Localized Erdős-Pósa Property for Subdivisions

Icey Siyi Ai*, Maria Chudnovsky*†, and Julien Codsi *†‡

Abstract

For a graph H, we say that H has the Erdős-Pósa property for subdivisions with function f, if for every graph G, either G contains (as a subgraph) $k+1$ pairwise disjoint subdivisions of H or there exists a set $X \subseteq G$ such that $G \setminus X$ contains no H-subdivision and $|X| \leq f(k)$. We show that every H that has the Erdős-Pósa property for subdivision also satisfies a localized version of the Erdős-Pósa property, as follows. Let H be an n-vertex graph with $m \geq 1$ edges that has the Erdős-Pósa property for subdivisions with function f, and let G be a graph that does not contain $k+1$ disjoint subdivisions of H. We demonstrate the existence of a set of at most k vertex disjoint subdivisions of H in G such that in their union, we can find a set X with the property that $G \setminus X$ contains no H-subdivision and $|X| \leq 2^{f(k)}mk + k(m-n)$.

1 Introduction

All graphs in this paper are finite and simple. For graphs G, H, we say that G *contains* H if there exists a subgraph of G that is isomorphic to H. Whenever a different notion of containment is used, it will be stated explicitly.

The interplay between packing and covering in combinatorial structures forms one of the richest areas of modern graph theory. Among the fundamental results in this domain, the Erdős-Pósa theorem [3] stands as a cornerstone, establishing a profound relationship between the maximum number of disjoint cycles in a graph and the minimum size of a feedback vertex set. Originally proved by Paul Erdős and Lajos Pósa in 1965, the theorem states the following:

Theorem 1 (Erdős-Pósa, 1965)**.** *There exists a function $f : \mathbb{N} \to \mathbb{R}^+$ such that for every integer $k \geq 1$ and for any graph G, at least one of the following holds:*

1. *G contains k vertex-disjoint cycles, or*

2. *There exists a subset $X \subseteq V(G)$ with $|X| \leq f(k)$ such that $G \setminus X$ is a forest.*

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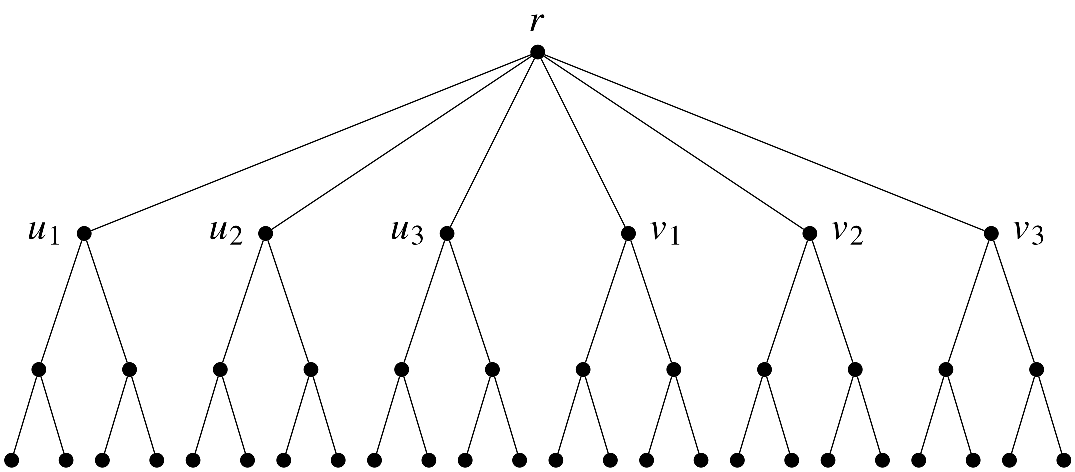

Figure 1: Tree T for the counter example of Question 1

Additionally, Erdős and Pósa showed that the optimal asymptotic bound for this function is $f(k) = O(k \log k)$. A natural question is whether this theorem concerning cycles can be extended to other structures. A graph H is a *minor* of a graph G if H can be obtained from G by a sequence of edge contractions, vertex deletions, and edge deletions. *G contains H as a minor* if there is a minor of G isomorphic to H. An *H-minor in G* is a subgraph of G that contains H as a minor. Robertson and Seymour [5] proved the following generalization of Theorem 1.

Theorem 2 (Robertson, Seymour, 1986)**.** *For every planar graph H, there exists a function $f_H(k)$ such that every graph G contains either k vertex-disjoint subgraphs each containing an H-minor, or a set X of at most $f_H(k)$ vertices such that $G \setminus X$ does not contain any H-minor.*

In this paper, we study variants of Theorem 2 where the set X is "localized" in a certain subgraph of G. First, we ask if it is possible to localize X in any given copy of an H-minor.

Question 1. *Does there exist, for every planar graph H, a constant $c(H)$ with the following property: If G is a graph with no two pairwise disjoint H-minors, then for every H-minor H' of G, there exists $X \subseteq V(H')$ such that $G \setminus X$ has no H-minor and $|X| \leq c(H)$?*

The answer to this question is negative. We present a counterexample here, due to Seymour [6]. Consider H as the tree T illustrated in Fig. 1, and consider G in Fig. 2. We leave it to the reader to verify that this construction indeed serves as a counterexample to Question 1. In light of this counterexample, we ask:

Question 2. *Does there exist, for every planar graph H, a constant $c(H)$ with the following property: If G is a graph with at least a H-minor, but no two pairwise disjoint H-minors, then there exists a H-minor H' in G and $X \subseteq V(H')$ such that $G \setminus X$ has no H-minor and $|X| \leq c(H)$?*

However, one quickly observes that by simply taking $H' = G$, this conjecture would reduce to the original Erdős-Pósa property for planar graphs and fails to capture any notion of localization. To formulate a more meaningful and structurally rich conjecture, we focus on subdivisions.

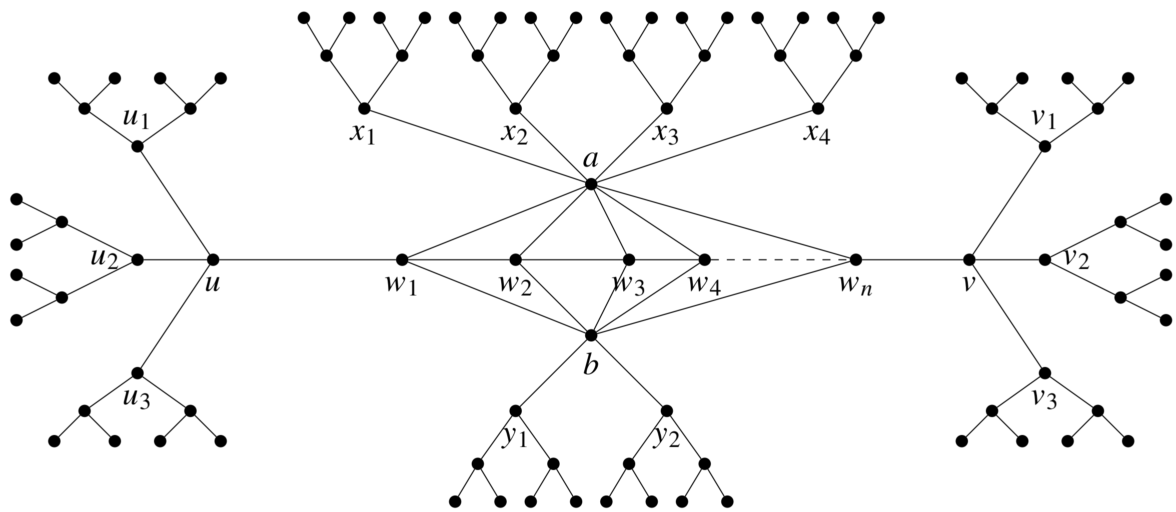

Figure 2: Graph G for the counter example of Question 1

Definition 1 (Subdivisions)**.** Let H be a graph. A *subdivision* of H consists of a graph $\widetilde{H}$ together with a mapping $\phi : V(H) \cup E(H) \longrightarrow V(\widetilde{H}) \cup \{\text{paths in } \widetilde{H}\}$ such that:

1. for each vertex $v \in V(H)$, $\phi(v)$ is a vertex of $\widetilde{H}$;
2. for each edge $e = \{u, v\} \in E(H)$, $\phi(e)$ is a path in $\widetilde{H}$ with endpoints $\phi(u)$ and $\phi(v)$;
3. for every $e, f \in E(H)$, $\phi(e)$ and $\phi(f)$ are vertex disjoint except at the endpoints; and
4. $\widetilde{H}$ is exactly the union of all the images $\phi(v)$ and $\phi(e)$.

The vertices $\phi(V(H))$ are called the *branch vertices*. We say that a graph G *contains a subdivision of* H (or an H*-subdivision*) if G contains a subgraph isomorphic to such a $\widetilde{H}$. When referring to an H-subdivision in G in this paper, we will *suppress the dependence on the specific mapping* ϕ *from the notation*.

We say that H has the *Erdős-Pósa property for subdivisions* if there exists a function $f_H(k)$ such that every graph G contains either k vertex-disjoint H-subdivisions, or a set Z of at most $f_H(k)$ vertices such that $G \setminus Z$ does not contain any H-subdivision. We call f_H the *Erdős-Pósa subdivision bound* for H. We prove

Theorem 3. *For every n-vertex graph H with $m \geq 1$ edges that has the Erdős-Pósa property for subdivisions with f_H as its Erdős-Pósa subdivision bound, the following holds. If a graph G does not contain $k + 1$ pairwise disjoint subdivisions of H, then there exists l with $0 \leq l \leq k$, such that we can find $H_1, H_2, \ldots, H_l$ pairwise disjoint subdivisions of H in G and a set*

$$X \subseteq \bigcup_{i=1}^{l} V(H_i),$$

with $|X| \leq 2^{f(k)} mk + k(m - n)$, and such that $G \setminus X$ is H-subdivision-free.

Qualitatively, Theorem 3 can be interpreted as an equivalence between graphs that have the Erdős-Pósa property for subdivisions and the "local" Erdős-Pósa property for subdivisions (where the set intersecting all the subdivisions is itself contained in the union of a few subdivisions).

This theorem can be specialized to families for which it is known that the Erdős-Pósa property for subdivisions holds. It is a well-known result that

Lemma 1 (from [1])**.** *Let H be a subcubic graph. A graph G contains H as a minor if and only if G contains H as a subdivision.*

Therefore, by combining Theorem 3 with Lemma 1 and Theorem 2, we get

Corollary 1. *For every n-vertex subcubic graph H with $m \geq 1$ edges. Let $f_H(k)$ be the bounding function as defined in Theorem 2. Then, if a graph G does not contain $k+1$ pairwise disjoint subdivisions of H, then there exists l with $0 \leq l \leq k$, such that we can find $H_1, H_2, \ldots, H_l$ pairwise disjoint subdivisions of H in G and a set*

$$X \subseteq \bigcup_{i=1}^{l} V(H_i),$$

with

$$|X| \leq 2^{f_H(k)} mk + k(m-n),$$

and such that $G \setminus X$ is H-subdivision-free.

The bounding function from Theorem 2 is quite large as it depends on the grid minor theorem [5]. For the family of subcubic trees, a more refined bound is known.

Theorem 4 (Dujmović, Joret, Micek, Morin, 2024 [2])**.** *Let F be a forest on t vertices and let t' be the maximum number of vertices in a component of F. For every positive integer k and every graph G, either G contains k pairwise vertex-disjoint subgraphs each having an F-minor, or there exists a set Z of at most $tk - t'$ vertices of G such that $G - Z$ has no F-minor.*

Corollary 2. *For every subcubic tree T with $|V(T)| = n > 1$, if a graph G does not contain $k+1$ pairwise disjoint subdivisions of T, then there exists l with $0 \leq l \leq k$, such that we can find $H_1, H_2, \ldots, H_l$ pairwise disjoint subdivisions of T in G and a set*

$$X \subseteq \bigcup_{i=1}^{l} V(H_i),$$

with

$$|X| \leq 2^{nk}(n-2)k,$$

and such that $G \setminus X$ is T-subdivision-free.

We note that the maximum degree bound in Corollary 2 is tight. It was proven in [7] (Theorem 4.2), that there exist trees with maximum degree 5 for which the Erdős-Pósa property for subdivision does not hold. Their construction can easily be modified to obtain trees with maximum degree 4.

2 The proof

We say that $P = p_1\text{-}\dots\text{-}p_z$ is a path in a graph G if $p_a p_b \in E(G)$ for every $a, b \in \{1, \dots, z\}$ where $|b-a| = 1$. The *interior* of P, denoted by P^*, is the set $P \setminus \{p_1, p_z\}$. A vertex v is an *internal vertex* of P if $v \in P^*$.

Let H be a graph with $|E(H)| = m$ and let k be an integer. Let G be a graph, and let S be a subgraph of G that is the disjoint union of k pairwise disjoint subdivisions of H. Let $V(G) = \{v_1, \dots, v_{|V(G)|}\}$ and order $V(G)$ as $v_1 < \dots < v_{|V(G)|}$. Let $X \subseteq S$ and $Y = \{y_1, \dots, y_{|Y|}\} \subseteq G \setminus S$ where $y_1 < y_2 < \dots < y_{|Y|}$. We say that the triple (S, X, Y) is a (k, H)-*hitting triple* if

- X contains all the branch vertices of S, and
- $G \setminus (X \cup Y)$ is H-subdivision free.

We say that a vertex v is *dangerous for* $y_i \in Y$ if $v \notin S$ and there exists a H-subdivision containing v and y_i but which does not intersect X. We say that a path P is *dangerous for* y_i if $p_z = y_i$ and every internal vertex of P is dangerous for y_i. Note that the property of being dangerous depends on X: as we vary X in the course of the proof, the set of dangerous vertices may change.

Let us partition S into paths whose endpoints are the vertices of X. We denote this set of paths by $\mathcal{P}(S, X, Y)$. We say that a path $P \in \mathcal{P}(S, X, Y)$ is *active for* y_i if there is a path dangerous for y_i from P^* to y_i. Let $N_i(S, X, Y)$ be the number of active paths for y_i.

Let us say that a (k, H)-hitting triple (S, X, Y) is *acceptable* if

$$|\mathcal{P}(S, X, Y)| \le 2^{f_H(k)-|Y|} mk.$$

Let

$$C = 2^{f_H(k)} mk.$$

We define the *score* of a (k, H)-hitting triple (S, X, Y) by

$$score(S, X, Y) = \sum_{i=1}^{|Y|} C^i N_i(S, X, Y).$$

Our strategy to prove Theorem 3 will be to show that there exists an acceptable (k, H)-hitting triple with a score of 0. To do so, we will show that, first, an acceptable (k, H)-hitting triple exists and, second, that it is possible to lower the score of an acceptable (k, H)-hitting triple with a non-zero score.

We will need the following:

Theorem 5 (Menger's Theorem [4])**.** *Let G' be a graph. Let $y \in V(G')$ and $A \subset V(G')$ with $y \notin A$; and let $j \ge 0$ be an integer. Then exactly one of the following holds:*

- *there are j paths in G' from y to A, pairwise disjoint except for y*
- *there is a separation (M, N) of G' with $|M \cap N| < j$ and such that $y \in M \setminus N$ and $A \subseteq N$.*

Proof of Theorem 3. The proof is by induction on k. The base case of $k = 0$ is trivial, and we take an empty set to be set X. We assume the result to be true for all integers up to and including k. Therefore, we can assume that G contains k pairwise disjoint H-subdivisions. Let $n = |V(H)|$ and $m = |E(H)|$.

(1) *For any (k, H)-hitting triple (S, X, Y), we have that $|\mathcal{P}(S, X, Y)| = |X| + k(m - n)$.*

We add the vertices while keeping track of the number of paths created. There are kn branch vertices, which create mk paths. Every subsequent vertex subdivides an existing path into two. Therefore, $|\mathcal{P}(S, X, Y)| = km + (|X| - kn) = |X| + k(m - n)$ which proves (1).

(2) *There exists an acceptable (k, H)-hitting triple.*

Since H has the Erdős-Pósa property for subdivisions, there exists a set $Z \subseteq V(G)$ of at most $f_H(k)$ vertices such that $G \setminus Z$ does not contain any $H-$subdivision. Let S_0 be a subgraph of G that is the disjoint union of k pairwise vertex-disjoint H-subdivisions in G. Let $X_0 = \{v : v \in S_0 \cap Z$ or v is a branch vertex in $S_0\}$ and $Y_0 = Z \setminus S_0$. We check that the triple (S_0, X_0, Y_0) is acceptable. By (1), we have

$$\begin{aligned}
|\mathcal{P}(S, X, Y)| &= |X_0| + k(m - n) \\
&\leq f_H(k) - |Y_0| + nk + k(m - n) \\
&= f_H(k) - |Y_0| + mk \\
&\leq 2^{f_H(k) - |Y_0|} mk.
\end{aligned}$$

Here, the last inequality used the fact that $m \geq 1$. This proves (2).

Among all acceptable (k, H)-hitting triple, let (S, X, Y) be chosen with minimal score and subject to that with Y inclusion-wise minimal (that is, if $Y' \subset Y$, then (S, X, Y') is not a (k, H)-hitting triple). Until stated otherwise, we will assume that $score(S, X, Y) > 0$ and so $|Y| > 0$. Let $d = |Y|$ and $Y = \{y_1, \cdots, y_d\}$ where $y_1 < y_2 < \ldots < y_d$. By the minimality of Y, $N_d(S, X, Y) > 0$.

We define three types of active paths for y_d.

(i) A path P for which there exist two vertex-disjoint (except at y_d) dangerous paths for y_d each with an end in P^*.

(ii) A path P for which there exists $x \in P^*$ such that there is no dangerous path for y_d with an end in $P^* \setminus x$.

(iii) A path P which is not of type (ii) for which there exists $v \in V(G)$ dangerous for y_d such that there is no dangerous path in $G \setminus v$ for y_d with an end in P^*.

Note that, by Theorem 5, every active path for y_d is of exactly one of the types (i), (ii), and (iii).

(3) *If there exists a path of type (i), there exist S', X' and Y' such that (S', X', Y') is a (k, H)-hitting triple, $|X'| = |X| + 1$, and $|Y'| = |Y| - 1$.*

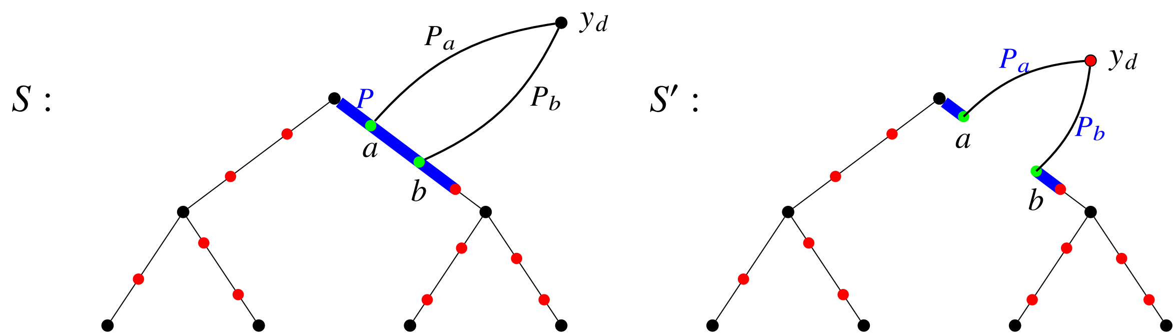

Figure 3: Operation for active paths of type (iii).

Let $a, b \in P^*$ such that there exist two vertex-disjoint (except at y_d) paths P_a and P_b from $\{a, b\}$ to y_d which are dangerous for y_d.

See Fig. 3 for an illustration. Setting $X' = X \cup \{y_d\}$, $Y' = Y \setminus \{y_d\}$, and $S' = (S \setminus a\text{-}P\text{-}b) \cup (P_a \cup P_b)$ with the same branch-vertices as S gives (3).

(4) *If there exists a path of type (ii), there exists X' such that (S, X', Y) is a (k, H)-hitting triple, $|X'| = |X| + 1$ and $N_d(S, X', Y) \leq N_d(S, X, Y) - 1$.*

Setting $X' = X \cup \{x\}$ subdivides P into two paths P_a and P_b. We claim that neither of P_a and P_b is active for y_d. Suppose that P_a is active for y_d. Then there exists a path Q dangerous for y_d (with respect to X') from P_a^* to y_d. Then $x \notin Q$, and every vertex of Q is also dangerous with respect to X. But now Q is a dangerous path from y_d to P^* and $x \notin Q$, a contradiction. This proves (4).

(5) *If there exists a path of type (iii), there exist X' and S' such that (S', X', Y) is a (k, H)-hitting triple, $|X'| = |X| + 1$ and $N_d(S', X', Y) \leq N_d(S, X, Y) - 1$.*

We apply Theorem 5 with $y = y_d$, $A = P^*$, $j = 2$ and G' as the subgraph of G induced by P^* and the dangerous vertices for y_d. As a result, there exists a separation (M, N) with $|M \cap N| = 1$ such that $y_d \in M$ and $P^* \subseteq N$. Among all such separations, we select (M, N) with M inclusion-wise maximal. Let $M \cap N = \{x\}$. We claim that there exist two vertex-disjoint paths P_a, P_b from x to some $a, b \in P^*$. Assume otherwise. Then, again by Theorem 5, we would find a separation (M', N') of N with $|M' \cap N'| = 1$ such that $x \in M' \setminus N'$ and $P^* \subseteq N'$. $(M \cup M', N')$ is now also a separation in G' with $|(M \cup M') \cap N')| = 1$ and with $|M \cup M'| > |M|$, contradicting the maximality of M. This proves the claim. We set $S' = (S \setminus a\text{-}P\text{-}b) \cup (P_a \cup P_b)$, and $X' = X \cup \{x\}$. See Fig. 4

for an illustration. Since all the dangerous paths for y_d with an end in P^* go through x, neither of the new paths is active for y_d. This proves (5).

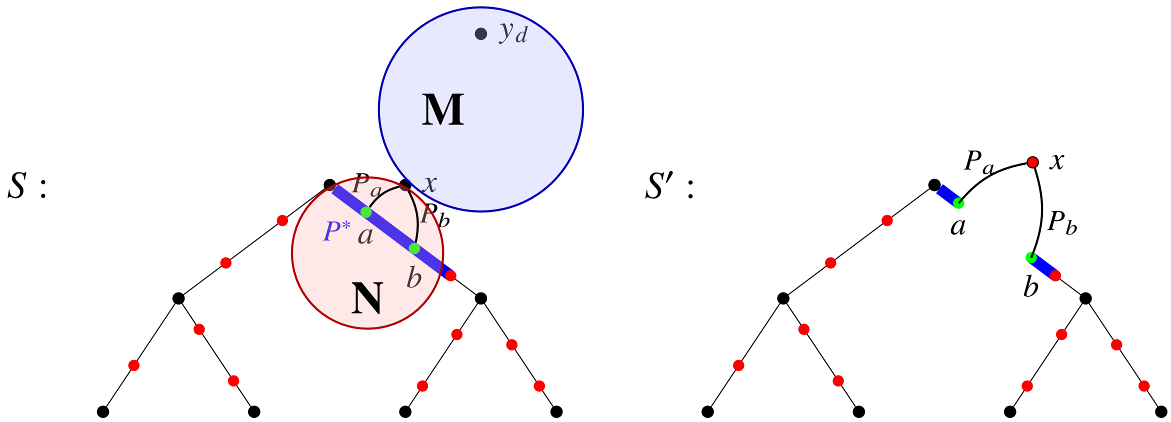

Figure 4: Rerouting from separation.

By applying repeatedly (3), (4) and (5), until there is no active path for y_d, we obtain (S^*, X^*, Y^*).

(6) (S^*, X^*, Y^*) *is acceptable.*

By (1), and since (S, X, Y) is acceptable, we have

$$\begin{aligned}
|\mathcal{P}(S^*, X^*, Y^*)| &= |X^*| + k(m-n) \\
&\leq N_d(S, X, Y) + |X| + k(m-n) \\
&= N_d(S, X, Y) + |\mathcal{P}(S, X, Y)| \\
&\leq 2|\mathcal{P}(S, X, Y)| \\
&\leq 2^{f_H(k)-(|Y|-1)} \\
&\leq 2^{f_H(k)-|Y^*|}.
\end{aligned}$$

This proves (6).

Finally, let us bound the change in the score from (S, X, Y) to (S^*, X^*, Y^*).

$$\begin{aligned} score(S, X, Y) - score(S^*, X^*, Y^*) &= \sum_{i=1}^{|Y|} C^i N_i(S, X, Y) - \sum_{i=1}^{|Y^*|} C^i N_i(S^*, X^*, Y^*) \\ &\geq \sum_{i=1}^{|Y|} C^i - \sum_{i=1}^{|Y^*|} C^i |\mathcal{P}(S^*, X^*, Y^*)| \\ &\geq \sum_{i=1}^{|Y|} C^i - \sum_{i=1}^{|Y^*|} C^{i+1} \\ &= C + \sum_{i=2}^{|Y|} C^i - \sum_{i=2}^{|Y^*|+1} C^i \\ &\geq C \end{aligned}$$

This contradicts the minimality of the score of (S, X, Y). Therefore, $score(S, X, Y) = 0$, which implies that $Y = \emptyset$ as any vertex with no active paths in Y can be removed to obtain an inclusion-wise smaller (k, H)-hitting triple.

Therefore, by (1) and (6),

$$\begin{aligned} |X| &= |\mathcal{P}(S, X, Y)| - k(m - n) \\ &\leq 2^{f_H(k)} mk - k(m - n) \end{aligned}$$

Thus, S and X satisfy the conclusion of the theorem. □

Remark 1. While Theorem 3 does not allow for the explicit specification of a set S of pairwise disjoint subdivisions in which the hitting set is localized, the proof nevertheless guarantees the existence of a hitting set contained in a family of subdivisions S' sharing the same branch vertices as S. Hence, a nontrivial degree of control over where the localization occurs is preserved. On the other hand, specifying a set S a priori for the localization is impossible, as one can modify the construction in Fig. 1 by splitting r into two vertices, and the graph in Fig. 2 remains a counterexample with subdivisions instead of minors.

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