Perfect divisibility and 2-divisibility

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Abstract

A graph G is said to be 2-divisible if for all (nonempty) induced subgraphs H of G, $V(H)$ can be partitioned into two sets A, B such that $\omega(A) < \omega(H)$ and $\omega(B) < \omega(H)$. A graph G is said to be perfectly divisible if for all induced subgraphs H of G, $V(H)$ can be partitioned into two sets A, B such that $H[A]$ is perfect and $\omega(B) < \omega(H)$. We prove that if a graph is (P_5, C_5)-free, then it is 2-divisible. We also prove that if a graph is bull-free and either odd-hole-free or P_5-free, then it is perfectly divisible.

1 Introduction

All graphs considered in this article are finite and simple. Let G be a graph. The complement G^c of G is the graph with vertex set $V(G)$ and such that two vertices are adjacent in G^c if and only if they are non-adjacent in G. For two graphs H and G, H is an *induced subgraph* of G if $V(H) \subseteq V(G)$, and a pair of vertices $u, v \in V(H)$ is adjacent if and only if it is adjacent in G. We say that G *contains* H if G has an induced subgraph isomorphic to H. If G does not contain H, we say that G is *H-free*. For a set $X \subseteq V(G)$ we denote by $G[X]$ the induced subgraph of G with vertex set X. For an integer $k > 0$, we denote by P_k the path on k vertices, and by C_k the cycle on k vertices. A *path in a graph* is a sequence $p_1 - \ldots - p_k$ (with $k \geq 1$) of distinct vertices such that p_i is adjacent to p_j if and only if $|i - j| = 1$. Sometimes we say that $p_1 - \ldots - p_k$ *is a* P_k. A *hole* in a graph is an induced subgraph that is isomorphic to the cycle C_k with $k \geq 4$, and k is the *length* of the hole. A hole is *odd* if k is odd, and *even* otherwise. The vertices of a hole can be numbered $c_1, \ldots, c_k$ so that c_i is adjacent to c_j if and only if $|i - j| \in \{1, k - 1\}$; sometimes we write $C = c_1 - \ldots - c_k - c_1$. An *antihole* in a graph is an induced subgraph that is isomorphic to C_k^c with $k \geq 4$, and again k is the *length* of the antihole. Similarly, an antihole is *odd* if k is odd, and *even* otherwise. The *bull* is the graph consisting of a triangle with two disjoint pendant edges. A graph is *bull-free* if no induced subgraph of it is isomorphic to the bull. The chromatic number of a graph G is denoted by $\chi(G)$ and the clique number by $\omega(G)$. A graph G is called *perfect* if for every induced subgraph H of G, $\chi(H) = \omega(H)$. For a set X of vertices, we will usually write $\chi(X)$ instead of $\chi(G[X])$, and $\omega(X)$

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instead of $\omega(G[X])$. If X is a set of vertices and x is a vertex, we will write $X + x$ for $X \cup \{x\}$.

A graph G is said to be 2-*divisible* if for all (nonempty) induced subgraphs H of G, $V(H)$ can be partitioned into two sets A, B such that $\omega(A) < \omega(H)$ and $\omega(B) < \omega(H)$. Hoàng and McDiarmid [5] defined the notion of 2-divisibility. They actually conjecture that a graph is 2-divisible if and only if it is odd-hole-free. A graph is said to be *perfectly divisible* if for all induced subgraphs H of G, $V(H)$ can be partitioned into two sets A, B such that $H[A]$ is perfect and $\omega(B) < \omega(H)$. Hoàng [4] introduced the notion of perfect divisibility and proved ([4]) that (banner, odd hole)-free graphs are perfectly divisible. A nice feature of proving that a graph is perfectly divisible is that we get a quadratic upper bound for the chromatic number in terms of the clique number. More precisely:

Lemma 1.1. Let G be a perfectly divisible graph. Then $\chi(G) \leq \binom{\omega(G)+1}{2}$.

Proof. Induction on $\omega(G)$. Let $\omega(G) = \omega$. Let $X \subseteq V(G)$ such that $G[X]$ is perfect and $\chi(G \setminus X) < \omega$. Since $G \setminus X$ is perfectly divisible, $\chi(G \setminus X) \leq \binom{\omega}{2}$. Since $G[X]$ is perfect, $\chi(X) \leq \omega$. Consequently, $\chi(G) \leq \chi(G \setminus X) + \chi(X) \leq \omega + \binom{\omega}{2} = \binom{\omega+1}{2}$. □

Analogously, 2-divisibility gives an exponential χ-bounding function.

Lemma 1.2. Let G be a 2-divisible graph. Then $\chi(G) \leq 2^{\omega(G)-1}$.

Proof. Induction on $\omega(G)$. Let $\omega(G) = \omega$. Let (A, B) be a partition of $V(G)$ such that $\omega(A) < \omega$ and $\omega(B) < \omega$. Now $\chi(A) \leq 2^{\omega-2}$ and $\chi(B) \leq 2^{\omega-2}$. Consequently, $\chi(G) \leq \chi(A)+\chi(B) \leq 2^{\omega-2}+2^{\omega-2} = 2^{\omega-1}$. □

We end the introduction by setting up the notation that we will be using. For a vertex v of a graph G, $N(v)$ will denote the set of neighbors of v (we write $N_G(v)$ if there is a risk of confusion). The closed neighborhood of v, denoted $N[v]$, is defined to be $N(v)+v$. We define $M(v)$ (or $M_G(v)$)to be $V(G) \setminus N[v]$. Let X and Y be disjoint subsets of $V(G)$. We say X is complete to Y if every vertex in X is adjacent to every vertex in Y. We say X is anticomplete to Y if every vertex in X is non-adjacent to every vertex in Y. A set $X \subseteq V(G)$ is a *homogeneous set* if $1 < |X| < |V(G)|$ and every vertex of $V(G) \setminus X$ is either complete or anticomplete to X. If G contains a homogeneous set, we say that G *admits a homogeneous set decomposition.*

This paper is organized as follows. In section 2 we prove that if a graph contains neither a P_5 nor a C_5, then it is 2-divisible. In Section 3 we prove that if a graph is bull-free and either odd-hole-free or P_5-free, then it is perfectly divisible.

2 (P_5, C_5)-free graphs are 2-divisible

We start with some definitions. Let G be a graph. $X \subseteq V(G)$ is said to be *connected* if $G[X]$ is connected, and *anticonnected* if $G^c[X]$ is connected. For $X \subseteq V(G)$, a *component* of X is a maximal connected subset of X, and an *anticomponent* of X is a maximal anticonnected subset of X.

The following lemma is used several times in the sequel.

Lemma 2.1. Let G be a graph. Let $C \subseteq V(G)$ be connected, and let $v \in V(G) \setminus C$ such that v is neither complete nor anticomplete to C. Then there exist $a, b \in C$ such that $v - a - b$ is a path.

Proof. Since v is neither complete nor anticomplete to C, it follows that both the sets $N(v) \cap C$ and $M(v) \cap C$ are non-empty. Since C is connected, there exist $a \in N(v) \cap C$ and $b \in M(v) \cap C$ such that $ab \in E(G)$. But now $v - a - b$ is the desired path. This completes the proof. $\square$

We are ready to prove the main result of this section.

Theorem 2.1. Every (P_5, C_5)-free graph is 2-divisible.

Proof. Let G be a (P_5, C_5)-free graph. We may assume that G is connected. Let $v \in V(G)$, let $N = N(v), M = M(v)$. Let $C_1, \cdots, C_t$ be the components of M.

(1) We may assume that there is i such that no vertex of N is complete to C_i.

For, otherwise, $X_1 = M + v, X_2 = N$ is the desired partition. This proves (1).

Let i be as in (1), we may assume that $i = 1$.

(2) There do not exist n_1, n_2 in N and m_1, m_2 in M such that n_1 is adjacent to m_1 and not to m_2, and n_2 is adjacent to m_2 and not to m_1, and n_1 is non-adjacent to n_2.

For, otherwise, $G[\{n_1, n_2, m_1, m_2, v\}]$ is a P_5 or a C_5. This proves (2).

(3) For every $i > 1$ there exists $n \in N$ complete to C_i.

For suppose that there does not exist $n \in N$ that is complete to C_2. For $i = 1, 2$ let $n_i \in N$ have a neighbor in C_i. Since C_1, C_2 are connected, by Lemma 2.1, there exist $a_i, b_i \in C_i$ such that $n_i - a_i - b_i$ is a path. Since $b_1 - a_1 - n_1 - a_2 - b_2$ is not a P_5, we deduce that $n_1 \neq n_2$, and therefore n_1 is complete or anticomplete to C_2, and n_2 is complete or anticomplete to C_1. By the choice of C_1 and the assumption, n_1 is anticomplete to C_2, and n_2 to C_1. By (2) n_1 is adjacent to n_2. But now $b_2 - a_2 - n_2 - n_1 - a_1$ is a P_5, a contradiction. This proves (3).

From the set of vertices in N that have a neighbor in C_1, choose one that has the maximum number of neighbors in M; call it n. (Such a vertex exists because G is connected.) Let $X_1 = N(n)$, and let $X_2 = V(G) \setminus X_1$. Clearly X_1 does not contain a clique of size $w(G)$. We claim that $\omega(X_2) < \omega(G)$, thus proving that (X_1, X_2) is a partition certifying 2-divisibility.

Suppose that there is a clique K of size $\omega(G)$ in X_2. Then $n \notin X$. By (3), $K \setminus (C_2 \cup \cdots \cup C_t) \neq \emptyset$.

(4) $K \nsubseteq C_1$.

For suppose that $K \subseteq C_1$. Then $K \subseteq C_1 \setminus N(n)$. Let D be the component of $C_1 \setminus N(n)$ containing K. Then some vertex $p \in N(n) \cap C_1$ has a neighbor in D. Since D contains a clique of size $\omega(G)$, p is not complete to D. Since D is connected, by Lemma 2.1, there exist $d_1, d_2 \in D$ such that $p - d_1 - d_2$ is a path. But now $d_2 - d_1 - p - n - v$ is a P_5, a contradiction. This proves (4).

It follows from (4) that K has a vertex $k_1 \in N \setminus X_1$, and a vertex $k_2 \in M \setminus X_1$. Then k_1 is non-adjacent to n, and k_2 is non-adjacent to n. But now by (2) $N(k_1) \cap M$ strictly contains $N(n) \cap M$, and in particular k_1 has a neighbor in C_1, contrary to the choice of n. This completes the proof. □

An easy consequence of this is

Corollary 2.1. Let G be a (P_5, C_5)-free graph. Then $\chi(G) \leq 2^{\omega(G)-1}$.

Proof. Follows from Theorem 2.1 and Lemma 1.2 □

3 Perfect divisibility in bull-free graphs

For an induced subgraph H of a graph G, a vertex $c \in V(G) \setminus V(H)$ that is complete to $V(H)$ is called a *center* for H. Similarly, a vertex $a \in V(G) \setminus V(H)$ that is anticomplete to $V(H)$ is called an *anticenter* for H. For a hole $C = c_1 - c_2 - c_3 - c_4 - c_5 - c_1$, an *$i$-clone* is a vertex adjacent to c_{i+1} and c_{i-1}, and not to c_{i+2}, c_{i-2} (in particular c_i is an i-clone). An *i-star* is a vertex complete to $V(C) \setminus c_i$, and non-adjacent to c_i. A *clone* is a vertex that is an i-clone for some i, and a *star* is a vertex that is an i-star for some i. We will need the following results from [2] and [3].

Theorem 3.1. (from [3]) If G is bull-free, and G has a P_4 with a center and an anticenter, then G admits a homogeneous set decomposition, or G contains C_5.

Theorem 3.2. (from [2]) If G is bull-free and contains an odd hole or an odd antihole with a center and an anticenter, then G admits a homogeneous set decomposition.

Theorem 3.3. (from [2]) If G is bull-free, then either G admits a homogeneous set decomposition, or for every $v \in V(G)$, either $G[N(v)]$ or $G[M(v)]$ is perfect.

The next two theorems refine Theorem 3.3 in the special cases we are dealing with in this paper.

Theorem 3.4. If G is bull-free and odd-hole-free, then either G admits a homogeneous set decomposition, or for every $v \in V(G)$ the graph $G[M(v)]$ is perfect.

Proof. We may assume that G does not admit a homogeneous set decomposition. Let $v \in V(G)$ such that $G[M(v)]$ is not perfect. Since G is odd-hole-free, by the strong perfect graph theorem [1], $G[M(v)]$ contains an odd antihole of length at least 7, and therefore a three-edge-path P with a center. Now v is an anticenter for P, and so by Theorem 3.1, G admits a homogeneous set decomposition, a contradiction. This proves the theorem. □

Theorem 3.5. If G is bull-free and P_5-free, then either G admits a homogeneous set decomposition, or for some $v \in V(G)$, $G[M(v)]$ is perfect.

Proof. By Theorem 3.4 we may assume that G contains a C_5, say $C = c_1 - c_2 - c_3 - c_4 - c_5 - c_1$. We may assume that G does not admit a homogeneous set decomposition.

(1) Let D be a hole of length 5, and let $v \notin V(D)$. Then v is a clone, a star, a center or an anticenter for D.

Since G has no P_5, v cannot have exactly one neighbor in D. Suppose that v has exactly two neighbors in D. Since G is bull-free, the neighbors are non-adjacent, so v is a clone. Suppose that v has exactly two non-neighbors in D. Since G is bull-free, the non-neighbors are adjacent, and v is a clone. The cases when v has 0, 4, 5 neighbors in D result in v being an anticenter, star, and a center for D, respectively. This proves (1).

(2) Let D be a hole of length 5 in G. Then there is no anticenter for D.

Suppose that v is an anticenter for D, we may assume that $D = C$. By Theorem 3.3 there is no center for D. Since G is connected, we may assume that v has a neighbor u such that u has a neighbor in $V(D)$. Let P be a path starting at u and with $V(P) \setminus u \subseteq V(D)$ with $|V(P)|$ maximum. Since $v - u - P$ is not a P_5, and v is not a center for P, it follows that for some i, v is adjacent to c_i and to c_{i+1}, but not to c_{i+2}. But now $G[\{c_i, c_{i+1}, c_{i+2}, u, v\}]$ is a bull, a contradiction. This proves (2).

(3) Let d_i and d'_i be i-clones non-adjacent to each other. Let v be adjacent to d_i and not to d'_i. Then v is a center for C, or v is an i-star for C, or v is an i-clone for C. Moreover, let D be the hole obtained from C by replacing c_i with d_i, and let D' be the hole obtained from C by replacing c_i with d'_i. It follows that either

- v is an i-clone for both D and D', or
- v is a center for D, and an i-star for D'.

We may assume that $i = 1$. If v is anticomplete to $\{c_2, c_5\}$, then we get a contradiction to (1) or (2) applied to v and D'. Thus we may assume that v is adjacent to c_2. Suppose that v is non-adjacent to c_5. By (1) applied to D, v is adjacent to c_3. But now $d'_1 - c_5 - d_1 - v - c_3$ is a P_5, a contradiction. Thus v is adjacent to c_5. By (1) applied to D', v is either complete or anticomplete to $\{c_3, c_4\}$. Now if v is anticomplete to $\{c_3, c_4\}$, then v is an i-clone; if v is complete to $\{c_3, c_4\}$ then v is a center or an i-star for C. This proves (3).

(4) There do not exist $d_1, d'_1, d_3, d'_3, v_1, v_3$ such that

- $\{d_1, d'_1\}$ is not complete to $\{d_3, d'_3\}$, and
- for $i = 1, 3$
  - d_i and d'_i are i-clones non-adjacent to each other, and
  - v_i is adjacent to d_i and non-adjacent to to d'_i, and
  - v_i is not an i-clone.

Observe that by (3), no vertex of $\{d_1, d'_1\}$ is mixed on $\{d_3, d'_3\}$ and the same with the roles of $1, 3$ exchanged. It follows that $\{d_1, d'_1\}$ is anticomplete to $\{d_3, d'_3\}$, and in particular $v_1, v_3 \notin \{d_1, d'_1, d_3, d'_3\}$. By (3) applied to the hole $d'_1 - c_2 - c_3 - c_4 - c_5 - d'_1$ and d_3, d'_3, it follows that v_3 is complete to $\{d_1, d'_1\}$. Similarly v_1 is complete to $\{d_3, d'_3\}$. In particular $v_1 \neq v_3$. But now $G[\{d'_1, v_3, d_1, v_1, d'_3\}]$ is either a bull or a P_5, in both cases a contradiction. This proves (4).

(5) There is not both a 1-clone non-adjacent to c_1, and a 3-clone non-adjacent to c_3.

For suppose that such clones exist. For $i = 1, 3$ let X_i be a maximal anticonnected set of i-clones with c_i in X_i. Then $|X_i| > 1$ for $i = 1, 3$. Since X_i is anticonnected, it follows from (3) that X_1 is anticomplete to X_3. Since $|X_1|, |X_3| > 1$, and G does not admit a homogeneous set decomposition, it follows that neither X_1 nor X_3 is a homogeneous set in G. Therefore for $i = 1, 3$ there exists $v_i \notin X_i$ with a neighbor and a non-neighbor in X_i. Then $v_i \notin X_1 \cup X_3$. Note that $X_i + v_i$ is anticonnected, and hence by the maximality of X_i, it follows that v_i is not an i-clone. By applying Lemma 2.1 in G^c with v_i and X_i for $i = 1, 3$, it follows that there exist $d_i, d_i' \in X_i$ such that d_i is non-adjacent to d_i', v_i is adjacent to d_i, and v_i is non-adjacent to d_i'. But now we get a contradiction to (4). This proves (5).

(6) For some i, $V(G) = N[c_i] \cup N[c_{i+2}]$ (here addition is *mod 5*).

Suppose that (6) is false. Since (6) does not hold with $i = 1$, (1), (2) and symmetry imply that we may assume that there is a 1-clone c_1' non-adjacent to c_1. Since (6) does not hold with $i = 5$, again by (1), (2) and symmetry we may assume that there is a 2-clone c_2' non-adjacent to c_2. Finally, since (6) does not hold with $i = 3$, by (1), (2) and symmetry we get a 3-clone c_3' non-adjacent to c_3. But this is a contradiction to (5). This proves (6).

Let i be as in (6); we may assume that $i = 1$. Suppose that $G[M(c_1)]$ is not perfect. Then, by the strong perfect graph theorem [1], $G[M(c_1)]$ contains an odd hole or an odd antihole H. But now c_3 is a center for H, and c_1 is an anticenter for H, contrary to Theorem 3.2. This proves the theorem. □

A graph G is *perfectly weight divisible* if for every non-negative integer weight function w on $V(G)$, there is a partition of $V(G)$ into two sets P, W such that $G[P]$ is perfect and the maximum weight of a clique in $G[W]$ is smaller than the maximum weight of a clique in G.

Theorem 3.6. A minimal non-perfectly weight divisible graph does not admit a homogeneous set decomposition.

Proof. Let G be such that all proper induced subgraphs of G are perfectly weight divisible. Let w be a weight function on $V(G)$. Let X be a homogeneous set in G, with common neighbors N and let $M = V(G) \setminus (X \cup N)$. Let G' be obtained from G by replacing X with a single vertex x of X with weight $w(x)$ equal to the maximum weight of a clique in $G[X]$. Let T be the maximum weight of a clique in G.

Let (P', W') be a partition of $V(G')$ corresponding to the weight w. Let (X_p, X_w) be a partition of X where $G[X_p]$ is perfect and the maximum weight of a clique in $G[X_w]$ is smaller than the maximum weight of a clique in $G[X]$. We construct a partition of $V(G)$.

Suppose first that $x \in W'$. Then let $P = P'$ and $W = W' \cup X$. Clearly this is a good partition. Now suppose that $x \in P'$. Let $P = (P' \setminus x) \cup X_p$ and let $W = W' \cup X_w$. By a theorem of [6], $G[P]$ is perfect. Suppose that W contains a clique K with weight T. Then $K \cap X_w$ is non-empty. Let K'

be a clique of maximum weight in X. Now $(K \setminus X_w) \cup K'$ is a clique in G with weight greater than T, a contradiction. This proves the theorem. □

We can now prove our main result:

Theorem 3.7. Let G be a bull-free graph that is either odd-hole-free or P_5-free. Then G is perfectly weight divisible, and hence perfectly divisible.

Proof. Let G be a minimal counterexample to the theorem. Then there is a non-negative integer weight function w on $V(G)$ for which there is no partition of $V(G)$ as in the definition of being perfectly weight divisible. Let U be the set of vertices of G with $w(v) > 0$, and let $G' = G[U]$. By theorems 3.4, 3.5, 3.6, G' has a vertex v such that $G'[M_{G'}(v)]$ is perfect. But now, since $w(v) > 0$, setting $P = M_{G'}(v) + v$ and $W = N_{G'}(v) \cup (V(G) \setminus U)$ we get a partition of $V(G)$ as in the definition of being perfectly weight divisible, a contradiction. This proves the theorem. □

Corollary 3.1. Let G be a bull-free graph that is either odd-hole-free or P_5-free. Then $\chi(G) \leq \binom{\omega(G)+1}{2}$.

Proof. Follows from Theorem 3.7 and Lemma 1.1. □

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