_e$, and has at most k connected components. Finally, since $G \setminus T \subseteq G_{i_e}$ and noting that $R_e = S_{i_e}$, $G_{i_e} \setminus R_e \in \mathcal{P}$, the heredity of $\mathcal{P}$ implies that $(G \setminus T) \setminus R_e \in \mathcal{P}$. Thus $G \setminus T$ is R_e-close to $\mathcal{P}$ for every $e \in [p]$, and the sets $R_1, \ldots, R_p$ satisfy the conclusion.

□

Next, we will specialize Lemma 4.5 to two specific hereditary properties. We need the following theorem of Chudnovsky, Codsi, E S, and Lokshtanov [18]. Their statement is phrased in terms of the *distance-r-independence number* rather than coverability, but a straightforward argument, given in the fourth paragraph of their preliminaries, shows the following.

Proposition 4.6. *Let t be a positive integer and let $G \in \mathcal{C}_t$ be an n-vertex graph. Then there exists a positive integer $d(t)$ and $\epsilon(t) \in (0,1]$, such that G has a tree decomposition (T,χ) where for every $x \in V(T)$, $\chi(x)$ is $(2^{d(t)\log^{1-\epsilon(t)} n}, 8)$-coverable in G.*

Remark. *For a positive integer t and a graph $G \in \mathcal{C}_t$, we use $k^{4.6}(t,n)$ to denote $\left\lceil 2^{d(t)\log^{1-\epsilon(t)} n} \right\rceil$, when t, n are clear from context, we simply write $k^{4.6}$. Similarly, we use $d^{4.6}$ to denote $d(t)$ and $\epsilon^{4.6}$ to denote $\epsilon(t)$. Finally, by decreasing $\epsilon^{4.6}$ if necessary, we assume throughout that $\epsilon^{4.6}$ is rational and belongs to $(0,1)$.*

For $\varepsilon \in (0,1)$, let $\mathcal{P}^{\varepsilon}_{\text{bal}}$ denote the hereditary property of being ε-balanced. We say that a graph G is *(k,r,ε)-separable* if, for every weight function w on $V(G)$, the labeled graph (G,w) is $(k,r,\mathcal{P}^{\varepsilon}_{\text{bal}})$-close. We say that a hereditary graph class $\mathcal{G}$ is *$(f(n),r,\varepsilon)$-separable* if every n-vertex graph $G \in \mathcal{G}$ is $(f(n),r,\varepsilon)$-separable.

Corollary 4.7. *Let t be a positive integer. Then $\mathcal{C}_t$ is $(4k^{4.6}(t,n), 8, \frac{1}{4})$-separable.*

Proof. Let $G \in \mathcal{C}_t$ be an n-vertex graph and let w be a weight function on $V(G)$. If $w(G) = 0$, then the empty set is a $(w, \frac{1}{4})$-balanced separator, so we may assume that $w(G) > 0$. Let (T,χ) be a tree decomposition of G as in Proposition 4.6. Applying Proposition 2.1 to (T,χ) and w gives a bag S_1 which is a $(w, \frac{1}{2})$-balanced separator in G. In particular, every component of $G \setminus S_1$ has weight at most $1/2$.

Let $C_1, \ldots, C_m$ be the components of $G \setminus S_1$ with weight more than $1/4$. Since these components are pairwise disjoint, we have $m \leq 3$. For each $i \in [m]$, define a weight function w_i on $G[C_i]$ by setting $w_i(v) = w(v)/w(C_i)$. Since $\mathcal{C}_t$ is hereditary, $G[C_i] \in \mathcal{C}_t$. Applying Proposition 4.6 to $G[C_i]$ and Proposition 2.1 to w_i, we obtain a set $S'_i \subseteq C_i$ which is a $(w_i, \frac{1}{2})$-balanced separator in $G[C_i]$ and is $(k^{4.6}(t,|C_i|), 8)$-coverable in $G[C_i]$. Since $|C_i| \leq n$, this implies that S'_i is $(k^{4.6}(t,n), 8)$-coverable in G.

Let $S := S_1 \cup \bigcup_{i=1}^{m} S'_i$. Then S is $(4k^{4.6}(t,n), 8)$-coverable in G, since S_1 and each S'_i can be covered by at most $k^{4.6}(t,n)$ balls of radius 8, and $m \leq 3$.

It remains to show that S is $(w, \frac{1}{4})$-balanced. Let C be a component of $G \setminus S$. If C is contained in a component of $G \setminus S_1$ that is not among $C_1, \ldots, C_m$, then $w(C) \leq 1/4$. Otherwise, $C \subseteq C_i \setminus S'_i$ for some $i \in [m]$. Since S'_i is $(w_i, \frac{1}{2})$-balanced in $G[C_i]$, we have $w_i(C) \leq 1/2$, and therefore $w(C) \leq w(C_i)/2 \leq 1/4$. Thus S is a $(w, \frac{1}{4})$-balanced separator in G. □

To make the results of this section more readily usable in the next sections, we introduce some definitions:

– Let ℓ be an integer. A *clean-set-packing* in G is a pair $\big(D, \{(S_i, C_i) : i \in [\ell]\}\big)$, where $D, S_i, C_i \subseteq V(G)$ for every $i \in [\ell]$, such that $S_i \cap (D \cup C_i) = \emptyset$ for every $i \in [\ell]$ and the sets $S_i \cup C_i$ and $S_j \cup C_j$ are disjoint and anticomplete in G whenever $i \neq j$. We write $D' := D \cup \bigcup_{i=1}^{\ell} C_i$ and call ℓ the *length* of the clean-set-packing.

– Let c, k be positive integers. A clean-set-packing is *k-thick* if for every $i \in [\ell]$, the subgraph of $G - (D' \setminus C_i)$ induced by $S_i \cup C_i$ has at most k connected components. It is *c-costly*, or has *cost c*, if $|D'| = c$.

– A clean-set-packing $(D, \{(S_i, C_i) : i \in [\ell]\})$ in G, is a *$(Y, \frac{1}{2})$-packing* if for every $i \in [\ell]$ the set S_i is a $(Y', \frac{1}{2})$-balanced separator in G', where $G' := G - D'$ and $Y' := Y \cap V(G')$.

Lemma 4.8. *Let t, ℓ be positive integers. There exists a positive integer $\mu = \mu(t)$ such that the following holds. Let $G \in \mathcal{C}_t$ be an n-vertex graph with $n \geq 2$ and $\omega(G) \leq \omega$, and let $Y \subseteq V(G)$. Set*

$$K := 4k^{4.6}(t, n), \qquad h' := \ell K(\lceil \log n \rceil + 1), \qquad \kappa := \big(2h'K(\omega + 1)\big)^{\mu}.$$

Then either G has a $(Y, \frac{1}{2})$-balanced separator of size at most κ, or G has a $(Y, \frac{1}{2})$-packing of length ℓ that is K-thick and has cost at most κ.

Proof. If $Y = \emptyset$, then the empty set is a $(Y, \frac{1}{2})$-balanced separator, so we may assume that $Y \neq \emptyset$. Define a weight function $w : V(G) \to \mathbb{R}_{\geq 0}$ by setting $w(v) = 1/|Y|$ for $v \in Y$, and $w(v) = 0$ otherwise. Regard G as the vertex-labeled graph (G, w), and let $\mathcal{P} := \mathcal{P}_{\mathrm{bal}}^{1/4}$.

We first check that every induced subgraph of G is $(K, 8, \mathcal{P})$-close. Let H be an induced subgraph of G. Since $\mathcal{C}_t$ is hereditary, $H \in \mathcal{C}_t$. By Corollary 4.7, applied to H and the restricted weight function $w|_{V(H)}$, there is a $(K, 8)$-coverable set $X \subseteq V(H)$ such that $H \setminus X \in \mathcal{P}$. Thus H is $(K, 8, \mathcal{P})$-close.

Since $\mathcal{C}_t \subseteq \mathcal{F}_t$, we can apply Lemma 4.5 with parameters $k = K$, $r = 8$, $p = \ell$, $s = \omega + 1$, and $a = h'$. Since $h' > \ell K \lceil \log n \rceil$, we obtain a set $T \subseteq V(G)$ with $|T| \leq \kappa$ such that either $G \setminus T \in \mathcal{P}$, or there exist sets $R_1, \ldots, R_\ell$, pairwise disjoint and anticomplete, such that

1. each R_i has at most K connected components,
2. each R_i is internally $(K, 8)$-coverable,
3. $(G \setminus T) \setminus R_i \in \mathcal{P}$ for every $i \in [\ell]$.

If $G \setminus T \in \mathcal{P}$, then T is a $(Y, \frac{1}{4})$-balanced separator, and hence a $(Y, \frac{1}{2})$-balanced separator. Thus we may assume that the second case holds. Set $Y' := Y \cap V(G \setminus T)$. If $|Y'| \leq |Y|/2$, then T is a $(Y, \frac{1}{2})$-balanced separator, so we may assume that $|Y'| > |Y|/2$.

For every $i \in [\ell]$, set $C_i := R_i \cap T$, $S_i := R_i \setminus T$, and let $D := T \setminus \bigcup_{i \in [\ell]} C_i$. Then $D' := D \cup \bigcup_{i \in [\ell]} C_i = T$, and $S_i \cup C_i = R_i$ for every i. It follows that $(D, \{(S_i, C_i) : i \in [\ell]\})$ is a K-thick clean-set-packing of length ℓ and cost at most κ.

It remains to check the separator condition. Since $(G \setminus T) \setminus R_i = (G \setminus D') \setminus S_i \in \mathcal{P}$, every component of $(G \setminus D') \setminus S_i$ contains at most $|Y|/4$ vertices of Y. Since $|Y'| > |Y|/2$, we have that $|Y|/4 \leq |Y'|/2$. Hence S_i is a $(Y', \frac{1}{2})$-balanced separator in $G \setminus D'$ for every $i \in [\ell]$. Therefore the clean-set-packing is a $(Y, \frac{1}{2})$-packing. □

Remark. *For positive integers t, ℓ, n, ω, with $n \geq 2$, we use $\kappa^{4.8}(t, \ell, n, \omega)$ to denote the function κ defined in Lemma 4.8. We also use $\mu^{4.8}(t)$ to refer to the constant implied by Lemma 4.8, and when t is clear from context, we simply write $\mu^{4.8}$.*

We say that a graph G is *(k, r, q)-slim-pair separable* if, for every q-slim pair $A, B \subseteq V(G)$, there exists a (k, r)-coverable A–B separator in G. A hereditary graph class $\mathcal{G}$ is *(k, r, q)-slim-pair separable* if every graph $G \in \mathcal{G}$ is (k, r, q)-slim-pair separable.

We now prove an analogue of Lemma 4.8 for A–B separators. To do so, we will need Corollary 4.9.

Corollary 4.9. *Let t, q be positive integers. Then $\mathcal{C}_t$ is $(qk^{4.6}(t,n), 8, q)$-slim-pair separable.*

The proof presented below follows the outline of a similar proof in [21]. We will need the following:

Proposition 4.10 (Erdős–Pósa property for subtrees of a tree [45])**.** *Let T be a tree and let $\mathcal{T}$ be a family of nonempty subtrees of T. For every positive integer q, either $\mathcal{T}$ contains q pairwise vertex-disjoint subtrees, or there exists a set $Z \subseteq V(T)$ with $|Z| < q$ such that Z intersects every subtree in $\mathcal{T}$.*

Proof of Corollary 4.9. Let $G \in \mathcal{C}_t$ be an n-vertex graph, and let $A, B \subseteq V(G)$ be q-slim in G. Let (T, χ) be a tree decomposition of G as in Proposition 4.6. Thus every bag is $(k^{4.6}(t,n), 8)$-coverable in G.

For every A–B path P in G, let $T_P = \{x \in V(T) : \chi(x) \cap V(P) \neq \emptyset\}$. Then T_P is a subtree of T. We claim that the family $\mathcal{T} = \{T_P : P \text{ is an } A\text{–}B \text{ path in } G\}$ contains no q pairwise vertex-disjoint subtrees. Indeed, if $T_{P_1}, \ldots, T_{P_q}$ are pairwise vertex-disjoint, then the paths $P_1, \ldots, P_q$ are pairwise vertex-disjoint and anticomplete, contradicting that A and B are q-slim. Indeed, a common vertex, or an edge between two of the paths, would be contained in some bag, forcing the corresponding subtrees to intersect.

By Proposition 4.10, there exists a set $Z \subseteq V(T)$ with $|Z| < q$ such that Z intersects every subtree in $\mathcal{T}$. Let

$$X = \bigcup_{z \in Z} \chi(z).$$

Then X intersects every A–B path in G, and hence X is an A–B separator. Since each bag $\chi(z)$ is $(k^{4.6}(t,n), 8)$-coverable in G and $|Z| < q$, the set X is $(qk^{4.6}(t,n), 8)$-coverable in G, as desired. □

Here again, we introduce some notation to make the following result more easily usable in the next sections. A clean-set-packing $(D, \{(S_i, C_i) : i \in [\ell]\})$ in G, is an *(A,B)-packing* if for every $i \in [\ell]$ the set S_i is an A'–B' separator in G', where $G' := G - D', A' := A \cap V(G')$, and $B' := B \cap V(G')$.

Lemma 4.11. *Let t, q, ℓ be positive integers. There exists a positive integer $\mu = \mu(t)$ such that the following holds. Let $G \in \mathcal{C}_t$ be an n-vertex graph with $n \geq 2$ and $\omega(G) \leq \omega$, and let $A, B \subseteq V(G)$ be q-slim in G. Set*

$$\tau := qk^{4.6}(t,n), \qquad h := \ell\tau(\lceil \log n \rceil + 1), \qquad c_{A,B} := \big(2h\tau(\omega+1)\big)^{\mu}.$$

Then either G has an A–B separator of size at most $c_{A,B}$, or G has an (A,B)-packing of length ℓ that is τ-thick and has cost at most $c_{A,B}$.

Proof. Label each vertex $v \in V(G)$ by

$$l(v) := \{\mathsf{A} : v \in A\} \cup \{\mathsf{B} : v \in B\},$$

and let $\mathcal{P} := \mathcal{P}_{\text{sep}}$.

We first check that every induced labeled subgraph of (G, l) is $(\tau, 8, \mathcal{P})$-close. Let H be such an induced labeled subgraph. Its underlying graph belongs to $\mathcal{C}_t$, and its inherited A- and B-labels correspond to $A \cap V(H)$ and $B \cap V(H)$. Moreover, these two sets are q-slim in H. By Corollary 4.9, there is a $(\tau, 8)$-coverable set $X \subseteq V(H)$ such that $H \setminus X \in \mathcal{P}$. Thus H is $(\tau, 8, \mathcal{P})$-close.

Since $\mathcal{C}_t \subseteq \mathcal{F}_t$, we can apply Lemma 4.5 with parameters $k = \tau$, $r = 8$, $p = \ell$, $s = \omega + 1$, and $a = h$. Since $h > \ell\tau \lceil \log n \rceil$, we obtain a set $T \subseteq V(G)$ with $|T| \leq c_{A,B}$ such that either $G \setminus T \in \mathcal{P}$, or there exist sets $R_1, \ldots, R_\ell$, pairwise disjoint and anticomplete, such that

1. each R_i has at most τ connected components,
2. each R_i is internally $(\tau, 8)$-coverable, and
3. $(G \setminus T) \setminus R_i \in \mathcal{P}$ for every $i \in [\ell]$.

In the first case, T is an A–B separator of size at most $c_{A,B}$. Thus we may assume that the second case holds. For every $i \in [\ell]$, set $C_i := R_i \cap T$, $S_i := R_i \setminus T$, and let $D := T \setminus \bigcup_{i\in[\ell]} C_i$. Then $D' := D \cup \bigcup_{i\in[\ell]} C_i = T$, and $S_i \cup C_i = R_i$ for every i. Since the sets $R_1, \ldots, R_\ell$ are pairwise disjoint and anticomplete, $(D, \{(S_i, C_i) : i \in [\ell]\})$ is a clean-set-packing of length ℓ.

It is τ-thick because, for every i, the graph induced by $S_i \cup C_i = R_i$ in $G \setminus (D' \setminus C_i)$ has at most τ connected components. Also, its cost is $|D'| = |T| \leq c_{A,B}$. Finally, since $(G \setminus T) \setminus R_i = (G \setminus D') \setminus S_i \in \mathcal{P}$,the set S_i is an A'–B' separator in $G \setminus D'$, where $A' := A \cap V(G \setminus D')$ and $B' := B \cap V(G \setminus D')$. Hence this is an (A, B)-packing. □

Remark. *For positive integers t, q, ℓ, n, ω, we use $\tau^{4.11}(q, t, n)$ and $c^{4.11}(t, q, \ell, n, \omega)$ to denote the functions τ and $c_{A,B}$ defined in Lemma 4.11. We also use $\mu^{4.11}(t)$ to refer to the constant implied by Lemma 4.11, and when t is clear from context, we simply write $\mu^{4.11}$.*

5 Graphs in $\mathcal{C}_t$ admit Strong Barriers.

In this section we show that every graph $G \in \mathcal{C}_t$, together with an (A, B)-packing Π, admits a strong-barrier with the additional properties stated in the main theorem of this section, Theorem 5.2. This combinatorial object is central to the arguments of Section 6. Let G be a graph, $A, B \subseteq V(G)$, and let $\Pi = (D, \{(S_i, C_i) : i \in [\ell]\})$ be an (A, B)-packing in G. Set $D' := D \cup \bigcup_{i=1}^{\ell} C_i$, $G' := G - D'$, $A' := A \cap V(G')$, and $B' := B \cap V(G')$. We make the following definitions, all relative to a partition (L, Z, R) of $V(G')$ and the (A, B)-packing Π.

- (L, Z, R) is a *strong-barrier* corresponding to Π in G if $A' \subseteq L$, $B' \subseteq R$, and every L–R path P in G' has a subpath that is again an L–R path, with the additional property that its interior is contained in Z.
- For $i \in [\ell]$, the *fragments* of $S_i \cup C_i$, written $\mathcal{F}_i$, are the connected components of $G[S_i \cup C_i]$. Furthermore, $\mathcal{F} := \bigcup_{i=1}^{\ell} \mathcal{F}_i$ is the set of *fragments* of Π. Since the sets $S_i \cup C_i$ for $i \in [\ell]$ pairwise do not touch, $\mathcal{F}$ is exactly the set of connected components of $G\big[\bigcup_{i=1}^{\ell}(S_i \cup C_i)\big]$.
- For a positive integer γ, a strong-barrier is *γ-broad* if the interior of every L–R path in G' intersects at least γ distinct fragments of $\mathcal{F}$.
- The *active components* of a strong-barrier, written $\mathcal{A}$, are the connected components of $G'[Z]$ with a neighbor in both L and R.
- (L, Z, R) is *(λ_1, λ_2)-light* if it satisfies $\big|\{F \in \mathcal{F} : \exists C \in \mathcal{A},\ V(F) \cap V(C) \neq \varnothing\}\big| \leq \lambda_2$ and $\big|\bigcup_{C\in\mathcal{A}} V(C)\big| \leq \lambda_1$.

We record the following theorem by Chudnovsky, Codsi, Fischer, and Lokshtanov [15], after which we state the main theorem of this section. Recall that two vertices u, v of a graph are called *false-twins* if $N(u) = N(v)$.

Proposition 5.1. *Corresponding to every positive integer t, there exists a positive integer $g = g(t)$ such that the following holds. Let G be a bipartite $K_{t,t}$-induced-minor-free graph with bipartition (A, B) such that no two vertices in B are false-twins. Then $|E(G)| \leq g|A|$.*

Remark. *For a positive integer t, we use $g^{5.1}(t)$ to denote the constant implied by Proposition 5.1.*

Theorem 5.2. *Let t be a positive integer and let $G \in \mathcal{C}_t$ be a graph on n vertices. Let k, γ, ℓ be positive integers with $\gamma \geq \tilde{t}^{3.5}$ and $\ell \geq 8\gamma$, let $A, B \subseteq V(G)$, and let Π be a k-thick (A,B)-packing of length ℓ in G. Set*

$$\lambda_1(\gamma, n, \ell) \;:=\; 16\gamma n/\ell \qquad \text{and} \qquad \lambda_2(\gamma, t, k) \;:=\; 16\,\gamma\,\tilde{t}^{3.5}\,g^{5.1}(\tilde{t}^{3.5})\,k.$$

Then there exists a strong-barrier corresponding to Π in G that is $(\lambda_1(\gamma,n,\ell),\ \lambda_2(\gamma,t,k))$-light and γ-broad.

In order to properly sketch a proof of this theorem, we need to fix some objects and make some definitions. For the rest of this section let us fix a positive integer t and an n-vertex graph $G \in \mathcal{C}_t$. Fix positive integers k, γ, ℓ with $\gamma \geq \tilde{t}^{3.5}$ and $\ell \geq 8\gamma$. Let $\lambda_1 := \lambda_1(\gamma, n, \ell)$ and $\lambda_2 := \lambda_2(\gamma, t, k)$. Also fix a pair of vertex subsets $A, B \subseteq V(G)$ and a k-thick (A,B)-packing $\Pi = (D, \{(S_i, C_i) : i \in [\ell]\})$ of length ℓ in G. As before, let $D' := D \cup \bigcup_{i=1}^{\ell} C_i$, $G' := G - D'$, $A' := A \cap V(G')$, and $B' := B \cap V(G')$. Now, for $v \in V(G')$, we define

$$d(v) \;:=\; \min\Big\{\ell,\ \min\big\{\big|\{i \in [\ell] : V(P) \cap S_i \neq \emptyset\}\big| : P \text{ is an } A'\text{–}v \text{ path in } G'\big\}\Big\}.$$

Here we use the convention $\min \varnothing = +\infty$; consequently $d(v) = \ell$ if no A'–v path exists. Let r be a positive integer. We denote by $\mathcal{B}_r$ the set $\{v \in V(G') : d(v) \leq r\}$, and write $\bar{\mathcal{B}}_r$ for $V(G') \setminus \mathcal{B}_r$. Define $\ell' := \left\lfloor \frac{\ell}{4\gamma} \right\rfloor$ and for future purposes, observe that $\ell \geq 8\gamma$ gives $\ell/\ell' = \ell/\lfloor \frac{\ell}{4\gamma} \rfloor \leq \ell/(\frac{\ell}{4\gamma} - 1) \leq \ell/(\frac{\ell}{8\gamma}) = 8\gamma$. For $i \in [\ell']$, we define the *i-th layer* to be $\mathcal{L}_i := \mathcal{B}_{2i\gamma} \setminus \mathcal{B}_{(2i-1)\gamma}$.

We make the following simple observations regarding our definitions: Since each S_i is an A'–B' separator in G', we have $d(v) = \ell$ for every $v \in B'$. Also, since the sets S_i are pairwise disjoint, every $v \in V(G')$ lies in at most one S_i. Hence $d(v) \leq 1$ for every $v \in A'$ and $d(v) \leq d(u) + 1$ for every edge uv of G'. Finally, the layers $\mathcal{L}_i$ for $i \in [\ell']$ pairwise do not touch in G': for $i < j$, every $u \in \mathcal{L}_i$ and $v \in \mathcal{L}_j$ satisfy $d(u) + \gamma \leq 2i\gamma + \gamma \leq (2j-1)\gamma < d(v)$. So $d(v) > d(u) + 1$, and hence u, v must be distinct and non-adjacent. With these definitions in hand, we can outline the proof of Theorem 5.2.

Proof Overview: We show in Lemma 5.3 that for every $i \in [\ell']$, the triple $(\mathcal{B}_{(2i-1)\gamma}, \mathcal{L}_i, \bar{\mathcal{B}}_{2i\gamma})$ is a γ-broad strong-barrier, so it suffices to find some $i \in [\ell']$ for which this barrier is (λ_1, λ_2)-light. For the λ_1 part, a simple averaging argument shows that at least half of the layers $\mathcal{L}_i$ have active components spanning at most λ_1 vertices in total, since the layers are pairwise disjoint and together contain at most n vertices. For the λ_2 part, we build an auxiliary graph H whose vertices correspond to the fragments of Π and the active components of the λ_1-light layers, with an edge between two vertices exactly when their corresponding sets touch in G. We show that H is $K_{\tilde{t}^{3.5},\tilde{t}^{3.5}}$-induced-minor-free, arguing that an induced $K_{\tilde{t}^{3.5},\tilde{t}^{3.5}}$-minor model in H would yield a $\tilde{\mathcal{K}}_{\tilde{t}^{3.5},\tilde{t}^{3.5}}$ in G, contradicting Lemma 3.5. Applying Proposition 5.1 to H then bounds its number of edges, and a further averaging argument over the λ_1-light layers produces one layer whose active components have low total degree in H, i.e. one that is also λ_2-light. One caveat is that Proposition 5.1 also requires the vertices representing active components to be free of false-twins, which need not hold in H — we get around this using γ-broadness to show every false-twin class among them has size less than $\tilde{t}^{3.5}$, so that retaining one representative from each class loses only a constant factor.

Now we move on to the formal statements and proofs.

Lemma 5.3. *For every $i \in [\ell']$, the triple $(\mathcal{B}_{(2i-1)\gamma}, \mathcal{L}_i, \bar{\mathcal{B}}_{2i\gamma})$ is a γ-broad strong-barrier.*

Proof. Fix $i \in [\ell']$. We have $A' \subseteq \mathcal{B}_1 \subseteq \mathcal{B}_{(2i-1)\gamma}$, since $d(v) \leq 1$ for every $v \in A'$ and $(2i-1)\gamma \geq 1$. Similarly $B' \subseteq \bar{\mathcal{B}}_{2\ell'\gamma} \subseteq \bar{\mathcal{B}}_{2i\gamma}$, since $d(v) = \ell$ for every $v \in B'$ and $2i\gamma < \ell$. For the second condition, let P be a $\mathcal{B}_{(2i-1)\gamma}$–$\bar{\mathcal{B}}_{2i\gamma}$ path in G'. Define u to be the last vertex of P that lies in $V(P) \cap \mathcal{B}_{(2i-1)\gamma}$ and let v be the first vertex of P after u that lies in $V(P) \cap \bar{\mathcal{B}}_{2i\gamma}$. Observe that both are well defined since the first and last vertices of P lie in $\mathcal{B}_{(2i-1)\gamma}$ and $\bar{\mathcal{B}}_{2i\gamma}$ respectively. Let P' be the u–v subpath of P, and observe that the interior of P' is contained in $\mathcal{L}_i$. Hence $(\mathcal{B}_{(2i-1)\gamma}, \mathcal{L}_i, \bar{\mathcal{B}}_{2i\gamma})$ is a strong-barrier.

Now to show that it is γ-broad, let P be a u–v path with $u \in \mathcal{B}_{(2i-1)\gamma}$ and $v \in \bar{\mathcal{B}}_{2i\gamma}$. Let P'' be an A'–u path witnessing $d(u)$, and let Q be an A'–v path within the A'–v walk obtained by concatenating P'' and P. Observe that Q intersects more than $2i\gamma$ distinct sets S_j as $d(v) > 2i\gamma$, while $V(P'')$ intersects at most $(2i-1)\gamma$ such sets as $d(u) \leq (2i-1)\gamma$, and v lies in at most one such set. Thus the interior of P intersects at least γ distinct sets S_j. Since the sets $S_j \cup C_j$ for $j \in [\ell]$ pairwise do not touch in G, each fragment lies in a single $S_j \cup C_j$. Consequently, we conclude that the interior of P intersects at least γ distinct fragments. □

For each $i \in [\ell']$, let $\mathcal{A}_i$ denote the set of active components of $(\mathcal{B}_{(2i-1)\gamma}, \mathcal{L}_i, \bar{\mathcal{B}}_{2i\gamma})$, which is a strong barrier by Lemma 5.3. Furthermore, for $I \subseteq [\ell']$, let $\mathcal{A}_I$ denote $\bigcup_{i \in I} \mathcal{A}_i$.

Lemma 5.4. *There exists $i \in [\ell']$ such that $(\mathcal{B}_{(2i-1)\gamma}, \mathcal{L}_i, \bar{\mathcal{B}}_{2i\gamma})$ is (λ_1, λ_2)-light.*

Proof. Let $I \subseteq [\ell']$ denote the set of $i \in [\ell']$ such that $\mathcal{L}_i$ satisfies $\left|\bigcup_{C \in \mathcal{A}_i} V(C)\right| \leq \lambda_1$. We first show that $|I| \geq \ell'/2$. Suppose for a contradiction that $|[\ell'] \setminus I| > \ell'/2$. The layers $\mathcal{L}_i$, $i \in [\ell']$, are pairwise disjoint subsets of $V(G')$ with $\bigcup_{C \in \mathcal{A}_i} V(C) \subseteq \mathcal{L}_i$ for every $i \in [\ell']$. Hence,

$$\sum_{i \in [\ell']} |\mathcal{L}_i| \quad \geq \quad \sum_{i \in [\ell'] \setminus I} \sum_{C \in \mathcal{A}_i} |V(C)| \quad > \quad |[\ell'] \setminus I| \cdot \lambda_1 \quad \geq \quad |[\ell'] \setminus I| \cdot \frac{2n}{\ell'} \quad > \quad \frac{\ell'}{2} \cdot \frac{2n}{\ell'} \quad = \quad n.$$

Define $\mathcal{A}_I := \bigcup_{i \in I} \mathcal{A}_i$. Construct an auxiliary graph H with vertex set $\mathcal{A}_I \cup \mathcal{F}$, joining two vertices of H by an edge whenever the vertex subsets of G they represent touch in G.

Claim 5.4.1. *H does not contain $K_{\tilde{t}^{3.5}, \tilde{t}^{3.5}}$ as an induced minor.*

Proof of the Claim. Suppose for contradiction that H contains $K_{\tilde{t}^{3.5}, \tilde{t}^{3.5}}$ as an induced minor, with sets $\{X_1, \ldots, X_{\tilde{t}^{3.5}}\}, \{Y_1, \ldots, Y_{\tilde{t}^{3.5}}\} \subseteq \mathcal{A}_I \cup \mathcal{F}$ forming the induced minor model. For $Z \subseteq \mathcal{A}_I \cup \mathcal{F}$, write $V_G(Z) := \bigcup_{v \in Z} V(v) \subseteq V(G)$, where $V(v)$ denotes the vertex subset of G corresponding to v. We show that $\mathcal{S} := \{V_G(X_1), \ldots, V_G(X_{\tilde{t}^{3.5}}), V_G(Y_1), \ldots, V_G(Y_{\tilde{t}^{3.5}})\}$ is a $\tilde{\mathcal{K}}_{\tilde{t}^{3.5}, \tilde{t}^{3.5}}$ in G.

- Every $v \in \mathcal{A}_I \cup \mathcal{F}$ corresponds to a connected vertex subset of G, and by definition of $E(H)$, if $u, v \in V(H)$ are adjacent then $V(u)$ and $V(v)$ touch and consequently $V(u) \cup V(v)$ is connected in G. Since $H[X_i]$ and $H[Y_i]$ are connected for every $i \in [\tilde{t}^{3.5}]$, applying this observation along a spanning tree of each of $H[X_i]$ and $H[Y_i]$ shows that $G[V_G(X_i)]$ and $G[V_G(Y_i)]$ are connected as well.
- Next, fix distinct $i, j \in [\tilde{t}^{3.5}]$. If $V_G(X_i)$ and $V_G(X_j)$ touch, then there exist $u \in X_i$ and $v \in X_j$ such that $V(u)$ and $V(v)$ touch. But then $uv \in E(H)$, which contradicts that X_i is anticomplete to X_j in H. Hence $V_G(X_i)$ and $V_G(X_j)$ do not touch. The same argument applied to $V_G(Y_i)$ and $V_G(Y_j)$ shows that they do not touch.
- Finally, fix $i, j \in [\tilde{t}^{3.5}]$ and observe that there must exist $u \in X_i$ and $v \in Y_j$ with $uv \in E(H)$. Thus $V(u)$ and $V(v)$ touch and consequently $V_G(X_i)$ and $V_G(Y_j)$ touch.

Thus $\mathcal{S}$ is a $\tilde{\mathcal{K}}_{\tilde{t}^{3.5},\tilde{t}^{3.5}}$ in G, contradicting Lemma 3.5. ◁

For this proof, we say that a vertex subset forms a *twin class* if it is a maximal set of pairwise false-twins.

Claim 5.4.2. *H is a bipartite graph with $\mathcal{A}_I$ and $\mathcal{F}$ forming the two sides of the bipartition, and every twin class that contains a vertex of $\mathcal{A}_I$ is contained in $\mathcal{A}_I$ and has cardinality less than $\tilde{t}^{3.5}$.*

Proof of the Claim. The elements of $\mathcal{F}$ are the connected components of $G\big[\bigcup_{i=1}^{\ell}(S_i \cup C_i)\big]$, hence pairwise do not touch in G, so $\mathcal{F}$ is independent in H. Likewise, for $i \in I$ the elements of $\mathcal{A}_i$ pairwise do not touch in G' by definition, and for distinct $i, j \in [\ell']$ the elements of $\mathcal{A}_i$ and $\mathcal{A}_j$ lie in $\mathcal{L}_i$ and $\mathcal{L}_j$ respectively and hence do not touch in G'. In either case, since G' is an induced subgraph of G, the elements do not touch in G either. So $\mathcal{A}_I = \bigcup_{i\in I}\mathcal{A}_i$ is also independent in H, and thus H is bipartite with $\mathcal{A}_I$ and $\mathcal{F}$ forming the two sides of the bipartition.

Now, fix $i \in I$ and $C \in \mathcal{A}_i$. Observe that C is connected in G' and has neighbors u, v in $\mathcal{B}_{(2i-1)\gamma}$ and $\bar{\mathcal{B}}_{2i\gamma}$ respectively. Consequently there is a $\mathcal{B}_{(2i-1)\gamma}$–$\bar{\mathcal{B}}_{2i\gamma}$ path in G' whose interior is contained in C. By Lemma 5.3, the barrier $(\mathcal{B}_{(2i-1)\gamma}, \mathcal{L}_i, \bar{\mathcal{B}}_{2i\gamma})$ is γ-broad, so the interior of this path, and hence C, intersects at least $\gamma \geq \tilde{t}^{3.5}$ distinct elements of $\mathcal{F}$. Consequently, since $\mathcal{F}$ is independent in H, any twin class that contains a vertex of $\mathcal{A}_I$ is contained in $\mathcal{A}_I$. Furthermore, since $\mathcal{A}_I$ and $\mathcal{F}$ are independent in H, if some twin class $T \subseteq \mathcal{A}_I$ of H had $|T| \geq \tilde{t}^{3.5}$, then T together with $\tilde{t}^{3.5}$ common neighbors of T in $\mathcal{F}$, induce a $K_{\tilde{t}^{3.5},\tilde{t}^{3.5}}$ in H, contradicting Claim 5.4.1. ◁

Claim 5.4.3. $|E(H)| \leq \tilde{t}^{3.5} g^{5.1}(\tilde{t}^{3.5})|\mathcal{F}|$.

Proof of the Claim. For each twin class $T \subseteq \mathcal{A}_I$ of H, fix a representative $C_T \in T$, and let $\mathcal{A}_I^\star := \{C_T : T \subseteq \mathcal{A}_I \text{ is a twin class of } H\}$. Let $H^\star := H[\mathcal{A}_I^\star \cup \mathcal{F}]$. As an induced subgraph of H, $H^\star$ is also bipartite with $\mathcal{A}_I^\star, \mathcal{F}$ forming the two parts of the bipartition. Furthermore, it contains no $K_{\tilde{t}^{3.5},\tilde{t}^{3.5}}$ induced minor by Claim 5.4.1. Since by definition no two vertices of $\mathcal{A}_I^\star$ are false-twins, we may apply Proposition 5.1 to $H^\star$ with $A := \mathcal{F}$ and $B := \mathcal{A}_I^\star$ and conclude that $|E(H^\star)| \leq g^{5.1}(\tilde{t}^{3.5})|\mathcal{F}|$. Finally, since every vertex of $\mathcal{A}_I$ has the same neighborhood in H as the representative of its twin class and since every such twin class has cardinality less than $\tilde{t}^{3.5}$ by Claim 5.4.2, we have

$$|E(H)| \;=\; \sum_{C\in\mathcal{A}_I} \deg_H(C) \;=\; \sum_{C_T\in\mathcal{A}_I^\star} |T|\cdot\deg_H(C_T) \;\leq\; \tilde{t}^{3.5}\sum_{T}\deg_H(C_T) \;\leq\; \tilde{t}^{3.5}g^{5.1}(\tilde{t}^{3.5})|\mathcal{F}|.$$

◁

Now, since $|I| \geq \ell'/2$, we conclude that there exists $i \in I$ such that:

$$\begin{aligned}
\big|\{F\in\mathcal{F} : \exists C\in\mathcal{A}_i,\ V(F)\cap V(C)\neq\varnothing\}\big| \;&\leq\; \frac{1}{|I|}\sum_{j\in I}\big|\{F\in\mathcal{F} : \exists C\in\mathcal{A}_j,\ V(F)\cap V(C)\neq\varnothing\}\big| \\
&\leq\; \frac{1}{|I|}\sum_{C\in\mathcal{A}_I}\deg_H(C) \;=\; \frac{|E(H)|}{|I|} \\
&\leq\; \frac{\tilde{t}^{3.5}g^{5.1}(\tilde{t}^{3.5})|\mathcal{F}|}{\ell'/2} \;\leq\; \frac{2\tilde{t}^{3.5}g^{5.1}(\tilde{t}^{3.5})\ell k}{\ell'} \\
&\leq\; 16\,\tilde{t}^{3.5}\,g^{5.1}(\tilde{t}^{3.5})\,\gamma k \;=\; \lambda_2.
\end{aligned}$$

Here, the second inequality holds since every $F \in \mathcal{F}$ with $V(F) \cap C \neq \emptyset$ for some $C \in \mathcal{A}_i$ touches C and is hence adjacent to it in H. The third inequality follows from Claim 5.4.3 together with $|I| \geq \ell'/2$; the next inequality holds since the (A, B)-packing $(D, \{(S_j, C_j) : j \in [\ell]\})$ is k-thick,

so that each $\mathcal{F}_j$ has at most k elements and $|\mathcal{F}| = \sum_{j=1}^{\ell} |\mathcal{F}_j| \leq \ell k$. The final inequality uses $\ell/\ell' \leq 8\gamma$. □

Combining Lemma 5.3 and Lemma 5.4 immediately implies the main theorem, which we restate below for convenience.

Theorem 5.2. *Let t be a positive integer and let $G \in \mathcal{C}_t$ be a graph on n vertices. Let k, γ, ℓ be positive integers with $\gamma \geq \tilde{t}^{3.5}$ and $\ell \geq 8\gamma$, let $A, B \subseteq V(G)$, and let Π be a k-thick (A,B)-packing of length ℓ in G. Set*

$$\lambda_1(\gamma, n, \ell) \;:=\; 16\gamma n/\ell \qquad \text{and} \qquad \lambda_2(\gamma, t, k) \;:=\; 16\,\gamma\,\tilde{t}^{3.5}\,g^{5.1}(\tilde{t}^{3.5})\,k.$$

Then there exists a strong-barrier corresponding to Π in G that is $(\lambda_1(\gamma, n, \ell),\ \lambda_2(\gamma, t, k))$-light and γ-broad.

Remark. *For positive integers γ, t, n, ℓ, k, define $\lambda_1^{5.2}(\gamma, n, \ell) := \lfloor 16\gamma n/\ell \rfloor$, and define $\lambda_2^{5.2}(\gamma, t, k)$ by the formula for $\lambda_2(\gamma, t, k)$ in Theorem 5.2. We remark that since the number of vertices in the active components is an integer, the conclusion of Theorem 5.2 holds with λ_1 replaced by $\lambda_1^{5.2}$ under the same hypotheses.*

6 Separating Slim Subsets.

For this section, fix $\gamma := \max\{p^{3.7}, \tilde{t}^{3.5}\}$. Let $T_t^{\omega}(n)$ denote the maximum over all graphs $G \in \mathcal{C}_t$ with $|V(G)| \leq n$ and $\omega(G) \leq \omega$, and over $q^{3.7}$-slim subsets $A, B \subseteq V(G)$, of the minimum size of an A–B separator in G. If the maximum is taken over an empty family, we set $T_t^{\omega}(n) = 0$. Consequently, the bounds on $T_t^{\omega}(n)$ are immediate for $t = 1$, so throughout the proofs in this section we assume $t \geq 2$. Now we state the main theorem of this section and provide a brief overview of the proof right below it.

Theorem 6.1. *For every positive integer t, there exists a positive integer $\beta(t)$ such that, for every positive integer ω and n,*

$$T_t^{\omega}(n) \;\leq\; \omega^{\mu^{4.11}} \cdot 2^{\beta(t)\log^{1-\epsilon^{4.6}/2} n}.$$

Proof Overview: The proof bounds $T_t^{\omega}(n)$ by a recursion, which we establish as follows. Given $G \in \mathcal{C}_t$ and $q^{3.7}$-slim sets $A, B \subseteq V(G)$, we invoke Lemma 4.11, which either produces a small A–B separator directly, in which case we are done, or produces a thick (A,B)-packing $\Pi = (D, \{(S_i, C_i)\}_{i\in[\ell]})$ of the desired length ℓ. In the latter case we apply Theorem 5.2 to Π, obtaining a strong-barrier (L, Z, R) that is both γ-broad and $(\lambda_1^{5.2}, \lambda_2^{5.2})$-light. Let Z' denote the set of vertices belonging to active components of the barrier. As before, let $D' := D \cup \bigcup_{i=1}^{\ell} C_i$ and $G' := G - D'$. For each fragment F, write $F' := V(F) \cap Z'$. For each pair of distinct fragments F_1, F_2 such that (F_1', F_2') is a $q^{3.7}$-slim pair in $G'[Z']$, choose a minimum F_1'–F_2' separator in $G'[Z']$. Let Ψ be the union of all these separators. In Claim 6.2.1, we use γ-broadness and strong $(p^{3.7}, q^{3.7})$-slimness to show that $D' \cup \Psi$ is an A–B separator in G. Since each of the smaller separators is computed inside a smaller instance of the same problem on the vertex set Z', this yields the recursive inequality of Lemma 6.2, bounding $T_t^{\omega}(n)$ in terms of $T_t^{\omega}(\lambda_1^{5.2})$.

The rest of the section is devoted to solving this recursion. Lemma 6.3 unrolls the recursion of Lemma 6.2 into a closed-form bound on $T_t^{\omega}(n)$, expressed in terms of the depth of the recursion $D_\ell(n)$ needed to reduce n below the base case threshold. Lemma 6.4 then fixes a specific choice of $\ell = \ell_t(n)$ and bounds $D_\ell(n)$ appropriately. Theorem 6.1 then follows by plugging in the bounds on $c^{4.11}$ and $\lambda_2^{5.2}$, together with the bound on $D_\ell(n)$ from Lemma 6.4, into Lemma 6.3.

We remark that throughout this section, the functions ℓ_t take values in the positive integers. Now we move on to the formal statements and proofs.

Lemma 6.2. *Let t, ω be positive integers, and let $q := q^{3.7}$. Let $\ell_t(n)$ be a non-decreasing function on the positive integers with $\ell_t(n) = o(n)$ such that there exists a positive integer n satisfying $\ell_t(n) > 256\gamma^2$. Let $n_0 = n_0(t)$ be the smallest positive integer such that $256\gamma^2 < \ell_t(n) \leq n$ for every integer $n \geq n_0$.*

Then $T_t^\omega(n)$ obeys the recursive inequality

$$T_t^\omega(n) \leq \begin{cases} n_0, & n < n_0, \\ \min\left\{n,\ c^{4.11}(t,q,\ell_t(n),n,\omega) + \lambda_2^{5.2}(\gamma,t,\tau^{4.11}(q,t,n))^2 \cdot T_t^\omega\left(\lambda_1^{5.2}(\gamma,n,\ell_t(n))\right)\right\}, & \text{otherwise.} \end{cases}$$

Proof. We prove this by induction on n. For the base case observe that if $n < n_0$, then the inequality holds trivially. Now, fix a positive integer $n \geq n_0$ and assume that the statement holds for all strictly smaller positive integers. Let $G \in \mathcal{C}_t$ and $A, B \subseteq V(G)$ be the graph and the pair of vertex subsets of G that realize $T_t^\omega(n)$.

Since adding isolated vertices outside $A \cup B$ preserves membership in $\mathcal{C}_t$, the clique bound, the q-slimness of (A, B), and the minimum size of an A–B separator, we may assume that $|V(G)| = n$. Since it is clear that $T_t^\omega(n) \leq n$, we focus on the second term. Let $\ell = \ell_t(n)$. By Lemma 4.11 applied to the tuple (G, A, B, q, ℓ), G either has an (A, B)-separator of size at most $c^{4.11}(t, q, \ell, n, \omega)$, or a $\tau^{4.11}(q, t, n)$-thick (A, B)-packing Π in G of length ℓ and cost at most $c^{4.11}(t, q, \ell, n, \omega)$. Assume the latter since in the former case, we are done. Now, as $\ell \geq \ell_t(n_0) \geq 8\gamma$ and as $\gamma \geq \tilde{t}^{3.5}$, we may apply Theorem 5.2 to obtain a strong-barrier (L, Z, R) corresponding to Π which is γ-broad and $(\lambda_1^{5.2}(\gamma, n, \ell), \lambda_2^{5.2}(\gamma, t, \tau^{4.11}(q, t, n)))$-light. Let $Z' := \bigcup_{C \in \mathcal{A}} V(C)$, where $\mathcal{A}$ denotes the active components of (L, Z, R). Let $\mathcal{F}$ denote the fragments of Π and let $\mathcal{F}' := \{F \in \mathcal{F} : V(F) \cap Z' \neq \emptyset\}$. Denote $D' := D \cup \bigcup_{i=1}^{\ell} C_i$, $G' := G - D'$. We consider the graph $G'[Z']$ define $F' := V(F) \cap Z'$ for each $F \in \mathcal{F}'$. Let

$$\hat{\mathcal{F}} := \{\{F_i, F_j\} : F_i, F_j \in \mathcal{F}',\ F_i \neq F_j \text{ and the pair } (F_i', F_j') \text{ is } q\text{-slim } G'[Z']\}.$$

Let $\Psi := \bigcup_{\{F_i, F_j\} \in \hat{\mathcal{F}}} \Psi_{ij}$, where, for each $\{F_i, F_j\} \in \hat{\mathcal{F}}$, Ψ_{ij} denotes the minimum F_i'–F_j' separator in $G'[Z']$.

Claim 6.2.1. *$D' \cup \Psi$ is an L–R separator in G.*

Proof of the Claim. To prove the claim, we only need to show that Ψ is an L–R separator in G'. Suppose not; then $G' - \Psi$ contains an L–R path P. By definition of active components, there must exist a subpath P' of P, with an endpoint in each of L and R, whose interior is contained in Z'. Then since (L, Z, R) is γ-broad, the interior of P' intersects at least $p^{3.7} \leq \gamma$ distinct fragments $F_1, \ldots, F_{p^{3.7}} \in \mathcal{F}'$ of Π. Let $\tilde{G} := G\left[Z' \cup \bigcup_{i=1}^{p^{3.7}} V(F_i)\right]$. Since $\mathcal{C}_t$ is hereditary, $\tilde{G} \in \mathcal{C}_t$. Furthermore, the sets in $\{V(F_1), \ldots, V(F_{p^{3.7}})\}$ pairwise do not touch, and are connected in $\tilde{G}$. Since $\tilde{G}$ is strongly-$(p^{3.7}, q^{3.7})$-slim by Lemma 3.7 and since $q = q^{3.7}$, there exists a pair $V(F_i), V(F_j)$ that is q-slim in $\tilde{G}$. Since $G'[Z']$ is an induced subgraph of $\tilde{G}$, each F_i'–F_j' path in $G'[Z']$ is also a $V(F_i)$–$V(F_j)$ path in $\tilde{G}$, so the pair F_i', F_j' is q-slim in $G'[Z']$. But then Ψ separates F_i' from F_j' in $G'[Z']$, contradicting that the interior of P' is a path in $G' - \Psi$ that intersects both F_i' and F_j'. ◁

Now, from Claim 6.2.1 since $A \setminus D' \subseteq L$ and $B \setminus D' \subseteq R$ it follows that,

$$\begin{aligned} T_t^{\omega}(n) &\leq |D'| + |\Psi| \\ &\leq c^{4.11}(t,q,\ell,n,\omega) + \sum_{\{F_i,F_j\}\in\hat{\mathcal{F}}} |\Psi_{ij}| \\ &\leq c^{4.11}(t,q,\ell,n,\omega) + \lambda_2^{5.2}(\gamma,t,\tau^{4.11}(q,t,n))^2 \cdot T_t^{\omega}\big(\lambda_1^{5.2}(\gamma,n,\ell)\big). \end{aligned}$$

The last inequality uses that (L,Z,R) is $\big(\lambda_1^{5.2}(\gamma,n,\ell), \lambda_2^{5.2}(\gamma,t,\tau^{4.11}(q,t,n))\big)$-light, which gives two bounds. First, $|\hat{\mathcal{F}}| \leq \lambda_2^{5.2}(\gamma,t,\tau^{4.11}(q,t,n))^2$, since $\hat{\mathcal{F}} \subseteq \binom{\mathcal{F}'}{2}$. Second, for every $\{F_i,F_j\} \in \hat{\mathcal{F}}$, $|\Psi_{ij}| \leq T_t^{\omega}\big(\lambda_1^{5.2}(\gamma,n,\ell)\big)$, since $G'[Z'] \in \mathcal{C}_t$ with $|Z'| \leq \lambda_1^{5.2}(\gamma,n,\ell)$ and $\omega(G'[Z']) \leq \omega$, and F_i', F_j' form a q-slim pair. □

We remark that the recursion in Lemma 6.2 is well defined, since $16\gamma \leq \lambda_1^{5.2}(\gamma,n,\ell_t(n)) < n$ whenever $n \geq n_0$. Now, let t be a positive integer, and let $\ell_t(n)$ be a non-decreasing function on the positive integers with $\ell_t(n) = o(n)$ such that there exists a positive integer n satisfying $\ell_t(n) > 256\gamma^2$. Let n_0 be as in Lemma 6.2 for this choice of ℓ_t. We define a function D_ℓ recursively as follows:

$$D_\ell(n) = \begin{cases} 0 & \text{if } n < n_0, \\ D_\ell(\lambda_1^{5.2}(\gamma,n,\ell_t(n))) + 1 & \text{otherwise.} \end{cases}$$

Note that this is well defined, since $\lambda_1^{5.2}(\gamma,n,\ell_t(n)) < n$ whenever $\ell > 16\gamma$.

Lemma 6.3. *Let t,ω be positive integers, and let $q := q^{3.7}$. Let $\ell_t(n)$ be a non-decreasing function on the positive integers with $\ell_t(n) = o(n)$ such that there exists a positive integer n satisfying $\ell_t(n) > 256\gamma^2$. Let n_0 be as in Lemma 6.2 for this choice of ℓ_t. Then $T_t^{\omega}(n)$ obeys the inequality*

$$T_t^{\omega}(n) \leq (n_0+1) \cdot c^{4.11}(t,q,\ell_t(n),n,\omega) \cdot \lambda_2^{5.2}(\gamma,t,\tau^{4.11}(q,t,n))^{2D_{\ell}(n)}.$$

Proof. For ease of exposition, we write $\ell = \ell_t(n)$, $c(n) := c^{4.11}(t,q,\ell,n,\omega)$, $\lambda_1(n) := \lambda_1^{5.2}(\gamma,n,\ell)$, and $\lambda_2(n) := \lambda_2^{5.2}(\gamma,t,\tau^{4.11}(q,t,n))$. To prove the lemma, for each positive integer N we define a function on the positive integers by

$$H_N(n) = \begin{cases} n_0, & \text{if } n < n_0, \\ c(N) + \lambda_2(N)^2 \cdot H_N\big(\lambda_1(n)\big), & \text{otherwise.} \end{cases}$$

Since the functions c and λ_2 are non-decreasing in n, $T_t^{\omega}(n) \leq H_N(n)$ for every positive integer N and every $n \in [N]$. Hence, to prove the lemma, it suffices to upper bound $H_N(N)$. To this end, we claim that

$$H_N(n) \leq (n_0+1) \cdot c(N) \cdot \lambda_2(N)^{2D_\ell(n)} - \frac{c(N)}{\lambda_2(N)^2 - 1}$$

for all positive integers n, N. The right-hand side is well defined since $\lambda_2(N) > 1$ for every N. We fix an arbitrary positive integer N and prove the claim for that N by induction on n. For the base case, observe that when $n < n_0$, since $\lambda_2(N) \geq 2$ and $c(N) \geq 1$, we have,

$$\begin{aligned} (n_0+1) \cdot c(N) \cdot \lambda_2(N)^{2D_\ell(n)} - \frac{c(N)}{\lambda_2(N)^2-1} &\geq (n_0+1) \cdot c(N) - c(N) \\ &\geq n_0 = H_N(n). \end{aligned}$$

For the inductive step, fix a positive integer $n \geq n_0$ and assume that the statement holds for all positive integers smaller than n. We have,

$$\begin{aligned}
H_N(n) &= c(N) + \lambda_2(N)^2 \cdot H_N\big(\lambda_1(n)\big) \\
&\leq c(N) + \lambda_2(N)^2 \cdot \left((n_0+1) \cdot c(N) \cdot \lambda_2(N)^{2D_\ell(\lambda_1(n))} - \frac{c(N)}{\lambda_2(N)^2 - 1} \right) \\
&\leq c(N) + (n_0+1) \cdot c(N) \cdot \lambda_2(N)^{2(D_\ell(\lambda_1(n))+1)} - c(N) \cdot \frac{\lambda_2(N)^2}{\lambda_2(N)^2 - 1} \\
&\leq (n_0+1) \cdot c(N) \cdot \lambda_2(N)^{2D_\ell(n)} - c(N) \left(\frac{\lambda_2(N)^2}{\lambda_2(N)^2 - 1} - 1 \right) \\
&\leq (n_0+1) \cdot c(N) \cdot \lambda_2(N)^{2D_\ell(n)} - \frac{c(N)}{\lambda_2(N)^2 - 1}.
\end{aligned}$$

This proves our claim and consequently the lemma. □

Lemma 6.4. *Let t be a positive integer, and let $\ell = \ell_t(n) := \left\lceil 2^{d^{4.6} \log^{1-\frac{\epsilon^{4.6}}{2}} n} \right\rceil$. Then $D_\ell(n)$ is well-defined and at most $\frac{4}{d^{4.6}\epsilon^{4.6}} \log^{\epsilon^{4.6}/2} n$.*

Proof. Since $O(\log^{1-\epsilon^{4.6}/2} n) \subseteq o(\log n)$, we have $\ell_t(n) = o(n)$, and since $\ell_t(n) \to \infty$ as $n \to \infty$, there exists a positive integer n such that $\ell_t(n) > 256\gamma^2$. Consequently $D_\ell(n)$ is well-defined. Let n_0 be as in Lemma 6.2 for this choice of ℓ_t. For each positive integer n, write $\lambda_1(n) := \lambda_1^{5.2}(\gamma, n, \ell_t(n)) = \lfloor 16\gamma n/\ell_t(n) \rfloor$. By definition of n_0 and monotonicity of ℓ_t, for every $n \geq n_0$ we have $\log(\ell_t(n)) \geq \log(\ell_t(n_0)) > 2\log(16\gamma)$, and hence

$$\log n - \log \lambda_1(n) \;\geq\; \log n - (\log(16\gamma) + \log n - \log(\ell_t(n))) \;=\; \log(\ell_t(n)) - \log(16\gamma) \;\geq\; \tfrac{1}{2}\log(\ell_t(n)).$$

We show by strong induction on n that $D_\ell(n) \leq \frac{4}{d^{4.6}\epsilon^{4.6}} \log^{\epsilon^{4.6}/2} n$ for every positive integer n. For the base case, observe that when $n < n_0$, we have $D_\ell(n) = 0 \leq \frac{4}{d^{4.6}\epsilon^{4.6}} \log^{\epsilon^{4.6}/2} n$, as required. For the inductive step, fix a positive integer $n \geq n_0$ and assume that the statement holds for all positive integers smaller than n. Now, we get the following inequality,

$$\begin{aligned}
\log^{\epsilon^{4.6}/2} n - \log^{\epsilon^{4.6}/2} \lambda_1(n) &\geq \tfrac{\epsilon^{4.6}}{2} \log^{\epsilon^{4.6}/2-1} n \cdot \big(\log n - \log \lambda_1(n)\big) \\
&\geq \tfrac{\epsilon^{4.6}}{2} \log^{\epsilon^{4.6}/2-1} n \cdot \tfrac{1}{2} \log(\ell_t(n)) \\
&\geq \tfrac{\epsilon^{4.6}}{2} \log^{\epsilon^{4.6}/2-1} n \cdot \tfrac{1}{2} d^{4.6} \log^{1-\epsilon^{4.6}/2} n \\
&= \tfrac{d^{4.6}\epsilon^{4.6}}{4}.
\end{aligned}$$

Here the first inequality applies the mean value theorem to the function $x \mapsto x^{\epsilon^{4.6}/2}$ on $[\log \lambda_1(n), \log n]$: since the derivative $\frac{\epsilon^{4.6}}{2} x^{\epsilon^{4.6}/2-1}$ is decreasing in x as $\epsilon^{4.6}/2 - 1 < 0$, it is minimized at $x = \log n$, giving the lower bound $\frac{\epsilon^{4.6}}{2} \log^{\epsilon^{4.6}/2-1} n \cdot (\log n - \log \lambda_1(n))$. The second uses $n \geq n_0$ combined with our earlier observation, and the last one substitutes $\log(\ell_t(n)) \geq d^{4.6} \log^{1-\epsilon^{4.6}/2} n$. Combining this with the definition of D_ℓ and the inductive assumption yields the following.

$$\begin{aligned}
D_\ell(n) &= D_\ell(\lambda_1(n)) + 1 \\
&\leq \tfrac{4}{d^{4.6}\epsilon^{4.6}} \log^{\epsilon^{4.6}/2}(\lambda_1(n)) + 1 \\
&\leq \tfrac{4}{d^{4.6}\epsilon^{4.6}} \left(\log^{\epsilon^{4.6}/2} n - \tfrac{d^{4.6}\epsilon^{4.6}}{4} \right) + 1 \\
&= \tfrac{4}{d^{4.6}\epsilon^{4.6}} \log^{\epsilon^{4.6}/2} n.
\end{aligned}$$

This completes the induction and thereby the proof of the lemma. □

Now we are ready to prove the main theorem of this section which we restate here for convenience.

Theorem 6.1. *For every positive integer t, there exists a positive integer $\beta(t)$ such that, for every positive integer ω and n,*

$$T_t^{\omega}(n) \quad \leq \quad \omega^{\mu^{4.11}} \cdot 2^{\beta(t)\log^{1-\epsilon^{4.6}/2} n}.$$

Proof. Let us assume without loss of generality that $n \geq 2$, since otherwise the theorem holds trivially. Fix t, let ω be a positive integer, set $q := q^{3.7}$, and let $\ell = \ell_t(n) := \left\lceil 2^{d^{4.6}\log^{1-\epsilon^{4.6}/2} n} \right\rceil$. Observe that $\ell_t(n) \to \infty$ as $n \to \infty$ and let n_0 be as in Lemma 6.2. By Lemma 6.3,

$$T_t^{\omega}(n) \quad \leq \quad (n_0+1) \cdot c^{4.11}(t,q,\ell,n,\omega) \cdot \lambda_2^{5.2}(\gamma, t, \tau^{4.11}(q,t,n))^{2D_\ell(n)}. \tag{†}$$

We begin by upper bounding $c^{4.11}(t,q,\ell,n,\omega)$:

$$\begin{aligned}
c^{4.11}(t,q,\ell,n,\omega) &= \Big(2h\cdot(\omega+1)\cdot\tau^{4.11}(q,t,n)\Big)^{\mu} \\
&= \Big(2\cdot\big(\ell\, qk^{4.6}(\lceil\log n\rceil+1)\big)\cdot qk^{4.6}\cdot(\omega+1)\Big)^{\mu} \\
&\leq c_1\cdot\omega^{\mu^{4.11}}\cdot\big((k^{4.6})^2\cdot\ell\cdot(\lceil\log n\rceil+1)\big)^{\mu^{4.11}} \\
&\leq \omega^{\mu^{4.11}}\cdot 2^{\log c_1 + \mu^{4.11}(2d^{4.6}\log^{1-\epsilon^{4.6}} n + d^{4.6}\log^{1-\epsilon^{4.6}/2} n + c_2\log(1+\log n))} \\
&\leq \omega^{\mu^{4.11}}\cdot 2^{c_3\,\log^{1-\epsilon^{4.6}/2} n} \qquad (1)
\end{aligned}$$

Here c_1, c_2 and c_3 are suitable constants that depend only on t. The last inequality uses the fact that $O(\log\log n) \subseteq o(\log^{1-\epsilon^{4.6}/2} n)$ and that $n \geq 2$. The rest of the transitions only involve substituting definitions or performing simplifications. Next, since $\gamma, \tilde{t}^{3.5}, g^{5.1}(\tilde{t}^{3.5})$ and q depend only on t and since $D_\ell(n) \leq \frac{4}{d^{4.6}\epsilon^{4.6}}\log^{\epsilon^{4.6}/2} n$, we have the following bound:

$$\begin{aligned}
\lambda_2^{5.2}(\gamma,t,\tau^{4.11}(q,t,n))^{2D_\ell(n)} &= \big(16\gamma\,\tilde{t}^{3.5}\,g^{5.1}(\tilde{t}^{3.5})\cdot qk^{4.6}\big)^{2D_\ell(n)} \\
&\leq \big(c_4\,2^{d^{4.6}\log^{1-\epsilon^{4.6}} n}\big)^{\frac{8}{d^{4.6}\epsilon^{4.6}}\log^{\epsilon^{4.6}/2} n} \\
&\leq 2^{c_5\,\log^{1-\epsilon^{4.6}/2} n}. \qquad (2)
\end{aligned}$$

Here c_4 and c_5 are suitable constants that depend only on t. Combining (1) and (2) via (†) gives

$$\begin{aligned}
T_t^{\omega}(n) &\leq (n_0+1)\cdot\omega^{\mu^{4.11}}\cdot 2^{c_3\,\log^{1-\epsilon^{4.6}/2} n}\cdot 2^{c_5\,\log^{1-\epsilon^{4.6}/2} n} \\
&\leq \omega^{\mu^{4.11}}\cdot 2^{\log(n_0+1)+(c_3+c_5)\log^{1-\epsilon^{4.6}/2} n} \\
&\leq \omega^{\mu^{4.11}}\cdot 2^{\beta(t)\log^{1-\epsilon^{4.6}/2} n}.
\end{aligned}$$

Here $\beta(t)$ is a suitably chosen constant that depends only on t. This completes the proof of the theorem. □

7 Proof of the Main Theorem.

In this section we complete the proof of Theorem 1.2, the main graph-theoretic result of the paper. We first prove Lemma 7.1, which establishes the existence of a $(Y, \frac{1}{2})$-balanced separator for graphs in $\mathcal{C}_t$, whose size depends on $\omega(G)$, via an argument similar to that of Lemma 6.2. We then use the results of Chudnovsky, E S, and Lokshtanov [20] to complete the proof of the theorem, bounding the tree-independence number of graphs in $\mathcal{C}_t$.

Lemma 7.1. *For every positive integer t, there exist positive integers $\nu(t), \mu(t)$ such that, for every graph $G \in \mathcal{C}_t$ on n vertices and every vertex subset $Y \subseteq V(G)$, G has a $(Y, \frac{1}{2})$-balanced separator of size at most $\omega(G)^{\mu(t)} \cdot 2^{\nu(t)\log^{1-\epsilon^{4.6}/2} n}$.*

Proof. Let us assume without loss of generality that $n \geq 2$, since otherwise the lemma holds trivially. Write $p := p^{3.7}$, $q := q^{3.7}$, and $\omega := \omega(G)$. By Lemma 4.8 applied to (G, Y) with $\ell = p$, G either has a $(Y, \frac{1}{2})$-balanced-separator of size at most $\kappa^{4.8}(t, p, n, \omega)$, or a $(Y, \frac{1}{2})$-packing $(D, \{(S_i, C_i) : i \in [p]\})$ of length p that is $4k^{4.6}$-thick and $\kappa^{4.8}(t, p, n, \omega)$-costly. Now, we prove an upper bound on $\kappa^{4.8}(t, p, n, \omega)$.

$$\begin{aligned}
\kappa^{4.8}(t,p,n,\omega) &= \left(2h' \cdot (\omega+1) \cdot 4k^{4.6}\right)^{\mu^{4.8}} \\
&= \left(2 \cdot \left(p\, 4k^{4.6}(\lceil \log n \rceil + 1)\right) \cdot 4k^{4.6} \cdot (\omega+1)\right)^{\mu^{4.8}} \\
&\leq c_1 \cdot \omega^{\mu^{4.8}} \cdot \left(k^{4.6}2 \cdot (\lceil \log n \rceil + 1)\right)^{\mu^{4.8}} \\
&\leq \omega^{\mu^{4.8}} \cdot 2^{\log c_1 + \mu^{4.8}(2d^{4.6}\log^{1-\epsilon^{4.6}} n + c_2 \log(1+\lceil \log n \rceil))} \\
&\leq \omega^{\mu^{4.8}} \cdot 2^{c_3 \log^{1-\epsilon^{4.6}} n} \qquad (1)
\end{aligned}$$

Here c_1, c_2, c_3 are suitable constants that depend only on t. The last inequality uses the fact that $O(\log\log n) \subseteq o(\log^{1-\epsilon^{4.6}} n)$. In view of (1), we may assume that the latter alternative in Lemma 4.8 holds. Set $D' := D \cup \bigcup_{i=1}^{p} C_i$, $G' := G - D'$, and $Y' = Y \cap V(G')$ so that each S_i is a $(Y', \frac{1}{2})$-balanced separator in G'. For every $i \in [p]$, let $\mathcal{F}_i$ denote the set of connected components of $G[S_i \cup C_i]$, which, with a slight abuse of notation, we call *fragments*. Since $(D, \{(S_i, C_i) : i \in [p]\})$ is $4k^{4.6}$-thick, we have $|\mathcal{F}_i| \leq 4k^{4.6}$ for every $i \in [p]$. Let $\mathcal{F} := \bigcup_{i\in[p]} \mathcal{F}_i$ and let $F' := F \cap V(G')$ for every $F \in \mathcal{F}$. Let

$$\hat{\mathcal{F}} \quad := \quad \{\{F_i, F_j\} \ : \ F_i, F_j \in \mathcal{F}, (F_i', F_j') \text{ is a } q\text{-slim pair in } G'\}.$$

For every $\{F_i, F_j\} \in \hat{\mathcal{F}}$, let Ψ_{ij} denote a minimum F_i'–F_j' separator in G', which by definition has size at most $T_t^{\omega}(n)$. Finally, let $\Psi := \bigcup_{\{F_i,F_j\}\in\hat{\mathcal{F}}} \Psi_{ij}$.

Claim 7.1.1. *$D' \cup \Psi$ is a $(Y, \frac{1}{2})$-balanced separator in G.*

Proof of the Claim. To prove the claim, we only need to show that Ψ is a $(Y', \frac{1}{2})$-balanced separator in G'. Suppose not; then $G' - \Psi$ contains a component K with $|Y' \cap V(K)| > \frac{1}{2}|Y'|$. Fix $i \in [p]$. Since S_i is a $(Y', \frac{1}{2})$-balanced separator in G', every component of $G' - S_i$ contains at most $\frac{1}{2}|Y'|$ vertices of Y'. If $V(K) \cap S_i = \emptyset$, then K is contained in a component of $G' - S_i$ containing more than $\frac{1}{2}|Y'|$ vertices of Y', a contradiction. Thus $V(K) \cap S_i \neq \emptyset$. Let $F_i \in \mathcal{F}_i$ be such that $K \cap F_i'$ is non-empty.

Define $\tilde{G} := G\big[V(G') \cup \bigcup_{i=1}^{p} V(F_i)\big]$. Since $\mathcal{C}_t$ is hereditary, $\tilde{G} \in \mathcal{C}_t$. Furthermore, the sets in $\{V(F_1), \ldots, V(F_p)\}$ pairwise do not touch and are connected in $\tilde{G}$. Since $\tilde{G}$ is strongly-$(p^{3.7}, q^{3.7})$-slim by Lemma 3.7 and since $q = q^{3.7}$ and $p = p^{3.7}$, there exists a pair $V(F_i), V(F_j)$ that is q-slim in $\tilde{G}$. Since G' is an induced subgraph of $\tilde{G}$, every F_i'–F_j' path in G' is also a $V(F_i)$–$V(F_j)$ path in $\tilde{G}$, so the pair F_i', F_j' is q-slim in G'. But then Ψ separates F_i' from F_j' in G', contradicting that K is a connected component of $G' - \Psi$ that intersects both F_i' and F_j'. ◁

Now, from Claim 7.1.1 it follows that, $D' \cup \Psi$ is a $(Y, \frac{1}{2})$-balanced separator in G of size at most $\kappa^{4.8}(t, p, n, \omega) + (4k^{4.6}p)^2 \cdot T_t^{\omega}(n)$, since there are at most $\sum_{i=1}^{p} |\mathcal{F}_i| \leq 4k^{4.6}p$ fragments in total, each

pair of which contributes a separator of size at most $T_t^\omega(n)$. Combining (1) with the upper bound on $T_t^\omega(n)$ from Theorem 6.1 yields

$$\begin{aligned}
|D' \cup \Psi| &\leq \kappa^{4.8}(t,p,n,\omega) \ + \ (4k^{4.6}p)^2 \cdot T_t^\omega(n) \\
&\leq \omega^{\mu^{4.8}} \cdot 2^{c_3 \log^{1-\epsilon^{4.6}} n} \ + \ (4k^{4.6}p)^2 \cdot \omega^{\mu^{4.11}} \cdot 2^{\beta(t)\log^{1-\epsilon^{4.6}/2} n} \\
&\leq \omega^{\mu^{4.8}} \cdot 2^{c_3 \log^{1-\epsilon^{4.6}} n} \ + \ c_4 \cdot 2^{2d^{4.6}\log^{1-\epsilon^{4.6}} n} \cdot \omega^{\mu^{4.11}} \cdot 2^{\beta(t)\log^{1-\epsilon^{4.6}/2} n} \\
&\leq \omega^{\mu(t)} \cdot 2^{c_3 \log^{1-\epsilon^{4.6}} n + \log c_4 + 2d^{4.6}\log^{1-\epsilon^{4.6}} n + \beta(t)\log^{1-\epsilon^{4.6}/2} n} \\
&\leq \omega^{\mu(t)} \cdot 2^{\nu(t)\log^{1-\epsilon^{4.6}/2} n},
\end{aligned}$$

Here c_4, $\mu(t)$ and $\nu(t)$ are suitable constants that depend only on t, which completes the proof of the lemma. □

To prove the main theorem, we need the following two results by Chudnovsky, E S, and Lokshtanov [20]. First we need some definitions. A family $\mathcal{F}$ of vertex subsets is a *(b,a)-container family* in G, if $\alpha(F) \leq a$ for every element F in $\mathcal{F}$, and for every vertex subset Z satisfying $\alpha(Z) \leq b$, there exists some element F of $\mathcal{F}$ such that $Z \subseteq F$. The *complement* of a graph G is denoted by $\overline{G}$ and is the graph with vertex set $V(G)$ satisfying $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$ for every pair of distinct $u, v \in V(G)$. For a graph G and positive integer k the graph *kG* is the graph obtained by taking k disjoint copies of G and making the copies anti-complete to each other.

Proposition 7.2 ([20]). *There exists an algorithm that takes as input (G, ω, k, b) where ω and k are positive integers, $b \leq \omega$ is a non-negative integer, and G is an n-vertex $\overline{kK_\omega}$-free graph, and outputs a (b,a)-container family $\mathcal{F}$ with $|\mathcal{F}| \leq f$ in time $f \cdot n^{O(1)}$, where*

$$f = (n+1)^{(2\omega\cdot\log(n)+k+b)^{k+b+1}} \qquad \text{and} \qquad a \leq (2\omega \cdot \log(n) + k + b)^{k+b+1}.$$

Given a graph G, a vertex subset $S \subseteq V(G)$, and a family $\mathcal{F}$ of vertex subsets of G satisfying $V(G) \subseteq \bigcup_{F\in\mathcal{F}} F$, we define the *cover number* of S using $\mathcal{F}$, denoted as $\mathrm{cov}_{\mathcal{F}}(S)$, to be the minimum cardinality of a subset $\mathcal{F}' \subseteq \mathcal{F}$ which satisfies $S \subseteq \bigcup_{F\in\mathcal{F}'} F$. Observe that $\mathrm{cov}_{\mathcal{F}}(S) \leq |S|$.

Proposition 7.3 ([20]). *Let G be an n-vertex graph, let f, a, and b be positive integers with $a \geq 2$, and let $\mathcal{F}$ be a (b,a)-container family in G. If $tw_\alpha(G) > 1020000 \cdot \lceil \log(2n) \cdot a^3 \cdot \log(a \cdot f + 4) \cdot f \rceil$, then there exists an induced subgraph $G' \subseteq G$ and an independent set $I \subseteq V(G')$ of size $680000 \cdot \lceil \log(2n) \cdot a^3 \cdot \log(a \cdot f + 4) \cdot f \rceil$ that satisfies the following properties:*

- *Every induced subgraph H of G' with $\alpha(H) \leq b$ satisfies*

$$|V(H)| \quad \leq \quad 42 \cdot b \cdot (\lceil \log 4|\mathcal{F}| \rceil + 680000 \cdot \lceil \log(2n) \cdot a^3 \cdot \log(a \cdot f + 4) \rceil).$$

- *For every $(I, \frac{1}{2})$-balanced separator S in G', we have that $\mathrm{cov}_{\mathcal{F}}(S) \geq f$.*

Now we are ready to prove the main structural theorem, which we restate for convenience.

Theorem 1.2. *For every positive integer t, there exists a positive integer $\nu'(t)$ and a rational number $\epsilon(t) \in (0,1)$ such that, for every graph $G \in \mathcal{C}_t$ on n vertices,*

$$tw_\alpha(G) \quad \leq \quad 2^{\nu'(t)\log^{1-\epsilon(t)} n}.$$

Proof. Let us assume without loss of generality that $n \geq 2$, since otherwise the theorem holds trivially. Let $G \in \mathcal{C}_t$ be a graph on n vertices. Set $a := \lceil (2t \cdot \log n + 3)^4 \rceil$ and let c_1 be the smallest constant such that $a \leq c_1 \log^4(n+1)$ for every integer $n \geq 2$, and observe that c_1 depends only on t. Since $G \in \mathcal{C}_t$, we have that G is $\overline{2K_t}$-free, and so Proposition 7.2 applied to the tuple $(G, t, 2, 1)$ implies the existence of a $(1, a)$-container family $\mathcal{F}$ in G satisfying $|\mathcal{F}| \leq (n+1)^a$. Define $\hat{\nu}(t)$ to be the smallest positive integer z such that, for every $n \geq 2$,

$$(\omega_n(z))^{\mu(t)} \cdot f_n(\nu(t)) \quad < \quad f_n(z),$$

where $\mu(t)$ and $\nu(t)$ are the constants in Lemma 7.1, and $f_n(z)$ and $\omega_n(z)$ are functions on the set of positive integers, defined as follows:

$$f_n(z) := \left\lceil 2^{z \log^{1-\epsilon^{4.6}/2} n} \right\rceil \text{ and } \omega_n(z) := 42 \cdot (\lceil a \cdot \log(n+1) + 2 \rceil + 680000 \cdot \lceil \log 2n \cdot a^3 \cdot \log(a\, f_n(z) + 4) \rceil)$$

Claim 7.3.1. *$\hat{\nu}(t)$ is well defined.*

Proof of the Claim. We need to show that the set of positive integers z that satisfy $(\omega_n(z))^{\mu(t)} \cdot f_n(\nu(t)) < f_n(z)$ is non-empty. First, observe that since $n \geq 2$,

$$\begin{aligned}
\omega_n(z) &= 42 \cdot \Big(\lceil a \cdot \log(n+1) + 2 \rceil \;+\; 680000 \cdot \lceil \log(2n) \cdot a^3 \cdot \log(a\, f_n(z) + 4) \rceil \Big) \\
&\leq 42 \cdot \lceil c_1 \log^5(n+1) + 2 \rceil \\
&\qquad + \; 42 \cdot 680000 \cdot \left\lceil \log(2n) \cdot c_1^3 \cdot \log^{12}(n+1) \cdot \log\Big(c_1 \log^4(n+1) \cdot \left\lceil 2^{z \log^{1-\epsilon^{4.6}/2} n} \right\rceil + 4 \Big) \right\rceil \\
&\leq c_2 \log^5(n+1) + \; c_3 \log^{13}(n+1) \cdot \big(\log c_1 + 4 \log\log(n+1) + (1 + z \log^{1-\epsilon^{4.6}/2} n) + 2 \big) \\
&\leq c_2 \log^5(n+1) \;+\; c_3 \log^{13}(n+1) \cdot \big(c_4 \cdot z \cdot \log n \big) \\
&\leq c_5 \cdot z \cdot \log^{14} n
\end{aligned}$$

Here c_2, c_3, c_4 and c_5 are suitably chosen constants that depend only on t. Now, using this inequality we obtain the following upper bound:

$$\begin{aligned}
(\omega_n)^{\mu(t)} \cdot f_n(\nu(t)) &\leq \big(c_5 \cdot z \cdot \log^{14} n \big)^{\mu(t)} \cdot \left\lceil 2^{\nu(t) \log^{1-\epsilon^{4.6}/2} n} \right\rceil \\
&\leq 2^{\mu(t)(\log c_5 + 14 \log\log n + \log z) + \nu(t) \log^{1-\epsilon^{4.6}/2} n + 1}
\end{aligned}$$

Since $O(\log\log n) \subseteq o(\log^{1-\epsilon^{4.6}/2} n)$ and since $O(\log z) \subseteq o(z)$, we conclude that there exists a large enough positive integer z such that, for every positive integer $n \geq 2$,

$$\mu(t)\,(\log c_5 \;+\; 14 \log\log n \;+\; \log z) \;+\; \nu(t) \log^{1-\epsilon^{4.6}/2} n + 1 \quad < \quad z \log^{1-\epsilon^{4.6}/2} n.$$

Thus, the set of positive integers z such that, $(\omega_n(z))^{\mu(t)} \cdot f_n(\nu(t)) < f_n(z)$ holds for every positive integer $n \geq 2$, is non-empty. Consequently $\hat{\nu}(t)$ is well defined. $\lhd$

Let us denote $\omega_n(\hat{\nu}(t))$ as ω_n and $f_n(\hat{\nu}(t))$ as f_n. Now, suppose for a contradiction that $tw_\alpha(G) > 1020000 \cdot \lceil \log(2n) \cdot a^3 \cdot \log(a\, f_n + 4) \cdot f_n \rceil$. Applying Proposition 7.3 to the tuple $(G, f_n, a, 1, \mathcal{F})$, we obtain an induced subgraph $G' \subseteq G$ and an independent set $I \subseteq V(G')$, such that every induced subgraph H of G' with $\alpha(H) \leq 1$ satisfies $|V(H)| \leq \omega_n$, and every $(I, \frac{1}{2})$-balanced separator S in G' satisfies $\mathrm{cov}_{\mathcal{F}}(S) \geq f_n$. Since a maximum clique H of G' satisfies $\alpha(H) \leq 1$ and $|V(H)| = \omega(G')$, the former property gives $\omega(G') \leq \omega_n$. Furthermore, $G' \in \mathcal{C}_t$ since $\mathcal{C}_t$ is hereditary. Thus, applying Lemma 7.1 to G' yields an $(I, \frac{1}{2})$-balanced separator S of size at most $\omega(G')^\mu \cdot f_{|V(G')|}(\nu(t))$, which

is at most $\omega_n^\mu \cdot f_n(\nu(t)) < f_n$. As $\mathcal{F}$ is a $(1,a)$-container family, this implies $\mathrm{cov}_{\mathcal{F}}(S) \le |S| < f_n$, which is a contradiction. Thus, we get,

$$\begin{aligned}
tw_\alpha(G) &\le 1020000 \cdot \lceil \log(2n) \cdot a^3 \cdot \log(af_n + 4) \cdot f_n \rceil \\
&\le 1020000 \cdot \left\lceil \log(2n) \cdot c_1^3 \cdot \log^{12}(n+1) \cdot \log\big(c_1 \cdot \log^4(n+1) \cdot \lceil 2^{\hat{\nu}(t)\log^{1-\epsilon^{4.6}/2} n} \rceil + 4\big) \cdot f_n \right\rceil \\
&\le c_2 \cdot \log^{13} n \cdot \Big(\log c_1 + 4 \log\log(n+1) + (\hat{\nu}(t) \log^{1-\epsilon^{4.6}/2} n + 1) + 2 \Big) \cdot f_n \\
&\le c_3 \cdot \log^{14} n \cdot \left\lceil 2^{\hat{\nu}(t)\log^{1-\epsilon^{4.6}/2} n} \right\rceil \\
&\le 2^{\log(c_3) + 14\log\log n + \hat{\nu}(t)\log^{1-\epsilon^{4.6}/2} n + 1}
\end{aligned}$$

Since $O(\log\log n) \subseteq o(\log^{1-\epsilon^{4.6}/2} n)$, we conclude that there exists a large enough positive constant $\nu'(t)$ depending only on t such that, for every $n \ge 2$,

$$\log c_3 \;+\; 14\log\log n \;+\; \hat{\nu}(t)\log^{1-\epsilon^{4.6}/2} n \;+\; 1 \quad\le\quad \nu'(t)\log^{1-\epsilon^{4.6}/2} n$$

Thus, setting $\nu'(t)$ as above and $\epsilon(t) = \epsilon^{4.6}/2$ completes the proof of the theorem. □

8 Tree Independence of Biclique-Induced-Minor-Free Graphs

In this section, we prove Theorem 1.3. Towards the proof, we need a few ingredients. First is the following separator theorem of Korhonen and Lokshtanov [54]. They state and prove it for an arbitrary pattern graph H excluded as an induced minor; we re-state here their separator theorem as it applies to $H = K_{t,t}$. Their proof relies on results of Lee [57] that unfortunately contains an error, but Davies et al. [35] have since proved a version of that result which saves the argument.

Proposition 8.1 ([35, 54]). *There exist a positive integer b_1 and a randomized polynomial-time algorithm that, given a graph G with at least two vertices and a positive integer t, outputs either an induced minor model of $K_{t,t}$ in G or a $(V(G), \frac{2}{3})$-balanced separator of G of size at most $b_1 t^3 \sqrt{|E(G)|}$.*

For the remainder of this section, fix b_1 to be the positive integer whose existence is implied by Proposition 8.1. The next lemma upgrades Proposition 8.1 from a balanced separator to a $(Y, \frac{1}{2})$-balanced separator for a prescribed set Y, at the cost of a $\log|V(G)|$ factor in the size.

Lemma 8.2. *There exists an absolute constant $b_1' > 0$ such that, for every positive integer t, every $K_{t,t}$-induced-minor-free graph G, and every vertex subset $Y \subseteq V(G)$ with $|Y| \ge 2$. the graph G has a $(Y, \frac{1}{2})$-balanced separator of size at most $b_1' t^3 \log|V(G)| \sqrt{|E(G)|}$.*

Proof. Write $n := |V(G)|$ and for ease of exposition let $b = b_1 t^3 \cdot \sqrt{|E(G)|}$. If $E(G) = \emptyset$, then $S = \emptyset$ suffices, so assume that $E(G) \neq \emptyset$. We prove by induction on $|C|$ that, for every nonempty $C \subseteq V(G)$ there is a set $S \subseteq C$ with $|S| \le b \cdot \big(\log_{3/2}|C| + 1\big)$ such that every connected component of $G[C] - S$ contains at most $\frac{1}{2}|Y|$ vertices of Y. The lemma then follows from this statement by taking $C := V(G)$, since

$$b \cdot \big(\log_{3/2} n + 1\big) \;\in\; O(b \log n) \;=\; O\big(t^3 \log n \cdot \sqrt{|E(G)|}\big),$$

which by definition implies the existence of the desired constant b_1'.

First, if no connected component of $G[C]$ contains more than $\frac{1}{2}|Y|$ vertices of Y, then $S := \emptyset$ suffices. Otherwise Proposition 8.1 provides a $(V(C), \frac{2}{3})$-balanced separator S_1 of $G[C]$ with $|S_1| \le b$.

Every connected component of $G[C] - S_1$ has at most $\frac{2}{3}|C|$ vertices, and since these components are pairwise disjoint, at most one of them contains more than $\frac{1}{2}|Y|$ vertices of Y. Now, if there is no such component, then $S := S_1$ suffices, as $|S_1| \leq b \leq b \cdot (\log_{3/2}|C| + 1)$. Otherwise let C' be the unique such component, so $|C'| \leq \frac{2}{3}|C|$ and in particular $|C'| < |C|$. By the induction hypothesis applied to C' there is a set $S_2 \subseteq C'$ with

$$|S_2| \;\leq\; b \cdot (\log_{3/2}|C'| + 1) \;\leq\; b \cdot (\log_{3/2}(\tfrac{2}{3}|C|) + 1) \;=\; b \log_{3/2}|C|$$

such that every component of $G[C'] - S_2$ contains at most $\frac{1}{2}|Y|$ vertices of Y. Set $S := S_1 \cup S_2$. Every connected component of $G[C] - S$ either is a component of $G[C] - S_1$ other than C', or is a component of $G[C'] - S_2$; in both cases it contains at most $\frac{1}{2}|Y|$ vertices of Y. Finally we have

$$|S| \;\leq\; |S_1| + |S_2| \;\leq\; b + b \log_{3/2}|C| \;=\; b \cdot (\log_{3/2}|C| + 1),$$

completing the induction. □

Finally, we use the following degree-boundedness result of Bourneuf, Bucić, Cook, and Davies [11].

Proposition 8.3 ([11]). *For every positive integer t there exist $b_2, d_1 > 0$ such that for every positive integer ω, every graph G that contains no subdivision of $K_{t,t}$ as an induced subgraph and no clique on more than ω vertices has average degree at most $b_2\,\omega^{d_1}$.*

For the rest of this section, for each positive integer t fix positive constants $b_2 = b_2(t)$ and $d_1 = d_1(t)$ as in Proposition 8.3. We are now ready to prove Theorem 1.3, which we re-state here for convenience.

Theorem 1.3. *For every positive integer t there exists a positive integer d such that for every $K_{t,t}$-induced-minor-free graph G, $\mathrm{tw}_\alpha(G) \leq O(\sqrt{n}(\log n)^d)$.*

Proof of Theorem 1.3. Let G be a $K_{t,t}$-induced-minor-free graph on n vertices. We define $a := \lceil (2t \cdot \log n + 3)^4 \rceil$ and let c_1 be the smallest constant such that $a \leq c_1 \log^4(n+1)$ for every positive integer n. Observe that c_1 depends only on t. Since G is $K_{t,t}$-induced-minor-free, we have that G is $\overline{2K_t}$-free, and so Proposition 7.2 applied to the tuple $(G, t, 2, 1)$ implies the existence of a $(1, a)$-container family $\mathcal{F}$ in G satisfying $|\mathcal{F}| \leq (n+1)^a$. Set

$$f_n := \left\lceil \sqrt{n}\,(\log n)^{7d_1+2} \right\rceil \text{ and } \omega_n := 42 \cdot \Big(\lceil a \cdot \log(n+1) + 2 \rceil + 680000 \cdot \lceil \log(2n) \cdot a^3 \cdot \log(a f_n + 4) \rceil \Big).$$

Furthermore, let n_0 be the smallest positive integer such that for all $n \geq n_0$ we have

$$f_n \;\leq\; \sqrt{n}\,(\log n)^{7d_1+2} + 1 \;\leq\; n.$$

Hence, there exists a positive constant c_2, depending only on t, such that for every $n \geq n_0$,

$$\log(a f_n + 4) \;\leq\; \log\big(c_1 n \log^4(n+1) + 4\big) \;\leq\; c_2 \log n.$$

Furthermore for such values of n we also conclude that

$$\begin{aligned}
\omega_n \;&\leq\; 42 \cdot \Big(\lceil a \cdot \log(n+1) + 2 \rceil \;+\; 680000 \cdot \lceil \log(2n) \cdot a^3 \cdot \log(a f_n + 4) \rceil \Big) \\
&\leq\; 42 \cdot \lceil c_1 \log^5(n+1) + 2 \rceil \;+\; 42 \cdot 680000 \cdot \lceil \log(2n) \cdot c_1^3 \log^{12}(n+1) \cdot c_2 \log n \rceil \\
&\leq\; 42 \cdot \big(c_1 \log^5(n+1) + 3 \big) \;+\; 42 \cdot 680000 \cdot \big(c_1^3 c_2 \log(2n) \cdot \log^{12}(n+1) \cdot \log n + 1 \big) \\
&\leq\; c_3 \log^5(n+1) \;+\; c_4 \log^{14}(n+1) \\
&\leq\; c_5 \log^{14} n.
\end{aligned}$$

Here c_3, c_4, c_5 are suitably chosen positive constants that depend only on t. Now, set

$$D \;:=\; b_1' t^3 \sqrt{\frac{b_2 c_5^{d_1}}{2}} \qquad \text{and} \qquad n_0' \;:=\; \max\bigl\{n_0, \bigl\lceil 2^{D+1} \bigr\rceil\bigr\}.$$

We may assume without loss of generality that $n \geq n_0'$, since otherwise the theorem holds trivially by increasing the implicit constant in the O-notation. Now, suppose for a contradiction that for such an n

$$tw_\alpha(G) \;>\; 1020000 \cdot \bigl\lceil \log(2n) \cdot a^3 \cdot \log(af_n + 4) \cdot f_n \bigr\rceil .$$

Applying Proposition 7.3 to the tuple $(G, f_n, a, 1, \mathcal{F})$, we obtain an induced subgraph $G' \subseteq G$ and an independent set $I \subseteq V(G')$, such that every induced subgraph H of G' with $\alpha(H) \leq 1$ satisfies $|V(H)| \leq \omega_n$, and every $(I, \frac{1}{2})$-balanced separator S in G' satisfies $\mathrm{cov}_{\mathcal{F}}(S) \geq f_n$.

Since a maximum clique H of G' satisfies $\alpha(H) \leq 1$ and $|V(H)| = \omega(G')$, the former property gives $\omega(G') \leq \omega_n$. Furthermore, G' is $K_{t,t}$-induced-minor-free since it is an induced subgraph of G. In particular, G' contains no subdivision of $K_{t,t}$ as an induced subgraph, and so Proposition 8.3 implies that the average degree of G' is at most

$$b_2 \omega(G')^{d_1} \;\leq\; b_2 \omega_n^{d_1} \;\leq\; b_2 c_5^{d_1} (\log n)^{14 d_1}.$$

Thus, applying Lemma 8.2 to G' with $Y := I$ yields an $(I, \frac{1}{2})$-balanced separator S of size

$$\begin{aligned}
|S| \;&\leq\; b_1' t^3 \cdot \log |V(G')| \cdot \sqrt{|E(G')|} \\
&\leq\; b_1' t^3 \cdot \log n \cdot \sqrt{\frac{b_2 c_5^{d_1}}{2} \cdot (\log n)^{14 d_1} \cdot n} \\
&\leq\; b_1' t^3 \sqrt{\frac{b_2 c_5^{d_1}}{2}} \cdot \sqrt{n}\, (\log n)^{7 d_1 + 1} \\
&=\; D \cdot \sqrt{n}\, (\log n)^{7 d_1 + 1} \\
&<\; \sqrt{n}\, (\log n)^{7 d_1 + 2} \\
&\leq\; f_n.
\end{aligned}$$

Here the strict inequality follows from $\log n \geq D + 1 > D$, by the choice of n_0'. As $\mathcal{F}$ is a $(1, a)$-container family, this implies $\mathrm{cov}_{\mathcal{F}}(S) \leq |S| < f_n$, which is a contradiction. Thus, we get,

$$\begin{aligned}
tw_\alpha(G) \;&\leq\; 1020000 \cdot \bigl\lceil \log(2n) \cdot a^3 \cdot \log(af_n + 4) \cdot f_n \bigr\rceil \\
&=\; O\bigl(\log n \cdot \log^{12} n \cdot \log n \cdot \sqrt{n}\, (\log n)^{7 d_1 + 2}\bigr) \\
&=\; O\bigl(\sqrt{n}\, (\log n)^{7 d_1 + 16}\bigr).
\end{aligned}$$

Thus, taking a positive integer $d \geq 7 d_1 + 16$ completes the proof of the theorem. □

9 Conclusion

We have shown that for every induced-minor-closed class $\mathcal{H}$ the tree-independence of $\mathcal{H}$ grows linearly in the number of vertices, or on the order of the square root of number of vertices up to polylogarithmic factors, or is sub-polynomial in the number of vertices. The tree-independence is

sub-polynomial whenever $\mathcal{H}$ excludes at least one complete bipartite graph and at least one wall. Our main theorem is a partial resolution of the conjecture of Chudnovsky et al. [20] that for every integer t the graphs that exclude the complete bipartite graph $K_{t,t}$ and the wall $W_{t\times t}$ as an induced minor have poly-logarithmic tree-independence number. A full resolution of this conjecture remains elusive.

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The tool materially affected the exposition across all sections of the paper. The authors have verified the correctness and originality of all content, including all mathematical statements, proofs, and references, and take complete responsibility for the same.

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