Hadwiger's conjecture for quasi-line graphs

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Abstract

A graph G is a quasi-line graph if for every vertex $v \in V(G)$, the set of neighbors of v in G can be expressed as the union of two cliques. The class of quasi-line graphs is a proper superset of the class of line graphs. Hadwiger's conjecture states that if a graph G is not t-colorable then it contains K_{t+1} as a minor. This conjecture has been proved for line graphs by Reed and Seymour [10]. We extend their result to all quasi-line graphs.

1 Introduction

Let G be a finite graph. Denote the set of vertices of G by $V(G)$ and the set of edges of G by $E(G)$. A *k-coloring* of G is a map $c : V(G) \to \{1, \ldots, k\}$ such that for every pair of adjacent vertices $v, w \in V(G)$, $c(v) \neq c(w)$. We may also refer to a k-coloring simply as a "coloring". The *chromatic number* of G, denoted by $\chi(G)$, is the smallest integer such that there is a $\chi(G)$-coloring of G.

For $v \in V(G)$, we denote the set of neighbors of v in G by $N_G(v)$, and for $X \subset V(G)$ we define $N_G(X) = \bigcup_{x \in X} N(x)$. For $X \subset V(G)$, let $G|X$ denote the subgraph of G induced on X and let $G \setminus X$ denote the subgraph of G induced on $V(G) \setminus X$. For G_1, G_2 induced subgraphs of G, let $G_1 \cup G_2 = G|(V(G_1) \cup V(G_2))$. We say that $X \subset V(G)$ is a *claw* if $G|X$ is isomorphic to the complete bipartite graph $K_{1,3}$. A graph G is then *claw-free* if no subset of $V(G)$ is a claw. We define a *path* P in G to be an induced connected subgraph of G such that either P is a one-vertex graph, or two vertices of P have degree one and all the others have degree two. The complement of G is the graph $\overline{G}$, on the same vertex set as G, and such that two vertices are adjacent in G if and only if they are non-adjacent in $\overline{G}$. A *hole* in a graph G is an induced cycle with at least four vertices. An *antihole* in G is a hole in $\overline{G}$. A hole (antihole) is *odd* if it has an odd number of vertices. A *clique* in G is a set of vertices of G that are all pairwise adjacent. A *stable set* in G is a clique in the complement of G. The *clique number* of G, denoted by $\omega(G)$, is the size of a maximum clique in G. The *complete graph* on t vertices, denoted by K_t, is a graph such that $|V(K_t)| = t$ and $V(K_t)$ is a clique. A *component* is a maximal connected subgraph of G. A set $S \subset V(G)$ is a *cutset* if $G \setminus S$ has more components than G. We say that S is a *clique cutset* if it is both a clique and a cutset.

We say that two subgraphs S_1, S_2 of G are adjacent if there is an edge between $V(S_1)$ and $V(S_2)$. A graph H is said to be a *minor* of a graph G if a copy of H can be obtained from G by deleting and/or contracting edges. Let H be a graph with $V(H) = \{v_1, \ldots, v_n\}$. Then H is a minor of G if and only if there are $|V(H)|$ non-null connected subgraphs $A_1, \ldots, A_n$ of G, such that $V(A_i \cap A_j) = \emptyset$, and A_i and A_j are adjacent if v_i is adjacent to v_j. We say that a graph G has a clique minor of size t if G has K_t as a minor.

In 1943, Hadwiger [7] conjectured that for every loopless graph G and every integer $t \geq 0$, either G is t-colorable, or G has a clique minor of size $t + 1$. In the same paper, Hadwiger proved his conjecture for $t \leq 3$. Six years earlier, Wagner [12] proved that the case $t = 4$ is equivalent to the *four color theorem*, which states that every planar graph admits a 4-coloring. The four color theorem was proved by Appel and Haken [1], [2] in 1977. In 1993, Seymour, Robertson, and Thomas [11] proved Hadwiger's conjecture for $t = 5$ also using the four color theorem. Thus, Hadwiger's conjecture is known to be true for $t \leq 5$ and remains unsolved for $t > 5$.

Hadwiger's conjecture has also been proved for some special classes of graphs. The *line graph* of a graph G, denoted by $L(G)$, is a graph whose vertices are the edges of G, and if $u, v \in E(G)$ then $uv \in E(L(G))$ if u and v share a vertex in G. In a recent work, Reed and Seymour [10] proved Hadwiger's conjecture for line graphs. In this paper, we prove Hadwiger's conjecture for a class of graphs that is a proper superset of line graphs and a proper subset of claw-free graphs, the set referred to as *quasi-line graphs*. A graph G is a quasi-line graph if for every vertex v, the set of neighbors of v can be expressed as the union of two cliques. Note that this is a partition of the vertex set of the neighborhood of v. The main result of this paper is the following:

Theorem 1.1. *Let G be a quasi-line graph with chromatic number χ. Then G has a clique minor of size χ.*

Our proof of Theorem 1.1 uses a structure theorem for quasi-line graphs that appears in [3]. The structure theorem asserts that every quasi-line graph belongs to one of two classes: the first is the class of the so-called “fuzzy circular interval graphs,” and the second is “compositions of fuzzy linear interval strips,” which is a generalization of line graphs. The word “fuzzy” in both cases refers to the presence of a certain structure in a graph that is called a “non-trivial homogeneous pair” (we give precise definitions in the next section). We also use the following result from [6]:

Theorem 1.2. *Let G be a quasi-line graph. Then $\chi(G) \leq \frac{3}{2}\omega(G)$.*

The remainder of this paper is organized as follows. In Section 2, we state the structure theorem for quasi-line graphs and all of the necessary definitions. In Section 3 and Section 4, we prove Theorem 1.1 for circular interval graphs and compositions of linear interval strips, respectively (these are precisely the quasi-line graphs that have no non-trivial homogeneous pairs). In Section 5, we use the results of the two previous sections and deal with non-trivial homogeneous pairs, to complete the proof of Theorem 1.1.

2 Structure theorem for quasi-line graphs

We start this section by introducing some definitions from [3] and [4] and then state the structure theorem of [3].

Let Σ be a circle and let $F_1, \dots, F_k$ be subsets of Σ, each homeomorphic to the closed interval $[0,1]$. Let V be a finite subset of Σ, and let G be the graph with vertex set V in which $v_1, v_2 \in V$ are adjacent if and only if $v_1, v_2 \in F_i$ for some i. Such a graph is called a *circular interval graph*. Let $\mathbb{F} = \{F_1, \dots, F_k\}$. Then we call the pair $(\Sigma, \mathbb{F})$ a *representation* of G. A subset $S \subset V$ is a *block* if $S = F_i \cap V$ for some $F_i \in \mathbb{F}$. We then call S *the block of* F_i. A *linear interval graph* is constructed in the same way as a circular interval graph except we take Σ to be a line instead of a circle. It is easy to see that all linear interval graphs are also circular interval graphs.

The structure theorem that we use states that there are two types of quasi-line graphs. The first subclass is a generalization of the class of circular interval graphs and we proceed to describe it below. Once again, we start with a few definitions.

Let X, Y be two subsets of $V(G)$ with $X \cap Y = \emptyset$. We say that X and Y are *complete* to each other if every vertex of X is adjacent to every vertex of Y, and we say that they are *anticomplete* if no vertex of X is adjacent to a member of Y. Similarly, if $A \subseteq V(G)$ and $v \in V(G) \setminus A$, then v is A-*complete* if it is adjacent to every vertex in A, and A-*anticomplete* if it has no neighbor in A. A pair (A, B) of disjoint subsets of $V(G)$ is called a *homogeneous pair* in G if for every vertex $v \in V(G) \setminus (A \cup B)$, v is either A-complete or A-anticomplete and either B-complete or B-anticomplete.

Let G be a circular interval graph with $V(G) = \{v_1, \dots, v_n\}$ in order clockwise. An edge joining v_j to v_k with $j < k$ is called a *maximal edge* if $\{v_j, v_{j+1}, \dots, v_k\}$ is a block. In this case the following operation produces another quasi-line graph: replace v_j and v_k by two cliques A and B, respectively, such that every member of A has the same neighbors as v_j and every member of B has the same neighbors as v_k in $V(G) \setminus \{v_j, v_k\}$, and the edges between A and B are arbitrary. The pair (A, B) is then a *homogeneous pair of cliques*.

Let (A, B) be a homogeneous pair of cliques in a circular interval graph. We say that (A, B) is *non-trivial* if there exists an induced 4-cycle in G with exactly two vertices in A and exactly two vertices in B. It is easy to see that if a fuzzy circular interval graph is not a circular interval graph, then it has a non-trivial homogeneous pair.

We proceed with the construction of graphs that belong to the second subclass of quasi-line graphs. A vertex $v \in V(G)$ is *simplicial* if the set of neighbors of v is a clique. A claw-free graph S together with two distinguished simplicial vertices a, b is called a *strip* (S, a, b), with *ends* a and

b. If S is a linear interval graph with $V(S) = \{v_1, \ldots, v_n\}$ in order and with $n > 1$, then v_1, v_n are simplicial. If either

- $N_S(a_1) \cap N_S(a_n) = \emptyset$, or
- $N_S(a_1) = N_S(a_n) = V(S) \setminus \{a_1, a_n\}$,

then (S, v_1, v_n) is a strip, called a *linear interval strip*. Let us call a strip (S, a, b) with $N_S(a) = N_S(b) = V(S) \setminus \{a, b\}$ a *line graph strip*. Since linear interval graphs are also circular interval graphs, we can define *fuzzy linear interval strips* by introducing homogeneous pairs of cliques in the same manner as before; with the exception that edges incident with the two ends of the strip cannot be replaced by homogeneous pairs.

Let (S, a, b) and (S', a', b') be two strips. Then they can be composed as follows. Let A, B be the set of neighbors of a, b in S respectively, and define A', B' analogously. Consider the disjoint union of $S \setminus \{a, b\}$ and $S' \setminus \{a', b'\}$, and make A complete to A' and B complete to B'.

This method of composing two strips described above can be used as follows. Let S_0 be a graph which is the disjoint union of complete graphs with $|V(S_0)| = 2n$. We arrange the vertices into pairs $(a_1, b_1), \ldots, (a_n, b_n)$, such that each pair meets two cliques of S_0. For $i = 1, \ldots, n$, let (S'_i, a'_i, b'_i) be a strip, Let S_i be the graph obtained by composing (S_{i-1}, a_i, b_i) and (S'_i, a'_i, b'_i). The resulting graph S_n is then called a *composition* of the strips (S'_i, a'_i, b'_i).

We are finally ready to state the structure theorem for quasi-line graphs [3] that we will use to prove our main result.

Theorem 2.1. *Let G be a connected, quasi-line graph. Then G is either a fuzzy circular interval graph or a composition of fuzzy linear interval strips.*

3 Circular Interval Graphs

We begin the proof of Theorem 1.1 by proving the result for circular interval graphs. Let G be a circular interval graph with $V(G) = \{v_1, \ldots, v_n\}$ in order clockwise. Let $v_i v_j$ be an edge with $i < j$ such that $v_i, v_{i+1}, \ldots, v_j$ are all contained in some block. Then we say that $S = \{v_i, v_{i+1}, \ldots, v_j\}$ is the set of vertices contained *non-strictly underneath* the edge $v_i v_j$. The *l*ength of an edge is the number of vertices non-strictly underneath it. We further say that v_i *h*as an edge of length $|S|$ in the clockwise direction and v_j has an edge of length $|S|$ in the counterclockwise direction.

Theorem 3.1. *Let G be a circular interval graph with representation $(\Sigma, \mathbb{F})$ where $\mathbb{F} = \{F_1, \ldots, F_k\}$ and such that $\chi(G) = \omega(G) + i$. Then G has a clique minor of size $\omega(G) + i$.*

Proof. Let $\omega = \omega(G)$, $\chi = \chi(G)$, and $n = |V(G)|$. We proceed by induction on n.

(1) For all $v \in V(G)$, we may assume $deg(v) \geq \omega + i - 1$.

Suppose there exists $v \in V(G)$ such that $deg(v) \leq \omega + i - 2$. If $\chi(G \setminus \{v\}) = \omega + i - 1$ then there is still a color left for v implying that $\chi(G) = \omega + i - 1$, which is a contradiction. Otherwise, $\chi(G \setminus \{v\}) = \omega + i$ and the theorem holds by induction. This proves (1).

(2) For every $v \in V(G)$, there is an edge of length $i + 1$ in each direction.

By (1), for every $v \in V(G)$ the sum of the lengths of maximal edges in each direction is at least $\omega + i + 1$, and the maximum length in either direction is ω. Therefore, the length in each direction must be at least $i + 1$. This proves (2).

(3) If there is no clique of size ω which is a block, then the theorem holds.

Suppose there is no clique of size ω which is a block. Let I_i be the block of F_i for $1 \leq i \leq m$ and let $\mathbb{I} = \{I_1, \ldots, I_m\}$. Let Q be a clique of size ω. Then there exist $a_1, a_2, a_3 \in Q$ such that a_1, a_2, a_3 are not all contained in the same block. Hence there exist $I_1, I_2, I_3 \in \mathbb{I}$ such that $a_1, a_2 \in I_1$, $a_1, a_3 \in I_2$, $a_2, a_3 \in I_3$ and $a_3 \notin I_1$, $a_2 \notin I_2$, $a_1 \notin I_3$. Then $V(G) = I_1 \cup I_2 \cup I_3$.

Choose I_1, I_2, I_3 such that

$$V(Q) \cap \bigcup_{1 \leq i < j \leq 3} (I_i \cap I_j)$$

is maximal.

If possible, permute I_1, I_2, I_3 so that $Q \cap (I_1 \setminus (I_2 \cup I_3)) \neq \emptyset$.

Let $\{i, j, k\} = \{1, 2, 3\}$. Let $J_i = I_i \setminus (Q \cup I_j \cup I_k)$, and let J_j and J_k be defined similarly.

Since $V(G) \setminus J_1 \subseteq N(a_3)$, it follows that $V(G) \setminus J_1$ is the union of two cliques, and hence by [8] can be colored with ω colors. Since $\chi = \omega + i$, it follows that $|J_1| \geq i$. Analogously, $|J_j| \geq i$ for $j = 2, 3$. Consider the graph induced on $J_1 \cup J_2 \cup J_3$ and let H be that graph with all the edges deleted except for those between the members of J_1 and the members of $J_2 \cup J_3$. Then H is a bipartite graph with bipartition $(J_1, J_2 \cup J_3)$.

Suppose first that H has a matching M of size i. We claim that the elements of M together with the members of Q form $\omega + i$ disjoint connected subgraphs of G that are pairwise adjacent. Since $J_1 \subseteq I_1$ is a clique, every two members if M are adjacent, and every member of M is adjacent to every member of $Q \cap I_1$. Since Q is a clique, every two members of Q are adjacent. Suppose some $m \in M$ is non-adjacent to some $q \in Q$. We may assume that m is the edge vw with $v \in J_1$ and $w \in J_2$, $q \in I_3 \setminus (I_1 \cup I_2)$, and q is anticomplete to $\{v, w\}$. But then, since G is a circular interval graph, it follows that q is non-adjacent to a_1, contrary to the fact that Q is a clique. This proves the claim, and so we may assume that no such matching M exists.

Consequently, there exists $v \in J_1$ such that $|N_H(v)| < i$. Then v has at most $\omega - 2$ neighbors in I_1, since $|I_1| < \omega$, and it has at most $i - 1$ neighbors in $J_2 \cup J_3$. Furthermore, v has no other neighbors in $I_2 \cap I_3$, since if it did, the structure of circular interval graphs would imply that either $J_2 \subset N_G(v)$ or $J_3 \subset N_G(v)$. But $|J_2|, |J_3| \geq i$ and v has fewer than i neighbors in $J_2 \cup J_3$, a contradiction. Since by (1), $deg(v) \geq \omega + i - 1$, it follows that v has a neighbor in $Q \setminus (I_1 \cup (I_2 \cap I_3))$, and, in particular, $Q \setminus (I_1 \cup (I_2 \cap I_3))$ is non-empty. From the choice of I_1, this implies that $Q \cap (I_1 \setminus (I_2 \cup I_3)) \neq \emptyset$. From the symmetry, we may assume that following $I_1 \setminus (I_2 \cup I_3)$ in the direction in which a_2 precedes a_3, starting at v, we encounter a vertex $q \in Q$. Since v is anticomplete to $I_2 \cap I_3$, it follows that for some $s \in \{1, \ldots, m\} \setminus \{1, 2, 3\}$, $(Q \cap I_3) \cup \{q\} \subseteq I_s$. But then

$$V(Q) \cap (\bigcup_{1 \leq i \leq 2} (I_i \cap I_3) \cup \{q\}) \subseteq V(Q) \cap \bigcup_{1 \leq i \leq 2} (I_i \cap I_s)$$

contrary to the choice of I_1, I_2, I_3. This proves (3).

So we may assume that there is a clique of size ω which is a block.

(4) For every $v \in V(G)$, $|V(G) \setminus N_G(v)| \geq i + 1$.

Suppose there exists $v \in V(G)$ such that $|V(G) \setminus N_G(v)| = j \leq i$. Let $N = N_G(v) \cup \{v\}$ and $M = V(G) \setminus N$. The graph $H = G|N$ is the complement of a bipartite graph and so by König's theorem [8] we can color it with ω colors. Since $|M| = j - 1$, we can color M with $j - 1$ colors. But now $\chi(G) \leq \omega + j - 1 < \omega + i$, which is a contradiction. This proves (4).

(5) If $n < \omega + 2i$, then the theorem holds.

Let $v \in V(G)$. By (1), $|N_G(v)| \geq \omega + i - 1$ and so $|V(G) \setminus N_G(v)| \leq i$, contradicting (4). This proves (5).

(6) If $n = \omega + 2i + j$ with $0 \leq j < i$, then the theorem holds.

Let $V(G) = \{b_1, \dots, b_i, a_1, \dots, a_\omega, c_1, \dots, c_i, d_1, \dots, d_j\}$ in order clockwise, such that $\{a_1, \dots, a_\omega\}$ is a clique.

(6.1) For $k \leq i$, a_k is non-adjacent to c_k.

Suppose otherwise. We claim that a_k is non-adjacent to one of $c_1, \dots, c_{k-1}$, for otherwise we would get a clique of size greater than ω. It follows that a_k is complete to the set $\{c_{k+1}, \dots, c_i, d_1, \dots, d_j, b_1, \dots, b_i\}$. But then $|V(G) \setminus N_G(v)| \leq i$, contradicting (4). This proves (6.1).

By symmetry, $a_{\omega-k+1}$ is non-adjacent to b_{i-k+1}. Define a coloring $c : V(G) \to \{1, \dots, \omega + j\}$ as follows:

$$\begin{aligned} c(a_k) &= k \\ c(c_k) &= k \\ c(b_k) &= \omega + k - i \\ c(d_k) &= \omega + k. \end{aligned}$$

Since $i \leq \frac{\omega}{2}$ by Theorem 1.2, it follows from (6.1) that this is a coloring of G. But now $\chi(G) \leq \omega + j$ and $j < i$, which is a contradiction. This proves (6).

So we may assume that $n = \omega + mi + j$ where $m \geq 3$ and $0 \leq j \leq i - 1$. Let $V(G) = \{a_1, \dots, a_\omega, b_1, \dots, b_k\}$ in order clockwise, such that $\{a_1, \dots, a_\omega\}$ is a clique. Let $A = \{a_1, \dots, a_\omega\}$ and $B = \{b_1, \dots, b_k\}$.

(7) If there exist i vertex disjoint paths $S_1, \dots, S_i$ such that the ends of S_t are b_t and $b_{(m-1)i+j+t}$ for $1 \leq t \leq i$, and $\bigcup_{j=1}^{i} V(S_j) \subseteq B$, then the theorem holds.

By (2), b_t is adjacent to $b_{t'}$ for $1 \leq t < t' \leq i$ and so $S_1, \dots, S_i$ are pairwise adjacent. By (1), for every $1 \leq s \leq \omega$ and $1 \leq t \leq i$, a_s is adjacent to at least one of b_t and $b_{(m-1)i+j+t}$. But now, since A is a clique, A together with $S_1, \dots, S_i$ form $\omega + i$ vertex disjoint connected subgraphs of G that are pairwise adjacent. This proves (7).

(8) If there is a clique of size $2i$ in B, then the theorem holds.

Let the clique be $K = \{b_{s+1}, \dots, b_{s+2i}\}$. Let I be a set of consecutive integers such that $I \subseteq \{1, \dots, k\}$ and $|I| \geq i$. Let $B' = \{b_i : i \in I\}$ and let $G' = G|B'$. We claim that G' is i-connected. Suppose not. Then there exists $X \subset B'$ such that $|X| < i$, $B' \setminus X = B_1 \cup B_2$, B_1 is anticomplete to B_2, and both B_1 and B_2 are non-empty. But then there exists $b_s \in B_1$ and $b_t \in B_2$ such that $|t - s| \leq i$. By symmetry, we may assume that $t > s$. But then b_t is non-adjacent to every $b_{s'}$ with $s' \leq s$, contradicting (2). This proves the claim.

Now by Menger's theorem [9] and the claim, there exist i vertex disjoint paths $P_1, \dots, P_i$ such that the ends of P_t are b_t and some vertex in $\{b_{s+1}, \dots, b_{s+i}\}$, and $V(P_t) \subseteq \{b_1, \dots, b_{s+i}\}$; and

i vertex disjoint paths $Q_1, \ldots, Q_i$ such that the ends of Q_t are $b_{(m-1)i+j+t}$ and some vertex of $\{b_{s+i+1}, \ldots, b_{s+2i}\}$, and $V(Q_t) \subseteq \{b_{s+i+1}, \ldots, b_k\}$ (note that some of P_t or Q_t may consist of just a single vertex). Since K is a clique, for every $1 \leq t \leq i$ the end of P_t in K is adjacent to the end of Q_t in K. Hence, there exist i vertex disjoint paths $S_1, \ldots, S_i$ such that the vertex set of S_t is $V(P_t) + V(Q_t)$. Therefore, the ends of S_t are b_t and $b_{(m-1)i+j+t}$, and $V(S_t) \subseteq B$. Now by (7), (8) follows.

(9) If m is even, then the theorem holds.

Since m is even, it follows that $m \geq 4$. Suppose first that for $2i+1 \leq s \leq 3i$, b_s is adjacent to b_{s+i+j}. By (2) and the assumption,

$$b_q - b_{i+q} - b_{2i+q} - b_{3i+q+j} - b_{4i+q+j} - \cdots - b_{(m-1)i+q+j}$$

is a path for $1 \leq q \leq i$. But now, by (7), the theorem holds. So we may assume that $b_s b_{s+i+j}$ is not an edge for some $s \geq 2i+1$. Since by (8) $b_s b_{s-2i+1}$ is not an edge, and by (1) $deg(b_s) \geq \omega + i - 1$, it follows that $(i+j-1) + 2i - 1 \geq \omega + i - 1$ and so $2i + j \geq \omega + 1$. But now we can define a coloring $c : V(G) \to \{1, \ldots, \omega + j\}$ as follows:

$$c(a_t) = t \text{ for } 1 \leq t \leq \omega$$

$$c(b_t) = \begin{cases} t - si & \text{for} \quad s = 0, 2, \ldots, m-2 \text{ and } 1 + si \leq t \leq 2i + si \\ \omega + t - mi & \text{for} \quad mi + 1 \leq t \leq mi + j \end{cases}$$

We claim that c is a coloring of G. The only colors used on more than one vertex are the colors $1, \ldots, 2i$. Let $1 \leq t \leq 2i$. Then the vertices $v \in V(G)$ with $c(v) = t$ are a_t and b_{t+is} for $s = 0, 2, \ldots, m-2$. It suffices to check that each one of these vertices is non-adjacent to the next one in clockwise order. By (8), b_{si+t} is non-adjacent to $b_{(s+2)i+t}$. Next, since there is no edge of length greater than ω a_t is non-adjacent to b_t, and since $2i + j > \omega$ it follows that $b_{(m-2)i+t}$ is non-adjacent to a_t. Hence, c is a coloring of G with $\omega + j < \omega + i$ colors, which is a contradiction. This proves (9).

In view of (9), we may assume from now on that m is odd.

(10) If $\omega \geq 3i$ and $m \geq 4$, then the theorem holds.

We have $\omega + i - 1 \geq 4i - 1$ and so every vertex has an edge of length at least $2i$ in some direction. For b_{2i} this implies that there is a clique of size $2i$ in B and so the result holds by (6). This proves (10).

(11) If $\omega \geq 3i$ and $m = 3$, then the theorem holds.

Let $c : V(G) \to \{1, \ldots, \omega + i - 1\}$ be a coloring defined as follows:

$$c(a_t) = t \text{ for } 1 \leq t \leq \omega$$

$$c(b_t) = \begin{cases} t & \text{for} \quad 1 \leq t \leq i + j + 1 \\ \omega + t - i - j - 1 & \text{for} \quad i + j + 2 \leq t \leq 2i + j \\ \omega + t - 3i - j & \text{for} \quad 2i + j + 1 \leq t \leq 3i + j \end{cases}$$

We claim that c is a coloring of G. The colors used on more than one vertex are $1, \ldots, i+j+1, \omega - i+1, \ldots, \omega$. We note that since $\omega \geq 3i$, it follows that $\omega - i + 1 > i + j + 1$. Let $1 \leq t \leq i + j + 1$. Then the vertices $v \in V(G)$ with $c(v) = t$ are a_t and b_t. We check that these vertices are non-adjacent. In the clockwise direction there is no edge between a_t and b_t because there is no clique of size greater than ω and in the counterclockwise direction there is no edge by (8). Now let $\omega - i + 1 \leq t \leq \omega$. Then the vertices $v \in V(G)$ with $c(v) = t$ are a_t and $b_{t+3i+j-\omega}$. By symmetry with the previous case these vertices are non-adjacent. Hence we can color G with fewer than ω colors, a contradiction. This proves (11).

(12) If $3i > \omega > 2i$, then the theorem holds.

Let $c : V(G) \to \{1, \dots, \omega + i - 1\}$ be a coloring defined as follows:

$$c(a_t) = t \text{ for } 1 \leq t \leq \omega$$

$$c(b_t) = \begin{cases} \omega + t & \text{for} & 1 \leq t \leq j \\ t - is - j & \text{for} & s = 0, 2, \dots, m-3 \text{ and } 1 + is + j \leq t \leq 2i + is + j \\ \omega + t - (m-1)i - j - 1 & \text{for} & (m-1)i + 1 + j \leq t \leq mi + j \end{cases}$$

Once again, we claim that c is a coloring of G. The colors used on more than one vertex are $1, \dots, 2i, \omega, \dots, \omega + j$. Let $1 \leq t \leq 2i$. Then the vertices $v \in V(G)$ with $c(v) = t$ are a_t and b_{t+is+j} for $s = 0, 2, \dots, m-3$. Once again, it suffices to check that each one of these is non-adjacent to the next one in clockwise order. By (8), b_{si+t} is non-adjacent to $b_{(s+2)i+t}$. Next, $b_{(m-3)i+j+t}$ is non-adjacent to a_t since $3i > \omega$ and there is no edge of length greater than ω. Similarly, a_t is non-adjacent to b_{t+j}. Now let $\omega + 1 \leq t \leq \omega + j$. The vertices with color t are $b_{t-\omega}$ and $b_{k+t-\omega-i+1}$. Since in the clockwise direction from $b_{t-\omega}$ there are more than $2i$ vertices between $b_{t-\omega}$ and $b_{k+t-\omega-i+1}$, there is no edge in that direction by (8). In the other direction, there are at least $\omega - 1$ vertices strictly between them, and therefore they are non-adjacent. Finally, the two vertices with color ω are a_ω and b_{k-i+1}. These two are non-adjacent for the same reasons as in the case $\omega + 1 \leq t \leq \omega + j$. Hence c is a coloring of G with $\omega + i - 1$ colors, which is a contradiction. This proves (12).

Now by Theorem 1.2 we may assume that $\omega = 2i$. (13) If $j < i - 1$ then the theorem holds.

Once again, we find a coloring of G with fewer than $\omega + i$ colors, thus obtaining a contradiction. Let c be a coloring defined as follows:

$$c(a_t) = t \text{ for } 1 \leq t \leq \omega$$

$$c(b_t) = \begin{cases} \omega + t & \text{for} & 1 \leq t \leq j + 1 \\ t - is - j - 1 & \text{for} & s = 0, 2, \dots, m-3 \text{ and } 2 + j + is \leq t \leq 2i + j + 1 + is \\ \omega + t - (m-1)i - 1 - j & \text{for} & (m-1)i + j + 2 \leq t \leq mi + j \end{cases}$$

We claim that c is a coloring of G. Since $\omega = 2i$ the colors used on more than one vertex are $1, \dots, \omega + j + 1$. Let $1 \leq t \leq \omega = 2i$. Then the vertices $v \in V(G)$ with $c(v) = t$ are a_t and b_{t+is+j} for $s = 0, 2, \dots, m-3$. It suffices to check that each of these is non-adjacent to the next one in clockwise order. By (8), b_{si+t} is non-adjacent to $b_{(s+2)i+t}$ and $b_{(m-3)i+j+t}$ is non-adjacent to a_t. Next, a_t is non-adjacent to b_{t+j} since there is no edge of length greater than ω. Now let $\omega + 1 \leq t \leq \omega + j + 1$. The vertices with color t are $b_{t-\omega}$ and $b_{k+t-\omega-i+1}$. Since $\omega = 2i$, there are at least $\omega - 1$ vertices between $b_{t-\omega}$ and $b_{k+t-\omega-i+1}$ in both directions, and so they are non-adjacent. Hence, c is a coloring of G with $\omega + j + 1 < \omega + i$ colors, which is a contradiction. This proves (13).

In view of (13), we may assume $j = i - 1$. It follows that

$$n = \omega + (m-1)i + 2i - 1.$$

We define the coloring c as follows:

$$c(a_t) = t \text{ for } 1 \leq t \leq \omega$$

$$c(b_t) = \begin{cases} t - is & \text{for} \quad s = 0, 2, \dots, m-3 \text{ and } 1 + is \leq t \leq 2i + is \\ t - (m-1)i + 1 & \text{for} \quad (m-1)i + 1 \leq t \leq (m+1)i - 1 \end{cases}$$

We verify that c is a coloring of G. Since $\omega = 2i$, there are $2i$ colors in this coloring. Let $2 \leq t \leq 2i$. The vertices with color t are a_t, b_{t+is} for $s = 0, 2, \dots, m-3$ and $b_{t+(m-1)i-1}$. By (8), b_{si+t} is non-adjacent to $b_{(s+2)i+t}$ for $s = 0, 2, \dots, m-3$ and $b_{t+(m-3)i}$ is non-adjacent to $b_{t+(m-1)i-1}$. Since $\omega = 2i$, $b_{t+(m-1)i-1}$ is non-adjacent to a_t and similarly a_t is non-adjacent to b_t. The same argument shows that the vertices colored 1 are also pairwise non-adjacent. Hence, c is a coloring of G with ω colors, which is a contradiction. This completes the proof of Theorem 3.1.

□

4 Compositions of linear interval strips

In this section we prove the main theorem for compositions of linear interval strips (meaning that every strip is a linear interval graph rather than a *f*uzzy linear interval graph). We begin with two lemmas.

Lemma 4.1. *Let G be a composition of linear interval strips each of which is a line-graph strip. Then G is a line-graph.*

Proof. Since G is a composition of line-graph strips, there exists a graph S_0 which is the disjoint union of complete graphs with $|V(S_0)| = 2n$, and so that the vertices of S_0 are arranged into pairs $(a_1, b_1), \dots, (a_n, b_n)$, each meeting two cliques of S_0; and a family of line-graph strips (S'_i, a'_i, b'_i) where $i \in \{1, \dots, n\}$; and $G = S_n$, where S_i is the graph obtained by composing (S_{i-1}, a_i, b_i) and (S'_i, a'_i, b'_i). Let H be the graph whose vertices are the complete subgraphs of S_0, and such that two vertices of H are adjacent if one of the pairs $(a_1, b_1), \dots, (a_n, b_n)$ meets both of them. Let e_i be the edge of H joining the two cliques of S_0 that contain a_i and b_i. Let H' be obtained from H by replacing e_i with $|V(S_i)'| - 2$ parallel edges for all $i \in \{1, \dots, n\}$. Then G is the line graph of H'. This proves Lemma 4.1. □

Lemma 4.2. *Let G be a quasi-line graph which is a counterexample to Theorem 1.1 with $|V(G)|$ minimum and let K_1, K_2 be two cliques in G. Then there exist $min(|K_1|, |K_2|)$ vertex disjoint paths between K_1 and K_2.*

Proof. Suppose not. Let S be a smallest cutset separating K_1 and K_2. Then Menger's Theorem [9] implies that $|S| < \min(|K_1|, |K_2|)$. It follows that there exists a partition (X_1, X_2) of $V(G) \setminus V(S)$ such that $K_i \subset X_i \cup S$ and there are no edges between X_1 and X_2. Let G_i be the graph obtained from $G|(X_i \cup S)$ by adding an edge $s_1 s_2$ for every pair of non-adjacent vertices $s_1, s_2 \in S$.

(1) *$max(\chi(G_1), \chi(G_2)) \geq \chi(G)$.*

Suppose not. Then there exist colorings of G_1, G_2 with fewer than $\chi(G)$ colors. We can permute the colors of these colorings so that they agree on S and from this obtain a coloring of G with fewer than $\chi(G)$ colors, which is a contradiction. This proves (1).

(2) For all $v \in S$, v has a neighbor in X_1 and in X_2.

Without loss of generality, suppose there exists $v \in S$ with no neighbor in X_1. Then if $v \notin K_1$ we can add v to X_2 and obtain a smaller cutset, $S \setminus \{v\}$ separating K_1 and K_2, contradicting the minimality of S. So $K_1 \subseteq S$. But $|S| < min(|K_1|, |K_2|)$, which is a contradiction. This proves (2).

(3) G_i is a quasi-line graph for $i = 1, 2$.

For $v \in X_i$, v has the same neighbors in G_i as in G and the edges between the neighbors in G_i are a superset of those in G. Hence, the neighbors of v in G_i are still the union of two cliques. For $v \in S$, we claim that the set of neighbors of v in X_i is a clique. For suppose v has two neighbors $x_1, x_1' \in X_1$ that are non-adjacent to each other. By (2), v has a neighbor $x_2 \in X_2$. But now x_1, x_1', x_2 are three pairwise non-adjacent vertices in the neighborhood of v in G, contrary to the fact that G is a quasi-line graph. This proves (3).

Without loss of generality, let $\chi(G_1) \geq \chi(G_2)$. Let $S = \{s_1, \ldots, s_n\}$ and let $\mathbb{P} = \{P_1, \ldots, P_n\}$ be $|S|$ vertex disjoint paths between S and K_2 in G_2 such that $s_i \in P_i$. Such paths exist by Menger's Theorem [9] and the minimality of S. Let $\phi : S \to \mathbb{P}$ be a bijection defined by $\phi(s_i) = P_i$.

By the minimality of $|V(G)|$, there exists a set $\mathbb{S}$ of $\chi(G_1)$ connected disjoint subgraphs of G_1 that are pairwise adjacent in G_1. For $H \in \mathbb{S}$ define $\psi(H)$ by

$$\psi(H) = (H \setminus S) \cup \bigcup_{s \in V(H) \cap S} \phi(S).$$

Then $\psi(H)$ is a subgraph of G. Define $\mathbb{Q} = \{\psi(H) : H \in \mathbb{S}\}$. Then $\mathbb{Q}$ is a set of $\chi(G_1) \geq \chi(G)$ connected disjoint subgraphs of G. We claim that the members of $\mathbb{Q}$ are pairwise adjacent. Suppose not. Choose $Q_1, Q_2 \in \mathbb{Q}$ that are not adjacent. For $i = 1, 2$, let H_i be the member of $\mathbb{S}$ such that $Q_i = \psi(H_i)$. Since K_2 is a clique in G, it follows that not both $V(Q_1)$ and $V(Q_2)$ contain a vertex of K_2, and therefore, not both $V(H_1)$ and $V(H_2)$ contain a vertex of S. Since H_1 and H_2 are adjacent, we deduce that there exist $h_1 \in V(H_1)$ and $h_2 \in V(H_2)$ such that not both h_1, h_2 are in S and h_1h_2 is an edge of G_1. But now by the definition of ψ and G_1, $h_1 \in V(Q_1)$, $h_2 \in V(Q_2)$ and h_1h_2 is an edge of G, contrary to the fact that Q_1 and Q_2 are non-adjacent. This proves the claim. Hence G has a clique minor of size $\chi(G)$, contrary to the fact that G is the minimal counterexample to Theorem 1.1. This completes the proof of Lemma 4.2. □

For two disjoint subsets U, W of $V(G)$ and a coloring c of G, let $m_c(U, W)$ denote the number of *repeated colors* on U and W (the number of colors i such that $i \in c(U) \cap c(W)$). We can now prove the main result of this section.

Theorem 4.3. *Let G be a connected, quasi-line graph with chromatic number χ and with no clique minor of size χ, and subject to that with $|V(G)|$ minimum. Then G is not a composition of linear interval strips.*

Proof. Suppose that G is a non-trivial composition of strips. Let $r > 0$ be an integer and let $\mathbb{F}$ be a family of strips (S_i, a_i, b_i) with $1 \leq i \leq r$ such that G is a composition of the members of $\mathbb{F}$. Let k be the number of members of $\mathbb{F}$ which are not line graph strips.

If $k = 0$, the result follows from [10] and Lemma 4.1. So we may assume $k > 0$ and (S_1, a_1, b_1) is not a line-graph strip. Let $A_1 = N_{S_1}(a_1)$, $B_1 = N_{S_1}(b_1)$, $A_2 = N_G(A_1) \setminus V(S_1)$, and $B_2 = N_G(B_1) \setminus V(S_1)$. Let $C_1 = V(S_1) \setminus (A_1 \cup B_1)$ and $C_2 = V(G) \setminus (V(S_1) \cup A_2 \cup B_2)$. Then $V(G) = A_1 \cup B_1 \cup C_1 \cup A_2 \cup B_2 \cup C_2$. Note that if $C_2 = \emptyset$, and $A_2 = B_2$, then G is a circular interval graph, and the theorem follows from Theorem 3.1. Therefore, we may assume that either $C_2 \neq \emptyset$, or $A_2 \neq B_2$.

Let n be the maximum size of a clique minor in G. Then $n < \chi$. Without loss of generality, we may assume that $|A_1 \cup A_2| \leq |B_1 \cup B_2|$. Then, by Lemma 4.2, there exist $|A_1 \cup A_2|$ vertex disjoint paths between $A_1 \cup A_2$ and $B_1 \cup B_2$ in G. From the definitions of A_1, A_2, B_1, and B_2 it follows that for $i = 1, 2$, $|A_i| \leq |B_i|$ and that there exist $|A_i|$ vertex disjoint paths from A_i to B_i in G_i.

Let G'_1 be the graph obtained from $G|(A_1 \cup B_1 \cup C_1 \cup A_2)$ by making A_2 complete to B_1. Then since there exist $|A_2|$ vertex disjoint paths between A_2 and B_2 in G_2, it follows that G'_1 is a minor of G. We claim that G'_1 is a quasi-line graph. For $v \in C_1$, v has the same neighbors in G'_1 as in G and the edges between those neighbors are likewise the same. For $v \in A_1$, we claim that the set of neighbors of v in $B_1 \cup C_1$ is a clique. For suppose v has two neighbors $v_1, v_2 \in B_1 \cup C_1$ that are non-adjacent to each other. Let $v_3 \in A_2$. Then v_1, v_2, v_3 are three pairwise non-adjacent vertices in the neighborhood of v in G, contrary to the fact that G is claw-free. This proves the claim. Therefore, for $v \in A_1$, the set of neighbors of v in G'_1 is the union of two cliques, namely $N_{G|(B_1 \cup C_1)}(v)$ and $(A_1 \cup A_2) \setminus \{v\}$. Similarly, for $v \in B_1$ the set of neighbors of v in $A_1 \cup C_1$ is a clique and the set of neighbors of v in G'_1 is the union of two cliques. Finally, for $v \in A_2$, the set of neighbors of v is the union of two cliques, namely $A_1 \cup A_2 \setminus \{v\}$ and B_1. This proves the claim.

Similarly, let G'_2 be the graph obtained from $G|(A_2 \cup B_2 \cup C_2 \cup A_1)$ by making A_1 complete to B_2. Then G'_2 is also a quasi-line graph and a minor of G. Since (S_1, a_1, b_1) is not a line-graph strip, it follows that $|V(G'_2)| < |V(G)|$; and since either $C_2 \neq \emptyset$, or $A_2 \neq B_2$, it follows that $|V(G'_1)| < |V(G)|$. Since G'_1 and G'_2 are minors of G, it follows that they contain no clique minors of size greater than n and so by the minimality of $|V(G)|$, it follows that $\chi(G'_1), \chi(G'_2) \leq n$. For $i = 1, 2$, let c'_i be an n-coloring of G'_i. Further, let $m_i = m_{c'_i}(A_i, B_i)$, $a_i = |A_i| - m_i$, and $b_i = |B_i| - m_i$. Then $m_1 + a_1 + b_1 + |A_2| = m_1 + a_1 + b_1 + m_2 + a_2 \leq n$ and $m_2 + a_2 + b_2 + |A_1| = m_2 + a_2 + b_2 + m_1 + a_1 \leq n$.

Suppose that $b_1 \leq a_2$. Then since $a_1 \leq b_1$ and $a_2 \leq b_2$, it follows that $a_1 \leq b_2$. Notice c'_1 induces an n-coloring c_1 of G_1 with $m_{c_1}(A_1, B_1) = m_1$. Let $T = \{1, \ldots, n\}$. Without loss of generality, $c_1(V(G_1)) \subseteq T$. Construct the following coloring c of G. For $v \in A_1 \cup B_1 \cup C_1$ let $c(v) = c_1(v)$. Next, use m_2 colors of $T \setminus c(A_1 \cup B_1)$ on both A_2 and B_2 (this is possible since $|c(A_1 \cup B_1)| = a_1 + b_1 + m_1$ and $a_1 + b_1 + m_1 + m_2 \leq n$). Next, use a_2 colors of $c(B_1) \setminus c(A_1)$ on the remaining vertices of A_2 and b_2 colors of $c(A_1) \setminus c(B_1)$ on the remaining vertices of B_2 (this is possible because $b_1 \leq a_2$ and $a_1 \leq b_2$). Now since $m_{c'_2}(A_2, B_2) = m_2$ it follows that the coloring constructed so far can be extended to an n-coloring of G_2 using the colors of T. We see that $c(A_2)$ is disjoint from $c(A_1)$ and $c(B_2)$ is disjoint from $c(B_1)$. Thus c is an n-coloring of G, a contradiction.

Hence, $b_1 > a_2$ and from symmetry $b_2 > a_1$. Let T be as before. We construct the following coloring c of G. We use $|B_1| + |B_2|$ distinct colors of T on $B_1 \cup B_2$ (this is possible since $B_1 \cup B_2$ is a clique in G). Next, we use m_1 colors of $c(B_1)$ and a_1 colors of $c(B_2)$ to color A_1. Then we use m_2 colors of $c(B_2) \setminus c(A_1)$ and a_2 colors of $c(B_1) \setminus c(A_1)$ to color A_2 (this is possible because $b_2 > a_1$ and $b_1 > a_2$). Now since $m_{c'_1}(A_1, B_1) = m_1$ we can extend c to an n-coloring of G_1 using the colors of T and since $m_{c'_2}(A_2, B_2) = m_2$ we can extend c to an n-coloring of G_2 using the colors of T. Once again we see that $c(A_2)$ is disjoint from $c(A_1)$ and $c(B_2)$ is disjoint from $c(B_1)$. But now c is an n-coloring of G, a contradiction. This proves Theorem 4.3. □

5 Homogeneous Pairs

We start with a lemma, that appears in [3], but we include its proof here, for completeness.

Lemma 5.1. *Let G be a quasi-line graph and let (A, B) be a homogeneous pair of cliques, such that A is not complete and not anticomplete to B. Let H be a graph obtained from G by arbitrarily changing the adjacency between some vertices of A and some vertices of B (all the other adjacencies remain unchanged). Then H is a quasi-line graph.*

Proof. Let C be the set of vertices of G that are A-complete and B-complete, D be the set of vertices of G that are A-complete and B-anticomplete, E the set of vertices of G that are A-anticomplete and B-complete, and F the set of vertices of G that are A-anticomplete and B-anticomplete. Let $v \in V(H)$. We need to show that the set $N_H(v)$ is the union of two cliques of H. We do so by considering the following cases:

1. $v \in D \cup E \cup F$. In this case $H|(N_H(v)) = G|(N_G(v))$ and hence $H|(N_H(v))$ is the union of two cliques since G is a quasi-line graph.

2. $v \in A \cup B$. From the symmetry, we may assume that $v \in A$. Let $B(v) = N_H(v) \cap B$. Then $N_H(v) = A \cup B(v) \cup C \cup D$. Let $a \in A$ be a vertex such that $N_G(a) \cap B \neq \emptyset$. Let $B(a) = N_G(a) \cap B$. Then $N_G(a) = A \cup B(a) \cup C \cup D$. Since G is a quasi-line graph, there exist two cliques, X_1, X_2, of G, such that $X_1 \cup X_2 = N_G(a)$. Since B is anticomplete to D, we may assume that $D \subseteq X_1$, and $B(a) \subseteq X_2$. This implies that $C = (C \cap X_1) \cup (C \cap X_2)$. Let $X'_1 = A \cup D \cup (C \cap X_1)$ and $X'_2 = B(v) \cup (C \cap X_2)$. Then X'_1, X'_2 are both cliques of H, and $V_H(v) = X'_1 \cup X'_2$, as required.

3. $v \in C$. First, we note that v has no neighbors in F. Suppose v has a neighbor $f \in F$. Since (A, B) is a non-trivial homogeneous pair in G, there exist $a \in A$ and $b \in B$ that are non-adjacent. But then f, a, b are three pairwise non-adjacent vertices in $N_G(v)$, contrary to the fact that G is a quasi-line graph. This implies that $N_H(v) \subseteq A \cup B \cup C \cup D \cup E$. We observe that $N_H(v) = N_G(v)$. Let $C(v) = C \cap N_H(v)$, $D(v) = D \cap N_H(v)$, $E(v) = E \cap N_H(v)$. Since G is a quasi-line graph, it follows that there exist two cliques X_1, X_2 of G, such that $A \cup B \cup C(v) \cup D(v) \cup E(v) = X_1 \cup X_2$. Let $a \in A$ and $b \in B$ be non-adjacent in G. We may assume that $a \in X_1$ and $b \in X_2$. Since X_1, X_2 are cliques, it follows that $D(v) \subseteq X_1$ and $E(v) \subseteq X_2$. Since $N_G(v) = X_1 \cup X_2$, it follows that $C(v) = (C(v) \cap X_1) \cup (C(v) \cap X_2)$. Let $X'_1 = A \cup D(v) \cup (C(v) \cap X_1)$ and $X'_2 = B \cup E(v) \cup (C(v) \cap X_2)$. then X'_1, X'_2 are both cliques of H, and $N_H(v) = X_1 \cup X_2$, as required.

This proves 5.1. □

Next we prove a lemma that allows us to handle non-trivial homogeneous pairs.

Lemma 5.2. *Let G be a quasi-line graph. Assume that G admits a non-trivial homogeneous pair of cliques. Then there exists a graph H with the following properties:*

1. *H is a quasi-line graph, and $|E(H)| < |E(G)|$.*

2. *For every coloring of H, there is a coloring of G with the same number of colors.*

3. *H is a minor of G.*

Proof. Let (A, B) be a non-trivial homogeneous pair of cliques in G. Let C be the set of vertices of G that are A-complete and B-complete, D be the set of vertices of G that are A-complete and B-anticomplete, E the set of vertices of G that are A-anticomplete and B-complete, and F the set of vertices of G that are A-anticomplete and B-anticomplete. We note that both D and E are cliques. Let $J = G|(A \cup B)$. Then $\overline{J}$ is bipartite. Let M be a maximum matching in $\overline{J}$ and let $|M| = m$. We claim that we can color J with $|A| + |B| - m$ colors. This follows from the fact that we can color the vertices of M with m colors and $|A \cup B \setminus V(M)| = |A| + |B| - 2m$.

By König's Theorem [8], $|M|$ equals the minimum size of a vertex cover of $\overline{J}$, that is, the minimum number of vertices hitting all edges of $\overline{J}$. Let X be a minimum vertex cover of $\overline{J}$. Then $A \setminus X$ is complete to $B \setminus X$ in G.

Let $A' = A \cap X$ and $B' = B \cap X$. Let H be the graph obtained from G by deleting the edges between the members of A' and the members of B and the edges between the members of B' and the members of A. Then $|E(H)| < |E(G)|$. Since (A, B) is a non-trivial homogeneous pair in G, it follows that A is not complete and not anticomplete to B in G, and therefore, by 5.1, H is a quasi-line graph, and thus the first assertion of the theorem holds.

To prove the second assertion of the lemma we need to show that every coloring of H can be used to obtain a coloring of G using the same number of colors. Let c_H be a coloring of H. Recall that $J = G|(A \cup B)$ can be colored using $|A| + |B| - m$ colors. Now since $(A \setminus A') \cup (B \setminus B')$ is a clique in H and $|A' \cup B'| = m$, it follows that every coloring of $H|(A \cup B)$, and in particular c_H, uses at least $|A| + |B| - m$ colors. Hence, at most m colors appear on both A and B. We construct a coloring of G as follows. We use each of the colors of c_H that appears on both A and B to color the vertices of $V(M)$ and the rest of the colors of c_H, which are all different, to color the remaining vertices in $A \cup B$. This yields a coloring of J. We keep the colors of the vertices of $V(G) \setminus (A \cup B)$ unchanged. The coloring just defined is a proper coloring of G, and it uses the same number of colors as c_H. This proves the second assertion of the lemma.

Since H is a subgraph of G the third assertion of the lemma follows. This completes the proof of Lemma 5.2.

□

We are now ready to prove the main result of this paper.

Proof of Theorem 1.1. Let G be a quasi-line graph. The proof is by induction on $|E(G)|$. We may assume that G is connected. If G has no non-trivial homogeneous pairs, then by Theorem 2.1, G is either a circular interval graph or a composition of linear interval strips, and the result follows from Theorem 3.1 and Theorem 4.3. Otherwise, let (A, B) be a non-trivial homogeneous pair in G. Let H be a graph as in Lemma 5.2. Inductively, since $|E(H)| < |E(G)|$, it follows that H has a clique minor of size $\chi(H)$. By property (2) of Lemma 5.2, $\chi(H) \geq \chi(G)$ and by property (3) of the same lemma, H is a minor of G, and so every clique minor of H is also a clique minor of G. Hence, G has a clique minor of size $\chi(H) \geq \chi(G)$. This completes the proof of the theorem.

□

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References

[1] K. Appel and W. Haken, Every planar map is four colorable. Part I: Discharging, *Illinois J. Math.* 21 (1977), 429-490.

[2] K. Appel, W. Haken and J. Koch, Every planar map is four colorable. Part II. Reducibility, *Illinois J. Math.* 21 (1977), 491–567.

[3] M. Chudnovsky and P. Seymour, The Structure of claw-free graphs, Surveys in Combinatorics 2005, 153-171, London Math. Soc. Lecture Note Ser., 327, Cambridge Univ. Press, Cambridge, 2005.

[4] M. Chudnovsky and P. Seymour, Claw-free graphs II. Circular interval graphs, *J. Comb. Theory, Ser. B* 98(2008), 249-290.

[5] M. Chudnovsky and P. Seymour, Claw-free graphs VII. Quasi-line graphs, *in preparation*

[6] M. Chudnovsky and A. Ovetsky, Coloring quasi-line graphs, *J. Graph Theory* 54(2007), 41-50.

[7] H. Hadwiger, *Ü*ber eine Klassifikation der Streckenkomplexe, *Vierteljschr. Naturforsch. Ges. Zürich*, 88 (1943), 133-142.

[8] D. König, "*Ü*ber Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre", *Math. Ann.* 77 (1916), 453-465.

[9] K. Menger, Zur allgemeinen Kurventheorie, *Fund. Math.* 10 (1927), 95-115.

[10] B. Reed and P. Seymour, Hadwiger's conjecture for line graphs, *European J. Combin.* 25 (2004), no. 6, 873-876.

[11] N. Robertson, P. Seymour, and R. Thomas, Hadwiger's conjecture for K_6-free graphs, *Combinatorica* 13 (1993), 279-361.

[12] K. Wagner, *Ü*ber eine Eigenschaft der ebenen Komplexe, *Math. Ann.* 114 (1937).