Coloring graphs with no induced five-vertex path or gem*

M. Chudnovsky† T. Karthick‡ P. Maceli§
Frédéric Maffray¶

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Abstract

For a graph G, let $\chi(G)$ and $\omega(G)$ respectively denote the chromatic number and clique number of G. We give an explicit structural description of (P_5, gem)-free graphs, and show that every such graph G satisfies $\chi(G) \leq \lceil \frac{5\omega(G)}{4} \rceil$. Moreover, this bound is best possible. Here a *gem* is the graph that consists of an induced four-vertex path plus a vertex which is adjacent to all the vertices of that path.

Keywords: P_5-free graphs; Chromatic number; Clique number; χ-boundedness.

1 Introduction

All our graphs are finite and have no loops or multiple edges. For any integer k, a *k-coloring* of a graph G is a mapping $\phi : V(G) \to \{1, \ldots, k\}$ such that any two adjacent vertices u, v in G satisfy $\phi(u) \neq \phi(v)$. A graph is *k-colorable* if it admits a k-coloring. The *chromatic number* $\chi(G)$ of a graph G is the smallest integer k such that G is k-colorable. A *clique* in a graph G is a set of pairwise adjacent vertices, and the *clique number* of G, denoted by $\omega(G)$, is the size of a maximum clique in G. Clearly $\chi(H) \geq \omega(H)$ for every induced subgraph H of G. A graph G is *perfect* if every induced subgraph H of G satisfies $\chi(H) = \omega(H)$. Following Gyárfás [10], we say that a class of graphs is *χ-bounded* if there is a function f (called a *χ-bounding function*) such that every member G of the class satisfies $\chi(G) \leq f(\omega(G))$. Thus the class of perfect graphs is χ-bounded with $f(x) = x$.

For any integer ℓ we let P_ℓ denote the chordless path on ℓ vertices and C_ℓ denote the chordless cycle on ℓ vertices. The *gem* is the graph that consists

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‡Computer Science Unit, Indian Statistical Institute, Chennai Centre, Chennai 600029, India. This research is partially supported by DST-SERB, Government of India under MATRICS scheme.

§Ithaca College, Ithaca NY 14850, USA.

¶Deceased on August 22, 2018.

of a P_4 plus a vertex adjacent to all vertices of the P_4. A *hole* (*antihole*) in a graph is an induced subgraph that is isomorphic to C_ℓ ($\overline{C_\ell}$) with $\ell \geq 4$, and ℓ is the length of the hole (antihole). A hole or an antihole is *odd* if ℓ is odd. Given a family of graphs $\mathcal{F}$, a graph G is $\mathcal{F}$*-free* if no induced subgraph of G is isomorphic to a member of $\mathcal{F}$; when $\mathcal{F}$ has only one element F we say that G is F-free; when $\mathcal{F}$ has two elements F_1 and F_2, we simply write G is (F_1, F_2)-free instead of $\{F_1, F_2\}$-free. If $\mathcal{F}$ is a finite family of graphs, and if $\mathcal{C}$ is the class of $\mathcal{F}$-free graphs which is χ-bounded, then by a classical result of Erdös [7], at least one member of $\mathcal{F}$ is a forest (see also [10]). Thus, for instance, the class of C_3-free (or triangle-free) graphs is not χ-bounded. We refer to [16, 17] for more results on χ-bounded classes of $\mathcal{F}$-free graphs, and we give below some of them which are related to our results.

Gyárfás [10] showed that the class of P_t-free graphs is χ-bounded. Gravier et al. [9] improved Gyárfás's bound slightly by proving that every P_t-free graph G satisfies $\chi(G) \leq (t-2)^{\omega(G)-1}$. It is well known that every P_4-free graph is perfect. The preceding result implies that every P_5-free graph G satisfies $\chi(G) \leq 3^{\omega(G)-1}$. The problem of determining whether the class of P_5-free graphs admits a polynomial χ-bounding function remains open, and it is remarked in [13] (without proof) that the known χ-bounding function f for such class of graphs satisfies $c(\omega^2/\log w) \leq f(\omega) \leq 2^\omega$. So the recent focus is on obtaining χ-bounding functions for some classes of P_5-free graphs. The first author and Sivaraman [6] showed that every (P_5, C_5)-free graph G satisfies $\chi(G) \leq 2^{\omega(G)-1}$, and that every ($P_5$, bull)-free graph G satisfies $\chi(G) \leq \binom{\omega(G)+1}{2}$. Schiermeyer [15] showed that every (P_5, H)-free graph G satisfies $\chi(G) \leq \omega(G)^2$, for some special graphs H. The second author with Arnab Char [3] showed that every (P_5, 4-wheel)-free graph G satisfies $\chi(G) \leq \frac{3}{2}\omega(G)$. Fouquet et al. [8] proved that there are infinitely many (P_5, $\overline{P_5}$)-free graphs G with $\chi(G) \geq \omega(G)^\alpha$, where $\alpha = \log_2 5 - 1$, and that every (P_5, $\overline{P_5}$)-free graph G satisfies $\chi(G) \leq \binom{\omega(G)+1}{2}$. The second author with Choudum and Shalu [4] studied the class of (P_5, gem)-free graphs and showed that every such graph G satisfies $\chi(G) \leq 4\omega(G)$. Later Cameron, Huang and Merkel [2] improved this result replacing 4ω with $\lfloor \frac{3\omega}{2} \rfloor$. We improve this result further and establish the best possible bound, as follows.

Theorem 1 *Let G be a (P_5, gem)-free graph. Then $\chi(G) \leq \lceil \frac{5\omega(G)}{4} \rceil$. Moreover, this bound is tight.*

The degree of a vertex in a graph G is the number of vertices adjacent to it. The maximum degree over all vertices in G is denoted by $\Delta(G)$. Clearly every graph G satisfies $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$. Reed [14] conjectured that every graph G satisfies $\chi(G) \leq \lceil \frac{\Delta(G)+\omega(G)+1}{2} \rceil$. Reed's conjecture is still open in general. It is shown in [11] that if a graph G satisfies $\chi(G) \leq \lceil \frac{5\omega(G)}{4} \rceil$, then $\chi(G) \leq \lceil \frac{\Delta(G)+\omega(G)+1}{2} \rceil$. So by Theorem 1, we immediately have the following theorem.

Theorem 2 *Let G be a (P_5, gem)-free graph. Then $\chi(G) \leq \lceil \frac{\Delta(G)+\omega(G)+1}{2} \rceil$. Moreover, this bound is tight.*

A *stable set* in a graph G is a set of pairwise nonadjacent vertices, and the *stability number* of G, denoted by $\alpha(G)$, is the size of a maximum stable set in G. It is folklore that every graph G satisfies $\chi(G) \geq \lceil \frac{|V(G)|}{\alpha(G)} \rceil$.

The bounds in Theorem 1 and in Theorem 2 are tight on the following example. Let G be a graph whose vertex-set is partitioned into five cliques $Q_1, \ldots, Q_5$ such that for each i mod 5, every vertex in Q_i is adjacent to every vertex in $Q_{i+1} \cup Q_{i-1}$ and to no vertex in $Q_{i+2} \cup Q_{i-2}$, and $|Q_i| = t$ for all i ($t > 0$). Then $|V(G)| = 5t$, $\Delta(G) = 3t - 1$, $\omega(G) = 2t$ and $\alpha(G) = 2$. Moreover, it is easy to check that G is (P_5, gem)-free. So by Theorem 1, $\chi(G) \leq \lceil \frac{5t}{2} \rceil$. Also, since $\chi(G) \geq \lceil \frac{|V(G)|}{\alpha(G)} \rceil$, we have $\chi(G) \geq \lceil \frac{5t}{2} \rceil$. So $\chi(G) = \lceil \frac{5t}{2} \rceil$.

Our proof of Theorem 1 uses the structure theorem for (P_5, gem)-free graphs (Theorem 3). Before stating it we recall some definitions.

Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. For any two subsets X and Y of $V(G)$, we denote by $[X, Y]$, the set of edges that has one end in X and other end in Y. We say that X is *complete* to Y or $[X, Y]$ is complete if every vertex in X is adjacent to every vertex in Y; and X is *anticomplete* to Y if $[X, Y] = \emptyset$. If X is singleton, say $\{v\}$, we simply write v is complete (anticomplete) to Y instead of writing $\{v\}$ is complete (anticomplete) to Y. For any $x \in V(G)$, let $N(x)$ denote the set of all neighbors of x in G; and let $\deg_G(x) := |N(x)|$. The neighborhood $N(X)$ of a subset $X \subseteq V(G)$ is the set $\{u \in V(G) \setminus X \mid u$ is adjacent to a vertex of $X\}$. If $X \subseteq V(G)$, then $G[X]$ denote the subgraph induced by X in G. A set $X \subseteq V(G)$ is a *homogeneous set* if every vertex in $V(G) \setminus X$ with a neighbor in X is complete to X. Note that in any gem-free graph G, for every $v \in V(G)$, $N(v)$ induces a P_4-free graph, and hence the subgraph induced by a homogeneous set in any connected graph G is P_4-free.

An *expansion* of a graph H is any graph G such that $V(G)$ can be partitioned into $|V(H)|$ nonempty sets Q_v, $v \in V(H)$, such that $[Q_u, Q_v]$ is complete if $uv \in E(H)$, and $[Q_u, Q_v] = \emptyset$ if $uv \notin E(H)$. An expansion of a graph is a *clique expansion* if each Q_v is a clique, and is a *P_4-free expansion* if each Q_v induces a P_4-free graph. See Figure 1 for examples.

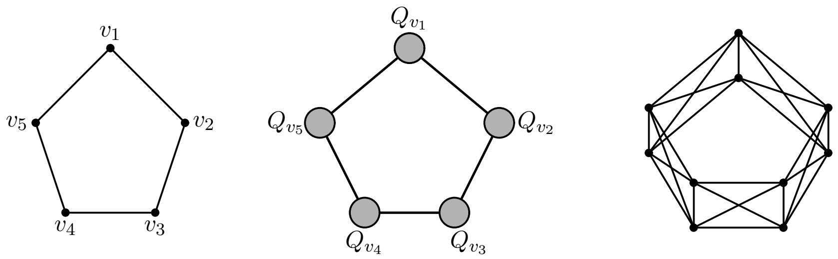

Figure 1: (a) A C_5. (b) Schematic representation of a P_4-free expansion of C_5 given in (a). Here, the shaded circles represent a collection of sets into which the vertex-set of the graph is partitioned. Each shaded circle means a nonempty set that induces a P_4-free subgraph. A solid line (the absence of a line) between any two circles means the respective sets are complete (anticomplete) to each other. (c) An example of a clique expansion of C_5 given in (a), where $|Q_{v_i}| = 2$ for each i.

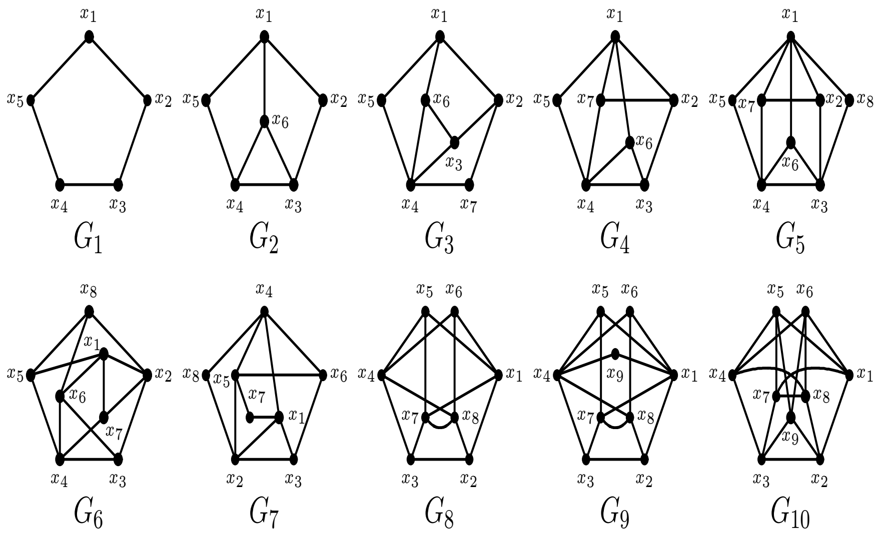

Figure 2: Basic graphs

Let $G_1, G_2, \ldots, G_{10}$ be the ten graphs shown in Figure 2. Clearly each of $G_1, \ldots, G_{10}$ is (P_5, gem)-free. Moreover, it is easy to check that any P_4-free expansion of a (P_5, gem)-free graph is (P_5, gem)-free.

Graph class $\mathcal{H}$: The class of connected (P_5, gem)-free graphs G such that $V(G)$ can be partitioned into seven nonempty sets $A_1, \ldots, A_7$ such that:

- Each A_i induces a P_4-free graph.
- $[A_1, A_2 \cup A_5 \cup A_6]$ is complete and $[A_1, A_3 \cup A_4 \cup A_7] = \emptyset$.
- $[A_3, A_2 \cup A_4 \cup A_6]$ is complete and $[A_3, A_5 \cup A_7] = \emptyset$.
- $[A_4, A_5 \cup A_6]$ is complete and $[A_4, A_2 \cup A_7] = \emptyset$.
- $[A_2, A_5 \cup A_6 \cup A_7] = \emptyset$ and $[A_5, A_6 \cup A_7] = \emptyset$.
- The vertex-set of each component of $G[A_7]$ is a homogeneous set.
- Every vertex in A_7 has a neighbor in A_6.

Now we can state our structural result.

Theorem 3 *Let G be a connected (P_5, gem)-free graph that contains an induced C_5. Then either $G \in \mathcal{H}$ or G is a P_4-free expansion of either G_1, G_2, $\ldots$, G_9 or G_{10}.*

We note that another structure theorem for (P_5, gem)-free graphs using a recursive construction is given by Brandstädt and Kratsch [1]. However, it seems difficult to use that theorem to get the bounds derived in this paper.

2 Proof of Theorem 3

Throughout this section, we use the following convention. We simply write v_1-v_2-v_3-v_4-v_5 to mean a P_5 with vertex-set $\{v_1, v_2, v_3, v_4, v_5\}$ and edge-set $\{v_1v_2, v_2v_3, v_3v_4, v_4v_5\}$. Also, we will say that the set $\{v_1, v_2, v_3, v_4, c\}$ induces a *gem*, if v_1-v_2-v_3-v_4 is a P_4, and c is complete to $\{v_1, v_2, v_3, v_4\}$.

Let G be a connected (P_5, gem)-free graph. Since G contains an induced C_5, there are five nonempty and pairwise disjoint sets $A_1, \ldots, A_5$ such that for each i modulo 5 the set A_i is complete to $A_{i-1} \cup A_{i+1}$ and anticomplete to $A_{i-2} \cup A_{i+2}$. Let $A := A_1 \cup \cdots \cup A_5$. We choose these sets such that A is maximal. From now on every subscript is understood modulo 5. Let $R := \{x \in V(G) \setminus A \mid x$ has no neighbor in $A\}$, and for each i, let

$$\begin{aligned} Y_i \;:=\; & \{x \in V(G) \setminus A \mid x \text{ is complete to } A_i, \text{ anticomplete to } A_{i-1} \cup A_{i+1}, \\ & \text{and } x \text{ has a neighbor in each of } A_{i-2} \text{ and } A_{i+2}, \text{ and } x \text{ is complete} \\ & \text{to one of } A_{i-2} \text{ and } A_{i+2}\}. \end{aligned}$$

Clearly, the sets Y_i's are pairwise disjoint. Moreover, we have the following.

Claim 3.1 $V(G) = A_1 \cup \cdots \cup A_5 \cup Y_1 \cup \cdots \cup Y_5 \cup R$.

Proof. Consider any $x \in V(G) \setminus (A \cup R)$. For each i let a_i be a neighbor of x in A_i (if any such vertex exists) and b_i be a nonneighbor of x in A_i (if any exists). Let $L := \{i \mid a_i$ exists$\}$. Then $L \neq \emptyset$ since $x \notin R$. Up to symmetry there are four cases:
(a) $L = \{i\}$ or $\{i, i+1\}$ for some i. Then x-a_i-b_{i-1}-b_{i-2}-b_{i-3} is a P_5, a contradiction.
(b) $L = \{i-1, i+1\}$ or $\{i-1, i, i+1\}$ for some i. Then x is complete to $A_{i-1} \cup A_{i+1}$, for otherwise x-a_{i+1}-b_{i+2}-b_{i-2}-b_{i-1} or x-a_{i-1}-b_{i-2}-b_{i+2}-b_{i+1} is a P_5, a contradiction. But then x can be added to A_i, contradicting the maximality of A.
(c) $L = \{i, i-2, i+2\}$ for some i. Then x is complete to A_i, for otherwise x-a_{i+2}-b_{i+1}-b_i-b_{i-1} is a P_5, and similarly x must be complete to one of A_{i-2} and A_{i+2}. So x is in Y_i.
(d) $|L| \geq 4$. Then $\{a_i, a_{i+1}, a_{i+2}, a_{i+3}, x\}$ induces a gem for some i, a contradiction. $\Diamond$

Claim 3.2 *For each i, $G[A_i]$ and $G[Y_i]$ are P_4-free.*

Proof. Since G is gem-free, the claim follows by the definitions of A_i and Y_i. $\Diamond$

Claim 3.3 *For each i we have $[Y_{i-1}, Y_{i+1}] = \emptyset$.*

Proof. Pick any $y \in Y_{i-1}$ and $z \in Y_{i+1}$. We know that y has neighbors $a_{i+1} \in A_{i+1}$ and $a_{i+2} \in A_{i+2}$, and z has a neighbor $a_{i-1} \in A_{i-1}$. Then $yz \notin E(G)$, for otherwise $\{a_{i-1}, z, a_{i+1}, a_{i+2}, y\}$ induces a gem, a contradiction. $\Diamond$

We say that a vertex in Y_i is *pure* if it is complete to $A_{i-2} \cup A_{i+2}$, and the set Y_i is *pure* if every vertex in Y_i is pure.

Claim 3.4 *Suppose that there exists a pure vertex in Y_i for some i. Then Y_i is pure.*

Proof. We may assume that $i = 1$ and let $p \in Y_1$ be pure. Suppose to the contrary that there exists a vertex $y \in Y_1$ which is not pure, say y has a nonneighbor $b_3 \in A_3$. So y is complete to A_4. Moreover, by the definition of Y_1, y has a neighbor $a_3 \in A_3$. Then $b_3a_3 \notin E(G)$, for otherwise b_3-a_3-y-a_1-a_5 is a P_5 for any $a_1 \in A_1$ and $a_5 \in A_5$. Also, for any $a_1 \in A_1$ and $a_4 \in A_4$, since $\{a_1, y, a_4, b_3, p\}$ does not induce a gem, we have $py \notin E(G)$. But, then for any $a_4 \in A_4$, $\{b_3, p, a_3, y, a_4\}$ induces a gem, a contradiction. ♢

Claim 3.5 *For each i, we have: either $[Y_i, A_{i+2}]$ is complete or $[Y_i, A_{i-2}]$ is complete.*

Proof. We may assume that $i = 1$. Suppose to the contrary that there exist vertices y_1 and y_2 in Y_1 such that y_1 has a nonneighbor $b_4 \in A_4$ and y_2 has a nonneighbor $b_3 \in A_3$. By the definition of Y_1, y_1 is complete to A_3, and has a neighbor $a_4 \in A_4$. Likewise, y_2 is complete to A_4, and has a neighbor $a_3 \in A_3$. Then $a_3b_3 \notin E(G)$, for otherwise b_3-a_3-y_2-a_1-a_5 is a P_5 for any $a_1 \in A_1$ and $a_5 \in A_5$. Also, for any $a_1 \in A_1$, since $\{a_1, y_1, a_3, b_4, y_2\}$ does not induce a gem, we have $y_1y_2 \notin E(G)$. But, then $\{b_3, y_1, a_3, y_2, a_4\}$ induces a gem, a contradiction. ♢

Claim 3.6 *Suppose that $[Y_i, A_{i-2}]$ is complete for some i. Let $A'_{i+2} = N(Y_i) \cap A_{i+2}$ and $A''_{i+2} = A_{i+2} \setminus A'_{i+2}$. Then: (i) $[A'_{i+2}, A''_{i+2}] = \emptyset$, and (ii) $[Y_i, A'_{i+2}]$ is complete.*

Proof. (i): Suppose to the contrary that there are adjacent vertices $p \in A'_{i+2}$ and $q \in A''_{i+2}$. Pick a neighbor of p in Y_i, say y. Clearly $yq \notin E(G)$. Then for any $a_i \in A_i$ and $a_{i-1} \in A_{i-1}$, q-p-y-a_i-a_{i-1} is a P_5, a contradiction. This proves item (i).

(ii): Suppose to the contrary that there are nonadjacent vertices $y \in Y_i$ and $p \in A'_{i+2}$. Pick a neighbor of p in Y_i, say y'. By the definition of Y_i, y has a neighbor in A'_{i+2}, say q. Pick any $a_{i-2} \in A_{i-2}$, $a_{i-1} \in A_{i-1}$ and $a_i \in A_i$. Now, $pq \notin E(G)$, for otherwise p-q-y-a_i-a_{i-1} is a P_5. Also, $yy' \notin E(G)$, for otherwise $\{p, a_{i-2}, y, a_i, y'\}$ induces a gem. Then since $\{y, q, y', p, a_{i-2}\}$ does not induce a gem, $qy' \notin E(G)$. But then p-y'-a_i-y-q is a P_5, a contradiction. This proves item (ii). ♢

Claim 3.7 *Suppose that Y_{i-2} and Y_{i+2} are both nonempty for some i. Let $A_i^- = N(Y_{i-2}) \cap A_i$ and $A_i^+ = N(Y_{i+2}) \cap A_i$. Then:*

(a) $[Y_{i-2}, Y_{i+2}]$ *is complete,* $A_i^- \cap A_i^+ = \emptyset$, *and* $[A_i^-, A_i^+] = \emptyset$,

(b) $[A_i \setminus (A_i^- \cup A_i^+), A_i^- \cup A_i^+] = \emptyset$,

(c) $[Y_{i-2}, A_{i+1} \cup A_i^-]$ *and* $[Y_{i+2}, A_{i-1} \cup A_i^+]$ *are complete,*

(d) $Y_{i-1} \cup Y_{i+1} = \emptyset$,

(e) Y_i *is pure,*

(f) One of the sets $A_i \setminus (A_i^- \cup A_i^+)$ and Y_i is empty.

Proof. Pick any $y \in Y_{i-2}$ and $z \in Y_{i+2}$. So y has neighbors $a_{i-2} \in A_{i-2}$, $a_{i+1} \in A_{i+1}$ and $a_i \in A_i$, and z has neighbors $a_{i+2} \in A_{i+2}$, $a_{i-1} \in A_{i-1}$ and $b_i \in A_i$.

(a): Now $yz \in E(G)$, for otherwise y-a_{i+1}-a_{i+2}-z-a_{i-1} is a P_5. Since this holds for arbitrary y, z, we obtain that $[Y_{i-2}, Y_{i+2}]$ is complete. Then $za_i \notin E(G)$, for otherwise $\{a_{i+1}, y, z, a_{i-1}, a_i\}$ induces a gem, and similarly $yb_i \notin E(G)$. In particular $a_i \neq b_i$; moreover $a_ib_i \notin E(G)$, for otherwise a_i-b_i-z-a_{i+2}-a_{i-2} is a P_5. Since this holds for any y, z, a_i, b_i, it proves item (a).

(b): Suppose that there are adjacent vertices $u \in A_i \setminus (A_i^- \cup A_i^+)$ and $v \in A_i^- \cup A_i^+$, say $v \in A_i^-$. Then u-v-y-a_{i-2}-a_{i+2} is a P_5, a contradiction. This proves item (b).

(c): Since y and z are not complete to A_i (by (a)), by Claim 3.5, $[Y_{i-2}, A_{i+1}]$ and $[Y_{i+2}, A_{i-1}]$ are complete. Also, by Claim 3.6(ii), $[Y_{i-2}, A_i^-]$ and $[Y_{i+2}, A_i^+]$ are complete. This proves item (c).

(d): If $Y_{i-1} \neq \emptyset$ then, by a similar argument as in the proof of (c) (with subscripts shifted by 1), $[Y_{i-2}, A_i]$ should be complete, which it is not. So $Y_{i-1} = \emptyset$, and similarly $Y_{i+1} = \emptyset$. This proves item (d).

(e): Consider any $x \in Y_i$ and suppose that it is not pure; up to symmetry x has a nonneighbor $b \in A_{i+2}$ and is complete to A_{i-2}. By Claim 3.3 we know that $xz \notin E(G)$. Then a_i-x-a_{i-2}-b-z is a P_5. This proves item (e).

(f): Suppose that there are vertices $b \in A_i \setminus (A_i^- \cup A_i^+)$ and $u \in Y_i$. By the definition of Y_i, we know that $bu \in E(G)$, and by Claim 3.3, $uy, uz \notin E(G)$. Then by item (c) and item (e), for any $a_{i-2} \in A_{i-2}$, b-u-a_{i-2}-y-z is a P_5, a contradiction. This proves item (f). ◊

Claim 3.8 *(i) Every vertex in R has a neighbor in Y_i, for some i. (ii) The vertex-set of each component of $G[R]$ is a homogeneous set, and hence each component of $G[R]$ is P_4-free.*

Proof. (i): Suppose to the contrary that there exists a vertex $r \in R$ which has no neighbor in Y_i for every i. Then since G is connected, by using Claim 3.1, there exists a vertex $r' \in R$ and an index $j \in \{1, 2, \ldots, 5\}$, $j \mod 5$ such that r' is adjacent to a vertex $y \in Y_j$ and that there is a shortest path P with end vertices r' and r in $G[R]$. Now the vertices of P together with $\{y, a_j, a_{j+1}\}$ induces a P_5, for any $a_j \in A_j$ and $a_{j+1} \in A_{j+1}$ which is a contradiction. So (i) holds.

(ii): Suppose that a vertex-set of a component T of $G[R]$ is not homogeneous. Then, since T is connected, there are adjacent vertices $u, t \in V(T)$ and a vertex $y \in V(G) \setminus V(T)$ with $yu \in E(G)$ and $yt \notin E(G)$. By Claim 3.1 we have $y \in Y_i$ for some i. Then t-u-y-a_i-a_{i+1} is a P_5, for any $a_i \in A_i$ and $a_{i+1} \in A_{i+1}$, a contradiction. So (ii) holds. ◊

Claim 3.9 *Suppose that there is any edge ry with $r \in R$ and $y \in Y_i$ for some i. Then y is pure and $Y_{i-1} \cup Y_{i+1} = \emptyset$. Moreover at most one of the sets Y_{i-2}, Y_{i+2} is nonempty, and R is complete to that nonempty set and to Y_i.*

Proof. Consider any edge ry with $r \in R$ and $y \in Y_i$. So y has a neighbor $a_j \in A_j$ for each $j \in \{i, i-2, i+2\}$. If y is not pure, then up to symmetry

y has a nonneighbor $b \in A_{i-2}$, and then r-y-a_i-a_{i-1}-b is a P_5 for any $a_{i-1} \in A_{i-1}$, a contradiction. So y is pure, and by Claim 3.7, $Y_{i-1} \cup Y_{i+1} = \emptyset$. Now suppose up to symmetry that there is a vertex $z \in Y_{i+2}$. By Claim 3.3, we have $yz \notin E(G)$. Then $rz \in E(G)$, for otherwise r-y-a_{i+2}-z-a_{i-1} is a P_5, for any $a_{i-1} \in A_{i-1} \cap N(z)$. Now by the same argument as above, z is pure, and by Claim 3.7, $Y_{i+1} \cup Y_{i+3} = \emptyset$. Since this holds for any z, the vertex r is complete to Y_{i+2}, and then by symmetry r is complete to Y_i; and by Claim 3.8(i) and the fact that G is connected, R is complete to $Y_i \cup Y_{i+2}$. $\Diamond$

It follows from the preceding claims that at most three of the sets $Y_1, \dots, Y_5$ are nonempty, and if $R \neq \emptyset$ then at most two of $Y_1, \dots, Y_5$ are nonempty. Hence we have the following cases:

(A) $R = \emptyset$ and $Y_2 \cup Y_3 \cup Y_5 = \emptyset$. Any of Y_1, Y_4 may be nonempty.
We may assume that both Y_1 and Y_4 are not empty, $[Y_1, A_3]$ is complete and $[Y_4, A_2]$ is complete. (Otherwise, using Claims 3.2, 3.5 and 3.6, it follows that G is a P_4-free expansion of either $G_1, G_2, \dots, G_6$ or G_9.) Suppose there exists $y_1 \in Y_1$ that has a nonneighbor $a_4 \in A_4$, and there exists $y_4 \in Y_4$ that has a nonneighbor $a_1 \in A_1$, then for any $a_3 \in A_3$, a_1-y_1-a_3-a_4-y_4 is a P_5 in G, a contradiction. So either Y_1 is pure or Y_4 is pure. Then by Claims 3.2, 3.5 and 3.6, we see that G is a P_4-free expansion of G_4, G_5 or G_6.

(B) $R = \emptyset$ and Y_2, Y_3 are both nonempty.
Then Claims 3.2 and 3.7 implies that G is a P_4-free expansion of either G_8, G_9 or G_{10}.

(C) $R \neq \emptyset$ and exactly one of $Y_1, \dots, Y_5$ is nonempty, say Y_1 is nonempty.
In this case, we show that $G \in \mathcal{H}$ as follows: Since $R \neq \emptyset$, by Claim 3.8(i), there exists a vertex $r \in R$ and a vertex $y \in Y_1$ such that $ry \in E(G)$. Then by Claim 3.9, y is a pure vertex of Y_1. So, by Claim 3.4, Y_1 is pure, and hence by Claims 3.2 and 3.8, we see that $G \in \mathcal{H}$.

(D) $R \neq \emptyset$ and exactly two of $Y_1, \dots, Y_5$ are nonempty.
In this case, by Claims 3.8 and 3.9 and up to symmetry we may assume that Y_1 and Y_4 are nonempty, all vertices in $Y_1 \cup Y_4$ are pure, and $[R, Y_1 \cup Y_4]$ is complete. Moreover, since G is gem-free, $G[R]$ is P_4-free. So by Claim 3.2, G is a P_4-free expansion of G_7.

This completes the proof of Theorem 3. $\square$

3 Bounding the chromatic number

We say that two sets *meet* if their intersection is not empty. In a graph G, we say that a stable set is *good* if it meets every clique of size $\omega(G)$. Moreover, we say that a clique K in G is a *t-clique* of G if $|K| = t$.

We use the following theorem often.

Theorem 4 ([12]) *Let G be a graph such that every proper induced subgraph G' of G satisfies $\chi(G') \le \lceil \frac{5}{4}\omega(G') \rceil$. Suppose that one of the following occurs:*

(i) G has a vertex of degree at most $\lceil \frac{5}{4}\omega(G) \rceil - 1$.

(ii) *G has a good stable set.*

(iii) *G has a stable set S such that $G - S$ is perfect.*

(iv) *For some integer $t \geq 5$ the graph G has t stable sets $S_1, \ldots, S_t$ such that $\omega(G - (S_1 \cup \cdots \cup S_t)) \leq \omega(G) - (t - 1)$.*

Then $\chi(G) \leq \lceil \frac{5}{4}\omega(G) \rceil$.

Given a graph G and a proper homogeneous set X in G, let G/X be the graph obtained by replacing X with a clique Q of size $\omega(X)$ (i.e., G/X is obtained from $G - X$ and Q by adding all edges between Q and the vertices of $V(G) \setminus X$ that are adjacent to X in G).

Lemma 1 ([11]) *In a graph G let X be a proper homogeneous set such that $G[X]$ is P_4-free. Then $\omega(G) = \omega(G/X)$ and $\chi(G) = \chi(G/X)$. Moreover, G has a good stable set if and only if G/X has a good stable set.*

For $k \in \{1, 2, \ldots, 10\}$, let $\mathcal{G}_k$ be the class of graphs that are P_4-free expansions of G_k, and let $\mathcal{G}_k^*$ be the class of graphs that are clique expansions of G_k. Let $\mathcal{H}^*$ be the class of graphs $G \in \mathcal{H}$ such that, with the notation as in Section 1, the five sets $A_1, A_2, \ldots, A_5$, and the vertex-set of each component of $G[A_7]$ are cliques.

The following lemma can be proved using Lemma 1, and the proof is very similar to that of Lemma 3.3 of [11], so we omit the details.

Lemma 2 *For every graph G in $\mathcal{G}_i$ ($i \in \{1, \ldots, 10\}$) (resp. G in $\mathcal{H}$) there is a graph G^* in $\mathcal{G}_i^*$ ($i \in \{1, \ldots, 10\}$) (resp. G^* in $\mathcal{H}^*$) such that $\omega(G) = \omega(G^*)$ and $\chi(G) = \chi(G^*)$. Moreover, G has a good stable set if and only if G^* has a good stable set.*

By Lemma 2 and Theorem 3, to prove Theorem 1, it suffices to consider the clique expansions of $G_1, G_2 \ldots, G_{10}$ and the members of $\mathcal{H}^*$.

3.1 Coloring clique expansions

Throughout this section, we will use the following notation:

Suppose that G is a clique expansion of $H \in \{G_1, \ldots, G_9\}$. So there is a partition of $V(G)$ into $|V(H)|$ nonempty cliques $Q_1, \ldots, Q_{|V(H)|}$, where Q_i corresponds to the vertex x_i of H. Since Q_i is nonempty for each $i \in \{1, \ldots, |V(H)|\}$, we may call x_i one vertex of Q_i. Moreover if $|Q_i| \geq 2$ we call x_i' one vertex of $Q_i \setminus \{x_i\}$, and if $|Q_i| \geq 3$ we call x_i'' one vertex of $Q_i \setminus \{x_i, x_i'\}$. We write, e.g., Q_{12} instead of $Q_1 \cup Q_2$ whenever $Q_1 \cup Q_2$ is a clique, Q_{123} instead of $Q_1 \cup Q_2 \cup Q_3$ whenever $Q_1 \cup Q_2 \cup Q_3$ is a clique, etc.

Theorem 5 *Let G be a clique expansion of either $G_1, \ldots, G_5$ or G_6, and assume that every proper induced subgraph G' of G satisfies $\chi(G') \leq \lceil \frac{5}{4}\omega(G') \rceil$. Then $\chi(G) \leq \lceil \frac{5}{4}\omega(G) \rceil$.*

Proof. Let G be a clique expansion of either $G_1, \ldots, G_5$ or G_6. Let $q = \omega(G)$. Recall that if G has a good stable set, then we can conclude the theorem using Theorem 4(ii).

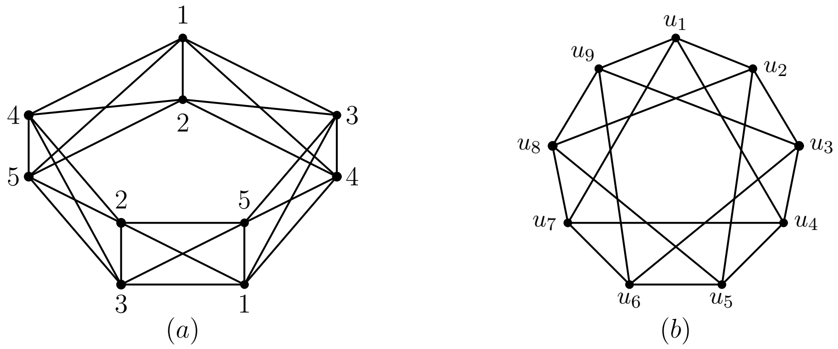

Figure 3: (a) A 5-coloring of a clique expansion of G_1, where for each $v \in V(G_1)$, Q_v is a clique of size 2, and (b) a graph isomorphic to G_{10}.

(I) Suppose that G is a clique expansion of G_1. (We refer to [11, 12] for alternate proofs.) We may assume that $|Q_i| \geq 2$, for each $i \in \{1, \ldots, 5\}$, otherwise if $|Q_1| = 1$ (say), then $G - \{x_1\}$ is perfect, as it is a clique expansion of a P_4, and we can conclude with Theorem 4(iii). Let X be a subset of $V(G)$ obtained by taking two vertices from Q_i for each $i \in \{1, \ldots, 5\}$. Then since $G[X]$ has no stable set of size 3, $\chi(G[X]) \geq \frac{|V(G[X])|}{\alpha(G[X])} = \frac{10}{2} = 5$, and since $\chi(G[X]) \leq 5$ (see Figure 3:(a)), we have $\chi(G[X]) = 5$. Moreover, $\omega(G - X) = q - 4$. So by hypothesis, we have $\chi(G) \leq \lceil \frac{5}{4}\omega(G - X) \rceil + 5 \leq \lceil \frac{5}{4} q \rceil$.

(II) Suppose that G is a clique expansion of G_2. Then $\{x_2, x_5, x_6\}$ is a good stable set of G, and we can conclude with Theorem 4(ii).

(III) Suppose that G is a clique expansion of G_3. Suppose that $|Q_5| \leq |Q_6|$. By hypothesis we can color $G - Q_5$ with $\lceil \frac{5}{4} q \rceil$ colors. Since Q_6 is complete to $Q_1 \cup Q_4$, which is equal to $N(Q_5)$, we can extend this coloring to Q_5, using for Q_5 the colors used for Q_6. Therefore let us assume that $|Q_5| > |Q_6|$. It follows that $|Q_{15}| > |Q_{16}|$, so Q_{16} is not a q-clique. Likewise we may assume that $|Q_7| > |Q_3|$, and consequently Q_{23} is not a q-clique. Therefore all q-cliques of G are in the set $\{Q_{12}, Q_{15}, Q_{27}, Q_{45}, Q_{47}, Q_{346}\}$.

If Q_{15} is not a q-clique, then $\{x_2, x_4\}$ is a good stable set of G, and we can conclude using Theorem 4(ii). Therefore we may assume that Q_{15} is a q-clique of G.

If Q_{45} is not a q-clique, then $\{x_1, x_3, x_7\}$ is a good stable set of G, and we can conclude using Theorem 4(ii). Therefore we may assume that Q_{45} is a q-clique of G.

If Q_{12} is not a q-clique, then $\{x_3, x_5, x_7\}$ is a good stable set of G, and we can conclude using Theorem 4(ii). Therefore we may assume that Q_{12} is a q-clique of G.

If Q_{47} is not a q-clique, then $\{x_2, x_5, x_6\}$ is a good stable set of G, and we can conclude using Theorem 4(ii). Therefore we may assume that Q_{47} is a q-clique of G.

If Q_{27} is not a q-clique, then $\{x_1, x_4\}$ is a good stable set of G, and we can conclude using Theorem 4(ii). Therefore we may assume that Q_{27} is a q-clique of G.

Thus the above properties imply that there is an integer a with $1 \le a \le q-1$ such that $|Q_2| = |Q_5| = |Q_7| = a$ and $|Q_1| = |Q_4| = q - a$. Since $|Q_7| > |Q_3|$, we have $a \ge 2$. Since $q = |Q_{27}| = 2a$, $a = \frac{q}{2}$. So q is even, $q \ge 4$ and $|Q_1| = |Q_2| = |Q_4| = |Q_5| = |Q_7| = \frac{q}{2} \ge 2$.

Now consider the five stable sets $\{x_1, x_3, x_7\}$, $\{x'_1, x_4\}$, $\{x_5, x_6, x'_7\}$, $\{x_2, x'_4\}$ and $\{x'_2, x'_5\}$. It is easy to see that their union U meets every q-clique four times. It follows that $\omega(G - U) = q - 4$, and we can conclude using Theorem 4(iv).

(IV) Suppose that G is a clique expansion of either G_4 or G_5. Suppose that $|Q_5| \le |Q_7|$. By hypothesis we can color $G - Q_5$ with $\lceil \frac{5}{4}q \rceil$ colors. Since Q_7 is complete to $Q_1 \cup Q_4$, which is equal to $N(Q_5)$, we can extend this coloring to Q_5, using for Q_5 the colors used for Q_7. Therefore let us assume that $|Q_5| > |Q_7|$. It follows that $|Q_{45}| > |Q_{47}|$, so Q_{47} is not a q-clique. Likewise we may assume that $|Q_5| > |Q_6|$ (for otherwise any $\lceil \frac{5}{4}q \rceil$-coloring of $G - Q_5$ can be extended to Q_5), and consequently Q_{16} is not a q-clique.

Therefore, if G is a clique expansion of G_4, all q-cliques of G are in the set $\{Q_{15}, Q_{23}, Q_{45}, Q_{127}, Q_{346}\}$, and if G is a clique expansion of G_5, all q-cliques of G are in the set $\{Q_{15}, Q_{18}, Q_{23}, Q_{45}, Q_{38}, Q_{127}, Q_{346}\}$.

Hence if G is a clique expansion of G_4, then $\{x_2, x_5, x_6\}$ is a good stable set of G, and if G is a clique expansion of G_5, then $\{x_2, x_5, x_6, x_8\}$ is a good stable set of G. In either case, we can conclude the theorem with Theorem 4(ii).

(V) Suppose that G is a clique expansion of G_6. Suppose that $|Q_8| \le |Q_1|$. By hypothesis we can color $G - Q_8$ with $\lceil \frac{5}{4}q \rceil$ colors. Since Q_1 is complete to $Q_2 \cup Q_5 \cup Q_6$, which is equal to $N(Q_8)$, we can extend this coloring to Q_8, using for Q_8 the colors used for Q_1. Therefore let us assume that $|Q_8| > |Q_1|$. It follows that $|Q_{68}| > |Q_{16}|$ and $|Q_{58}| > |Q_{15}|$, and consequently Q_{16} and Q_{15} are not q-cliques. Likewise we may assume that $|Q_5| > |Q_6|$ (for otherwise any $\lceil \frac{5}{4}q \rceil$-coloring of $G - Q_5$ can be extended to Q_5), and consequently Q_{68} is not a q-clique. Therefore all q-cliques of G are in the set $\{Q_{23}, Q_{28}, Q_{45}, Q_{47}, Q_{58}, Q_{127}, Q_{346}\}$.

If Q_{23} is not a q-clique, then $\{x_1, x_4, x_8\}$ is a good stable set of G, and we can conclude using Theorem 4(ii). Therefore we may assume that Q_{23} is a q-clique of G.

If Q_{28} is not a q-clique, then $\{x_3, x_5, x_7\}$ is a good stable set of G, and we can conclude using Theorem 4(ii). Therefore we may assume that Q_{28} is a q-clique of G.

If Q_{58} is not a q-clique, then $\{x_2, x_4\}$ is a good stable set of G, and we can conclude using Theorem 4(ii). Therefore we may assume that Q_{58} is a q-clique of G.

If Q_{45} is not a q-clique, then $\{x_3, x_7, x_8\}$ is a good stable set of G, and we can conclude using Theorem 4(ii). Therefore we may assume that Q_{45} is a q-clique of G.

Now we claim that Q_{47} is not a q-clique. Suppose not. Then the above properties imply that there is an integer a with $1 \le a \le q-1$ such that $|Q_2| = |Q_5| = |Q_7| = a$ and $|Q_3| = |Q_4| = |Q_8| = q - a$. Since $|Q_{346}| = |Q_6| + 2(q-a) \le q$, we have $|Q_6| \le 2a - q$. Also, since $|Q_{127}| = |Q_1| + 2a \le q$, we have $|Q_1| \le q - 2a$. However, $2 \le |Q_{16}| \le (q-2a) + (2a-q) = 0$ which is a contradiction. So Q_{47} is not a q-clique. Then $\{x_2, x_5, x_6\}$ is a good stable set of G, and we can conclude the theorem with Theorem 4(ii). □

Theorem 6 *Let G be a clique expansion of G_7, and assume that every proper induced subgraph G' of G satisfies $\chi(G') \le \lceil \frac{5}{4}\omega(G')\rceil$. Then $\chi(G) \le \lceil \frac{5}{4}\omega(G)\rceil$.*

Proof. Let $q = \omega(G)$. Suppose that $|Q_7| \le |Q_2|$. By hypothesis we can color $G - Q_7$ with $\lceil \frac{5}{4}q\rceil$ colors. Since Q_2 is complete to $Q_1 \cup Q_5$, which is equal to $N(Q_7)$, we can extend this coloring to Q_7, using for Q_7 the colors used for Q_2. Therefore let us assume that $|Q_7| > |Q_2|$; and similarly, that $|Q_8| > |Q_5|$. It follows that $|Q_{25}| < |Q_{57}|$, so Q_{25} is not a q-clique of G. By symmetry, Q_{14} is not a q-clique of G. Therefore all q-cliques of G are in the set $\mathcal{Q} = \{Q_{17}, Q_{28}, Q_{36}, Q_{48}, Q_{57}, Q_{123}, Q_{456}\}$.

If Q_{123} is not a q-clique, then $\{x_6, x_7, x_8\}$ is a good stable set of G, and we can conclude using Theorem 4(ii). Therefore we may assume that Q_{123}, and similarly Q_{456}, is a q-clique of G.

If Q_{36} is not a q-clique, then $\{x_1, x_5, x_8\}$ is a good stable set of G, and we can conclude using Theorem 4(ii). Therefore we may assume that Q_{36} is a q-clique of G.

If Q_{17} is not a q-clique, then $\{x_3, x_5, x_8\}$ is a good stable set of G, and we can conclude using Theorem 4(ii). Therefore we may assume that Q_{17}, and similarly each of Q_{57}, Q_{28} and Q_{48}, is a q-clique of G.

Hence $\mathcal{Q}$ is precisely the set of all q-cliques of G. It follows that there are integers a, b, c with $a = |Q_1|$, $b = |Q_2|$, $c = |Q_3|$, $a + b + c = q$, and then $|Q_7| = q - a$, $|Q_5| = a$, $|Q_8| = q - b$, $|Q_4| = b$, hence $|Q_6| = c$. Since $q = |Q_{36}| = 2c$, it must be that q is even and $c = \frac{q}{2}$, so $|Q_3| = |Q_6| = \frac{q}{2}$.

Since each of Q_1, Q_2, Q_3 is nonempty we have $q \ge 3$, and since q is even, $q \ge 4$. Hence $|Q_3|, |Q_6| \ge 2$ (so the vertices x'_3 and x'_6 exist). Since Q_2 and Q_3 are nonempty, and $|Q_3| = \frac{q}{2}$, we have $a < \frac{q}{2}$, so $|Q_7| = q - a > \frac{q}{2}$, so $|Q_7| \ge 3$ (and so the vertices x'_7 and x''_7 exist). Likewise $|Q_8| \ge 3$ (and so the vertices x'_8 and x''_8 exist). We observe that the clique Q_{14} satisfies $|Q_{14}| = a+b = \frac{q}{2} \le q-2$ since $q \ge 4$. Likewise $|Q_{25}| \le q - 2$.

Now consider the five stable sets $\{x_3, x_4, x_7\}$, $\{x_1, x_6, x_8\}$, $\{x'_3, x_5, x'_8\}$, $\{x'_6, x_2, x'_7\}$ and $\{x''_7, x''_8\}$. It is easy to see that their union U meets every q-clique (every member of $\mathcal{Q}$) four times, and that it meets each of Q_{14} and Q_{25} twice. It follows (since $|Q_{14}|, |Q_{25}| \le q-2$) that $\omega(G-U) = q-4$, and we can conclude using Theorem 4(iv). □

Theorem 7 *Let G be a clique expansion of either G_8, G_9 or G_{10}, and assume that every proper induced subgraph G' of G satisfies $\chi(G') \le \lceil \frac{5}{4}\omega(G')\rceil$. Then $\chi(G) \le \lceil \frac{5}{4}\omega(G)\rceil$.*

Proof. Let G be a clique expansion of either G_8, G_9 or G_{10}. Let $q = \omega(G)$.

(I) Suppose that G is a clique expansion of G_8. Suppose that $|Q_2| \le |Q_7|$. By hypothesis we can color $G - Q_2$ with $\lceil \frac{5}{4}q\rceil$ colors. Since Q_7 is complete to $Q_1 \cup Q_3 \cup Q_8$, which is equal to $N(Q_2)$, we can extend this coloring to Q_2, using for Q_2 the colors used for Q_7. Therefore let us assume that $|Q_2| > |Q_7|$; and similarly, that $|Q_3| > |Q_8|$. It follows that $|Q_{28}| > |Q_{78}|$, so Q_{78} is not a q-clique of G. Likewise $|Q_{23}| > |Q_{37}|$, so Q_{37} is not a q-clique of G, and similarly Q_{28} is not a q-clique. Therefore all q-cliques of G are in the set $\{Q_{12}, Q_{16}, Q_{23}, Q_{34}, Q_{45}, Q_{157}, Q_{468}\}$.

If Q_{45} is not a q-clique, then $\{x_1, x_3, x_8\}$ is a good stable set of G, and we can conclude with Theorem 4(ii). Hence we may assume that Q_{45}, and similarly Q_{16}, is a q-clique. Also Q_{12} is a q-clique, for otherwise $\{x_3, x_5, x_6\}$ is a good stable set, and similarly Q_{34} is a q-clique.

Now we claim that Q_{23} is not a q-clique of G. Suppose not. Then the above properties imply that there is an integer a with $1 \leq a \leq q-1$ such that $|Q_1| = |Q_3| = |Q_5| = a$ and $|Q_2| = |Q_4| = |Q_6| = q - a$. However we have $q \geq |Q_{157}| > 2a$ and $q \geq |Q_{468}| > 2(q-a)$, hence $2q > 2a + 2(q-a)$, a contradiction. So Q_{23} is not a q-clique of G. But, then $\{x_1, x_4\}$ is a good stable set of G, and we can conclude the theorem with Theorem 4(ii).

(II) Now suppose that G is a clique expansion of G_9. Then a similar argument, as in the case of G_8, shows that, we may assume that Q_{28}, Q_{37} and Q_{78} are not q-cliques (we omit the details). Likewise we may assume that $|Q_9| > |Q_5|$ (for otherwise any $\lceil \frac{5}{4}q \rceil$-coloring of $G - Q_9$ can be extended to Q_9), and consequently Q_{45} is not a q-clique; and similarly Q_{16} is not a q-clique.

Then Q_{19} is a q-clique, for otherwise $\{x_2, x_4, x_7\}$ is a good stable set, and similarly Q_{49} is a q-clique. Also Q_{12} is a q-clique, for otherwise $\{x_3, x_5, x_6, x_9\}$ is a good stable set; and similarly Q_{34} is a q-clique. And Q_{23} is a q-clique, for otherwise $\{x_1, x_4\}$ is a good stable set.

The properties given in the preceding paragraph imply that q is even and that $|Q_1| = |Q_2| = |Q_3| = |Q_4| = |Q_9| = \frac{q}{2}$. We now distinguish two cases.

First suppose that $q = 4k$ for some $k \geq 1$. Hence $\lceil \frac{5}{4}q \rceil = 5k$. Let A, B, C, D, E be five disjoint sets of colors, each of size k. We color the vertices in Q_1 with the colors from $A \cup B$, the vertices in Q_2 with $C \cup D$, the vertices in Q_3 with $E \cup A$, the vertices in Q_4 with $B \cup C$, and the vertices in Q_9 with $D \cup E$. Thus we obtain a $5k$-coloring of $G[Q_1 \cup Q_2 \cup Q_3 \cup Q_4 \cup Q_9]$. We can extend it to the rest of the graph as follows. Since Q_{157} is a clique, and $|Q_1| = \frac{q}{2} = 2k$, we have $|Q_5| + |Q_7| \leq 2k$, hence either $|Q_5| \leq k$ or $|Q_7| \leq k$. Likewise, we have either $|Q_6| \leq k$ or $|Q_8| \leq k$. This yields (up to symmetry) three possibilities:
(i) $|Q_5| \leq k$ and $|Q_6| \leq k$. Then we can color Q_5 with colors form E, Q_6 with colors from D, Q_7 with colors from $C \cup D$, and Q_8 with colors from $A \cup E$.
(ii) $|Q_5| \leq k$ and $|Q_8| \leq k$. Then we can color Q_5 with colors form E, Q_6 with colors from $D \cup E$, Q_7 with colors from $C \cup D$, and Q_8 with colors from A. (The case where $|Q_6| \leq k$ and $|Q_7| \leq k$ is symmetric.)
(iii) $|Q_7| \leq k$ and $|Q_8| \leq k$. Then we can color Q_5 and Q_6 with colors from $D \cup E$, Q_7 with colors from C, and Q_8 with colors from A.

Now suppose that $q = 4k + 2$ for some $k \geq 1$. Hence $\lceil \frac{5}{4}q \rceil = 5k + 3$. Let A, B, C, D, E and $\{z\}$ be six disjoint sets of colors, with $|A| = |B| = |C| = k$ and $|D| = |E| = k + 1$. So these are $5k + 3$ colors. We color the vertices in Q_1 with the colors from $C \cup D$, the vertices in Q_2 with $A \cup E$, the vertices in Q_3 with $B \cup D$, the vertices in Q_4 with $C \cup E$, and the vertices in Q_9 with $A \cup B \cup \{z\}$. Thus we obtain a $5k+3$-coloring of $G[Q_1 \cup Q_2 \cup Q_3 \cup Q_4 \cup Q_9]$. We can extend it to the rest of the graph as follows. Since Q_{157} is a clique, and $|Q_1| = \frac{q}{2} = 2k + 1$, we have $|Q_5| + |Q_7| \leq 2k + 1$, hence either $|Q_5| \leq k$ or $|Q_7| \leq k$ (and in any case $\max\{|Q_5|, |Q_7|\} \leq 2k$). Likewise, we have either $|Q_6| \leq k$ or $|Q_8| \leq k$ (and $\max\{|Q_6|, |Q_8|\} \leq 2k$). This yields (up to symmetry) three possibilities:

(i) $|Q_5| \le k$ and $|Q_6| \le k$. Then we can color Q_5 with colors form B, Q_6 with colors from A, Q_7 with colors from $A \cup E$, and Q_8 with colors from $B \cup D$.
(ii) $|Q_5| \le k$ and $|Q_8| \le k$. Then we can color Q_5 with colors form B, Q_6 with colors from $A \cup B$, Q_7 with colors from $A \cup E$, and Q_8 with colors from D. (The case where $|Q_6| \le k$ and $|Q_7| \le k$ is symmetric.)
(iii) $|Q_7| \le k$ and $|Q_8| \le k$. Then we can color Q_5 and Q_6 with colors from $A \cup B$, Q_7 with colors from E, and Q_8 with colors from D.

(III) Finally suppose that G is a clique expansion of G_{10}. We view G_{10} as the graph with nine vertices $u_1, \ldots, u_9$ and edges $u_i u_{i+1}$ and $u_i u_{i+3}$ for each i modulo 9; see Figure 3:(b). For each i let Q_i be the clique of G that corresponds to u_i, and let u_i be one vertex of Q_i. As usual for the clique $Q_1 \cup Q_2$, we write Q_{12} instead of $Q_1 \cup Q_2$, etc. We make two observations.

Observation 1: If for some i the three cliques $Q_{i,i+1}$, $Q_{i+1,i+2}$ and $Q_{i+2,i+3}$ are not q-cliques, then $\{u_{i+4}, u_{i+6}, u_{i+8}\}$ is a good stable set of G, and we can conclude using Theorem 4(ii). $\diamond$

Observation 2: If for some i we have $|Q_{i-1}| \le \frac{q}{3}$ and $|Q_{i+1}| \le \frac{q}{3}$, then $|Q_i| \ge \frac{2q}{3}$. Indeed suppose (for $i = 1$) that $|Q_9| \le \frac{q}{3}$, $|Q_2| \le \frac{q}{3}$ and $|Q_1| < \frac{2q}{3}$. Then Q_{19} and Q_{12} are not q-cliques, so, by Observation 1, we may assume that Q_{89} and Q_{23} are q-cliques. Hence $|Q_8| \ge \frac{2q}{3}$, and consequently, since Q_{58} is a clique, $|Q_5| \le \frac{q}{3}$, and since Q_{78} is a clique, $|Q_7| \le \frac{q}{3}$; and similarly $|Q_3| \ge \frac{2q}{3}$, and consequently $|Q_4| \le \frac{q}{3}$ and $|Q_6| \le \frac{q}{3}$. But then Q_{45}, Q_{56} and Q_{67} are not q-cliques, so we can conclude as in Observation 1. $\diamond$

Now, since Q_{147} is a clique, we have $|Q_i| \le \frac{q}{3}$ for some $i \in \{1,4,7\}$; and similarly $|Q_j| \le \frac{q}{3}$ for some $j \in \{2,5,8\}$, and $|Q_k| \le \frac{q}{3}$ for some $k \in \{3,6,9\}$. Up to symmetry this implies one the following three cases:
(a) $|Q_1|, |Q_2|, |Q_3| \le \frac{q}{3}$. Then we can conclude using Observation 2.
(b) $|Q_1|, |Q_2|, |Q_6| \le \frac{q}{3}$. Then Q_{12} is not a q-clique, so, by Observation 1, we may assume that one of Q_{91} and Q_{23}, say Q_{91} is a q-clique. Hence $|Q_9| \ge \frac{2q}{3}$, and consequently $|Q_3| \le \frac{q}{3}$. But then we are in case (a) again.
(c) $|Q_1|, |Q_3|, |Q_5| \le \frac{q}{3}$. By Observation 2 we have $|Q_2| \ge \frac{2q}{3}$ and $|Q_4| \ge \frac{2q}{3}$, and consequently $|Q_8| \le \frac{q}{3}$ and $|Q_7| \le \frac{q}{3}$. Then Q_7, Q_8 and Q_3 are like in case (b).

This completes the proof of the theorem. □

3.2 Coloring the graph class $\mathcal{H}^*$

Recall that $\mathcal{H}^*$ is the class of graphs $G \in \mathcal{H}$ such that, with the notation as in Section 1, the five sets $A_1, A_2, \ldots, A_5$, and the vertex-set of each component of $G[A_7]$ are cliques.

Theorem 8 *Let $G \in \mathcal{H}^*$ and assume that every proper induced subgraph G' of G satisfies $\chi(G') \le \lceil \frac{5}{4}\omega(G') \rceil$. Then $\chi(G) \le \lceil \frac{5}{4}\omega(G) \rceil$.*

Proof. Let $q = \omega(G)$. Let $T_1, T_2, \ldots, T_k$ be the components of $G[A_7]$. For each $i \in \{1, \ldots, 5\}$ and for each $j \in \{1, \ldots, k\}$: let x_i be one vertex of A_i, and let t_j be one vertex of $V(T_j)$. Moreover if $|A_i| \ge 2$ we call x'_i one vertex of $A_i \setminus \{x_i\}$, if $|V(T_i)| \ge 2$ we call t^1_i one vertex of $V(T_i) \setminus \{t_i\}$, if $|V(T_i)| \ge 3$ we call t_i^2 one vertex of $V(T_i) \setminus \{t_i, t_i^1\}$, and if $|V(T_i)| \geq 4$ we call t_i^3 one vertex of $V(T_i) \setminus \{t_i, t_i^1, t_i^2\}$.

Suppose that $|A_2| \leq \omega(G[A_6])$. Then by hypothesis, $G - A_2$ can be colored with $\lceil \frac{5}{4}q \rceil$ colors, and since A_6 is complete to $A_1 \cup A_3$ which is equal to $N(A_2)$, we can extend this coloring to A_2 by using the colors of A_6 on A_2. So we may assume that $|A_2| > \omega(G[A_6])$. Likewise, $|A_5| > \omega(G[A_6])$. So it follows that no clique of $A_1 \cup A_6$ is a q-clique of G.

Now consider the stable set $S := \{x_2, x_5, t_1, \ldots, t_k\}$. We may assume that S is not a good stable set of G (otherwise, we can conclude with Theorem 4(ii)). So there is a maximum clique Q of G contained in $A_3 \cup A_4 \cup A_6$. Further, it follows that for every maximum clique Q of G with $Q \cap S = \emptyset$, we have $A_3 \cup A_4 \subset Q$.

If $A_1 \cup A_2$ is not a q-clique, then $\{x_3, x_5, t_1, \ldots, t_k\}$ is a good stable set of G, and we can conclude using Theorem 4(ii). So we may assume that $A_1 \cup A_2$ is a q-clique of G. Likewise, $A_1 \cup A_5$ is a q-clique of G.

If $A_2 \cup A_3$ is not a q-clique, then $\{x_1, x_4, t_1, \ldots, t_k\}$ is a good stable set of G, and we can conclude using Theorem 4(ii). So we may assume that $A_2 \cup A_3$ is a q-clique of G. Likewise, $A_4 \cup A_5$ is a q-clique of G.

The above properties imply that there is an integer a with $1 \leq a \leq q - 1$ such that $|A_1| = |A_3| = |A_4| = a$ and $|A_2| = |A_5| = q - a$. Moreover, every q-clique of G either contains $A_i \cup A_{i+1}$, for some $i \in \{1, \ldots, 5\}$, i modulo 5, or contains T_j, for some $j \in \{1, \ldots, k\}$.

Now if $|V(T_j)| \leq 2a$, for some j, then by hypothesis, $G - V(T_j)$ can be colored with $\lceil \frac{5}{4}q \rceil$ colors. Since $|A_3 \cup A_4| = 2a$, $V(T_j)$ is anticomplete to $A_3 \cup A_4$, $N(V(T_j)) \subseteq A_6$, and since A_6 is complete to $A_3 \cup A_4$, we can extend this coloring to $V(T_j)$ by using the colors of $A_3 \cup A_4$ on $V(T_j)$. So, we may assume that, for each $j \in \{1, \ldots, k\}$, $|V(T_j)| > 2a$.

If $a = 1$, then $\deg_G(x_2) = 2 \leq \lceil \frac{5}{4}q \rceil - 1$, and we can conclude with Theorem 4(i). So we may assume that $a \geq 2$.

Thus for each $j \in \{1, \ldots, k\}$, we have $|V(T_j)| > 4$. Also, since $q - a > \omega(G[A_6])$, we have $q - a \geq 2$.

Now consider the five stable sets $\{x_1, x_3, t_1, t_2 \ldots, t_k\}$, $\{x_3', x_5, t_1^1, t_2^1 \ldots, t_k^1\}$, $\{x_2, x_5', t_1^2, t_2^2, \ldots, t_k^2\}$, $\{x_2', x_4, t_1^3, t_2^3 \ldots, t_k^3\}$, and $\{x_1', x_4'\}$. It is easy to see that their union U meets every q-clique of G four times. It follows that $\omega(G - U) = q - 4$, and we can conclude using Theorem 4(iv). □

Proof of Theorem 1. Let G be any (P_5, gem)-free graph. We prove the theorem by induction on $|V(G)|$. If G is perfect, then $\chi(G) = \omega(G)$ and the theorem holds. So we may assume that G is not perfect, and that G is connected. Since a P_5-free graph contains no hole of length at least 7, and a gem-free graph contains no antihole of length at least 7, it follows from the Strong Perfect Graph Theorem [5] that G contains a hole of length 5. That is, G contains a C_5 as an induced subgraph. By Lemma 2 and Theorem 3 that it suffices to consider the clique expansions of $G_1, G_2, \ldots, G_{10}$ and the members of $\mathcal{H}^*$. Now the result follows directly by the induction hypothesis and from Theorems 5, 6, 7 and 8. This completes the proof of Theorem 1. □

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