Tree-independence number VII. Excluding a star.

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Abstract

We prove that for every fixed integer s and every planar graph H, the class of H-induced-minor-free and $K_{1,s}$-induced-subgraph-free graphs has polylogarithmic tree-independence number. This is a weakening of a conjecture of Dallard, Krnc, Kwon, Milanič, Munaro, Štorgel, and Wiederrecht.

1 Introduction

Tree decompositions and treewidth are among the most influential concepts in structural graph theory. Intuitively, a tree decomposition is a hierarchical decomposition of a graph G into sets called *bags*. If these sets are all small (i.e., G has small treewidth), then G is "tree-like" and thus "simple;" see Section 2 for a formal definition.

Since its birth, the notion of treewidth was closely related to graph minors. (A graph H is a minor of a graph G if it can be obtained from G by deleting vertices and edges, and contracting edges.) This close relation is witnessed by the following landmark result of Robertson and Seymour [24], usually referred to as *Grid Minor Theorem.*

Theorem 1.1 (Robertson, Seymour [24]). *For every planar graph H there exists an integer c_H such that every graph that does not contain a minor isomorphic to H has treewidth at most c_H.*

On the other hand, any class that does not exclude a planar graph contains all planar graphs which have unbounded treewidth. Thus, Theorem 1.1 provides a full characterization of minor-closed classes that have bounded treewidth.

While graphs that exclude some fixed graph as a minor are necessarily sparse, it turns out that tree decompositions can also find application in the study of well-behaved classes of dense graphs. A class of graphs is *hereditary* if it is closed under vertex deletion. Let G and H be graphs. We say that H is an *induced subgraph* of G if it can be obtained from G by removing vertices. If H is not an induced subgraph of G, then G is H-free. We say that H is an *induced minor* of G if H can be obtained from an induced subgraph of G by contracting edges (and repeatedly deleting parallel edges obtained in the process).

In recent years a lot of attention attention was devoted to the study of treewidth of hereditary graph classes. Again, the question is the same: Which substructures should one exclude

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to obtain a class of bounded treewidth? Despite significant progress on this question [2–6, 11], we are still quite far from a full resolution. However, the answer is known if we additionally assume that the maximum vertex degree is bounded. Indeed, Korhonen [22] proved the following analogue of Theorem 1.1, which was earlier conjectured by Aboulker et al. [1].

Theorem 1.2 (Korhonen [22])**.** *For every integer Δ and a planar graph H there exists an integer $c_{\Delta,H}$ such that every graph of maximum degree at most Δ that does not contain an induced minor isomorphic to H has treewidth at most $c_{\Delta,H}$.*

Another way of dealing with dense graphs is to redefine how we measure the quality of a tree decomposition. Instead of saying that a graph is "simple" if it has a tree decomposition where each bag is small, we can instead ask for tree decompositions where every bag induces a subgraph of "simple structure." For example chordal graphs are precisely the ones that admit a tree decomposition where every bag is a clique. This leads to the notion of *tree-independence number*, another graph parameter associated with tree decompositions, introduced independently by Yolov [25] and by Dallard, Milanič, and Štorgel [18]. Intuitively, the tree-independence number of G, is the minimum k such that G has a tree decomposition where no bag contains $k+1$ pairwise non-adjacent vertices. For example, aforementioned chordal graphs are precisely graphs with tree-independence number 1.

Much of the research on tree-independence number revolves around trying to characterize graph families where this parameter is bounded, or at least grows slowly as a function of the size of the graph. In this spirit, Dallard, Krnc, Kwon, Milanič, Munaro, Štorgel, and Wiederrecht [16] suggested the following "dense" analogue of Theorem 1.2. (For integers s, t, by $K_{s,t}$ we denote the complete bipartite graph with sides of a bipartition of size s and t.)

Conjecture 1.3 ([16])**.** *For every integer s and every planar graph H there exists an integer $c_{s,H}$ such that every graph which is H-induced minor-free and $K_{1,s}$-free has tree independence number at most $c_{s,H}$.*

The conjecture has been confirmed only for very restricted cases [8, 13, 16, 21]. In this short note we prove a polylogarithmic version of Conjecture 1.3.

Theorem 1.4. *For every integer s and every planar graph H there exists an constant $c_{s,H}$ such that every n-vertex graph which is H-induced minor-free and $K_{1,s}$-free has tree-independence number at most $\log^{c_{s,H}} n$.*

2 Notation and tools

Graphs. An *independent set* is a subset of vertices of $V(G)$ which are pairwise non-adjacent. The *independence number* of set $A \subseteq V(G)$, denoted by $\alpha(A)$, is the size of the largest independent set in $G[A]$.

A *clique* in G is a set of vertices of G that are pairwise adjacent. The *clique number* of a graph G, denoted by $\omega(G)$, is the number of vertices in a largest clique of G.

We will use the following bound for the off-diagonal Ramsey number.

Theorem 2.1 (Ramsey [23], see also Erdős-Szekeres [19])**.** *For all $s, t \in \mathbb{N}$, every graph on at least t^s vertices has either a clique of cardinality t or an independent set of cardinality $s+1$.*

An $r \times r$ hexagonal grid is denoted as $W_{r\times r}$. The following result is folklore, see e.g. [7, Theorem 12].

Theorem 2.2. *For every planar graph H there exist $r \in \mathbb{N}$ such that H is an induced minor of $W_{r\times r}$.*

Tree decompositions. A tree decomposition $\mathcal{T}$ of a graph G is a pair (T, β) where T is a tree and β is a function assigning each node of T a non-empty subset $V(G)$ such that the following conditions are satisfied:

1. For each vertex v of $V(G)$ a subset of nodes $\{x \in V(T) \mid v \in \beta(x)\}$ induces a non-empty subtree;
2. For each edge uv of $E(G)$ there exists a node $x \in V(T)$ such that $u, v \in \beta(x)$.

The *width* of a tree decomposition $\mathcal{T} = (T, \beta)$ is equal to $\max_{x \in V(T)} |\beta(x)| - 1$. The *treewidth* of a graph G is a minimal width over all tree decompositions of G and is denoted as $\mathrm{tw}(G)$. The *independence number* of a tree decomposition $\mathcal{T} = (T, \beta)$ is equal to $\max_{x \in V(T)} \alpha(\beta(x))$. The *tree-independence number* of a graph G is a minimal independence number over all tree decomposition of G and is denoted as $\text{tree-}\alpha(G)$.

2.1 Building blocks

The proof of relies on three results from the literature. We start the definitions necessary to state these results. The first one is a theorem describing properties of graphs that contains a large complete bipartite graph as an induced minor.

A *constellation*, defined in [12], is a graph $\mathfrak{c}$ in which there is an independent set $I_\mathfrak{c}$ such that each connected component in $\mathfrak{c} - I_\mathfrak{c}$ is a path and each vertex $v \in I_\mathfrak{c}$ has at least one neighbor in each connected component in $\mathfrak{c} - I_\mathfrak{c}$. An (s, ℓ)*-constellation* is a constellation $\mathfrak{c}$ where $|I_\mathfrak{c}| = s$ and there are ℓ connected components in $\mathfrak{c} - I_\mathfrak{c}$. We can now state the first theorem that we need.

Theorem 2.3 (Chudnovsky, Hajebi, Spirkl [9, Theorem 1.3]). *For all $\ell, r, q \in \mathbb{N}$, there is a constant $t \in \mathbb{N}$ such that if G is a graph with an induced minor isomorphic to $K_{t,t}$, then one of the following holds.*

1. *There is an induced minor of G isomorphic to $W_{r \times r}$.*

2. *There is an (q, ℓ)-constellation in G.*

For $\lambda \in \mathbb{N}$, we say that a graph G is λ*-separable* if for all pairs of vertices $u,\ v$ of $V(G)$, which are distinct and non-adjacent, there is no set of λ pairwise internally disjoint paths in G from u to v. The next result that we use is the following:

Theorem 2.4 (Hajebi [20, Theorem 3.2 for $\kappa = 2$]). *For every planar graph H and every t there exists $d \in \mathbb{N}$ such that for all $\lambda \in \mathbb{N}$, if G is a λ-separable graph with no induced minor isomorphic to H or $K_{t,t}$, then $\mathrm{tw}(G) \leqslant (2(\omega(G) + 1))^d$.*

We also need:

Theorem 2.5 (Chudnovsky, Lokshtanov, Satheeshkumar [14]). *Let $\mathcal{C}$ be a hereditary class. Then the following are equivalent:*

1. *There exists a positive constant c_1 such that for every graph $G \in \mathcal{C}$ on $n \geqslant 3$ vertices we have $\text{tree-}\alpha(G) \leqslant (\log n)^{c_1}$.*

2. *There exists a positive constant c_2 such that for every graph $G \in \mathcal{C}$ on $n \geqslant 3$ vertices we have $\text{tree-}\alpha(G) \leqslant (\omega(G) \log n)^{c_2}$.*

3. *There exists a positive constant c_3 such that for every graph $G \in \mathcal{C}$ on $n \geqslant 3$ vertices we have $\mathrm{tw}(G) \leqslant (\omega(G) \log n)^{c_3}$.*

3 Proof of Theorem 1.4

Theorem 1.4. *For every integer s and every planar graph H there exists an constant $c_{s,H}$ such that every n-vertex graph which is H-induced minor-free and $K_{1,s}$-free has tree-independence number at most $\log^{c_{s,H}} n$.*

Proof. Given H and s, let us consider any n-vertex graph G which is H-induced-minor-free and $K_{1,s}$-free. Let us denote the clique number of G as ω. Since H is planar, by Theorem 2.2 there exists r such that H is an induced minor of $W_{r\times r}$. Thus, G excludes $W_{r\times r}$ as an induced minor.

Since G is $K_{1,s}$-free, it follows that no induced subgraph of G is a $(1,s)$-constellation. Applying Theorem 2.3 with $q=1$ and $\ell=s$, we deduce that there is $t\in\mathbb{N}$ (that depends on H and s only), such that G is $K_{t,t}$-induced-minor-free.

Denote by Δ the maximum degree of a vertex in G. For every vertex $v\in V(G)$ there is no independent set of size s or a clique of size ω inside $N(v)$. Thus by Theorem 2.1 we get that $\Delta(G)<\omega^s$. Since for every pair of vertices u,v in G there exist at most Δ pairwise vertex disjoint paths from u to v, it follows that G is $(\Delta(G)+1)$-separable. Consequently, G is ω^s-separable

Since G is ω^s-separable and $K_{t,t}$-induced-minor-free, Theorem 2.4 implies that there exists d that depends only on t and H, and therefore only on s and H, such that

$$\mathrm{tw}(G)\leqslant\left(2\omega^s(\omega+1)\right)^d.$$

Finally, by Theorem 2.5 we get that $\text{tree-}\alpha(G)\leqslant\log^c n$ where c is a constant that depends only on s and H. This completes the proof. □

4 Conclusion

In this note we proved that Conjecture 1.3 is "morally true," i.e., it holds up to factors polylogaritmic in the number of vertices. The full resolution of the conjecture is wide open.

Let us remark that Dallard et al. [17] made another conjecture about tree-independence number – they suggested that every hereditary class where treewidth is bounded in terms of the clique number, has bounded tree-independence number. This conjecture was recently refuted by Chudnovsky and Trotignon [15]. However, shortly after that Chudnovsky, Lokshtanov, and Satheeshkumar [14] proved that the conjecture is "morally true" (again, up to polylogarithmic factors), see Theorem 2.5.

Let us conclude the paper with recalling yet another conjecture by Dallard, Krnc, Kwon, Milanič, Munaro, Štorgel, and Wiederrecht [16], closely related to Conjecture 1.3.

Conjecture 4.1 ([16]). *Let $\mathcal{S}$ denote the family of forests where every component has at most three leaves. For every $S\in\mathcal{S}$ and every integer t there exists $c_{S,t}$ such that every graph which is S-induced-minor-free and $K_{t,t}$-free has tree-independence number $C_{S,t}$.*

Interestingly, as shown by Chudnovsky et al. [10], this conjecture is also "morally true," i.e., all such graphs have tree-independence number polylogarithmic in the number of vertices.

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