Obstructions for three-coloring graphs without induced paths on six vertices

Maria Chudnovsky*[1], Jan Goedgebeur†[2], Oliver Schaudt[3], and Mingxian Zhong[4]

[1] *Princeton University, Princeton, NJ 08544, USA. E-mail: mchudnov@math.princeton.edu*

[2] *Ghent University, Ghent, Belgium. E-mail: jan.goedgebeur@ugent.be*

[3] *Universität zu Köln, Köln, Germany. E-mail: schaudto@uni-koeln.de*

[4] *Columbia University, New York, NY 10027, USA. E-mail: mz2325@columbia.edu*

Abstract

We prove that there are 24 4-critical P_6-free graphs, and give the complete list. We remark that, if H is connected and not a subgraph of P_6, there are infinitely many 4-critical H-free graphs. Our result answers questions of Golovach et al. and Seymour.

1 Introduction

A *k-coloring* of a graph $G = (V, E)$ is a mapping $c : V \to \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for all edges $uv \in E$. If a k-coloring exists, we say that G is *k-colorable*. We say that G is *k-chromatic* if it is k-colorable but not $(k-1)$-colorable. A graph is called *k-critical* if it is k-chromatic, but every proper subgraph is $(k-1)$-colorable. For example, the class of 3-critical graphs is the family of all chordless odd cycles. The characterization of critical graphs is a notorious problem in the theory of graph coloring, and also the topic of this paper.

Since it is NP-hard to decide whether a given graph admits a k-coloring, assuming $k \geq 3$, there is little hope of giving a characterization of the $(k+1)$-critical graphs that is useful for algorithmic purposes. The picture changes if one restricts the structure of the graphs under consideration.

Let a graph H and a number k be given. An *H-free* graph is a graph that does not contain H as an induced subgraph. We say that a graph G is *k-critical H-free* if G is H-free, k-chromatic, and every H-free proper subgraph of G is $(k-1)$-colorable. Note that a *k-critical H-free* graph is not necessarily a k-critical graph. In this paper we stick to the case of 4-critical graphs; these graphs we may informally call *obstructions.*

Bruce et al. [3] proved that there are exactly six 4-critical P_5-free graphs, where P_t denotes the path on t vertices. Randerath et al. [17] have shown that the only 4-critical P_6-free graph without a triangle is the Grötzsch graph (i.e., the graph F_{18} in Fig. 2). More recently, Hell and Huang [11] proved that there are four 4-critical P_6-free graphs without induced four-cycles.

In view of these results, Golovach et al. [10] posed the question of whether the list of 4-critical P_6-free graphs is finite (cf. *Open Problem 4* in [10]). In fact, they ask whether there is a certifying algorithm for the 3-colorability problem in the class of P_6-free graphs, which is an immediate consequence of the finiteness of the list. Our main result answers this question affirmatively.

Theorem 1. *There are exactly 24 4-critical P_6-free graphs.*

*Partially supported by NSF grant DMS-1265803.

†Supported by a Postdoctoral Fellowship of the Research Foundation Flanders (FWO).

These 24 graphs, which we denote here by F_1-F_{24}, are shown in Fig. 1 and 2. The list contains several familiar graphs, e.g., F_1 is K_4, F_2 is the 5-wheel, F_3 is the Moser-spindle, and F_{18} is the Grötzsch graph. The adjacency lists of these graphs can be found in the Appendix. Let $\mathcal{L} = \{F_1, \ldots, F_{24}\}$.

We also determined that there are exactly 80 4-vertex-critical P_6-free graphs (details on how we obtained these graphs can be found in the Appendix). Table 1 gives an overview of the counts of all 4-critical and 4-vertex-critical P_6-free graphs. All of these graphs can also be obtained from the *House of Graphs* [2] by searching for the keywords "4-critical P6-free" or "4-vertex-critical P6-free" where several of their invariants can be found.

In Section 8 we show that there are infinitely many 4-critical P_7-free graphs using a construction due to Pokrovskiy [16]. Note that there are infinitely many 4-critical claw-free graphs. For example, this follows from the existence of 4-regular bipartite graphs of arbitrary large girth (cf. [13] for an explicit construction of these), whose line graphs are then 4-chromatic. Also, there are 4-chromatic graphs of arbitrary large girth, which follows from a classical result of Erdős [6]. This together with Theorem 1 yields the following dichotomy theorem, which answers a question of Seymour [18].

Theorem 2. *Let H be a connected graph. There are finitely many 4-critical H-free graphs if and only if H is a subgraph of P_6.*

We will next give a sketch of the proof of our main result, thereby explaining the structure of this paper. An extended abstract of this paper appeared at SODA 2016 [5].

Vertices	Critical graphs	Vertex-critical graphs
4	1	1
6	1	1
7	2	7
8	3	6
9	4	16
10	6	34
11	2	3
12	1	1
13	3	9
16	1	2
total	24	80

Table 1: Counts of all 4-critical and 4-vertex-critical P_6-free graphs

1.1 Sketch of the proof

Given a 4-critical P_6-free graph G, our aim is to show that it is contained in $\mathcal{L}$. Our proof is based on the contraction (and uncontraction) of a particular kind of subgraph called *tripod*. Tripods have been used before in the design of 3-coloring algorithms for P_7-free graphs [1]. In Section 2 tripods are defined, and it is shown that contracting a maximal tripod to a single triangle is a *safe* operation for our purpose.

Our proof considers two cases.

1. If all maximal tripods of G are just single triangles, G is diamond-free. Here, a *diamond* is the graph obtained by removing an edge from K_4. We determine all 4-critical $(P_6, \text{diamond})$-free graphs in Section 3 and verify that they are in $\mathcal{L}$. Our proof is computer-aided and builds on a substantial strengthening of a method by Hoàng et al. [12].

2. If G contains a maximal tripod that is not a triangle, the structural analysis is more involved. We further distinguish two cases.

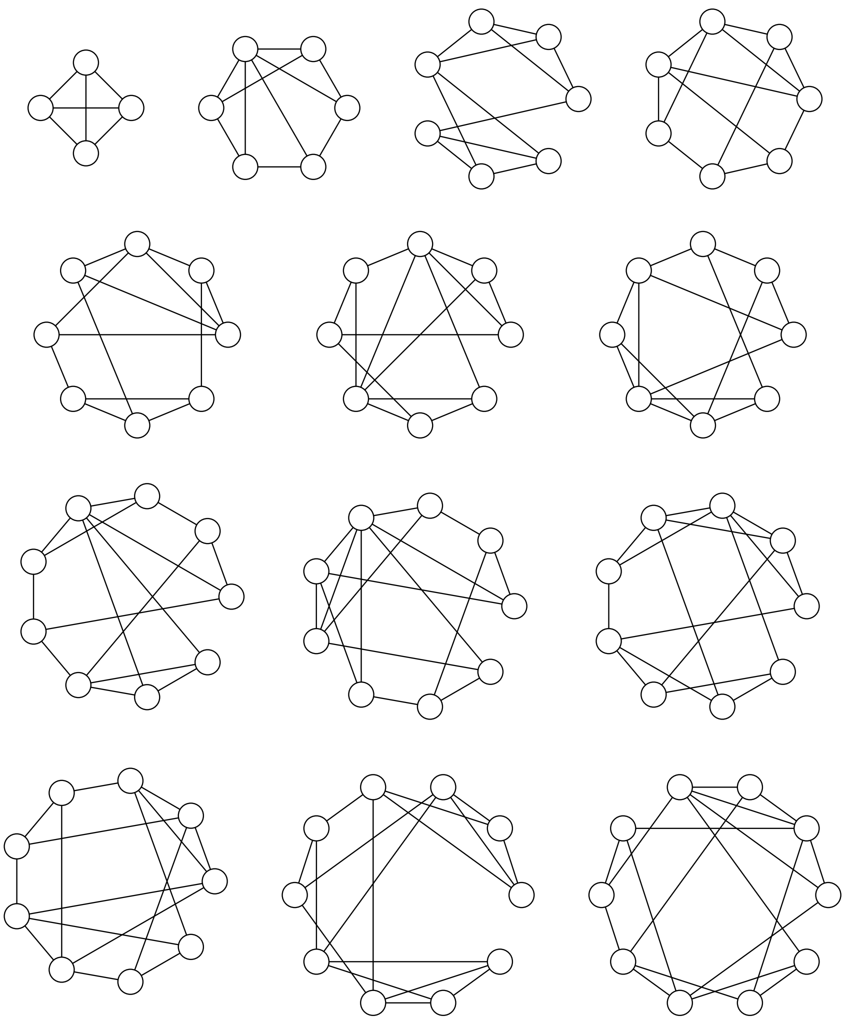

Figure 1: The graphs F_1 to F_{13}, in reading direction

(a) If G has a vertex x with neighbors in all three sets of the tripod, an exhaustive computer search shows that G has at most 18 vertices. This is done in Section 5. The algorithm used is completely different from the one used in the (P_6, diamond)-free case. It mimics the way that a tripod can be traversed, thereby applying a set of strong pruning rules that exploit the minimality of the obstruction.

    In Section 6 we show that a 4-critical P_6-free graph on at most 28 vertices is a member of $\mathcal{L}$ using a computer search similar to the one in Section 3. This time, we allow the graphs to contain diamond, but discard it if it has more than 28 vertices. From this result we see that G must be a member of $\mathcal{L}$ since $|V(G)| \leq 18$.

(b) If G does not contain such a vertex, a rigorous structural analysis (presented in Section 4) combined with an inductive argument using the previous results proves that $|V(G)| \leq 28$. Like in case (a), it follows that $G \in \mathcal{L}$.

    The analysis in Section 4 does not rely on a computer search.

We wrap up the whole proof in Section 7.

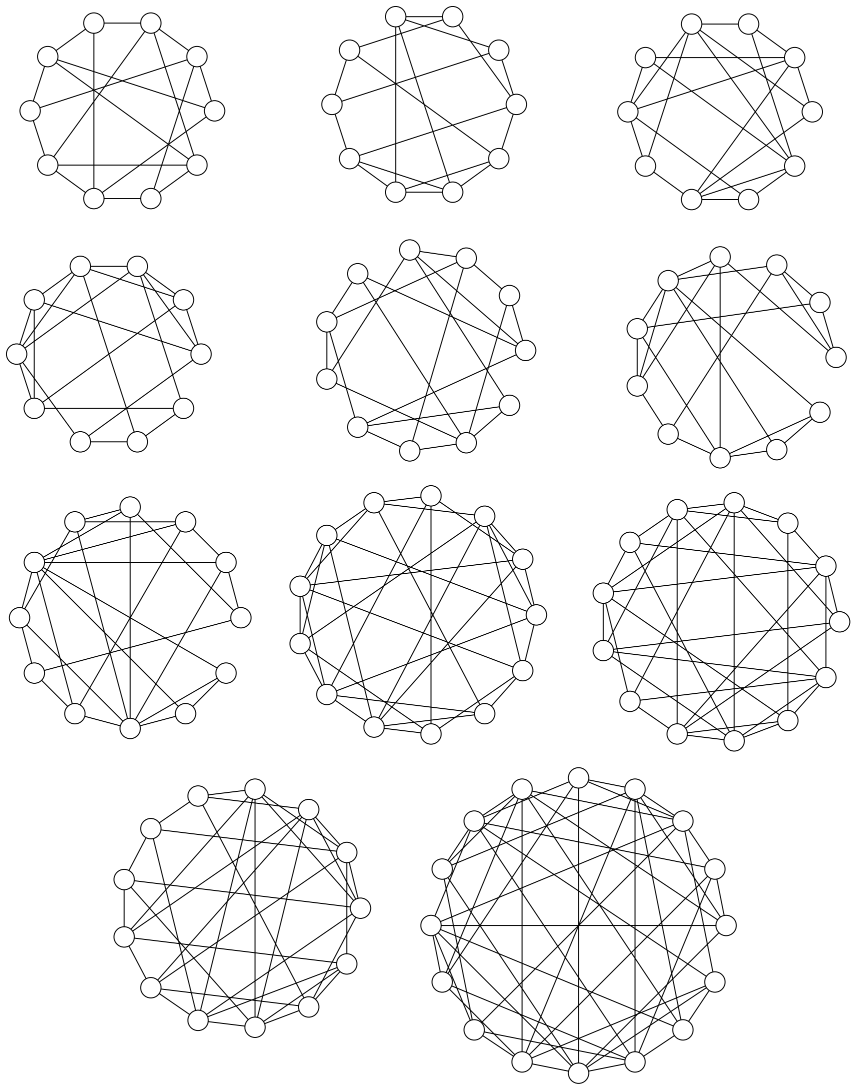

Figure 2: The graphs F_{14} to F_{24}, in reading direction

As mentioned earlier, in Section 8 we show that there are infinitely many 4-critical P_7-free graphs, which results in our dichotomy theorem.

2 Tripods

Let G be a graph, and let $A \subseteq V(G)$ and $b \in V(G) \setminus A$. We say that b is *complete* to A if b is adjacent to every vertex of A, and b is *anticomplete* to A if b is non-adjacent to every vertex of A. If b has both a neighbor and a non-neighbor in A, then b is *mixed on* A. For $B \subseteq V(G) \setminus A$, B is *complete* to A if every vertex of B is complete to A, and B is *anticomplete* to A if every vertex of B is anticomplete to A.

A *tripod* in a graph G is a triple $T = (A_1, A_2, A_3)$ of disjoint stable sets with the following properties:

(a) $A_1 \cup A_2 \cup A_3 = \{v_1, \ldots, v_k\}$;

(b) $v_i \in A_i$ for $i = 1, 2, 3$;

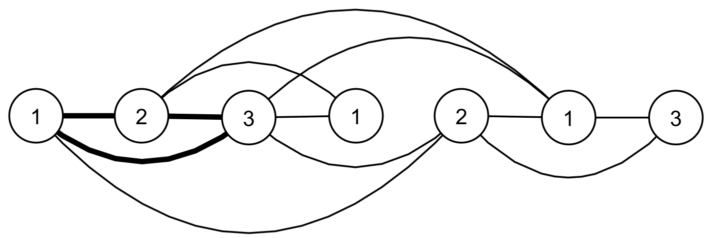

Figure 3: A tripod T with vertex set $V(T) = \{v_1, \dots, v_7\}$ from left to right. The bold triangle is the root of T. The numbers in the vertices denote the unique coloring (up to permutation of colors) of T.

(c) $v_1v_2v_3$ is a triangle, the *root* of T; and

(d) for all $i \in \{1,2,3\}$, $\{\ell, k\} = \{1,2,3\} \setminus \{i\}$, and $j \in \{4,\dots,k\}$ with $v_j \in A_i$, the vertex v_j has neighbors in both $\{v_1, \dots, v_{j-1}\} \cap A_\ell$ and $\{v_1, \dots, v_{j-1}\} \cap A_k$.

Assuming that G admits a 3-coloring, it follows right from the definition above that each A_i is contained in a single color class. Moreover, since $v_1v_2v_3$ is a triangle, A_1, A_2, A_3 are pairwise contained in distinct color classes.

An illustration of a tripod is given in Fig. 3.

To better reference the ordering of the tripod, we put $t(v_1) = t(v_2) = t(v_3) = 0$, and $t(v_i) = i - 3$ for all $4 \leq i \leq k$. For each $u \in A_i$, let $n_j(u)$ be the neighbor v of u in A_j with $t(v)$ minimum, where $i, j \in \{1,2,3\}, i \neq j$. We write $T(t) = G|\{v \in V(T) : t(v) \leq t\}$, i.e., the subgraph induced by G on the vertex set $\{v \in V(T) : t(v) \leq t\}$. Moreover, we write T_i for the graph $G|(A_j \cup A_k)$ where $\{i, j, k\} = \{1,2,3\}$, and finally $T_i(t)$ for the graph $G|\{v \in A_j \cup A_k : t(v) \leq t\}$.

We call a tripod (A_1, A_2, A_3) *maximal* in a given graph if no further vertex can be added to any set A_i without violating the tripod property.

2.1 Contracting a tripod

By *contracting* a tripod (A_1, A_2, A_3) we mean the operation of identifying each A_i to a single vertex a_i, for all $i = 1,2,3$. We then make a_i adjacent to the union of neighbors of the vertices in A_i, for all $i = 1,2,3$.

The neighborhood of a vertex v in a graph G is denoted $N_G(v)$. If G is clear from the context we might also omit G in the subscript. If U is a subset of the vertex set of G, we denote the induced subgraph on the set U by $G|U$.

Lemma 3. *Let G be a graph with a maximal tripod T such that no vertex of G has neighbors in all three classes of T. Let G' be the graph obtained from G by contracting T. Then the following holds.*

(a) The graph G is 3-colorable if and only if G' is 3-colorable, and

(b) if G is P_6-free, G' is P_6-free.

Proof. Assertion (a) follows readily from the definition of a tripod, so we just prove (b). For this, suppose that G is P_6-free but G' contains an induced P_6, say $P = v_1\text{-}\dots\text{-}v_6$. Let $T = (A_1, A_2, A_3)$, and let a_i be the vertex of G' the set A_i is contracted to, for $i = 1,2,3$.

Since P is an induced path, it cannot contain all three of a_1, a_2, a_3. Moreover, if P contains neither of a_1, a_2, a_3, then G contains a P_6, a contradiction.

Suppose that P contains, say, a_1 and a_2. We may assume that $a_1 = v_i$ and $a_2 = v_{i+1}$ for some $1 \leq i \leq 3$. If $i = 1$, pick $b \in A_1$ and $c \in A_2 \cap N_G(v_3)$ with minimum distance in T_3. Otherwise, if $i \geq 2$, pick $b \in A_1 \cap N_G(v_{i-1})$ and $c \in A_2 \cap N_G(v_{i+2})$ again with minimum distance in T_3. In both cases, let Q be the shortest path between b and c in T_3. Such a path exists since T_i is connected for $i = 1,2,3$ by the definition of a tripod. Due to the choice of b and c, the induced path v_1-...-b-Q-c-...-v_6 is induced in G, which means G contains a P_6, a contradiction.

So, we may assume that P contains only one of a_1, a_2, a_3, say $v_i = a_1$ for some $1 \leq i \leq 3$. We obtain an immediate contradiction if $i = 1$, so suppose that $i \geq 2$. Since v_{i+2} is not contained in T, we may assume that v_{i+2} is anticomplete to A_2 in G. Pick $b \in A_1 \cap N_G(v_{i-1})$ and $c \in A_1 \cap N_G(v_{i+1})$ such that the distance in T_3 between b and c is minimum. Let Q be a shortest path in T_3 between b and c. Since $v_i v_{i+2} \notin E(G')$, v_{i+2} is anticomplete to A_1 and thus to $V(Q)$ in G. If $b = c$, then v_1-...-v_{i-1}-b-v_{i+1}-...-v_6 is a P_6 in G, a contradiction. Otherwise, the induced path v_{i-1}-b-Q-c-v_{i+1}-v_{i+2} is induced in G and contains at least six vertices, which is also contradictory. □

3 Diamond-free obstructions

Recall that a diamond is the graph obtained by removing an edge from K_4. After successively contracting all maximal tripods in a graph, we are left with a diamond-free graph. In this section we prove the following statement.

Lemma 4. *There are exactly six 4-critical (P_6, diamond)-free graphs.*

These graphs are F_1, F_{11}, F_{14}, F_{16}, F_{18}, and F_{24} in Fig. 1 and 2.

The proof of Lemma 4 is computer-aided and builds upon a method recently proposed by Hoàng et al. [12]. With this method they have shown that there is a finite number of 5-critical (P_5, C_5)-free graphs. The idea is to automatize the large number of necessary case distinctions, resulting in an exhaustive enumeration algorithm. Since we have to deal with a graph class which is substantially less structured, we need to significantly extend their method.

3.1 Preparation

In order to prove Lemma 4, we make use of the following tools.

Let G be a k-colorable graph. We define the *k-hull* of G, denoted G_k, to be the graph with vertex set $V(G)$ where two vertices u, v are adjacent if and only if there is no k-coloring of G where u and v receive the same color. Note that G_k is a simple supergraph of G, since adjacent vertices can never receive the same color in any coloring. Moreover, G_k is k-colorable.

It is easy to see that a k-critical graph cannot contain two distinct vertices, u and v say, such that $N(u) \subseteq N(v)$. The following observation is a proper generalization of this fact. If U is a vertex subset of a graph G, we denote by $G - U$ the graph obtained from G by deleting the vertices in U.

Lemma 5. *Let $G = (V, E)$ be a k-vertex-critical graph and let U, W be two non-empty disjoint vertex subsets of G. Let $H := (G - U)_{k-1}$. If there exists a homomorphism $\phi : G|U \mapsto H|W$, then $N_G(u) \setminus U \not\subseteq N_H(\phi(u))$ for some $u \in U$.*

Note that, in the statement of Lemma 5, H is well-defined since G is k-vertex-critical.

Proof of Lemma 5. Suppose that $N_G(u) \setminus U \subseteq N_H(\phi(u))$ for all $u \in U$. Fix some $(k-1)$-coloring c of H. In particular, for each $u \in U$, the color of $\phi(u)$ is different from that of any member of $N_H(\phi(u))$.

We now extend c to a $(k-1)$-coloring of G by giving any $u \in U$ the color $c(\phi(u))$. It suffices to show that this is a proper coloring. Clearly there are no conflicts between any two vertices of U, since ϕ is a homomorphism. Let $u \in U$ and $v \in N_G(u) \setminus U$ be arbitrary. Since $N_G(u) \setminus U \subseteq N_H(\phi(u))$, $c(v) \neq c(\phi(u))$, and so u and v receive distinct colors. But this contradicts with the assumption that G is a k-vertex-critical graph. □

We make use of Lemma 5 in the following way. Assume that G is a $(k-1)$-colorable graph that is an induced subgraph of some k-vertex-critical graph G'. Pick two non-empty disjoint vertex subsets $U, W \subseteq V$ of G, and let $H := (G - U)_{k-1}$. Assume there exists a homomorphism $\phi : G|U \mapsto H|W$ such that

$N_G(u) \setminus U \subseteq N_H(\phi(u))$ for all $u \in U$. Then there must be some vertex $x \in V(G') \setminus V(G)$ which is adjacent to some $u \in U$ but non-adjacent to $\phi(u)$ in G'. Moreover, x is non-adjacent to $\phi(u)$ in the graph $(G' - U)_{k-1}$.

We also make use of the following well-known fact.

Lemma 6. *A k-vertex-critical graph has minimum degree at least $k-1$.*

Another fact we need is the following.

Lemma 7. *Any $(P_6, \text{diamond})$-free 4-critical graph other than K_4 contains an induced C_5.*

Proof. By the Strong Perfect Graph Theorem [4], every 4-critical graph different from K_4 must contain an odd hole or an anti-hole as an induced subgraph. A straightforward argumentation shows that only the 5-hole, C_5, can possibly appear. □

3.2 The enumeration algorithm

Generally speaking, our algorithm (Algorithm 1 together with Algorithm 2) constructs a graph G' with $n+1$ vertices from a graph G with n vertices by adding a new vertex and connecting it to vertices of G in all possible ways. So, all graphs constructed from G contain G as an induced subgraph. Since 3-colorability and $(P_6, \text{diamond})$-freeness are both hereditary properties, we do not need to expand G if it is not 3-colorable, contains a P_6 or a diamond.

We use Algorithm 1 below to enumerate all $(P_6, \text{diamond})$-free 4-critical graphs. In order to keep things short, we use the following conventions for a graph G. We call a pair (u, v) of distinct vertices for which $N_G(u) \subseteq N_{(G-u)_3}(v)$ *similar vertices*. Similarly, we call a 4-tuple (u, v, u', v') of distinct vertices with $uv, u'v' \in E(G)$ such that $N_G(u) \setminus \{v\} \subseteq N_{(G-\{u,v\})_3}(u')$ and $N_G(v) \setminus \{u\} \subseteq N_{(G-\{u,v\})_3}(v')$ *similar edges*. Finally, we define *similar triangles* in an analogous fashion.

Algorithm 1 Generate $(P_6, \text{diamond})$-free 4-critical graphs

```
Let 𝓕 be an empty list
Add K_4 to the list 𝓕
Construct(C_5)  // i.e. perform Algorithm 2
return 𝓕
```

We now prove that Algorithm 1 is correct.

Lemma 8. *Assume that Algorithm 1 terminates, and outputs the list of graphs $\mathcal{F}$. Then $\mathcal{F}$ is the list of all $(P_6, \text{diamond})$-free 4-critical graphs.*

Proof. In view of lines 1 and 3 of Algorithm 2, it is clear that all graphs of $\mathcal{F}$ are 4-critical $(P_6, \text{diamond})$-free. So, it remains to prove that $\mathcal{F}$ contains all $(P_6, \text{diamond})$-free 4-critical graphs. To see this, we first prove the following claim.

Claim 1. *For every $(P_6, \text{diamond})$-free 4-critical graph F other than K_4, Algorithm 2 applied to C_5 generates an induced subgraph of F with i vertices for every $5 \leq i \leq |V(F)|$.*

We prove this inductively, as an invariant of our algorithm. Due to Lemma 7, we know that F contains an induced C_5, so the claim holds true for $i = 5$.

So assume that the claim is true for some $i \geq 5$ with $i < |V(F)|$. Let G be the induced subgraph of F with $|V(G)| = i$. First assume that G contains similar vertices (u, v). We put $U = \{u\}$, $W = \{v\}$, $H = (F - u)_3$. Then, by Lemma 5, $N_F(u) \setminus U \nsubseteq N_H(v)$. Hence, there is some vertex $x \in V(F) \setminus V(G)$ which is adjacent to u in F, but not to v in H. Following the statement of line 10, Construct$(F|(V(G) \cup \{x\}))$ is called. We omit the discussion of the lines 16 and 20, as they are analogous.

So assume that G contains a vertex u of degree at most 2. Then, since the minimum degree of any 4-vertex-critical graph is at least 3, there is some vertex $x \in V(F) \setminus V(G)$ adjacent to u. Following the statement of line 26, Construct$(F|(V(G) \cup \{x\}))$ is called.

Algorithm 2 Construct(Graph G)

1: **if** G is $(P_6, \text{diamond})$-free AND not generated before **then**
2: **if** G is not 3-colorable **then**
3: **if** G is 4-critical P_6-free **then**
4: add G to the list $\mathcal{F}$
5: **end if**
6: **return**
7: **else**
8: **if** G contains similar vertices (u, v) **then**
9: **for** every graph H obtained from G by attaching a new vertex x and incident edges in all possible ways, such that $ux \in E(H)$, but $vx \notin E((H-u)_3)$ **do**
10: Construct(H)
11: **end for**
12: **else if** G contains a vertex u of degree at most 2 **then**
13: **for** every graph H obtained from G by attaching a new vertex x and incident edges in all possible ways, such that $ux \in E(H)$ **do**
14: Construct(H)
15: **end for**
16: **else if** G contains similar edges (u, v, u', v') **then**
17: **for** every graph H obtained from G by attaching a new vertex x and incident edges in all possible ways, such that $ux \in E(H)$ and $u'x \notin E((H-\{u,v\})_3)$, or $vx \in E(H)$ and $v'x \notin E((H-\{u,v\})_3)$ **do**
18: Construct(H)
19: **end for**
20: **else if** G contains similar triangles (u, v, w, u', v', w') **then**
21: **for** every graph H obtained from G by attaching a new vertex x and incident edges in all possible ways, such that $ux \in E(H)$ and $u'x \notin E((H-\{u,v,w\})_3)$, $vx \in E(H)$ and $v'x \notin E((H-\{u,v,w\})_3)$, or $wx \in E(H)$ and $w'x \notin E((H-\{u,v,w\})_3)$ **do**
22: Construct(H)
23: **end for**
24: **else**
25: **for** every graph H obtained from G by attaching a new vertex x and incident edges in all possible ways **do**
26: Construct(H)
27: **end for**
28: **end if**
29: **end if**
30: **end if**

Finally, if none of the above criteria apply to G, the algorithm attaches a new vertex to G in all possible ways, and calls Construct for all of these new graphs. Since $|V(F)| > |V(G)|$, among these graphs there is some induced subgraph of F, and of course this graph has $i+1$ vertices. This completes the proof of Claim 1.

Given that the algorithm terminates and K_4 is added to the list $\mathcal{F}$, Claim 1 implies that $\mathcal{F}$ must contain all 4-critical $(P_6, \text{diamond})$-free graphs. □

We implemented this algorithm in C with some further optimizations. To make sure that no isomorphic graphs are accepted (cf. line 1 of Algorithm 2), we use the program `nauty` [14, 15] to compute a canonical form of the graphs. We maintain a list of the canonical forms of all non-isomorphic graphs which were generated so far and only accept a graph if it was not generated before (and then add its canonical form to the list).

Our program does indeed terminate (in about 2 seconds), and outputs the six graphs F_1, F_{11}, F_{14}, F_{16}, F_{18}, and F_{24} from Fig. 1 and 2. Together with Lemma 8 this proves Lemma 4. Let us stress the fact that in order for the algorithm to terminate, all proposed expansion rules are needed.

Table 2 shows the number of non-isomorphic graphs generated by the program. The source code of the program can be downloaded from [7] and in the Appendix we describe how we extensively tested the correctness of our implementation.

The second and third author also extended this algorithm which allowed to determine all k-critical graphs for several other cases as well (see [9]).

$\|V(G)\|$	5	6	7	8	9	10	11	12	13	14	15	16
# graphs generated	1	4	16	55	130	230	345	392	395	279	211	170
$\|V(G)\|$	17	18	19	20	21	22	23	24	25	26	27	28
# graphs generated	112	95	74	53	40	32	20	15	12	3	1	0

Table 2: Counts of the number of non-isomorphic (P_6, diamond)-free graphs generated by our implementation of Algorithm 1

4 Uncontracting a triangle to a tripod

Let G be a P_6-free graph that is not 3-colorable. Assume that G contains a maximal tripod $T = (A_1, A_2, A_3)$ with $A_1 \cup A_2 \cup A_3 = \{v_1, \ldots, v_k\}$. The aim of this section is to prove the following statement.

Lemma 9. *Let G' be obtained from G by contracting a maximal tripod (A_1, A_2, A_3) to a triangle $\{a_1, a_2, a_3\}$. Let $H' \in \mathcal{L}$ be an induced subgraph of G'. If $H' = K_4$, assume that $|V(H') \cap \{a_1, a_2, a_3\}| < 3$. Then there exists an induced subgraph H of G that is not 3-colorable with at most $|V(H')| + 12$ vertices.*

In Section 5 a similar statement is proved for the case when $H' = K_4$ and $a_1, a_2, a_3 \in V(H')$. To construct the graph H as in Lemma 9 we replace each of a_1, a_2, a_3 by a subset of A_1, A_2, A_3 (call these subsets C_1, C_2, C_3 respectively) such that every vertex of $N_{H'}(a_i) \setminus \{a_1, a_2, a_3\}$ has a neighbor in C_i. We then add a few more vertices from $A_1 \cup A_2 \cup A_3$, to ensure that in every 3-coloring of H each of the sets C_1, C_2, C_3 is monochromatic, and no colors appear in two of them.

The proof of Lemma 9 is organized as follows. The first several claims (Claim 2—Claim 6) are technical tools we need later. In Claim 7 we show that for every i we can construct $C'_i \subseteq A_i$ and add at most two more vertices so that every vertex of $N_{H'}(a_i) \setminus \{a_1, a_2, a_3\}$ has a neighbor in C'_i, and C'_i is monochromatic in every 3-coloring of the resulting graph. Claim 7 needs a few technical assumptions; in Claim 8 we show that the assumptions of Claim 7 hold. In Claim 9 we analyze how the sets C'_i from Claim 7 interact for two different values of i, again, under certain technical assumptions. In Claim 10 we show that the assumptions of Claim 9 hold except in two special cases. Finally, we deal with the two cases not covered by Claim 9, and use Claim 9 to produce H.

We now describe the proof of Lemma 9 in detail. Let C be a hole in G. A *leaf* for C is a vertex $v \in V(G) \setminus V(C)$ with exactly one neighbor in $V(C)$. Similarly, a *hat* for C is a vertex in $V(G) \setminus V(C)$ with exactly two neighbors $u, v \in V(C)$, where u is adjacent to v.

The following observation is immediate from the fact that G is P_6-free.

Claim 2. *No C_6 in G has a leaf or a hat.*

Claim 3. *The graph $T_i(t)$ is connected, for all $i \in \{1, 2, 3\}$ and $0 \leq t \leq k$.*

Proof. This follows readily from the definition of a tripod. □

Claim 4. *Let $a \in A_1$, and let $y, z \in V(G) \setminus (A_1 \cup A_2 \cup A_3)$ such that a-y-z is an induced path, and z is anticomplete to $A_2 \cup A_3$. Then $(A_2 \cup A_3) \setminus N(a)$ is stable, and in particular, for $i = 2, 3$ there exist $n_i \in N(a) \cap A_i$ such that n_2 is adjacent to n_3.*

Proof. By the maximality of the tripod, y is anticomplete to $A_2 \cup A_3$. Suppose there are $p_i \in A_i \setminus N(a)$, $i = 2, 3$, such that p_2 is adjacent to p_3. Since by 3 T_1 is connected, we can choose p_2, p_3 such that, possibly exchanging A_2 and A_3, p_2 has a neighbor q_3 in $A_3 \cap N(a)$. But now z-y-a-q_3-p_2-p_3 is a P_6, a contradiction. Since T_3 is connected, the second statement of the theorem follows. □

A *2-edge matching* are two disjoint edges ab, cd where ad, cb are non-edges.

Claim 5. *Let X be a stable set in $V(G) \setminus (A_1 \cup A_2 \cup A_3)$, such that for every $x, x' \in X$ there exists $p \in V(G) \setminus (A_1 \cup A_2 \cup A_3)$ such that p is anticomplete to A_1 and adjacent to exactly one of x, x'. Assume that there is a 2-edge matching $ax, a'x'$ between A_1 and X. Then*

(a) there do not exist $n_2 \in A_2$ and $n_3 \in A_3$ such that $\{a, a'\}$ is complete to $\{n_2, n_3\}$, and

(b) there exists $a'' \in A_1$, with $t(a'') < \max(t(a), t(a'))$ such that a'' is complete to $X \cap (N(a) \cup N(a'))$.

Proof. Suppose $ax, a'x'$ is such a matching. We may assume that xp is an edge. Let P be an induced path from a to a' with interior in $A_2 \cup A_3$. Such a path exists since T_1 is connected, and both a, a' have neighbors in $A_2 \cup A_3$. By the maximality of the tripod, $\{a, a'\}$ is anticomplete to $A_2 \cup A_3$. If P has at least three edges, then x-a-P-a'-x' is a P_6, so we may assume that a, a' have a common neighbor $n_2 \in A_2$. If p is non-adjacent to n_2, then p-x-a-n_2-a'-x' is a P_6, a contradiction. So p is adjacent to n_2, and therefore p has no neighbor in A_3. By symmetry, a, a' have no common neighbor in A_3, and so (a) follows.

Since a, a' do not have a common neighbor in A_3, there is an induced path a-b-c-d-a' from a to a' in T_2. It follows from the maximality of the tripod that $(N(a) \cup N(a')) \cap X$ is anticomplete to $A_2 \cup A_3$. Since z-a-b-c-d-a' and a-b-c-d-a'-z' are not a P_6 for any $z \in (N(a) \cap X) \setminus N(a')$, and $z' \in (N(a') \cap X) \setminus N(a)$, we deduce that c is complete to $((N(a) \cap X) \setminus N(a')) \cup ((N(a') \cap X) \setminus N(a))$. We may assume that there exists $x'' \in X \cap N(a) \cap N(a')$ such that c is non-adjacent to x'', for otherwise (b) holds. Now if p is non-adjacent to x'', then p-x-c-d-a'-x'' is a P_6, and if p is adjacent to x'', then p-x''-a-b-c-x' is a P_6, in both cases a contradiction. This proves (b). □

Claim 6. *Let X, Y be two disjoint stable sets in $V(G) \setminus (A_1 \cup A_2 \cup A_3)$ such that every vertex of $X \cup Y$ has a neighbor in A_1. Moreover, assume that the following assertions hold.*

1. *For every $x \in X$ and $y \in Y$, either*
    - *(i) x is adjacent to y,*
    - *(ii) x has a neighbor in $V(G)$ that is anticomplete to A_1, or*
    - *(iii) y has a neighbor in $V(G)$ that is anticomplete to A_1.*
2. *For every $x, x' \in X$ there exists $p \in V(G) \setminus (A_1 \cup A_2 \cup A_3)$ such that*
    - *(i) p is anticomplete to A_1, and*
    - *(ii) p is adjacent to exactly one of x, x'.*
3. *The above assertion holds for Y in an analogous way.*
4. *Let $u, v \in X \cup Y$ be distinct and non-adjacent. Then $N(u) \setminus A_1$ and $N(v) \setminus A_1$ are incomparable.*

Then either

(a) there is a vertex $p \in A_1$ which is complete to $X \cup Y$, or

(b) there exist $c, d \in A_1$, $p \in A_2$ and $q \in A_3$, such that p and q are adjacent, c is complete to X, d is complete to Y, and $\{c, d\}$ is complete to $\{p, q\}$.

Proof. After deleting all vertices of $V(G) \setminus (X \cup Y \cup A_2 \cup A_3)$ with a neighbor in A_1 (this does not change the hypotheses or the outcomes), we may assume that no vertex of $V(G) \setminus (A_2 \cup A_3 \cup X \cup Y)$ has a neighbor in A_1.

There exist $a, b \in A_1$ such that a is complete to X, and b is complete to Y. (1)

To see (1), it is enough to show that a exists, by symmetry. So, suppose that such an a does not exist. Pick $a \in A_1$ with $N(a) \cap X$ maximal, and note that a is not complete to X by assumption. Let $x' \in X \setminus N(a)$, and let $a' \in A_1 \cap N(x')$. By the maximality of $N(a) \cap X$, there exists $x \in N(a) \cap X$ such that a' is non-adjacent to x. Now $ax, a'x'$ is a 2-edge matching. But now by Claim 5(b), there exists $a'' \in A_1$ complete to $(N(a) \cap X) \cup x'$, contrary to the choice of a. This proves (1).

We may assume that no vertex of A_1 is complete to $X \cup Y$, for otherwise Claim 6(a) holds. Moreover, we may assume that there exist $x \in X$, and $y \in Y$ such that ax, by is a 2-edge matching. We choose a, b with $t(a) + t(b)$ minimum, and subject to that x and y are chosen adjacent if possible.

There is no $p \in A_1$, with $t(p) < \max(t(a), t(b))$ such that p is complete to $(X \setminus N(b)) \cup (Y \setminus N(a))$. (2)

Suppose such a p exists. We may assume that $t(a) > t(b)$, and hence $t(p) < t(a)$. By the choice of a and b, p is not complete to X, and so there is a 2-edge matching between $\{b, p\}$ and X. Thus by Claim 5(b), there exists a vertex p' with $t(p') < \max(t(b), t(p)) < t(a)$ that is complete to X, again contrary to the choice of a and b. This proves (2).

Either a is adjacent to $n_2(b)$, or b is adjacent to $n_2(a)$. (3)

Suppose that this is false. We may assume that $t(n_2(a)) > t(n_2(b))$. Let P be an induced path from $n_2(a)$ to $n_2(b)$ in $T_3(t(n_2(a)))$. Then $n_2(a)$ is the unique neighbor of a in P. Since a-$n_2(a)$-P-$n_2(b)$ is not a P_6, we deduce that P has length two, say $P = n_2(a)$-p-$n_2(b)$. Moreover, since x'-a-$n_2(a)$-p-$n_2(b)$-b is not a P_6 for any $x' \in X \setminus N(b)$, we know that $X \setminus N(b)$ is complete to p (recall that by the maximality of the tripod x' is anticomplete to A_2). Finally, since y'-b-$n_2(b)$-p-$n_2(a)$-a is not a P_6 for any $y' \in Y \setminus N(a)$, p is complete to $Y \setminus N(a)$. But since $p \in T_3(t(n_2(a)))$, we know that $t(p) < t(a) \leq \max(t(a), t(b))$, contrary to (2). This proves (3).

By (3) and using the symmetry between A_2 and A_3, we deduce that for $i = 2, 3$ there exists $n_i \in A_i$ such that $\{a, b\}$ is complete to $\{n_2, n_3\}$, and each n_i is the smallest neighbor of one of a, b in A_i w.r.t. their value of t. We may assume that n_2 is non-adjacent no n_3, for otherwise Claim 6(b) holds.

Let $z \in V(G) \setminus (A_1 \cup A_2 \cup A_3 \cup X \cup Y)$ be anticomplete to A_1. Then z is not mixed on any non-edge with one end in $X \setminus N(b)$ and the other in $Y \setminus N(a)$. In particular, either x is adjacent to y, or some $z \in V(G) \setminus (A_1 \cup A_2 \cup A_3 \cup \{x, y\})$ is complete to $\{x, y\}$ and anticomplete to A_1. (4)

Suppose z is mixed on a non-edge x', y' with $x' \in X \setminus N(b)$, and $y' \in Y \setminus N(a)$. From the maximality of the tripod, we may assume that z is anticomplete to A_2. Now one of the induced paths z-x'-a-n_2-b-y' and z-y'-b-n_2-a-x' is a P_6, a contradiction. The second statement follows from assumption 1. This proves (4).

By symmetry, we may assume that $t(n_2) > t(n_3)$, and that $n_2 = n_2(a)$. Thus, there is an induced path n_2-n_3'-c-n_3 in $T_1(t(n_2))$. Hence $t(c) < t(n_2)$, and so a is non-adjacent to c.

Vertex a is adjacent to n_3', and b has a neighbor in the set $\{c, n_3'\}$. (5)

Suppose first that x is adjacent to y. If a is non-adjacent to n_3', then y-x-a-n_2-n_3'-c is a P_6, a contradiction. Moreover, if b is anticomplete to $\{c, n_3'\}$, then x-y-b-n_2-n_3'-c is a P_6, a contradiction. So we may assume that x is non-adjacent to y, and thus, by the choice of x and y, deduce that $X \setminus N(b)$ is anticomplete to $Y \setminus N(a)$.

Now it follows from (4) that every $z \in V(G) \setminus (A_1 \cup A_2 \cup A_3 \cup X \cup Y)$ that is anticomplete to A_1 and that has a neighbor in $(X \setminus N(b)) \cup (Y \setminus N(a))$ is already complete to $(X \setminus N(b)) \cup (Y \setminus N(a))$. By assumption 2.(ii), we deduce that $X \setminus N(b) = \{x\}$, and similarly $Y \setminus N(a) = \{y\}$. Moreover, since x is non-adjacent to

y, it follows from assumption 4 and (4) that there exist $x' \in X \cap N(b)$ and $y' \in Y \cap N(a)$ such that xy' and yx' are edges. Now if a is non-adjacent to n'_3, then y-x'-a-n_2-n'_3-c is a P_6, and if b is anticomplete to $\{c, n'_3\}$, then x-y'-b-n_2-n'_3-c is a P_6, in both cases a contradiction. This proves (5).

If b is adjacent to n'_3, then (b) holds, and thus we may assume the opposite. By (5), b is adjacent to c. Since x-a-n'_3-c-b-y is not a P_6, we deduce that x is adjacent to y. Similarly, $X \setminus N(b)$ is complete to $Y \setminus N(a)$.

Let $d = n_1(n'_3)$. Then $t(d) \leq t(n_2) < t(a)$, and therefore $a \neq d$. Since d-n'_3-a-x-y-b is not a P_6, we deduce that d is complete to one of $X \setminus N(a)$ and $Y \setminus N(b)$.

By (2), d is not complete to both $X \setminus N(b)$ and $Y \setminus N(a)$. Suppose first that d is complete to $X \setminus N(b)$. Then there is some $y' \in Y \setminus N(a)$ that is non-adjacent to d. Since n'_3-d-x-y'-b-n_3 is not a P_6, we deduce that d is adjacent to n_3. Since $t(d) < t(a)$, d is not complete to X, and so there is $x' \in X \cap N(b)$ that is non-adjacent to d. Since x'-b-c-n'_3-d-x is not a P_6, d is adjacent to c. But dx, bx' is a 2-edge matching between $\{d, b\}$ and X, and $\{d, b\}$ is complete to $\{c, n_3\}$, contrary to Claim 5(a).

This proves that d is not complete to $X \setminus N(b)$, and thus d is complete to $Y \setminus N(a)$ and has a non-neighbor $x' \in X \setminus N(b)$. Suppose that d is non-adjacent to n_2. Since $t(n_2(d)) < t(d) \leq t(n_2)$, we deduce that $t(n_2(d)) < t(n_2)$, and a is non-adjacent to $n_2(d)$ (since $n_2 = n_2(a)$). But now $n_2(d)$-d-y-x'-a-n_2 is a P_6, a contradiction. This proves that d is adjacent to n_2.

Since $\{a, d\}$ is complete to $\{n_2, n'_3\}$, we deduce that there is no 2-edge matching between Y and $\{a, d\}$, by Claim 5(a). But then d is complete to Y, and (b) holds, since n_2 is adjacent to n'_3. This completes the proof. □

Claim 7. *Let G' be the graph obtained from G by contracting (A_1, A_2, A_3) to a triangle $a_1a_2a_3$. Let H' be an induced subgraph of G' with $a_1 \in V(H')$. Assume that no two non-adjacent neighbors of a_1 dominate each other in H'. Moreover, assume also that for every $v \in V(H')$, either*

1. *$N_{H'}(v) = X' \cup Y'$, each of X', Y' is stable,*
   - *(i) for every $x \in X'$ and $y \in Y'$, either*
     - *(A) x is adjacent to y,*
     - *(B) x has a neighbor in $V(H') \setminus (N_{H'}(v) \cup \{v\})$, or*
     - *(C) y has a neighbor in $V(H') \setminus (N_{H'}(v) \cup \{v\})$;*
   - *(ii) for every $x, x' \in X$ there exists $p \in V(H') \setminus \{v\}$ such that p is non-adjacent to v, and p is adjacent to exactly one of x, x';*
   - *(iii)* (1ii) *holds for Y in an analogous way.*
2. *$N_{H'}(v)$ is a triangle, or*
3. *$N_{H'}(v)$ induces a C_5.*

Then either

- *(a) some $a \in A_1$ is complete to $N'_H(a_1) \setminus \{a_2, a_3\}$; or*
- *(b) assumption 1 holds, and no vertex of A_1 is complete to $N_{H'}(a_1) \setminus \{a_2, a_3\}$, and there exist $a, b \in A_1$, $n_2 \in A_2$, and $n_3 \in A_3$ such that a is complete to $X' \setminus \{a_2, a_3\}$, b is complete to $Y' \setminus \{a_2, a_3\}$, $\{a, b\}$ is complete to $\{n_2, n_3\}$, and n_2 is adjacent to n_3;*
- *(c) assumption 2 or 3 holds, and G contains a non-3-colorable graph with seven or eight vertices; or*
- *(d) assumption 2 or 3 holds, there exists a set $A \subseteq A_1$, with $|A| \leq 3$, $n_2 \in A_2$, and $n_3 \in A_3$ such that every vertex of $N_{H'}(a_1)$ has a neighbor in A, A is complete to $\{n_2, n_3\}$, and n_2 is adjacent to n_3.*

Moreover, suppose a_2, a_3 are in $V(H')$. If (a) *holds, let $H = G|((V(H') \setminus \{a_1\}) \cup \{a\})$. Then H is isomorphic to H'. If* (b) *holds, let $H = G|((V(H') \setminus \{a_1\}) \cup \{a, b, n_1, n_2\})$. Then in every coloring of H, a and b have the same color. If* (d) *holds, let $H = G|((V(H') \setminus \{a_1\}) \cup A \cup \{n_1, n_2\})$. Then in every coloring of H, A is monochromatic.*

In all cases, H is 3-colorable if and only if H' is.

Proof. Suppose (a) does not hold.

Assume first that assumption 1 holds for a_1. Let $X = X' \setminus \{a_2, a_3\}$ and $Y = Y' \setminus \{a_2, a_3\}$. We now quickly check that the assumptions of Claim 6 hold for A_1, X, Y (in G).

- Every vertex $v \in X \cup Y$ has a neighbor in A_1, since every such v is adjacent to a_1 in H'.
- Assumption 1 of Claim 6 follows from assumption 1.(i) of Claim 7.
- Assumption 2 holds since there is such a p by assumption 1.(ii) of Claim 7. Since p is non-adjacent to a_1, we deduce that $p \notin \{a_2, a_3\}$, and so $p \in V(G) \setminus (A_1 \cup A_2 \cup A_3)$, as desired.
- Assumption 3 of Claim 6 follows analogously.
- Assumption 4 of Claim 6 is seen like this: $N(u) \setminus \{a_1\}$ and $N(v) \setminus \{a_1\}$ are incomparable in H', and $\{u, v\}$ is anticomplete to $\{a_2, a_3\}$ by the maximality of the tripod.

Now Claim 7 follows from Claim 6.

Next assume that assumption 2 holds for a_1, and $N(a_1) = \{x_1, x_2, x_3\}$. We claim that $\{x_1, x_2, x_3\} \cap \{a_2, a_3\} = \emptyset$. Suppose not; we may assume that $x_1 = a_2$, and $x_3 \notin \{a_2, a_3\}$. Then, in G, x_3 has both a neighbor in A_1 and a neighbor in A_2, contrary to the maximality of the tripod.

Assume first that there exist $b_1, b_2, b_3 \in A_1$ such that b_i is complete to $\{x_j, x_k\}$ (where $\{1, 2, 3\} = \{i, j, k\}$). Since (a) does not hold, b_i is non-adjacent to x_i, $i = 1, 2, 3$. If some $n_2 \in A_2$ is complete to $\{b_1, b_2, b_3\}$, then (c) holds. So we may assume that there is a 2-edge matching from A_2 to $\{b_1, b_2, b_3\}$, say $n_2b_1, n_2'b_2$. But then n_2-b_1-x_2-x_1-b_2-n_2' is a P_6, a contradiction. So we may assume that no vertex of A_1 is adjacent to both x_1 and x_2. For $i = 1, 2$, let c_i' be the smallest vertex in A_1 adjacent to x_i w.r.t. their value of t. By Claim 6 applied with $X = \{x_1\}$ and $Y = \{x_2\}$, and since no vertex of A_1 is adjacent to both x_1 and x_2, we deduce that there exist a neighbor c_i of x_i, and vertices $n_2 \in A_2$ and $n_3 \in A_3$, such that $\{c_1, c_2\}$ is complete to $\{n_2, n_3\}$, and n_2 is adjacent to n_3. If x_3 is adjacent to one of c_1, c_2, then (d) holds, so we may suppose this is not the case. Let c_3 be a neighbor of x_3 in A. We may assume that c_3 is non-adjacent to x_1. Now c_3-x_3-x_1-c_1-n_2-c_2 is not a P_6, and so c_3 is adjacent to n_2. Similarly, c_3 is adjacent to n_3. But now (d) holds. This finishes the case when assumption 2 holds.

Finally, assume that 3 holds. Let $N_{H'}(a_1) = \{x_1, \ldots, x_5\} = X$, where x_1-x_2-$\ldots$-x_5-x_1 is a C_5. Since $H'|X$ is connected, the maximality of the tripod implies that $\{a_2, a_3\} \cap X = \emptyset$. Let A be a minimum size subset of A_1 such that each of $x_1, \ldots, x_5$ has a neighbor in A. Since every $a \in A$ has a neighbor in A_2, we deduce that every $a \in A$ has two non-adjacent neighbors in X, due to P_6-freeness. We may assume that $|A| > 1$, or (c) holds, and so every $a \in A$ is either a *clone* (i.e., has two non-adjacent or three consecutive neighbors in X), a *star* (i.e., has four neighbors in X), or a *pyramid* for $G|X$ (i.e., has three neighbors in X, one of which is non-adjacent to the other two).

Suppose some $a \in A$ is a clone. We may assume a is adjacent to x_2 and x_5. If a is mixed on $A_2 \cup A_3$, then, since T_1 is connected, there is an induced path a-p-q where $p, q \in A_2 \cup A_3$. There is also an induced path a-x_2-x_3-x_4, so q-p-a-x_2-x_3-x_4 is a P_6, a contradiction. So a is complete to $A_2 \cup A_3$. If at most one vertex of A is not a clone and $|A| \leq 3$, then (using n_2, n_3 from Claim 4 applied to the unique vertex of A that is not a clone, if one exist), outcome (d) holds. So we may assume that if $|A| \leq 3$, then there are at least two non-clones in A.

We claim that a is adjacent to x_1. Suppose that this is false, and let $b \in A$ be adjacent to x_1. By the minimality of A, b is not complete to $\{x_2, x_5\}$. Since b has two non-adjacent neighbors in X, by symmetry we may assume that b is adjacent to x_4. If b is adjacent to x_3, then, by the minimality of A, $A = \{a, b\}$ and b is the unique non-clone in A, so b is non-adjacent to x_3. Now $|A \setminus \{a, b\}| = 1$, and so b is not a clone. Therefore b is adjacent to x_2.

By the minimality of A, b is non-adjacent to x_5. Let $c \in A$ be adjacent to x_3. Then $A = \{a, b, c\}$. By the minimality of A, c is non-adjacent to x_5, and to at least one of x_1, x_4. But now c is a clone, and b is the unique non-clone in A, a contradiction. So a is adjacent to x_1. This implies that $A = \{a, b, c\}$, b is adjacent to x_4 but not to x_3, c is adjacent to x_3 but not x_4, neither of b, c is a clone, and (by the minimality of A) no vertex of A_1 is complete to $\{x_3, x_4\}$. By Claim 6, there exist $b', c' \in A_1$, $n_2 \in A_2$ and $n_3 \in A_3$, such that

$b'x_4$ and $c'x_5$ are edges, n_2 is adjacent to n_3, and $\{b', c'\}$ is complete to $\{n_2, n_3\}$. Now (d) holds. So we may assume that A does not contain a clone.

If $A = \{a, b\}$ and there exist $x, y, z \in X$ such that z-a-x-y-b or a-x-y-b-z is an induced path, then (b) *holds.* (6)

Since p-a-x-y-b-q is not a P_6 for any $p, q \in A_2$, we deduce that either $N(a) \cap A_2 \subseteq N(b) \cap A_2$, or $N(b) \cap A_2 \subseteq N(a) \cap A_2$, and the same holds in A_3. Since we may assume (b) does not hold, Claim 4 implies that, up to symmetry, there exist $n_2 \in N(a) \cap A_2$ and $n_3 \in N(b) \cap A_3$ such that a is non-adjacent to n_3, and b is non-adjacent to a_2. Then n_2 is adjacent to n_3 (or n_2-a-x-y-b-n_3 is a P_6). But now z-a-n_2-n_3-b-y or z-b-n_3-n_2-a-x is a P_6, a contradiction. This proves (6).

Suppose some $a \in A$ is a star, say a is adjacent to $x_1, \dots, x_4$, and not to x_5. Let $b \in A$ be adjacent to x_5. Then we know that $A = \{a, b\}$. If b is adjacent to both x_1 and x_4, then (c) holds, and so we may assume that b is non-adjacent to x_1. Since b is not a clone, b is adjacent to x_2. If b is adjacent to x_3, then (c) holds, so b is non-adjacent to x_3; since b is not a clone, b is adjacent to x_4. But now (6) holds with $x = x_1$, $y = x_5$ and $z = x_3$. So we may assume that no $a \in A$ is a star, and so every vertex of A is a pyramid.

Let $a \in A$. We may assume that a is adjacent to x_1, x_3, x_4 and not to x_2, x_5. Let $b \in A$ be adjacent to x_2. If $N(b) \cap X = \{x_2, x_4, x_5\}$, then (b) holds by (6) applied with $x = x_3$, $y = x_2$ and $z = x_5$. If $N(b) \cap X = \{x_2, x_3, x_5\}$, then we obtain the previous case by exchanging the roles of a and b. So we may assume that $N(b) \cap X = \{x_1, x_2, x_4\}$.

Hence, there exists $c \in A \setminus \{a, b\}$ adjacent to x_5 with $N(c) \cap X = \{x_1, x_3, x_5\}$. But now every $x \in X$ has a neighbor in $A \setminus \{a\}$, contrary to the minimality of A. This shows how the statement of Claim 7 follows from assumption 3, completing the proof. ☐

Claim 8. *Every graph $H' \in \mathcal{L}$ satisfies the assumptions of Claim 7.*

Proof. Since H' is a minimal obstruction to 3-coloring, H' has no dominated vertex, meaning any two neighborhoods of vertices are incomparable. Let $v \in V(H')$. If $N(v)$ is not bipartite, then v contains a triangle or C_5, and so $V(H') = \{v\} \cup N(v)$, and assumptions 2 or 3 of Claim 7 hold. So $N(v)$ is bipartite with a bipartition (X, Y).

We implemented a straightforward program which we used to verify that assumption 1 of Claim 7 indeed holds for all 24 4-critical P_6-free graphs from Theorem 1 where $N(v)$ is bipartite. The source code of this program can be downloaded from [8]. ☐

Claim 9. *Let G' be obtained from G by contracting (A_1, A_2, A_3) to a triangle $a_1a_2a_3$. Let H' be an induced subgraph of G', with $a_1, a_2 \in V(H')$. For $i = 1, 2$, let $Z_i = N(a_i) \setminus \{a_1, a_2, a_3\}$.*

Assume that

1. *no two non-adjacent neighbors of a_1 dominate each other, and no two non-adjacent neighbors of a_2 dominate each other, and*

2. *$H'|N(a_1)$ and $H'|N(a_2)$ are bipartite.*

Then Claim 7(a) *or Claim 7*(b) *holds for each of a_1, a_2. If Claim 7*(a) *holds for a_1, let c_1 be the vertex a of Claim 7*(a)*, set $A = \{c_1\}$ and $Z = \emptyset$. If Claim 7*(b) *holds for a_1, let $a, b, n_2(a_1), n_3(a_1)$ be the vertices as in Claim 7*(b)*. Moreover, set $A = \{a, b\}$, and $Z = \{n_2(a_1), n_3(a_1)\}$.*

If Claim 7(a) *holds for a_2, let c_2 be the vertex a of Claim 7*(a)*, set $C = \{c_2\}$, and $W = \emptyset$. If Claim 7*(b) *holds for a_2, let $c, d, n_1(a_2), n_3(a_2)$ be the vertices as in Claim 7*(b)*, set $C = \{c, d\}$, and $W = \{n_1(a_2), n_3(a_2)\}$. C.*

One of the following holds.

(a) Outcome Claim 7(a) *holds for a_1, there is $c \in C$, and an induced path c_1-c'-a'-c in $T_3(t)$ where $t = \max(t(c_1), t(c))$, such that a' is complete to Z_1. Or the analogous statement holds for a_2.*

(b) There is an edge between A and C. In this case let $H = (H' \setminus \{a_1, a_2\}) \cup A \cup C \cup Z \cup W$.

(c) *In H', there is an induced path a_1-q_1-q_2-a_2, and a vertex complete to $\{a_1, q_1, q_2\}$ or to $\{a_2, q_2, q_1\}$.In this case let $H = (H' \setminus \{a_1, a_2\}) \cup A \cup C \cup Z \cup W$.*

(d) *There are adjacent vertices $n_1 \in A_1$ and $n_2 \in A_2$, such that n_1 is complete to C, n_2 is complete to A, and some vertex $s \in A_3$ is complete to $A \cup C \cup \{n_1, n_2\}$. In this case let $H = (H' \setminus \{a_1, a_2\}) \cup A \cup C \cup \{n_1, n_2, s\}$.*

In each the cases (b), (c), (d), *in every 3-coloring of H, A and C are monochromatic, and no color appears in both A and C.*

In all cases, H is 3-colorable if and only if H' is.

Proof. By assumption 2 Claim 7(a) or Claim 7(b) holds for each of a_1, a_2. We may assume that no vertex of $V(G) \setminus (Z_1 \cup A_2 \cup A_3)$ has a neighbor in A_1, and no vertex of $V(G) \setminus (Z_2 \cup A_1 \cup A_3)$ has a neighbor in A_2 (otherwise we may delete such vertices from G without changing the hypotheses or the outcomes).

Moreover, we may assume that A is anticomplete to C, as otherwise (b) holds. Pick $a \in A$ and $c \in C$. Let $t = \max(t(a), t(c))$, and let c-a'-c'-a be an induced path from a to c in $T_3(t)$. If possible, we choose a' to be complete to C, and c' complete to A.

Assuming (a) does not hold, we derive the following.

$$\text{Vertex } a' \text{ is not complete to } Z_1, \text{ and } c' \text{ is not complete to } Z_2. \tag{7}$$

We also make use of the following fact.

$$\text{Vertex } c' \text{ is complete to } A, \text{ and } a' \text{ to } C. \tag{8}$$

To see this, suppose c' is not complete to A. Then $A = \{a, b\}$, and c' is non-adjacent to b. By the choice of c', we deduce that $n_2(a_1)$ is non-adjacent to a' (otherwise we may replace c' with $n_2(a_1)$). Now b-$n_2(a_1)$-a-c'-a'-c is a P_6, a contradiction. Similarly, a' is complete to C. This proves (8).

$$\text{Let } p \in Z_1 \text{ be non-adjacent to } a'. \text{ Then } p \text{ has no neighbor in } V(H') \setminus (\{a_1, a_2, a_3\} \cup Z_1 \cup Z_2), \text{ and } p \text{ has a neighbor } q \in Z_1. \tag{9}$$

Since a_2 does not dominate p, p has a neighbor $q \in H'$ non-adjacent to a_2. Then in G, q is anticomplete to A_2. Let $z \in A$ be adjacent to p. If q is not in Z_1, then q is anticomplete to A_1, and so, by (8), q-p-z-c'-a'-c is a P_6 in G, a contradiction. This proves (9).

By (7), (9) and the symmetry between A_1 and A_2, there exist $p, q \in Z_1$ and $s, t \in Z_2$ such that pq, st are edges, a' is non-adjacent to p, and c' is non-adjacent to s. Let $r \in A$ be adjacent to p, and let $u \in C$ be adjacent to s. Since p-r-c'-a'-u-s is not a P_6, we deduce that p is adjacent to s.

Let D be the following C_6: r-c'-a'-u-s-p-r.

$$\text{Vertex } p \text{ is complete to } A, \text{ and } s \text{ is complete to } C. \tag{10}$$

Suppose p has a non-neighbor $r' \in A$. Then, since A is anticomplete to C, r' is a leaf for D, in contradiction to Claim 2. Similarly, s is complete to C. This proves (10).

By (10), we may assume that r is adjacent to q, and u is adjacent to t. If q is adjacent to s, then (c) holds, which we may assume not to be the case. Similarly, t is non-adjacent to p. Since q, t are not hats for D, by Claim 2, we deduce that q is adjacent to a', and t to c'.

Let $d \in A_3$ be adjacent to a'. If d is non-adjacent to c', then d-a'-c'-t-s-p is a P_6, a contradiction. So d is adjacent to c'. Since by Claim 2 d is not a hat for D, we deduce that d is adjacent to at least one of r, u. Suppose that d is adjacent to r and not to u. Then p-r-d-a'-u-s-p is a C_6, and t is a hat for it, again contrary to Claim 2. This proves that d is complete to $\{r, u\}$. Similarly d is complete to $A \cup C$ and (d) holds.

It is now easy to verify that the last assertion of Claim 9 holds. This completes the proof. □

By W_5 we denote the graph that is C_5 plus a vertex adjacent to all vertices of that C_5.

Claim 10. *Every $H \in \mathcal{L}$ except K_4 and W_5 satisfies the assumptions of Claim 9.*

Proof. Let $H \in \mathcal{L}$. Since H is minimal non-3-colorable, H has no dominated vertices, and so assumption 1 of Claim 9 holds. If $H|N(v)$ is not bipartite for some $v \in V(H)$, then $H|N(v)$ contains a triangle or a C_5, and so $H = K_4$ or $H = W_5$. □

We can now prove our main statement of this section.

Proof of Lemma 9. We may assume that at least one of a_1, a_2, a_3 is in $V(H')$. If $|V(H') \cap \{a_1, a_2, a_3\}| = 1$, we are done using Claim 7 and Claim 8, so we may assume that $|V(H') \cap \{a_1, a_2, a_3\}| \geq 2$. Note that if $H' = K_4$, every edge is in a triangle, and if $H' = W_5$, then every triangle is in a diamond. Hence, the maximality of (A_1, A_2, A_3) implies that $H' \neq K_4, W_5$.

By Claim 10, H' satisfies the assumptions of Claim 9. We define the sets C_i, W_i, N_i, Z_i for $i \in \{1, 2, 3\}$. If $a_i \notin V(H')$, let $C_i = W_i = Z_i = \emptyset$, and let $N_j = \emptyset$ for every $j \neq i$. Now suppose that $a_1 \in V(H')$. Let $C_1 = A$ be as in Claim 7(a) or Claim 7(b). (Observe that there may be several possible choices of C_1.) If Claim 7(b) holds for a_1, let $W_1 = \{n_2, n_3\}$ (in the notation of Claim 7(b)), if If Claim 7(a) holds for a_1, let $W_1 = \emptyset$. Let Z_1 be the set of neighbors of a_1 in $H' \setminus \{a_1, a_2, a_3\}$. Suppose $a_2 \in V(H')$. If Claim 9(d) holds for a_1a_2, define $N_3 = \{n_1, n_2, s\}$ in the notation of Claim 9(d). If Claim 9(a), Claim 9(b) or Claim 9(c) holds for a_1a_2, let $N_3 = \emptyset$. Note that in all cases $|N_3| \leq 3$. Define $C_2, C_3, W_2, W_3, N_1, N_2, Z_2, Z_3$ similarly.

Next we define the sets D_1, D_2, D_3. If Claim 9(d) holds for the pair a_1a_2 or for the pair a_1a_3, we set $D_1 = N_2 \cup N_3$, and otherwise we set $D_1 = W_1$.

As usual, we may assume $V(G) = A_1 \cup A_2 \cup A_3 \cup (V(H') \setminus \{a_1, a_2, a_3\})$.

We analyze the possible outcomes of Claim 9.

Let us call outcomes Claim 9(b), Claim 9(c), Claim 9(d) *good*. Suppose first that a good outcome holds for each pair a_1a_2, a_2a_3, a_1a_3 contained in $V(H')$.

Let
$$H = (H' \setminus \{a_1, a_2, a_3\}) \cup (C_1 \cup C_2 \cup C_3 \cup D_1 \cup D_2 \cup D_3).$$

- If Claim 7(a) holds for each of a_1, a_2, a_3, then $|V(H)| \leq |V(H')| + 9$, as follows.

  We observe that in this case $|C_i| = 1$ and $|W_i| = 0$ for each i, and therefore $D_1 \cup D_2 \cup D_3 = N_1 \cup N_2 \cup N_3$. Since $|N_i| \leq 3$ for every i, we have that $|V(H)| \leq |V(H')| - 3 + 3 + 9 = |V(H')| + 9$.

- If Claim 7(a) holds for exactly two of a_1, a_2, a_3, then $|V(H)| \leq |V(H')| + 12$, as follows.

  Assume Claim 7(a) holds for a_1 and a_2. Then Claim 7(b) holds for a_3. It follows that $|C_1| = |C_2| = 1$, $W_1 = W_2 = \emptyset$, and $|C_3| = |W_3| = 2$. Therefore $D_1 \cup D_2 \cup D_3 \subseteq N_1 \cup N_2 \cup N_3 \cup W_3$, and so $|D_1 \cup D_2 \cup D_3| \leq 11$. Consequently, $|V(H)| \leq |V(H')| - 3 + 4 + 11 = |V(H')| + 12$.

- If Claim 7(a) holds for exactly one of a_1, a_2, a_3, then $|V(H)| \leq |V(H')| + 12$, as follows.

  Assume Claim 7(a) holds for a_1. Then Claim 7(b) holds for exactly a_2, a_3. It follows that $|C_1| = 1$, $W_1 = \emptyset$ and $|C_2| = |W_2| = |C_3| = |W_3| = 2$. We claim that $D_1 \cup D_2 \cup D_3 \leq 10$. Since $W_1 = \emptyset$, we deduce that $D_1 \subseteq N_2 \cup N_3$. If $N_1 \neq \emptyset$. Then $D_2 \cup D_3 \subseteq N_1 \cup N_2 \cup N_3$, and so, since $|N_i| \leq 3$ for every i, $|D_1 \cup D_2 \cup D_3| \leq |N_1 \cup N_2 \cup N_3| \leq 9$. Thus we may assume that $N_1 = \emptyset$. Then $D_1 \cup D_2 \cup D_3 \subseteq N_2 \cup N_3 \cup W_2 \cup W_3$, and again $|D_1 \cup D_2 \cup D_3| \leq 10$. Consequently, $|V(H)| \leq |V(H')| - 3 + 5 + 10 = |V(H')| + 12$.

- If Claim 7(b) holds for all of a_1, a_2, a_3, then $|V(H)| \leq |V(H')| + 12$, as follows.

  Since Claim 7(b) holds for each of a_1, a_2, a_3, it follows that for every $|C_i| = |W_i| = 2$ for every i.

  We show that $|D_1 \cup D_2 \cup D_3| \leq 9$. Suppose first that $N_1 \neq \emptyset$ and $N_2 \neq \emptyset$. Then $D_2 \cup D_3 \cup D_3 \subseteq N_1 \cup N_2 \cup N_3$, and the claim follows since $|N_i| \leq 3$ for every i. This we may assume that $N_2 = N_3 = \emptyset$.

  Next assume that $N_1 \neq \emptyset$. Then $D_2 = D_3 = N_1$, and $D_1 = W_1$, and so $|D_1 \cup D_2 \cup D_3| \leq 5$.

  Finally, if $N_1 = N_2 = N_3 = \emptyset$, then $D_i = W_i$ for every i, and thus $|D_1 \cup D_2 \cup D_3| \leq 6$.

  Thus, in all cases $|V(H)| \leq |V(H')| - 3 + 6 + 9 = |V(H')| + 12$.

Now we may assume that for least one of the pairs a_1a_2, a_2a_3, a_1a_3 contained in H' no good outcome holds. Consequently, for at least one of the pairs a_1a_2, a_2a_3, a_1a_3 contained in H' Claim 9(a) holds. Observe that this is true for every choice of C_1, C_2, C_3 as above.

Suppose that $V(H') \cap \{a_1, a_2, a_3\} = \{a_1, a_2\}$. Then Claim 9(a) holds for the pair a_1a_2. In the notation of Claim 9, we may assume that a' is complete to $N_{H'}(a_1)$. Then replacing C_1 with $\{a'\}$, we observe that outcome Claim 9(b) holds for the pair a_1a_2, a contradiction. Thus we may assume that $a_1, a_2, a_3 \in V(H')$.

Permuting the indices if necessary, there exist $b_2, b_3 \in A_1$, and $C_2 \subseteq A_2$, $C_3 \subseteq A_3$ such that the following holds.

- *$\{b_2, b_3\}$ is complete to Z_1,*
- *C_2 and C_3 are as in Claim 7(a) or Claim 7(b),*
- *b_2 has a neighbor in C_2 and none in C_3,* (11)
- *b_3 has a neighbor in C_3 and none in C_2, and*
- *one of the good outcomes holds for the pair C_2, C_3.*
- *b_2 and b_3 have a common neighbor in A_2 or A_3.*

In order to prove (11), we first prove that a certain condition is sufficient for (11).

If there exist $C_i' \subseteq A_i$ as in Claim 7(a) or Claim 7(b) such that there is an edge between C_1' and C_2', and an edge between C_2' and C_3', then (11) holds. (12)

To see this, apply Claim 9 with $A = C_1'$ and $C = C_3'$. If one of the good outcomes holds, then a good outcome holds for all three pairs among C_1', C_2', C_3', and so we may assume that this is not the case. There is symmetry between C_1' and C_3', so we may assume that $|C_1'| = 1$ and that there is an induced path c_1'-c_3''-c_1''-c_3' in T_2, where $\{c_1'\} = C_1'$, $c_3' \in C_3'$, and c_1'' is complete to Z_1. If c_1'' has a neighbor in C_2, or c_1' has a neighbor in C_3, then a good outcome holds for all pairs among $\{c_1''\}, C_2', C_3'$ or C_1', C_2', C_3'. Hence, we may assume that this is not the case. Now (11) holds, and this proves (12).

We may assume that Claim 9(a) holds for the pair C_2, C_3. By modifying C_2, C_3 we may assume that there is an edge between C_2 and C_3 and outcome Claim 9(b) holds for (C_2, C_3). If a good outcome holds for both C_1, C_2, and C_1, C_3, then a good outcome holds for all three pairs, so we may assume that this is not the case.

So, assume that outcome (a) holds when Claim 9 is applied to C_1, C_2. If there is $c_1 \in A_1$ that is complete to Z_1 and has a neighbor in C_2, then (11) holds by (12). So we may assume that there is a vertex $c_2' \in A_2$ that is complete to Z_2, and an induced path c_1-c_2'-c_1'-c_2 in T_3, where $c_1 \in C_1$ and $C_2 = \{c_2\}$. If a good outcome holds for C_1, C_3, then either (11) holds, or a good outcome holds for all three pairs among $C_1, \{c_2\}, C_3$ or $C_1, \{c_2'\}, C_3$.

So, we may assume that Claim 9(a) holds for C_1, C_3. By the symmetry between C_1 and C_3, we may assume that there is $d_1 \in A_1$ and an induced path c_1-c_3'-d_1-c_3 where $c_3 \in C_3$, $C_1 = \{c_1\}$, and d_1 is complete to Z_1. But now there is an edge between C_3 and $\{d_1\}$, and between C_3 and C_2, and (11) follows from (12). This proves (11).

If Claim 9(b) or Claim 9(c) holds for the pair C_2, C_3, let

$$H = G|((V(H') \setminus \{a_1, a_2, a_3\}) \cup \{b_2, b_3\} \cup C_2 \cup C_3 \cup W_2 \cup W_3),$$

and let

$$H'' = G|((V(H') \setminus \{a_1, a_2, a_3\}) \cup \{b_2, b_3\} \cup C_2 \cup C_3).$$

If Claim 9(d) holds for the pair C_2, C_3, let

$$H = G|((V(H') \setminus \{a_1, a_2, a_3\}) \cup \{b_2, b_3\} \cup C_2 \cup C_3 \cup N_1),$$

and let

$$H'' = G|((V(H') \setminus \{a_1, a_2, a_3\}) \cup \{b_2, b_3\} \cup C_2 \cup C_3).$$

Then $|V(H)| \leq |V(H')| + 7$, and so we may assume that H is 3-colorable.

Let us call a 3-coloring of H'' *promising* if C_2 is monochromatic, C_3 is monochromatic, and no color appears in both of C_2, C_3. We observe that by Claim 7 and Claim 9, every 3-coloring of H gives a promising 3-coloring of H''. Since H' is not 3-colorable, in every promising coloring of H'' the vertices b_2 and b_3 receive different colors.

Let c be a 3-coloring of H. We may assume that $c(b_i) = i$, c is constantly 1 or 3 on C_2, and c is constantly 1 or 2 on C_3. Then $c(z) = 1$ for every $z \in Z_1$. If c is 1 on C_2, then we recolor b_2 with color 3, and get a coloring of H', a contradiction. So we may assume that c is 3 on C_2, and c is 2 on C_3. If no vertex of Z_2 has color 1, we recolor C_2 with color 1, and recolor b_2 with color 3. We obtain coloring of H with b_2, b_3 colored in the same color, a contradiction. So, for some $z_2 \in Z_2$, $c(z_2) = 1$. Similarly, for some $z_3 \in Z_3$, $c(z_3) = 1$.

For $i = 2, 3$ let Z'_i be the set of all vertices $z \in Z_i$ with $c(z_i) = 1$. Then $Z_1 \cup Z'_2 \cup Z'_3$ is a stable set. Let $c_i \in C_i$ be adjacent to b_i.

$$Z'_2 \textit{ is anticomplete to } V(G) \setminus (Z_2 \cup A_2). \tag{13}$$

Suppose $p \in V(G) \setminus (Z_2 \cup A_2)$ has a neighbor $z_2 \in Z'_2$. Then $p \notin Z_1$. Let $c'_2 \in C_2$ be adjacent to z_2. Suppose first that b_2 is non-adjacent to c'_2. Then $c'_2 \neq c_2$. Let $n_1 \in A_1$ be complete to $\{c_2, c'_2\}$, a possible choice by b. Now p-z_2-c'_2-n_1-c_2-b_2 is a P_6, a contradiction. So c'_2 is adjacent to b_2. Let $n_2 \in A_2$ be adjacent to b_2 and b_3 (as in (11), with the roles of A_2 and A_3 exchanged). Then p-z_2-c'_2-b_2-n_2-b_3 is a P_6, again a contradiction. This proves (13).

Now, by (13), we can recolor H'' by putting $c'(C_2) = 1$ and $c'(Z'_2) = 3$, $c'(b_2) = 3$, which yields a 3-coloring of H', a contradiction. This completes the proof. □

In Section 7 we use Lemma 9 (together with Lemma 10, which is the analogue of Lemma 9 for the case when $H' = K_4$) to prove the main result of the paper.

5 Obstructions that are 1-vertex extensions of a tripod

In this section, we prove the following statement.

Lemma 10. *Let G be a 4-critical P_6-free graph. Assume that there is a tripod $T = (A_1, A_2, A_3)$ in G and some vertex x which has a neighbor in each A_i, $i = 1, 2, 3$. Then $|V(G)| \leq 18$.*

To see this, let G, $T = (A_1, A_2, A_3)$, and x be as in Lemma 10. Let a_1, a_2, a_3 be the root of T. It is clear that $V(G) = V(T) \cup \{x\}$. We call G a *1-vertex extension of a tripod.*

5.1 Preparation

We may assume that the ordering $A_1 \cup A_2 \cup A_3 = \{v_1, \ldots, v_k\}$ has the following property.

Claim 11. *Let $u \in A_\ell$ and $v \in A_k$ for some $\ell, k \in \{1, 2, 3\}$. Moreover, let $\{\ell, \ell', \ell''\} = \{1, 2, 3\}$ and $\{k, k', k''\} = \{1, 2, 3\}$. Assume that* $\max(t(n_{k'}(v)), t(n_{k''}(v))) < \max(t(n_{\ell'}(u)), t(n_{\ell''}(u)))$. *Then $t(v) < t(u)$.*

Let b_i be the neighbor of x in A_i with $t(b_i)$ maximum, for all $i = 1, 2, 3$. We may assume that $t(b_1) > t(b_2) > t(b_3)$.

Claim 12. *We may assume that $N(x) \cap A_1 = \{b_1\}$ and $N(x) \cap A_i = \{b_i\}$ for some $i \in \{2, 3\}$.*

Proof. Since $G|(V(T(t(b_1))) \cup \{x\})$ is 4-chromatic we know that $V(G) = V(T(t(b_1))) \cup \{x\}$. In particular, $N(x) \cap A_1 = \{b_1\}$.

To see the second statement, assume that $|N(x) \cap A_2|, |N(x) \cap A_3| \geq 2$. Suppose for a contradiction that $|N(b_1) \cap A_2|, |N(b_1) \cap A_3| \geq 2$, and let u be the vertex in the set $\{b_2, b_3, n_2(b_1), n_3(b_1)\}$ with $t(u)$ maximum. Then $G - u$ is still 4-chromatic, a contradiction.

So we may assume that $|N(b_1) \cap A_i| = 1$ for some $i \in \{2, 3\}$. Note that $T' = (A_1 \setminus \{b_1\} \cup \{x\}, A_2, A_3)$ is a tripod. Consequently, b_1 has neighbors in all three classes of T'. Since $|N(b_1) \cap (A_1 \cup \{x\})| = |N(b_1) \cap A_i| = 1$, we are done. □

5.2 The enumeration algorithm

Consider the following way of traversing the tripod T. Initially, the vertices b_1, b_2, b_3 are labeled *active*, and all other vertices are unlabeled. Then, we label the vertices a_1, a_2, a_3 as *inactive*. Consequently, if $b_3 = a_3$, say, then b_3 is labeled inactive.

Iteratively, pick an active vertex, say $u \in A_i$ with $\{i, j, k\} = \{1, 2, 3\}$. Make $n_j(u)$ and $n_k(u)$ active, unless they are labeled already, whether active or inactive. Then label u as inactive and re-iterate, picking another active vertex, if possible.

Claim 13. *Regardless of which active vertex is picked in the successive steps, this procedure terminates and, moreover, every vertex of T is visited during this procedure.*

Proof. Clearly this procedure terminates when there is no active vertex left. Since every vertex is labeled active at most once, this proves the first assertion.

Assume now the procedure has terminated. The latter assertion follows from the fact that, if W is the collection of inactive vertices, $G|W$ is already a tripod. Thus, since $b_1, b_2, b_3 \in W$, $G|(W \cup \{x\})$ is 4-chromatic and so $G|(W \cup \{x\}) = G$, due to the choice of G. □

Instead of traversing a given tripod, we use this method to enumerate all possible 4-critical P_6-free 1-vertex extensions of a tripod. The idea is to successively generate the possible subgraphs induced by the labeled vertices only. This is done by Algorithm 3. Starting from all relevant graphs on the vertex set $\{x, b_1, b_2, b_3, a_1, a_2, a_3\}$, we iteratively add new vertices, mimicking the iterative labeling procedure mentioned above. The following list contains all of these start graphs.

Claim 14. *We may assume that the graph $G' := G|\{x, b_1, b_2, b_3, a_1, a_2, a_3\}$ has the following properties.*

(a) If $b_1 = a_1$, then $G = G'$ is K_4.

(b) If $b_1 \neq a_1$ and $b_2 = a_2$, then $b_3 = a_3$. Moreover,

$$E(G') \supseteq \{xb_1, xa_2, xa_3, a_1a_2, a_1a_3, a_2a_3\} := F$$
$$E(G') \subseteq F \cup \{b_1a_2, b_1a_3\}.$$

(c) If $b_1 \neq a_1$, $b_2 \neq a_2$ and $b_3 = a_3$, then

$$E(G') \supseteq \{xb_1, xb_2, xa_3, a_1a_2, a_1a_3, a_2a_3\} := F$$
$$E(G') \subseteq F \cup \{xa_2, b_1a_2, b_1b_2, b_1a_3, b_2a_1, b_2a_3\}.$$

(d) If $b_1 \neq a_1$, $b_2 \neq a_2$ and $b_3 \neq a_3$, then

$$E(G') \supseteq \{xb_1, xb_2, xb_3, a_1a_2, a_1a_3, a_2a_3\} := F$$
$$E(G') \subseteq F \cup \{xa_2, xa_3, b_1a_2, b_1b_2, b_1a_3, b_1b_3, b_2a_1, b_2a_3, b_2b_3, b_3a_1, b_3a_2\}.$$

Proof. This follows readily from our assumption $t(b_3) < t(b_2) < t(b_1)$ with Claim 11 and Claim 12. □

In our algorithm, we do not only consider graphs, but rather tuples containing a graph together with its list of vertex labels and a linear vertex ordering. The algorithm is split into three parts.

- Algorithm 3 initializes all relevant tuples according to Claim 14.
- Algorithm 4 is the main procedure, where a certain tuple is extended in all possible relevant ways. This corresponds to a labeling step in our tripod traversal algorithm.

  The list Act is the list of currently active vertices. In each step of the traversal algorithm, an active vertex is picked. Then one neighbor each from the two other tripod classes is added to the set of active vertices (unless they have been visited before). Consequently, Algorithm 4 adds up to two new vertices to the tripod that correspond to these two neighbors. According to Claim 13, it does not matter which active vertex is picked next.

  The list Ord maintained by the algorithm corresponds to the ordering proposed by Claim 11. Algorithm 4 implicitly enumerates all orderings that obey the properties of Claim 11. Whenever the ordering of the vertices of the partial tripod generated so far does not obey the properties listed in Claim 11, we may prune.
- Algorithm 5 is a subroutine we use to prune tuples we do not need to consider. We call a tuple *prunable* if Algorithm 5 applied to it returns the value *false*.

  The criteria for a prunable tuple we apply (in that order) are as follows:

  - an induced P_6,
  - the graph is not 3-colorable,
  - one of the properties of Claim 12 is violated,
  - the ordering Ord does not obey Claim 11, and
  - the partial tripod enumerated so far cannot be extended to a 4-vertex-critical 1-vertex extension of a tripod.

We now come to the correctness proof of these algorithms.

Lemma 11. *Assume that Algorithm 3 terminates and does never generate a tuple whose graph has $k+1$ or $k+2$ vertices, for some $k \geq 4$. Then any 4-critical P_6-free graph which is a 1-vertex extension of a tripod has at most k vertices.*

To see this, let G be a 4-critical P_6-free graph other than K_4 that is a 1-vertex extension of a tripod, with the notation from above. We need the following claim.

Claim 15. *There is a sequence of tuples $\Gamma^i = (G^i = (V^i, E^i), A_1^i, A_2^i, A_3^i, Ord^i, Act^i)$, $i = 0, \dots, r$, and a way of traversing the tripod T in r steps, in the way described above, for which the following holds, after possibly renaming vertices. Let $V(i)$ be set of all labeled vertices after the i-th iteration of the traversal, together with x, and let $Act(i)$ be the set of vertices which are active after the i-th iteration of the traversal, for $i = 0, \dots, r$.*

(a) At some point during the algorithm, Expand(Γ^0) is called.

(b) During the procedure Expand(Γ^i), Γ^{i+1} is generated and so Expand(Γ^{i+1}) is called, for all $i = 0, \dots, r-1$.

(c) The following holds, for all $i = 0, \dots, r$.

(i) $G|V(i) = G^i$, and in particular $A_j \cap V(i) = A_j^i$, for all $j = 1, 2, 3$,

(ii) $Act^i = Act(i)$, and

(iii) for any two $u, v \in V(i)$ with $t(u) < t(v)$, $u <_{Ord^i} v$.

Proof. Since G is not K_4 we may assume that $b_1 \neq a_1$, by Claim 14.

If $b_2 = a_2$, then Claim 14 implies $b_3 = a_3$, and Γ^0 is generated by Algorithm 3. Here, $\Gamma^0 = (G^0 = (V^0, E^0), A_1^0, A_2^0, A_3^0, \mathrm{Ord}^0, \mathrm{Act}^0)$ with

Algorithm 3 Generate 4-critical P_6-free 1-vertex extension of a tripod

$V := \{x, b_1, a_1, a_2, a_3\}$ // in this case, $b_2 = a_2$ and $b_3 = a_3$
$E^{\text{must}} := \{xb_1, xa_2, xa_3, a_1a_2, a_1a_3, a_2a_3\}$
$E^{\text{may}} := \{b_1a_2, b_1a_3\}$
$\text{Ord} := (a_3, a_2, a_1, b_1, x)$ and $\text{Act} := \{b_1\}$
$A_1 := \{a_1, b_1\}$, $A_2 := \{a_2\}$, and $A_3 := \{a_3\}$
for each $E \subseteq E^{\text{must}} \cup E^{\text{may}}$ with $E^{\text{must}} \subseteq E$ **do**
  $\text{Expand}(G = (V, E), A_1, A_2, A_3, \text{Ord}, \text{Act})$
end for
$V := \{x, b_1, b_2, a_1, a_2, a_3\}$ // in this case, $b_2 \neq a_2$ and $b_3 = a_3$
$E^{\text{must}} := \{xb_1, xb_2, xa_3, a_1a_2, a_1a_3, a_2a_3\}$
$E^{\text{may}} := \{xa_2, b_1b_2, b_1a_2, b_1a_3, b_2a_1, b_2a_3\}$
$\text{Ord} := (a_3, a_2, a_1, b_2, b_1, x)$ and $\text{Act} := \{b_1, b_2\}$
$A_1 := \{a_1, b_1\}$, $A_2 := \{a_2, b_2\}$, and $A_3 := \{a_3\}$
for each $E \subseteq E^{\text{must}} \cup E^{\text{may}}$ with $E^{\text{must}} \subseteq E$ **do**
  $\text{Expand}(G = (V, E), A_1, A_2, A_3, \text{Ord}, \text{Act})$
end for
$V := \{x, b_1, b_2, b_3, a_1, a_2, a_3\}$ // in this case, $b_2 \neq a_2$ and $b_3 \neq a_3$
$E^{\text{must}} := \{xb_1, xb_2, xb_3, a_1a_2, a_1a_3, a_2a_3\}$
$E^{\text{may}} := \{xa_2, xa_3, b_1b_2, b_1b_3, b_2b_3, b_1a_2, b_1a_3, b_2a_1, b_2a_3, b_3a_1, b_3a_2\}$
$\text{Ord} := (a_3, a_2, a_1, b_3, b_2, b_1, x)$ and $\text{Act} := \{b_1, b_2, b_3\}$
$A_1 := \{a_1, b_1\}$, $A_2 := \{a_2, b_2\}$, and $A_3 := \{a_3, b_3\}$
for each $E \subseteq E^{\text{must}} \cup E^{\text{may}}$ with $E^{\text{must}} \subseteq E$ **do**
  $\text{Expand}(G = (V, E), A_1, A_2, A_3, \text{Ord}, \text{Act})$
end for

- $V^0 = \{a_1, b_1, a_2, a_3, x\}$ and $E^0 = E(G|V^0)$,
- $A^0_1 = \{a_1, b_1\}$, $A^0_2 = \{a_2\}$, and $A^0_3 = \{a_3\}$, and
- $\text{Ord}^0 = (a_3, a_2, a_1, b_1, x)$, and $\text{Act}^0 = \{b_1\}$.

By Claim 14, $E^{\text{must}} \subseteq E^0 \subseteq E^{\text{must}} \cup E^{\text{may}}$. The cases when $a_2 \neq b_2$ but $a_3 = b_3$ resp. $a_3 \neq b_3$ are dealt with similarly. This proves (c) for $i = 0$.

For the inductive step assume that for some $s \in \{0, \ldots, r-1\}$ the tuple Γ^s has the properties mentioned in (c). We first prove that Γ^{s+1} is generated while $\text{Expand}(\Gamma^s)$ is processed, and that Γ^{s+1} has the properties mentioned in (c).

First we discuss why Algorithm 5 returns *true* on the input Γ^s. Clearly $G^s = G|V(s) \neq G$ is 3-colorable and P_6-free, and so the if-conditions in lines 4 and 1 both do not apply. Also, the if-conditions in the lines 7 and 10 does not apply to Γ^s due to Claim 12 applied to G together with (c).(i) in the case $i = s$.

During the steps 13-23, the if-condition in line 19 never applies due to Claim 11. To see this, pick two distinct vertices $u, v \in (V^s \setminus \{x\})$ with $u < v$ and $u \notin \text{Act}^s$. Let $\{i, j, k\} = \{1, 2, 3\}$ be such that $u \in A^s_i$, let u_j be the $<_{\text{Ord}^s}$-minimal neighbor of u in A^s_j, and let u_k be defined accordingly, let $\{i', j', k'\} = \{1, 2, 3\}$ be such that $v \in A^s_{i'}$, and let $v_{j'}$ be the $<_{\text{Ord}^s}$-minimal neighbor of v in $A^s_{j'}$, if existent, and let $v_{k'}$ be defined accordingly.

Due to property (c).(iii), $t(u) < t(v)$. Since $u \in V^s \setminus \text{Act}^s$, we know that $u \in V(s) \setminus \text{Act}(s)$, by (c).(i). Thus, $n_j(u), n_k(u) \in V(s)$. Moreover, by (c).(i), $n_j(u) = u_j$ and $n_k(u) = u_k$. Now, if $v_{j'}, v_{k'}$ both exist and $v_{j'}, v_{k'} <_{\text{Ord}^s} u_r$ for some $r \in \{j, k\}$, then in particular $t(n_{j'}(v)), t(n_{k'}(v)) < t(n_r(u))$, in contradiction to Claim 11.

Finally, Γ^s is not pruned in the lines 24-34 since $G - u$ is 3-colorable for every $u \in V$.

Now we argue why Γ^{s+1} is constructed and carries the desired properties. If $s = 0$, the case is clear, so we may assume that $s > 0$. Say that, in the procedure $\text{Expand}(\Gamma^s)$, vertex u is picked in line 4 of Algorithm 4.

Algorithm 4 Expand(Graph $G = (V, E)$, Set A_1, Set A_2, Set A_3, List Ord, Set Act)

if not Feasible(G, A_1, A_2, A_3, Ord, Act) **then**
  return
end if
pick a vertex u from the set Act and let $\{i, j, k\} = \{1, 2, 3\}$ be such that $u \in A_i$
let u_j be the $<_{\text{Ord}}$-minimal neighbor of u in A_j, if existent, and let u_k be defined accordingly
  // we write $u <_{\text{Ord}} v$ whenever u appears before v in the list Ord
let v_j, v_k be two entirely new vertices
for all ways of inserting v_j and v_k into the list Ord such that
  (a) $a_1 <_{\text{Ord}} v_j, v_k <_{\text{Ord}} u$,
  (b) $v_j <_{\text{Ord}} u_j$, if existent, and $v_k <_{\text{Ord}} u_k$, if existent
 do

$$\begin{aligned} E^* := &\{wv_j : w \in A_i \cup A_k \text{ is active}\} \cup \{wv_k : w \in A_i \cup A_j \text{ is active}\} \cup \{xv_j, xv_k\} \cup \{v_j, v_k\} \\ &\cup \{wv_j : w \in A_i \cup A_k \text{ is inactive and has a neighbor } w' \in A_j \text{ with } w' <_{\text{Ord}} v_j\} \\ &\cup \{wv_k : w \in A_i \cup A_k \text{ is inactive and has a neighbor } w' \in A_k \text{ with } w' <_{\text{Ord}} v_k\} \end{aligned}$$

  for all subsets E' of E^* **do**
    $A'_i := A_i$, $A'_j := A_j \cup \{v_j\}$, $A'_k := A_k \cup \{v_k\}$, and $\text{Act}' := (\text{Act} \setminus \{u\}) \cup \{v_j, v_k\}$
    let Ord′ be Ord where v_j and v_k are inserted in the position we currently consider
    Expand$((V \cup \{v_j, v_k\}, E \cup E'), A'_1, A'_2, A'_3, \text{Ord}', \text{Act}')$
  end for
end for
for $r = j, k$ **do**
  if u_r is existent and $u_r <_{\text{Ord}} u$ **then**
    let $\{r, s\} = \{j, k\}$
    for all ways of inserting v_s into the list Ord such that $a_1 <_{\text{Ord}} v_s <_{\text{Ord}} u$ **do**

$$\begin{aligned} E^* := &\{wv_s : w \in A_i \cup A_r \text{ is active}\} \cup \{xv_s\} \\ &\cup \{wv_s : w \in A_i \cup A_r \text{ is inactive and has a neighbor } w' \in A_s \text{ with } w' <_{\text{Ord}} v_s\} \end{aligned}$$

      for all subsets E' of E^* **do**
        $A'_i := A_i$, $A'_s := A_s \cup \{v_s\}$, $A'_r := A_r$, and $\text{Act}' := (\text{Act} \setminus \{u\}) \cup \{v_s\}$
        let Ord′ be Ord where v_s is inserted in the position we currently consider
        Expand$((V \cup \{v_s\}, E \cup E'), A'_1, A'_2, A'_3, \text{Ord}', \text{Act}')$
      end for
    end for
  end if
end for
if both u_j and u_k exist and $u_j, u_k <_{\text{Ord}} u$ **then**
  Expand$(G, A_1, A_2, A_3, \text{Ord}, \text{Act} \setminus \{u\})$
end if

Let us say that $u \in A_i^s$, where $\{i, j, k\} = \{1, 2, 3\}$. In the traversal procedure, $n_j(u)$ and $n_k(u)$ are now visited and made active, in case they are not in $V(s)$ already.

Let us first assume that $n_j(u), n_k(u) \notin V(s)$, and let v_j, v_k be the two entirely new vertices picked in

Algorithm 5 Feasible(Graph $G = (V, E)$, Set A_1, Set A_2, Set A_3, List Ord, Set Act)

1: **if** G contains a P_6 **then**
2:     **return false**
3: **end if**
4: **if** G is not 3-colorable **then**
5:     **return false**
6: **end if**
7: **if** x has at least two neighbors in A_1 **then**
8:     **return false**
9: **end if**
10: **if** x has at least two neighbors in A_2 and at least two neighbors in A_3 **then**
11:     **return false**
12: **end if**
13: **for** any two distinct vertices $u, v \in (V \setminus \{x\})$ with $u <_{\text{Ord}} v$ **do**
14:     **if** $u \notin \text{Act}$ **then**
15:         let $\{i, j, k\} = \{1, 2, 3\}$ be such that $u \in A_i$
16:         let u_j be the $<_{\text{Ord}}$-minimal neighbor of u in A_j, and let u_k be defined accordingly
17:         let $\{i', j', k'\} = \{1, 2, 3\}$ be such that $v \in A_{i'}$
18:         let $v_{j'}$ be the $<_{\text{Ord}}$-minimal neighbor of v in $A_{j'}$, if existent, and let $v_{k'}$ be defined accordingly
19:         **if** the following hold:
(a) $\{u_j, u_k\} \not\subseteq \{a_1, a_2, a_3\}$,
(b) $v_{j'}$ and $v_{k'}$ both exist, and
(c) $v_{j'}, v_{k'} <_{\text{Ord}} u_r$ for some $r \in \{j, k\}$
        then
20:             **return false**
21:         **end if**
22:     **end if**
23: **end for**
24: **for** each $u \in (V \setminus \{x\})$ **do**
25:     $W := \{v \in V : v <_{\text{Ord}} u\}$
26:     $B_i := A_i \cap W$ for each $i = 1, 2, 3$
27:     **while** there is a vertex $v \in V \setminus (B_1 \cup B_2 \cup B_3 \cup \{u\})$ with neighbors in at least two of B_1, B_2, B_3 **do**
28:         **if** v has neighbors in all three of B_1, B_2, B_3 **then**
29:             **return false**
30:         **else**
31:             $B_i := B_i \cup \{v\}$, where B_i is the set that v does not have neighbors in
32:         **end if**
33:     **end while**
34: **end for**
35: **return true**

line 6. Due to the definition of tripods, $t(a_1) < t(n_j(u)), t(n_k(u)) < t(u)$, and

$$t(n_\ell(u)) < \min(\{t(w) : w \in N_G(u) \cap A_\ell\} \cup \{\infty\}) \text{ for } \ell = j, k.$$

Consequently, the algorithm considers in line 7 inserting the two new vertices v_j and v_k into Ord^s such that (c).(iii) holds, where we identify v_j with $n_j(u)$ and v_k with $n_k(u)$. Moreover, E^* in line 8 contains all edges incident to $n_j(u)$ and $n_k(u)$ in $G|V(s)$, due to the definition of $n_j(u)$ and $n_k(u)$. Due to steps 10 and 11, the tuple Γ^{s+1} is indeed generated, and $\text{Expand}(\Gamma^{s+1})$ is called, where

- $G^{s+1} = G|(V(s) \cup \{n_j(u), n_k(u)\}) = G|V(s+1)$, and in particular $A_i^{s+1} = A_i^s = V(s) \cap A_i =$

$V(s+1) \cap A_i$, and $A_\ell^{s+1} = A_\ell^s \cup \{v_\ell = n_\ell(u)\} = V(s+1) \cap A_\ell$ for $\ell = j, k$,

- $\mathrm{Act}^{s+1} = (\mathrm{Act}^{s+1} \setminus \{u\}) \cup \{v_j, v_k\} = (\mathrm{Act}(s) \setminus \{u\}) \cup \{n_j(u), n_k(u)\} = \mathrm{Act}(s+1)$, and
- for any two vertices $u, v \in V(s+1)$ with $t(u) < t(v)$, $u <_{\mathrm{Ord}^{s+1}} v$.

The cases when $n_j(u)$ and/or $n_k(u)$ have been active before are handled analogously. This completes the proof of Claim 15. □

Next we derive Lemma 11.

Proof of Lemma 11. Like above, Γ^r is not pruned in step 1 during the procedure of Expand(Γ^r). Since $G^r = G|V(r) = G$, G is indeed generated by the algorithm. As $|V(G^s)|+2 \geq |V(G^{s+1})|$ for all $s = 0, \ldots, r-1$, G has at most k vertices. □

We implemented this set of algorithms in C with some further optimizations. A crucial detail is how the active vertex is picked in line 4 of Algorithm 4. The following choice seemed to terminate most quickly.

- If the graph which is currently expanded has at most 12 vertices, we pick the Ord-maximal active vertex in line 4.
- If the graph has more than 12 vertices, we pick the active vertex for which the number of non-prunable tuples generated from it is minimum. This is done by trying to extend every active vertex once without iterating any further and counting the number of non-prunable tuples generated.

With this choice, our program does indeed terminate (in about 60 hours) and the largest non-prunable generated graph has 18 vertices. Together with Lemma 11, we arrive at Lemma 10. Table 3 shows the number of non-prunable tuples generated by the program.

$\lvert V(G)\rvert$	5	6	7	8	9
# non-prunable tuples	3	67	2,010	11,726	81,523
$\lvert V(G)\rvert$	10	11	12	13	14
# non-prunable tuples	388,190	1,234,842	3,380,785	10,669,960	16,322,798
$\lvert V(G)\rvert$	15	16	17	18	19, 20
# non-prunable tuples	137,031	49,506	2,865	330	0

Table 3: Counts of the number of non-prunable tuples generated by our implementation of Algorithm 3

In order to be sure the algorithm is implemented correctly, we also modified the program so it collects all 4-critical graphs found along the way, similar to line 3 of Algorithm 2. As expected, all 4-critical P_6-free 1-vertex extensions of a tripod in $\mathcal{L}$ were found. In the Appendix we describe in more detail how we tested the correctness of our implementation and the source code of the program can be downloaded from [8].

6 Obstructions up to 28 vertices

In this section we prove the following result.

Lemma 12. *Let G be a 4-critical P_6-free graph. If $|V(G)| \leq 28$, then G is contained in $\mathcal{L}$.*

For the proof of this result, we run the enumeration algorithm of Section 1, with the following modifications. In line 1 of Algorithm 2, we do not discard a graph if it contains a diamond, only when it is not P_6-free. Moreover, we discard a graph if it contains more than 28 vertices. This procedure terminates exactly with the list $\mathcal{L}$ (note that the largest graph in $\mathcal{L}$ has 16 vertices). Table 4 shows the number of graphs generated by the algorithm on each relevant number of vertices. This computation took approximately 9 CPU years on a cluster.

$\|V(G)\|$	5	6	7	8	9	10
# graphs generated	1	7	45	253	1,385	5,402
$\|V(G)\|$	11	12	13	14	15	16
# graphs generated	12,829	24,802	36,435	41,422	42,769	46,176
$\|V(G)\|$	17	18	19	20	21	22
# graphs generated	54,001	70,205	99,680	145,968	233,687	382,762
$\|V(G)\|$	23	24	25	26	27	28
# graphs generated	696,462	1,430,280	3,002,407	6,410,184	13,703,206	30,764,536

Table 4: Counts of the number of P_6-free graphs generated by our implementation of Algorithm 1 without testing for induced diamonds

7 Proof of Theorem 1

Let G be a 4-critical P_6-free graph. Suppose that $G \notin \mathcal{L}$ and that $|V(G)|$ is minimal with respect to this property. If G is diamond-free, Lemma 4 implies $G \in \mathcal{L}$, a contradiction. We may thus assume that there is a maximal tripod $T = (A_1, A_2, A_3)$ in G which is not just a triangle.

Suppose that there is some vertex $x \in V(G) \setminus V(T)$ with a neighbor in each A_i, $i = 1, 2, 3$. Then $V(G) = V(T) \cup \{x\}$, and so $|V(G)| \leq 18$ by Lemma 10. By Lemma 12, $G \in \mathcal{L}$, a contradiction.

So, we may assume that no vertex has a neighbor in all three classes of T. Let G' be the graph obtained by contracting T in G. By Lemma 3 we know that G' is P_6-free and not 3-colorable. We may thus pick a 4-critical P_6-free subgraph H of G'.

Since G was chosen to have a minimal number of vertices among all 4-critical P_6-free graphs not in $\mathcal{L}$, we may assume that $H \in \mathcal{L}$. Let T' be the triangle in G' obtained by contracting T. If H is a K_4, $|V(H) \cap V(T')| \leq 2$, since no vertex of G has a neighbor in all three classes of T. Since $H \in \mathcal{L}$, $|V(H)| \leq 16$. Now Lemma 9 and the minimality of G imply that $|V(G)| \leq |V(H)| + 12 \leq 28$. Consequently, by Lemma 12, $G \in \mathcal{L}$.

8 P_7-free obstructions

This section is devoted to the following unpublished observation by Pokrovskiy [16].

Lemma 13. *There are infinitely many 4-critical P_7-free graphs.*

In the proof we construct an infinite family of 4-vertex-critical P_7-free graphs, i.e., P_7-free graphs which are 4-chromatic but every proper induced subgraph is 3-colorable. This means that there is also an infinite number of 4-critical P_7-free graphs. Note that, indeed, not all members of our family are 4-critical P_7-free.

Proof of Lemma 13. Consider the following construction. For each $r \geq 1$, G_r is a graph defined on the vertex set $v_0, \ldots, v_{3r}$. The graph G_5 is shown in Fig. 4. A vertex v_i, where $i \in \{0, 1, \ldots, 3r\}$, is adjacent to v_{i-1}, v_{i+1}, and v_{i+3j+2}, for all $j \in \{0, 1, \ldots, r-1\}$. Here and throughout the proof, we consider the indices to be taken modulo $3r + 1$.

First we observe that, up to permuting the colors, there is exactly one 3-coloring of $G_r - v_0$. Indeed, we may w.l.o.g. assume that v_i receives color i, for $i = 1, 2, 3$, since $\{v_1, v_2, v_3\}$ forms a triangle in G_r. Similarly, v_4 receives color 1, v_5 receives color 2 and so on. Finally, v_{3r} receives color 3. Since the coloring was forced, our claim is proven.

In particular, G_r is not 3-colorable, since v_0 is adjacent to all of v_1, v_2, v_{3r}. As the choice of v_0 was arbitrary, we know that G_r is 4-vertex-critical.

It remains to prove that G_r is P_7-free. Suppose that $P = x_1$-x_2-$\ldots$-x_7 is an induced P_7 in G_r. To simplify the argumentation, we assume G_r to be equipped with the proper coloring described above. That is, v_0 has color 4, and, for all $i = 0, \ldots, r-1$ and $j = 1, 2, 3$, the vertex v_{3i+j} is colored with color j. Let X_i denote the set of vertices of color i, for $i = 1, 2, 3, 4$.

If $r \leq 2$, $|V(G_r)| \leq 7$, and so we are done since obviously G_2 is not isomorphic to P_7. Therefore, we may assume $r \geq 3$ and, since G_r is vertex-transitive, w.l.o.g. $v_0 \notin V(P)$. Hence, P is an induced P_7 in the graph $H := G_r - v_0$, which we consider from now on.

First we suppose that some vertices of P appear consecutively in the ordering $v_1, \ldots, v_{3r}$. That is, w.l.o.g. $x_i = v_j$ and $x_{i+1} = v_{j+1}$ for some $i \in \{1, \ldots, 6\}$ and $j \in \{1, \ldots, 3r-1\}$. Since P is an induced path, we know that neither of v_{j-1} and v_{j+2}, if existent, are contained in P. Thus, we may assume that $j = 1$, and so $v_3 \notin V(P)$. Recall that $N_H(v_1) \setminus \{v_2\} = X_3$ and $N_H(v_2) \setminus \{v_3\} = X_1$. Thus, $|N_H(x_i) \cap V(P)| \leq 2$ implies $|X_3 \cap V(P)| \leq 1$, and similarly $|N_H(x_{i+1}) \cap V(P)| \leq 2$ implies $|X_1 \cap V(P)| \leq 2$. Therefore, $|X_2 \cap V(P)| = 4$, which means that $x_1, x_3, x_5, x_7 \in X_2$. But this is a contradiction to the fact that $N_H(v_1) \setminus \{v_2\} = X_3$.

Hence, no two vertices of P appear consecutively in the ordering $v_1, \ldots, v_{3r}$. For simplicity, let us say that a vertex v_i is *left of* (*right of*) a vertex v_j if $i < j$ (if $i > j$). We now know the following. Let $x \in V(P)$ be left of $y \in V(P)$. Then $xy \in E$ if and only if $x \in X_1$ and $y \in X_3$, $x \in X_2$ and $y \in X_1$, or $x \in X_3$ and $y \in X_2$. Below we make frequent use of this fact without further reference.

W.l.o.g. $x_1 \in X_1$ and $x_2 \in X_2$. In particular, x_2 is left of x_1. We now distinguish the possible colorings of the remaining vertices of P, obtaining a contradiction in each case.

Case 1. $x_3 \in X_1$.

In this case, x_3 must be right of x_2.

Case 1.1. $x_4 \in X_2$.

In this case, x_4 is right of x_1, and in turn x_3 is right of x_4. Hence, x_5 cannot be in X_1, since then it must be right of x_4 but left of x_2. So, $x_5 \in X_3$, and thus x_5 is between x_2 and x_1.

Case 1.1.1. $x_6 \in X_1$.

Then x_6 must be left of x_2. If $x_7 \in X_2$, it must be left of x_6 but right of x_3, a contradiction. Otherwise if $x_7 \in X_3$, it must be left of x_1 but right of x_4, another contradiction.

Case 1.1.2. $x_6 \in X_2$.

In this case x_6 must be right of x_3. If $x_7 \in X_1$, it must be left of x_4 but right of x_6, a contradiction. Otherwise if $x_7 \in X_3$, it must be left of x_1 but right of x_4, again a contradiction.

Case 1.2. $x_4 \in X_3$.

In this case, x_4 is right of x_3, and in turn x_1 is right of x_4.

Case 1.2.1. $x_5 \in X_1$.

So, x_5 must be left of x_2. Hence, x_6 cannot be in X_2, since then x_6 must be left of x_5 and right of x_1. Thus, $x_6 \in X_3$, which means that x_6 is between x_2 and x_3.

If $x_7 \in X_1$, it must be left of x_6 but right of x_1, a contradiction. Otherwise if $x_7 \in X_2$, it must be left of x_4 but right of x_1, another contradiction.

Case 1.2.2. $x_5 \in X_2$.

So, x_5 must be right of x_1. Clearly $x_6 \notin X_1$, for then it must be left of x_2 but right of x_5. So, $x_6 \in X_3$, and thus x_6 is between x_2 and x_3.

If $x_7 \in X_1$, it must be left of x_6 but right of x_4, a contradiction. Otherwise if $x_7 \in X_2$, it must be left of x_4 but right of x_1, another contradiction.

Case 2. $x_3 \in X_3$.

In this case, x_3 must be left of x_2.

Case 2.1. $x_4 \in X_1$.

Then x_4 is left of x_3, and thus also x_1 and x_2.

If $x_5 \in X_2$, x_5 must be left of x_4 but right of x_1, a contradiction. So, $x_5 \in X_3$. Then x_5 must be between x_2 and x_1. If $x_6 \in X_2$, it must be right of x_5 but left of x_3, a contradiction. So, $x_6 \in X_1$, and thus x_6 must be between x_3 and x_2.

If $x_7 \in X_2$, it must be left of x_6 but right of x_1, a contradiction. Hence, $x_7 \in X_3$. But now x_7 must be right of x_6 and left of x_4, another contradiction.

Case 2.2. $x_4 \in X_2$.

Then x_4 must be right of x_1.

If $x_5 \in X_1$, it must be right of x_4 but left of x_2, a contradiction. So, $x_5 \in X_3$, and thus x_5 is between x_2 and x_1.

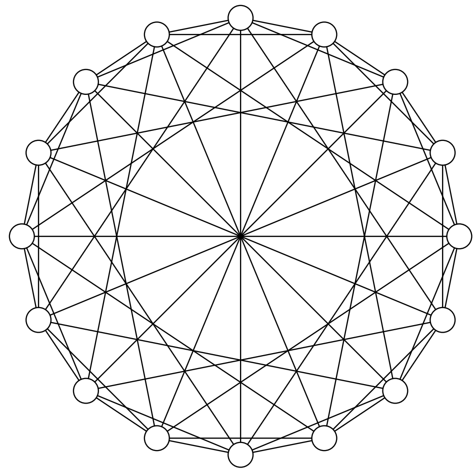

Figure 4: A circular drawing of G_5

If $x_6 \in X_1$, it must be between x_3 and x_2. If, moreover, $x_7 \in X_2$, x_7 is left of x_6 but right of x_1, a contradiction. Similarly, if $x_7 \in X_3$, x_7 is left of x_1 but right of x_4, another contradiction.

We thus know $x_6 \in X_2$. But then x_6 must be left of x_3 and right of x_5, a contradiction.

Summing up, G_r is P_7-free, and this completes the proof. □

We also modified Algorithm 2 to generate 4-critical P_7-free graphs. As one would expect, the number of obstructions is much larger than in the P_6-free case. Table 5 contains the counts of all 4-critical and 4-vertex-critical P_7-free graphs up to 15 vertices.

Vertices	Critical graphs	Vertex-critical graphs
4	1	1
6	1	1
7	2	7
8	5	8
9	21	124
10	99	2,263
11	212	1,771
12	522	6,293
13	679	15,064
14	368	4,521
15	304	2,914
≤ 15	2,214	32,967

Table 5: Counts of all 4-critical and 4-vertex-critical P_7-free graphs up to 15 vertices

Acknowledgments

We thank Alexey Pokrovskiy for pointing us to the existence of an infinite family of P_7-free obstructions.

Several of the computations for this work were carried out using the Stevin Supercomputer Infrastructure at Ghent University.

References

[1] F. Bonomo, M. Chudnovsky, P. Maceli, O. Schaudt, M. Stein, and M. Zhong, *Three-coloring and list three-coloring graphs without induced paths on seven vertices*, to appear in Combinatorica, `http://doi.org/10.1007/s00493-017-3553-8`, 2017.

[2] G. Brinkmann, K. Coolsaet, J. Goedgebeur, and H. Mélot, *House of Graphs: a database of interesting graphs*, Discrete Applied Mathematics **161** (2013), no. 1-2, 311–314, Available at `http://hog.grinvin.org/`.

[3] D. Bruce, C.T. Hoàng, and J. Sawada, *A certifying algorithm for 3-colorability of P_5-free graphs*, Proceedings of the 20th International Symposium on Algorithms and Computation, Springer-Verlag, 2009, pp. 594–604.

[4] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, *The Strong Perfect Graph Theorem*, Annals of Mathematics **164** (2006), no. 1, 51–229.

[5] Maria Chudnovsky, Jan Goedgebeur, Oliver Schaudt, and Mingxian Zhong, *Obstructions for three-coloring graphs with one forbidden induced subgraph*, Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2016, Arlington, VA, USA, January 10-12, 2016, 2016, pp. 1774–1783.

[6] P. Erdős, *Graph theory and probability*, Canadian Journal of Mathematics **11** (1959), 34–38.

[7] J. Goedgebeur, Homepage of generator for 4-critical P_t-free graphs: `http://caagt.ugent.be/criticalpfree/`.

[8] ———, Homepage of generator for 4-critical P_t-free 1-vertex extensions of tripods: `http://caagt.ugent.be/tripods/`.

[9] J. Goedgebeur and O. Schaudt, *Exhaustive generation of k-critical $\mathcal{H}$-free graphs*, Journal of Graph Theory **87(2)** (2018), 188–207.

[10] P.A. Golovach, M. Johnson, D. Paulusma, and J. Song, *A survey on the computational complexity of coloring graphs with forbidden subgraphs*, Journal of Graph Theory **84(4)** (2017), 331–363.

[11] P. Hell and S. Huang, *Complexity of coloring graphs without paths and cycles*, LATIN 2014: Theoretical Informatics, Springer, 2014, pp. 538–549.

[12] C.T. Hoàng, B. Moore, D. Recoskie, J. Sawada, and M. Vatshelle, *Constructions of k-critical P_5-free graphs*, Discrete Applied Mathematics **182** (2015), 91–98.

[13] F. Lazebnik and V.A. Ustimenko, *Explicit construction of graphs with an arbitrary large girth and of large size*, Discrete Applied Mathematics **60** (1995), 275–284.

[14] B.D. McKay, *nauty User's Guide (Version 2.5)*, Technical Report TR-CS-90-02, Department of Computer Science, Australian National University. The latest version of the software is available at `http://cs.anu.edu.au/~bdm/nauty`.

[15] B.D. McKay and A. Piperno, *Practical graph isomorphism, II*, Journal of Symbolic Computation **60** (2014), 94–112.

[16] A. Pokrovskiy, *private communication.*

[17] B. Randerath, I. Schiermeyer, and M. Tewes, *Three-colorability and forbidden subgraphs. II: polynomial algorithms*, Discrete Mathematics **251** (2002), 137–153.

[18] P. Seymour, *Barbados workshop on graph coloring and structure, 2014.*

[19] N. Sloane, The on-line encyclopedia of integer sequences: `http://oeis.org/`.

Appendix 1: Correctness testing

Since several results obtained in this paper rely on computations, it is very important that the correctness of our programs has been thoroughly verified to minimize the chance of programming errors. In the following subsections we explain how we tested the correctness of our implementations.

Since all of our consistency tests passed, we believe that this is strong evidence for the correctness of our implementations.

Appendix 1.1: Correctness testing of critical P_t-free graph generator

We performed the following consistency tests to verify the correctness of our generator for k-critical P_t-free graphs (i.e. Algorithm 1). The source code of this program can be downloaded from [7].

- We applied the program to generate critical graphs for cases which were already settled before in the literature and verified that our program indeed obtained the same results. More specifically we verified that our program yielded exactly the same results in the following cases:
  - There are six 4-critical P_5-free graphs [3].
  - There are eight 5-critical (P_5, C_5)-free graphs [12].
  - The Grötzsch graph is the only 4-critical (P_6, C_3)-free graph [17].
  - There are four 4-critical (P_6, C_4)-free graphs [11].
- We developed an independent generator for k-critical P_t-free graphs by starting from the program `geng` [14, 15] (which is a generator for all graphs) and adding pruning routines to it for colorability and P_t-freeness. This generator cannot terminate, but we were able to independently verify the following results with it:
  - We executed this program to generate all 4-critical $(P_6, \text{diamond})$-free graphs up to 16 vertices and it indeed yielded the same 6 critical graphs from Lemma 4.
  - We executed this program to generate all 4-critical and 4-vertex-critical P_6-free graphs up to 16 vertices and it indeed yielded the same graphs from Theorem 1 and Table 1.
  - We executed this program to generate all 4-critical and 4-vertex-critical P_7-free graphs up to 13 vertices and it indeed yielded the same graphs from Table 5.
- We modified our program to generate all P_t-free graphs and compared it with the known counts of P_t-free graphs for $t = 4, 5$ on the On-Line Encyclopedia of Integer Sequences [19] (i.e. sequences A000669 and A078564).
- We modified our program to generate all k-colorable graphs and compared it with the known counts of k-colorable graphs for $k = 3, 4$ on the On-Line Encyclopedia of Integer Sequences [19] (i.e. sequences A076322 and A076323).
- We determined all k-vertex-critical graphs in two independent ways and both methods yielded exactly the same results:
  1. By modifying line 3 of Algorithm 2 so it tests for k-vertex-criticality instead of k-criticality.
  2. By recursively adding edges in all possible ways to the set of critical graphs (as long as the graphs remain k-vertex-critical) and testing if the resulting graphs are P_t-free.

Appendix 1.2: Correctness testing of tripod generator

We performed the following consistency tests to verify the correctness of our generator for 4-critical P_6-free 1-vertex extensions of tripods (i.e. Algorithm 3). The source code of this program can be downloaded from [8].

- We wrote a program to test if a graph is a 1-vertex extension of a tripod and applied it to the 24 4-critical P_6-free graphs from Theorem 1. 11 of those graphs are 1-vertex extensions of a tripod (i.e. F_1, F_2, F_4, F_6, F_7, F_9, F_{10}, F_{17}, F_{21}, F_{22} and F_{23}). We verified that our implementation of Algorithm 3 indeed yielded exactly those 11 graphs which are a 1-vertex extension of a tripod (except K_4).
- We used Algorithm 1 to generate all 4-critical P_7-free graphs up to 14 vertices. There are 1910 such graphs and 595 of them are 1-vertex extensions of a tripod (see Table 6 for details). We modified our implementation of Algorithm 3 to generate 4-critical P_7-free 1-vertex extensions of tripods and executed it up to 14 vertices. We verified that this indeed yields exactly those 595 graphs which are a 1-vertex extension of a tripod (except K_4).

Vertices	Critical graphs	1-vertex extensions
4	1	1
6	1	1
7	2	1
8	5	4
9	21	14
10	99	56
11	212	87
12	522	141
13	679	196
14	368	94
≤ 14	1,910	595

Table 6: Counts of 4-critical P_7-free graphs up to 14 vertices and the number of those graphs which are 1-vertex extensions of a tripod

Appendix 2: Adjacency lists

This section contains the adjacency lists of the 24 4-critical P_6-free graphs from Theorem 1. The graphs are listed in the same order as in Fig. 1 and 2.

- Graph F_1: {0 : 1 2 3; 1 : 0 2 3; 2 : 0 1 3; 2 : 0 1 3}
- Graph F_2: {0 : 2 3 5; 1 : 3 4 5; 2 : 0 4 5; 3 : 0 1 5; 4 : 1 2 5; 5 : 0 1 2 3 4}
- Graph F_3: {0 : 2 4 5; 1 : 3 5 6; 2 : 0 4 6; 3 : 1 5 6; 4 : 0 2 6; 5 : 0 1 3; 6 : 1 2 3 4}
- Graph F_4: {0 : 3 4 5; 1 : 3 5 6; 2 : 4 5 6; 3 : 0 1 4 6; 4 : 0 2 3 6; 5 : 0 1 2; 6 : 1 2 3 4}
- Graph F_5: {0 : 3 4 5; 1 : 4 6 7; 2 : 5 6 7; 3 : 0 6 7; 4 : 0 1 5; 5 : 0 2 4; 6 : 1 2 3 7; 7 : 1 2 3 6}
- Graph F_6: {0 : 3 5 6; 1 : 4 5 7; 2 : 5 6 7; 3 : 0 6 7; 4 : 1 6 7; 5 : 0 1 2; 6 : 0 2 3 4 7; 7 : 1 2 3 4 6}
- Graph F_7: {0 : 3 4 5 7; 1 : 4 5 6; 2 : 5 6 7; 3 : 0 6 7; 4 : 0 1 7; 5 : 0 1 2; 6 : 1 2 3 7; 7 : 0 2 3 4 6}
- Graph F_8: {0 : 3 5 7; 1 : 4 7 8; 2 : 5 6 7; 3 : 0 6 8; 4 : 1 7 8; 5 : 0 2 8; 6 : 2 3 8; 7 : 0 1 2 4; 8 : 1 3 4 5 6}

- Graph F_9: {0 : 4 5 8; 1 : 4 7 8; 2 : 5 6 8; 3 : 6 7 8; 4 : 0 1 6 8; 5 : 0 2 7; 6 : 2 3 4 8; 7 : 1 3 5; 8 : 0 1 2 3 4 6}
- Graph F_{10}: {0 : 4 5 7; 1 : 4 7 8; 2 : 5 6 7; 3 : 6 7 8; 4 : 0 1 6 8; 5 : 0 2 8; 6 : 2 3 4 8; 7 : 0 1 2 3; 8 : 1 3 4 5 6}
- Graph F_{11}: {0 : 3 4 5 8; 1 : 4 5 6; 2 : 5 6 7 8; 3 : 0 6 7; 4 : 0 1 7 8; 5 : 0 1 2; 6 : 1 2 3 8; 7 : 2 3 4; 8 : 0 2 4 6}
- Graph F_{12}: {0 : 3 6 9; 1 : 4 6 7; 2 : 5 7 8; 3 : 0 6 9; 4 : 1 8 9; 5 : 2 7 8; 6 : 0 1 3 8; 7 : 1 2 5 9; 8 : 2 4 5 6; 9 : 0 3 4 7}
- Graph F_{13}: {0 : 4 6 9; 1 : 5 6 8; 2 : 6 8 9; 3 : 7 8 9; 4 : 0 7 8; 5 : 1 7 9; 6 : 0 1 2 7; 7 : 3 4 5 6; 8 : 1 2 3 4 9; 9 : 0 2 3 5 8}
- Graph F_{14}: {0 : 4 5 7 9; 1 : 5 6 7; 2 : 6 7 8; 3 : 7 8 9; 4 : 0 6 8; 5 : 0 1 8 9; 6 : 1 2 4 9; 7 : 0 1 2 3; 8 : 2 3 4 5; 9 : 0 3 5 6}
- Graph F_{15}: {0 : 4 5 8 9; 1 : 4 7 8 9; 2 : 5 6 8; 3 : 6 7 8; 4 : 0 1 6 9; 5 : 0 2 7; 6 : 2 3 4 9; 7 : 1 3 5; 8 : 0 1 2 3; 9 : 0 1 4 6}
- Graph F_{16}: {0 : 5 6 7; 1 : 5 6 9; 2 : 5 8 9; 3 : 6 7 8; 4 : 7 8 9; 5 : 0 1 2 7 8; 6 : 0 1 3 8 9; 7 : 0 3 4 5 9; 8 : 2 3 4 5 6; 9 : 1 2 4 6 7}
- Graph F_{17}: {0 : 3 5 6 9; 1 : 4 6 8; 2 : 5 6 7 8 9; 3 : 0 7 8 9; 4 : 1 7 9; 5 : 0 2 7 8; 6 : 0 1 2; 7 : 2 3 4 5; 8 : 1 2 3 5 9; 9 : 0 2 3 4 8}
- Graph F_{18}: {0 : 5 6 10; 1 : 5 9 10; 2 : 6 7 10; 3 : 7 8 10; 4 : 8 9 10; 5 : 0 1 7 8; 6 : 0 2 8 9; 7 : 2 3 5 9; 8 : 3 4 5 6; 9 : 1 4 6 7; 10 : 0 1 2 3 4}
- Graph F_{19}: {0 : 4 6 7 10; 1 : 5 9 10; 2 : 6 8 9 10; 3 : 7 8 9 10; 4 : 0 8 9; 5 : 1 9 10; 6 : 0 2 7; 7 : 0 3 6; 8 : 2 3 4 10; 9 : 1 2 3 4 5; 10 : 0 1 2 3 5 8}
- Graph F_{20}: {0 : 5 10 11; 1 : 6 7 10 11; 2 : 6 9 10 11; 3 : 7 8 10 11; 4 : 8 9 10 11; 5 : 0 10 11; 6 : 1 2 8 10; 7 : 1 3 9; 8 : 3 4 6 11; 9 : 2 4 7; 10 : 0 1 2 3 4 5 6; 11 : 0 1 2 3 4 5 8}
- Graph F_{21}: {0 : 4 6 7 10 11; 1 : 5 6 7 8 11; 2 : 6 8 10 11 12; 3 : 7 8 9 10 11 12; 4 : 0 8 9 12; 5 : 1 9 10 11 12; 6 : 0 1 2 9 10 12; 7 : 0 1 3 12; 8 : 1 2 3 4; 9 : 3 4 5 6 11; 10 : 0 2 3 5 6; 11 : 0 1 2 3 5 9; 12 : 2 3 4 5 6 7}
- Graph F_{22}: {0 : 4 6 8 9 11 12; 1 : 5 6 7 10 11 12; 2 : 6 7 8 9 11 12; 3 : 9 10 11 12; 4 : 0 7 10 11 12; 5 : 1 8 9 12; 6 : 0 1 2 10; 7 : 1 2 4 9; 8 : 0 2 5 10 11; 9 : 0 2 3 5 7 10; 10 : 1 3 4 6 8 9; 11 : 0 1 2 3 4 8; 12 : 0 1 2 3 4 5}
- Graph F_{23}: {0 : 4 6 7 9 10; 1 : 5 7 8 9; 2 : 6 7 9 10 11; 3 : 7 8 9 10 11 12; 4 : 0 8 9 10 11 12; 5 : 1 10 11 12; 6 : 0 2 8 11 12; 7 : 0 1 2 3 11 12; 8 : 1 3 4 6; 9 : 0 1 2 3 4 12; 10 : 0 2 3 4 5; 11 : 2 3 4 5 6 7; 12 : 3 4 5 6 7 9}
- Graph F_{24}: {0 : 4 8 13 14 15; 1 : 5 8 10 14 15; 2 : 6 8 9 10 15; 3 : 7 8 9 10 11; 4 : 0 9 10 11 12; 5 : 1 9 11 12 13; 6 : 2 11 12 13 14; 7 : 3 12 13 14 15; 8 : 0 1 2 3 11 12 13; 9 : 2 3 4 5 13 14 15; 10 : 1 2 3 4 12 13 14; 11 : 3 4 5 6 8 14 15; 12 : 4 5 6 7 8 10 15; 13 : 0 5 6 7 8 9 10; 14 : 0 1 6 7 9 10 11; 15 : 0 1 2 7 9 11 12}