Forbidding anticomplete planar minors: Induced Erdős–Pósa property and Maximum Independent Set in QP *

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Abstract

The Erdős–Pósa theorem asserts that every graph G with no k disjoint cycles contains a set X of $f(k)$ vertices such that $G \setminus X$ has no cycle. Robertson and Seymour showed that this *Erdős–Pósa property* also holds for H-minor models of any planar graph H. Equivalently, if G has no k minor models of H pairwise at distance at least 1 (i.e. disjoint), then one can remove $f(k, H)$ balls of radius 0 (i.e. vertices) to make the graph H-minor free.

We show that this coarse graph theory point of view generalizes to distance at least 2 versus radius 1 balls, yielding the *induced Erdős–Pósa property* for planar minors. Namely, every graph G which does not contain k pairwise non-adjacent minor models of a planar graph H (we say that G is *kH-free*) can be made H-minor free by removing $f(k, H)$ neighborhoods. The proof relies on the fact that sparse kH-free graphs have linearly many independent large protrusions. The same method gives that sparse kH-free graphs can be made H-minor free by deleting $O(\log n)$ vertices (and thus have logarithmic treewidth). This gives a quasi-polynomial algorithm for the Maximum Independent Set problem for kH-free graphs.

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1 Introduction

The Erdős–Pósa theorem [20] asserts that every graph G with no k disjoint cycles contains a set X of $f(k)$ vertices such that $G \setminus X$ is a forest. In other words, the minimum feedback vertex set is functionally equivalent to the maximum packing of cycles. This bridged duality gap is referred to as the *Erdős–Pósa property* (EPP), and the term is now used for a vast collection of analogous results relating the packing and hitting number of more general objects. It often has direct algorithmic applications. For instance, Maximum Independent Set (MIS) is polynomial-time solvable in graphs without k disjoint cycles, since their feedback vertex set, and hence their treewidth, is bounded.

To this day, variants of cycles have constituted a rich playing ground for the Erdős–Pósa property, in terms of both positive and negative results. On the positive side, the EPP is known to hold for cycles with length constraints [45, 34], holes [30], cycles passing through prescribed vertices [29, 40], and even oriented cycles [42]. On the negative side, odd cycles in an Escher Wall pairwise intersect yet their hitting set is unbounded [41], induced cycles of length at least 5 do not have the EPP [30], and neither do cycles of prime lengths [26]. If we wish to generalize the EPP to more complex objects, one can observe that cycles are K_3-minor models. It is then natural to ask if minor models of some fixed graph H have the EPP. This is not always the case since toroidal grids do not contain two disjoint K_5-minor models, whereas the minimum hitting set of K_5-minor models has unbounded size. Arguably, the most striking result in this field is that H-minor models have the EPP if and only if H is planar. This was shown by Robertson and Seymour [43], and the "if" implication is a direct consequence of the grid minor theorem: either a graph has a large grid minor, and it contains k disjoint H-minor models, or it has bounded treewidth, in which case a simple divide and conquer argument gives a hitting set of bounded size.

A recent thread in graph structure has been to generalize classical results to the *coarse* setting. For hitting versus packing problems, such analogues look at replacing disjointness conditions with large distance ones, and vertices with small radius balls. In that direction, a natural coarse EPP for cycles was conjectured in [3], and also by Chudnovsky and Seymour (see also [25]): either a graph G contains a packing of k cycles at pairwise distance at least d, or there exist $f(k)$ balls of radius $g(d)$ hitting all cycles of G. Strikingly, this was very recently confirmed by Dujmović, Joret, Micek and Morin [18], who showed the above with $g(d) = 19(d-1)$. Aiming to push this result to more general structures, Albrechtsen and Davies [4, 7.6] recently conjectured the coarse EPP for planar minors (analogous to [43]): if G does not contain k minor models of H at pairwise distance at least d, then $f(k)$ balls of radius $c_H d$ suffice to hit all H-minor models. As we will see, Theorem 1 confirms in a strong sense their conjecture for $d = 2$. A similar conjecture was made in [3] corresponding to the case $d = 2$ above, but asking to pack *induced* minor models of a planar graph[1]. Unfortunately, coarse analogues of Menger's theorem, fundamental to much of classical graph structure, were recently ruled out, disproving a conjecture posed in [25] and [5]. Indeed, Nguyen, Scott and Seymour [36] (see also [37]) showed that for any $d \geq 3, k \geq 3$, and for every N, there exist graphs with no k X, Y-paths at pairwise distance at least d, but no N balls of radius N whose deletion disconnects X from Y.

While the lack of a Menger analogue seems to predict a bleak future for coarse graph theory, the case $d = 2$ (asking for non-adjacent paths) remains open even with balls of radius 1. It appears that a wide range of classical results could still be generalized to this *induced setting*, relating objects constrained by non-adjacency with (closed) neighborhoods[2]. For instance, Gartland and Lokshtanov [24] conjecture that for any planar H, graphs forbidding H as an induced

[1] Another positive hitting versus packing result is the recent proof of a coarse Gallai theorem [17].

[2] It is often easy to construct counter-examples ruling out a relation between usual (vertex) hitting sets and induced structure, see [15], and the first natural relaxation is to allow for neighborhoods.

minor admit balanced separators consisting of few neighborhoods. Abrishami, Czyżewska, Kluk, Pilipczuk, Pilipczuk, Rzążewski [2] further conjecture that classes with such separators admit tree-decompositions whose bags can be covered by few bounded radius balls, which was recently confirmed in several regimes [13, 12]. The latter question is still open for radius 1, that is, when asking to cover bags by few neighborhoods, see the conclusion of [13], which combined with a weighted version of [24] may yield an induced variant of the grid minor theorem. Coming back to packing versus hitting problems, the most pressing question in the induced setting is the induced Menger conjecture, formulated in [28], stating that graphs with no k pairwise non-adjacent paths between two vertex sets admit a set of $f(k)$ neighborhoods disconnecting them.

The *induced Erdős–Pósa property* asks for a set of pairwise non-adjacent copies of an object, or a bounded number of neighborhoods hitting all copies (a *dominating set* of the object). In this setting, the first proof of an induced EPP was obtained by Ahn, Gollin, Huynh and Kwon [3] for (non-induced) cycles of length at least t with a dominating set of size $O(tk \log k)$. This is particularly remarkable as the quasilinear dependence on k matches the EPP for cycle packing (for fixed t). In view of the fact that the classical EPP for cycles generalizes to planar minors [43] and [16], we ask the question for the induced EPP: do graphs forbidding k independent copies of H admit a hitting set of H-minor models consisting of $f(k)$ neighborhoods?

Our first result is that planar minors admit the induced Erdős–Pósa property. We say a graph G is *kH-free* if it does not contain k pairwise non-adjacent (*independent*) H-minor models.

Theorem 1. *Given a planar graph H, there exists a function f such that for every kH-free graph G, there exists $X \subseteq V(G)$ with $|X| \leq f(k)$ such that $G \setminus N[X]$ is H-minor free.*

To the best of our knowledge, Theorem 1 is one of the few results on the EPP for such a general family of objects. Apart from planar minors admitting the EPP [43], Liu [33] showed that *all* minors admit the "half-integral" EPP (see also [38]), and Amiri, Kawarabayashi, Kreutzer and Wollan [6] showed the EPP for butterfly (or topological) minors of the cylindrical grid in digraphs. In particular, Theorem 1 confirms the planar coarse EPP conjecture of [16] for $d = 2$ in a strong form. Let us also note that [15] very recently considered the induced EPP for induced *objects*, showing the induced EPP for long holes and long theta induced minors. They conjecture more generally that for any planar H, induced minor models of H satisfy the induced EPP (a strenghtening of [3]), which corresponds to replacing "minor" with "induced minor" in Theorem 1.

The obstacles to coarse graph theory are not the only motivation to understand the induced setting. On the algorithmic side, the tractability of hard problems in classes forbidding some induced substructure has a long history. The most studied problem there is Maximum Independent Set, both for exact algorithms and approximations [14, 46, 39, 27], with the most general conjecture that planar H-induced minor free graphs admit a polynomial-time algorithm for MIS [24]. When non-adjacency is imposed *between* objects, as for kH-freeness, some geometric problems naturally involve instances that forbid independent substructures. For example, given as input some finite $X \subseteq \mathbf{R}^3$, the computation of a maximum size subset of X whose elements are pairwise at distance less than 1 admits an EPTAS, see [9] and [10]. The algorithm consists of computing the Maximum Independent Set in the distance at least one graph, and crucially relies on the fact that such graphs forbid two independent odd cycles. In this vein, one of the basic open questions is the computation of MIS in polynomial time for graphs without k non-adjacent cycles (i.e. kK_3-free graphs). A quasi-polynomial time (QP) algorithm was proposed in [11].

Our main algorithmic result generalizes the quasi-polynomiality of MIS to graphs forbidding k independent planar minors.

Theorem 2. *Let H be a planar graph and k be an integer. There is an $n^{O(\log n)}$ time algorithm computing MIS in kH-free graphs.*

The MIS algorithm for kK_3-free graphs in [11] is based on the fact that sparse kK_3-free graphs (i.e. avoiding some fixed $K_{r,r}$ subgraph) have logarithmic feedback vertex set (and thus logarithmic treewidth). Therefore MIS can be computed by dynamic programming in polynomial time. Unfortunately, reducing to the sparse case incurs a QP branching preprocess. Our QP algorithm for MIS in kH-free graphs follows exactly these lines. We start with a reduction to the sparse case via QP branching, and then apply our second main result: sparse kH-free graphs have a logarithmic H-hitting set.

Theorem 3. *Given a planar graph H, there exists a function f such that for every kH-free graph G on n vertices with no $K_{r,r}$ subgraph, G admits an H-hitting set of size at most $f(k,r)\log n$.*

Since removing an H-hitting set leaves a graph with bounded treewidth, this result directly implies that sparse kH-free graphs have logarithmic treewidth. For more classes having the logarithmic treewidth property, see [1].

Regarding Theorem 2, let us highlight one of the tantalizing questions of this domain, which is a weakening of [24, Conjecture 1.4.2], and remains open in the $2K_3$-free case.

Conjecture 4. *For any planar H and any k, the Maximum Independent Set problem is polynomial-time solvable in the class of kH-free graphs.*

Another interesting algorithmic question regarding kH-free graphs is that of recognition. There, a polynomial time algorithm for $H = K_3$ was shown by Nguyen, Scott and Seymour [35].

Structure of the paper We end this introduction with overviews of Theorem 1, Theorem 3 and Theorem 2, all crucially based on the *linear protrusion property* (Theorem 13). In Section 2 we introduce the main definitions and provide a deeper overview of our strategy. In Section 3, we formally define protrusions and their reduction, and Section 4 shows the linear protrusion property. In Section 5, we show the induced EPP for planar minors, first for connected H, then in general. Section 6 establishes the logarithmic H-hitting set for sparse graphs, and derives the QP algorithm for MIS in kH-free graphs.

1.1 Overview of the induced Erdős–Pósa property for planar minors

For the sake of exposition, we assume throughout this introduction that H is connected. A natural strategy to show that a (huge) bounded number of neighborhoods hit all H-minor models in a kH-free graph is to consider a minimum counterexample G, and argue that some reduction can be performed. Our first step is to reduce to the sparse case. Say that the *H-girth* of G is the size of the smallest H-minor model in G (the K_3-girth then corresponds to the usual girth). Note that if there is a small H-minor model $\mathcal{H}$ in G, the graph $G \setminus N[\mathcal{H}]$ is $(k-1)H$-free, thus G is not a minimum counterexample (since we can choose a huge bound). Thus we can assume that G has arbitrarily high H-girth.

Now, an important remark is that there are sparse kH-free graphs with unbounded treewidth. Examples of high girth and high treewidth graphs with no two independent cycles (later on called *death stars* for unknown reasons) can be found in [11]. So G can still be rather complex, and we need to reduce it further. Isolated or degree one vertices can be deleted, so we can assume that the minimum degree is 2. Then, a natural toy-problem is the following: how to reduce a kK_3-free graph with high girth?

It turns out that such graphs have long proper paths, i.e. with internal vertices of degree 2 (see technical overview in Section 2.3). This shows Theorem 1 for kK_3-free graphs, since one can

contract an edge in a long proper path, contradicting the minimality of the counterexample. More generally, every kH-free graph with large H-girth has a p-vertex cutset A partitioning $V(G) \setminus A$ into two arbitrarily large sides V_1, V_2 (for some fixed p, see Section 2.3). Note that if both V_1 and V_2 contain an H-minor model, then they are both $(k-1)H$-free, contradicting the minimality of the counterexample (if the bound is chosen larger than twice the bound for the $(k-1)H$-free case plus p). Thus we can assume that $G[V_1]$ is arbitrarily large, H-minor free (thus of bounded treewidth) and has boundary size at most p. This is the definition of a p-protrusion P, with boundary A, a very popular tool in FPT algorithms, see [21], [22], and [31]. The key-property of large p-protrusions is that they can be reduced while preserving some properties of G. Therefore, similarly to the contraction of a proper path in the kK_3-case, the protrusion P can be reduced, thus contradicting the minimality of G. The full proof of existence of large p-protrusions can be found in Theorem 13.

Let us hint at how to reduce the p-protrusion $G[V_1]$. The crucial observation is that no minimum size set X such that $N[X]$ hits all H-minor models intersects V_1 on more than p vertices, otherwise swapping $X \cap V_1$ for the boundary of V_1 would be a smaller hitting set. So we can encode all the possible types of intersections of $G[V_1]$ and the optimal solution in a bounded way. Therefore if V_1 is too large, it can be replaced by a smaller graph. The technical overview of this protrusion reduction argument is discussed in Section 2.2, and then proved in Section 3.

1.2 Overview of the logarithmic hitting set, and of the QP algorithm for MIS

To compute MIS, we follow our usual recipe and preprocess our input graph G to reduce it to (not too many) sparse instances. Let us illustrate this for $2H$-free graphs. Since a small H-minor model $\mathcal{H}$ (say of size at most s) is adjacent to all other H-minor models, there is a vertex v of $\mathcal{H}$ which is a neighbor of at least a fraction of $1/s$ of the number of small H-minor models. Thus, deleting $N[v]$ reduces the total number of small H-minor models by a factor $1 - 1/s$ (and there are at most n^s of them). Therefore, choosing v as a branching vertex to compute MIS leads to a recursion tree with a quasi-polynomial number of leaves, each leaf corresponding to a graph without a small H-minor model. We now need to solve MIS for sparse kH-free graphs.

The key is Theorem 3, which shows that sparse kH-free graphs have a logarithmic size hitting set X for all H-minor models. With this in hand, we just run over all possible sets X and all possible intersections I of X with a MIS, and extend I to $G \setminus X$ (which has bounded treewidth since H is planar) using dynamic programming. This again takes quasi-polynomial time. Therefore the main difficulty is to prove Theorem 3.

The first step is to observe that the proof of Theorem 13, which shows the existence of a large p-protrusion in kH-free graphs with large H-girth, yields in fact a *linear* number of independent large p-protrusions (see the overview in Section 2.3). Ignoring the at most $k-1$ protrusions containing an H-minor model, we may assume a linear number of them have H-minor free interiors. Then, the minimum size L of our p-protrusions is set in such a way that each of them can be reduced to a smaller one while preserving both kH-freeness and the size of a minimum H-hitting set (see the overview in Section 2.2). We can then reduce this linear set of protrusions all at once, in order to form a new graph G' which has the same size of minimum H-hitting set. Note that the number of vertices of G' is a fraction of $|V(G)|$ since we strictly reduced a linear number of sets. The problem is that the H-girth of G' may have decreased, and we cannot apply Theorem 13 again. Crucially, note that we are not in the process of *computing* an H-hitting set of G, but just showing that one of logarithmic size exists. We can then focus on G' and forget G.

To iterate our reduction, we must now restore the H-girth of G' through a sparsification step, by deleting only a bounded number of vertices. This is made possible by the fact that G' cannot become too dense. In particular, if the value of p is chosen (among other properties) such that $K_{p,p}$

contains an H-minor model, the graph G also does not contain $K_{2p,2p}$ (since this would be a small H-minor model). We claim our parallel p-protrusion reduction from G to G' also leaves G' without $K_{2p,2p}$ subgraphs (though it may contain a $K_{p,p}$). Indeed, $K_{2p,2p}$ cannot cross the boundary of a p-protrusion, since deleting p vertices cannot disconnect it. It also cannot live inside a replacement p-protrusion, as this would yield a $K_{p,p}$, thus an H minor in its interior, while reductions preserve H-minor-freeness. Therefore, while we may not control the H-girth of G', at least G' is still somewhat sparse.

Now, our sparsification step (which is very classical in this domain) consists in deleting a bounded number of vertices to reduce G' to a new graph with no small H-minor model. If we can achieve this, repeating the process (reduction then sparsification) iterates $O(\log n)$ times the deletion of a bounded set of vertices, showing that our starting graph G has a logarithmic size H-hitting set. To justify this sparsification step, we show that if a kH-free graph has no $K_{2p,2p}$ subgraph, then one can delete $f(k,\ell,p)$ vertices to suppress all small H-minor models. It suffices to observe that a maximum size N packing of small H-minor models induces a graph with $\Omega(N^2)$ edges and $O(N)$ vertices: every subset of k disjoint small H-minor models contains an edge joining two of them. Such a quadratic density gives a $K_{2p,2p}$ if N is large enough. So N is bounded, and therefore deleting at most N small H-minor models leaves G' with large H-girth.

Concluding remarks. Our proofs virtually never consider the structure of H, barely that of G, and do not effectively construct any hitting or dominating set. This is graph theory without graphs, just consisting of parameter tuning. It is quite surprising to obtain some results and algorithms without any insight on the fine-grained structure of kH-free graphs, apart from the crucial linear protrusion property. Compared to the original EPP of planar graphs, in which treewidth fully explains the phenomenon, this study does not (yet?) reveal any new complexity or width parameter hidden behind induced packing of planar graphs. Still, the linear protrusion property can be a useful tool for other graph classes.

2 Preliminaries and overview of the linear protrusion property

2.1 Preliminaries

We generally assume in this paper that $G = (V, E)$ is a graph on n vertices. When X is a subset of vertices of G, we denote by $G[X]$ the subgraph of G *induced* by X. Given a partition $\mathcal{A} = A_1, \ldots, A_h$ of V, the quotient of G by $\mathcal{A}$, denoted $G/\mathcal{A}$, is the graph obtained by contracting each A_i into a single vertex, adjacent to all neighbors of A_i. We denote by $N(v)$ the neighborhood of a vertex v, and by $N[v]$ the closed neighborhood $N(v) \cup \{v\}$. The neighborhood of a set of vertices X is $N(X) := (\bigcup_{x\in X} N(x)) \setminus X$ and its closed neighborhood is $N[X] := \bigcup_{x\in X} N[x]$.

Two disjoint sets of vertices X, Y in a graph G are *independent* if there is no edge xy such that $x \in X$ and $y \in Y$. We also call *independent* a family $\mathcal{F}$ of pairwise independent subsets. When all subsets in $\mathcal{F}$ are singletons, we speak of an *independent set* of G. It is a *maximum independent set* (MIS) of G if it has maximum size.

A *class* $\mathcal{C}$ of graphs is always supposed to be closed under taking induced subgraphs. A class $\mathcal{C}$ is *sparse* if there exists r such that no graph in $\mathcal{C}$ contains $K_{r,r}$ as a subgraph. A class $\mathcal{C}$ is *δ-degenerate* if every graph in $\mathcal{C}$ on n vertices has at most δn edges. A class $\mathcal{C}$ has *linear neighborhood complexity* with parameter c if for every graph G in $\mathcal{C}$ and for every non empty subset X of vertices of G, the size of $\{N(v)\cap X : v \in V(G)\}$ is at most $c|X|$. In other words, the number of distinct neighborhoods of vertices of G in X is linear in the size of X.

Given some graph H, an *H-minor model* in G is a set $\mathcal{H}$ of $h := |V(H)|$ disjoint connected subsets $\{X_1, \ldots, X_h\}$ of G such that $G[\cup\mathcal{H}]/\mathcal{H}$ contains H as a (spanning) subgraph. When $G[\cup\mathcal{H}]/\mathcal{H}$ is isomorphic to H, we speak of an *induced minor model*. A graph G is *H-minor free* if it does not contain H as a minor. By extension, a class $\mathcal{C}$ is *H-minor free* if all graphs in $\mathcal{C}$ are H-minor free. In a graph G, an *H-hitting set* is a set X of vertices such that $G \setminus X$ is H-minor free. We denote by $\tau_H(G)$ the minimum size of an H-hitting set. An *H-dominating set* is a set X of vertices such that $G \setminus N[X]$ is H-minor free. We denote by $\gamma_H(G)$ the minimum size of an H-dominating set.

Given some graph H and an integer k, a graph G is *kH-free* if for any set $\mathcal{H}_1, \ldots, \mathcal{H}_k$ of disjoint H-minor models in G, there are some $i \neq j$ such that there exists an edge between $\mathcal{H}_i$ and $\mathcal{H}_j$. We will also say that there are no k disjoint *independent* H-minor models in G. By extension, a class $\mathcal{C}$ is *kH-free* if all G in $\mathcal{C}$ are kH-free.

Theorem 5. *For every planar graph H, H-minor free graphs have bounded treewidth.*

Proof. This follows from Robertson and Seymour grid minor theorem [43]. □

Theorem 6. *Every sparse kH-free class $\mathcal{C}$ is degenerate and has linear neighborhood complexity.*

Proof. Dvořák [19] showed that every sparse class not inducing some subdivision of a fixed graph F has bounded expansion; therefore it is degenerate, and moreover, it has linear neighborhood complexity, as shown by Gajarský et al. [23]. The conclusion follows since $\mathcal{C}$ does not induce a subdivision of the disjoint union of k independent copies of H. □

Theorem 7. *For every planar graph H, there exists a function g such that $\tau_H(G) \leq g(k, d)$ for every kH-free graph G with maximum degree d.*

Proof. Korhonen [32] showed that there exists f such that every graph with maximum degree d and treewidth at least $f(N, d)$ contains an $N \times N$ grid as an induced minor. To show that g exists, we set N such that every $N \times N$ grid induced minor contains k independent H-minor models. It follows that G has a tree-decomposition T with width less than $f(N, d)$. When H is connected, T has a bag B such that every component of $G \setminus B$ is $(k-1)H$-free. This follows by the standard argument orienting the edges of T towards a side containing $(k-1)H$. Indeed, if for some edge of the tree-decomposition, removing its corresponding adhesion (the intersection between its two bags) leaves two sides that are $(k-1)H$-free, and by induction in each, minor models of H can all be hit with $g(k-1, d)$ vertices. This yields a hitting set of size at most $f(N, d) + 2g(k-1, d)$. Otherwise, if no such edge exists, orient all edges of T towards the (unique) side containing $(k-1)H$, and note the other side is always H-free. Then, the bag corresponding to the sink of this orientation is a H-hitting set of size at most $f(N, d)$.

We now deal with the case where H has connected components $H_1, \ldots, H_C$ with $C \geq 2$, and assume that we have shown a function $g(k, d)$ for any H with fewer than C components. We can assume that G contains $2f(N, d)kC$ disjoint (even anticomplete) H_1 minors, and fix such a collection, otherwise we conclude with the connected case. We will show below how to delete at most $f(N, d)(kC-1)$ vertices so that the resulting graph contains at least kC connected components containing such an H_1, call those *good*. Let us show this is enough. Assume that once these vertices are deleted, G contains k independent copies of $H \setminus H_1$. The union of these copies intersects at most $k(C-1)$ good components. Therefore, at least k good components are not intersected, and appending the corresponding k independent copies of H_1 forms k independent copies of H. Therefore, it must be that the deletion of the $f(N, d)(kC-1)$ vertices, G became $k(H \setminus H_1)$-free, and we can conclude by induction.

We now show how to split G into enough good components. First observe the following: if at any point during the process, a connected component F of G contains at least $f(N,d)+1$ of our copies of H_1, then it can be separated into at least two good components by some S. Indeed, take a tree-decomposition of F of width less than $f(N,d)$. Consider every edge of the tree T, and if removing its corresponding adhesion (intersection of its two bags) leaves both sides with one of our initial H_1, take S to be the adhesion, and we are done. We may therefore orient every edge towards the side containing at least one copy of H_1, which exists since the adhesion intersects at most $f(N,d)$ copies. Now, consider a sink t of this orientation, and note that its bag B_t also intersects at most $f(N,d)$ copies, therefore at least one copy of H_1 is disjoint from B_t. Since H_1 is connected, this copy must lie entirely on a side corresponding to a component of $T-t$ (outside the adhesion), which would direct an edge outwards from t, and is absurd. The important points here are that removing S from G: increases the number of good components by at least one, and decreases our collection of disjoint copies of H_1 by at most $f(N,d)$.

We can now apply the following process. As long as G does not contain kC good components and it contains a component F with at least $f(N,d)+1$ copies of H_1, separate F into at least two good components as above. After $i \geq 1$ iterations, at most $if(N,d)$ of our original copies of H_1 have been hit, and the number of good components is at least $i+1$. Therefore, the process ends after at most $kC-1$ steps, meaning at least $(kC+1)f(N,d)$ copies of H_1 survive. Now, if all good components contain at most $f(N,d)$ copies, then there must be at least $kC+1$ good components, so in any case we have obtained kC good components. □

We now introduce some parameters related to our planar graph H. We set these (global) variables for the rest of the paper.

- h is the number of vertices of H and m is the maximum between h and $|E(H)|$. Note then that H is a minor of the complete bipartite graph $K_{m,m}$.
- t is the maximum treewidth of an H-minor free graph, and $p := m+2t$ will be the boundary size of our protrusions.
- By Theorem 6, kH-free graphs G with no $K_{m,m}$ subgraph have linear neighborhood complexity. Let c be such that $|\{N(v) \cap S : v \in V\}| \leq c|S|$ for every non empty $S \subseteq V(G)$.
- By Theorem 6, there exists δ such that every kH-free graphs with no $K_{m,m}$ subgraph are δ-degenerate, thus have average degree at most 2δ.

2.2 Protrusion reduction

We formally define protrusion reduction in Section 3. This subsection is just a gentle introduction to the main ideas since the protrusion literature can be a little bit intimidating and technical. A nice introduction to the topic (in the case of topological minors) can be found in [7], see also [8] for kernelization aspects. Fortunately, the main point is based on a very simple idea, which is basically the pigeonhole principle. The difficulty is to express the fact that the number of different holes for the pigeons is bounded. This is generally done using MSOL-formulas up to some bounded depth, but we will try to make this paper both kH-free and MSOL-free.

A p-protrusion $P=(A,B)$ in a graph G is such that $|A| \leq p$, $N(B) \subseteq A$ and $G[B]$ is "simple", which generally means bounded treewidth. The treewidth requirement will be satisfied in our case, but irrelevant for our proofs, so let us simply forget it. When we say that P contains a minor, we mean its interior B does. For instance, isolated vertices are 0-protrusions, degree 1 vertices are

1-protrusions, and a path P all of whose internal vertices have degree 2 (called *proper path*) is a 2-protrusion. All these objects are generally treated (both in graph theory and algorithms) in the same way: we delete or contract them to find an equivalent instance of the problem. Finding a reduction becomes much less clear when a very large set B is isolated by, say, 10 vertices. Fortunately, in our minimum counterexample strategy, we just focus on the existence of a reduction, so we can drop the (tedious) question of *how* to find them efficiently. Let us now focus on why large protrusions can be reduced.

The key-question is: does the underlying problem depend (or not) on a bounded size set of "properties" of $G[A \cup B]$? One way to capture such properties is through the concept of "folios", developed in the following, which capture possible rooted structures of $G[A \cup B]$. The type of $G[A, B]$ is thus determined by its folio; and folios have bounded size, so the total number of possible types of $G[A \cup B]$ (with respect to these properties) is at most some N. In particular, for every type i, if we let m_i be the minimum size of a p-protrusion of type i and let L be the maximum of $m_i + 1$, where $i = 1, \ldots, N$, then every p-protrusion of size at least L can be reduced to a smaller equivalent one.

The concept of folios was first introduced by Robertson and Seymour [44] in the context of minors, and we generalize it here to induced minors. Let us call f-folio of P the set of all induced minors of size at most f one can find in $G[A \cup B]$, with the additional information of the respective intersections of the connected parts of the minor models with A. The main point to check (and a good warm-up) is that if we replace a p-protrusion B by a p-protrusion with the same $(p + f)$-folio to form a new graph G', then G and G' have the same induced minors on at most f vertices.

Assuming for the sake of exposition that H is connected, recall (Section 1.1) that an H-minor free p-protrusion B cannot intersect a minimum H-hitting set on more than p elements. We now define the (s, f)-deletion folio as the set of all f-folios one can obtain in $G[(A \cup B) \setminus X]$ by deleting a set X of size at most s with the additional information $A \cap X$. Observe that the hitting set size τ_H is invariant when we replace an H-minor free p-protrusion by one with the same $(p, h + p)$-deletion folios.

To get the same invariance for the H-dominating set γ_H, we now need to encode all f-folios one can obtain by deleting a set $X \subseteq A$ of size at most s and the *closed neighborhood* of a set Y of size at most s. We need then to specify X and all possible intersections $Y \cap A$ and $N[Y] \cap A$. This is still a bounded size information, which we call (s, f)-neighborhood-deletion folio (the exact definition is slightly more technical, see Section 3). Again, the parameter γ_H is invariant when replacing an H-minor free p-protrusion by one with the same neighborhood-deletion folio (with suitable parameters).

Therefore, bounding γ_H for all kH-free graphs G reduces to showing that there exists a large p-protrusion. We now turn to the central tool of this paper, how to find a linear number of large protrusions.

2.3 Linear Protrusion Property

We illustrate how our method works in the cycle case, and discuss afterward its generalization to any planar H. The goal is to overview the proof of Theorem 13, which asserts that for every L, there is a bound g_L such that whenever a kH-free graph has H-girth more than g_L, it has a linear number of independent p-protrusions of size at least L.

Assume now that G is kK_3-free with minimum degree 2 (vertices with degree at most 1 can be easily treated separately). Our goal is to show that if G has (high) girth g, then it contains a linear number of long proper paths.

The trick is to pick some very large d and partition $V(G)$ into the vertices Y of degree at most

d and the vertices X of degree more than d. Since G is degenerate (large girth), the ratio $|Y|/|X|$ is arbitrarily close to 1. Thus, the graph $G[Y]$ is kK_3-free with degree at most d so by Theorem 7, it satisfies $\tau_{K_3}(G[Y]) \leq C$ for some constant C. Hence there is a subset Z of size at most C inside Y such that $G[Y \setminus Z]$ is K_3-minor free, thus is a forest. Observe that if the number of vertices of G is much larger than C, we still have $|Y \setminus Z|/|X \cup Z|$ arbitrarily close to 1. Rename now $X \cup Z$ as X and $Y \setminus Z$ as Y. Note that every leaf in $G[Y]$ has a neighbor in X since the minimum degree is 2.

Unfortunately, due to parameter tuning, the full proof for cycles is not much simpler than the general kH-free case. However, since $G[Y]$ is a forest, the argument is a little bit simpler to present. We discuss three possible examples for $G[Y]$ illustrating the main parts of the proof of Theorem 13. These three cases correspond to a trichotomy lemma for trees. Either a tree T on N nodes has $o(N)$ leaves and then almost all vertices have degree 2, thus there are linearly many long proper paths (our first example). Or T has $\Omega(N)$ leaves in which case either it has linearly many vertex-disjoint leaf to leaf paths (our second example), or there is an $o(N)$ subset of nodes of T whose deletion leaves connected components containing at most one leaf of T (our last example).

1. $G[Y]$ has linearly many (much more than $|X|$) vertex-disjoint long proper paths (P_i) (possibly obtained by splitting longer paths). Half of them have bounded size (since they are linearly many). If half of these latter have at least two neighbors on X, we obtain many (still much more than $|X|$) short (X, X)-paths (P_i'). Contracting these paths does not create C_4 since the girth is very high, so the linear neighborhood complexity property still holds. But then two of these P_i''s must share endvertices in X, contradicting the high girth hypothesis. So a subfamily (P_i'') consisting of a quarter of the family (P_i) has at most one neighbor on X. Since at most one edge connects P_i'' to X due to the girth hypothesis, each P_i'' path contains a proper path of G of size at least $|P_i''|/2$.

2. $G[Y]$ has many (much more than $|X|$ and n/g) disjoint connected subsets (S_i) each with at least two leaves. This would happen for instance in a binary tree. Then half of these sets have size much less than g. Hence contracting all S_i gives a C_4-free graph G' which is still kK_3-free. The contradiction appears since G' has linear neighborhood complexity and there are too many S_i, each with degree at least 2 on X.

3. $G[Y]$ has few internal vertices and many leaves. This happens for instance in a caterpillar of length $\sqrt{n}$ with internal vertices of degree $\sqrt{n}$. To solve this case, we have to change our point of view and consider the set S of leaves of $G[Y]$ versus the rest of the graph. Since each leaf has another neighbor in X, and the size of S is almost n, we contradict the linear neighborhood complexity on $V \setminus S$, or C_4-freeness.

The full proof of Theorem 13 follows the lines of the cycle case. The first step is to show that almost all of G consists of an induced subgraph $G[Y]$ with treewidth at most t. Then Theorem 12 gives a linear set of large independent $2t$-protrusions in $G[Y]$ (this is the previous trichotomy result applied to the decomposition tree). Finally, as for the path case discussed for cycles, a fraction of these protrusions provides a linear set of large p-protrusions of G.

To sum-up, a trichotomy argument shows that graphs with treewidth t have a linear number of large independent $2t$-protrusions. Then, since a kH-free graph G with large H-girth almost entirely consists of a graph with treewidth at most t, one can lift the linear protrusion property from T to G.

3 Protrusions

A *p-boundaried graph* $P = (A, B)$ is a graph on vertex set $A \cup B$, where $A \cap B = \emptyset$, and the vertices of A are enumerated by $v_1, \ldots, v_p$. We call A the *boundary* of P and $|B|$ the *size* of P. We say that two p-boundaried graphs $P = (A, B)$ and $P' = (A', B')$, where $A = \{v_1, \ldots, v_p\}$ and $A' = \{v'_1, \ldots, v'_p\}$, are *compatible* if the function $\phi(v_i) = v'_i$ for all $i = 1, \ldots, p$ is an isomorphism from $P[A]$ to $P'[A']$.

The *signature* of a subset X of vertices of P is $\sigma(X) := \{i \in [p] : v_i \in X\}$, that is the set of indices of the intersection $X \cap A$. Given some graph F on the set of vertices $[f] := \{1, \ldots, f\}$ and an induced minor model $\mathcal{M} = \{X_1, \ldots, X_f\}$ of F in $P = (A, B)$, we call *signature* of $\mathcal{M}$ the f-tuple of disjoint sets $\sigma(\mathcal{M}) := (\sigma(X_1), \ldots, \sigma(X_f))$. We say that $(F, \sigma(\mathcal{M}))$ is a *boundaried induced minor* of P and its size is f. Since a given graph F can appear in at most $(f+1)^{|A|}$ distinct boundaried induced minors of P, the set of all boundaried induced minors of P with size at most f, called the *f-folio* of P, is therefore bounded. To illustrate the notion, note that two p-boundaried graphs with the same p-folios are compatible, as one can consider the induced minor consisting of the p singleton vertices of their respective boundaries.

A *p-protrusion* $P = (A, B)$ in a graph G is a boundaried graph induced by two disjoint sets of vertices: the *boundary* A, such that $|A| \leq p$, and the *interior* B, such that $N_G(B) \subseteq A$. We will often assume that the boundary is of size exactly p. We say that a protrusion $P = (A, B)$ *contains* an H-minor model if its interior $G[B]$ does (unlike folios which also consider A). Let $A = \{v_1, \ldots, v_p\}$ be the boundary of a p-protrusion $P = (A, B)$ of G. Given another boundaried graph $P' = (A', B')$ compatible with P (where $A' = \{v'_1, \ldots, v'_p\}$), one can *replace* P by P' in G to form a new graph G' as follows: delete B from G, add a disjoint independent copy of P', and identify A with A' by contracting all pairs $\{v_i, v'_i\}$. We then say that G' is obtained by *protrusion replacement* from G, or *protrusion reduction* when the size of B' is less than the size of B. For simplicity, when P is replaced by P', we will often assume that A' is equal to A.

Lemma 8. *Let G be a graph with a p-protrusion $P = (A, B)$ and G' be obtained by replacing P with $P' = (A, B')$ with the same $(f+p)$-folio. Then G and G' have the same induced minors of size at most f.*

Proof. We just have to show that if F on vertex set $[f]$ is an induced minor of G, then it is an induced minor of G'. This is sufficient by symmetry, as G is obtained from G' by replacing P' with P. We consider a minor model $\{Y_1, \ldots, Y_f\}$ of F in G and denote by X_i the intersection of Y_i and $V \setminus (A \cup B)$ for all $i = 1, \ldots, f$. We denote by $\mathcal{Z}_i = \{Z_i^1, \ldots, Z_i^{k_i}\}$ the connected components of $G[Y_i \cap (A \cup B)]$. If $Y_i \cap A = \emptyset$, then since $G[Y_i]$ is connected, we have $k_i = 0$ (Y_i disjoint from P) or $k_i = 1$ (Y_i inside B). If $Y_i \cap A \neq \emptyset$, note that $k_i \leq |Y_i \cap A|$. In particular we always have $k_i \leq |Y_i \cap A| + 1$, and therefore $k_1 + \cdots + k_f \leq f + p$. Since P and P' have the same $(f+p)$-folios, there exist disjoint corresponding connected subsets W_i^j in $G'[A \cup B']$ such that $W_i^j \cap A = Z_i^j \cap A$ for all $j \in [k_i]$. Moreover there is an edge between W_i^j and W_a^b if and only if an edge exists between Z_i^j and Z_a^b, for all $j \in [k_i]$ and $b \in [k_a]$.

We now set $Y'_i := X_i \cup W_i^1 \cup \cdots \cup W_i^{k_i}$ for all $i = 1, \ldots, f$. Since $G[Y_i]$ is connected, $G'[Y'_i]$ is also connected. Also, $G[Y_i \cup Y_j]$ and $G'[Y'_i \cup Y'_j]$ are both connected or both not connected, for all $i, j \in [f]$. In particular, $\{Y'_1, \ldots, Y'_f\}$ forms an induced F-minor model in G'. □

Corollary 9. *Let G be a kH-free graph with a p-protrusion $P = (A, B)$ and G' be obtained by replacing P with $P' = (A, B')$ with the same $(kh+p)$-folio. Then G' is kH-free.*

Proof. The property of being H-minor free can be expressed as being H'-induced minor free for all possible supergraphs H' of H on h vertices. Similarly, the property of being kH-free can be

expressed as a bounded number of forbidden induced minors on kh vertices (all independent unions of k supergraphs of H). □

We now extend Corollary 9 in order to perform protrusion replacements which not only preserve kH-freeness but also the size of a minimum H-hitting set in G. The first difficulty here is to encode the fact that one can remove some subset X of at most s vertices. This can be done by considering all possible folios one can obtain by removing such an X (subject to all possible intersections $A \cap X$ with the boundary A). The second difficulty is that a p-protrusion P can intersect a minimum H-hitting set on many vertices. The key-remark is that if P does not contain H as a minor (by this we mean that $P[B]$ is H-minor free), then a minimum H-hitting set cannot have more than p vertices in P, since just picking the boundary would be a better choice. Note that we use the connectivity of H for this.

Given a subset of vertices X of some boundaried graph $P = (A, B)$, where A is $\{v_1, \ldots, v_p\}$, we denote by $P \setminus X = (A \setminus X, B \setminus X)$ the boundaried graph in which $A \setminus X = \{v_1, \ldots, v_p\} \setminus X$ is enumerated in the same order as in A. The *(s, f)-deletion folio* of $P = (A, B)$ is the set of all possible triples $(r, \sigma(Y), \mathcal{F})$ where $r \leq s$, Y is a subset of vertices of size r in $A \cup B$ and $\mathcal{F}$ is the f-folio of $P \setminus Y$.

Lemma 10. *Let H be a connected graph, and G be a kH-free graph with a p-protrusion $P = (A, B)$ such that $G[B]$ is H-minor free. Let G' be obtained by replacing P by $P' = (A, B')$ with the same $(kh + p)$-folio and the same $(p, h + p)$-deletion folio. Then G' is kH-free and $\tau_H(G) = \tau_H(G')$.*

Proof. By Corollary 9, the graph G' is also kH-free. We only show $\tau_H(G) \leq \tau_H(G')$, the reverse direction follows by swapping the roles of G and G'. Consider a minimum H-hitting set X' of G'. Since P and P' have the same h-folio, $G'[B']$ is also H-minor free. Since H is connected, then X' satisfies $|X' \cap (A \cup B')| \leq p$, otherwise $(X' \setminus B') \cup A$ would be a smaller hitting set. Therefore $Y' = X' \cap (A \cup B')$ has size at most p.

Since P and P' have the same $(p, h + p)$-deletion folio (applied to Y'), there exists $Y \subseteq A \cup B$ such that $Y \cap A = Y' \cap A$ and $|Y| = |Y'|$, and such that the boundaried graphs $Q = P \setminus Y$ and $Q' = P' \setminus Y'$ have the same $(h+p)$-folio. We now set $X := (X' \setminus Y') \cup Y$. Note that $G \setminus X$ is obtained by replacing the protrusion Q' in $G' \setminus X'$ by Q. Since $G' \setminus X'$ is H-minor free, by Corollary 9 (applied with $k = 1$), the graph $G \setminus X$ is also H-minor free. We have $\tau_H(G) \leq |X| = |X'| = \tau_H(G')$. □

We need to generalize Lemma 10 to the parameter γ_H. We define for this the *(s, f)-neighborhood-deletion folio* of a boundaried graph $P = (A, B)$ to be the set of all f-folios one can obtain by deleting both a subset of A and the closed neighborhood of a set of vertices of size at most s with a prescribed intersection and neighborhood intersection with A. Formally, it is the set of all possible 6-tuples $(a, b, \sigma(X), \sigma(Y), \sigma(N[Y]), \mathcal{F})$ where $a \leq s$, $b \leq s$, X is a subset of A of size a, Y is a subset of vertices of size b in $A \cup B$ and $\mathcal{F}$ is the f-folio of $P \setminus (X \cup N[Y])$.

Lemma 11. *Let H be a connected graph, and G be a kH-free graph with a p-protrusion $P = (A, B)$ such that $G[B]$ is H-minor free, and G' be obtained by replacing P by $P' = (A, B')$ with the same $(kh + p)$-folio and the same $(p, h + p)$-neighborhood-deletion folio. Then G' is kH-free and $\gamma_H(G) = \gamma_H(G')$.*

Proof. The proof follows the lines of Lemma 10. Again, by Corollary 9, since P and P' have the same $kh + p$-folio, G' is also kH-free. Now, assume that D' is a minimum H-dominating set of G', and let us build an H-dominating set of G of the same size. Since P and P' have the same h-folio, $G'[B']$ is also H-minor free, and again because H is connected, $Y' = D' \cap (A \cup B')$ has size at most p.

We denote by X' the set $N(D' \setminus (A \cup B')) \cap A$, which is the set of vertices of A dominated by D' from the outside of P'. Since P and P' have the same $(p, h+p)$-neighborhood-deletion folio (applied to the couple of sets X', Y'), there exists $Y \subseteq A \cup B$ and $X \subseteq A$ such that $X \cap A = X' \cap A$, $Y \cap A = Y' \cap A$, $N[Y] \cap A = N[Y'] \cap A$, $|Y| = |Y'|$, $|X| = |X'|$, and such that the boundaried graphs $Q = P \setminus (X \cup N[Y])$ and $Q' = P' \setminus (X' \cup N[Y'])$ have the same $(h+p)$-folio. Note that $X = X'$ since $X, X' \subseteq A$ and $X \cap A = X' \cap A$.

We set $D := (D' \setminus Y') \cup Y$. Note that $G \setminus N[D]$ is obtained by replacing the protrusion Q' in $G' \setminus N[D']$ by Q. Since $G' \setminus N[D']$ is H-minor free, by Corollary 9 (applied with $k = 1$), the graph $G \setminus N[D]$ is also H-minor free. We have $\gamma_H(G) \leq |D| = |D'| = \gamma_H(G')$. □

4 Linear Protrusion Property

Recall that a p-protrusion $P = (A, B)$ satisfies $|A| \leq p$. Two protrusions $P = (A, B)$ and $P' = (A', B')$ in a graph G are *independent* if B and B' are disjoint and independent (no edge between them). An *independent* family of p-protrusions of G is a set of pairwise independent p-protrusions of G. A class $\mathcal{C}$ of graphs has the (p, L)-*Linear Protrusion Property* (for short (p, L)-LPP) if there exists $c_L > 0$ such that every graph G in $\mathcal{C}$ has an independent family of p-protrusions $\mathcal{P}$ where $|\mathcal{P}| \geq \lfloor c_L n \rfloor$ and every protrusion of $\mathcal{P}$ has size at least L. Moreover, $\mathcal{C}$ has the *p-Strong Linear Protrusion Property* (for short p-SLPP) if for every L, the class $\mathcal{C}$ has the (p, L)-LPP.

Theorem 12. *Graphs with treewidth t have the $2t$-SLPP.*

Proof. For every integer L, we want to show the existence of $c_L > 0$ such that every graph G with treewidth at most t has $\lfloor c_L \cdot n \rfloor$ pairwise independent $2t$-protrusions of size at least L. Note that we just have to show this property for large enough values of L, so we can assume $L \geq 2t + 2$.

Consider a tree-decomposition T of G which is rooted at some bag B_0 (possibly empty when G is disconnected). Given a bag B_i, we denote by T_i the subtree of T rooted at B_i and by G_i the subgraph of G induced by the vertices of the union of the bags of T_i. We also denote by G_i^* the induced subgraph $G_i \setminus B_i$. We can assume without loss of generality that all the graphs G_i are connected (apart possibly G_0) and that all B_i are pairwise incomparable with respect to inclusion.

We consider the subtree T' of T consisting of the bags B_i such that $|V(G_i)| \geq L + t + 1$. We denote by b the number of nodes of T' and by ℓ its number of leaves. We denote by G' the graph induced by the union of the vertices of the bags of T'. Recall that graphs with treewidth at most t have linear neighborhood complexity (for some constant $c_t \geq 1$). We now set $c_L := 1/40c_t(t+1)L^3$.

We discuss the values of b and ℓ, with three main cases.

- Assume $\ell \geq n/30c_t(t+1)L^3$. Consider a leaf B_i of T'. Observe that $N(G_i^*) \subseteq B_i$ and $|V(G_i^*)| \geq |V(G_i)| - |B_i| \geq L$. Thus $N(G_i^*)$ has size at most $t + 1 \leq 2t$ and therefore $P_i = (N(G_i^*), G_i^*)$ is a $2t$-protrusion of size at least L. In particular, by selecting a protrusion for each leaf B_i, we have found our independent family of size $\lfloor c_L \cdot n \rfloor$.

- Assume that $b \geq n/4c_t(t+1)L^2$ and $\ell < n/30c_t(t+1)L^3$. The number ℓ' of nodes of T' with no child in T' (leaves) or more than one child in T' is at most 2ℓ. So $\ell' \leq 2\ell < n/15c_t(t+1)L^3$. A path P of T' (seen as a rooted path going away from the root) is *proper* if all its nodes have one child in T' and P is maximal for this property. To a proper path P, one can associate the (private) child bag B_P of T' of the last node of P. Note that by maximality, B_P does not have a unique child in T', hence the number of proper paths is at most ℓ'. Each proper path P of T' can be partitioned into $\lfloor |V(P)|/2L \rfloor$ subpaths of length $2L$, and some leftover set of nodes with size less than $2L$. The collection of all these subpaths of length $2L$ form a family

$\mathcal{F} = \{P'_1, \ldots, P'_N\}$. A node of T' is *good* if it belongs to $\bigcup \mathcal{F}$. Each of these paths P'_i of T' starts with a bag B_i which has a parent node B'_i and ends with a bag B_j which has a unique child node B'_j in T'. Each P'_i can be seen as a protrusion on vertex set $V(G_i) \setminus (V(G'_j) \setminus B_j)$ whose boundary $A_i := (B_i \cap B'_i) \cup (B_j \cap B'_j)$ has size at most $2t$. The set of vertices of P'_i which are not in A_i has size at least $2L - 2t \geq L$. We then just have to show that N is large enough.

To get a lower bound on N, note that at most $2L\ell'$ nodes of proper paths are not used in $\mathcal{F}$ (the leftover of the partition modulo $2L$). The number of good nodes is at least

$$b - \ell' - 2L\ell' \geq b - 3L\ell' \geq n/4c_t(t+1)L^2 - n/5c_t(t+1)L^2 \geq n/20c_t(t+1)L^2.$$

In particular, $N > n/40c_t(t+1)L^3$ since the good nodes are grouped into subsets of size $2L$. We have found our independent family of size $\lfloor c_L \cdot n \rfloor$.

- Assume now that $b < n/4c_t(t+1)L^2$, in particular G' has at most $n/4c_tL^2$ vertices. We denote by $\mathcal{B}$ the set of connected components of $G \setminus V(G')$. By definition of T', the size of each component in $\mathcal{B}$ is at most $L+t+1$. So the size of $\mathcal{B}$ is at least $(n - n/4c_tL^2)/(L+t+1) > n/2L$ since $L + t + 1 \leq 3L/2$ and $1 - 1/4c_tL^2 > 3/4$.

  Consider now the graph $G/\mathcal{B}$ in which every connected component in $\mathcal{B}$ is contracted to a single vertex. Note that $\bigcup \mathcal{B}/\mathcal{B}$ is an independent set of $G/\mathcal{B}$ denoted by B. The neighborhood A of B in $G/\mathcal{B}$ has size at most $n/4c_tL^2$ since it is contained in $V(G')$. Partition the elements of B into $C_1, \ldots, C_r$ according to their neighborhoods in A. By the linear neighborhood complexity property of $G/\mathcal{B}$ (which has treewidth at most t), we have $r \leq n/4L^2$. Refine further the partition (C_i) into a partition (D_i) in which each C_i is partitioned into some subsets of size L and a subset of size $|C_i| \mod L$. Denote by $\mathcal{D}$ the subcollection of subsets D_i with size exactly L, and set $d := |\mathcal{D}|$. Each D_i corresponds to a subset of vertices U_i of G obtained by uncontracting each element b_j of B in D_i to the corresponding set B_j of $\mathcal{B}$. Note that $N(U_i)$ in G has size at most t otherwise $D_i, N(D_i)$ would be a $K_{L,|N(D_i)|}$ subgraph in $G/\mathcal{B}$, and thus $U_i, N(U_i)$ would be a $K_{L,|N(D_i)|}$-minor model in G, contradicting the fact that the treewidth of G is at most t. So $\mathcal{P} = \{(N(U_1), U_1), \ldots, (N(U_d), U_d)\}$ is an independent family of t-protrusions each with size at least L.

  We just have to obtain a lower bound on d. The set B has size at least $n/2L$ and the leftover vertices (not appearing in the D_i) are at most $Lr \leq Ln/4L^2 = n/4L$. So at least $n/4L$ vertices belong to the U_i, and since each U_i has size L we have $d \geq n/4L^2$, and our conclusion. □

We can now show the linear protrusion property for kH-free graphs with no small H-minor model, which is the backbone of both Theorem 1 and Theorem 3.

Theorem 13. *There exists an integer p such that for every integer L, there exists g_L such that the class of kH-free graphs with H-girth more than g_L has the (p, L)-LPP.*

Proof. An equivalent task is to find N, p, c_L and g_L such that every kH-free graph G with at least N vertices and H-girth more than g_L has a family $\mathcal{P}$ of $\lfloor c_L n \rfloor$ pairwise independent p-protrusions of size at least L. Indeed, to translate into a statement without a lower bound on n, we just have to set a new parameter $c'_L = \min(c_L, 1/N)$. The introduction of N gives more flexibility for parameter tuning. Let us recall them:

- m is the maximum of $|V(H)|$ and $|E(H)|$. Note that H is therefore a minor of $K_{m,m}$.

- t is the maximum treewidth of an H-minor free graph, and $p := 2t + m$.
- By Theorem 6, there exists c such that kH-free graphs with no $K_{m,m}$ subgraph have neighborhood complexity at most c. Hence $|\{N(v) \cap S : v \in V\}| \leq c|S|$ for every non-empty $S \subseteq V(G)$.
- By Theorem 6, there exists δ such that kH-free graphs with no $K_{m,m}$ subgraph have average degree at most 2δ.
- By Theorem 12, there is a constant $\varepsilon > 0$ such that every graph T with treewidth at most t has a family of $\varepsilon|V(T)|$ independent $2t$-protrusions with size at least $12ctL^2$.

We assume that G has H-girth more than $g_L := \max(32m/\varepsilon, 4m/3ctL\varepsilon)$. In particular G does not contain any $K_{m,m}$ subgraph, and therefore its average degree is at most 2δ. We also set our parameter $c_L := \min(\varepsilon/8, ct\varepsilon/2)$.

Consider some integer d (to be chosen later) and partition G into the set Y of vertices of degree at most d and $X := V \setminus Y$. Observe that the size of X is at most $2\delta n/d$ by maximum average degree. By Theorem 7, since the graph $G[Y]$ has maximum degree d and is kH-free, it contains a subset X' of size s' only depending on d and k such that $G[Y] \setminus X'$ is H-minor free, and thus has treewidth at most t.

Set $N = ds'/2\delta$, and thus, since $n \geq N$, we have $s' \leq 2\delta n/d$. We rename both $X \cup X'$ as X and $Y \setminus X'$ as Y. Note that since $|X| \leq 4\delta n/d$, the ratio $|Y|/|X|$ can be made arbitrarily large by a suitable choice of d and N. We then assume from now on that we have both $\varepsilon|Y| \geq 8cm|X|$ and $\varepsilon t|Y| \geq |X|$.

By Theorem 12, there is a family $\mathcal{P}$ of $8\ell := \varepsilon|Y|$ independent $2t$-protrusions of $G[Y]$, each of size at least $12ctL^2$. We distinguish two cases.

- There are 4ℓ protrusions of $\mathcal{P}$ with a connected component of size at least L. Let us call $\mathcal{P}'$ the collection of 4ℓ protrusions obtained from $\mathcal{P}$ by selecting a single component of size at least L in each. If 2ℓ protrusions of $\mathcal{P}'$ have neighborhoods in X of size at most m, we have our conclusion since they consist of $\varepsilon|Y|/4 \geq \varepsilon n/8 \geq c_L n$ many p-protrusions of G since $|Y| \geq n/2$. Let us reach a contradiction in the other case. Assume that 2ℓ protrusions $\mathcal{P}''$ of $\mathcal{P}'$ have neighborhoods of size more than m in X. Let $\mathcal{B}'' := \{B_1, B_2, \ldots, B_\ell\}$ be the interior of the ℓ smallest protrusions of $\mathcal{P}''$. Each B_i has size at most n/ℓ, otherwise the union of $\mathcal{P}'$ would exceed n. Thus for each i we have $|B_i| \leq n/\ell = 8n/\varepsilon|Y| \leq 16/\varepsilon$ since $|Y| \geq n/2$. Consider the induced graph $G[(\bigcup_{i=1}^{\ell} B_i) \cup X]$ and contract each B_i to a single vertex b_i to form a new graph G'. Note that G' is kH-free since it is an induced minor of G. Note also that G' has no $K_{m,m}$ subgraph since this would be an H-minor model with size at most $32m/\varepsilon$. Since b_i has at least m neighbors in X and $\ell > cm|X|$, by linear neighborhood complexity (and pigeonhole principle), there are m vertices b_i with the same neighborhood. This gives a $K_{m,m}$ and a contradiction.
- There are 4ℓ protrusions of $\mathcal{P}$ with no connected component of size at least L. Splitting these protrusions into connected components, we form a family $\mathcal{P}'$ of at least $12ctL \cdot 4\ell$ connected independent $2t$-protrusions. Note that the boundary C of $\mathcal{P}'$ in $G[Y]$ has size at most $8\ell t$ since each protrusion of $\mathcal{P}$ has boundary size at most $2t$. Since the size of X is at most $8\ell t$, the boundary C' of $\mathcal{P}'$ in G has size at most $16\ell t$. Consider $\mathcal{P}''$ consisting of the smallest $24ctL\ell$ protrusions of $\mathcal{P}'$ and observe that each of their size is at most $n/24ctL\ell = 8n/24ctL\varepsilon|Y| \leq 2/3ctL\varepsilon$ since $|Y| \geq n/2$. Contract each protrusion B_i of $\mathcal{P}''$ to a single vertex b_i and observe that the resulting graph G' is kH-free with no $K_{m,m}$ (since this would

give an H-minor model with size $4m/3ctL\varepsilon$). By linear neighborhood complexity, the number of distinct neighborhoods of the b_i's in G' is at most $c|C'| \leq 16ct\ell$. Now, we greedily group the b_i's with identical neighborhoods into subsets B'_j of size L to form a family of protrusions $\mathcal{S} = \{B'_1, \ldots, B'_s\}$. The remaining b_i's which cannot be grouped are at most L times the number of distinct neighborhoods, that is $16ctL\ell$. The number of b_i's is $24ctL\ell$, so at least $8ctL\ell$ of them belong to some B'_j, in particular $s \geq 8ct\ell \geq ct\varepsilon|Y| \geq ct\varepsilon n/2 \geq c_L n$. Each B'_j has neighborhood size less than m, otherwise G' would contain a $K_{L,m}$ subgraph, which contradicts $K_{m,m}$-freeness (assuming w.l.o.g $L \geq m$). To conclude observe that the (uncontracted) B_j's forms a family of $ct\varepsilon n/2$ many independent m-protrusions of size at least L. This is our solution.

□

5 Induced Erdős–Pósa property for planar minors

We begin by showing the Erdős–Pósa property for connected graphs H.

Theorem 14. *For any connected planar graph H, there exists a function f such that every kH-free graph G satisfies $\gamma_H(G) \leq f(k)$.*

Proof. We use our definition for m and t and $p = 2t + m$. Let us set L to be such that for every p-boundaried graph $P = (A, B)$ of size at least L, there exists a p-boundaried graph $P' = (A', B')$ with smaller size, same $(kh+p)$-folio, and same $(p, h+p)$-neighborhood-deletion folio. Such a bound L exists since there is only a bounded number of different types.

By Theorem 13, there exists g_L and c_L such that every kH-free graph G with H-girth more than g_L contains $\lfloor c_L n \rfloor$ many p-protrusions of size at least L.

We show $f(2) := \max(g_L, 2/c_L)$. Assume for contradiction that G is a $2H$-free graph with minimum size such that $\gamma_H(G) > f(2)$. In particular $n > 2/c_L$. If G contains an H-minor model $\mathcal{H}$ of size at most g_L, then $G \setminus N[V(\mathcal{H})]$ is H-minor free, a contradiction. Otherwise, by Theorem 13, there are $\lfloor c_L n \rfloor > 2$ independent p-protrusions of size at least L. Since G is $2H$-free, one of them, say $P = (A, B)$, is such that $G[B]$ is H-minor free. But then there exists a p-boundaried graph $P' = (A', B')$ with smaller size, same $(2h+p)$-folio, and same $(p, h+p)$-neighborhood-deletion folio. Thus by Lemma 11, the (smaller) graph G' obtained by replacing P by P' is $2H$-free and satisfies $\gamma_H(G') = \gamma_H(G)$, a contradiction to the minimality of G.

By applying the same argument, we obtain more generally that $f(k) = \max(k/c_L, g_L + f(k-1))$. □

To handle disconnected H, we need the following straightforward adaptation of Lemma 11, allowing to reduce protrusions when they contain no minor of *any* connected component of H.

Lemma 15. *Let H be a planar graph, and G be a kH-free graph with a p-protrusion $P = (A, B)$ such that $G[B]$ is C-minor free for every connected component C of H. Then, the graph G' obtained by replacing P by $P' = (A, B')$ with the same $(kh + p)$-folio and the same $(p, h + p)$-neighborhood-deletion folio is kH-free and $\gamma_H(G) = \gamma_H(G')$.*

Proof. The proof follows that of Lemma 11 line for line, except for the end of the first paragraph justifying $|Y'| \leq p$. Recall we take a minimum H-dominating set D' in G'. Now, since P and P' have the same h-folio, $G'[B']$ is C-minor free for every connected component C of H. We claim $Y' = D' \cap (A \cup B')$ still has size at most p, otherwise $D'' = (D' \setminus B') \cup A$ is strictly smaller, and it

is still H-dominating. Indeed, any H-minor model in $G' - N[D'']$ would have to lie entirely outside B' (as it forbids any component), thus outside P', but then this copy could not have been hit by D' in G'. The rest of the proof follows Lemma 11. □

With Lemma 15 in hand, we can show the induced Erdős–Pósa property for all planar graphs.

Theorem 1. *Given a planar graph H, there exists a function f such that for every kH-free graph G, there exists $X \subseteq V(G)$ with $|X| \leq f(k)$ such that $G \setminus N[X]$ is H-minor free.*

Proof. We show the existence of $f(k, H)$ by induction on k and the number of components d of H. The case $d = 1$ for any k corresponds to Theorem 14. Consider $d \geq 2$ and assume that the result holds for every planar graph H with fewer than d connected components. Let H be a planar graph with connected components $H_1, ..., H_d$. Recall that h denotes the number of vertices of H. Set now L as well as g_L, c_L as in the proof of Theorem 14, assume without loss of generality $L \geq p$. Define $f(k, H) = \max(\max_i f(k, H \setminus V(H_i)) + 8dk/c_L, g_L + f(k-1, H))$. Let G be a graph of minimum size n that is kH-free, and such that $\gamma_H(G) > f(k, H)$. If G has H-girth at most g_L, we reduce to the $(k-1)H$-free case, for a bound of $g_L + f(k-1, H)$. Otherwise, we apply Theorem 13 to get a set of $\lfloor c_L n \rfloor$ independent p-protrusions of size at least L.

We first deal with the case where at least $c_L n/2$ of these protrusions contain a minor model of some H_i in their interior. Then, by the pigeonhole principle, at least $c_L n/2d$ protrusions contain an H_1-minor model, without loss of generality. Out of those, take the half smallest, such that each have size at most $4d/c_L$. If $k \geq c_L n/4d$, n is at most $4dk/c_L$, so we have our conclusion by taking the whole $V(G)$ in our H-dominating set. If not, we consider a subset $\{P_i = (A_i, B_i) : i = 1, \ldots, k\}$ of these latter protrusions. Let then $X_1 = \bigcup_{i \leq k}(A_i \cup B_i)$, and note that $|X_1| \leq k(p + 4d/c_L) \leq 8kd/c_L$ since $p \leq L \leq 4d/c_L$. Observe now that the graph $G \setminus X_1$ is $k(H \setminus V(H_1))$-free, because X_1 contains k independent copies of H_1 with no neighbors in $G \setminus X_1$. By induction, it admits a minimum $(H \setminus V(H_1))$-dominating set X' of size at most $f(k, H \setminus V(H_1))$. Then, $X = X' \cup X_1$ is a dominating set of $(H \setminus V(H_1))$-minor models, thus of H-minor models. Moreover $|X| \leq f(k, H \setminus V(H_1)) + 8dk/c_L$, which concludes this case.

We may now assume that at least $c_L n/2$ protrusions do not contain a minor of any H_i. If $n < 2/c_L$, then the whole set $V(G)$ is a H-dominating set of size at most $f(k, H)$ as desired, so we may assume that there exists at least one such protrusion $P = (A, B)$. By definition of L, there is a p-boundaried graph $P' = (A', B')$ with smaller size, same $(kh + p)$-folio, and same $(p, h + p)$-neighborhood-deletion folio. Then, replacing P with P' yields by Lemma 15 a graph G' that is still kH-free and satisfies $\gamma_H(G') = \gamma_H(G)$, contradicting the minimality of G. □

6 Logarithmic size hitting set and Maximum Independent Set

We now prove the existence of a logarithmic size hitting set for sparse kH-free graphs. We first need a sparsification lemma which asserts that in every sparse kH-free graph G, there is a bounded number of vertices which intersects all small H-minor models.

Lemma 16. *For every g and r, there exists $h(g, r)$ such that every kH-free graph G on n vertices with no $K_{r,r}$ subgraph has a set of vertices X with size at most $h(g, r)$ such that $G \setminus X$ has H-girth more than g.*

Proof. Consider a kH-free graph G with no $K_{r,r}$ subgraph and a disjoint union $\mathcal{H}_1, \ldots, \mathcal{H}_N$ of H-minor models in G each with size at most g. We assume also that N is as large as possible. Call G' the subgraph induced by $\bigcup \mathcal{H}_i$. The graph G'' obtained by contracting in G' each $\mathcal{H}_i$ to a

single vertex has maximum independent set at most $k-1$. By Turán's theorem, G'' has at least $(k-1)\binom{\lfloor \frac{N}{k-1} \rfloor}{2}$ edges. So the number of edges of G' is $\Omega(N^2/k)$ and its number of vertices is at most gN. Therefore G' has a quadratic number of edges. Since graphs without $K_{r,r}$ subgraphs have average degree $O(n^{1-1/r})$, the number of vertices of G' is bounded by some value $h(g,r)$. By maximality of N, deleting $V(G')$ from G leaves a graph with H-girth more than g. □

6.1 Logarithmic hitting set

Theorem 17. *For any connected planar graph H, every kH-free graph G on n vertices with no $K_{r,r}$ subgraph satisfies $\tau_H(G) = O_{r,k}(\log n)$.*

Proof. Again $p = m + 2t$. Let us set L to be such that for every p-boundaried graph $P = (A, B)$ of size at least L there exists a p-boundaried graph $P' = (A', B')$ with smaller size, same $(kh+4p)$-folio, and same $(p, h+p)$-deletion folio. Such a bound L exists since there is only a bounded number of different types.

By Theorem 13, there exists g_L and c_L such that every kH-free graph G with H-girth more than g_L contains $\lfloor c_L n \rfloor$ many independent p-protrusions of size at least L.

We now proceed by induction on n and show the existence of a constant C such that every kH-free graph on n vertices without $K_{2p,2p}$ subgraph satisfies $\tau_H(G) \leq C \log n$.

By Lemma 16, there exists $X \subseteq V(G)$ with size at most $h(g_L, 2p)$ such that $G_1 := G \setminus X$ has H-girth more than g_L. We denote by n_1 the number of vertices of G_1. By Theorem 13, there are $\lfloor c_L n_1 \rfloor$ independent p-protrusions in G_1 of size at least L. Less than k of these protrusions contain an H-minor model. If $n_1 \leq 2k/c_L$, the number of vertices of G is at most $2k/c_L + h(g_L, 2p)$, and we set C such that $C \log n \geq 2k/c_L + h(g_L, 2p)$. Otherwise, by the choice of L we can reduce all the (at least $c_L n_1/2$ many) protrusions without H-minor model to smaller ones with the same $(kh+4p)$-folio and same $(p, h+p)$-deletion folio. This (iterated or parallel) sequence of reductions terminates on a graph G_2, which by Lemma 10 is still kH-free and satisfies $\tau_H(G_2) = \tau_H(G_1)$.

Observe that every protrusion $P = (A, B)$ which is reduced (to some $P' = (A, B')$) is such that B does not contain any H-minor model, and thus in particular $G_1[A \cup B]$ does not contain a $K_{2p,2p}$-minor model. Since P and P' have the same $(kh+4p)$-folio, $G_2[A \cup B']$ does not contain a $K_{2p,2p}$ subgraph. Also, G_1 and G_2 coincide outside of the protrusions, and a $K_{2p,2p}$ cannot cross the boundary of a protrusion, so the graph G_2 has no $K_{2p,2p}$-subgraph. Therefore by induction, $\tau_H(G_1) = \tau_H(G_2) \leq C \log(|V(G_2)|)$.

Since $|V(G_2)| \leq (1 - c_L/2)n_1$, we have $\tau_H(G_1) \leq C \log(n_1) + C \log(1 - c_L/2)$. Therefore $\tau_H(G) \leq h(g_L, 2p) + C \log(n_1) + C \log(1 - c_L/2)$. We just have to choose C large enough such that $h(g_L, 2p) + C \log(1 - c_L/2) \leq 0$ to satisfy the induction step.

For the general case, in which G has no $K_{r,r}$ subgraph, we apply Lemma 16 to find a set X of size at most $h(4p, r)$ such that $G \setminus X$ has no $K_{2p,2p}$ subgraph. In particular $\tau_H(G) \leq h(4p, r) + C \log n$. □

We now move to the proof for all planar graphs H, where we will use the generalized protrusion reduction property given by the straightforward adaptation of Lemma 10 to disconnected H (see Lemma 15).

Theorem 3. *Given a planar graph H, there exists a function f such that for every kH-free graph G on n vertices with no $K_{r,r}$ subgraph, G admits an H-hitting set of size at most $f(k,r) \log n$.*

Proof. The argument is similar to the proof of Theorem 1. Either some connected component H_1 of H appears in a linear number of p-protrusions, and therefore one can find k independent copies of H_1 in a bounded size set X_1, allowing to reduce to the $k(H \setminus H_1)$-free case. Or there is a linear

number of p-protrusions which do not contain any component of H, and we can reduce all of them, thus reducing the size by some fixed factor. □

6.2 Quasi-polynomial-time algorithm for Maximum Independent Set

Using Theorem 3, we are now ready to show our algorithm for MIS, which follows the lines of [11].

Theorem 2. *Let H be a planar graph and k be an integer. There is an $n^{O(\log n)}$ time algorithm computing MIS in kH-free graphs.*

Proof. By Theorem 3, we just have to reduce the problem from the general case to the sparse case. Indeed, if an H-hitting set X has logarithmic size, we can guess it in QP-time, and then guess its intersection I with a MIS in polynomial time. Since $G \setminus (X \cup N[I])$ is H-minor free, its treewidth is at most t, hence one can compute a maximum independent set I' in linear time. We finally output the maximum of $|I| + |I'|$ for all choices of I and X.

Let us now reduce to the sparse case. We construct the graph G^{2m} whose vertices are all the (possibly pairwise intersecting) copies of $K_{m,m}$ subgraphs of G. There is an edge K, K' in G^{2m} if there is an edge connecting a vertex of K to a vertex of K' (which is the case when $K \cap K' \neq \emptyset$). Observe that G^{2m} has no independent set of size k, so there is a vertex K with degree at least $|V(G^{2m})|/k$. This vertex K corresponds to a copy of $K_{m,m}$ in G, so there is a vertex v of this copy which is adjacent to at least a $1/2km$ fraction of the vertices of G^{2m}.

Note that v can be found in polynomial time $O(n^{4m})$, which is the cost of constructing G^{2m}. Now, if one starts a branching process on v to compute MIS in G, the branch $v \notin MIS$ recurses on $G \setminus v$ and the branch $v \in MIS$ recurses on $G \setminus N[v]$. In the latter case the number of $K_{m,m}$ decreases by a $1/2km$ fraction. This branching process ends when the current input graph has no $K_{m,m}$ subgraph. Since one branch decreases by 1 and the other decreases by a fraction (of n^{2m}), the leaves in this recursion tree correspond to $0, 1$ words of length at most n in which the number of 1 is logarithmic. Hence this preprocess reduces G to a quasi-polynomial $n^{O(\log n)}$ number of instances without $K_{m,m}$ subgraphs, in which we can apply the algorithm in the sparse case. □

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