FAR-APART ERDŐS–PÓSA PROPERTY OF LONG CYCLES[♀]

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Abstract. We prove that there exist functions $f : \mathbb{N}^2 \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ such that for all positive integers k, d, and $\ell \geqslant 3$, every graph G either contains k cycles of length at least ℓ that are pairwise at distance greater than d, or admits a subset of vertices X with $|X| \leqslant f(k,\ell)$ such that $G - B_G(X, g(d))$ contains no cycle of length at least ℓ, where $B_G(X,r)$ denotes the ball of radius r around X. This generalises a theorem of Dujmović, Joret, Micek, and Morin (2024), which established the $\ell = 3$ case. Moreover, we prove that the theorem holds with $f(k,\ell) \in \mathcal{O}(\ell k \log k)$ and $g(d) \in \mathcal{O}(d)$. The linear bound on g is best possible, while the bound on f is optimal as a function of k for every fixed ℓ. In particular, for $\ell = 3$ our result improves the previous bound of $\mathcal{O}(k^{18} \operatorname{polylog} k)$ by Dujmović et al. The linear bound on g is best possible, while the bound on f is optimal in k for every fixed ℓ, and optimal in ℓ for every fixed $k \geqslant 2$.

1 Introduction

In 1965, Erdős and Pósa [20] proved that there exists a function $f(k)$ such that for every positive integer k and every graph G, either G contains k vertex-disjoint cycles, or there is a subset X of vertices of G with $|X| \leqslant f(k)$ such that $G - X$ has no cycles. This celebrated result has become one of the cornerstones of graph theory and has inspired a vast literature on the *Erdős–Pósa property* for a wide range of combinatorial objects.

Many extensions of the Erdős–Pósa theorem impose additional constraints on the cycles under consideration while retaining the same packing-versus-covering paradigm. Well-known constraints and generalisations include: long cycles [10, 14, 21, 37], where

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each cycle has length at least ℓ; S-cycles [14, 30, 31, 33, 39], where the cycles must intersect a prescribed set S; modularity constraints [6, 16, 27, 30, 31, 34, 40, 44–46], where the lengths of cycles have remainder r modulo m. Another key ingredient in Erdős–Pósa type theorems is the mode of packing, for example, minors [1, 6, 8, 11, 19, 22, 23, 35, 42], edge-disjointness [7, 12, 13, 15, 25], half-integral packing [26, 29, 32, 36, 41].

1.1 Far-Apart Erdős–Pósa Property of Cycles

The following theorem of Dujmović, Joret, Micek, and Morin [18], which we call the "far-apart Erdős–Pósa property of cycles", may be viewed as an extension of the Erdős–Pósa theorem where the cycles being "pairwise far apart" is the mode of packing. Here and throughout the paper, $B_G(X, r)$ denotes the set of vertices at distance at most r from X in G.

Theorem 1 ([18])**.** *There exist functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$, such that for all positive integers k and d, and for every graph G, either G contains k cycles that are pairwise at distance greater than d in G, or there exists a subset X of vertices of G with $|X| \leqslant f(k)$ such that $G - B_G(X, g(d))$ has no cycles.*

Theorem 1 was conjectured in 2024 independently by Chudnovsky and Seymour[1] and by Ahn, Gollin, Huynh, and Kwon [2]. Furthermore, Ahn et al. [2] solved the cases $k = 2$ and arbitrary d as well as $d = 1$ and arbitrary k. They also pointed out that balls are needed in Theorem 1, even for the $d = 1$ case, since in complete graphs, no pair of cycles have distance more than 1, yet an unbounded number of vertices are needed to hit all cycles.

Long cycles, that is, cycles of length at least $\ell \geqslant 3$, form one of the most well-studied instances of the Erdős–Pósa property, with a long line of work in the classical setting of vertex-disjoint packings. A 1988 result of Thomassen [44] implies that for every ℓ, long cycles have the Erdős–Pósa property: every graph either contains k vertex-disjoint cycles of length at least ℓ, or a set of at most $f(k, \ell)$ vertices meeting all of them. Thomassen's argument gives $f(k, \ell) \in 2^{\ell^{\mathcal{O}(k)}}$. The bound was improved to $\mathcal{O}(\ell k^2)$ by Birmelé, Bondy, and Reed [10], then to $\mathcal{O}(\ell k \log k)$ by Fiorini and Herinckx [21], and finally to the asymptotically optimal $\Theta(\ell k + k \log k)$ by Mousset, Noever, Škorić, and Weissenberger [37]. The following theorem, which is our main result, generalises this line of work to the far-apart setting, and equally generalises Theorem 1 to long cycles, with optimal bounds for every fixed ℓ.

Theorem 2. *There exist functions $f : \mathbb{N}^2 \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$, such that for all integers k, d, and ℓ with $k \geqslant 1$, $d \geqslant 1$, and $\ell \geqslant 3$, and for every graph G, either G contains k cycles of length at least ℓ that are pairwise at distance greater than d in G, or there exists a subset X of vertices of G with $|X| \leqslant f(k, \ell)$ such that $G - B_G(X, g(d))$ has no cycles of length at least ℓ. Furthermore, this holds with $f(k, \ell) \in \mathcal{O}(\ell k \log k)$ and $g(d) \in \mathcal{O}(d)$.*

We note that Ahn, Gollin, Huynh, and Kwon [3] recently proved this result for $d = 1$.

[1] The problem was discussed in March 2024 at the Barbados Graph Theory Workshop held at the Bellairs Research Institute of McGill University.

Let us now comment on the optimality of the bounds in Theorem 2. First, f necessarily depends on both k and ℓ since, even for $d = 1$, we have $f(k,\ell) \in \Omega(\ell k + k \log k)$, where the constants are allowed to depend on $g(1)$. To see this, let $\ell_0 := \ell/(2g(1)+1)$, and let G be a graph with at most $k-1$ pairwise vertex-disjoint cycles each of length at least ℓ_0 such that any hitting set for such cycles has size $\Omega(\ell_0 k + k \log k)$. These graphs exist by the '$d = 0$ case' of the problem (which is not covered by Theorem 2): there, balls are not needed, and the optimal size of a hitting set was determined to be $\Theta(\ell k + k \log k)$ by Mousset et al. [37]. Now, let H be obtained from G by subdividing each edge $2g(1)$ times. Thus, every pair of vertices of G has distance greater than $2g(1)$ in H, and so each ball $B_H(x, g(1))$ contains a unique vertex y_x of G. Moreover, H has at most $k-1$ pairwise vertex-disjoint cycles each of length at least ℓ. If, for some $X \subseteq V(H)$, $B_H(X, g(1))$ hits all such cycles in H, then $\{y_x : x \in X\}$ hits all cycles of length at least ℓ_0 in G: otherwise there is a cycle of length at least ℓ in H that is disjoint from $B_H(\{y_x : x \in X\}, 2g(1))$, and hence is not hit by $B_H(X, g(1))$. Therefore $f(k,\ell) \in \Omega(\ell_0 k + k \log k)$, which implies the desired bound $f(k,\ell) \in \Omega(\ell k + k \log k)$ since $g(1)$ is a constant. Second, this $\Omega(\ell k + k \log k)$ bound also establishes optimality of the bound $f(k,\ell) \in \mathcal{O}(\ell k \log k)$ in each variable: it is optimal in k for every fixed ℓ, and optimal in ℓ for every fixed $k \geqslant 2$. Third, the bound $g(d) \in \mathcal{O}(d)$ in Theorem 2 is best possible since $g(d) \geqslant d$ is necessary in any such result.[2]

We now explain how Theorem 2 is obtained, that is, how we generalise Theorem 1 to cycles of length at least ℓ while attaining optimal bounding functions for fixed ℓ. At a high level, our proof follows the same general framework as that of [18]. However, some key ingredients break down when moving beyond $\ell = 3$, and others stand in the way of optimal bounds.

The proof of Theorem 1 in [18] exploits the case $\ell = 3$ in an essential way: hitting all cycles means leaving a forest behind. The argument in [18] is built around cycles whose surrounding balls induce unicyclic subgraphs, culminating in an application of the Helly property for subtrees of a tree (via the Gyárfás–Lehel theorem). For general ℓ, this structure is simply not available. Namely, a graph with no cycle of length at least ℓ need not resemble a forest and may be locally dense. Our proof instead relies on two coarser ingredients. First, by a theorem of Birmelé [9], graphs with no cycle of length at least ℓ have treewidth less than $\ell - 1$. Thus the forests from [18] become graphs of bounded treewidth, and the Helly-type argument must be carried out not on these graphs themselves but on the trees underlying their tree-decompositions. Second, since unicyclic neighbourhoods have no analogue for general ℓ, we introduce the notion of the *span* of a cycle C, which

[2] *Proof.* We show that there is no function f such that Theorem 2 holds with f and $g(d) = d - 1$. Assume for a contradiction that there is a function f. Let $s := f(2,\ell)$. Construct a graph G as follows: Write $V(K_{2s+1}) = [2s+1]$ and let K^*_{2s+1} be the graph obtained from K_{2s+1} by replacing each edge ij with a path P_{ij} between i and j, and of length d. Let $\{C_1, \dots, C_{2s+1}\}$ be a collection of pairwise vertex-disjoint cycles each of length ℓ such that $V(C_i) \cap V(K^*_{2s+1}) = \{i\}$ for all $i \in [2s+1]$. Let $G := K^*_{2s+1} \cup C_1 \cup \dots \cup C_{2s+1}$. Since $G - [2s+1]$ is a forest, every cycle in G has a vertex in $[2s+1]$. Moreover, since distinct $i, j \in [2s+1]$ have $\mathrm{dist}_G(i,j) = d$, G does not have a d-packing of two cycles each of length at least ℓ. To reach a contradiction, we now show that for every set $X \subseteq V(G)$ with $|X| \leqslant s$, $G - B_G(X, d-1)$ has a cycle of length at least ℓ. Notice that each vertex $v \in V(G)$ either lies in $V(C_i)$ for some $i \in [2s+1]$, or is an inner vertex of P_{ij} for some $i, j \in [2s+1]$. Then $B_G(v, d-1)$ intersects either one or two cycles in $\{C_1, \dots, C_{2s+1}\}$ since each P_{ij} has length d. It follows that $B_G(X, d-1) = \bigcup_{v \in X} B_G(v, d-1)$ intersects at most $2|X| \leqslant 2s$ cycles in $\{C_1, \dots, C_{2s+1}\}$, hence missing at least one of them, as required. □

measures how far apart along C the BFS-projections of the endpoints of nearby edges can be. Cycles of small span provide exactly the coarse control that unicyclic balls provided for $\ell = 3$: edges near C project onto short intervals of C. A final difficulty is quantitative: a direct implementation of this strategy produces balls of radius $\mathcal{O}(d\ell)$, and keeping the radius at $\mathcal{O}(d)$, independent of ℓ, requires an additional padding argument that spreads a small hitting set on the cycles of the packing into a slightly larger but well-distributed one.

Regarding the bounding function f, the proof in [18] yields $f(k) \in \mathcal{O}(k^{18} \operatorname{polylog} k)$, largely because it invokes a theorem of Gyárfás and Lehel [28] on the approximate Helly property of subgraphs of a tree with a bounded number of components. Replacing this ingredient with a theorem of Alon [5] (see Theorem 5 in Section 2) already lowers the bound to $\mathcal{O}(k^3 \log k)$, but no further: the approach in [18] applies local arguments to every pair of cycles in a packing of up to $k-1$ cycles, and therefore cannot produce a subquadratic bound. We instead analyze all cycles in the packing at once, and in a global manner. Altogether this yields Theorem 2 with $f(k,\ell) \in \mathcal{O}(\ell k \log k)$ and $g(d) \in \mathcal{O}(d)$ and, as a special case, the optimal bound $f(k) \in \mathcal{O}(k \log k)$ in Theorem 1, where optimality follows from standard lower bounds for the Erdős–Pósa theorem.

Theorems 1 and 2 are part of a larger body of work. The *far-apart Erdős–Pósa property*, as we call it, asks for either a large packing of objects that are pairwise far apart, or a small number of small-radius balls that hit all of the objects. Another example is the following "coarse Gallai theorem" of Distel, Giocanti, Hodor, Legrand-Duchesne, and Micek [17]: for every graph G and every set A of its vertices, either G contains k non-trivial paths that are pairwise at distance at least d, each with both of its endpoints in A, or there is a set X of at most $f(k)$ vertices such that every such path meets $B_G(X, g(k,d))$. The radius of the balls depends on k, in contrast with Theorem 2 where it is $\mathcal{O}(d)$ only.

1.2 Coarse Motivations

From the point of view of graph minor theory, one can restate the Erdős–Pósa theorem as follows: There exists a function $f(k)$ such that for every positive integer k, and for every graph G, either G contains k vertex-disjoint subgraphs each containing a K_3 minor, or there exists a subset X of vertices of G with $|X| \leqslant f(k)$ such that $G - X$ has no K_3 minor. Our far-apart Erdős–Pósa property (from Section 1.1) is inspired by a new line of research around the Erdős–Pósa theorem, where the usual notion of minors is replaced with that of "fat" minors.

This notion comes from a recently developed field of research called *coarse graph theory* that has its roots in the pioneering work of Georgakopoulos and Papasoglu [24], who studied graph minors from the point of view of coarse geometry. Quoting Georgakopoulos and Papasoglu, the general goal of coarse graph theory is

"to view graphs 'from far away' to see their large-scale geometry and its implications."

This novel viewpoint led to a wealth of new and exciting questions and conjectures in graph theory that have attracted the attention of researchers in the last few years. This includes a coarse Menger conjecture (disproved recently by Nguyen, Scott, and Seymour [38]), a coarse grid conjecture (disproved recently by Albrechtsen and Davies [4]), and the

following coarse Erdős–Pósa conjecture.[3] (Necessary definitions are given below.)

Conjecture 3 (Coarse Erdős–Pósa conjecture [24])**.** *There exist functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$, with $g(d) \in \mathcal{O}(d)$, such that, for all positive integers k and d, and for every graph G, either G contains k minor-models of K_3 that are d-fat and pairwise at distance more than d, or there exists a subset X of vertices of G with $|X| \leqslant f(k)$ such that $G - B_G(X, g(d))$ has no d-fat K_3 minor-model.*

To explain the notion of fat minor-models, it will be helpful to first consider the following way of defining the usual notion of minor-models: A *minor-model* of a graph H in a graph G is a collection of vertex-disjoint connected subgraphs $\{M_v : v \in V(H)\}$ and a collection of internally-disjoint paths $\{P_e : e \in E(H)\}$ such that for every edge $e = uv$ of H, the path P_e has one endpoint in M_u and the other endpoint in M_v, and none of its inner vertices is in any subgraph M_w with $w \in V(H)$. Now, given a positive integer d, a minor-model of H in G is *d-fat* if it satisfies the following extra properties:

- M_u and M_v are at distance at least d in G for every two distinct vertices $u, v \in V(H)$;
- P_e and P_f are at distance at least d in G for every two distinct edges $e, f \in E(H)$, and
- M_u and P_e are at distance at least d in G for every vertex $u \in V(H)$ and edge $e \in E(H)$ not incident to u.

Informally, every two objects from the definition of minor-model are required to be far apart in G, except when they are required to intersect as per the original definition of minor. It might be helpful to think of fat minors as minors that would survive even if some disjoint balls of bounded radius were contracted into vertices. Alternatively, at a very intuitive level, the difference between minors and fat minors is that minors are essentially subgraphs that are allowed to be spread out, whereas fat minors are required to be spread out for the purpose of being "seen" even when looking at the graph from far away.

Theorem 2 and Conjecture 3 are closely related but incomparable statements: neither immediately implies the other. A first difference lies in how the parameters interact. In Conjecture 3, a single parameter d governs everything: the fatness of the minor-models, the distances between them, and, as a consequence, the lengths of the cycles involved, since every d-fat minor-model of K_3 contains a cycle of length at least $6d$. In Theorem 2, by contrast, the length threshold ℓ and the distance parameter d are independent of each other. A second difference lies in the nature of the objects being packed. A d-fat K_3 minor-model is intrinsically spread out: many pairs of its vertices are required to be at distance at least d in G. A cycle of length at least ℓ carries no such requirement: any two of its vertices may be close, or even adjacent, in G. For this reason, Conjecture 3 does not imply Theorem 2, even in the regime $\ell \leqslant 6d$: a hitting set for d-fat K_3 minor-models need not come close to any long cycle. (For instance, large complete graphs contain no 2-fat K_3 minor-model but many long cycles.) Conversely, Theorem 2 does not imply Conjecture 3, since a packing of pairwise far-apart long cycles need not yield even a single fat K_3 minor-model.

[3] Very recently, a proof of the conjecture by Sandra Albrechtsen, Marthe Bonamy, Romain Bourneuf, and James Davies was announced at the Focused Workshop on Erdős–Pósa problems held in March 2026 in Będlewo, Poland.

This paper is organised as follows. In Section 2, we introduce the main tools we use in our proofs; in Section 3, we state our key definitions; in Section 4, we sketch the main proof; and in Section 5, we prove Theorem 2.

2 Tools

Our proof makes use of three tools. The first is a key ingredient of Simonovits's [43] proof of the Erdős–Pósa theorem, which was published in 1967 just two years after the original paper. The second is a beautiful statement generalising the Helly property of a family of subtrees in a fixed host tree, which is due to Alon [5] (improving on an earlier result of Gyárfás and Lehel [28]). The last is a theorem of Birmelé [9] which says that graphs with no long cycles have bounded treewidth.

For all positive integers k, define[4]

$$s(k) := \begin{cases} \lceil 4k(\log k + \log\log k + 4)\rceil & \text{if } k \geqslant 2, \\ 2 & \text{if } k = 1. \end{cases}$$

Theorem 4 (Simonovits [43]). *Let k be a positive integer and let G be a graph with all vertices of degree 2 or 3. If G contains at least $s(k)$ vertices of degree 3, then G contains k pairwise vertex-disjoint cycles.*

If a graph has at least one vertex, then it is *non-null*.

Theorem 5 (Alon [5]). *For all positive integers k and c, for every tree T and every collection $\mathcal{A}$ of non-null subgraphs of T each having at most c components, either*

(a) *$\mathcal{A}$ has k pairwise vertex-disjoint members, or*
(b) *there exists a subset $X \subseteq V(T)$ with $|X| \leqslant 2c^2(k-1)$ such that $X \cap V(A) \neq \varnothing$ for all $A \in \mathcal{A}$.*

Theorem 6 (Birmelé [9]). *For every integer $\ell \geqslant 3$, every graph containing no cycle of length at least ℓ has treewidth less than $\ell - 1$.*

3 Definitions

We denote by $\mathbb{N}$ the set of positive integers. For each positive integer k, we write $[k]$ as a compact form of $\{1,\dots,k\}$. For a set S, let $\binom{S}{2}$ be the set of all subsets of S that have size 2.

We consider simple, finite, and undirected graphs. Formally, a graph is a pair (V,E) consisting of a finite set of *vertices* V with $V \cap \binom{V}{2} = \varnothing$, and a set of *edges* $E \subseteq \binom{V}{2}$.

Let G be a graph. We use the notations $V(G)$ and $E(G)$ to denote the vertex set of G and the edge set of G, respectively. A graph H is a *subgraph* of G, written as $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph of G is *spanning* if its vertex set equals $V(G)$. For a set $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G with vertex set S and edge set $\binom{S}{2} \cap E(G)$, and $G - S := G[V(G) \setminus S]$. For a set $Z \subseteq \binom{V(G)}{2}$, $G \cup Z$ is the graph with vertex set $V(G)$ and edge set $E(G) \cup Z$; and $G - Z$ is the graph with vertex set $V(G)$ and edge set $E(G) \setminus Z$. For two subsets A and B of $V(G)$, we say that an edge uv of G with $u \in A$ and $v \in B$ is *between* A

[4] Here and throughout, $\log x$ denotes the base-2 logarithm of x.

and B. For subgraphs H_1 and H_2 of G, $H_1 \cup H_2$ is the graph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$. For a collection $\mathcal{H}$ of subgraphs of G, $\bigcup \mathcal{H}$ is the graph with vertex set $\bigcup_{H \in \mathcal{H}} V(H)$ and edge set $\bigcup_{H \in \mathcal{H}} E(H)$.

A *path* is a graph with the vertex set $\{v_0, \dots, v_k\}$ and the edges $\{v_{i-1}v_i : i \in [k]\}$, where $k \geqslant 0$ and $v_0, \dots, v_k$ are pairwise distinct. We often refer to a path by a natural sequence of its vertices, writing, say, $v_0 \cdots v_k$ or equivalently $v_k \cdots v_0$. Writing a path as $v_0 \cdots v_k$ fixes an underlying orientation of the path, where v_0 is the first vertex and v_k is the last vertex. We say that v_0 and v_k are the *endpoints* of $v_0 \cdots v_k$. We also say that $v_0 \cdots v_k$ *starts* at v_0 and *ends* at v_k. Given a path P and two vertices v, w of P, we denote by vPw the subpath of P from v to w.

Suppose A and B are vertices or sets of vertices in a graph G. A path P with $P \subseteq G$ is an *(A, B)-path* in G if P starts at (a vertex in) A and ends at (a vertex in) B, and the inner vertices of P are disjoint from $A \cup B$.

A *cycle* is a graph with at least three vertices such that deleting any one edge results in a path. A *walk* in a graph G is a sequence $v_0 v_1 \cdots v_k$ of vertices of G, with $k \geqslant 0$, such that $v_{i-1}v_i \in E(G)$ for every $i \in [k]$. Note that $v_0, \dots, v_k$ are not necessarily distinct. Often we speak of a walk $v_0 v_1 \cdots v_k$ as a graph; in such cases, we are referring to the graph with vertex set $\{v_0, \dots, v_k\}$ and edge set $\{v_{i-1}v_i : i \in [k]\}$.

For walks, paths or edges $P = v_0 v_1 \cdots v_s$ and $Q = v_s v_{s+1} \cdots v_t$, with $0 \leqslant s \leqslant t$, let $P \cup Q$ be the walk (or path if it is one) whose underlying sequence of vertices is $v_0 v_1 \cdots v_t$.

A *forest* is a graph with no cycles. A *tree* is a connected forest.

A *tree-decomposition* of a graph G is a function β from the vertex set of some tree T to the power set of $V(G)$, such that for every vertex $v \in V(G)$, the set $\{t \in V(T) : v \in \beta(t)\}$ induces a non-null tree in T; and for every edge $uv \in E(G)$, there exists $t \in V(T)$ with $\{u, v\} \subseteq \beta(t)$. The *width* of β is $\max_{t \in V(T)} |\beta(t)| - 1$. The *treewidth* of G, denoted by $\mathrm{tw}(G)$, is the minimum width of a tree-decomposition of G.

The length, $\mathrm{len}(P)$, of a path P is the number of edges in P. Similarly, the length of a cycle C is the number of edges in C and is denoted $\mathrm{len}(C)$. The *distance* between two vertices x and y of G, denoted by $\mathrm{dist}_G(x, y)$, is the minimum length of an (x, y)-path in G if there is one, or ∞ if there is none. For non-empty subsets X and Y of $V(G)$, define $\mathrm{dist}_G(X, Y) := \min\{\mathrm{dist}_G(x, y) : (x, y) \in X \times Y\}$. For an integer $r \geqslant 0$ and a vertex x of G, the *ball of radius r around x* is $B_G(x, r) := \{y \in V(G) : \mathrm{dist}_G(x, y) \leqslant r\}$. For subsets X of $V(G)$, let $B_G(X, r) := \bigcup_{x \in X} B_G(x, r)$.

Let G be a graph and let d be a nonnegative integer. A collection $\mathscr{C}$ of subgraphs of G is a *d-packing* if $\mathrm{dist}_G(V(A), V(B)) > d$ for every pair of distinct $A, B \in \mathscr{C}$.

3.1 Projection, Span, and Padding

Let G be a graph and let H be a subgraph of G that is isomorphic to a disjoint union of cycles and contains at least one vertex from each component of G. An *H-supported BFS-spanning subgraph* of G is an edge-minimal spanning subgraph U of G such that $\mathrm{dist}_U(V(H), v) = \mathrm{dist}_G(V(H), v)$ for every vertex $v \in V(G)$. Observe that U is a spanning forest in G and each component of U contains exactly one vertex of H, which we call the

root of the component. For each $v \in V(U)$ the *U-projection* of v, denoted by $\mathrm{proj}_U(v)$, is the root of the component of U that contains v; and $U\langle v\rangle$ is the unique path in U between v and $\mathrm{proj}_U(v)$. The *(H,U)-span* of an edge vw in G and the *(H,U)-span* of G are defined as follows:

$$\mathrm{span}(G,H,U,vw) := \mathrm{dist}_H(\mathrm{proj}_U(v),\mathrm{proj}_U(w)),$$

$$\mathrm{span}(G,H,U) := \max(\{\mathrm{span}(G,H,U,vw) : vw \in E(G)\} \setminus \{\infty\}).$$

The *H-span* of G is the number $\max_U \mathrm{span}(G,H,U)$, where the maximum is taken over all H-supported BFS-spanning subgraphs of G.

The intuition behind span is that it measures how far apart the BFS-projections of the endpoints of an edge are along a cycle in H. A large finite span therefore gives rise to a long cycle. Indeed, suppose that $\ell \geqslant 3$ and $\ell \leqslant \mathrm{span}(G,H,U,uv) < \infty$, and write $x = \mathrm{proj}_U(u)$ and $y = \mathrm{proj}_U(v)$. Then x and y lie on the same cycle of H, and $U\langle u\rangle \cup uv \cup U\langle v\rangle$ is an (x,y)-path whose edges and inner vertices lie outside H. Joining this path with an (x,y)-path on that cycle produces a cycle of length at least ℓ, since each of these paths in H has length at least $\mathrm{dist}_H(x,y) = \mathrm{span}(G,H,U,uv) \geqslant \ell$.

Lemma 7. *Let $d \geqslant 1$ and $\ell \geqslant 3$ be integers. Let G be a graph and let C be a cycle in G of length at least ℓ. There exists a cycle C' in G of length at least ℓ such that either*

(a) *$V(C') \subseteq B_G(V(C),3d)$ and the length of C' is at most $6d+2$, or*
(b) *$V(C') \subseteq B_G(V(C),2d)$ and the C'-span of $G[B_G(V(C'),d)]$ is at most ℓ.*

Proof. We say that a pair (Q,P) is *good* if Q is a non-null path in C, P is a path in G of length at most $4d+1$, and $Q \cup P$ is a cycle of length at least ℓ. Observe that, for any edge e of C, $(C-e,e)$ is a good pair. Among all good pairs, let (Q,P) be one that minimises $\mathrm{len}(Q)$.

Let $D = Q \cup P$ and let $G_D := G[B_G(V(D),d)]$. Since $Q \subseteq C$ and $\mathrm{len}(P) \leqslant 4d+1$, we have that $V(D) \subseteq B_G(V(C),2d)$. Thus, if $\mathrm{span}(G_D,D,U) \leqslant \ell$ for every D-supported BFS-spanning subgraph U of G_D, then (b) holds for $C' = D$ and there is nothing to prove.

Therefore, we assume there is a D-supported BFS-spanning subgraph U of G_D and an edge vw of G_D with $\infty > \mathrm{span}(G_D,D,U,vw) > \ell$. Let $R := U\langle v\rangle \cup vw \cup U\langle w\rangle$. Since $\mathrm{span}(G_D,D,U,vw) > \ell \geqslant 3$, $vw \notin E(D)$ and $V(U\langle v\rangle) \cap V(U\langle w\rangle) = \varnothing$, thus R is a path whose edges and inner vertices do not appear in D. Also observe that $\mathrm{len}(R) = \mathrm{len}(U\langle v\rangle) + 1 + \mathrm{len}(U\langle w\rangle) \leqslant 2d+1$. Now let $x := \mathrm{proj}_U(v)$ and $y := \mathrm{proj}_U(w)$ be the endpoints of R. The argument splits into cases depending on the locations of x and y in $D = Q \cup P$.

If $x \in V(P)$ and $y \in V(P)$, then consider the unique cycle C' in $R \cup P$. Since $\mathrm{dist}_D(x,y) > \ell$, we know that $\mathrm{len}(C') \geqslant \ell$. On the other hand, $\mathrm{len}(C') \leqslant \mathrm{len}(R) + \mathrm{len}(P) \leqslant 2d+1+4d+1 = 6d+2$. Finally, $V(R) \subseteq B_G(V(P),d)$ and $V(P) \subseteq B_G(V(C),2d)$ which implies $V(C') \subseteq B_G(V(C),3d)$. This proves that (a) holds.

Otherwise, we may assume without loss of generality that $x \in V(Q) \setminus V(P)$. Then, the graph $D \cup R$ has two cycles that contain R. One of these cycles, D', contains at most $\lfloor \frac{1}{2}\mathrm{len}(P) \rfloor \leqslant 2d$ edges of P. Note also that $\mathrm{len}(D') \geqslant \mathrm{dist}_D(x,y) > \ell$. Let $Q' := D' \cap Q$ and $P' := D' \cap (P \cup R)$. Note that x lies in Q' so Q' is non-null. Also, observe that x has two neighbours in Q (since $x \notin V(P)$) and D' contains only one of these two. Hence,

$\mathrm{len}(Q') < \mathrm{len}(Q)$. Finally, $\mathrm{len}(P') = \mathrm{len}(P \cap D') + \mathrm{len}(R) \leqslant 2d + 2d + 1 = 4d + 1$. Altogether, (Q', P') is a good pair and $\mathrm{len}(Q') < \mathrm{len}(Q)$, which contradicts the choice of (Q, P).

This concludes the proof of the lemma. □

Next next lemma is needed to define padding of sets, which is notion similar to a ball, except it is more sparsely distributed.

Lemma 8. *For every cycle C, for every $y \in V(C)$, and for every pair of positive integers (α, β), there exists a set $Z \subseteq V(C)$ such that*

(i) $y \in Z$;
(ii) $|Z| \leqslant 2\beta + 1$;
(iii) $B_C(Z, \lfloor \alpha/2 \rfloor) = B_C(y, \alpha\beta + \lfloor \alpha/2 \rfloor)$.

Proof. Fix an orientation of C. Let $v_0 v_1 \cdots v_{\alpha\beta+\lfloor \alpha/2 \rfloor}$ be the walk in C such that $v_0 = y$, and v_{i+1} is the successor of v_i in the clockwise cyclic ordering of C for all i. Similarly, let $u_0 u_1 \cdots u_{\alpha\beta+\lfloor \alpha/2 \rfloor}$ be the walk in C such that $u_0 = y$, and u_{i+1} is the successor of u_i in the anticlockwise cyclic ordering of C for all i. We show that $Z := \{v_\alpha, v_{2\alpha}, \ldots, v_{\alpha\beta}\} \cup \{y\} \cup \{u_\alpha, u_{2\alpha}, \ldots, u_{\alpha\beta}\}$ satisfies the outcomes of the lemma. Observe that (i) and (ii) are immediate. Also, since $B_C(Z, \lfloor \alpha/2 \rfloor) = \{v_{\alpha\beta+\lfloor \alpha/2 \rfloor}, \ldots, v_2, v_1, y, u_1, u_2, \ldots, u_{\alpha\beta+\lfloor \alpha/2 \rfloor}\} = B_C(y, \alpha\beta + \lfloor \alpha/2 \rfloor)$, (iii) also holds. □

For cycles C, vertices $y \in V(C)$, and positive integers α, β, let $\mathrm{pad}(C, y, \alpha, \beta)$ denote the corresponding set Z constructed in Lemma 8. See Figure 1 for examples of padding.

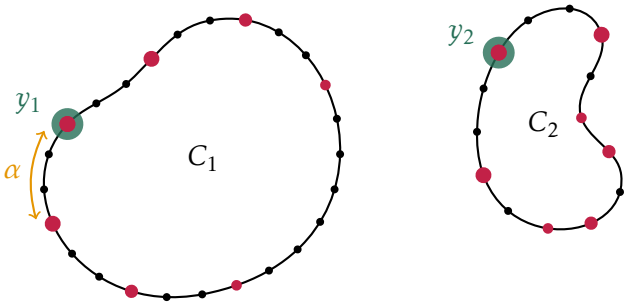


Figure 1: Two examples of padded sets $\mathrm{pad}(C_i, y_i, 3, 3)$ represented by the red vertices. In the proof of Lemma 8, padding a vertex y in some cycle C gives, in each direction from y, a sequence of β vertices along C such that consecutive vertices have α edges between them. Notice that in C_1, the corresponding sequences from y_1 are visually separate. On the other hand, in C_2, the corresponding sequences from y_2 appear to interlace around C_2.

Let H be a disjoint union of cycles $C_1 \cup \cdots \cup C_p$, let $Y \subseteq V(H)$, and let (α, β) be a pair of positive integers. Define

$$\mathrm{pad}(H, Y, \alpha, \beta) := \bigcup_{i \in [p]} \bigcup_{y \in Y \cap V(C_i)} \mathrm{pad}(C_i, y, \alpha, \beta).$$

The outcomes of Lemma 8 generalise to $\mathrm{pad}(H, Y, \alpha, \beta)$ in the following way:

Corollary 9. *If H is a disjoint union of cycles $C_1 \cup \cdots \cup C_p$, $Y \subseteq V(H)$, and (α, β) is a pair of positive integers, then*

(i) $Y \subseteq \mathrm{pad}(H, Y, \alpha, \beta) \subseteq V(H)$;
(ii) $|\mathrm{pad}(H, Y, \alpha, \beta)| \leqslant |Y|(2\beta + 1)$;
(iii) $B_H(\mathrm{pad}(H, Y, \alpha, \beta), \lfloor \alpha/2 \rfloor) = B_H(Y, \alpha\beta + \lfloor \alpha/2 \rfloor)$.

Proof. Note that Lemma 8 immediately implies (i) and (ii). To see that (iii) holds, observe that $B_H(\mathrm{pad}(H, Y, \alpha, \beta), \lfloor \alpha/2 \rfloor) = \bigcup_{i \in [p]} \bigcup_{y \in Y \cap V(C_i)} B_{C_i}(\mathrm{pad}(C_i, y, \alpha, \beta), \lfloor \alpha/2 \rfloor)$. Then Lemma 8 implies $B_H(\mathrm{pad}(H, Y, \alpha, \beta), \lfloor \alpha/2 \rfloor) = \bigcup_{i \in [p]} \bigcup_{y \in Y \cap V(C_i)} B_{C_i}(y, \alpha\beta + \lfloor \alpha/2 \rfloor) = B_H(Y, \alpha\beta + \lfloor \alpha/2 \rfloor)$, as required. □

4 Proof Sketch

We now present a proof sketch of Theorem 2. Note that we deliberately simplify many technical aspects of our proof to help convey the main ideas. In particular, some of the definitions are informal and do not exactly match those given in the formal proof.

Let G be a graph that does not have a d-packing of k cycles each of length at least ℓ. A cycle is 'long' if it has length at least ℓ. Thus we need to construct a set $X \subseteq V(G)$ of size $\mathcal{O}(\ell k \log k)$ such that $G - B_G(X, \mathcal{O}(d))$ has no long cycles. We build up $X = X_0 \cup X_1 \cup X_2 \cup X_3 \cup X_4$ in steps.

The proof begins by choosing a maximal d-packing $\mathcal{Z}$ of 'medium' cycles, which are long cycles with length at most $6d + 2$, and choosing a maximal $2d$-packing $\mathscr{C}$ of long cycles such that the C-span of $G[B_G(V(C), d)]$ is at most ℓ for all $C \in \mathscr{C}$. Note that by the starting assumption, both $\mathcal{Z}$ and $\mathscr{C}$ have size less than k. Lemma 7 says that in a small-radius ball around each long cycle, there is either a medium cycle or a long cycle that satisfies the span condition in the definition of $\mathscr{C}$. Consequently, every long cycle in G intersects $B_G(V(\bigcup \mathcal{Z}), 4d) \cup B_G(V(\bigcup \mathscr{C}), 4d)$ by the maximality of both $\mathcal{Z}$ and $\mathscr{C}$. By choosing one vertex from each member of $\mathcal{Z}$, we obtain a set of vertices X_0 such that $B_G(V(\bigcup \mathcal{Z}), 4d) \subseteq B_G(X_0, 7d + 1)$.

If $\mathscr{C} = \varnothing$, then X_0 already suffices, so assume that $\mathscr{C} \neq \varnothing$. The next two steps hit all long cycles in $G[B_G(V(\bigcup \mathscr{C}), 6d)]$. Set $R = 6d + \lceil d/2 \rceil$ and $G_R = G[B_G(V(\bigcup \mathscr{C}), R)]$. Fix a $\bigcup \mathscr{C}$-supported BFS-spanning subgraph U of G_R. Call an edge of G_R 'large-span' if its $(\bigcup \mathscr{C}, U)$-span is greater than ℓ; this includes edges of infinite span.

We first control the set W of endpoints of large-span edges in G_R. This is done in Lemma 10. For each such edge $e = uv$, let $P_e = U\langle u \rangle \cup uv \cup U\langle v \rangle$ and let $\mathrm{proj}_U(e) = \{\mathrm{proj}_U(u), \mathrm{proj}_U(v)\}$. Choose a maximal set I of large-span edges such that, for all distinct $e, f \in I$,

$$\mathrm{dist}_G(V(P_e), V(P_f)) > d \quad \text{and} \quad \mathrm{dist}_{\bigcup \mathscr{C}}(\mathrm{proj}_U(e), \mathrm{proj}_U(f)) > d\ell.$$

If $2|I| \geqslant s(k)$, then these paths, along with the cycles in $\mathscr{C}$, form a graph G' with all vertices of degree 2 or 3 and at least $s(k)$ degree-3 vertices. By Simonovits' theorem (Theorem 4), G' has k vertex-disjoint cycles. The local small-span condition on $\mathscr{C}$ and the two separation

conditions ensure that these cycles are long and form a d-packing in G, a contradiction. Therefore $|I| < s(k)/2$. By maximality, for every large-span edge $e \notin I$, there is an edge $f \in I$ such that either P_e and P_f are within distance d in G, or their projected endpoint sets are within distance $d\ell$ along $\bigcup\mathscr{C}$. Projecting the endpoints of the edges in I to $\bigcup\mathscr{C}$ and padding them using Corollary 9 handles both cases: padding converts the latter closeness along the cycles into distance $\mathcal{O}(d)$ from a padded vertex. This produces a set $X_1 \subseteq V(\bigcup\mathscr{C})$ with $|X_1| \leqslant \mathcal{O}(s(k)\ell)$ such that $W \subseteq B_G(X_1, \mathcal{O}(d))$.

Next, we hit all long cycles in $G[B_G(V(\bigcup\mathscr{C}), 6d)]$ by our first application of Lemma 15, which is a Helly-type statement for families of connected subgraphs whose edges have small span. Note that we need only hit long cycles in $G[B_G(V(\bigcup\mathscr{C}), 6d)]$ whose edges have small span because W was controlled in the previous step. This application of Lemma 15 is carried out in Lemma 16, where it is shown that the packing outcome of Lemma 15 would produce a d-packing of k long cycles. Therefore, the hitting outcome of Lemma 15 applies. This, along with appropriate padding via Corollary 9, produces a set of vertices $X_2 \subseteq V(\bigcup\mathscr{C})$ with $|X_2| \leqslant \mathcal{O}(\ell k)$ such that $B_G(X_2, \mathcal{O}(d))$ hits the relevant cycles.

After applying Lemma 10 and Lemma 16, the remaining long cycles contain subpaths that leave $B_G(V(\bigcup\mathscr{C}), 5d)$ and later return to it. We call these excursions 'ears'. For an ear $E = v_0 \cdots v_s$, let $W_E = U\langle v_0\rangle \cup E \cup U\langle v_s\rangle$ be the ear together with its two BFS legs. Let M be a ball of radius $\mathcal{O}(d)$ around $X_0 \cup X_1 \cup X_2$ containing the regions controlled so far. An ear E is 'admissible' if W_E avoids $B_G(M, d)$. By restricting attention to cycles outside a sufficiently large neighbourhood of M, we ensure that all their ears are admissible. We say that an ear E is 'problematic' if W_E meets only one $C \in \mathscr{C}$ and there is a short cycle in $W_E \cup C$. A family of non-problematic admissible ears whose walks W_E are pairwise far apart in G and whose projections are pairwise far apart along $\bigcup\mathscr{C}$ is amenable to the Simonovits argument in Lemma 10. The next step hits these non-problematic admissible ears.

Enumerate $\mathscr{C}$ as $C_1, C_2, \ldots, C_p$. We organise the surviving regions of G into subgraphs $F_{-1}, F_0, F_1, \ldots, F_p$, where F_{-1} is on $\bigcup\mathscr{C}$, F_0 is far from $\bigcup\mathscr{C}$, and F_i is the region around C_i for all $i \in [p]$. The previous steps guarantee that these subgraphs have no long cycles, hence they have bounded treewidth by Theorem 6. Moreover, F_{-1} is a forest, since the previously chosen neighbourhoods meet every cycle in $\mathscr{C}$. The subgraphs F_i may overlap, so we take disjoint copies of them and fix a tree-decomposition (T, β) of their disjoint union. We choose this decomposition so that bags corresponding to F_{-1} have at most two vertices, while the other bags have at most $\ell - 1$ vertices. For each non-problematic admissible ear E, we represent a suitable region $\Psi^\star(E)$ around E as a subgraph $T(\Psi^\star(E))$ of T that has a bounded number of components. Next, we apply a generalised Helly property for trees by Alon (Theorem 5) to the family $\{T(\Psi^\star(E))\}$. This results in a large packing or a small hitting set for the family. From the packing outcome, we obtain many non-problematic admissible ears with the two separation properties above. If sufficiently many of their walks W_E contain cycles, these already yield a d-packing of k long cycles. Otherwise, sufficiently many of the walks are paths, to which we apply the Simonovits argument. Either possibility contradicts our starting assumption. Thus the hitting outcome of Theorem 5 applies, giving a set $X^\star$ of $\mathcal{O}(s(k))$ vertices of T that meets every $T(\Psi^\star(E))$. We pull the bags at $X^\star$ back to G and pad only the vertices coming from the copies of F_{-1}. Each such bag has at most two vertices, so padding contributes $\mathcal{O}(\ell)$ vertices per bag. The other bags

already have $\mathcal{O}(\ell)$ vertices and are used without padding. This gives a set $X_3 \subseteq V(G)$ with $|X_3| \leqslant \mathcal{O}(\ell\, s(k))$ such that $B_G(X_3, 6d)$ hits every non-problematic admissible ear.

Finally, we use a second application of Lemma 15 to deal with the remaining long cycles whose ears are all problematic. We build an auxiliary graph G_{Aux} from G_R by adding, for every problematic admissible ear, a 'new' edge joining its endpoints. Since by definition problematic ears enclose a short cycle, each added edge has small span in G_{Aux}. For each remaining long cycle D, we replace its ears by the corresponding new edges, obtaining a connected subgraph of G_{Aux}. We then let H_D be the subgraph of G_{Aux} induced by the radius-$\lceil d/2 \rceil$ neighbourhood of its vertex set. This enlargement is used to show that disjoint members H_D correspond to cycles D that are at distance greater than d in G. We show that $\{H_D\}$ is a family of connected subgraphs of G_{Aux} whose edges have small span in G_{Aux}. Hence we may apply Lemma 15 in G_{Aux}. In the packing outcome, we obtain k pairwise vertex-disjoint members of $\{H_D\}$, which lifts to a d-packing of k long cycles in G, a contradiction. Therefore, the hitting outcome of Lemma 15 holds, giving a small set on $\bigcup\mathscr{C}$ whose suitable neighbourhoods meet every H_D. After padding, this gives a set $X_4 \subseteq V(\bigcup\mathscr{C})$ with $|X_4| \leqslant \mathcal{O}(\ell k)$ that is close to all remaining long cycles in G.

5 Proof of the Main Result

Lemma 10. *Let k, d, r, R, ℓ be positive integers with $R \geqslant r > d$ and $\ell \geqslant 3$. Let G be a graph, let $\mathscr{C}$ be a non-empty $2d$-packing of cycles in G each of length at least ℓ, let U be a $\bigcup\mathscr{C}$-supported BFS-spanning subgraph of $G_R := G[B_G(V(\bigcup\mathscr{C}), R)]$, and let W be the set of endpoints of all edges $e \in E(G_R)$ with $\mathrm{span}(G_R, \bigcup\mathscr{C}, U, e) > \ell$. If the $\bigcup\mathscr{C}$-span of $G_d := G[B_G(V(\bigcup\mathscr{C}), d)]$ is at most ℓ, then either*

(a) *G contains a d-packing of k cycles each of length at least ℓ, or*
(b) *there exists $X \subseteq V(\bigcup\mathscr{C})$ with $|X| \leqslant (s(k)-1)(2\lceil \ell/2 \rceil + 1)$ such that*

$$W \cup \bigcup_{\{C,D\} \in \binom{\mathscr{C}}{2}} (B_G(V(C), r) \cap B_G(V(D), r)) \subseteq B_G(W, r-d-1) \subseteq B_G(X, R+r+d).$$

Proof. Let $\mathcal{E} := \{e \in E(G_R) : \mathrm{span}(G_R, \bigcup\mathscr{C}, U, e) > \ell\}$. (Thus, W is the set of endpoints of edges in $\mathcal{E}$.) For each $e = uv \in \mathcal{E}$, let

$$P_e := U\langle u \rangle \cup uv \cup U\langle v \rangle.$$

Since $\mathrm{span}(G_R, \bigcup\mathscr{C}, U, uv) > \ell \geqslant 3$, $uv \notin E(\bigcup\mathscr{C})$ and $V(U\langle u \rangle) \cap V(U\langle v \rangle) = \varnothing$, thus P_e is a path whose edges and inner vertices do not appear in $\bigcup\mathscr{C}$.

For each edge $e = uv \in \mathcal{E}$, let $\mathrm{proj}_U(e) := \{\mathrm{proj}_U(u), \mathrm{proj}_U(v)\}$. Let H be the auxiliary graph with vertex set $\mathcal{E}$, where two distinct elements $e, f \in V(H)$ are adjacent if and only if

$$\mathrm{dist}_G(V(P_e), V(P_f)) \leqslant d \quad \text{or} \quad \mathrm{dist}_{\bigcup\mathscr{C}}(\mathrm{proj}_U(e), \mathrm{proj}_U(f)) \leqslant d\ell.$$

Let I be a maximal independent set in H and consider the graph

$$G' := \bigcup\mathscr{C} \cup \bigcup_{e \in I} P_e.$$

Recall that $\mathscr{C}$ is a collection of pairwise vertex-disjoint cycles. Also since $\mathrm{dist}_G(V(P_e), V(P_f)) > d$ for all distinct $e, f \in I$, $(P_e : e \in I)$ is a collection of pairwise vertex-disjoint paths. Furthermore, no edge or inner vertex of P_e appears in $\bigcup\mathscr{C}$ for all $e \in \mathcal{E}$. Therefore, every vertex $v \in V(G')$ satisfies $\deg_{G'}(v) \in \{2, 3\}$, and the degree-3 vertices of G' are precisely the endpoints of paths in $(P_e : e \in I)$. Hence G' has exactly $2|I|$ degree-3 vertices.

The argument splits into the following two cases: **(1)** $2|I| \geqslant s(k)$, in which case it is shown that (a) holds, and **(2)** $2|I| \leqslant s(k) - 1$, in which case it is shown that (b) holds.

Case (1) $2|I| \geqslant s(k)$: A *$\mathscr{C}$-segment* is a path S in $\bigcup\mathscr{C}$ between an endpoint of P_e and an endpoint of P_f for some $e, f \in I$, such that S has at least one edge and the inner vertices of S have degree 2 in G'.

Claim 11. *Every cycle in G' has length at least ℓ.*

Proof of Claim. Every cycle in $\mathscr{C}$ has length at least ℓ and every cycle in G' that is not in $\mathscr{C}$ contains a $\mathscr{C}$-segment. Thus, it suffices to show that every $\mathscr{C}$-segment has length greater than ℓ. Let S be a $\mathscr{C}$-segment whose endpoints x and y correspond with an endpoint x of P_e and an endpoint y of P_f for some $e, f \in I$. If $e = f$, then $\mathrm{len}(S) \geqslant \mathrm{span}(G_R, \bigcup\mathscr{C}, U, e) > \ell$ since $e \in \mathcal{E}$. If $e \neq f$, then $\mathrm{len}(S) \geqslant \mathrm{dist}_{\bigcup\mathscr{C}}(x, y) \geqslant \mathrm{dist}_{\bigcup\mathscr{C}}(\mathrm{proj}_U(e), \mathrm{proj}_U(f)) > d\ell \geqslant \ell$. ∎

Claim 12. *For every pair of vertex-disjoint cycles D and D' in G', $\mathrm{dist}_G(V(D), V(D')) > d$.*

Proof of Claim. Assume for a contradiction that there exists a pair of vertex-disjoint cycles D and D' in G' such that $\mathrm{dist}_G(V(D), V(D')) \leqslant d$. Consider any $u \in V(D)$ and $u' \in V(D')$ with $\mathrm{dist}_G(u, u') \leqslant d$.

If $u \in V(P_e)$ and $u' \in V(P_f)$ for some $e, f \in I$ with $P_e \subseteq D$ and $P_f \subseteq D'$, then since D and D' are vertex-disjoint, $e \neq f$. Therefore $\mathrm{dist}_G(u, u') \geqslant \mathrm{dist}_G(V(P_e), V(P_f)) > d$, a contradiction. Hence it may be assumed that $u \notin V(P_e)$ for every $e \in I$ with $P_e \subseteq D$. Consequently, there exists a unique $C \in \mathscr{C}$ such that $u \in V(C)$.

Let P be a shortest (u, u')-path in G. Since $\mathrm{dist}_G(u, u') \leqslant d$, $\mathrm{len}(P) \leqslant d$, so $V(P) \subseteq B_G(V(C), d)$. In particular, since $\mathscr{C}$ is a $2d$-packing in G, $u' \in B_G(V(C), d)$ and $u' \notin B_G(V(\bigcup\mathscr{C}) \setminus V(C), d)$. Therefore $\mathrm{proj}_U(u') \in V(C)$, $V(U\langle u'\rangle) \subseteq B_G(V(C), d)$, and $\mathrm{len}(U\langle u'\rangle) \leqslant d$. Furthermore, either $u' \in V(C)$ or u' is an inner vertex of P_e for some $e \in I$ with $P_e \subseteq D'$. In the former case, $\mathrm{proj}_U(u') = u' \in V(D')$, and in the latter case $\mathrm{proj}_U(u') \in V(U\langle u'\rangle) \subseteq V(P_e) \subseteq V(D')$. Hence $\mathrm{proj}_U(u') \in V(D')$ in both cases.

Let $v_0, v_1, \ldots, v_s$ be the sequence of vertices along P such that $v_0 = u$, $v_s = u'$, and $s \leqslant d$. Since $V(P) \subseteq B_G(V(C), d) \subseteq V(G_d)$, an assumption of the lemma implies that $\mathrm{span}(G_R, \bigcup\mathscr{C}, U, e) = \mathrm{span}(G_d, \bigcup\mathscr{C}, U[V(G_d)], e) \leqslant \ell$ for every $e \in E(P)$. Therefore:

$$\begin{aligned}\mathrm{dist}_C(u, \mathrm{proj}_U(u')) = \mathrm{dist}_{\bigcup\mathscr{C}}(u, \mathrm{proj}_U(u')) &\leqslant \sum_{i=1}^{s} \mathrm{dist}_{\bigcup\mathscr{C}}(\mathrm{proj}_U(v_{i-1}), \mathrm{proj}_U(v_i)) \\ &= \sum_{i=1}^{s} \mathrm{span}(G_R, \bigcup\mathscr{C}, U, v_{i-1}v_i) \leqslant s\ell \leqslant d\ell. \qquad (*)\end{aligned}$$

Now consider a shortest $(u, \mathrm{proj}_U(u'))$-path S in C. Recall that $u \in V(D)$ and $\mathrm{proj}_U(u') \in V(D')$. Let x be the first vertex in S starting from u such that $x \in V(P_e)$ for some $e \in I$ with $P_e \subseteq D$. Let y be the first vertex in S starting from $\mathrm{proj}_U(u')$ such that $y \in V(P_f)$ for some $f \in I$ with $P_f \subseteq D'$. Since D and D' are vertex-disjoint, both vertices x and y exist and $e \neq f$. Note that $\mathrm{dist}_C(x,y) \leqslant \mathrm{dist}_C(u, \mathrm{proj}_U(u')) \leqslant d\ell$ by $(*)$. On the other hand, $x \in V(C) \cap V(P_e) \subseteq \mathrm{proj}_U(e)$, $y \in V(C) \cap V(P_f) \subseteq \mathrm{proj}_U(f)$, and e and f are distinct elements of I together imply $\mathrm{dist}_C(x,y) \geqslant \mathrm{dist}_{\bigcup \mathscr{C}}(\mathrm{proj}_U(e), \mathrm{proj}_U(f)) > d\ell$, a contradiction. ∎

Since $2|I| \geqslant s(k)$, G' has at least $s(k)$ degree-3 vertices. So Theorem 4 implies that G' contains k pairwise vertex-disjoint cycles. By Claim 11, each of these cycles has length at least ℓ. By Claim 12, these cycles form a d-packing in G. Thus outcome (a) holds.

Case (2) $2|I| \leqslant s(k) - 1$: Let

$$Y := \bigcup_{e \in I} \mathrm{proj}_U(e) \quad \text{and} \quad X := \mathrm{pad}(\bigcup \mathscr{C}, Y, 2d, \lceil \ell/2 \rceil).$$

Note that $Y \subseteq V(\bigcup \mathscr{C})$ and $|Y| = 2|I| \leqslant s(k) - 1$. Then by Corollary 9, $X \subseteq V(\bigcup \mathscr{C})$ and $|X| \leqslant |Y|(2\lceil \ell/2 \rceil + 1) \leqslant (s(k) - 1)(2\lceil \ell/2 \rceil + 1)$.

Claim 13. $W \subseteq B_G(X, R + 2d + 1)$.

Proof of Claim. Consider any $e \in \mathcal{E}$ and let v be an endpoint of e. Note that $e \in V(H)$. If $e \in I$, then $\mathrm{proj}_U(v) \in \mathrm{proj}_U(e) \subseteq Y \subseteq X$, so $v \in B_G(X, R)$ as desired. Hence it may be assumed that $e \in V(H) \setminus I$. Since I is a maximal independent set in H, there exists $f \in I$ such that $ef \in E(H)$. Then there are two cases, **(i)** $\mathrm{dist}_G(V(P_e), V(P_f)) \leqslant d$, or **(ii)** $\mathrm{dist}_{\bigcup \mathscr{C}}(\mathrm{proj}_U(e), \mathrm{proj}_U(f)) \leqslant d\ell$.

Case (i) $\mathrm{dist}_G(V(P_e), V(P_f)) \leqslant d$: Let $u \in V(P_e)$ and $u' \in V(P_f)$ such that $\mathrm{dist}_G(u,u') = \mathrm{dist}_G(V(P_e), V(P_f))$, and let P be a shortest (u,u')-path in G. Let Q be the walk in G starting from $\mathrm{proj}_U(u')$ and going to u' along P_f, then following P to u, and finally going to v along P_e. Let s be the length of $\mathrm{proj}_U(u')Qu'$. Since U is a $\bigcup \mathscr{C}$-supported BFS-spanning subgraph of G_R, $\mathrm{dist}_G(V(\bigcup \mathscr{C}), u') = s$. Then $s \leqslant \mathrm{dist}_G(V(\bigcup \mathscr{C}), u) + \mathrm{dist}_G(u,u')$, which implies $\mathrm{dist}_G(V(\bigcup \mathscr{C}), u) \geqslant s - d$. Since each endpoint of e is at distance at most R from $V(\bigcup \mathscr{C})$ in G, the length of uQv is at most $R + 1 - (s - d)$. (The +1 accounts for the fact that uQv may contain the edge e.) It follows that Q has length at most $s + d + R + 1 - (s - d) = R + 2d + 1$. Recall that, since $f \in I$, $\mathrm{proj}_U(u') \in \mathrm{proj}_U(f) \subseteq Y$, thus $v \in B_G(\mathrm{proj}_U(u'), R + 2d + 1) \subseteq B_G(Y, R + 2d + 1) \subseteq B_G(X, R + 2d + 1)$, as desired.

Case (ii) $\mathrm{dist}_{\bigcup \mathscr{C}}(\mathrm{proj}_U(e), \mathrm{proj}_U(f)) \leqslant d\ell$: Since $\mathrm{proj}_U(f) \subseteq Y$, $\mathrm{dist}_{\bigcup \mathscr{C}}(w, Y) \leqslant d\ell \leqslant 2d\lceil \ell/2 \rceil$ for some $w \in \mathrm{proj}_U(e)$. Then Corollary 9 implies that $\mathrm{dist}_{\bigcup \mathscr{C}}(w, X) \leqslant d$. Hence $v \in B_G(w, R + 1) \subseteq B_G(X, R + d + 1)$, as desired. ∎

Claim 14. $\bigcup_{\{C,D\} \in \binom{\mathscr{C}}{2}} (B_G(V(C), r) \cap B_G(V(D), r)) \subseteq B_G(W, r - d - 1)$.

Proof of Claim. By an assumption of the lemma, $r > d$. Let $p := |\mathscr{C}|$ and enumerate $\mathscr{C}$ as $C_1, C_2, \ldots, C_p$. For each $i \in [p]$, let $U_i := U[B_U(V(C_i), r)]$. Then $(V(U_i) : i \in [p])$ is a partition of $B_G(V(\bigcup \mathscr{C}), r)$.

Consider each $a \in \bigcup_{\{C,D\}\in\binom{\mathscr{C}}{2}} (B_G(V(C),r) \cap B_G(V(D),r))$. Since $a \in B_G(V(\bigcup\mathscr{C}),r)$, there exists a unique $i \in [p]$ such that $a \in V(U_i)$. Therefore $a \in B_G(V(C_i),r) \cap B_G(V(\bigcup\mathscr{C}) \setminus V(C_i),r)$ and $a \notin V(\bigcup\mathscr{C}) \setminus V(C_i)$. Let Q be a shortest $(a, V(\bigcup\mathscr{C}) \setminus V(C_i))$-path in G. Note that Q has an edge and $V(Q) \subseteq B_G(V(\bigcup\mathscr{C}),r)$. Let $u_a v_a$ be the first edge on Q such that $u_a \in V(U_i)$ and $v_a \notin V(U_i)$. Thus $v_a \in V(U_j)$ for some $j \in [p] \setminus \{i\}$. Therefore $u_a v_a$ is an edge of G_R with $\operatorname{span}(G_R, \bigcup\mathscr{C}, U, u_a v_a) = \infty$, so $\{u_a, v_a\} \subseteq W$. If $u_a v_a$ is one of the last d edges of Q, then $u_a \in B_G(V(\bigcup\mathscr{C}) \setminus V(C_i), d)$, implying $u_a \notin V(U_i)$, a contradiction. Hence $\operatorname{len}(aQu_a) \leqslant \operatorname{len}(Q) - (d+1) \leqslant r - d - 1$ and $a \in B_G(u_a, r-d-1) \subseteq B_G(W, r-d-1)$, as desired. ∎

Claim 13 and Claim 14 together imply

$$W \cup \bigcup_{\{C,D\}\in\binom{\mathscr{C}}{2}} (B_G(V(C),r) \cap B_G(V(D),r)) \subseteq B_G(W, r-d-1) \subseteq B_G(X, R+r+d),$$

hence (b) holds.

This concludes the proof of the lemma. □

Lemma 15. *Let k, r, ℓ be positive integers with $\ell \geqslant 3$. Let G be a graph, let $\mathscr{C}$ be a non-empty collection of pairwise vertex-disjoint cycles in G, and let U be a $\bigcup\mathscr{C}$-supported BFS-spanning subgraph of $G_r := G[B_G(V(\bigcup\mathscr{C}), r)]$. If $\mathcal{H}$ is a collection of non-null connected subgraphs of G_r such that $\operatorname{span}(G_r, \bigcup\mathscr{C}, U, e) \leqslant \ell$ for every $e \in E(\bigcup\mathcal{H})$, then either*

(i) *$\mathcal{H}$ contains k pairwise vertex-disjoint members, or*
(ii) *there exists $Y \subseteq V(\bigcup\mathscr{C})$ with $|Y| \leqslant k-1+|\mathscr{C}|$ such that $B_U(B_{\bigcup\mathscr{C}}(Y, \lfloor \ell/2 \rfloor), r) \cap V(H) \neq \varnothing$ for all $H \in \mathcal{H}$.*

Proof. For each $H \in \mathcal{H}$, let $\operatorname{proj}_U(H) := \{\operatorname{proj}_U(v) : v \in V(H)\}$.

Consider each $H \in \mathcal{H}$. For each $uv \in E(H)$, since $\operatorname{span}(G_r, \bigcup\mathscr{C}, U, uv)$ is finite, there exists $C \in \mathscr{C}$ such that $\{\operatorname{proj}_U(u), \operatorname{proj}_U(v)\} \subseteq V(C)$. Moreover, since H is connected and $\mathscr{C}$ is a collection of pairwise vertex-disjoint cycles, there exists a unique $C(H) \in \mathscr{C}$ such that $\operatorname{proj}_U(H) \subseteq V(C(H))$ (this is true even when $E(H) = \varnothing$). Hence, we may define

$$S_H := C(H)[B_{C(H)}(\operatorname{proj}_U(H), \lfloor \ell/2 \rfloor)].$$

Since the projections of u and v have distance at most $\ell \leqslant 2\lfloor \ell/2 \rfloor + 1$ in $C(H)$, the union of $B_{C(H)}(\operatorname{proj}_U(u), \lfloor \ell/2 \rfloor)$ and $B_{C(H)}(\operatorname{proj}_U(v), \lfloor \ell/2 \rfloor)$ is connected in $C(H)$. Consequently, S_H is connected, and so each S_H is either equal to $C(H)$ or a path in $C(H)$.

For each $C \in \mathscr{C}$, arbitrarily choose a vertex $v_C \in V(C)$. Let $\mathcal{H}'$ be the set of all $H \in \mathcal{H}$ such that S_H avoids $Y_0 := \{v_C : C \in \mathscr{C}\}$. Then $(S_H : H \in \mathcal{H}')$ is a family of subpaths in the disjoint union of paths $\bigcup\mathscr{C} - Y_0$, so the Helly property for intervals implies that either **(1)** $(S_H : H \in \mathcal{H}')$ has k pairwise vertex-disjoint members, or **(2)** there exists $Z \subseteq V(\bigcup\mathscr{C} - Y_0)$ with $|Z| \leqslant k-1$ such that $Z \cap V(S_H) \neq \varnothing$ for all $H \in \mathcal{H}'$.

Case (1) $(S_H : H \in \mathcal{H}')$ has k pairwise vertex-disjoint members: Then there exists $\mathcal{P} \subseteq \mathcal{H}'$ with $|\mathcal{P}| = k$ such that $V(S_A) \cap V(S_B) = \varnothing$ for all distinct $A, B \in \mathcal{P}$. Therefore $\operatorname{proj}_U(A) \cap$

$\mathrm{proj}_U(B) = \varnothing$, which implies $V(A) \cap V(B) = \varnothing$. Hence $\mathcal{P}$ is a collection of k pairwise vertex-disjoint members of $\mathcal{H}$, so (i) holds.

Case (2) there exists $Z \subseteq V(\bigcup \mathscr{C} - Y_0)$ with $|Z| \leqslant k-1$ such that $Z \cap V(S_H) \neq \varnothing$ for all $H \in \mathcal{H}'$: Hence $Y := Y_0 \cup Z$ is a subset of $V(\bigcup \mathscr{C})$ that intersects $V(S_H)$ for all $H \in \mathcal{H}$, and $|Y| \leqslant k - 1 + |\mathscr{C}|$. In particular, for each $H \in \mathcal{H}$, there exists $v \in V(H)$ such that $\mathrm{proj}_U(v) \in B_{\bigcup \mathscr{C}}(Y, \lfloor \ell/2 \rfloor)$. Thus $v \in B_U(B_{\bigcup \mathscr{C}}(Y, \lfloor \ell/2 \rfloor), r)$, which implies that (ii) holds.

This concludes the proof of the lemma. □

Lemma 16. *Let k, d, r, ℓ be positive integers with $r \geqslant \lceil d/2 \rceil$ and $\ell \geqslant 3$. Let $R := r + \lceil d/2 \rceil$, let G be a graph, let $\mathscr{C}$ be a non-empty d-packing of cycles in G each of length at least ℓ, let U be a $\bigcup \mathscr{C}$-supported BFS-spanning subgraph of $G_R := G[B_G(V(\bigcup \mathscr{C}), R)]$, and let W be the set of endpoints of all edges $e \in E(G_R)$ such that $\mathrm{span}(G_R, \bigcup \mathscr{C}, U, e) > \ell$. Then either*

(a) *G contains a d-packing of k cycles each of length at least ℓ, or*
(b) *there exists $Y \subseteq V(\bigcup \mathscr{C})$ with $|Y| \leqslant 2(k-1)$ such that, by letting $G_r := G[B_G(V(\bigcup \mathscr{C}), r)]$ and $X := \mathrm{pad}(\bigcup \mathscr{C}, Y, 2\lceil d/2 \rceil + 1, \lceil \ell/2 \rceil)$, we have that*

$$G_r - \Big(B_G(W, \lfloor d/2 \rfloor) \cup B_{\bigcup \mathscr{C} \cup U}(X, R) \Big)$$

has no cycles of length at least ℓ.

Proof. If $|\mathscr{C}| \geqslant k$, then (a) holds, hence it may be assumed that $|\mathscr{C}| \leqslant k-1$. Let $\mathcal{D}$ be the set of all cycles in $G_r - B_G(W, \lfloor d/2 \rfloor)$ of length at least ℓ. For each $D \in \mathcal{D}$, let U_D be a D-supported BFS-spanning subgraph of $G[B_G(V(D), \lceil d/2 \rceil)]$, and let $H_D := D \cup U_D$.

Claim 17. $\mathrm{span}(G_R, \bigcup \mathscr{C}, U, e) \leqslant \ell$ *for every $D \in \mathcal{D}$ and every $e \in E(H_D)$.*

Proof of Claim. Consider any $D \in \mathcal{D}$ and any edge $e \in E(H_D)$. Without loss of generality, $e = uv$ where $u \in B_G(V(D), \lceil d/2 \rceil - 1)$. Since $\lceil d/2 \rceil - 1 \leqslant \lfloor d/2 \rfloor$ and $B_G(V(D), \lfloor d/2 \rfloor) \cap W = \varnothing$, $u \notin W$. This along with $uv \in E(G_R)$ implies that $\mathrm{span}(G_R, \bigcup \mathscr{C}, U, uv) \leqslant \ell$, as desired. ■

Now $\mathcal{H} := \{H_D : D \in \mathcal{D}\}$ is a collection of non-null connected subgraphs of G_R that, by Claim 17, satisfies $\mathrm{span}(G_R, \bigcup \mathscr{C}, U, e) \leqslant \ell$ for every $e \in E(\bigcup \mathcal{H})$. Hence we may proceed by cases depending on the outcome of Lemma 15.

Case (i) $\mathcal{H}$ contains k pairwise vertex-disjoint members: Then there exists $\mathcal{P} \subseteq \mathcal{D}$ with $|\mathcal{P}| = k$ such that $V(H_A) \cap V(H_B) = \varnothing$ for all distinct $A, B \in \mathcal{P}$. Recall that $V(H_D) = V(U_D) = B_G(V(D), \lceil d/2 \rceil)$ for all $D \in \mathcal{D}$, thus $\mathcal{P}$ is a $(2\lceil d/2 \rceil)$-packing of k cycles in G each of length at least ℓ, so (a) holds.

Case (ii) there exists $Y \subseteq V(\bigcup \mathscr{C})$ with $|Y| \leqslant k-1+|\mathscr{C}|$ such that for $Z := B_{\bigcup \mathscr{C}}(Y, \lfloor \ell/2 \rfloor)$, $B_U(Z, R) \cap V(H) \neq \varnothing$ for all $H \in \mathcal{H}$: Since $|\mathscr{C}| \leqslant k-1$, $|Y| \leqslant 2(k-1)$. Let $X := \mathrm{pad}(\bigcup \mathscr{C}, Y, 2\lceil d/2 \rceil + 1, \lceil \ell/2 \rceil)$. The following shows that $B_G(W, \lfloor d/2 \rfloor) \cup B_{\bigcup \mathscr{C} \cup U}(X, R)$ meets every cycle in G_r of length at least ℓ, which implies that (b) holds. Consider any cycle D in G_r of length at least ℓ. It may be assumed that D has no vertex in $B_G(W, \lfloor d/2 \rfloor)$. Then $D \in \mathcal{D}$. Let $u \in B_U(Z, R) \cap V(H_D)$ as promised by the premise of the present case. Since $Z \subseteq V(\bigcup \mathscr{C})$,

there exists $z \in Z$ such that $z = \mathrm{proj}_U(u)$. Since $u \in V(H_D)$, there exists $v \in V(D)$ and a path $q_0 q_1 \cdots q_s$ in H_D such that $q_0 = v$, $q_s = u$, and $s \leqslant \lceil d/2 \rceil$. Then by Claim 17,

$$\begin{aligned}\mathrm{dist}_{\bigcup \mathscr{C}}(\mathrm{proj}_U(v), z) &\leqslant \sum_{i=1}^{s} \mathrm{dist}_{\bigcup \mathscr{C}}(\mathrm{proj}_U(q_{i-1}), \mathrm{proj}_U(q_i)) \\ &= \sum_{i=1}^{s} \mathrm{span}(G_R, \bigcup \mathscr{C}, U, q_{i-1}q_i) \leqslant s\ell \leqslant \lceil d/2 \rceil \ell.\end{aligned}$$

Furthermore, since $z \in Z = B_{\bigcup \mathscr{C}}(Y, \lfloor \ell/2 \rfloor)$,

$$\mathrm{dist}_{\bigcup \mathscr{C}}(\mathrm{proj}_U(v), Y) \leqslant \mathrm{dist}_{\bigcup \mathscr{C}}(\mathrm{proj}_U(v), z) + \mathrm{dist}_{\bigcup \mathscr{C}}(z, Y) \leqslant \lceil d/2 \rceil \ell + \lfloor \ell/2 \rfloor \leqslant (2\lceil d/2 \rceil + 1)\lceil \ell/2 \rceil.$$

Since $X = \mathrm{pad}(\bigcup \mathscr{C}, Y, 2\lceil d/2 \rceil + 1, \lceil \ell/2 \rceil)$, Corollary 9 implies $\mathrm{dist}_{\bigcup \mathscr{C}}(\mathrm{proj}_U(v), X) \leqslant \lceil d/2 \rceil$. Finally, since $v \in V(D) \subseteq V(G_r)$, $v \in B_U(\mathrm{proj}_U(v), r)$, thus $v \in B_{\bigcup \mathscr{C} \cup U}(X, r + \lceil d/2 \rceil)$. Consequently, D has a vertex in $B_{\bigcup \mathscr{C} \cup U}(X, R)$, so (b) holds.

This concludes the proof of the lemma. □

We now prove a quantitative version of Theorem 2.

Theorem 18. *Let $f : \mathbb{N}^2 \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ be the following functions:*

$$f(k, \ell) = (51s(k) + 104k - 155)(2\lceil \ell/2 \rceil + 1) + k - 1, \qquad g(d) = 20d.$$

For all positive integers k, d, ℓ with $\ell \geqslant 3$, for every graph G, either G has a d-packing of k cycles each of length at least ℓ, or there exists $X \subseteq V(G)$ with $|X| \leqslant f(k, \ell)$ and $G - B_G(X, g(d))$ has no cycle of length at least ℓ.

Proof. If G has a d-packing of k cycles each of length at least ℓ, then there is nothing to prove. Therefore we assume that G does not have a d-packing of k cycles each of length at least ℓ.

Let $\mathscr{Z}$ be a maximal d-packing of cycles in G such that $\ell \leqslant \mathrm{len}(D) \leqslant 6d + 2$ for all $D \in \mathscr{Z}$. Let $\mathscr{C}$ be a maximal $2d$-packing of cycles in G such that $\mathrm{len}(C) \geqslant \ell$ and the C-span of $G[B_G(V(C), d)]$ is at most ℓ for all $C \in \mathscr{C}$.

Note that $\mathscr{Z}$ and $\mathscr{C}$ have size at most $k - 1$, and each may be the empty set.

Claim 19. *Every cycle in G of length at least ℓ contains a vertex in*

$$B_G(V(\bigcup \mathscr{Z}), 4d) \cup B_G(V(\bigcup \mathscr{C}), 4d).$$

Proof of Claim. Assume for a contradiction that there exists a cycle C in G such that C has length at least ℓ and contains no vertex in $B_G(V(\bigcup \mathscr{Z}), 4d) \cup B_G(V(\bigcup \mathscr{C}), 4d)$. Proceed by cases depending on the outcomes of Lemma 7.

Case (a) there exists a cycle C' in G of length at least ℓ such that $V(C') \subseteq B_G(V(C), 3d)$ and the length of C' is at most $6d + 2$: Then $V(C') \cap B_G(V(\bigcup \mathscr{Z}), d) = \varnothing$, therefore $\mathscr{Z} \cup \{C'\}$ contradicts the maximality of $\mathscr{Z}$.

Case (b) there exists a cycle C' in G of length at least ℓ such that $V(C') \subseteq B_G(V(C), 2d)$ and the C'-span of $G[B_G(V(C'), d)]$ is at most ℓ: Then $V(C') \cap B_G(V(\bigcup \mathscr{C}), 2d) = \varnothing$, therefore $\mathscr{C} \cup \{C'\}$ contradicts the maximality of $\mathscr{C}$. ∎

Let $Z := B_G(V(\bigcup \mathcal{Z}), 4d)$ and let X_0 be a set consisting of one vertex from each cycle in $\mathcal{Z}$. Then

$$|X_0| \leqslant k - 1. \tag{1}$$

Furthermore, since each cycle in $\mathcal{Z}$ has length at most $6d + 2$,

$$Z \subseteq B_G(X_0, 7d + 1). \tag{2}$$

If $\mathscr{C} = \varnothing$, then (1) and (2), along with Claim 19, imply that the theorem holds. Hence from this point on we assume that $\mathscr{C} \neq \varnothing$ and thus, $\bigcup \mathscr{C}$-span is now well-defined for $G[B_G(V(\bigcup \mathscr{C}), t)]$ for each $t \geqslant 0$.

Let

$$G_d := G[B_G(V(\bigcup \mathscr{C}), d)].$$

Claim 20. *The $\bigcup \mathscr{C}$-span of G_d is at most ℓ.*

Proof of Claim. Suppose U is a $\bigcup \mathscr{C}$-supported BFS-spanning subgraph of G_d and suppose $e \in E(G_d)$. First consider the case that both endpoints of e lie in $B_G(V(C), d)$ for some $C \in \mathscr{C}$. Let $G_C := G[B_G(V(C), d)]$. Since $\mathscr{C}$ is a $2d$-packing in G, $S := U[V(G_C)]$ is a C-supported BFS-spanning subgraph of G_C such that $\operatorname{span}(G_C, C, S, e) = \operatorname{span}(G_d, \bigcup \mathscr{C}, U, e)$. Furthermore since $\operatorname{span}(G_C, C, S, e)$ is finite and the C-span of G_C is at most ℓ (by definition of $C \in \mathscr{C}$), $\operatorname{span}(G_C, C, S, e) \leqslant \ell$. Thus $\operatorname{span}(G_d, \bigcup \mathscr{C}, U, e) \leqslant \ell$. On the other hand, if e is an edge between $B_G(V(C), d)$ and $B_G(V(D), d)$ for distinct $C, D \in \mathscr{C}$, then $\operatorname{span}(G_d, \bigcup \mathscr{C}, U, e) = \infty$. It follows that the $(\bigcup \mathscr{C}, U)$-span of G_d is at most ℓ. Moreover, since U was an arbitrary $\bigcup \mathscr{C}$-supported BFS-spanning subgraph of G_d, the $\bigcup \mathscr{C}$-span of G_d is at most ℓ, as desired. ∎

Let

$$r := 6d,$$
$$G_r := G[B_G(V(\bigcup \mathscr{C}), r)],$$
$$R := r + \lceil d/2 \rceil,$$
$$G_R := G[B_G(V(\bigcup \mathscr{C}), R)].$$

Let U be a $\bigcup \mathscr{C}$-supported BFS-spanning subgraph of G_R.

Let W be the set of all endpoints of edges $e \in E(G_R)$ such that $\operatorname{span}(G_R, \bigcup \mathscr{C}, U, e) > \ell$. Since $\mathscr{C}$ is a $2d$-packing in G_R, every edge $e \in E(G[B_G(V(C), d)])$ has finite $(\bigcup \mathscr{C}, U)$-span for all $C \in \mathscr{C}$. Thus Claim 20 implies that every edge $e \in E(G[B_G(V(C), d)])$ satisfies $\operatorname{span}(G_R, \bigcup \mathscr{C}, U, e) \leqslant \ell$.

Therefore for all $C \in \mathscr{C}$,

$$W \cap B_G(V(C), d - 1) = \varnothing. \tag{3}$$

Let

$$I := \bigcup_{\{C,D\}\in\binom{\mathscr{C}}{2}} B_G(V(C),r)\cap B_G(V(D),r).$$

By Claim 20 the $\bigcup\mathscr{C}$-span of G_d is at most ℓ. Since $R \geqslant r > d$, Lemma 10 implies that there exists $X_1 \subseteq V(\bigcup\mathscr{C})$ such that

$$|X_1| \leqslant (s(k)-1)(2\lceil \ell/2\rceil + 1), \tag{4}$$

and

$$W \cup I \subseteq B_G(W, r-d-1) \subseteq B_G(X_1, R+r+d). \tag{5}$$

Since $\mathscr{C}$ is a $2d$-packing, it is a d-packing. Therefore since $r \geqslant \lceil d/2\rceil$, Lemma 16 implies that there exists $Y_2 \subseteq V(\bigcup\mathscr{C})$ with $|Y_2| \leqslant 2(k-1)$ such that, by letting

$$X_2 := \mathrm{pad}(\bigcup\mathscr{C}, Y_2, 2\lceil d/2\rceil + 1, \lceil \ell/2\rceil),$$

and

$$L := B_G(W, \lfloor d/2\rfloor) \cup B_{\bigcup\mathscr{C}\cup U}(X_2, R),$$

we have that

$$G_r - L \text{ has no cycles of length at least } \ell. \tag{6}$$

Then by Corollary 9,

$$|X_2| \leqslant |Y_2|(2\lceil \ell/2\rceil + 1) \leqslant 2(k-1)(2\lceil \ell/2\rceil + 1). \tag{7}$$

Let

$$F_{-1} := \bigcup\mathscr{C} - L,$$
$$F_0 := G - \Big(Z \cup B_G(V(\bigcup\mathscr{C}), r-2d)\Big),$$
$$F_0^- := G - \Big(Z \cup B_G(V(\bigcup\mathscr{C}), r-d)\Big).$$

Then F_0^- is an induced subgraph of F_0. Moreover by Claim 19,

$$F_0 \text{ and } F_0^- \text{ contain no cycle of length at least } \ell. \tag{8}$$

Let $p := |\mathscr{C}|$ and enumerate $\mathscr{C}$ as $C_1, C_2, \ldots, C_p$. For each $i \in [p]$ let

$$F_i := G[B_G(V(C_i), r)] - L,$$
$$F_i^- := G[B_G(V(C_i), r-d)] - L.$$

Then F_i^- is an induced subgraph of F_i. Moreover, (6) guarantees that for all $i \in [p]$,

$$F_i \text{ contains no cycle of length at least } \ell. \tag{9}$$

Let

$$M := B_G(X_0 \cup X_1 \cup X_2, R+r+d), \tag{10}$$

then by (2) and (5),

$$\begin{aligned} Z \cup L \cup I &\subseteq B_G(X_0, 7d+1) \cup B_G(W, \lfloor d/2 \rfloor) \cup B_{\bigcup \mathscr{C} \cup U}(X_2, R) \cup B_G(X_1, R+r+d) \\ &\subseteq B_G(X_0, 7d+1) \cup B_G(W, r-d-1) \cup B_G(X_2, R) \cup B_G(X_1, R+r+d) \\ &= B_G(X_0, 7d+1) \cup B_G(X_2, R) \cup B_G(X_1, R+r+d) \\ &\subseteq M. \end{aligned} \tag{11}$$

For each $i \in [p]$, define $U_i := U[B_U(V(C_i), r-d)]$. Then $\bigcup_{i\in[p]} U_i$ is a forest and the sets $V(U_1), V(U_2), \dots, V(U_p)$ form a partition of $B_G(V(\bigcup \mathscr{C}), r-d)$. See Figure 2 for a summary of the many key definitions given so far.

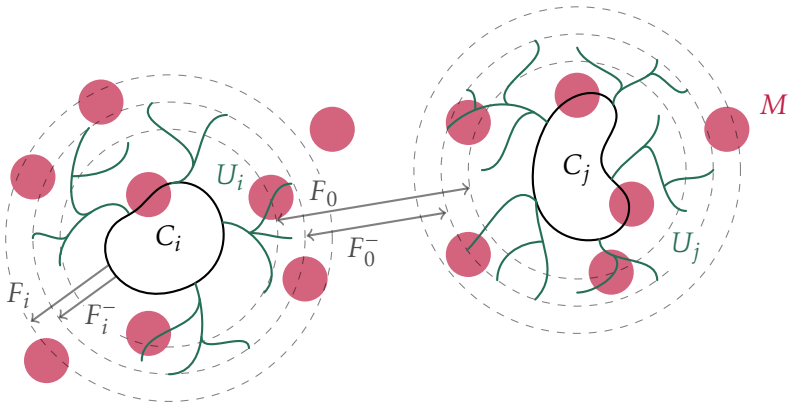


Figure 2: Simplified drawing showing the interplay between many of the key definitions given so far.

For each pair $i, j \in [p]$, a path $P := v_0 v_1 \cdots v_s$ in G with $s \geqslant 2$ is an *(i,j)-ear* if

- $v_0 \in V(U_i)$ and $v_s \in V(U_j)$;
- $v_1 \cdots v_{s-1}$ is a path in F_0^-.

An *ear* is an (i,j)-ear for some $i, j \in [p]$. The *first leg* of P is the path $P_1 := U\langle v_0 \rangle$, and the *second leg* of P is the path $P_2 := U\langle v_s \rangle$. Therefore P_1 is the unique path in U_i between v_0 and $\mathrm{proj}_U(v_0)$, and P_2 is the unique path in U_j between v_s and $\mathrm{proj}_U(v_s)$. Note that $V(P_1) \cap V(P_2)$ may be non-empty for (i,j)-ears where $i = j$. Let

$$\mathrm{proj}_U(P) := \{\mathrm{proj}_U(v_0), \mathrm{proj}_U(v_s)\}.$$

Then $\mathrm{proj}_U(P)$ is a non-empty subset of $V(\bigcup \mathscr{C})$. Figure 3 shows typical examples of ears and their projections. Define the walk

$$W_P := P_1 \cup P \cup P_2.$$

Note that the first and second legs of P have exactly $r-d$ edges, whereas P may be arbitrarily long. P is said to be *admissible* if $V(W_P) \cap B_G(M, d) = \varnothing$.

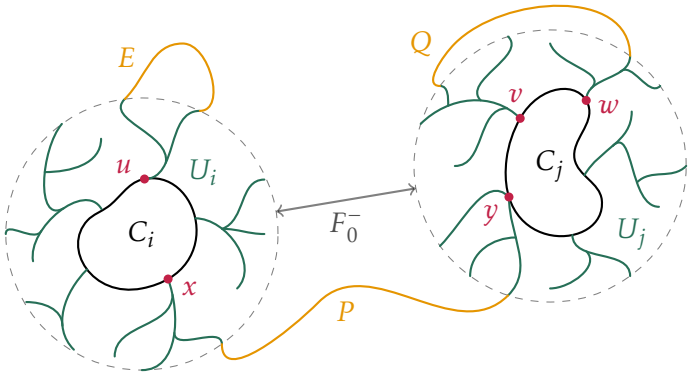


Figure 3: E is an (i,i)-ear with $\mathrm{proj}_U(E) = \{u\}$; Q is a (j,j)-ear with $\mathrm{proj}_U(Q) = \{v,w\}$; and P is an (i,j)-ear with $\mathrm{proj}_U(P) = \{x,y\}$.

Suppose $P := v_0v_1\cdots v_s$ is an admissible (i,j)-ear and let $P_0 := v_1\cdots v_{s-1}$. For each $m \in \{-1,0,1,\dots,p\}$, let

$$\Psi_m(P) := \begin{cases} \varnothing & \text{if } m \notin \{-1,0,i,j\}, \\ B_{\bigcup\mathscr{C}-L}(\mathrm{proj}_U(P), d\ell) & \text{if } m = -1, \\ B_G(V(P_0), d) & \text{if } m = 0, \\ B_G(V(W_P)\cap V(U_m), d) & \text{if } m \in \{i,j\}. \end{cases}$$

Claim 21. *If P is an admissible ear, then*

(i) *$\Psi_{-1}(P) \subseteq V(F_{-1})$, and $F_{-1}[\Psi_{-1}(P)]$ has at most two components;*
(ii) *$G[\Psi_0(P)]$ is a connected subgraph of F_0;*
(iii) *for all $m \in [p]$, $G[\Psi_m(P)]$ is a subgraph of F_m and has c_m components where $c_m \in \{0,1,2\}$. Moreover, $\sum_{m\in[p]} c_m \leqslant 2$.*

Proof of Claim. Write $P = v_0v_1\cdots v_s$ and let $i,j \in [p]$ such that P is an (i,j)-ear. Let P_1 and P_2 be the first and second legs of P respectively, and let $P_0 := v_1\cdots v_{s-1}$. Let $u_0 := \mathrm{proj}_U(v_0)$ and $u_s := \mathrm{proj}_U(v_s)$. Recall that $Z \cup L \subseteq M$ by (11).

To prove (i), it suffices to show that $B_{C_i-L}(u_0, d\ell)$ is a connected set of vertices in $C_i - L$, and $B_{C_j-L}(u_s, d\ell)$ is a connected set of vertices in $C_j - L$. However this is trivial since each is a ball centred at some vertex.

Next, we prove (ii). Since P is admissible, $B_G(V(W_P), d)\cap M = \varnothing$. Thus $P_0 \subseteq W_P$ and $Z \subseteq M$ together imply $B_G(V(P_0), d)\cap Z = \varnothing$. Furthermore since $P_0 \subseteq F_0^-$, $V(P_0)\cap B_G(V(\bigcup\mathscr{C}), r-d) = \varnothing$. Hence $B_G(V(P_0), d)$ and $Z \cup B_G(V(\bigcup\mathscr{C}), r-2d)$ are disjoint. Therefore, since P_0 is connected, $G[B_G(V(P_0), d)]$ is a connected subgraph of $G-(Z\cup B_G(V(\bigcup\mathscr{C}), r-2d))$. In other words, $G[\Psi_0(P)]$ is a connected subgraph of F_0.

It remains to prove (iii). Let $m \in [p]$. Note that $m \neq 0$. If $m \notin \{i,j\}$, then $\Psi_m(P) = \varnothing$, implying that $G[\Psi_m(P)] \subseteq F_m$ and $c_m = 0$. Therefore it suffices to show that $B_G(V(P_1), d)$ is a connected set of vertices in F_i, and $B_G(V(P_2), d)$ is a connected set of vertices in F_j.

We only prove the first statement as the second is symmetric. Since P is admissible and $L \subseteq M$, $B_G(V(W_P), d) \cap L = \varnothing$, thus $B_G(V(P_1), d) \cap L = \varnothing$. Furthermore, since $V(P_1) \subseteq V(U_i) \subseteq B_G(V(C_i), r-d)$, $B_G(V(P_1), d) \subseteq B_G(V(C_i), r) \setminus L$. Therefore, since P_1 is connected, $B_G(V(P_1), d)$ is a connected set of vertices in $G[B_G(V(C_i), r)] - L$. In other words, $B_G(V(P_1), d)$ is a connected set of vertices in F_i as desired. ∎

Claim 22. *If P and Q are admissible ears such that $\Psi_{-1}(P) \cap \Psi_{-1}(Q) = \varnothing$, then*

$$\operatorname{dist}_{\bigcup \mathscr{C}}(\operatorname{proj}_U(P), \operatorname{proj}_U(Q)) > d\ell.$$

Proof of Claim. Proceed by contraposition and suppose that $\operatorname{dist}_{\bigcup \mathscr{C}}(\operatorname{proj}_U(P), \operatorname{proj}_U(Q)) \leqslant d\ell$. Let S be a shortest path in $\bigcup \mathscr{C}$ between $\operatorname{proj}_U(P)$ and $\operatorname{proj}_U(Q)$, and let $C \in \mathscr{C}$ be the cycle that contains S. The following shows that $V(S) \cap L = \varnothing$, which implies that $\operatorname{dist}_{\bigcup \mathscr{C} - L}(\operatorname{proj}_U(P), \operatorname{proj}_U(Q)) \leqslant d\ell$, hence $\Psi_{-1}(P) \cap \Psi_{-1}(Q) \neq \varnothing$ as required.

We first show that each component of $C[V(C) \cap L]$ has more than $d\ell + 1$ vertices: Recall that $X_2 = \operatorname{pad}(\bigcup \mathscr{C}, Y_2, \alpha, \beta)$, where $\alpha = 2\lceil d/2 \rceil + 1$ and $\beta = \lceil \ell/2 \rceil$, and recall that $L = B_G(W, \lfloor d/2 \rfloor) \cup B_{\bigcup \mathscr{C} \cup U}(X_2, R)$. Since $\lfloor d/2 \rfloor \leqslant d - 1$, (3) implies $B_G(W, \lfloor d/2 \rfloor) \cap V(\bigcup \mathscr{C}) = \varnothing$, thus $V(\bigcup \mathscr{C}) \cap L = V(\bigcup \mathscr{C}) \cap B_{\bigcup \mathscr{C} \cup U}(X_2, R)$. Moreover, since each component of U attaches to $\bigcup \mathscr{C}$ only at its root, $V(\bigcup \mathscr{C}) \cap B_{\bigcup \mathscr{C} \cup U}(X_2, R) = B_{\bigcup \mathscr{C}}(X_2, R)$, and so $V(\bigcup \mathscr{C}) \cap L = B_{\bigcup \mathscr{C}}(X_2, R)$. Now $B_{\bigcup \mathscr{C}}(X_2, \lfloor \alpha/2 \rfloor) = B_{\bigcup \mathscr{C}}(Y_2, \alpha\beta + \lfloor \alpha/2 \rfloor)$ by Corollary 9, therefore since $R \geqslant \lfloor \alpha/2 \rfloor$, we have $V(\bigcup \mathscr{C}) \cap L = B_{\bigcup \mathscr{C}}(Y_2, \alpha\beta + R)$. It follows that each component of $(\bigcup \mathscr{C})[V(\bigcup \mathscr{C}) \cap L]$ has at least $2(\alpha\beta + R) + 1 > d\ell + 1$ vertices. In particular, each component of $C[V(C) \cap L]$ has more than $d\ell + 1$ vertices.

Assume for a contradiction that $V(S) \cap L \neq \varnothing$. Since P and Q are admissible and $L \subseteq M$ by (11), the set of endpoints of S is disjoint from L. Therefore there exists a component of $C[V(C) \cap L]$ that is a subgraph of S, so $d\ell \geqslant \operatorname{len}(S) = |V(S)| - 1 > d\ell$, a contradiction. ∎

Claim 23. *If P and Q are admissible ears such that $\Psi_m(P) \cap \Psi_m(Q) = \varnothing$ for all $m \in \{0, 1, \ldots, p\}$, then*

$$\operatorname{dist}_G(V(W_P), V(W_Q)) > d.$$

Proof of Claim. Write $P = v_0 v_1 \cdots v_s$ and let $i, j \in [p]$ such that P is an (i, j)-ear. Let P_1 and P_2 be the first and second legs of P respectively, and let $P_0 \coloneqq v_1 \cdots v_{s-1}$. Similarly, write $Q = w_0 w_1 \cdots w_h$ and let $i', j' \in [p]$ such that Q is an (i', j')-ear. Let Q_1 and Q_2 be the first and second legs of Q respectively, and let $Q_0 \coloneqq w_1 \cdots w_{h-1}$.

Proceed by contraposition and suppose that $\operatorname{dist}_G(V(W_P), V(W_Q)) \leqslant d$. Let $v \in V(W_P)$ and $w \in V(W_Q)$ such that $\operatorname{dist}_G(v, w) \leqslant d$. Then either **(1)** $v \in V(P_0)$ or $w \in V(Q_0)$, or **(2)** $v \notin V(P_0)$ and $w \notin V(Q_0)$.

Case (1) $v \in V(P_0)$ or $w \in V(Q_0)$: Without loss of generality assume the latter. Then it is immediate that $v \in B_G(w, d) \subseteq \Psi_0(Q)$, so it suffices to show that $v \in \Psi_0(P)$. If $v \in V(P_0)$, then $v \in \Psi_0(P)$. Otherwise v lies in the first or second leg of P. Without loss of generality $v \in V(P_1)$. Let $S \coloneqq v P_1 v_0$. Recall that $w \in V(Q_0) \subseteq V(F_0^-)$. Also, recall that P_1 is a shortest path in G between $V(\bigcup \mathscr{C})$ and v_0, and $\operatorname{len}(P_1) = r - d$, thus $B_G(v, \operatorname{len}(S)) \subseteq B_G(V(\bigcup \mathscr{C}), r - d)$. Hence $B_G(v, \operatorname{len}(S))$ is disjoint from $V(F_0^-)$. Therefore since $w \in B_G(v, d) \cap V(F_0^-)$, $\operatorname{len}(S) \leqslant$

$d-1$. Then $S \cup v_0v_1$ is a path between v and $V(P_0)$ of length at most d, which implies that $v \in B_G(V(P_0), d) = \Psi_0(P)$, as desired.

Case (2) $v \notin V(P_0)$ and $w \notin V(Q_0)$: Without loss of generality assume that $v \in V(P_1)$ and $w \in V(Q_1)$. It is immediate that $w \in B_G(v,d) \subseteq \Psi_i(P)$, so it suffices to show that $w \in \Psi_i(Q)$. By Claim 21, $\Psi_i(P) \subseteq V(F_i)$, thus $w \in B_G(V(C_i), r)$. This, along with $w \in V(Q_1) \subseteq V(U_{i'}) \subseteq B_G(V(C_{i'}), r)$, shows that $w \in B_G(V(C_i), r) \cap B_G(V(C_{i'}), r)$. Now since Q is admissible and $w \in V(W_Q)$, $w \notin M$. In particular $w \notin I$ by (11). Consequently, $i = i'$. Hence $w \in B_G(V(Q_1), d) \subseteq \Psi_{i'}(Q) = \Psi_i(Q)$, as desired. ∎

Claim 24. *If P is an admissible ear and there exist sets $S \subseteq V(\bigcup \mathscr{C})$ and $S' \subseteq V(G)$ such that $S \cap \Psi_{-1}(P) \neq \varnothing$ or $S' \cap \bigcup_{m=0}^{p} \Psi_m(P) \neq \varnothing$, then $B_G(\mathrm{pad}(\bigcup \mathscr{C}, S, 2d, \lceil \ell/2 \rceil) \cup S', r) \cap V(P) \neq \varnothing$.*

Proof of Claim. First, suppose that $S \cap \Psi_{-1}(P) \neq \varnothing$. Then $S \cap B_{\bigcup \mathscr{C}}(\mathrm{proj}_U(P), d\ell) \neq \varnothing$. In other words, there exists $u \in \mathrm{proj}_U(P)$ such that $\mathrm{dist}_{\bigcup \mathscr{C}}(u, S) \leqslant d\ell \leqslant 2d\lceil \ell/2 \rceil$. Then Corollary 9 implies $\mathrm{dist}_{\bigcup \mathscr{C}}(u, \mathrm{pad}(\bigcup \mathscr{C}, S, 2d, \lceil \ell/2 \rceil)) \leqslant d$. Moreover, since $B_G(u, r-d) \cap V(P) \neq \varnothing$, $B_G(\mathrm{pad}(\bigcup \mathscr{C}, S, 2d, \lceil \ell/2 \rceil), r) \cap V(P) \neq \varnothing$ as required.

Next, suppose that $S' \cap \bigcup_{m=0}^{p} \Psi_m(P) \neq \varnothing$. Since $\bigcup_{m=0}^{p} \Psi_m(P) \subseteq B_G(V(W_P), d)$, $S' \cap B_G(V(W_P), d) \neq \varnothing$. Then since $V(W_P) \subseteq B_G(V(P), r-d)$, $S' \cap B_G(V(P), r) \neq \varnothing$, hence $B_G(S', r) \cap V(P) \neq \varnothing$ as required. ∎

Let $F^\star_{-1}, F^\star_0, \ldots, F^\star_p$ be pairwise vertex-disjoint graphs such that $F^\star_i$ is isomorphic to F_i for all $i \in \{-1, 0, \ldots, p\}$. Note that unlike $(F^\star_{-1}, F^\star_0, \ldots, F^\star_p)$, the graphs in $(F_{-1}, F_0, \ldots, F_p)$ may intersect each other. For each $i \in \{-1, 0, \ldots, p\}$ let $\pi_i : V(F_i) \to V(F^\star_i)$ be an isomorphism from F_i to $F^\star_i$. Let $F^\star = F^\star_{-1} \cup \bigcup_{i=0}^{p} F^\star_i$.

Let β_{-1} and $\beta_{\geqslant 0}$ be minimum-width tree-decompositions of $F^\star_{-1}$ and $\bigcup_{i=0}^{p} F^\star_i$ respectively. Let T_{-1} and $T_{\geqslant 0}$ be the underlying trees of β_{-1} and $\beta_{\geqslant 0}$ respectively, which we may assume to be vertex-disjoint. Let T be a tree that contains $T_{-1} \cup T_{\geqslant 0}$ as a spanning subgraph. Then $\beta := \beta_{-1} \cup \beta_{\geqslant 0}$ is a tree-decomposition of $F^\star$. Now $\bigcup \mathscr{C} - L$ is a forest by (6), thus β_{-1} has width at most 1. Furthermore, Theorem 6, (8), and (9) together imply that $\beta_{\geqslant 0}$ has width less than $\ell - 1$. Therefore for all $t \in V(T)$,

$$|\beta(t)| \leqslant \begin{cases} 2 & \text{if } t \in V(T_{-1}), \\ \ell - 1 & \text{if } t \in V(T_{\geqslant 0}). \end{cases} \tag{12}$$

For subsets $S \subseteq V(F^\star)$, let $T(S) := T[\{t \in V(T) : \beta(t) \cap S \neq \varnothing\}]$.

An ear P is *i-problematic*, for some $i \in [p]$, if it is an (i,i)-ear and there exists a cycle in $W_P \cup C_i$ of length less than ℓ. An ear is *problematic* if it is i-problematic for some $i \in [p]$. An ear is *non-problematic* if it is not problematic. For non-problematic admissible ears P, let $\Psi^\star(P) := \bigcup_{m=-1}^{p} \pi_m(\Psi_m(P))$. Let $\mathcal{A}$ be the set of all graphs $T(\Psi^\star(P))$ where P is a non-problematic admissible ear. By Claim 21, $F^\star[\Psi^\star(P)]$ has at most five components for every non-problematic admissible ear P. Therefore $\mathcal{A}$ is a collection of subgraphs of T each with at most five components.

Let $k^\star$ be the minimum integer such that $2k^\star \geqslant s(k) + 2(k-1)$. Proceed by cases depending on the outcome of Theorem 5 when applied to T, $\mathcal{A}$, and $k^\star$.

Case (a) $\mathcal{A}$ has $k^\star$ pairwise vertex-disjoint members: Then there exists a collection $\mathcal{N}$ of $k^\star$ non-problematic admissible ears such that $(T(\Psi^\star(P)) : P \in \mathcal{N})$ is a collection of pairwise vertex-disjoint graphs. Thus $(\Psi^\star(P) : P \in \mathcal{N})$ is a collection of pairwise disjoint sets of vertices of $F^\star$. Consequently, for all distinct $P, Q \in \mathcal{N}$ and all $m \in \{-1, 0, \dots, p\}$ we have $\Psi_m(P) \cap \Psi_m(Q) = \varnothing$. Then by Claim 22 and Claim 23, for all distinct $P, Q \in \mathcal{N}$,

$$\operatorname{dist}_{\bigcup \mathscr{C}}(\operatorname{proj}_U(P), \operatorname{proj}_U(Q)) > d\ell. \tag{13}$$

and

$$\operatorname{dist}_G(V(W_P), V(W_Q)) > d, \tag{14}$$

For each $P \in \mathcal{N}$, either W_P is a path or W_P contains a cycle. Let $\mathcal{P}$ be the set of all $P \in \mathcal{N}$ such that W_P is a path.

Claim 25. *G contains a d-packing of $|\mathcal{N} \setminus \mathcal{P}|$ cycles each of length at least ℓ.*

Proof of Claim. For each ear $E \in \mathcal{N} \setminus \mathcal{P}$, let D_E be a cycle in W_E. Therefore, (14) implies that $\{D_E : E \in \mathcal{N} \setminus \mathcal{P}\}$ is a d-packing of $|\mathcal{N} \setminus \mathcal{P}|$ cycles in G. Now consider each $E \in \mathcal{N} \setminus \mathcal{P}$. Since W_E contains a cycle, E must be an (i,i)-ear for some $i \in [p]$. Then since E is non-problematic, every cycle in $W_E \cup C_i$ has length at least ℓ, implying $\operatorname{len}(D_E) \geqslant \ell$. ∎

If $|\mathcal{N} \setminus \mathcal{P}| \geqslant k$, then by Claim 25 G contains a d-packing of k cycles each of length at least ℓ, a contradiction. Hence $|\mathcal{N} \setminus \mathcal{P}| \leqslant k-1$, which along with $|\mathcal{N}| = k^\star \geqslant s(k)/2 + k - 1$ implies that $2|\mathcal{P}| \geqslant s(k)$.

Consider the graph

$$G' := \bigcup \mathscr{C} \cup \bigcup_{E \in \mathcal{P}} W_E.$$

Recall that $\mathscr{C}$ is a collection of pairwise vertex-disjoint cycles. Moreover, (14) implies that $(W_E : E \in \mathcal{P})$ is a collection of pairwise vertex-disjoint paths. Furthermore, no edge or inner vertex of W_E appears in $\bigcup \mathscr{C}$ for all $E \in \mathcal{P}$. Therefore every vertex $v \in V(G')$ satisfies $\deg_{G'}(v) \in \{2, 3\}$, and the degree-3 vertices of G' are precisely the endpoints of paths in $(W_E : E \in \mathcal{P})$. Hence G' has $2|\mathcal{P}| \geqslant s(k)$ degree-3 vertices.

Any path in G' whose endpoints are degree-3 vertices and whose inner vertices are degree-2 is a *segment*. Any segment is either a W_E for some $E \in \mathcal{P}$ (called a *$\mathcal{P}$-segment*) or is a path in $\bigcup \mathscr{C}$ between an endpoint of W_P and an endpoint of W_Q for some $P, Q \in \mathcal{P}$ (called a *$\mathscr{C}$-segment*).

Claim 26. *Every cycle in G' has length at least ℓ.*

Proof of Claim. Every cycle in $\mathscr{C}$ has length at least ℓ. Now consider any cycle D in G' that is not in $\mathscr{C}$. Then D contains a $\mathcal{P}$-segment. If D contains exactly one $\mathcal{P}$-segment, then there exists $P \in \mathcal{P}$ and $i \in [p]$ such that P is an (i,i)-ear and $D \subseteq W_P \cup C_i$. Since P is non-problematic, every cycle in $W_P \cup C_i$ has length at least ℓ, implying $\operatorname{len}(D) \geqslant \ell$. On the other hand, if D contains more than one $\mathcal{P}$-segment, then there exist distinct $P, Q \in \mathcal{P}$ and

a $\mathscr{C}$-segment S between an endpoint of W_P and an endpoint of W_Q such that $S \subseteq D$. Thus (13) implies $\text{len}(D) > \text{len}(S) \geqslant \text{dist}_{\bigcup \mathscr{C}}(\text{proj}_U(P), \text{proj}_U(Q)) > d\ell \geqslant \ell$. ∎

We remark that the following proof is similar to Claim 12.

Claim 27. *For every pair of vertex-disjoint cycles D and D' in G', $\text{dist}_G(V(D), V(D')) > d$.*

Proof of Claim. Assume for a contradiction that there exists a pair of vertex-disjoint cycles D and D' in G' such that $\text{dist}_G(V(D), V(D')) \leqslant d$. Consider any $u \in V(D)$ and $u' \in V(D')$ with $\text{dist}_G(u, u') \leqslant d$.

If $u \in V(W_P)$ and $u' \in V(W_Q)$ for some $P, Q \in \mathcal{P}$ with $W_P \subseteq D$ and $W_Q \subseteq D'$, then since D and D' are vertex-disjoint, $P \neq Q$. Therefore $\text{dist}_G(u, u') \geqslant \text{dist}_G(V(W_P), V(W_Q)) > d$ by (14), a contradiction. Hence it may be assumed that $u \notin V(W_P)$ for every $P \in \mathcal{P}$ with $W_P \subseteq D$. Consequently, there exists $C \in \mathscr{C}$ such that $u \in V(C)$.

Let P be a shortest (u, u')-path in G. Since $\text{dist}_G(u, u') \leqslant d$, $\text{len}(P) \leqslant d$, so $V(P) \subseteq B_G(V(C), d)$. In particular, since $\mathscr{C}$ is a $2d$-packing in G, $u' \in B_G(V(C), d)$ and $u' \notin B_G(V(\bigcup \mathscr{C}) \setminus V(C), d)$. Therefore $\text{proj}_U(u')$ exists and is a vertex of C. Furthermore, either $u' \in V(C)$ or u' lies in the first or second leg of some $Q := q_0 q_1 \cdots q_h \in \mathcal{P}$ with $W_Q \subseteq D'$. In the former case, $\text{proj}_U(u') = u' \in V(D')$, and in the latter case $\text{proj}_U(u') \in V(U\langle u' \rangle) \subseteq V(U\langle q_0 \rangle \cup U\langle q_h \rangle) \subseteq V(W_Q) \subseteq V(D')$. Hence $\text{proj}_U(u') \in V(D')$ in both cases.

Let $v_0, v_1, \ldots, v_s$ be the sequence of vertices along P such that $v_0 = u$, $v_s = u'$, and $s \leqslant d$. Since $V(P) \subseteq B_G(V(C), d) \subseteq V(G_d)$, then Claim 20 implies that $\text{span}(G_R, \bigcup \mathscr{C}, U, e) = \text{span}(G_d, \bigcup \mathscr{C}, U[V(G_d)], e) \leqslant \ell$ for every $e \in E(P)$. Therefore:

$$\begin{aligned} \text{dist}_C(u, \text{proj}_U(u')) = \text{dist}_{\bigcup \mathscr{C}}(u, \text{proj}_U(u')) &\leqslant \sum_{i=1}^{s} \text{dist}_{\bigcup \mathscr{C}}(\text{proj}_U(v_{i-1}), \text{proj}_U(v_i)) \\ &= \sum_{i=1}^{s} \text{span}(G_R, \bigcup \mathscr{C}, U, v_{i-1}v_i) \leqslant s\ell \leqslant d\ell. \quad (15) \end{aligned}$$

Now consider a shortest $(u, \text{proj}_U(u'))$-path S in C. Recall that $u \in V(D)$ and $\text{proj}_U(u') \in V(D')$. Let x be the first vertex in S starting from u such that $x \in V(W_Q)$ for some $Q \in \mathcal{P}$ with $W_Q \subseteq D$. Let y be the first vertex in S starting from $\text{proj}_U(u')$ such that $y \in V(W_E)$ for some $E \in \mathcal{P}$ with $W_E \subseteq D'$. Since D and D' are vertex-disjoint, both vertices x and y exist and $Q \neq E$. Note that $\text{dist}_C(x, y) \leqslant \text{dist}_C(u, \text{proj}_U(u')) \leqslant d\ell$ by (15). On the other hand, $x \in V(C) \cap V(W_Q) \subseteq \text{proj}_U(Q)$, $y \in V(C) \cap V(W_E) \subseteq \text{proj}_U(E)$, Q and E are distinct elements of $\mathcal{P} \subseteq \mathcal{N}$, and (13) together imply $\text{dist}_C(x, y) \geqslant \text{dist}_{\bigcup \mathscr{C}}(\text{proj}_U(Q), \text{proj}_U(E)) > d\ell$, a contradiction. ∎

Since G' is a graph with all vertices of degree 2 or 3 and contains at least $s(k)$ degree-3 vertices, Theorem 4 implies that G' contains k pairwise vertex-disjoint cycles. By Claim 26, each of these cycles has length at least ℓ, and by Claim 27, these cycles form a d-packing in G. Hence G contains a d-packing of k cycles each of length at least ℓ, a contradiction.

Case (b) there exists a subset $X^\star \subseteq V(T)$ with $|X^\star| \leqslant 50(k^\star - 1)$ such that $X^\star \cap V(A) \neq \varnothing$ for all $A \in \mathcal{A}$: Recall that $\mathcal{A}$ is the set of all graphs $T(\Psi^\star(P))$ where P is a non-problematic

admissible ear. Let $B^\star_{-1} := \bigcup_{t\in X^\star\cap V(T_{-1})}\beta(t)$ and $B^\star_{\geqslant 0} := \bigcup_{t\in X^\star\cap V(T_{\geqslant 0})}\beta(t)$. Consider any non-problematic admissible ear P. The definition of $X^\star$ implies that $(B^\star_{-1}\cup B^\star_{\geqslant 0})\cap\Psi^\star(P)\neq\varnothing$. Therefore $B^\star_{-1}\cap\pi_{-1}(\Psi_{-1}(P))\neq\varnothing$ or $B^\star_{\geqslant 0}\cap\bigcup_{m=0}^{p}\pi_m(\Psi_m(P))\neq\varnothing$. Let $S_{-1} := \pi_{-1}^{-1}(B^\star_{-1})$ and $S_{\geqslant 0} := \bigcup_{m=0}^{p}\pi_m^{-1}(B^\star_{\geqslant 0})$. Then $S_{-1}\subseteq V(\bigcup\mathscr{C})$, $S_{\geqslant 0}\subseteq V(G)$, and $S_{-1}\cap\Psi_{-1}(P)\neq\varnothing$ or $S_{\geqslant 0}\cap\bigcup_{m=0}^{p}\Psi_m(P)\neq\varnothing$. Consequently, by letting $X_3 := \mathrm{pad}(\bigcup\mathscr{C}, S_{-1}, 2d, \lceil\ell/2\rceil)\cup S_{\geqslant 0}$, Claim 24 implies that

$$B_G(X_3, r)\cap V(P)\neq\varnothing \text{ for every non-problematic admissible ear } P. \tag{16}$$

By Corollary 9,

$$\begin{aligned}|X_3| &\leqslant |\mathrm{pad}(\textstyle\bigcup\mathscr{C}, S_{-1}, 2d, \lceil\ell/2\rceil)| + |S_{\geqslant 0}|\\ &\leqslant |S_{-1}|(2\lceil\ell/2\rceil+1)+|S_{\geqslant 0}|.\end{aligned}$$

Since $|S_{-1}|\leqslant|B^\star_{-1}|$ and $|S_{\geqslant 0}|\leqslant|B^\star_{\geqslant 0}|$, we have

$$|X_3|\leqslant|B^\star_{-1}|(2\lceil\ell/2\rceil+1)+|B^\star_{\geqslant 0}|.$$

By (12),

$$|X_3|\leqslant 2|X^\star\cap V(T_{-1})|(2\lceil\ell/2\rceil+1)+(\ell-1)|X^\star\cap V(T_{\geqslant 0})|.$$

Since $|X^\star\cap V(T_{-1})|+|X^\star\cap V(T_{\geqslant 0})| = |X^\star|\leqslant 50(k^\star-1)$ and $\ell-1\leqslant 2\lceil\ell/2\rceil+1$,

$$\begin{aligned}|X_3| &\leqslant (2|X^\star|-|X^\star\cap V(T_{\geqslant 0})|)(2\lceil\ell/2\rceil+1)\\ &\leqslant 100(k^\star-1)(2\lceil\ell/2\rceil+1).\end{aligned}$$

Since $2(k^\star-1)\leqslant s(k)+2(k-1)-1$,

$$|X_3|\leqslant 50(s(k)+2(k-1)-1)(2\lceil\ell/2\rceil+1). \tag{17}$$

Claim 28. *For every cycle D in G of length at least ℓ, either*

(i) *D contains a vertex in $B_G(M\cup X_3, r)$, or*
(ii) *D is the union of pairwise internally disjoint paths $Q_1\cup\cdots\cup Q_{2t}$ with $t\geqslant 1$, such that*
- *the start vertex of Q_i is the end vertex of Q_{i-1} for all $i\in[2t]$, where the indices are interpreted cyclically;*
- *$V(Q_1\cup Q_3\cup\cdots\cup Q_{2t-1}) = V(D)\cap B_G(V(\bigcup\mathscr{C}), r-d)$;*
- *$\{Q_2, Q_4, \ldots, Q_{2t}\}$ is a collection of problematic admissible ears.*

Proof of Claim. Consider any cycle D in G of length at least ℓ. If D contains a vertex in $B_G(M\cup X_3, r)$, then (i) holds. Hence it may be assumed that $V(D)\cap B_G(M\cup X_3, r) = \varnothing$. Recall that $Z\cup L\subseteq M$ by (11), hence $V(D)\cap Z$ and $V(D)\cap L$ are empty.

As a preliminary step towards (ii), we show that $V(D)\subseteq(B_G(V(\bigcup\mathscr{C}), r-d)\setminus L)\cup V(F_0^-)$, $V(D)\not\subseteq B_G(V(\bigcup\mathscr{C}), r-d)\setminus L$, and $V(D)\not\subseteq V(F_0^-)$. Then, since $G[B_G(V(\bigcup\mathscr{C}), r-d)\setminus L]$ and F_0^- are vertex-disjoint subgraphs of G, and $V(D)$ intersects $B_G(V(\bigcup\mathscr{C}), r-2d)$ by Claim 19, it will follow that D is the union of pairwise internally disjoint paths $Q_1\cup\cdots\cup Q_{2t}$ with $t\geqslant 1$ such that

- the start vertex of Q_i is the end vertex of Q_{i-1} for all $i \in [2t]$, where the indices are interpreted cyclically;
- $V(Q_1 \cup Q_3 \cup \cdots \cup Q_{2t-1}) = V(D) \cap B_G(V(\bigcup \mathscr{C}), r-d)$;
- for all $i \in [t]$, $\mathrm{len}(Q_{2i}) \geqslant 2$ and the inner vertices of Q_{2i} lie in $V(F_0^-)$.

Since $V(D) \cap Z = \varnothing$, $V(D) \subseteq B_G(V(\bigcup \mathscr{C}), r-d) \cup V(F_0^-)$. Furthermore, since $V(D) \cap L = \varnothing$, $V(D) \subseteq (B_G(V(\bigcup \mathscr{C}), r-d) \setminus L) \cup V(F_0^-)$. Now (6) implies $V(D) \nsubseteq B_G(V(\bigcup \mathscr{C}), r-d) \setminus L$, and since F_0^- is an induced subgraph of G, (8) implies $V(D) \nsubseteq V(F_0^-)$. This completes the preliminary step.

To show (ii), it now suffices to show that $\{Q_2, Q_4, \ldots, Q_{2t}\}$ is a collection of problematic admissible ears. Consider any $Q \in \{Q_2, Q_4, \ldots, Q_{2t}\}$ and let Q' be the path obtained from Q by deleting its endpoints. Since $V(Q') \subseteq V(F_0^-)$ and F_0^- is an induced subgraph of G, Q' is a path in F_0^-. Therefore since the endpoints of Q lie in $B_G(V(\bigcup \mathscr{C}), r-d) = \bigcup_{i \in [p]} V(U_i)$, Q is an ear. To see that Q is admissible, note that since $Q \subseteq D$ and $V(D) \cap B_G(M, r) = \varnothing$, $V(Q) \cap B_G(M, r) = \varnothing$, thus $V(W_Q) \subseteq B_G(V(Q), r-d)$ implies that $V(W_Q) \cap B_G(M, d) = \varnothing$. Finally, if Q is non-problematic, then (16) implies $B_G(X_3, r) \cap V(Q) \neq \varnothing$, therefore $B_G(X_3, r) \cap V(D) \neq \varnothing$, a contradiction. ∎

Let $\mathcal{E}$ be the set of all pairs $uv \in \binom{V(G)}{2}$ such that there exists a problematic admissible ear whose set of endpoints is $\{u, v\}$. Consider the auxiliary graph $G_{\mathrm{Aux}} \coloneqq G_R \cup \mathcal{E}$.

Let $\mathcal{D}$ be the set of cycles D in G that satisfy $B_G(M \cup X_3, r + \lfloor d/2 \rfloor) \cap V(D) = \varnothing$ and $\mathrm{len}(D) \geqslant \ell$. For each $D \in \mathcal{D}$, let $H_D \coloneqq G_{\mathrm{Aux}}[B_{G_{\mathrm{Aux}}}(V(D) \cap B_G(V(\bigcup \mathscr{C}), r-d), \lceil d/2 \rceil)]$. Observe that by decomposing $D \in \mathcal{D}$ into a union of paths $Q_1 \cup \cdots \cup Q_{2t}$ as per Claim 28, we may write

$$H_D = G_{\mathrm{Aux}}[B_{G_{\mathrm{Aux}}}(V(Q_1 \cup Q_3 \cup \cdots \cup Q_{2t-1}), \lceil d/2 \rceil)]. \tag{18}$$

Claim 29. *For every pair $D, D' \in \mathcal{D}$ with $V(H_D) \cap V(H_{D'}) = \varnothing$, $\mathrm{dist}_G(V(D), V(D')) > d$.*

Proof of Claim. Assume for a contradiction that there exists a pair $D, D' \in \mathcal{D}$ with $V(H_D) \cap V(H_{D'}) = \varnothing$ such that $\mathrm{dist}_G(V(D), V(D')) \leqslant d$. Recall that $B_G(M \cup X_3, r) \cap V(D) = \varnothing$ and $B_G(M \cup X_3, r) \cap V(D') = \varnothing$, hence we may write $D = Q_1 \cup \cdots \cup Q_{2t}$ and $D' = Q'_1 \cup \cdots \cup Q'_{2s}$ according to Claim 28. Let $i \in [2t]$ and $j \in [2s]$ such that $\mathrm{dist}_G(V(Q_i), V(Q'_j)) \leqslant d$. Let P be a shortest $(V(Q_i), V(Q'_j))$-path in G. Let u be the endpoint of P in Q_i and let u' be the endpoint of P in Q'_j.

The following shows that if E is a path in $Q_i \cup P \cup Q'_j$ with $\mathrm{len}(E) \geqslant 2$ and $V(E) \cap B_G(V(\bigcup \mathscr{C}), r-d)$ equals the set of endpoints of E, then E is a problematic admissible ear: Since P is a (D, D')-path in G of length at most d and $V(D \cup D') \cap B_G(M \cup X_3, r + \lfloor d/2 \rfloor) = \varnothing$, $V(D \cup P \cup D') \cap B_G(M \cup X_3, r) = \varnothing$, and so $V(Q_i \cup P \cup Q'_j) \cap B_G(M \cup X_3, r) = \varnothing$. Then

$$V(E) \cap B_G(M \cup X_3, r) = \varnothing. \tag{19}$$

Let E' be the path obtained from E by deleting its endpoints. Then (19) and (11) imply $V(E') \cap Z = \varnothing$, thus E' is a path in F_0^-. Therefore since the endpoints of E lie in $B_G(V(\bigcup \mathscr{C}), r-d)$, E is an ear. Next, since $V(W_E) \subseteq B_G(V(E), r-d)$, (19) implies $V(W_E) \cap B_G(M, d) = \varnothing$, thus E is admissible. Therefore (19) and (16) imply that E is problematic.

Let $V := V(Q_1 \cup Q_3 \cup \cdots \cup Q_{2t-1})$ and $V' := V(Q'_1 \cup Q'_3 \cup \cdots \cup Q'_{2s-1})$. Observe that since $V(H_D) \cap V(H_{D'}) = \varnothing$, $\mathrm{dist}_{G_{\mathrm{Aux}}}(V, V') \geqslant 2\lceil d/2 \rceil + 1 > d$.

We now show that i is even, j is even, u is an inner vertex of Q_i, and u' is an inner vertex of Q'_j. By symmetry it suffices to prove that i is even and u is an inner vertex of Q_i. Assume for a contradiction that i is odd or u is an endpoint of Q_i. Then u' is an inner vertex of Q'_j and j is even, otherwise $d \geqslant \mathrm{len}(P) \geqslant \mathrm{dist}_{G_R}(V, V') \geqslant \mathrm{dist}_{G_{\mathrm{Aux}}}(V, V') > d$, a contradiction. Note that $u \in B_G(V(\bigcup \mathscr{C}), r-d)$ and $u' \notin B_G(V(\bigcup \mathscr{C}), r-d)$, thus $u \neq u'$. Let x be an endpoint of Q'_j. Let $y \in V(P - u') \cap B_G(V(\bigcup \mathscr{C}), r-d)$ such that the path $E := xQ'_j u' \cup u'Py$ has minimum possible length. Such a y exists since u is a candidate and $u \neq u'$. Since Q'_j is an ear, the inner vertices of Q'_j are disjoint from $B_G(V(\bigcup \mathscr{C}), r-d)$, therefore $V(E) \cap B_G(V(\bigcup \mathscr{C}), r-d) = \{x, y\}$. Moreover, $\mathrm{len}(E) \geqslant 2$ since E has an inner vertex u'. Consequently, E is a problematic admissible ear, and xy is an edge of G_{Aux}. Then $xy \cup yPu$ is a (V', V)-path in G_{Aux} of length at most d, a contradiction. Hence i is even, j is even, u is an inner vertex of Q_i, and u' is an inner vertex of Q'_j.

Next, notice that if Q_i and $Q_{j'}$ intersect, $Q_i \cup Q_{j'}$ contains a path E between an endpoint of Q_i and an endpoint of $Q_{j'}$ whose inner vertices avoid $B_G(V(\bigcup \mathscr{C}), r-d)$. Since $V \cap V' = \varnothing$, Q_i and $Q_{j'}$ do not intersect at their endpoints, thus $\mathrm{len}(E) \geqslant 2$, implying that E is a problematic admissible ear. Thus there is an edge of G_{Aux} between V and V', so $d \geqslant 1 \geqslant \mathrm{dist}_{G_R}(V, V') \geqslant \mathrm{dist}_{G_{\mathrm{Aux}}}(V, V')$, a contradiction. Hence we may assume from now on that Q_i and $Q_{j'}$ are vertex disjoint.

To obtain the final contradiction of the claim, we show that there exist edges $e, e' \in E(G_{\mathrm{Aux}})$, such that e is between an endpoint of Q_i and an inner vertex of P, and e' is between an endpoint of Q'_j and an inner vertex of P. This will imply $d \geqslant \mathrm{dist}_{G_{\mathrm{Aux}}}(V, V')$, a contradiction. By symmetry, it suffices to prove the existence of e. Let Q be a shortest path in $Q_i \cup P \cup Q'_j$ between the set of endpoints of Q_i and the set of endpoints of Q'_j. Then the endpoints of Q lie in $B_G(V(\bigcup \mathscr{C}), r-d)$. If $\mathrm{len}(Q) \leqslant 1$, then $d \geqslant 1 \geqslant \mathrm{dist}_{G_R}(V, V') \geqslant \mathrm{dist}_{G_{\mathrm{Aux}}}(V, V')$, a contradiction. Hence $\mathrm{len}(Q) \geqslant 2$. If $V(Q) \cap B_G(V(\bigcup \mathscr{C}), r-d)$ equals the set of endpoints of Q, then Q is a problematic admissible ear and there exists an edge of G_{Aux} between the endpoints of Q, which implies $d \geqslant 1 \geqslant \mathrm{dist}_{G_{\mathrm{Aux}}}(V, V')$, a contradiction. Hence it may be assumed that some inner vertex of Q lies in $B_G(V(\bigcup \mathscr{C}), r-d)$. Since $Q \subseteq Q_i \cup P \cup Q'_j$ and Q_i and Q'_j are ears, such a vertex lies in $V(P)$. Let x be the endpoint of Q in Q_i. Let $y \in V(P) \cap B_G(V(\bigcup \mathscr{C}), r-d)$ such that the path $E := xQu \cup uPy$ has minimum possible length. Since the inner vertices of Q_i and Q'_j are disjoint from $B_G(V(\bigcup \mathscr{C}), r-d)$, u and u' are not in $B_G(V(\bigcup \mathscr{C}), r-d)$, thus y is an inner vertex of P. Furthermore, $V(E) \cap B_G(V(\bigcup \mathscr{C}), r-d) = \{x, y\}$. Also, $\mathrm{len}(E) \geqslant 2$ since E has an inner vertex u. It follows that E is a problematic admissible ear, and $e := xy$ is an edge of G_{Aux} between an endpoint of Q_i and an inner vertex of P, as required. ∎

Claim 30. *U is a $\bigcup \mathscr{C}$-supported BFS-spanning subgraph of G_{Aux}.*

Proof of Claim. Recall that for every edge $e \in \mathcal{E}$, both endpoints of e have distance exactly $r-d$ from $V(\bigcup \mathscr{C})$ in G. Hence $\mathrm{dist}_{G \cup \mathcal{E}}(V(\bigcup \mathscr{C}), v) = \mathrm{dist}_G(V(\bigcup \mathscr{C}), v)$ for all $v \in V(G \cup \mathcal{E}) = V(G)$. Consequently, every $\bigcup \mathscr{C}$-supported BFS-spanning subgraph of $G[B_G(V(\bigcup \mathscr{C}), R)]$

is a $\bigcup\mathscr{C}$-supported BFS-spanning subgraph of $(G\cup\mathcal{E})[B_{G\cup\mathcal{E}}(V(\bigcup\mathscr{C}),R)]$. Therefore since $G[B_G(V(\bigcup\mathscr{C}),R)]=G_R$ and $(G\cup\mathcal{E})[B_{G\cup\mathcal{E}}(V(\bigcup\mathscr{C}),R)]=G_R\cup\mathcal{E}=G_{\text{Aux}}$, U is a $\bigcup\mathscr{C}$-supported BFS-spanning subgraph of G_{Aux}. ∎

Claim 31. *For every $D\in\mathcal{D}$, H_D is a non-null connected subgraph of G_{Aux} such that every edge $e\in E(H_D)$ satisfies* $\text{span}(G_{\text{Aux}},\bigcup\mathscr{C},U,e)\leqslant\ell$.

Proof of Claim. Consider any $D\in\mathcal{D}$ and write $D=Q_1\cup\cdots\cup Q_{2t}$ as per Claim 28. First note that $\varnothing\neq V(Q_1\cup Q_3\cup\cdots\cup Q_{2t-1})=V(D)\cap B_G(V(\bigcup\mathscr{C}),r-d)\subseteq V(G_R)=V(G_{\text{Aux}})$. Now since $\{Q_2,Q_4,\ldots,Q_{2t}\}$ is a collection of problematic admissible ears, for each $i\in[t]$ there exists an edge in G_{Aux} between the endpoints of Q_{2i}. Consequently, $V(Q_1\cup Q_3\cup\cdots\cup Q_{2t-1})$ is a non-empty connected set of vertices in G_{Aux}. Then by (18), H_D is a non-null connected subgraph of G_{Aux}. We now show that $\text{span}(G_{\text{Aux}},\bigcup\mathscr{C},U,e)\leqslant\ell$ for every edge $e\in E(H_D)$. Consider any edge $e\in E(H_D)$. Since $H_D\subseteq G_{\text{Aux}}=G_R\cup\mathcal{E}$, either **(1)** $e\in\mathcal{E}$ or **(2)** $e\in E(G_R)$.

Case (1) $e\in\mathcal{E}$: Then for some $i\in[p]$ there exists an i-problematic admissible ear E that has the same endpoints as e. Then E is an (i,i)-ear and there exists a cycle in $W_E\cup C_i$ of length less than ℓ. Therefore since $\text{len}(C_i)\geqslant\ell$, the endpoints of the walk W_E have distance less than ℓ in C_i, which implies $\text{span}(G_{\text{Aux}},\bigcup\mathscr{C},U,e)<\ell$ as required.

Case (2) $e\in E(G_R)$: Let $V:=V(D)\cap B_G(V(\bigcup\mathscr{C}),r-d)$. Recall that $W\subseteq M$ by (5) and (10). We show that an endpoint of e is not in W, which implies that $\text{span}(G_{\text{Aux}},\bigcup\mathscr{C},U,e)=\text{span}(G_R,\bigcup\mathscr{C},U,e)\leqslant\ell$ as required. Recall that $e\in E(H_D)$. Let P be a shortest path in G_{Aux} between V and the set of endpoints of e. Then $\text{len}(P)\leqslant\lceil d/2\rceil$ and $e\notin E(P)$. Let v be the endpoint of e in P. If $P\subseteq G_R$, then $B_G(M,r+\lfloor d/2\rfloor)\cap V(D)=\varnothing$ implies that v is not in $B_G(M,r+\lfloor d/2\rfloor-\text{len}(P))$. Therefore $v\notin W$ as required. On the other hand, $P\nsubseteq G_R$, so P contains an edge from $\mathcal{E}$. Let $xy\in E(P)\cap\mathcal{E}$ such that $\text{dist}_P(v,\{x,y\})$ is minimum. Then $\text{dist}_G(v,\{x,y\})\leqslant\lceil d/2\rceil$. Let E be a problematic admissible ear whose set of endpoints is $\{x,y\}$. Since E is admissible, $V(W_E)\cap B_G(M,d)=\varnothing$, thus $\{x,y\}\subseteq V(W_E)$ implies that $\{x,y\}\cap B_G(W,d)=\varnothing$. Therefore since $\text{dist}_G(v,\{x,y\})\leqslant\lceil d/2\rceil$, $v\notin W$ as required. ∎

Now, $\mathscr{C}$ is a non-empty collection of pairwise vertex-disjoint cycles in $G\cup\mathcal{E}$ and, by Claim 30, U is a $\bigcup\mathscr{C}$-supported BFS-spanning subgraph of G_{Aux}. Note that the earlier definition of G_{Aux} is equivalent to $(G\cup\mathcal{E})[B_{G\cup\mathcal{E}}(V(\bigcup\mathscr{C}),R)]$. Furthermore by Claim 31, $\mathcal{H}:=\{H_D:D\in\mathcal{D}\}$ is a collection of non-null connected subgraphs of G_{Aux} such that $\text{span}(G_{\text{Aux}},\bigcup\mathscr{C},U,e)\leqslant\ell$ for every $e\in E(\bigcup\mathcal{H})$. Hence we may proceed by cases depending on the outcome of Lemma 15.

Case (i) $\mathcal{H}$ contains k pairwise vertex-disjoint members: Then there exists $\mathcal{D}'\subseteq\mathcal{D}$ with $|\mathcal{D}'|=k$ such that $V(H_A)\cap V(H_B)=\varnothing$ for all distinct $A,B\in\mathcal{D}'$. By Claim 29, $\mathcal{D}'$ is a d-packing of k cycles in G, and by definition of $\mathcal{D}$, each of these cycles has length at least ℓ. Hence G contains a d-packing of k cycles each of length at least ℓ, a contradiction.

Case (ii) there exists $Y\subseteq V(\bigcup\mathscr{C})$ with $|Y|\leqslant k-1+|\mathscr{C}|$ such that for $X_4':=B_{\bigcup\mathscr{C}}(Y,\lfloor\ell/2\rfloor)$, $B_U(X_4',R)\cap V(H)\neq\varnothing$ for all $H\in\mathcal{H}$. Since $|\mathscr{C}|\leqslant p\leqslant k-1$, $|Y|\leqslant 2(k-1)$. Let $X_4:=\text{pad}(\bigcup\mathscr{C},Y,2\lceil d/2\rceil+1,\lceil\ell/2\rceil)$. Then by Corollary 9,

$$|X_4|\leqslant|Y|(2\lceil\ell/2\rceil+1)\leqslant 2(k-1)(2\lceil\ell/2\rceil+1). \tag{20}$$

Let $X := X_0 \cup X_1 \cup X_2 \cup X_3 \cup X_4$. Then (1, 4, 7, 17, 20) imply

$$\begin{aligned}|X| &\leqslant |X_0| + |X_1| + |X_2| + |X_3| + |X_4| \\ &\leqslant k - 1 + (s(k)-1)(2\lceil \ell/2 \rceil + 1) + 2(k-1)(2\lceil \ell/2 \rceil + 1) \\ &\qquad + 50(s(k) + 2(k-1) - 1)(2\lceil \ell/2 \rceil + 1) + 2(k-1)(2\lceil \ell/2 \rceil + 1) \\ &= (51s(k) + 104k - 155)(2\lceil \ell/2 \rceil + 1) + k - 1 \\ &= f(k,\ell). \end{aligned} \tag{21}$$

By (10),

$$\begin{aligned} B_G(M \cup X_3, r + \lfloor d/2 \rfloor) \cup B_G(X_4, r - \lfloor d/2 \rfloor) &\subseteq B_G(X, R + 2r + d + \lfloor d/2 \rfloor) \\ &= B_G(X, 20d) \\ &= B_G(X, g(d)). \end{aligned} \tag{22}$$

Claim 32. *$G - B_G(X, g(d))$ has no cycle of length at least ℓ.*

Proof of Claim. Consider any cycle D in G of length at least ℓ. We show that $B_G(X, g(d)) \cap V(D) \neq \varnothing$. If $B_G(M \cup X_3, r + \lfloor d/2 \rfloor) \cap V(D) \neq \varnothing$, then (22) implies $B_G(X, g(d)) \cap V(D) \neq \varnothing$ as required. Hence it may be assumed that $B_G(M \cup X_3, r + \lfloor d/2 \rfloor) \cap V(D) = \varnothing$, in other words, $D \in \mathcal{D}$. Let $u \in B_U(X_4', R) \cap V(H_D)$ as promised by the case (ii) above. Since $X_4' \subseteq V(\bigcup \mathscr{C})$, there exists $x \in X_4'$ such that $x = \mathrm{proj}_U(u)$. Since $u \in V(H_D) = B_{G_{\mathrm{Aux}}}(V(D) \cap B_G(V(\bigcup \mathscr{C}), r - d), \lceil d/2 \rceil)$, there exists $v \in V(D) \cap B_G(V(\bigcup \mathscr{C}), r - d)$ and a path $q_0 q_1 \cdots q_s$ in H_D such that $q_0 = v$, $q_s = u$, and $s \leqslant \lceil d/2 \rceil$. Then by Claim 31,

$$\begin{aligned} \mathrm{dist}_{\bigcup \mathscr{C}}(\mathrm{proj}_U(v), x) &\leqslant \sum_{i=1}^{s} \mathrm{dist}_{\bigcup \mathscr{C}}(\mathrm{proj}_U(q_{i-1}), \mathrm{proj}_U(q_i)) \\ &= \sum_{i=1}^{s} \mathrm{span}(G_{\mathrm{Aux}}, \bigcup \mathscr{C}, U, q_{i-1}q_i) \leqslant s\ell \leqslant \lceil d/2 \rceil \ell. \end{aligned}$$

Furthermore, since $x \in X_4' = B_{\bigcup \mathscr{C}}(Y, \lfloor \ell/2 \rfloor)$,

$$\mathrm{dist}_{\bigcup \mathscr{C}}(\mathrm{proj}_U(v), Y) \leqslant \mathrm{dist}_{\bigcup \mathscr{C}}(\mathrm{proj}_U(v), x) + \mathrm{dist}_{\bigcup \mathscr{C}}(x, Y) \leqslant \lceil d/2 \rceil \ell + \lfloor \ell/2 \rfloor \leqslant (2\lceil d/2 \rceil + 1)\lceil \ell/2 \rceil.$$

Since $X_4 = \mathrm{pad}(\bigcup \mathscr{C}, Y, 2\lceil d/2 \rceil + 1, \lceil \ell/2 \rceil)$, Corollary 9 implies $\mathrm{dist}_{\bigcup \mathscr{C}}(\mathrm{proj}_U(v), X_4) \leqslant \lceil d/2 \rceil$. Therefore since $v \in B_G(V(\bigcup \mathscr{C}), r - d)$, $v \in B_U(\mathrm{proj}_U(v), r - d)$, thus $v \in B_{\bigcup \mathscr{C} \cup U}(X_4, r - \lfloor d/2 \rfloor)$. Recall that $v \in V(D)$, thus we have shown that D has a vertex in $B_G(X_4, r - \lfloor d/2 \rfloor)$. Then by (22), $B_G(X, g(d)) \cap V(D) \neq \varnothing$, as desired. ∎

Claim 32 and (21) imply that the theorem holds. □

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AI Statement

The authors proved all results in this paper and wrote the manuscript. They used GPT-6 Astra for minor language edits, typographical corrections. Before journal submission, they asked GPT-6 Astra for a review of the manuscript for errors and inconsistencies. The review identified no major errors. The authors corrected all minor inconsistencies it identified and also simplified the proofs in several places.

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