Unavoidable induced subgraphs in graphs with complete bipartite induced minors

Maria Chudnovsky[1], Meike Hatzel[2], Tuukka Korhonen[3],
Nicolas Trotignon[4], and Sebastian Wiederrecht[5]

[1]Princeton University, Princeton, NJ 08544, USA
[2]National Institute of Informatics, Tokyo, Japan
[3]University of Bergen, Norway
[4]Univ Lyon, EnsL, CNRS, LIP, F-69342, Lyon Cedex 07, France
[5]KAIST, Daejeon, South Korea

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Abstract

We prove that if a graph contains the complete bipartite graph $K_{134,12}$ as an induced minor, then it contains a cycle of length at most 12 or a theta as an induced subgraph. With a longer and more technical proof, we prove that if a graph contains $K_{3,4}$ as an induced minor, then it contains a triangle or a theta as an induced subgraph. Here, a *theta* is a graph made of three internally vertex-disjoint chordless paths $P_1 = a \ldots b$, $P_2 = a \ldots b$, $P_3 = a \ldots b$, each of length at least two, such that no edges exist between the paths except the three edges incident to a and the three edges incident to b.

A consequence is that excluding a grid and a complete bipartite graph as induced minors is not enough to guarantee a bounded tree-independence number, or even that the treewidth is bounded by a function of the size of the maximum clique, because the existence of graphs with large treewidth that contain no triangles or thetas as induced subgraphs is already known (the so-called layered wheels).

1 Introduction

Graphs in this paper are finite and simple. A graph H is an *induced subgraph* of a graph G if H can be obtained from G be repeatedly deleting vertices. It

is an *induced minor* of G if H can be obtained from G be repeatedly deleting vertices and contracting edges. It is a *minor* of G if H can be obtained from G be repeatedly deleting vertices, deleting edges and contracting edges. We denote by K_t the complete graph on t vertices and by $K_{a,b}$ the complete bipartite graph with sides of size a and b. The $(k \times k)$-grid is the graph whose vertices are the pairs (i, j) of integers such that $1 \leq i, j \leq k$ and where (i, j) is adjacent to (i', j') if and only if $|i - i'| + |j - j'| = 1$.

The tree-independence number was introduced in [3]. It is defined via tree-decompositions similarly to treewidth, except that the number associated to each bag of a tree-decomposition is the maximum size of an independent set of the graph induced by the bag, instead of its number of vertices (we omit the full definition for the sake of brevity). It attracted some attention lately, in particular because for each class of graphs with bounded tree-independence number, there exists polynomial time algorithms for maximum independent set and other related problems [3, 5, 6, 11].

The celebrated grid minor theorem of Robertson and Seymour [7] states that there exists a function $f \colon \mathbb{N} \to \mathbb{N}$ so that any graph with treewidth at least $f(k)$ contains a $(k \times k)$-grid as a minor. The motivation for our work is the quest for a similar theorem with "tree-independence number" instead of "treewidth".

Here, the natural containment relation should be "induced minor" instead of "minor", since the tree-independence number is monotone under taking induced minors but not under taking minors. The list of unavoidable graphs arising from a large tree-independence number should contain at least large grids and large complete bipartite graphs, which are known to have unbounded independence-tree number. Our main result is that this list is not complete. Indeed, we prove that a construction called *layered wheel*, first defined in [8], contains no (5×5)-grid and no $K_{3,4}$ as an induced minor, while having arbitrarily large tree-independence number.

Outline of the proof

We do not need to define layered wheels, which is good since the definition is a bit long. We just need some of their properties and some preliminary definitions to state them. A *theta* is a graph made of three internally vertex-disjoint chordless paths $P_1 = a \ldots b$, $P_2 = a \ldots b$, $P_3 = a \ldots b$, each of length at least two, such that no edges exist between the paths except the three edges incident to a and the three edges incident to b. A graph is *theta-free* if it does contain a theta as an induced subgraph, and more generally, a graph is *H-free* whenever it does not contain H (when H is a graph) or any graph

2

in H (when H is a class of graphs, such as thetas) as an induced subgraph. The only property of layered wheels that we need is the following theorem that states their existence.

Theorem 1.1 (see [8]). *For all integers $t \geq 1$ and $k \geq 3$, there exists a theta-free graph of girth k that contains K_t as a minor.*

Since containing K_t as a minor implies having treewidth at least t, layered wheels provide theta-free graphs of arbitrarily large girth and treewidth. Their tree-independence number is also arbitrarily large, since the treewidth of a triangle-free graph with tree-independence number at most t is at most $R(3, t + 1) - 2$ where $R(a, b)$ denotes the classical Ramsey number, see [3]. To fulfill our goal, it therefore remains to prove that layered wheels do not contain large grids or complete bipartite graphs as induced minors. As far as we can see, this is non-trivial because even if layered wheels are precisely defined, checking directly that they do not contain some $(k \times k)$-grid or $K_{r,s}$ as an induced minor seems to be tedious, at least according to our several attempts. An indication of this is that some layered wheels do contain $K_{3,3}$ as an induced minor, which is not obvious, see Fig. 1 (this figure is meaningful only with the precise definition of a layered wheel).
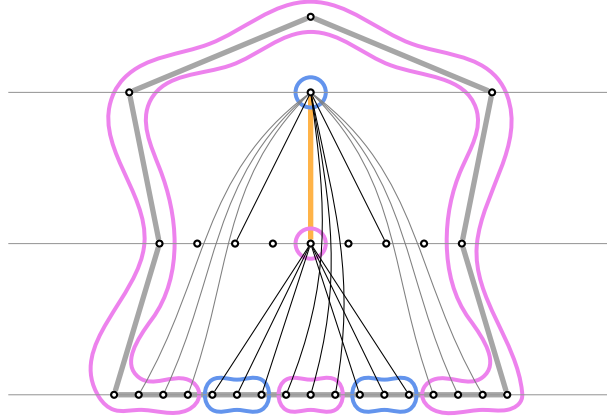

Figure 1: A $K_{3,3}$ induced minor in a layered wheel.

Our approach is therefore less direct: we study what induced subgraphs are forced by the presence of a large grid or complete bipartite graph as an induced minor as we explain now.

Complete bipartite graphs

The list of induced subgraphs forced by the presence of $K_{2,3}$ as an induced minor is already known and not very difficult to obtain, see [4]. But this is not enough for our purpose since containing $K_{2,3}$ as an induced minor does not imply anything we can use. By a short argument, we first prove the following.

Theorem 1.2. If a graph G contains $K_{134,12}$ as an induced minor, then G contains a cycle of length at most 12 or a theta as an induced subgraph.

The advantage of Theorem 1.2 is its short proof. But its statement is far from being optimal, as shown by the following, that we obtain by a more careful structural study.

Theorem 1.3. If a graph G contains $K_{3,4}$ as an induced minor, then G contains a triangle or a theta as an induced subgraph.

Once Theorem 1.2 (or 1.3) is proved, checking that layered wheels contain no $K_{134,12}$ (or no $K_{3,4}$) as an induced minor becomes trivial since by Theorem 1.1, they do not contain short cycles and thetas as induced subgraphs. Our results therefore avoid some tedious checking, but we also believe that they are of independent interest.

Note that Theorem 1.3 is best possible in several ways. First, thetas need to be excluded because the subdivisions of $K_{3,4}$ provide triangle-free graphs that obviously contain $K_{3,4}$ as an induced minor. Triangles must be excluded because of line graphs of subdivisions of $K_{3,4}$. A less obvious construction is represented in Fig. 2, showing that $K_{3,4}$ cannot be replaced by $K_{3,3}$ in Theorem 1.3.

In fact, we prove a more general result than Theorem 1.3. Before stating it, we need some definitions.

A *prism* is a graph made of three vertex-disjoint chordless paths $P_1 = a_1 \ldots b_1$, $P_2 = a_2 \ldots b_2$, $P_3 = a_3 \ldots b_3$, such that $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are triangles and no edges exist between the paths except those of the two triangles and either:

- P_1, P_2 and P_3 all have length at least 1, or

- one of P_1, P_2, P_3 has length 0 and each of the other two has length at least 2.

Note that allowing a path of length zero in prisms (for instance if $a_1 = b_1$) is not standard, but natural in our context. A prism with a path of
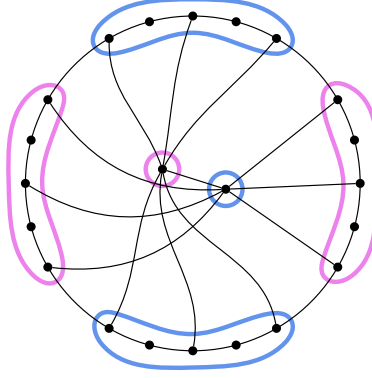
4

Figure 2: A (triangle, theta)-free graph containing $K_{3,3}$ as an induced minor

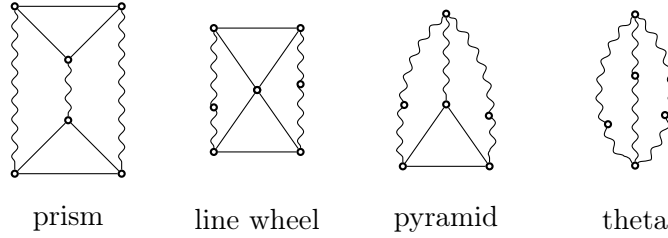length zero is sometimes referred to as a *line wheel*, but we do not use this terminology here.

prism   line wheel   pyramid   theta

Figure 3: The different 3-path configurations.

A *pyramid* is a graph made of three chordless paths $P_1 = a \ldots b_1$, $P_2 = a \ldots b_2$, $P_3 = a \ldots b_3$, each of length at least one, and two of them with length at least two, vertex-disjoint except at a, and such that $\{b_1, b_2, b_3\}$ is a triangle and no edges exist between the paths except those of the triangle and the three edges incident to a.

A *hole* in a graph is a chordless cycle of length at least 4. A graph that is either a theta, a prism or a pyramid is called a *3-paths configuration*, or *3PC* for short. Observe that a graph G is a 3PC if and only if there exist three pairwise internally vertex disjoint paths P_1, P_2, P_3 such that $V(G) = V(P_1) \cup V(P_2) \cup V(P_3)$ and for every $i \neq j \in \{1, 2, 3\}$, $V(P_i) \cup V(P_j)$ induces a hole.

Theorem 1.4. If a graph G contains $K_{3,4}$ as an induced minor, then G

5

contains a 3-path configuration as an induced subgraph.

Theorem 1.4 clearly implies Theorem 1.3 because a 3PC that is not a theta contains a triangle. Proving Theorem 1.4 is just slightly longer than Theorem 1.3, but is also interesting in its own right for the following reason. A hole is *even* if it has an even number of vertices. Computing the maximum independent set of an even-hole-free graph in polynomial time is a well known open question. Hence, understanding the tree-independence number of even-hole-free graphs would be interesting. Moreover, Theorem 1.4 implies directly the existence of even-hole-free graphs of large tree-independence number that contain no large grids and no $K_{3,4}$ as induced minors. Indeed, [8] not only provides the layered wheels that we already mentioned, but a variant, called *even-hole-free layered wheels*, whose existence is stated in the next theorem.

Theorem 1.5 (see [8]). For all integers $t \geq 1$, there exists a $(K_4$, even hole, 3PC)-free graph that contains K_t as a minor.

So, the simple counter-part of the Robertson and Seymour grid theorem, that would state that graphs with large tree-independence number should contain a large grid or a large complete bipartite graph as an induced minor is false, even when we restrict ourselves to even-hole-free graphs. Even-hole-free layered wheels provide a counter-example (note that being K_4-free ensures that the high treewidth implies a high tree-independence number, see [3]).

Theorem 1.4 is best possible in some sense since we cannot get rid of thetas in the statement, which are needed because of subdivisions of $K_{3,4}$ that are easily seen to contain thetas. Also prisms are needed because the line graph of a theta is a prism, so line graphs of subdivisions of $K_{3,4}$ contain prisms. Observe that we have to allow a path of length zero in a prism because of the graph depicted in Fig. 4 that contains $K_{3,4}$ as an induced minor while the only 3PC in it is a prism with a path of length zero. Maybe pyramids are are not needed in the statement, but we need them in the proof for reasons that we now explain.

The proof of Theorem 1.4 relies on a precise description of how $K_{3,3}$ can be contained as an induced minor in a 3PC-free graph, see Lemma 5.2. The description is too long to be stated in the introduction, but Fig. 5 provides representations of the different possible situations.

Observe that excluding pyramids is necessary in Lemma 5.2, because of the graph presented in Fig. 6, that contains $K_{3,3}$ as induced minor, while the only 3PC in it is a pyramid.
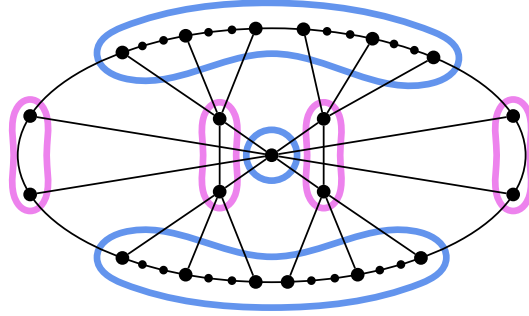
Figure 4: $K_{3,4}$ as an induced minor while all 3PC's are prisms with a path of length zero
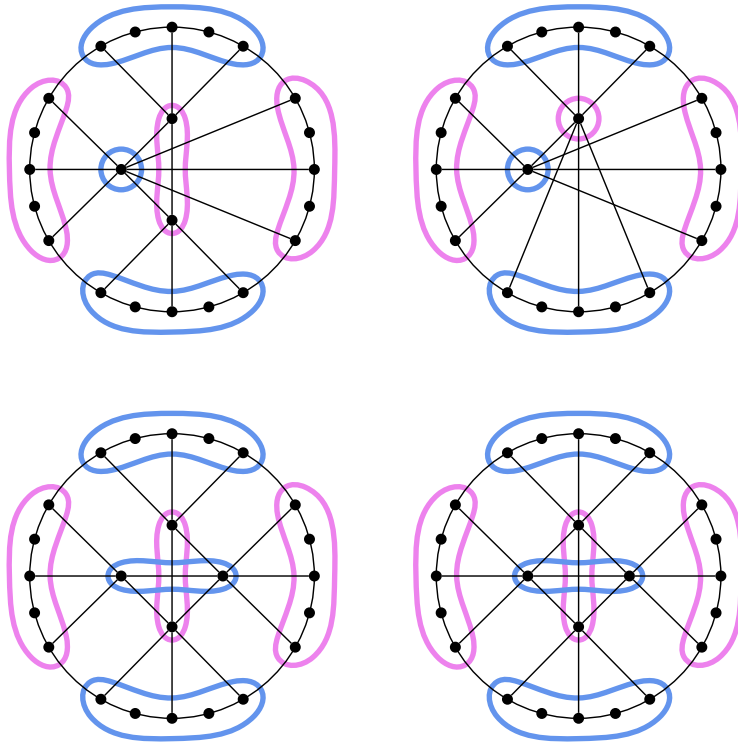

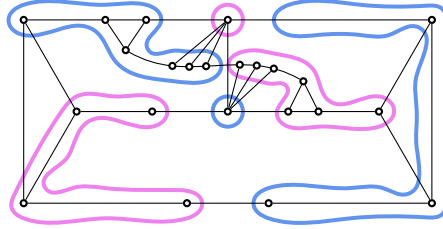Figure 5: $K_{3,3}$ as an induced minor in 3PC-free graphs

7

Figure 6: $K_{3,3}$ as an induced minor in a (theta, prism)-free graph

Grids

Grids are easier to handle than complete bipartite graphs as shown by the next lemma whose proof is less involved than the proofs of Theorems 1.2, 1.3 or 1.4. We do not know if a (4×4)-grid as an induced minor is enough to guarantee the presence of a 3PC as an induced subgraph.

Lemma 1.6. If a graph G contains a (5×5)-grid as an induced minor, then G contains a 3-path configuration as an induced subgraph.

Checking that layered wheels (and their even-hole-free variant) contain no (5×5)-grid as an induced minor is then easy by Theorem 1.1 and Theorem 1.5.

Outline of the paper

In Section 2, we prove some technical lemmas about how a connected induced subgraph of some graph G sees the rest of G. These lemmas are more or less known already. We reprove them for the sake of completeness and because we could not find them with the precise statement that we need. Note that they are needed for all the other results. In Section 3, we prove Lemma 1.6. In Section 4, we prove Theorem 1.2. In Section 5, we prove Theorem 1.4. We conclude the paper by Section 6 that presents several open questions.

Notation

Let G be a graph and X and Y be disjoint sets of vertices of G. We say that X is *complete* to Y if every vertex of X is adjacent to every vertex of Y, X is *anticomplete* to Y if every no vertex of X is adjacent to a vertex of Y. We say that X *sees* Y if there exist $x \in X$ and $y \in Y$ such that

8

$xy \in E(G)$. Note that the empty set is complete (and anticomplete) to every set of vertices of G.

By *path* we mean a sequence of vertices $p_1 \ldots p_k$ such that for all $1 \le i < j \le k$, $p_i p_j \in E(G)$ if and only if $j = i + 1$. Therefore, what we call *path* for the sake of brevity is sometimes referred to as *chordless path* or *induced path* in a more standard notation. We use the notation aPb to denote the subpath of P from a to b (possibly $a = b$ since a path may consist of a single vertex).

When we deal with a theta with two vertices of degree 3 u and v, we say that the theta is *from u to v*. We use similar terminology for prisms and pyramids that are, respectively, from a triangle to a triangle and from a vertex to a triangle.

To avoid too heavy notation, we allow some abuse. Typically, we do not distinguish between a set of vertices in graph G and the subgraph of G that it induces. For instance, when P is a path and v is a vertex in a graph G, we denote by $P \setminus v$ either $V(P) \setminus \{v\}$ or $G[V(P) \setminus \{v\}]$. Also, we may say that a set of vertices C of G is *connected* when the correct statement should be that $G[C]$ is connected. We hope that this improves the readability without causing any confusion.

2 Types

Let A, X, Y and Z be four disjoint sets of vertices in a graph G. We distinguish three different types the set A can have, see Fig. 7 for an illustration.

We say that A is of *type path centered at Y* with respect to X, Y and Z if there exists a path $P = x \ldots z$ in A such that x sees X, z sees Z, P sees Y, $P \setminus x$ is anticomplete to X and $P \setminus z$ is anticomplete to Z.

Similarly we define A being of *type path centered at X* and being of *type path centered at Z*. We say that A is of *type path* with respect to X, Y and Z if it is of type path centered at either X, Y or Z. Additionally, if A is of type path centered at a set W, we call W a *center for A*.

Observe that when A is of type path centered at Y and a unique vertex of P sees Y, and moreover this vertex is x, then A is also centered at X. In particular, if $x = z$, then A is centered at X, Y and Z.

We say that A is of *type claw* with respect to X, Y and Z if there exists in A a vertex a and three paths $P = a \ldots x$, $Q = a \ldots y$ and $R = a \ldots z$ such that $V(P) \cap V(Q) \cap V(R) = \{a\}$, $P \setminus a$, $Q \setminus a$ and $R \setminus a$ are pairwise anticomplete, x sees X, y sees Y, z sees Z, $(P \cup Q \cup R) \setminus x$ is anticomplete to X, $(P \cup Q \cup R) \setminus y$ is anticomplete to Y and $(P \cup Q \cup R) \setminus z$ is anticomplete
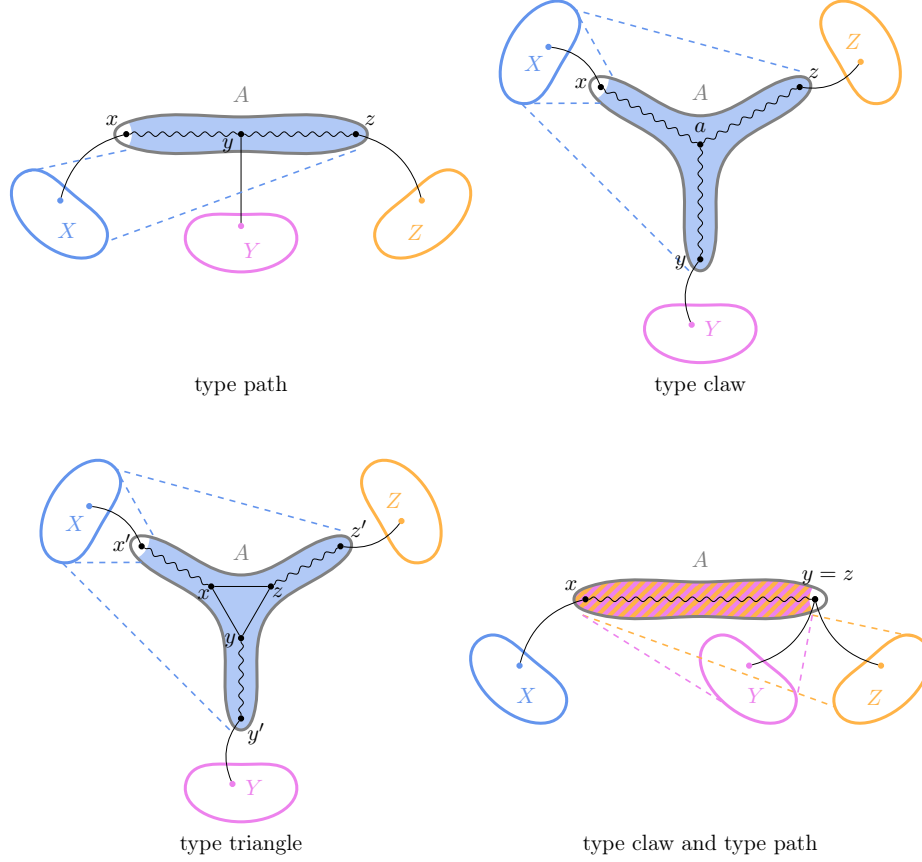
Figure 7: We illustrate the three types and a degenerate case in which type path and type claw overlap. For the three types we by way of example illustrate how X is not connected to anything in A but x. For the degenerate it is depicted that Y and Z are not connected to anything in A but the vertex $y = z$.

to Z.

Observe that possibly $a = x$, $a = y$ or $a = z$ (in fact, possibly two or three of these equalities hold). Observe that if $a = x$, then A is not only of type claw, but also of type path centered at X, this case is additionally depicted in Fig. 7.

We say that A is of *type triangle* with respect to X, Y and Z if there exists in A three vertex-disjoint paths $P = x \ldots x'$, $Q = y \ldots y'$ and $R = z \ldots z'$ such that the only edges between P, Q and R are xy, yz and yz, x' sees X,

10

y' sees Y, z' sees Z, $(P \cup Q \cup R) \setminus x$ is anticomplete to X, $(P \cup Q \cup R) \setminus y$ is anticomplete to Y and $(P \cup Q \cup R) \setminus z$ is anticomplete to Z.

Observe that each of P, Q and R is possibly of length 0.

Lemma 2.1. Let G be a graph and A, X, Y and Z be disjoint sets of vertices of G. If A is connected and A sees X, Y and Z, then A is of type path, of type claw, or of type triangle with respect to X, Y and Z.

Proof. Suppose that A is not of type path. Let $P = x \ldots z$ be a path in A such that x sees X, z sees Z, $P \setminus x$ is anticomplete to X and $P \setminus z$ is anticomplete to Z. Note that such a path exists (consider for instance a shortest path from the vertices that see X to the vertices that see Z). Since A is not of type path centered at Y, P is anticomplete to Y. Let $Q = y \ldots y'$ be path in A disjoint from P and such that y sees Y, y' sees P, $Q \setminus y$ is anticomplete to Y and $Q \setminus y'$ is anticomplete to P. Note again that such a path exists (consider for instance a shortest path from the vertices that see Y to the vertices that see P). We suppose that P and Q are chosen subject to the minimality of $V(P) \cup V(Q)$.

Let a be the neighbor of y' in P closest to x along P and a' be the neighbor of y' in P closest to z along P (possibly $a = a'$ or $aa' \in E(G)$).

If Q sees both X and Z, then consider vertex u in Q such that yQu sees both X and Z, and choose u closest to y along Q. The path yQu shows that A is of type path (centered at X or Z, possibly both, possibly also centered at Y if $u = y$), a contradiction. Hence, we may assume up to symmetry that Q is anticomplete to Z. If Q sees X, then the path $yQy'a'Pz$ shows that A is of type path centered at X, a contradiction. Hence, Q is anticomplete to X and Z.

If $a \neq a'$ and $aa' \notin E(G)$, then consider the path $P' = xPay'a'Pz$. If $y = y'$, then because of P', A is of type path centered at Y, a contradiction. So, $y \neq y'$ and the paths P' and $Q \setminus y'$ contradict the minimality of $V(P) \cup V(Q)$.

Hence $a = a'$ or $aa' \in E(G)$. If $a = a'$, then the three paths aPx, $ay'Qy$ and aPz show that A is of type claw with respect to X, Y and Z. If $aa' \in E(G)$, then the three paths aPx, Q and $a'Pz$ show that A is of type triangle with respect to X, Y and Z. $\square$

The next lemma is implicitly about the presence of $K_{2,3}$ as an induced minor in a graph. In [4], it is proved along similar lines that a graph contains $K_{2,3}$ as an induced minor if and only if it contains some configuration from a restricted list as an induced subgraph (the so-called thetas, pyramids, long prisms and broken wheels, not worth defining here).

Lemma 2.2. Let G be a 3PC-free graph and A, B, X, Y and Z be disjoint connected subsets of $V(G)$. If X, Y and Z are pairwise anticomplete, A and B are anticomplete to each other, and each of A and B sees each of X, Y and Z, then A and B are of type path with respect to X, Y and Z.

Proof. Suppose for a contradiction that A (the argument for B is the same by symmetry) is not of type path with respect to X, Y and Z. Hence, by Lemma 2.1, A we may consider the two cases below.

Case 1: A is of type claw (and not of type path). So, there exists a vertex $a \in A$ and three paths $P = a \ldots x$, $Q = a \ldots y$ and $R = a \ldots z$ like in the definition of type claw. Moreover, $a \neq x$, $a \neq y$ and $a \neq z$ for otherwise, A would be of type path.

If B is of type claw, then there exist a vertex $b \in B$ and three paths $P' = b \ldots x'$, $Q' = b \ldots y'$ and $R = b \ldots z'$ like in the definition of type claw. Consider a shortest path P'' from x to x' with interior in X, and let $Q'' = y \ldots y'$ and $R'' = z \ldots z'$ be defined similarly through Y and Z. The nine paths P, Q, R, P', Q', R', P'', Q'' and R'' form a theta from a to b, a contradiction.

If B is of type triangle, a similar contradiction is found because of the existence of a pyramid in G.

So B is not of type claw or triangle. By Lemma 2.1, B is of type path, and up to symmetry we suppose that it is centered at Y. Let $P' = x' \ldots z'$ be a path like in the definition of type path. Consider a shortest path P'' from x to x' with interior in X, and let $R'' = z \ldots z'$ be defined similarly through Z. Let $Q'' = b \ldots b'$ be a shortest path in Y such that $by \in E(G)$ and b' sees P'. Let a' be the neighbor of b' in P' closest to x' along P'. Let a'' be the neighbor of b' in P' closest to z' along P'. If $a' = a''$, then B is of type claw, a contradiction. If $a'a'' \in E(G)$, then the seven paths P, Q, R, P', P'', Q'' and R'' form a pyramid from a to $b'a'a''$, a contradiction. Hence $a \neq a'$ and $aa' \notin E(G)$. So the paths P, Q, R, $x'P'a'$, $a''P'z'$, P'', Q'' and R'' form a theta from a to b', a contradiction.

Case 2: A is of type triangle.

The proof is almost the same as in Case 1, so we just sketch it. If B is of type claw, then G contains a pyramid. If B is of type triangle, then G contains a prism. And if B is of type path, then G contains a prism or a pyramid. □

Let G and H be two graphs. An *induced minor model* of H in G is a collection of pairwise disjoint sets $\{X_v\}_{v \in V(H)}$, called the *branch sets* of the model, such that

12

- $X_v \subseteq V(G)$ for all $v \in V(H)$,

- X_v induces a connected subgraph of G for every $v \in V(H)$, and

- X_u sees X_v (in G) if and only if $uv \in E(H)$.

It is well known and easy to check that G contains a graph isomorphic to H as an induced minor if and only if there exists an induced minor model of H in G. We identify an induced minor model $\{X_v\}_{v \in V(H)}$ of H in G with the graph $H' := G[\bigcup_{v \in V(H)} X_v]$. Please note that the definition of the branch sets is not uniquely determined by H'.

An induced minor model $H' \subseteq G$ of H is *minimal* if $H' \setminus a$ does not contain an induced minor isomorphic to H for all $a \in V(H')$.

We denote by $K_{2,3}^*$ the graph obtained from $K_{2,3}$ by subdividing every edge once, i.e., $K_{2,3}^*$ is the graph with two degree-3 vertices and three paths of length four between them.

Lemma 2.3. If a graph contains $K_{2,3}^*$ as an induced minor, then it contains a 3PC.

Proof. Suppose for a contradiction that a 3PC-free graph G contains $H = K_{2,3}^*$ as an induced minor. Denote by a and b the degree-3 vertices and let $ap_1p_2p_3b$, $aq_1q_2q_3b$ and $ar_1r_2r_3b$ be the three paths of H. Consider a minimal induced minor model $\{X_v\}_{v \in V(H)}$ of H in G. For all $v \in V(H) \setminus \{a, b\}$, v has two neighbors u and w in H and by minimality, X_v is a path $P = v' \ldots v''$ such that v' see X_u, v'' sees X_w, $P \setminus v'$ is anticomplete to X_u and $P \setminus v''$ is anticomplete to X_w. It follows that if we set $A = X_a \cup X_{p_1} \cup X_{p_2} \cup X_{q_1} \cup X_{q_2} \cup X_{r_1} \cup X_{r_2}$, $X = X_{p_3}$, $Y = X_{q_3}$, $Z = X_{r_3}$ and $B = X_b$, A cannot be of type path with respect to X, Y and Z. Indeed, since $X_{p_1} \cup X_{p_2}$, $X_{q_1} \cup X_{q_2}$ and $X_{r_1} \cup X_{r_2}$ all induce paths of length at least 1 in G, A cannot contain a path with vertices seeing X, Y and Z. Hence, A, B, X, Y and Z contradict Lemma 2.2. $\qed$

3 Proof of Lemma 1.6

We here prove Lemma 1.6 that we restate.

Lemma 1.6. If a graph G contains a (5×5)-grid as an induced minor, then G contains a 3-path configuration as an induced subgraph.

Proof. If a graph contains a (5×5)-grid as an induced minor, then it contains $K_{2,3}^*$ as an induced minor, because the (5×5)-grid contains $K_{2,3}^*$ as an induced subgraph. The result therefore follows from Lemma 2.3. $\qed$

4 Proof of Theorem 1.2

Lemma 4.1. Let G and H be graphs and $\{X_v\}_{v \in V(H)}$ a minimal induced minor model of H in G. For every $v \in V(H)$, the graph $G[X_v]$ does not contain cycles longer than the degree of v.

Proof. First, for every neighbor u of v in H, let us mark a single vertex in X_v that is adjacent to a vertex in X_u. If there exists a connected induced proper subgraph of $G[X_v]$ that contains all of the marked vertices, we contradict the minimality of the induced minor model.

Suppose that $G[X_v]$ contains a cycle of length longer than the degree of v, and let $C \subseteq X_v$ be the vertices of this cycle. We say that a vertex $w \in C$ is *necessary* if either w is marked or there exists a connected component Y of $G[X_v \setminus C]$ that contains a marked vertex and whose only neighbor in C is the vertex w. Because $|C|$ is larger than the number of marked vertices, there exists a vertex in C that is not necessary, let $w \in C$ be such a vertex. We remove from X_v the vertex w and every component of $G[X_v \setminus C]$ whose only neighbor in C is w. All of the remaining vertices in X_v are still connected to $C \setminus \{w\}$ and contain all of the marked vertices, so we get a connected induced proper subgraph of $G[X_v]$ that contains all of the marked vertices, which is a contradiction. $\qquad\square$

Lemma 4.1 is useful for asserting that $G[X_v]$ is a tree, which in turn is useful for obtaining degree-1 vertices in $G[X_v]$. We make use of degree-1 vertices with the following lemma.

Lemma 4.2. Let $\{X_v\}_{v \in V(H)}$ be a minimal induced minor model of a graph H in a graph G, and let $v \in V(H)$. For every vertex $w \in X_v$ whose degree is one in $G[X_v]$, there exists $u \in V(H) \setminus \{v\}$ so that w is the only neighbor of X_u in X_v.

Proof. Otherwise, we could remove w from X_v and contradict the minimality of $\{X_v\}_{v \in V(H)}$. $\qquad\square$

We say that such a branch set X_u is *private* to the vertex w. We may now prove Theorem 1.2 which we restate below.

Theorem 1.2. If a graph G contains $K_{134,12}$ as an induced minor, then G contains a cycle of length at most 12 or a theta as an induced subgraph.

Proof. Let $p = 12$ and $t = 2\binom{p}{2} + 2 = 134$. Let G be a graph with girth at least $p + 1$, and let $\{X_v\}_{v \in V(K_{t,p})}$ be a minimal induced minor model of $K_{t,p}$ in G. We denote the branch sets corresponding to vertices of $K_{t,p}$ on

the side with t vertices by $A_1, \ldots, A_t$ and on the other side by $B_1, \ldots, B_p$. Note that G is triangle-free.

First suppose that there are two branch sets A_i, A_j of size $|A_i|, |A_j| \leq 2$. In this case, there must be a vertex $v \in A_i$ that is adjacent to at least $p/2$ different branch sets on the other side, and a vertex $u \in A_j$ that is adjacent to at least $p/4 = 3$ different branch sets on the other side that v is also adjacent to. Fix three different branch sets B_a, B_b, B_c that both v and u are adjacent to. Now, by selecting in each branch set B_a, B_b, B_c a shortest path from a neighbor of v to a neighbor of u, we obtain a theta from u to v.

We may therefore assume that at least $t-1$ of the branch sets $A_1, \ldots, A_t$ contain at least three vertices. By Lemma 4.1 and because G has girth at least $p + 1$, for all branch sets $A_1, \ldots, A_t$ the induced subgraph $G[A_i]$ is a tree, and for all such sets with $|A_i| \geq 3$, the tree must have two non-adjacent leaves v and u. By Lemma 4.2, both v and u have a private branch set on the other side, so we can label each branch set A_i with $|A_i| \geq 3$ with an unordered pair $\{B_j, B_k\}$ so that B_j is private to v and B_k is private to u.

Now, because $t - 1 = 2\binom{p}{2} + 1$, there must exist three branch sets A_a, A_b, and A_c that are labeled with the same pair $\{B_j, B_k\}$. By contracting both B_j and B_k into a single vertex and taking the paths in A_a, A_b, and A_c between the corresponding leaves, we obtain $K_{2,3}^*$ as an induced minor. By Lemma 2.3, G must contain a theta since the theta is the only triangle-free 3PC. $\qquad \square$

5   Proof of Theorem 1.4

We start with an improvement of Lemma 2.2.

Lemma 5.1. Let G be a 3PC-free graph and $k \geq 2$ be an integer. Suppose that X, Y, Z and A_1, $\ldots$, A_k are disjoint connected subsets of $V(G)$, X, Y and Z are pairwise anticomplete, A_1, $\ldots$, A_k are pairwise anticomplete, and for every $i \in \{1, \ldots, k\}$, A_i sees X, Y and Z.

Then the A_i's are all of type path with respect to X, Y and Z and furthermore, one of X, Y and Z is a center for all of them.

Proof. By Lemma 2.2 applied to $A = A_i$, $B = A_j$ for some $j \neq i$ and X, Y and Z, we see that all A_i's are of type path with respect to X, Y and Z. It remains to prove that they all share a common center.

We denote by $\tau(A_i)$ the sets of all elements $U \in \{X, Y, Z\}$ such that A_i is centered at U. Note that for all $i \in \{1, \ldots, k\}$, $\tau(A_i)$ is nonempty. It is enough to prove that for all $i, j \in \{1, \ldots, k\}$, either $\tau(A_i) \subseteq \tau(A_j)$

or $\tau(A_j) \subseteq \tau(A_i)$. Indeed, this implies that the sets $\tau(A_i)$ are linearly ordered by the inclusion, so that $\cap_{i \in \{1,\ldots,k\}} \tau(A_i)$ is non-empty and contains the common center that we are looking for.

So suppose for a contradiction that $A = A_i$ and $B = A_j$ are such that $\tau(A_i)$ and $\tau(A_j)$ are inclusion-wise incomparable. So, up to symmetry, A is centered at X and not at Y while B is centered at Y and not at X.

Since A is centered at X, there exists $P = u \ldots u'$ in A such that u sees Y, u' sees Z, P sees X, $P \setminus u$ is anticomplete to Y and $P \setminus u'$ is anticomplete to Z. Since B is centered at Y, there exists $Q = v \ldots v'$ in B such that v sees X, v' sees Z, Q sees Y, $Q \setminus v$ is anticomplete to X and $Q \setminus v'$ is anticomplete to Z.

Let $R = z \ldots z'$ be a shortest path in Z such that $u'z \in E(G)$ and $z'v' \in E(G)$. Let $S = y \ldots y'$ be a shortest path in Y such that y sees Q and $y'u \in E(G)$. Let $T = x \ldots x'$ be a shortest path in X such that x sees P and $x'v \in E(G)$. Observe that each of P, Q, R, S and T can be of length 0.

Let a be the neighbor of x in P closest to u along P. Let a' be the neighbor of x in P closest to u' along P. Let b be the neighbor of y in Q closest to v along Q. Let b' be the neighbor of y in Q closest to v' along Q. See Figure 8.
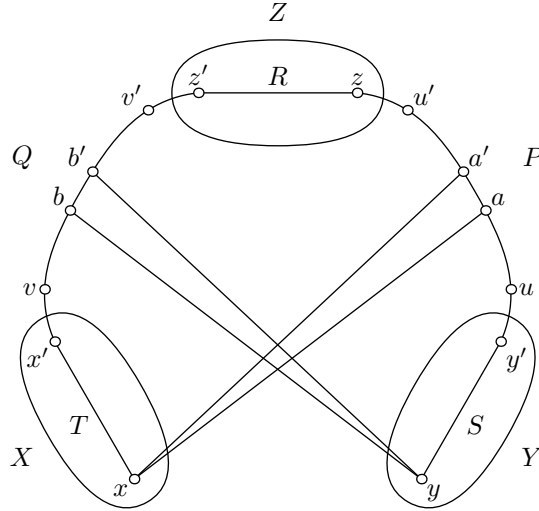

Figure 8: Paths P, Q, R, S and T in the proof of Lemma 5.1

Suppose first that $a = a'$. Observe that $a = a' \neq u$ for otherwise A would be centered at Y, contrary to our assumption. Hence, $ay \notin E(G)$. If $b = b'$,

16

then P, Q, R, S and T form a theta from a to b, so $b \neq b'$. If $bb' \in E(G)$, then P, Q, R, S and T form a pyramid from a to ybb'. If $bb' \notin E(G)$, then P, vQb, $b'Qv'$, R, S and T form a theta from a to y (because $ay \notin E(G)$ as already noted). Hence $a \neq a'$, and symmetrically we can prove that $b \neq b'$.

Suppose now $aa' \in E(G)$. If $bb' \in E(G)$, then P, Q, R, S and T form a prism from xaa' to ybb'. If $bb' \notin E(G)$, then P, vQb, $b'Qv'$, R, S and T form a pyramid from y to xaa'. Hence $aa' \notin E(G)$, and symmetrically we can prove that $bb' \notin E(G)$.

We are left with the case where $a \neq a'$, $aa' \notin E(G)$, $b \neq b'$ and $bb' \notin E(G)$. Then uPa, $a'Pu'$, vQb, $b'Qv'$, R, S and T form a theta from x to y. $\qquad \square$

The following lemma describes what happens when $K_{3,3}$ is an induced minor of some 3PC-free graph. It is worth noting that it has a true converse that we do not need to state formally. More precisely, if in any graph six paths A, B, C, P, Q and R satisfy all the properties described in Lemma 5.2, then they form a model for a $K_{3,3}$ induced minor, and moreover the graph that they induce can be checked to be 3PC-free, see Fig. 5.

Note that the statement of Lemma 5.2 is not completely symmetric. Namely, A, B and C are assumed to be minimal while X, Y and Z are not. This yields a slightly stronger statement which is needed for the application in the proof of Theorem 1.4.

Lemma 5.2. Let G be a 3PC-free graph and A, B, C, X, Y and Z be connected disjoint subsets of $V(G)$ such that X, Y and Z are pairwise anticomplete, A, B and C are pairwise anticomplete and each of A, B and C sees each of X, Y and Z. Suppose that no connected proper subset of A (resp. B and C) sees each of X, Y and Z. Then, there exist six vertex-disjoint paths $A' = a \ldots a'$, $B' = b \ldots b'$, $C' = c \ldots c'$, $P = p \ldots p'$, $Q = q \ldots q'$ and $R = r \ldots r'$ in G such that:

- Each of A', B' and C' *is equal to* exactly one of A, B or C.

- Each of X, Y and Z *contains* exactly one of P, Q or R.

- $H = aA'a'rRr'c'C'cp'Ppa$ is a hole.

- $B' \setminus b$ (resp. $B' \setminus b'$, $Q \setminus q$, $Q \setminus q'$) is anticomplete to P (resp. R, A', C').

- B' and Q both have length at most 1 (so they each contain at most two vertices).

- b (resp. b', q, q') has at least three neighbors in P (resp. R, A', C'). In particular, each of P, R, A' and C' contains at least three vertices.

- B' is complete to Q, or $G[B' \cup Q]$ has four vertices and five edges.

Proof. By Lemma 5.1, A, B and C are of type path with respect to X, Y and Z, centered at some $Y' \in \{X, Y, Z\}$. By Lemma 5.1, X, Y and Z are of type path with respect to A, B and C, centered at some $B' \in \{A, B, C\}$. We let X', Z', A' and C' be such that $\{A', B', C'\} = \{A, B, C\}$ and $\{X', Y', Z'\} = \{X, Y, Z\}$.

So, A' contains some path that sees X', Y' and Z' as in the definition of type path centered at Y', but by the assumption about the minimality of A, B or C, we see that this path is in fact A' itself. So A' is equal to exactly one of A, B or C. The arguments works also with B and C, so that $A' = a \ldots a'$, $B' = b \ldots b'$, $C' = c \ldots c'$, each of a, b and c sees X', each of a', b' and c' sees Z', each of A', B' and C' sees Y', each of $A' \setminus a$, $B' \setminus b$ and $C' \setminus c$ is anticomplete to X' and each of $A' \setminus a'$, $B' \setminus b'$ is and $C' \setminus c'$ is anticomplete to Z'.

Also X' contains a path $P = p \ldots p'$ such that p sees A', p' sees C', $P \setminus p$ is anticomplete to A', $P \setminus p'$ is anticomplete to C' and P sees B'. Note that we cannot claim that $X' = P$, since we made no assumption about the minimality of X. But since $A' \setminus a$ is anticomplete to X' and $P \setminus p$ is anticomplete to A', the only possible edge between P and A' is pa. Similarly, $p'c$ is the only edge between P and C'. Moreover, P sees B', and since $B \setminus b$ is anticomplete to X', we know that b sees P (and not only X').

Similarly, Z' contains a path $R = r \ldots r'$ with $ra' \in E(G)$, $r'c' \in E(G)$, $R \setminus r$ is anticomplete to A', $R \setminus r'$ is anticomplete to C' and such that b' sees R. Observe that A', P, C' and R form a hole H.

Also Y' is of type path with respect to A, B and C and centered at B'. So, Y' contains a path $Q = q \ldots q'$ such that q sees A', q' sees C', $Q \setminus q$ is anticomplete to A', $Q \setminus q'$ is anticomplete to C' and Q sees B'.

Let α be the neighbor of q in A' closest to a along A'. Let α' be the neighbor of q in A' closest to a' along A. Let β be the neighbor of b in P closest to p along P. Let β' be the neighbor of b in P closest to p' along P. Let γ be the neighbor of q' in C' closest to c along C'. Let γ' be the neighbor of q' in C' closest to c' along C'. Let δ be the neighbor of b' in R closest to r along R. Let δ' be the neighbor of b' in R closest to r' along R. See Fig. 9.

Suppose that B' has length at least 2. Then B and H contains a theta, a prism or a pyramid, namely from β (if $\beta = \beta'$) or $b\beta\beta'$ (if $\beta\beta' \in E(G)$)
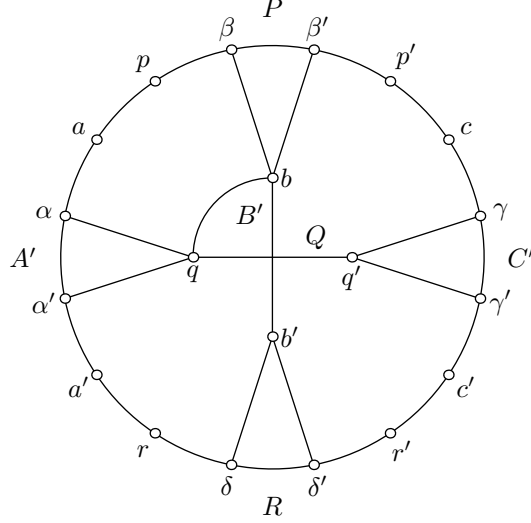
18

Figure 9: Paths A', B', C', P, Q and R in the proof of Lemma 5.2

or b (otherwise), to δ (if $\delta = \delta'$) or $b'\delta\delta'$ (if $\delta\delta' \in E(G)$) or b' (otherwise). Hence B' has length at most 1, meaning that either $b = b'$ or $bb' \in E(G)$. Similarly, Q has length at most 1 and $q = q'$ or $qq' \in E(G)$.

Suppose that $\beta = \beta'$. Then $b = b'$ for otherwise B and H contain a theta (if $\delta = \delta'$), or a pyramid from β to $b'\delta\delta'$ (if $\delta\delta' \in E(G)$), or a theta from β to b' (otherwise). Since B' sees Q, b has a neighbor in Q. If b has a unique neighbor in Q, say q up to symmetry (so either $q = q'$, or $q \neq q'$ and $bq' \notin E(G)$), then the three paths βbq, $\beta PpaA'\alpha q$ and $\beta Pp'cC'\gamma q'q$ form a theta from β to q. So, b has two neighbors in Q. Hence, the three paths βb, $\beta PpaA'\alpha q$ and $\beta Pp'cC'\gamma q'$ form a pyramid from β to bqq'. We proved that $\beta \neq \beta'$.

Suppose that $\beta\beta' \in E(G)$. Then $b = b'$ for otherwise B and H contains a pyramid from δ to $b\beta\beta'$ (if $\delta = \delta'$), a prism from $b\beta\beta'$ to $b'\delta\delta'$ (if $\delta\delta' \in E(G)$) or a pyramid from b' to $b\beta\beta'$ (otherwise). Since B' sees Q, b has a neighbor in Q. If b has a unique neighbor in Q, say q up to symmetry (so either $q = q'$, or $q \neq q'$ and $bq' \notin E(G)$), then the three paths qb, $q\alpha A'apP\beta$ and $qq'\gamma C'cp'P\beta'$ form a pyramid from q to $b\beta\beta'$. So, b has two neighbors in Q. Hence, the three paths b, $q\alpha A'apP\beta$ and $q'\gamma C'cp'P\beta'$ form a prism from bqq' to $b\beta\beta'$. We proved that $\beta\beta' \notin E(G)$. This implies that b has at least three neighbors in P for otherwise H and b would form a theta from β to β'.

We proved b has at least three neighbors in P. By a symmetric argument,

we can prove that b' (resp. q, q') has at least three neighbors in R (resp. A', C'). It remains to prove that B' is complete to Q, or $B' \cup Q$ induces a graph with four vertices and five edges. So suppose that B' is not complete to Q.

Since B' sees Q and B' is not complete to Q, there is at least one edge and at least one non-edge with ends in B and Q. So, there must be a vertex, either in B or Q, that is incident to such an edge and such a non-edge. Up to symmetry, we assume that this vertex is b and $bq \in E(G)$ (so $bq' \notin E(G)$ and $q \neq q'$). If $b = b'$, or if $b \neq b'$ and $G[B' \cup Q]$ has only three edges (namely bb', qq' and bq), then bqq', $b\beta'Pp'cC'\gamma q'$ and $bb'\delta'Rr'c'C'\gamma'q'$ form a theta from b to q'. We proved that $b \neq b'$ and $G[B' \cup Q]$ has at least four edges. So, $G[B' \cup Q]$ has four vertices and it remains to prove that it has exactly five edges. So suppose for a contradiction that $G[B' \cup Q]$ has exactly four edges.

If $b'q' \in E(G)$ (and therefore $b'q \notin E(G)$), then bqq', $bb'q'$ and $b\beta'Pp'cC'\gamma q'$ form a theta from b to q'. If $b'q \in E(G)$ (and therefore $b'q' \notin E(G)$), then $q'q$, $q'\gamma C'cp'P\beta'b$ and $q'\gamma'C'c'r'R\delta'b'$ form a pyramid from q' to $bb'q$. $\qquad\square$

Lemma 5.3. Let G be a 3PC-free graph and A, B, C, X, Y and Z be connected disjoint subsets of $V(G)$ such that X, Y and Z are pairwise anticomplete, A, B and C are pairwise anticomplete, and each of A, B and C sees each of X, Y and Z. Suppose that no connected proper subset of A (resp. B and C) sees each of X, Y and Z. Then exactly one of A, B and C contains at most 2 vertices.

Proof. Follows directly from Lemma 5.2. $\qquad\square$

We may now prove Theorem 1.4 that we restate.

Theorem 1.4. If a graph G contains $K_{3,4}$ as an induced minor, then G contains a 3-path configuration as an induced subgraph.

Proof. Suppose for a contradiction that a 3PC-free graph G contains $K_{3,4}$ has an induced minor. So, G contains seven disjoint connected sets X, Y, Z, A, B, C and D such that X, Y and Z are pairwise anticomplete, A, B, C and D are pairwise anticomplete, and each of X, Y and Z sees each of A, B, C and D. We suppose that these sets are chosen subject to the minimality of $A \cup B \cup C \cup D$. It follows that no proper connected subset of A (resp. B, C, D) sees X, Y and Z (note that it is important here that no assumption is made about the minimality of X, Y and Z).

By Lemma 5.3 applied to A, B, C, X, Y and Z, exactly one of A, B and C has size at most 2, say $|A| \leq 2$, $|B| > 2$ and $|C| > 2$. Hence, by

Lemma 5.3 applied to B, C, D, X, Y and Z, we have $|D| \leq 2$. Hence, A, B, D, X, Y and Z contradict Lemma 5.3. $\square$

6   Open questions

We need pyramids in Theorem 1.4 only for the sake of the precise description of Lemma 5.2, see Fig. 6. We therefore wonder whether pyramids in Theorem 1.4 are really needed. More precisely, we do not know whether a (theta, prism)-free graph that contains $K_{3,4}$ as an induced minor exists.

A *wheel* is a graph made of a hole called the *rim* together with a vertex called the *center* that has at least three neighbors on the rim. It is *even* if the center has an even number of neighbors in the rim. It is well known (and easy to check) that even-hole-free graphs contain no prisms, no thetas and no even wheels as induced subgraphs, since each of these configurations implies the presence of an even hole. Conversely, many theorems about even-hole-free graphs suggest that (theta, prism, even wheel)-free graphs, that are called *odd signable graphs*, capture the essentials structural properties of even-hole-free graphs, see [9, 10]. We believe that the following is true.

Conjecture 6.1. If G is an odd signable graph (in particular if G is an even-hole-free graph), then G does not contain $K_{3,3}$ as an induced minor.

Observe that we do not know whether the even-hole-free layered wheels contain $K_{3,3}$ as an induced minor. Conjecture 6.1 would prove that they do not. We also propose the following.

Conjecture 6.2. If G contains K_6 as a minor, then G contains a triangle (as a subgraph) or G contains $K_{3,3}$ as an induced minor.

In [1], it is proved that triangle-free odd-signable graphs (in particular (triangle, even-hole)-free graphs) have treewidth at most 5 and therefore do not contain K_6 as a minor. So, provided that Conjecture 6.1 is true, Conjecture 6.2 is just a more precise statement.

Here are some remarks about Conjecture 6.2. It is false with a K_5 assumption instead of a K_6 assumption, see Fig. 10 where an (even hole, triangle)-free graph with a K_5 minor, first discovered in [2], is represented. It is false with a "$K_{3,4}$ as an induced minor or triangle" conclusion because of the layered wheels. Provided that Conjecture 6.1 is true, it is false with a "$K_{3,3}$ as an induced minor or 3PC as an induced subgraph" conclusion, or with a "$K_{3,3}$ as an induced minor or K_4 as subgraph" conclusion, because of the even-hole-free layered wheels that are pyramid-free and K_4-free by

Theorem 1.5. Conjecture 6.2 would therefore provide a statement that is best possible in many ways.
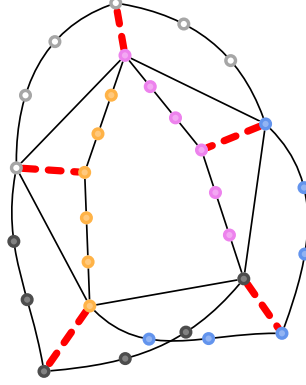

Figure 10: A K_5 minor in an (even hole, triangle)-free graph. To see even hole-freeness first note that no even hole can contain a fat red edge.

It might be interesting to study the implications of a $K_{3,3} - e$ induced minor in an even-hole free graph, where $K_{3,3} - e$ is the graph obtained from $K_{3,3}$ by removing one edge. In Fig. 11, an even-hole-free graph that contains $K_{3,3} - e$ as an induced minor is represented, and we observe that this graph plays an important role in the structural study of even-hole-free graphs, see [9]. In Fig. 12, another example of a graph that contains $K_{3,3} - e$ as an induced minor is represented, and we observe that this graph contains an even wheel.
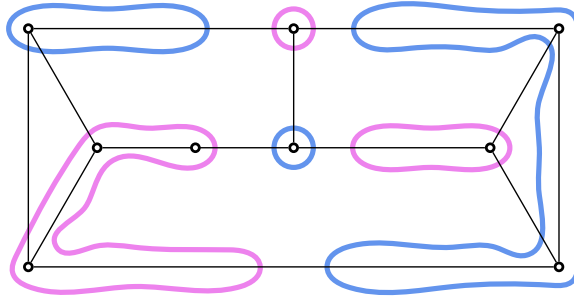
Figure 11: A $K_{3,3} - e$ induced minor in an (even hole, triangle)-free graph

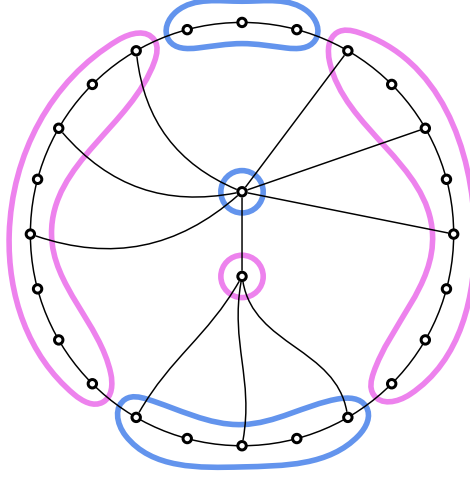More generally, we believe that studying implications between different

Figure 12: A $K_{3,3}-e$ induced minor in a graph without triangles and without thetas.

containment relations for different kinds of graphs might have more applications. For instance, this approach is used in [4] to design a polynomial time algorithm that decides whether an input graph contains $K_{2,3}$ as an induced minor. We wonder what is the complexity of detecting $K_{3,3}$ as an induced minor.

7 Acknowledgement

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