Tree Independence Number IV. Even-hole-free Graphs*

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Abstract

We prove that the tree independence number of every even-hole-free graph is at most polylogarithmic in its number of vertices. More explicitly, we prove that there exists a constant $c > 0$ such that for every integer $n > 1$ every n-vertex even-hole-free graph has a tree decomposition where each bag has stability (independence) number at most $c \log^{10} n$. This implies that the Maximum Weight Independent Set problem, as well as several other natural algorithmic problems that are known to be NP-hard in general, can be solved in quasi-polynomial time if the input graph is even-hole-free. The quasi-polynomial complexity will remain the same even if the exponent of the logarithm is reduced to 1 (which would be asymptotically best possible).

1 Introduction

A graph G is *even-hole-free* if G does not contain a cycle of even length as an induced subgraph. Here an *induced subgraph* of a graph G is a graph that can be obtained from G by deleting vertices (and all edges incident to the deleted vertices), the *length* of a cycle (or path) is the number of edges in it, and a *hole* in G in an induced subgraph that is a cycle of length at least four. The class of even-hole-free graphs has attracted much attention due to its somewhat tractable, yet quite rich structure [17, 20, 46]. In particular the structure of even-hole-free graphs has some similarities [46] with the structure of perfect graphs, which by the strong perfect graph theorem [16] are precisely the graphs that are odd-hole-free and whose complement is odd-hole-free. In addition to their structure, much effort was put into designing efficient algorithms for even-hole-free graphs (to solve problems that are NP-hard in general). This is discussed in the survey [46], while [1, 12, 19, 38] provide examples of more recent work. We now consider some of the problems that have received the most attention on even-hole free graphs.

A vertex set S of a graph G is *stable* (or *independent*) if no two vertices of S have an edge between them. A *clique* is a set S of vertices such that every pair of vertices of S has an edge between them. A *proper k-coloring* of G is a partition of the vertex set of G into (at most) k independent sets. In the Maximum Weight Independent Set (Maximum Weight

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Clique) problem, the input is a graph G and a weight function that assigns to each vertex an integer weight. The task is to find a stable set (clique) S in G of maximum weight. In the k-Coloring problem, the input is a graph G, the task is to determine whether G has a proper k-coloring. Finally, in the Coloring problem, the input is a graph G and the task is to determine the minimum k such that G has a proper k-coloring. All of the above mentioned problems are known to be NP-hard [32, 36]. On even-hole-free graphs, Maximum Weight Clique is known to be polynomial-time solvable [46]. The questions of whether or not there exist polynomial time algorithms for Maximum Weight Independent Set and Coloring remain open. This is in stark contrast to perfect graphs, for which polynomial time algorithms for these problems have been known since 1981 [34].

This discrepancy is somewhat surprising, to explain why (and to state our main result) we need to define tree decompositions and treewidth. For a graph $G = (V(G), E(G))$, a *tree decomposition* (T, χ) of G consists of a tree T and a map $\chi : V(T) \to 2^{V(G)}$ with the following properties:

- For every $v_1v_2 \in E(G)$, there exists $t \in V(T)$ with $v_1, v_2 \in \chi(t)$.
- For every $v \in V(G)$, the subgraph of T induced by $\{t \in V(T) \mid v \in \chi(t)\}$ is non-empty and connected.

The *width* of a tree decomposition (T, χ), denoted by $\text{width}(T, \chi)$, is $\max_{t \in V(T)} |\chi(t)| - 1$. The *treewidth* of G, denoted by $\text{tw}(G)$, is the minimum width of a tree decomposition of G. Bounded treewidth is a useful graph property from both a structural [41] and an algorithmic [10] perspective. For example Maximum Weight Independent Set, Maximum Weight Clique, k-Coloring (for every fixed k) and a host of other problems are known to admit $O(c^t n)$ time algorithms on graphs of treewidth at most t [21, 28, 31], while Coloring is known to admit $O(t^{t+O(1)}n)$ time algorithms on graphs of treewidth at most t.

From the perspective of tree decompositions and treewidth, perfect graphs appear much more intractable than even-hole-free graphs. On one hand there exist triangle-free (a triangle is a clique on 3 vertices) perfect graphs whose treewidth is *linear* in the number of vertices in the graph: the complete bipartite graph $K_{t,t}$, consisting of two stable sets L and R of size t with an edge connecting every vertex in L with every vertex in R, is an example. On the other hand, triangle-free even-hole-free graphs have constant treewidth [12]. Sintiari and Trotignon [44] give a construction of arbitrarily large K_4-free (K_t is the clique on t vertices) even-hole-free graphs whose treewidth is *logaritmic* in the number of vertices. This led Sintiari and Trotignon [44] to conjecture that for every t there exists a constant c_t such that every n-vertex K_t-free and even-hole-free graph has treewidth at most $c_t \log n$. This conjecture was very recently confirmed by Chudnovsky et al. [13]. The logarithmic treewidth bound of Chudnovsky et al. [13] immediately implies that k-Coloring can be solved in polynomial time for every fixed k on even-hole-free graphs (here we do not need to assume a bound on the clique number).

An early step of the proof of Chudnovsky et al. [13] is to prove that even-hole-free graphs admit "dominated balanced separators". More precisely they show that there exists a constant c such that every even-hole-free graph contains a vertex set S of size at most c such that every connected component of $G - N[S]$ has at most $n/2$ vertices. Here $N[S]$ is the *closed neighborhood* of S, namely the set of all vertices in S and all vertices with at least one neighbor in S. A fairly direct consequence of this result (based on an argument of Chudnovsky et al. [15]) is that for every $\epsilon > 0$ there exists a $(1 + \epsilon)$-approximation algorithm for Maximum Weight Independent Set which runs in quasi-polynomial time. Here an algorithm runs in *quasi-polynomial* time if the running time is upper bounded by $2^{O(\log^c n)}$ for some constant c. Nevertheless, this is fully consistent with Maximum Weight Independent Set on even-hole-free graphs being NP-hard, and so the complexity of Maximum Weight Independent Set and Coloring on even-hole-free graphs remain open.

Our results. We prove a structural result which implies that Maximum Weight Independent Set, as well as a host of other problems, admit quasi-polynomial time algorithms on even-hole-free graphs. To state our main result we need one more notion, that of tree independence number. This is a relatively new width parameter, defined by Dallard, Milanič and Štorgel [25], in the second of a series of papers [22, 23, 24, 25, 26] aiming to identify graphs whose large treewidth can be completely explained by the presence of a large clique. The *tree independence number* of a tree decomposition (T, χ) is the maximum over all $t \in V(T)$ of the maximum stable set size of the subgraph $G[\chi(t)]$ of G induced by $\chi(t)$. The tree independence number of a graph G is the minimum tree independence number of a tree decomposition of G. We are now ready to state our main result.

Theorem 1.1. *There exists a constant c such that for every integer $n > 1$ every n-vertex even-hole-free graph has tree independence number at most $c \log^{10} n$.*

Since the only construction of even-hole-free graphs with large treewidth known to date is the construction of [44], where all graphs have clique number at most four and have treewidth logarithmic in the number of vertices, we do not know if the bound of Theorem 1.1 is asymptotically tight, or whether the exponent of $\log n$ can be reduced. Dallard et al. [22] gave an algorithm that takes as input a graph G and integer k, runs in time $2^{O(k^2)} n^{O(k)}$ and either outputs a tree decomposition of G with tree independence number at most $8k$, or determines that the tree independence number of G is larger than k. Using this algorithm, Theorem 1.1 can be made constructive in the sense that there exists an algorithm which takes as input an even-hole-gree graph, runs in time $2^{O(\log^{20} n)}$ and computes a tree decomposition of G with tree independence $O(\log^{10} n)$.

Theorem 1.1 implies the main result of [13] with a $O(\log^{10} n)$ instead of logarithmic bound on the treewidth. Indeed, let G be a K_t-free even-hole free graph, and let (T, χ) be the tree decomposition obtained from Theorem 1.1. We claim that this decomposition has width $O(\log^{10} n)$. This follows from the fact that every even-hole-free graph on at least $2\alpha t$ vertices either has a clique of size at least t or a stable set of size at least α [17]. Thus every bag of the decomposition must have size at most $c \log^{10} n \cdot 2t$.

A number of problems can be solved efficiently when a tree decomposition of the input graph of low independence number is given as input. This is discussed in more detail in [14] and [39]. We will not repeat the discussion here, but only list some of the problems that can be solved in quasi-polynomial time in the class of even-hole-free graphs as a direct consequence of Theorem 1.1, the above mentioned approximation algorithm of Dallard et al. [22], and existing algorithms when a tree decomposition of the input graph of low independence number (or low width) is given as input.

Theorem 1.2. *For every integer $k \geq 0$, the following problems admit quasi-polynomial time time algorithms on even-hole free graphs:*

- *Maximum Weight Independent Set,*
- *Weighted Feedback Vertex Set,*
- *Weighted Odd cycle transversal,*
- *Maximum Weight Induced Path,*
- *Maximum Weight Induced Matching,*
- *Maximum Weight Induced Subgraph of Treewidth at most k,*
- *k-coloring.*

Resolving the complexity status of Maximum Weight Independent Set on even-hole-free graphs has been stated as an open problem a number of times [2, 12, 19, 35, 38], and the complexity of Feedback Vertex Set on even-hole-free graphs has been posed at least once [45]. Even though Theorem 1.2 does not fully resolve these open problems, it offers a partial resolution in the following sense. If an NP-hard problem has a quasi-polynomial-time algorithm then every problem in NP has a quasi-polynomial-time algorithm. Thus Theorem 1.2 implies that, unless a highly unexpected complexity theoretic collapse occurs, none of the above problems are NP-hard in even-hole-free graphs. It remains an intriguing and challenging open problem to design *polynomial* time algorithms for the problems in Theorem 1.2 (with the exception of k-COLORING, for which a polynomial time algorithm was recently found [13]) on even-hole-free graphs. Polynomial time algorithms for these problems would require significant new ideas. Indeed, even if the exponent of $\log n$ in Theorem 1.1 were reduced to 1, the resulting algorithms would still take quasi-polynomial (rather than polynomial) time.

It is worth noting that Feedback Vertex Set, Maximum Weight Induced Path, and Maximum Weight Induced Matching are all NP-hard on bipartite graphs, and therefore on perfect graphs. This partially confirms the intuition that even-hole-free graphs should be more algorithmically tractable than perfect graphs.

All of the results of Theorem 1.2 (except for k-Coloring) can be derived from the following theorem. The theorem uses Counting Monadic Second Order Logic (CMSO_2), which is a useful formalism to express properties of graphs and vertex and edge sets [11, 39]. We refer the reader to Lima et al. [39] for an introduction to CMSO_2 logic.

Theorem 1.3. *There is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every integer ℓ and $CMSO_2$ formula ϕ, there exists an algorithm that takes as input an even-hole-free graph G and a weight function $w : V(G) \to \mathbb{N}$, runs in time $(f(\phi, \ell)n)^{O(\ell \log^{10} n)}$ and outputs a maximum weight vertex subset S such that $G[S]$ has treewidth at most ℓ and $G[S] \models \phi$.*

Theorem 1.3 is obtained from Theorem 1.1 in (exactly) the same way as Chudnovsky et al. [14] obtain their algorithmic consequences (Theorem 8.2 of [14]) from their bound on the tree independence number (Theorem 1.2 of [14]) of 3PC-free graphs. The algorithm for Odd Cycle Transversal in Theorem 1.2 follows from Theorem 1.3 because on even-hole-free graphs, Odd Cycle Transversal and Feedback Vertex Set are the same problem.

The quasi-polynomial time algorithm of Theorem 1.2 for k-Coloring follows from the fact that K_{k+1} is not k-colorable, the $O(k^{t+O(1)}n)$ time algorithm for k-Coloring on graphs of treewidth t [21], and the $O(\log^{10} n)$ treewidth bound on K_t-free even-hole free graphs which follows from Theorem 1.1)

The list of problems in Theorem 1.2 for which Theorem 1.3 implies a quasi-polynomial-time algorithm is by no means exhaustive. For an example Theorem 1.3 also yields a quasi-polynomial time algorithm to recognize even-hole-free graphs (since we can encode in CMSO_2 that $G[S]$ is a cycle of even length). We refer the reader to [14] and [39].

Comparison with the Algorithmic Consequences of [13]: It is worth comparing the algorithmic consequences of Theorem 1.1 with those of the logarithmic treewidth bound for K_t-free graphs in [13]. The logarithmic treewidth bound for K_t-free even-hole-free graphs in [13] typically leads to *polynomial* time algorithms for problems on K_t-free even-hole-free graphs. With the exception of k-Coloring, for which this leads to a polynomial time algorithm on even-hole-free graphs, for other problems of Theorem 1.2 the results on K_t-free even-hole-free graphs only lead to polynomial (or quasi-polynomial) time approximation schemes on even-hole-free graphs. On the other hand, Theorem 1.1 readily leads to quasi-polynomial time *exact* algorithms on even-hole-free graphs (and hence shows that the considered problems are unlikely to be NP-hard), but does not appear to give any meaningful polynomial time algorithms (neither exact nor approximation).

Organization of the paper. In Section 2 we define the notation and basic definitions used in the paper. In Section 3 we give a brief outline of the proof of Theorem 1.1. The remainder of the paper concerns the proof of Theorem 1.1.

2 Preliminaries

All graphs in this paper are finite and simple and all logarithms are base 2. We begin with some standard definitions (see, for example, [13]). Let $G = (V(G), E(G))$ be a graph. In this paper, we use induced subgraphs and their vertex sets interchangeably. For a graph G and vertex set S, the graph $G \setminus S$ is the graph obtained from G by deleting all vertices in S and all edges incident to at least one vertex in S. The subgraph of G *induced by* S is denoted by $G[S]$ and defined as $G[S] = G \setminus (V(G) \setminus S)$. For graphs G, H we say that G *contains* H if H is isomorphic to an induced subgraph of G. We say that G is *H-free* if G does not contain H. For a set $\mathcal{H}$ of graphs, G is *$\mathcal{H}$-free* if G is H-free for every $H \in \mathcal{H}$.

Let $v \in V(G)$. Let $X \subseteq V(G)$. We denote by $N_G(X)$ the set of all vertices in $V(G) \setminus X$ with at least one neighbor in X. We also define $N_G[X] = N_G(X) \cup X$. When $X = \{v\}$, we write $N_G(v)$ for $N_G(\{v\})$ and $N_G[v]$ for $N_G[\{v\}]$. If there is no danger of confusion, we omit the subscript G. If H is an induced subgraph of G, then $N_H(X) = N(X) \cap H$ and $N_H[X] = N_H(X) \cup X$. Let $Y \subseteq V(G)$ be disjoint from X. We say X is *complete* to Y if every vertex in X is adjacent to every vertex in Y in G, and X is *anticomplete* to Y if there are no edges between X and Y.

A *path* in a graph is an *induced* subgraph that is a path. Given a path P with ends a, b, the *interior* of P, denoted by P^*, is the set $P \setminus \{a, b\}$. The *length* of a path or a hole is the number of edges in it. A hole is *even* if its length is even. A graph is *even-hole-free* it contains no even holes.

The *stability (or independence) number* $\alpha(G)$ of G is the maximum size of a stable set in G. A related parameter, the *clique cover number* $\kappa(G)$ of G, is the smallest number of cliques whose union equals $V(G)$.

Next, we define a slight generalization of even-hole-free graphs; we need the following definitions (see, for example, [13]). A *theta* is a graph consisting of three internally vertex-disjoint paths $P_1 = a\text{-}\cdots\text{-}b$, $P_2 = a\text{-}\cdots\text{-}b$, and $P_3 = a\text{-}\cdots\text{-}b$, each of length at least 2, such that P_1^*, P_2^*, P_3^* are pairwise anticomplete. We call a and b the *ends* of the theta and P_1, P_2, P_3 the *paths* of the theta. A *near-prism* is a graph consisting of two triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, and three paths P_i from a_i to b_i for $1 \leq i \leq 3$, and such that $P_i \cup P_j$ is a hole for all distinct $i, j \in \{1, 2, 3\}$. It follows that P_1^*, P_2^*, P_3^* are pairwise disjoint and anticomplete to each other, $|\{a_1, a_2, a_3\} \cap \{b_1, b_2, b_3\}| \leq 1$, and if $|\{a_1, a_2, a_3\} \cap \{b_1, b_2, b_3\}| = 1$, then two of the paths have length at least 2. Moreover, the only edges between P_i and P_j are a_ia_j and b_ib_j. A *prism* is a near-prism whose triangles are disjoint. A *wheel* in G is a pair (H, x) where H is a hole and x is a vertex with at least three neighbors in H. The vertex x is called the *center* of the wheel (H, x). A wheel (H, x) is *even* if x has an even number of neighbors on H. Let $\mathcal{C}$ be the class of (C_4, theta, prism, even wheel)-free graphs (these are sometimes called "C_4-free odd-signable graphs"). Every even-hole-free graph belongs to $\mathcal{C}$. For every integer $t \geq 1$, let $\mathcal{C}_t$ be the class of all graphs in $\mathcal{C}$ with no clique of size t.

A wheel (H, x) is a *universal wheel* if x is complete to H. A wheel (H, x) is a *twin wheel* if $N(x) \cap H$ induces a path of length two. A wheel (H, x) is a *short pyramid* if $|N(x) \cap H| = 3$ and x has exactly two adjacent neighbors in H. A wheel is *proper* if it is neither a twin wheel nor a short pyramid.

A *pyramid* is a graph consisting of a vertex a and a triangle $\{b_1, b_2, b_3\}$, and three paths P_i from a to b_i for $1 \leq i \leq 3$, such that $P_i \cup P_j$ is a hole for all distinct $i, j \in \{1, 2, 3\}$. It follows that $P_1 \setminus a, P_2 \setminus a, P_3 \setminus a$ are pairwise disjoint, and the only edges between them are of the form b_ib_j. It also follows that at most one of P_1, P_2, P_3 has length exactly 1. We say that a is the *apex* of the pyramid and that $b_1b_2b_3$ is the *base* of the pyramid.

3 Proof Outline

We give here the main ideas of the proof. We will not concern ourselves with the exponent of $\log n$ in our bound and simply show that the tree independence number of G is polylogarithmic in n. The high level scaffolding of the proof is quite similar to the proof of [13] that K_t-free even-hole-free graphs have logarithmic treewidth. However most of the individual high level pieces of the proof differ substantially from the corresponding step in [13] and require significant new ideas.

The proof of [13] that every K_t-free graph has treewidth $O(\log n)$ goes as follows: first it is proved that even-hole-free graphs admit "dominated balanced separators". In particular there exists a constant c such that every even-hole-free graph contains a vertex set S of size at most c such that every connected component of $G - N[S]$ has at most $n/2$ vertices. This fact is then used to show that, in order to obtain the treewidth bound, it is sufficient to show that every pair of non-adjacent vertices a, b in a K_t-free even-hole free graph can be separated from each other by a set of size at most $c_t \log n$ for a constant c_t depending only on t. The most technical part of the argument is then to show the existence of such small a-b separators.

The first step of the proof of [13], that even-hole-free graphs admit dominated balanced separators, does not use the assumption that G is K_t-free and hence we can use it here. Using the dominated balanced separator bound we then show an analog of the second step of [13] tailored to tree independence number rather than treewidth: we show that in order to obtain the tree independence number bound, it is sufficient to prove that every pair of non-adjacent vertices a, b in an even-hole-free graph can be separated from each other by a set with independence number $O(\log^{O(1)} n)$. Here we cannot re-use the proof from [13] because that proof crucially uses the assumption that G is K_t-free. However we can directly apply a similar step from [14]. Thus, "all" that remains is to prove the following statement:

Theorem 3.1. *There exists a constant c with the following property. Let G be an even-hole-free graph with $|V(G)| = n$, and let $a, b \in V(G)$ be non-adjacent. Then there is a set $X \subseteq V(G) \setminus \{a, b\}$ with $\kappa(X) \leq c \log^8 n$ and such that every component of $G \setminus X$ contains at most one of a, b.*

In Theorem 3.1, the clique cover number κ of the separator X is bounded rather than the independence number α because that is what naturally comes out of the proof. Obtaining a separator X with polylogarithmic independence number would also have been sufficient. Let us now not worry too much about the exponent 8 in the statement of Theorem 3.1, and simply aim for a polylogarithmic bound on κ.

Towards this goal we employ a recent tool of Korchemna et al. [37], who provide a "max flow-min cut" like theorem for clique separators. Applying this theorem to even-hole-free graphs (and using that even-hole-free graphs are $K_{2,2}$-free and therefore have only polynomially many maximal cliques [30]) we get one of two outcomes. Either we get an a-b separator X with a polylogaritmic upper bound on $\kappa(X)$, this is the desired outcome, or we get a set of f paths $P_1, \ldots, P_f$ from a to b such that no clique of $G - \{a, b\}$ intersects more than $O(\log n)$ of the paths. Here f can be chosen to be an arbitrarily large polylogarithmic function of n (and the larger we choose f, the worse bound we get on $\kappa(X)$).

Observe that the graph G' induced by all of these a-b paths $P_1, \ldots, P_f$ is K_t-free, where $t = \Omega(\log n)$. Indeed, every vertex of a clique K in G' must be on some a-b path (by the definition of G'), and no path can contain more than 2 vertices of K (since the path is induced), so K must intersect at least $|K|/2$ paths. It follows that G' is K_t-free where $t = \Omega(\log n)$.

If only the constant c_t in the $c_t \log n$ bound on the size of an a-b separator from [13] depended polynomially on t we would be done! Indeed, we could then have chosen f to be so large that $f \geq (c_t \cdot \log n) \cdot \log^2 n$, and get an a-b separator X in G' of size at most $c_t \cdot \log n$. But then some vertex of X belongs to at least $\log^2 n$ of the paths, contradicting that no clique meets more than $O(\log n)$ of them.

Unfortunately the constant c_t in the proof of [13] does not depend polynomially on t, and it does not look like an easy task to improve the dependence to a polynomial in t. However, the argument above works just fine even if c_t is not a constant but rather a polynomial in t and $\log n$. Hence, in order to prove Theorem 3.1 it is sufficient to prove that, for every K_t-free graph, every pair a, b of vertices can be separated by a set X of size polynomial in t and $\log n$. This is precisely the approach that we take, proving Theorem 13.1. This is the most involved part of our arguments.

Separating Two Vertices We now sketch the main ideas of the proof of Theorem 13.1: that for every K_t-free graph every pair a and b of vertices can be separated by a set X of size polynomial in t and $\log n$. We start by sketching how we would have liked the proof to work, point to where this approach breaks, and then outline how the proof actually works.

We wish to separate a from b. We will concentrate on the neighborhood $N(a)$ of a. Let D be the component of $G - N[a]$ that contains b, we can safely ignore the vertices in $N(a) \setminus N(D)$ and focus on the vertices in $N(D)$. As long as $N(D)$ contains a clique that covers at least 0.1% of $N(D)$ we can add this clique to X since we only can do this step $O(\log n)$ many times. After this step, "many" (i.e., at least 99%) of vertex triples x_1, x_2, x_3 in $N(a) \cap N(D)$ are stable (this follows from, e.g., [17]).

For a stable triple x_1, x_2, x_3 let D' be an inclusion minimal connected subset of D that contains neighbors of x_1, x_2 and x_3, and let $H = \{a, x_1, x_2, x_3\} \cup D'$. A simple case analysis shows that $\{a, x_1, x_2, x_3\} \cup D'$ is either a pyramid or a wheel. We show in Section 4 that if H is a pyramid, then either there is a clique K in D that separates at least two vertices of $\{x_1, x_2, x_3\}$ from b in $(\{x_1, x_2, x_3\} \cup D)$, or there is a clique K in D that has an "almost as good" separation effect (the precise formulation of this "almost as good" effect is cumbersome, and we skip it here). Additionally there are two exceptional cases for which we are not able to obtain this outcome: H could grow to a *loaded pyramid* or an *extended near-prism* (see Section 4 for definitions).

Suppose that at least 1% of the stable triples x_1, x_2, x_3 in $N(a) \cap N(D)$ there is a clique K_{x_1,x_2,x_3} that separates at least two of them from b in $(\{x_1, x_2, x_3\} \cup D)$ (or does the morally equivalent job). One of the main structural insights in this paper is that in this case we can conclude that there is a single set K in D such that $\kappa(K)$ is constant and no component of $D - K$ sees more than 99% of $N(D)$. This kind of local-to-global transition is usually very hard to force when one works with families of graphs defined by forbidden induced subgraphs. Our arguments here only rely on G being C_4-free, so we expect for this technique to be applicable in other contexts in the future.

Whenever we get a K as above we win – we can just add it to our separator X, and again we will only do this $O(\log n)$ many times before $|N(D)|$ drops to 0 and a is separated from b. A similar outcome can be derived using structural arguments from [17] when H is a pyramid that grows to an extended near-prism for a sufficiently large proportion of stable triples in $N(D)$.

The problem with this approach is that it gets stuck whenever 99% of the stable triples $\{x_1, x_2, x_3\}$ satisfy that H is a wheel or grows to a loaded pyramid. When this problem occurs a large fraction of the vertices of $N(D)$ are *hubs*. We say that a vertex v of G is a *hub* if v is the center of a proper wheel, or the "corner" of a loaded pyramid (again, see Section 4) and we denote by $\text{Hub}(G)$ the set of all hubs of G. We remark that our definition of hubs is not precisely the same as the definition of hubs in [13], although the role hubs play in [13] and here are similar. When *none* of the vertices in $N(a)$ (and therefore $N(D)$) are hubs an argument quite similar to the one outlined above works, and we are able to obtain an a-b separator X with polylogarithmic κ. We call this the "hub-free" case.

The remainder of the proof then consists of reducing the general case to the hub-free case. The reduction is based on the "central bag" method, developed in [7] and [6] and also used in [13]. Since G is C_4-free it follows that $N(a) \cap N(b)$ is a clique, and we may add $N(a) \cap N(b)$

to X at the cost of increasing $|X|$ by t (recall that G is K_t-free). From now on we assume that $N(a) \cap N(b)$ is empty.

Since every even-hole-free graph has a vertex whose neighborhood is the union of two cliques, it follows that every induced subgraph of G has average degree upper bounded by $O(t)$. Thus, the set $\text{Hub}(G)$ contains a stable set S_1 of size at least $\Omega(\frac{|\text{Hub}(G)|}{t})$ such that the degree of each vertex in S_1 in $G[\text{Hub}(G)]$ is at most $O(t)$. We show that for every hub v in a graph in $\mathcal{C}$, $G - N[v]$ is disconnected. When v is a wheel center this was already known, for loaded pyramid corners we prove it in Section 6 (we skip this proof in the overview).

For each $v \in S_1$, since $v \notin N(a) \cap N(b)$ there is a component D_v of $G \setminus N[v]$ that contains at least one of $\{a, b\}$. We claim that $N[D_v]$ must contain both $\{a, b\}$ or we are already done! Indeed, suppose $N[D_v]$ contains b but not a. Then a set X that separates v from b also separates a from b, but v only has $O(t)$ hubs in its neighborhood. So adding these t hubs to X leaves us with the task of separating v from b, but now v has no hubs in its neighborhood and we are in the hub-free case (and therefore done).

Thus, if we are not done yet, then for every vertex v in S_1 there is a component D_v such that $\{a, b\} \subseteq N[D_v]$. We define the *central bag* to be $\beta = \bigcap_{v \in S_1}(N[D_v] \cup \{v\})$ (the actual definition of the central bag is subtly different). Observe that both a and b are in the central bag β.

There are now two key observations behind the central bag method. The first (and easy) one is that no vertex $v \in S_1$ can be a hub in β. Indeed v cannot be a hub in $N[D_v]$ since $N[D_v] - N(v) = D_v$ is connected, contradicting that the neighborhood of every hub is a separator. Since $\beta \subseteq D_v$ it follows that v cannot be a hub in β either. When the more nuanced definition is used, the proof is slightly more involved.

This observation means that the number of hubs in β is smaller by a linear fraction than the number of hubs in G. Thus, by induction on $\log(|\text{Hub}(G)|)$, we can find an a-b separator Y in the central bag β of polylogarithmic size. If we can grow Y to an a-b separator X in G incurring an *additive* polylogarithmic cost, then the induction goes through and we are able to upper bound the total size of X by $(\log n)^{O(1)}$. In particular the depth of the induction is logaritmic in n, so if $|X|$ grows by an additive term of $(\log n)^{O(1)}$ in each inductive step this is ok, but if X grows by a factor 1.01 in each step then $|X|$ ends up being polynomial in n.

The second (and more involved) component of the central bag method is to show that Y can indeed be grown to X as prescribed above. This incurs a cost which is proportional to (essentially) $|Y \cap S_1|$, because for each vertex s in $|Y \cap S_1|$ we add all of its $O(t)$ hub neighbors to X and then separate s from either a or b using the hub-free case.

Unfortunately the inductive step which gives us Y does not give us any guarantees on the size of $|Y \cap S_1|$, so in the worst case $Y \cap S_1$ could be almost as big as Y itself. Then the cost of turning Y into a separator X in G would incur at least a constant *multiplicative* cost, which would be too expensive. We therefore apply an additional "pivot" step where Y is changed so that $|Y \cap S_1|$ is small. This pivot step again relies on the hub-free case as well as a second application of the local-global transition mentioned above. This concludes the proof outline.

We note that most of the results are proved for the slightly more general class of graphs $\mathcal{C}$, rather than even-hole-free graphs. However, Section 7 deals with even-hole-free graphs only. It is very likely that the proofs there do in fact work in the more general setting, but we did not verify the details. Theorem 13.2 is another fact we need that has only been proved for even-hole-free graphs (and not for C_4-free odd-signable graphs); once again it is likely to generalize, but we have not checked it carefully. These are the two reasons for the fact that our main theorem applies to even-hole-free graphs only.

3.1 Organization of the Proof

The pieces of the proof appear in a different order than in the outline. Specifically each piece is proved before it is used. In Section 4 we prove Theorem 4.3, which allows us to generate clique separators from pyramids that do not grow to an extended near-prisms. In Sections 5 and 6 we define hubs and prove Theorem 6.5, that every for every hub v, $N[v]$ separates the graph (in a particular way). In Section 7 we prove Theorem 7.2 which allows us to decompose graphs that contain pyramids that *do* grow to an extended near-prisms. In Section 8 we prove Theorem 8.3: that if we have sufficiently many clique separators of the type that are the output of Theorem 4.3, then there is a bounded κ size set whose removal substantially separates the graph. In Section 9 we prove Theorem 9.1 which shows how to deal with the case where we have a vertex a, and many of the stable triples x_1, x_2, x_3 in the neighborhood of the component D of $G - N(a)$ that contains b are contained in a pyramid that grows to an extended near-prism. In Section 10 we handle the hub-free case and prove Theorem 10.1, that if a does not have any hub neighbors, then a and b can be separated with $O(\log n)$ cliques. Section 11 contains all of the elements needed for the central bag method, with the exception of the "pivot" step where the separator Y is changed to (mostly) avoid S_1. In Section 12 we prove Theorem 12.1, which does the aforementioned pivot step. In Section 13 we apply the central bag method, putting together the results from Sections 10, 11 and 12 to prove Theorem 13.1, that every pair of non-adjacent vertices in a K_t-free even-hole-free graph can be separated by $(t \log n)^{O(1)}$ cliques. In Section 14 we prove Theorem 14, that every pair of non-adjacent vertices in an even-hole-free graph can be separated by $(\log n)^{O(1)}$ cliques. Finally, in Section 15 we use Theorem 14 to prove Theorem 1.1.

4 Jumps on pyramids

The goal of this section is to prove Theorem 4.3, which asserts the existence of well-structured cutsets that separate the neighbors of the apex of a pyramid. Theorem 4.3 is then used to produce the "local cutsets" in Theorem 8.3.

We start with some definitions. Let $G \in \mathcal{C}$ and let Σ be a pyramid in G with apex a, base $b_1b_2b_3$ and paths P_1, P_2, P_3. We say that $X \subseteq \Sigma$ is *local (in Σ)* if $X \subseteq P_i$ for some $i \in \{1, 2, 3\}$, or $X \subseteq \{b_1, b_2, b_3\}$. Let $P = p_1\text{-}\cdots\text{-}p_k$ be a path with $P \cap \Sigma = \emptyset$. P is a *corner path for b_1* if p_1 is adjacent to b_2, b_3, p_k has a neighbor in $P_1 \setminus b_1$, and there are no other edges from $\Sigma \setminus b_1$ to P. A corner path for b_2 and b_3 is defined similarly. We say that P is a *corner path for Σ* if P is a corner path for b_1, b_2 or b_3. If $v \in G \setminus \Sigma$, and v is not a corner path for Σ, and $N_\Sigma(v)$ is not local, we say that v is *major (for Σ)*.

A *loaded pyramid* in a graph G is a pair $\Pi = (\Sigma, P)$ where Σ is pyramid with apex a, base $b_1b_2b_3$ and paths P_1, P_2, P_3, a is adjacent to b_2 (hence $|P_2| = 2$), and $P = p_1\text{-}\cdots\text{-}p_k$ is a path such that

- $P \cap \Sigma = \emptyset$;
- p_1 is adjacent to b_2;
- p_k has a neighbor in P_1^*;
- P_3 is anticomplete to P (and in particular a is anticomplete to P);
- b_2 is anticomplete to $P \setminus p_1$; and
- $P_1 \setminus b_1$ is anticomplete to $P \setminus p_k$.

In this case, we say b_2 is a *loaded pyramid corner*, and denote the loaded pyramid as a pair (Π, b_2). We also use the notation Π to denote the vertex set $P \cup \Sigma$.

Recall that a *wheel* (H, x) in G is a pair where H is a hole and x is a vertex with at least three neighbors in H; the vertex x is called the *center* of the wheel. A wheel (H, x) is *proper* if it is neither a twin wheel nor a short pyramid. We say that a vertex v of G is a *hub* if v is a proper wheel center or a loaded pyramid corner, and we denote by $\text{Hub}(G)$ the set of all hubs of G.

An *extended near-prism*, defined in [17], is a graph obtained from a near-prism by adding one extra edge, as follows. Let P_1, P_2, P_3 be as in the definition of a near-prism, and let $a \in P_1^*$ and $b \in P_2^*$; and add an edge ab. (It is important that a, b do not belong to the triangles.) If the two triangles of the extended near-prism are disjoint, we also call it an *extended prism.* We call ab the *cross-edge* of the extended near-prism (or of the extended prism). We start with two lemmas:

Theorem 4.1. *Let $G \in \mathcal{C}$. Let Σ be a pyramid in G with apex a, base $b_1b_2b_3$ and paths P_1, P_2, P_3. Let p be a major vertex for Σ. Then one of the following holds:*

1. *p is adjacent to a and at least two of the neighbors of a in Σ;*
2. *p is adjacent to a, and p is a hub;*
3. *$\Sigma \cup p$ is an extended prism whose cross-edge contains a; or*
4. *(Σ, p) is a loaded pyramid, and so one of b_1, b_2, b_3 is a loaded pyramid corner.*

Proof. We may assume that the last three outcomes of Theorem 4.1 do not hold. First we prove that p is adjacent to a. Suppose not. Assume first that p has a neighbor in each of P_1, P_2, P_3. Since p is not a corner path for Σ and $N_\Sigma(p)$ is not local, it follows that p has neighbors in the interiors of at least two of P_1, P_2, P_3. Consequently, there is a theta with ends a and p the interiors of whose paths are subpaths of P_1, P_2, P_3, a contradiction. Thus we may assume that p is anticomplete to at least one of P_1, P_2, P_3, say p is anticomplete to P_3.

(1) *If p is non-adjacent to a, then for $i = 1, 2$, we have $N_{P_i}(p) \neq \{b_i\}$.*

Suppose $N_{P_1}(p) = \{b_1\}$. Since p is major, p has a neighbor in $P_2 \setminus b_2$. Since p is non-adjacent to a, and $\Sigma \cup p$ is not a loaded pyramid, it follows that b_1 is non-adjacent to a. But now we get a theta with ends b_1, a and paths b_1-P_1-a, b_1-p-P_2-a and b_1-b_3-P_3-a, a contradiction. This proves (1).

By (1) and since p is major and we have assumed that p is non-adjacent to a, p has both a neighbor in $P_1 \setminus b_1$, and a neighbor in $P_2 \setminus b_2$. Let H be the hole formed by P_1 and P_2.

(2) *If p is non-adjacent to a, then p has exactly three neighbors in H, and two of them are consecutive.*

Suppose that p has two non-adjacent neighbors in P_1. Then there exists a path P_1' from p to a, and a path P_1'' from p to b_1, both with interior in P_1 and such that $P_1' \setminus p$ is anticomplete to $P_1'' \setminus p$. Now we get a theta with ends p, a and paths p-P_1'-a, p-P_1''-b_1-b_3-P_3-a and a path from p to a with interior in $P_2 \setminus b_2$, a contradiction. This proves that p has either one or two consecutive neighbors in P_1, and the same for P_2. Since (H, p) is not an even wheel, and $H \cup p$ is not a theta, (2) follows.

We may assume that $N_{P_1}(p) = \{t\}$, $N_{P_2}(p) = \{q, r\}$, where P_2 traverses a, q, r, b_2 is this order. By (1), $t \neq b_1$. It follows from (2) that q is adjacent to r. Since $\Sigma \cup p$ is not an extended prism, it follows that t is non-adjacent to a. But now there is a theta with ends t, a and paths t-P_1-a, t-p-q-P_2-a and t-P_1-b_1-b_3-P_3-a, a contradiction. This proves that p is adjacent to a.

To complete to proof of 4.1, assume that p is anticomplete to $N_{P_1 \cup P_2}(a)$. Since p is major, p has a neighbor in $H \setminus a$. Since $p \notin \text{Hub}(G)$ and $H \cup p$ is not a theta, it follows that p has exactly two neighbors in $H \setminus a$, and they are adjacent. Since p is not a corner path for b_3, it follows that $N_H(p) \neq \{a, b_1, b_2\}$, and so we may assume that $N_H(p) \subseteq P_1$. Since p is major, we deduce that p has a neighbor in $P_3 \setminus a$. Let H' be the hole $P_1 \cup P_3$. Since $p \notin \text{Hub}(G)$, it follows that (H', p) is not a proper wheel. Consequently, $N_{P_3}(p) = \{a\} \cup N_{P_3}(a)$. But now (H', p) is an even wheel, a contradiction. ■

Theorem 4.2. *Let $G \in \mathcal{C}$. Let Σ be a pyramid in G with apex a, base $b_1b_2b_3$ and paths P_1, P_2, P_3. Assume that $N_G(a) \subseteq \Sigma$. Let P be a path in $G \setminus \Sigma$. Then one of the following holds.*

1. *$N_\Sigma(P)$ is local in Σ;*
2. *P contains a major vertex for Σ;*
3. *P contains a corner path for Σ;*
4. *There exist distinct $i, j \in \{1, 2, 3\}$ and a subpath $Q = q_1$-$\cdots$-q_m of P such that*
   - *$N_\Sigma(q_1) \subseteq P_i$;*
   - *q_1 has a unique neighbor in P_i and $N_{P_i}(q_1) = N_{P_i}(a)$;*
   - *$N_\Sigma(q_m) \subseteq P_j$;*
   - *q_m has exactly two neighbors x, y in P_j; x is adjacent to y, and $a \notin \{x, y\}$; and*
   - *there are no other edges between Σ and Q;*

   In particular, a is contained in the cross-edge of an extended prism.
5. *There is an $i \in \{1, 2, 3\}$ and a subpath $Q = q_1$-$\cdots$-q_m of P such that (Σ, P) is a loaded pyramid with loaded pyramid corner b_i.*

Proof. Let $P = p_1$-$\cdots$-p_k. Suppose for a contradiction that none of the outcomes hold. We may assume that $N_\Sigma(P)$ is not local, and that $N_\Sigma(X)$ is local for every proper subpath X of P. Since no vertex of P is major, and no subpath of P is a corner path, it follows that $k \geq 2$.

Since $N_\Sigma(P) \not\subseteq \{b_1, b_2, b_3\}$, by symmetry, we may assume that

- $N_\Sigma(p_1) \subseteq P_1$ and p_1 has a neighbor in $P_1 \setminus b_1$;
- p_k has a neighbor in $P_2 \setminus a$, and either $N_\Sigma(p_k) \subseteq P_2$, or $N_\Sigma(p_k) \subseteq \{b_1, b_2, b_3\}$; and
- $\Sigma \setminus b_1$ is anticomplete to P^*.

We first show:

(3) *We have $N_\Sigma(p_k) \subseteq P_2 \cup \{b_1\}$.*

Suppose not. Then $N_\Sigma(p_k) \subseteq \{b_1, b_2, b_3\}$. Since p_k has a neighbor in $P_2 \setminus a$, it follows that p_k is adjacent to b_2. Since $P \cup (P_1 \setminus b_1)$ contains a path from p_k to a, and since P does not contain a corner path for b_1, it follows that p_k is non-adjacent to b_3. This proves (3).

(4) $N_{P_2}(p_k) \neq \{b_2\}$.

Suppose that $N_{P_2}(p_k) = \{b_2\}$. By (3) p_k is non-adjacent to b_3. If b_2 is non-adjacent to a, then there is a theta in G with ends b_2, a and paths P_2, b_2-b_3-P_3-a and b_2-p_k-P-p_1-P_1-a, a contradiction; so b_2 is adjacent to a. Now since a has no neighbor in P, (Σ, P) is a loaded pyramid with base $b_1b_2b_3$, apex a, and paths P_1, P_2, P_3, and the fifth outcome holds, a contradiction. This proves (4).

(5) *There is no edge from b_1 to $P \setminus p_1$.*

Suppose for a contradiction that b_1 has a neighbor in $P \setminus p_1$. Since $N_\Sigma(P \setminus p_1)$ is local, it follows from (3) that $N_\Sigma(p_k) \subseteq \{b_1, b_2\}$, contrary to (4), and (5) follows.

By (5), it follows that Σ is anticomplete to P^* and that $N_\Sigma(p_k) \subseteq P_2$. Traversing P_1 from b_1 to a, let x_1 be the first neighbor of p_1. Traversing P_2 from b_2 to a, let x_2 be the first neighbor of p_k. Then $H = p_1$-P-p_k-x_2-P_2-b_2-b_1-P_1-x_1-p_1 is a hole in G (since it contains at least the four distinct vertices b_1, b_2, p_1, p_k). For $i = 1, 2$, let z_i be the neighbor of x_i in P (and thus $z_i \in \{p_1, p_k\}$).

(6) *For $i = 1, 2$, either there is a vertex y_i in P_i with x_i adjacent to y_i and $N(z_i) \cap P_i = \{x_i, y_i\}$; or x_i is the only neighbor of z_i in P_i and x_i is adjacent to a.*

Since p_1 has a neighbor in $P_1 \setminus b_1$, and since by (3) and (4) p_k has a neighbor in $P_2 \setminus b_2$, it follows that for $i = 1, 2$, there is a path from every vertex of P to a with interior in $(P \cup P_i) \setminus b_i$.

If x_i is the only neighbor of z_i in P_i, and x_i is non-adjacent to a, then we find a theta with ends x_i, a in G and paths x_i-P_i-a, x_i-P_i-b_i-b_3-P_3-a, and a path whose interior is contained in $(P \cup P_{3-i}) \setminus b_{3-i}$ given by the claim of the previous paragraph applied to z_{3-i}, a contradiction.

Thus we may assume that z_i has two non-adjacent neighbors in P_i. Let y_i be the neighbor of z_i along P_i closest to a. Since $y_i \neq a$, there is a theta with ends z_i, a and paths z_i-y_i-P_i-a, z_i-x_i-b_i-b_3-P_3-a, and a path whose interior is contained in $(P \cup P_{3-i}) \setminus b_{3-i}$ given by the claim of the first paragraph applied to z_{3-i}, a contradiction. Now (6) follows.

If the first outcome of (6) holds for both $i = 1$ and $i = 2$, we get a prism with triangles $x_1y_1z_1$ and $x_2y_2z_2$ and paths y_1-P_1-a-P_2-y_2, P and x_1-P_1-b_1-b_2-P_2-x_2, a contradiction. If the second outcome of (6) holds for both $i = 1$ and $i = 2$, then we get a theta with ends x_1, x_2 and paths x_1-P_1-a-P_2-x_2, P and x_1-P_1-b_1-b_2-P_2-x_2, a contradiction. Thus we may assume that the first outcome holds for $i = 1$ and the third outcome holds for $i = 2$. But now the fourth outcome of Theorem 4.2 holds, a contradiction. ■

We need an additional definition: given a graph G, $x, y \in V(G)$, a path P from x to y, and a non-empty set $A \subseteq P$, we define the (P, y)-*last vertex of* A to be the vertex $a \in A$ such that a is the unique vertex of A in the subpath of P from a to y.

We now prove the main result of this section.

Theorem 4.3. *Let $G \in \mathcal{C}$. Let Σ be a pyramid in G with apex a, base $b_1b_2b_3$ and paths P_1, P_2, P_3. For each i, let Q_i be a subpath of P_i, with ends a and x_i, such that all internal vertices of Q_i have degree two in G. Let $b \in V(G) \setminus (Q_1 \cup Q_2 \cup Q_3)$. Assume that*

- $N_G(a) = N_\Sigma(a)$;
- $\{x_1, x_2, x_3\}$ *is a stable set;*
- b *is non-adjacent to* x_1, x_2, x_3;
- a *does not belong to a cross-edge of an extended near-prism in G; and*
- $\mathrm{Hub}(G) \cap \{x_1, x_2, x_3\} = \emptyset$.

Let $D = G \setminus (Q_1 \cup Q_2 \cup Q_3)$. Then there is a clique $K \subseteq D$ and distinct $i, j \in \{1, 2, 3\}$ such that one of the following holds: (For $i \in \{1, 2, 3\}$ let D_i be the union of components of $D \setminus K$ such that $N(x_i) \cap D_i \neq \emptyset$.)

1. $b \notin K \cup D_i \cup D_j$*; or*

2. We have $b \notin N[D_i]$*. Moreover, there is a set* $D'_j = D'_j(x_1x_2x_3)$ *of vertices such that:*

- D'_j *is not a clique;*
- D'_j *is complete to* K*;*
- *There is a vertex* $q = q(x_1x_2x_3)$ *with the following property. Either* $b \in K$ *and* $q = b$*; or there exists* $k \in \{1,2,3\} \setminus \{i\}$ *such that* x_k *is complete to* $K \cup D'_j$ *and* $q = x_k$*;*
- *For every* $v \in D'_j$*, there is a path* P *in* $D \cup \{x_j\}$ *from* b *to* x_j *such that* $P \cap N(q) \neq \emptyset$ *and the* (P, x_j)*-last vertex in* $P \cap N(q)$ *is* v*; and*
- *For every path* P *in* $D \cup \{x_j\}$ *from* b *to* x_j*, we have* $P \cap N(q) \neq \emptyset$*, and the* (P, x_j)*-last vertex of* $P \cap N(q)$ *is complete to* K*.*

Proof. Let $B_1, C_1, B_2, C_2, B_3, C_3$ be pairwise disjoint subsets of $G \setminus a$ with the following properties:

- the sets B_1, B_2, B_3 are all pairwise complete to each other;
- the sets C_1, C_2, C_3 are pairwise anticomplete to each other;
- for distinct $i, j \in \{1,2,3\}$, the set B_i is anticomplete to C_j:
- for every i, every vertex of B_i is an end of a path to x_i with interior in C_i;
- for every i, one of the following holds:
  - $x_i \in C_i$ and every vertex of $C_i \setminus Q_i$ is in the interior of a path from some vertex of B_i to x_i; or
  - $x_i = b_i$, and $B_i = \{x_i\}$, and $C_i = Q_i^*$.
- For every i, we have $b_i \in B_i$ and $P_i \setminus \{b_i, a\} \subseteq C_i$.

Subject to these properties, we choose the sets with $W = \{a\} \cup \bigcup_{i=1}^{3}(B_i \cup C_i)$ maximal.

(7) *Let* D *be a component of* $G \setminus W$*. Then either* $N(D) \subseteq B_1 \cup B_2 \cup B_3$*, or there exists* i *such that* $N(D) \subseteq B_i \cup C_i$*.*

Suppose not and let D be a component of $G \setminus W$ violating the statement. Then there exist $i \in \{1,2,3\}$ and a path $P = p_1\text{-}\cdots\text{-}p_k$ in D such that p_1 has a neighbor in C_i and p_k has a neighbor in $W \setminus (B_i \cup C_i \cup \{a\})$. We may assume that P is chosen with k as small as possible and that $i = 1$. Let $c_1 \in C_1$ be a neighbor of p_1. Let P'_1 be a path from $b'_1 \in B_1$ to a with interior in C_1 and such that $c_1 \in P'_1$. Let $c_2 \in B_2 \cup C_2 \cup B_3 \cup C_3$ be a neighbor of p_k; choose $c_2 \notin B_2 \cup B_3$ if possible. We may assume that $c_2 \in B_2 \cup C_2$. Let P'_2 be a path from $b'_2 \in B_2$ to a with interior in C_2 and such that $c_2 \in P'_2$. Let $P'_3 = P_3$. Now $\Sigma' = P'_1 \cup P'_2 \cup P'_3 \cup \{a\}$ is a pyramid with apex a an base $b'_1b'_2b_3$. We apply Theorem 4.2 to Σ' and P. Since P is not local for Σ' and $N_G(a) = N_\Sigma(a)$ and $N_G(a) \cap \text{Hub}(G) = \emptyset$ (because each neighbor of a ethre has degree 2 in G or is in $\{x_1, x_2, x_3\}$), and a is not contained in a cross-edge of an extended near-prism in G, it follows that one of the following holds:

- P contains a major vertex for Σ'; or
- P contains a corner path for Σ';

Theorem 4.1 implies that P does not contain a major vertex for Σ', and therefore P contains a corner path for Σ'. By the minimality of k, it follows that P is a corner path for Σ'; consequently $N_{\Sigma' \setminus \{b'_1\}}(p_k) = \{b'_2, b_3\}$, $N_{\Sigma' \setminus \{b'_1\}}(p_1) \subseteq B_1 \cup C_1$, and there are no other edges between P and $\Sigma' \setminus b'_1$. In particular, p_1 has a neighbor in C_1, and so $x_1 \in C_1$.

We claim that P is anticomplete to $C_2 \cup C_3$ and $P \setminus p_k$ is anticomplete to $B_2 \cup B_3$. By the minimality of k, it follows that $P \setminus p_k$ is anticomplete to $C_2 \cup C_3 \cup B_2 \cup B_3$. From the choice of c_2, it follows that p_k is anticomplete to $C_2 \cup C_3$. This proves the claim.

Now let $i \in \{2, 3\}$ and let Σ'' be obtained for Σ' by replacing the path P'_i by an arbitrary path from some $b''_i \in B_i$ to a with interior in C_i. Then P is not local for Σ''. Applying Theorems 4.2 and 4.1 to Σ'' and P, we deduce that P is a corner path for Σ''. It follows that p_k is adjacent to b''_i. Since b''_i was chosen arbitrarily, we conclude that p_k is complete to $B_2 \cup B_3$. But now, we can replace B_1 by $B_1 \cup p_k$ and C_1 by $C_1 \cup (P \setminus p_k)$, contradicting the maximality of W. This proves (7).

Let F be the union of the components D of $G \setminus W$ with $N(D) \subseteq B_1 \cup B_2 \cup B_3$. Let F_i be the union of the components D of $G \setminus (W \cup F)$ such that $N(D) \subseteq B_i \cup C_i$. By (7), $G \setminus W = F_1 \cup F_2 \cup F_3 \cup F$ and the sets F_1, F_2, F_3, F are pairwise disjoint and anticomplete to each other. We may assume that $b \in B_3 \cup C_3 \cup F_3 \cup F$. If $K = B_1 \cup B_2$ is a clique and $x_1, x_2 \notin B_1 \cup B_2$, then for $i \in \{1, 2\}$, we have $D_i = (C_i \cup F_i) \setminus Q_i$ and outcome (2)(a) holds. Thus we may assume that either B_2 is not a clique or $B_2 = \{x_2\}$.

(8) *We may assume that $B_2 = \{x_2\}$ or $B_3 = \{x_3\}$.*

Suppose not; by symmetry, we may assume also that $B_1 \neq \{x_1\}$. Then B_2 is not a clique. Since G is C_4-free, it follows that $B_1 \cup B_3$ is a clique. Let $K = B_1 \cup B_3$. Now $D_1 = (C_1 \cup F_1) \setminus Q_1$ and $D_2 \subseteq (C_2 \cup F_2 \cup B_2 \cup F) \setminus Q_2$ and $D_3 = (C_3 \cup F_3) \setminus Q_3$.

If $b \in C_3 \cup F_3$, then outcome (2)(a) of the theorem holds with $i = 1$ and $j = 2$, and if $b \in F$, then outcome (2)(a) of the theorem holds with $i = 1$ and $j = 3$. Thus we may assume that $b \in B_3$. Let $i = 1$ and $j = 2$. Then b is anticomplete to D_1. Since every path from b to x_2 with interior in D contains exactly one vertex of B_2 and exactly one vertex in $N(b)$, and B_2 is not a clique, and for every vertex v in B_2 there is a path from v to x_2 with interior in C_2, it follows that outcome (2)(b) holds with $D'_j = B_2$ and $q = b$. This proves (8).

Since b is non-adjacent to a, it follows that if $B_3 = \{x_3\}$, then $C_3 = Q^*_3$ and $F_3 = \emptyset$, and so $b \in F$ and in particular, $b \in B_2 \cup C_2 \cup F_2 \cup F$; so there is symmetry between 2 and 3 in this case.

Therefore, we may assume that $B_2 = \{x_2\}$. Since $\{x_1, x_2, x_3\}$ is a stable set, it follows that $x_1, x_3 \notin B_1 \cup B_3$. Since b is non-adjacent to x_2, it follows that $b \in C_3 \cup F_3 \cup F$.

(9) *$B_1 \cup B_3$ is not a clique.*

Suppose that $B_1 \cup B_3$ is a clique, and let $K = B_1 \cup B_3$. Then $D_1 = (C_1 \cup F_1) \setminus Q_1$ and $D_3 = (C_3 \cup F_3) \setminus Q_3$ and $D_2 \subseteq F$. If $b \in F$, then outcome (1) of the theorem holds with $i = 1$ and $j = 3$. If $b \in C_3 \cup F_3$, then outcome (1) of the theorem holds with $i = 1$ and $j = 2$. This proves (9).

(10) *B_1 is a clique.*

Suppose not. It follows that B_3 is a clique. Let $K = B_3$. Then $D_3 = (C_3 \cup F_3) \setminus Q_3$ and $D_1, D_2 \subseteq (C_1 \cup B_1 \cup F_2 \cup F) \setminus (Q_1 \cup Q_2)$. If $b \in C_3 \cup F_3$, then outcome (1) of the theorem holds with $i = 1$ and $j = 2$. It follows that $b \in F$. Let R be the component of F containing b. Let $M = N(R) \cap B_1$. If M is a clique, then outcome (1) of the theorem holds with $K = B_3 \cup M$ and $i = 1$ and $j = 3$.

It follows that M is not a clique. Now outcome (2) of the theorem holds with $i = 3$ and $j = 1$ as well as $K = B_3$ and $D'_1 = M$ and $q = x_2$. We show that the last two bullets of (2) hold:

- Let $v \in M$. Let P be a path from b to v in $R \cup \{v\}$. Let Q be a path from v to x_1 with $Q^* \subseteq C_1$. Then $P' = b\text{-}P\text{-}v\text{-}Q\text{-}x_1$ is a path from b to x_1 in $D \cup \{x_1\}$. Then, v is the (P', x_1)-last vertex in $P' \cap N(q)$, and the second-to-last bullet of (2) holds.
- Let P be a path from b to x_1 in $D \cup \{x_1\}$. Traversing P from b to x_1, let v be the last vertex of P which is not in $C_1 \cup F_1$. It follows that $v \in B_1$, and so v is complete to K, and the last bullet holds.

(11) *B_3 is a clique.*

Suppose not. It follows that B_1 is a clique. Let $K = B_1$. Then $D_1 = (C_1 \cup F_1) \setminus Q_1$ and $D_2, D_3 \subseteq (C_3 \cup B_3 \cup F_2 \cup F) \setminus (Q_2 \cup Q_3)$. Suppose first that $b \in F$. Then there is symmetry between 1 and 3, and the result follows from (10). It follows that $b \in F_3 \cup C_3$. Let R be the component of $(F_3 \cup C_3) \setminus Q_3$ containing b. Then $M = N_D(R) \subseteq B_3$. If M is a clique, then outcome (2)(a) holds with $K = B_1 \cup M$ and $i = 1$ and $j = 2$. So we may assume that M is not a clique. Now we let $i = 1$, $j = 2$, $q = x_2$, and $D'_2 = M$. It follows that outcome (2) holds. This proves (11).

Since B_1 is complete to B_3, together, (9), (10), and (11) yield a contradiction; this concludes the proof. ■

5 Star cutsets from wheels

The following well-known definitions appear, for example, in [7]. A *cutset* $C \subseteq V(G)$ of G is a set of vertices such that $G \setminus C$ is disconnected. A *star cutset* in a graph G is a cutset $S \subseteq V(G)$ such that either $S = \emptyset$ or for some $x \in S$, $S \subseteq N[x]$.

Let G be a graph and let $X, Y, Z \subseteq V(G)$. We say that X *separates* Y from Z if no component of $G \setminus X$ meets both Y and Z. Recall that a *wheel* (H, x) of G consists of a hole H and a vertex x that has at least three neighbors in H, and a wheel is *proper* if t is neither a twin wheel nor a short pyramid. A *sector* of (H, x) is a path P of H whose ends are distinct and adjacent to x, and such that x is anticomplete to P^*. A sector P is a *long sector* if P^* is non-empty. A wheel (H, x) is a *universal wheel* if x is complete to H. The following result was observed in [7] based on results of [8, 42, 43] and stated in this form in [13]; it shows that proper wheels force star cutsets in graphs in $\mathcal{C}$.

Theorem 5.1 (Abrishami, Chudnovsky, Vušković [7]; see also [13])**.** *Let $G \in \mathcal{C}$ and let (H, v) be a proper wheel in G. Then there is no component D of $G \setminus N[v]$ such that $H \subseteq N[D]$.*

In particular, we need the following:

Theorem 5.2 (Addario-Berry, Chudnovsky, Havet, Reed, Seymour [8], da Silva, Vušković [42])**.** *Let $G \in \mathcal{C}$ and let (H, x) be a proper wheel in G that is not a universal wheel. Let x_1 and x_2 be the endpoints of a long sector Q of (H, x). Let W be the set of all vertices h in $H \cap N(x)$ such that the subpath of $H \setminus \{x_1\}$ from x_2 to h contains an even number of neighbors of x, and let $Z = H \setminus (Q \cup N(x))$. Let $N' = N(x) \setminus W$. Then, $N' \cup \{x\}$ is a cutset of G that separates Q^* from $W \cup Z$.*

6 Star cutsets from loaded pyramids

The main theorem of this section is the following.

Theorem 6.1. *Let $G \in \mathcal{C}$. Suppose that G contains a loaded pyramid $\Pi = (\Sigma, P)$ with $a, b_1, b_2, b_3, P_1, P_2, P_3$ as in the definition. Then there is no connected component D of $G \setminus N[b_2]$ with $\Pi \subseteq N[D]$.*

In order to prove Theorem 6.1, we first prove the following:

Theorem 6.2. *Let $G \in \mathcal{C}$. Suppose that G contains a loaded pyramid $\Pi = (\Sigma, P)$ with $a, b_1, b_2, b_3, P_1, P_2, P_3$ as in the definition. Moreover, assume that D is a connected component of $G \setminus N[b_2]$ such that neither $N[D] \cap (\Pi \setminus P_3)$ nor $N[D] \cap (P_3 \setminus \{a\})$ is empty. Then one of the following holds.*

- *We have $N[D] \cap (\Pi \setminus P_3) = \{b_1\}$ and $N[D] \cap (P_3 \setminus \{a\}) = \{b_3\}$; or*
- *There is a proper wheel (H, b_2) in G with two long sectors Γ_1 and Γ_3 such that Γ_1^* contains the neighbor of a in P_1 and $\Gamma_3 = P_3$.*

In order to prove Theorem 6.2, first we need to verify its assertion for two special types of loaded pyramids, as follows. For a loaded pyramid (Σ, P) in a graph G, we say (Σ, P) is of *type 1* if $N_{P_1}(p_k) \subseteq N_{P_1}[b_1]$. We say that (Σ, P) is of *type 2* if b_1 is anticomplete to P and p_k has exactly two neighbors $x, y \in P_1$, and x is adjacent to y.

Theorem 6.3. *Let $G \in \mathcal{C}$. Suppose that G contains a loaded pyramid $\Pi = (\Sigma, P)$ of type 1 with $a, b_1, b_2, b_3, P_1, P_2, P_3, P$ as in the definition. Moreover, assume that D is a connected component of $G \setminus N[b_2]$ such that neither $N[D] \cap (\Pi \setminus P_3)$ nor $N[D] \cap (P_3 \setminus \{a\})$ is empty. Then one of the following holds.*

- *We have $N[D] \cap (\Pi \setminus P_3) = \{b_1\}$ and $N[D] \cap (P_3 \setminus \{a\}) = \{b_3\}$; or*
- *There is a proper wheel (H, b_2) in G with two long sectors Γ_1 and Γ_3 such that Γ_1^* contains the neighbor of a in P_1 and $\Gamma_3 = P_3$.*

Proof. Suppose not. Then since the first bullet of Theorem 6.3 does not hold, there exists a shortest path $Q = q_1\text{-}\cdots\text{-}q_t$ in $G \setminus N[b_2]$ with $t \geq 1$ where

- q_1 has a neighbor in $P_3 \setminus \{a\}$;
- q_t has a neighbor in $\Pi \setminus P_3$; and
- either q_1 has a neighbor in P_3^* or q_t has a neighbor in $P_1^* \cup P$.

From the minimality of Q, it follows that Q^* is anticomplete to $(\Sigma \cup P) \setminus \{a, b_1, b_3\}$. Let c be the neighbor of b_1 in P_1. Then $c \neq a$, and since G is C_4-free, c is non-adjacent to a.

(12) $N(q_1) \cap P_3^* \neq \emptyset$.

For otherwise we have $N(q_1) \cap (P_3 \setminus \{a\}) = \{b_3\}$. Consequently q_t has a neighbor in $P_1^* \cup P$, and so by the minimality of Q, $Q \setminus \{q_1\}$ is anticomplete to $P_3 \setminus \{a\}$. Traversing Q starting at q_1, let q be the the first vertex with a neighbor in $P' = (P_1 \setminus b_1) \cup P$, and traversing the path P' starting at a, let x be the first vertex adjacent to q. Then $H = a\text{-}P'\text{-}x\text{-}q\text{-}Q\text{-}q_1\text{-}b_3\text{-}P_3$ is a hole in G. Note that $\{a, b_3\} \subseteq N(b_2) \cap H \subseteq \{a, b_3, p_1\}$. Since $H \cup \{b_2\}$ is not a theta in G, it follows that $N(b_2) \cap H = \{a, b_3, p_1\}$. But then (H, b_2) is a wheel in G satisfying the second bullet of Theorem 6.3, a contradiction. This proves (12).

(13) *$N(b_1) \cap (Q \setminus q_t) = \emptyset$, and q_t has a neighbor in $P_1^* \cup P$.*

Suppose not. Traversing Q starting at q_t, let q be the last vertex adjacent to b_1. Since (13) does not hold, it follows that q has no neighbor in $P_1^* \cup P$. Note that by (12), $q\text{-}Q\text{-}q_1 \cup (P_3 \setminus b_3)$ is connected, and so contains an induced path R from q to a. But then $P_1 \cup R \cup \{b_2\}$ is a theta with ends a, b_1 in G, which is impossible. This proves (13).

(14) $N(Q) \cap (P \cup \{b_1\}) \neq \emptyset$.

Suppose not. Then q_t has a neighbor in $P_1 \setminus b_1$. Traversing $P_1 \setminus b_1$ starting at a, let x be the last vertex with a neighbor in Q. It follows that $x \neq a$. By (13) x is adjacent to q_t and anticomplete to $Q \setminus q_t$. Traversing P_3 starting at a, let y be the last vertex adjacent to q_1; hence $y \neq a$. Then both $H_1 = (P_1 \setminus b_1) \cup P$ and $H_2 = c\text{-}P_1\text{-}x\text{-}q_t\text{-}Q\text{-}q_1\text{-}y\text{-}P_3\text{-}b_3\text{-}b_2\text{-}p_1\text{-}P\text{-}p_k\text{-}c$ are holes in G. Note that b_1 has at least two non-adjacent neighbors in H_1 and $N(b_1) \cap H_2 = (N(b_1) \cap H_1) \cup \{b_3\}$. So one of (H_1, b_1) and (H_2, b_1) is an even wheel in G, a contradiction. This proves (14).

(15) $N(q_t) \cap P \neq \emptyset$.

Suppose for a contradiction that q_t is anticomplete to P. Recall that by the minimality of t, $Q \setminus q_t$ is also anticomplete to P. Thus, Q is anticomplete to P, and so by (14), we have $N(b_1) \cap Q \neq \emptyset$. This, together with (13), implies that q_t is adjacent to b_1. Since $q_t\text{-}b_1\text{-}b_2\text{-}a\text{-}q_t$ is not a C_4 in G, q_t is nonadjacent to a. Moreover, note that by (13), it follows that q_t has a neighbor in P_1^*. Traversing $P_1 \setminus b_1$ starting at c, let x be the last vertex adjacent to q_t. Also, by (12), $(P_3 \setminus b_3) \cup Q$ is connected, and so contains a path R from q_t to a (note that R has more than one edge, as q_t is not adjacent to a). Now, if $x \neq c$, then we get a theta with ends a, q_t and paths $a\text{-}P_1\text{-}x\text{-}q_t$, R and $q_t\text{-}b_1\text{-}b_2\text{-}a$, a contradiction. Therefore, we have $x = c$. But now we get a theta with ends c, a and paths $c\text{-}P_1\text{-}a$, $c\text{-}p_k\text{-}P\text{-}p_1\text{-}b_2\text{-}a$ and $c\text{-}q_t\text{-}R\text{-}a$, again a contradiction. This proves (15).

In particular, (15) implies that b_3 is anticomplete to $Q \setminus q_1$. Henceforth, we denote the hole $(P_1 \setminus b_1) \cup P$ in G by K. Also, in view of (15), let $i \in \{1, \dots, k\}$ be minimum such that q_t is adjacent to p_i. Denote the neighbor of a in P_1 by w. Then $w \neq c$.

(16) $|N(q_t) \cap K| \geq 2$.

Suppose not. Then p_i is the unique neighbor of q_t in K. By (12), $(P_3 \setminus b_3) \cup Q$ is connected, and so contains a path R from q_t to a. But now $K \cup R$ is a theta with ends a, p_i in G, which is impossible. This proves (16).

(17) *q_t has a neighbor in $K \setminus (N[a] \cup \{p_1\})$.*

Suppose not. Then $i = 1$. Since $q_t\text{-}p_1\text{-}b_2\text{-}a\text{-}q_t$ is not a C_4 in G, it follows that q_t is non-adjacent to a. Now $N_K(q_t) = \{p_1, w\}$ and so $K \cup q_t$ is a theta, a contradiction. This proves (17).

In view of (17) there is a path P' from q_t to b_1 with interior in $K \setminus \{b_2, a, p_1, w\}$. It follows from (13) and the minimality of t that q_t is the only vertex of P'^* with a neighbor in $Q \setminus q_t$.

(18) $N(a) \cap Q = \emptyset$.

Suppose not. By (12), $(P_3 \setminus a) \cup \{q_1\}$ is connected, and so contains an induced path R from b_3 to q_1. Now $H_1 = Q \cup P' \cup R$ and $H_2 = Q \cup b_2\text{-}p_1\text{-}P\text{-}p_i \cup R$ are holes in G. Also, we have $N(a) \cap H_1 = N(a) \cap (Q \cup R)$ and $N(a) \cap H_2 = (N(a) \cap H_1) \cup \{b_2\}$. If $|N(a) \cap (Q \cup R)| \geq 3$, then either (H_1, a) or (H_2, a) is an even wheel in G, which is impossible. Also, if $|N(a) \cap (Q \cup R)| = 1$, then $H_2 \cup \{a\}$ is a theta in G, which is impossible. It follows that $|N(a) \cap (Q \cup R)| = 2$. Since $H_1 \cup \{a\}$ is not a theta in G, the two neighbors of a in H_1 are adjacent and contained in $(Q \cup R) \setminus \{b_3\}$. But now $H_1 \cup \{a, b_2\}$ is a prism in G, which is impossible. This proves (18).

If q_1 has exactly one neighbor x in P_3, then by (12), $x \in P_3^*$ and, using (18) we get a theta with ends x, b_2 and paths x-P_3-a-b_2, x-q_1-Q-q_t-p_i-P-b_2, and x-P_3-b_3-b_2, a contradiction. Thus, traversing P_3 starting at b_3, we may assume y and z to be the first and the last neighbor of q_1 in P_3, respectively, and $y \neq z$. If z is non-adjacent to y, using (18) we get a theta with ends b_2, q_1 and paths q_1-z-P_3-a-b_2, q_1-Q-q_t-p_i-P-b_2, and q_1-y-P_3-b_3-b_2, a contradiction. So z and y are adjacent. But now, again using (18), we get a near-prism with triangles $b_1b_2b_3$ and q_1zy and paths b_1-P'-q_t-Q-q_1, b_2-a-P_3-z and b_3-P_3-y, a contradiction. ∎

Theorem 6.4. *Let $G \in \mathcal{C}$. Suppose that G contains a loaded pyramid $\Pi = (\Sigma, P)$ of type 2 with $a, b_1, b_2, b_3, P_1, P_2, P_3, P$ as in the definition. Moreover, assume that D is a connected component of $G \setminus N[b_2]$ such that neither $N[D] \cap (\Pi \setminus P_3)$ nor $N[D] \cap (P_3 \setminus \{a\})$ is empty. Then one of the following holds.*

- *We have $N[D] \cap (\Pi \setminus P_3) = \{b_1\}$ and $N[D] \cap (P_3 \setminus \{a\}) = \{b_3\}$; or*
- *There is a proper wheel (H, b_2) in G with two long sectors Γ_1 and Γ_3 such that Γ_1^* contains the neighbor of a in P_1 and $\Gamma_3 = P_3$.*

Proof. Suppose not. Assume that P_1 traverses b_1, x, y, a in this order, where x and y are the two neighbors of q_t in P_1. Then $b_1 \neq x$. Let $P_1' = b_1$-P_1-x, $P_1'' = y$-P_1-a. We write $P = p_1$-$\cdots$-p_k where p_1 is adjacent to b_2, and p_k has a neighbor in P_1^*.

Since the first bullet of Theorem 6.4 does not hold, there exists a shortest path $Q = q_1$-$\cdots$-q_t in $G \setminus N[b_2]$ with $t \geq 1$ where

- q_1 has a neighbor in $P_3 \setminus \{a\}$;
- q_t has a neighbor in $\Pi \setminus P_3$; and
- either q_1 has a neighbor in P_3^* or q_t has a neighbor in $P_1^* \cup P$.

From the minimality of Q, it follows that Q^* is anticomplete to $(\Sigma \cup P) \setminus \{a, b_1, b_3\}$. Let c be the neighbor of b_1 in P_1. Then $c \neq a$, and since G is C_4-free, c is non-adjacent to a.

(19) *q_t has a neighbor in $P \cup P_1'$.*

Suppose not. Then $N_{P_1}(q_t) \subseteq P_1''$, and there is a path R from y to b_3 with $R \subseteq P_1'' \cup Q \cup P_3$. Now we get a prism with triangles p_kxy and $b_1b_2b_3$ and paths P, P_1' and R, a contradiction. This proves (19).

(20) *q_1 has a neighbor in P_3^*.*

Suppose not. Then $N_{P_3}(q_1) \subseteq \{b_3, a\}$. Suppose first that either a has a neighbor in Q, or q_t has a neighbor in $(P_1 \setminus b_1) \cup (P \setminus p_1)$. Then there is a path R from q_1 to a with $R^* \subseteq Q \cup (P_1 \setminus b_1) \cup (P \setminus p_1)$. Now we get a theta with ends b_3, a and paths P_3, b_3-b_2-a and b_3-q_1-R-a, a contradiction.

It follows that a is anticomplete to Q, and $N_{P_1 \cup P}(q_t) \subseteq \{p_1, b_1\}$. From the choice of Q, and since G is C_4-free, it follows that $N_{P_1 \cup P}(q_t) = \{p_1\}$. Now $H = b_3$-Q-p_1-P-y-P_1''-a-P_3-b is a hole, and so (H, b_2) is a proper wheel satisfying the second outcome of the theorem. This proves (20).

It follows from (20) and the minimality of t that b_1 is anticomplete to $Q \setminus q_t$. Let v be the neighbor of q_1 in P_3 closest to b_3. Let v' be the neighbor of q_1 in P_3 closest to a.

(21) q_t *has a neighbor in* $P_1^* \cup P$.

Suppose not. Then $N_{P_1 \cup P}(q_t) \subseteq \{b_1, a\}$. By (20) there is a path R from q_t to a with $R^* \subseteq Q \cup (P_3 \setminus b_3)$. Now we get a theta with ends b_1, a and paths P_1, b_1-b_2-a and b_1-q_t-R-a, a contradiction. This proves (21).

It follows from (21) and the minimality of t that b_3 is anticomplete to $Q \setminus q_1$.

(22) a *is anticomplete to* Q.

Suppose not. Let R_1 be a path from q_1 to b_1 with $R_1^* \subseteq Q \cup P_1' \cup P$. If possible, choose R_1 so that $p_1 \notin R_1$. Observe that $Q \subseteq R_1$ and a is anticomplete to $R_1 \setminus Q$. Let H_1 be the hole b_1-R_1-q_1-v-P_3-b_3-b_1. By (21) there is a path R_2 from b_2 to q_1 with $R_2^* \subseteq Q \cup P \cup (P_1' \setminus b_1)$. Observe that $Q \subseteq R_2$ and a is anticomplete to $R_2 \setminus Q$. Let H_2 be the hole b_2-R_2-q_1-v-P_3-b_3-b_2. Now $N_{H_2}(a) = N_{H_1}(a) \cup \{b_2\}$. Since neither of (H_1, a), (H_2, a) is a theta or an even wheel in G, it follows that a has exactly two neighbors uw in Q, and u is adjacent to w. We may assume that Q traverses q_1, u, w, q_t in this order. If $p_1 \notin R_1$, we get a prism with triangles auw and $b_1b_2b_3$ and paths ab_2, u-Q-q_1-v-P_3-b_3 and w-R_1-b_1, a contradiction. This proves that $p_1 \in R_1$, and therefore $N_{P \cup P_1'}(q_t) = \{p_1\}$. Now let H_3 be the hole b_1-P_1'-x-p_k-P-p_1-q_t-Q-w-a-P_3-b_3-b_1. Then $N_{H_3}(b_2) = \{b_1, b_3, p_1, a\}$ and so (H_3, b_2) is an even wheel, a contradiction. This proves (22).

(23) $N(q_t) \cap (P_1 \cup P)$ *is a clique of size at least 2.*

Suppose not. Since G is C_4-free, q_t is non-adjacent to at least one of b_1 and the neighbor p_1 of b_2 in P. Suppose that there are two paths R_1, R_2 from q_t to a, both with interior in $P_1 \cup P \cup b_2$ and such that R_1^* is anticomplete to R_2^*. Then we get a theta with ends q_t, a and paths R_1, R_2 and q_t-Q-v'-P_3-a, a contradiction. It follows that q_t has a unique neighbor w in $P_1 \cup P \cup b_2$. By (19), w is non-adjacent to a. Now there are two path R_1, R_2 from w to a with interior in $P_1 \cup P \cup b_2$ and such that R_1^* is anticomplete to R_2^*, and we get a theta with ends w, a and paths R_1, R_2 and w-q_t-Q-v'-P_3-a, a contradiction. This proves (23).

If follows from (19) and (23) that $N_{P_1''}(q_t) \subseteq \{y\}$.

(24) q_t *is adjacent to* y.

Suppose not. Let $N_{P_1 \cup P}(q_t) = \{u, w\}$, and we may assume that $P_1' \cup P$ traverses b_1, u, w, p_1 in this order. Then there is a prism with triangles uwq_t and $b_1b_2b_3$, two of whose paths have interior in $P_1' \cup P$, and the third one is q_t-Q-q_1-v-P_3-b_3, a contradiction. This proves (24).

(25) q_t *is non-adjacent to* x.

Suppose q_t is adjacent to x. Since we do not get a prism with triangles xyq_t and $b_1b_2b_3$ and paths x-P_1-b_1, y-P_1-a-b_2 ad q_t-Q-q_1-v-P_3-b_3, it follows that $v = v'$ is adjacent to a. But now we get a theta with ends b_2, v and paths b_2-a-v, b_2-p_1-P-p_k-x-q_t-Q-q_1-v (x is omitted if p_k is adjacent to q_t) and b_2-b_3-P_3-v. This proves (25).

By (23), (24), and (25), we deduce that $N_{P \cup P_1}(q_t) = \{y, p_k\}$.

Suppose that y is non-adjacent to a. Then G contains a theta with ends a, y and paths P_1'', y-P_1'-b_1-b_2-a, and y-q_t-Q-q_1-v'-P_3-a, a contradiction. This proves that y is adjacent to a. Since G is C_4-free, it follows that p_k is non-adjacent to b_2. Now, since by (20) $v' \neq b_3$, there is a theta with ends p_k, b_2 and paths p_k-x-P_1'-b_1-b_2, p_k-P-p_1-b_2 and p_k-q_t-Q-q_1-v'-P_3-a-b_2. ■

Having proved Theorems 6.3 and 6.4, we can give a proof of Theorem 6.2, below.

Proof of Theorem 6.2. Suppose for a contradiction that there exists a loaded pyramid $\Pi = (\Sigma, P)$ in G with $a, b_1, b_2, b_3, P_1, P_2, P_3, P$ as in the definition, and a connected component D of $G \setminus N[b_2]$ for which neither $N[D] \cap (\Pi \setminus P_3)$ nor $N[D] \cap (P_3 \setminus \{a\})$ is empty, such that neither of the two bullets of Theorem 6.2 hold. Also, let a' be the neighbor of a in P_1, and subject to the above properties and a' being the neighbor of a in P_1, let $|P_1|$ be minimal. We claim:

(26) *We have $N(p_k) \cap P_1$ is a clique.*

Suppose not. Traversing P_1 from b_1 to a, let x and y be the first and the last neighbor of p_k in P_1, respectively. Then x and y are distinct and non-adjacent, and we have $y \neq a$. If $k = 1$, then $H = b_1$-P_1-x-p_k-y-P_1-a-P_3-b_3-b_1 is a hole in G and b_2 has exactly four neighbors in H, namely a, b_1, b_3 and p_k. But then (H, b_2) is an even wheel in G, which is impossible. So $k > 1$ and consequently b_2 is not adjacent to p_k. Now, replacing the path P_1 in the pyramid Σ by the path $P_1' = b_1$-P_1-x-p_k-y-P_1-a we obtain a pyramid Σ' in G, where $\Pi' = (\Sigma', P \setminus p_k)$ is a loaded pyramid in G with $|P_1'| < |P_1|$ and D is a connected component of $G \setminus N[b_2]$ for which neither $N[D] \cap (\Pi' \setminus P_3)$ nor $N[D] \cap (P_3 \setminus \{a\})$ is empty, such that neither of the two bullets of Theorem 6.2 hold. This violates the choice of Π, and hence proves (26).

(27) *The vertex b_1 is anticomplete to P.*

Suppose not. Let i be maximum such that b_1 is adjacent to p_i. Also, traversing P_1 from b_1 to a, let y be the last neighbor of p_k in P_1. Then we have $y \in P_1 \setminus \{a, b_1\}$. If y is adjacent to b_1, then $N_{P_1}(p_k) \subseteq N_{P_1}(b_1)$ and Π is a loaded pyramid of type 1, which together with Theorem 6.3 implies that the one of the two bullets of Theorem 6.2 holds, a contradiction. Thus, y is not adjacent to b_1. Suppose that $i = 1$. Then $H = b_1$-p_1-P-y-P_1-a-P_3-b_3-b_1 is a hole in G and b_2 has exactly four neighbors in H, namely a, b_1, b_3 and p_1. But then (H, b_2) is an even wheel, which is impossible. So $i > 1$, and consequently b_2 is not adjacent to p_i. Now, replacing the path P_1 in the pyramid Σ by the path $P_1' = b_1$-p_i-P-p_k-y-P_1-a we obtain a pyramid Σ' in G, where $\Pi' = (\Sigma', p_1$-P-$p_{i-1})$ is a loaded pyramid in G with $|P_1'| < |P_1|$, and D is a connected component of $G \setminus N[b_2]$ for which neither $N[D] \cap (\Pi' \setminus P_3)$ nor $N[D] \cap (P_3 \setminus \{a\})$ is empty, such that neither of the two bullets of Theorem 6.2 hold. This violates the choice of Π, and hence proves (27).

Now, if p_k has exactly one neighbor x in P_1, then $x \neq b_1$ and so by (27), $P_1 \cup P \cup \{b_2\}$ is a theta with ends b_2, x in G, a contradiction. Therefore, by (26), p_k has exactly two neighbors in P_1, which are adjacent. Also, by (27), b_1 is anticomplete to P, and in particular b_1 is not adjacent to p_k. Therefore, we may assume that $N(p_k) \cap P_1 = \{x, y\} \subseteq P_1 \setminus \{a, b_1\}$ where P_1 traverses b_1, x, y, a in this order. But now Π is loaded pyramid of type 2, which along with Theorem 6.4 implies that the one of the two bullet of Theorem 6.2 holds, a contradiction. This completes the proof of Theorem 6.2. ■

We now prove Theorem 6.1.

Proof. Suppose not, and let D be a component of $G \setminus N[b_2]$ such that $\Pi \subseteq N[D]$. By Theorem 6.2 we deduce that there is a proper wheel (H, b_2) in G with two long sectors Γ_1 and Γ_3 such that Γ_1^* contains the neighbor of a in P_1 and $\Gamma_3 = P_3$. But now we get a contradiction to Theorem 5.2. ■

From Theorem 5.1 and Theorem 6.1 we deduce:

Theorem 6.5. *Let $G \in \mathcal{C}$ and let (H, v) be a proper wheel or a loaded pyramid in G. Then there is no component D of $G \setminus N[v]$ such that $H \subseteq N[D]$.*

7 Tree strip systems

In this section we summarize results from [17] that allow us to deal with cross-edges of near-prisms in an even-hole-free graph. Recall that an *extended near-prism* is a graph obtained from a near-prism by adding one extra edge, as follows. Let P_1, P_2, P_3 be as in the definition of a near-prism, and let $a \in P_1^*$ and $b \in P_2^*$; and add an edge ab. We call ab the *cross-edge* of the extended near-prism. Next we explain a theorem from [17] that describes the structure of graphs with an extended near prism. We start with several definitions from [17].

Let T be a tree with at least 3 leaves. A *leaf* of T is a vertex of degree exactly one, and a *leaf-edge* is an edge incident with a leaf. Let (A', B') be a bipartition of T, and assume that for every $v \in V(T)$, there is at most one component C of $T \setminus v$ such that $A' \cap C = \emptyset$, and at most one such that $B' \cap C = \emptyset$. (Note that every component C of $T \setminus v$ contains a leaf of T and therefore meets at least one of A', B'.) Since $|V(T)| \geq 3$, each leaf-edge is incident with only one leaf; let A be the set of leaf-edges incident with a leaf in A', and define B similarly. Let $L(T)$ be the line-graph of T, that is the vertex set of $L(T)$ is the edge set of T, and two edges of T are adjacent in $L(T)$ if they share an end in T. Add to $L(T)$ two more vertices a, b and the edge ab, and make a complete to A and b complete to B, forming a graph $H(T)$ with vertex set $E(T) \cup \{a, b\}$. We say that $H(T)$ is an *extended tree line-graph*, and ab is its *cross-edge*.

Every extended near-prism is an extended tree line-graph, where the corresponding tree has four leaves and exactly two vertices of degree three.

A *branch-vertex* of a tree is a vertex of degree different from two (thus, leaves are branch-vertices). A *branch* of a tree T is a path P of T with distinct ends u, v, both branch-vertices, such that every vertex of P^* has degree two in T. Every edge of T belongs to a unique branch.

Let T be a tree, and let U be the set of branch-vertices of T; and make a new tree J with vertex set U by making $u, v \in U$ adjacent in J if there is a branch of T with ends u, v. We call J the *shape* of T. Thus J has no vertices of degree two; and T is obtained from J by replacing each edge by a path of positive length.

Let A, B, C be subsets of $V(G)$, with $A, B \neq \emptyset$ and disjoint from C, and let $S = (A, B, C)$. A rung of S, or an *S-rung*, is a path p_1-$\cdots$-p_k of $G[A \cup B \cup C]$ such that $p_1 \in A$, $p_k \in B$ and $p_2, \ldots, p_{k-1} \in C$, and if $k > 1$ then $p_1 \notin B$ and $p_k \notin A$. (If $A \cap B \neq \emptyset$, then $k = 1$ is possible.) If every vertex in $A \cup B \cup C$ belongs to an S-rung, we call S a *strip*.

Let J be a tree with at least three vertices. M is a *J-strip system* in a graph G consists of:

- for each edge $e = uv$ of J, a subset $M_{uv} = M_{vu} = M_e$ of $V(G)$; and
- for each $v \in V(J)$, a subset M_v of $V(G)$

satisfying the following conditions:

- the sets M_e $(e \in E(J))$ are pairwise disjoint;
- for each $u \in V(J)$, $M_u \subseteq \bigcup_{v \in N_J(u)} M_{uv}$;
- for each $uv \in E(J)$, $(M_{uv} \cap M_u, M_{uv} \cap M_v, M_{uv} \setminus (M_u \cup M_v))$ is a strip;
- if $uv, wx \in E(J)$ with u, v, w, x all distinct, then there are no edges between M_{uv} and M_{wx};

- if $uv, uw \in E(J)$ with $v \neq w$, then $M_u \cap M_{uv}$ is complete to $M_u \cap M_{uw}$, and there are no other edges between M_{uv} and M_{uw}.

A rung of the strip $(M_{uv} \cap M_u, M_{uv} \cap M_v, M_{uv} \setminus (M_u \cup M_v))$ will be called an *e-rung* or *uv-rung*. (the dependence on M and J is left implicit, for the sake of brevity.) Let $V(M)$ denote the union of the sets M_e $(e \in E(J))$.

Let J be a tree, let M be a J-strip system in G, and let (α, β) be a partition of the set of leaves of J. We say an edge ab of G is a *cross-edge* for M with *partition* (α, β) if:

- J has no vertex of degree two, and at least three vertices;
- for every vertex $s \in V(J)$, s has at most one neighbor in α, and at most one in β;
- $a, b \notin V(M)$; and
- a is complete to $\bigcup_{u \in \alpha} M_u$, and a has no other neighbors in $V(M)$; b is complete to $\bigcup_{u \in \beta} M_u$, and b has no other neighbors in $V(M)$.

Under these circumstances, a leaf $v \in \alpha$ is called and *a-leaf*, and a leaf $v \in \beta$ is a *b-leaf*.

Let M be a J-strip system in G with cross-edge ab and partition (α, β). We say $X \subseteq V(M) \cup \{a, b\}$ is *local* if either:

- $X \subseteq M_e$ for some $e \in E(J)$;
- $X \subseteq M_u$ for some $u \in V(J)$; or
- X contains a and not b, and $X \setminus \{a\} \subseteq M_u$ for some leaf $u \in \alpha$; or X contains b and not a, and $X \setminus \{a\} \subseteq M_u$ for some leaf $u \in \beta$.

Next we describe two maximizations:

- We start with an even-hole-free graph G, and an edge ab of G, such that there is an extended tree line-graph $H(T)$ that is an induced subgraph of G, with cross-edge ab. Subject to this we choose T with as many branches as possible, that is, such that its shape J has $|E(J)|$ maximum.
- Then we choose a J-strip system M in G with the same cross-edge ab, with $V(M)$ maximal.

In these circumstances (J, M) is said to be *optimal for ab*.

We will need the following special case of Theorem 4.2 of [17] (here we have corrected a typo that occurred in the statement in [17]):

Theorem 7.1 (Chudnovsky, Seymour [17])**.** *Let ab be an edge of an even-hole-free graph G, and let (J, M) be optimal for ab. Assume that no vertex of G is adjacent to both a and b. Then for every connected induced subgraph F of $G \setminus (M \cup \{a, b\})$:*

- *if not both a, b have neighbors in $V(F)$, then the set of vertices in $V(M) \cup \{a, b\}$ with a neighbor in $V(F)$ is local;*
- *if both a, b have neighbors in $V(F)$, then there exists a leaf t of J such that every vertex of $V(M)$ with a neighbor in $V(F)$ belongs to M_t.*

Theorem 7.1 assumes that G is even-hole-free, rather than $G \in \mathcal{C}$. It is likely that the proof works under the more general assumption, we but we have not verified the details. The last result of this section is a slight strengthening of Theorem 7.1:

Theorem 7.2. *Let ab be an edge of an even-hole-free graph G, and let (J, M) be optimal for ab. Let (α, β) be the partition such that ab is a cross-edge for M with partition (α, β). Assume that*

- $|\beta| \geq 2$;
- *no vertex of G is adjacent to both a and b;*
- $a, b \notin \mathrm{Hub}(G)$*; and*
- $G \setminus N[a]$ *is connected.*

Let $F \subseteq G \setminus (M \cup \{a, b\})$ be connected. Then a is anticomplete to F, and the set of vertices in $M \cup \{b\}$ with a neighbor in F is local.

Proof. Let F be a component of $G \setminus (M \cup \{a, b\})$; notice that it suffices to prove Theorem 7.2 for such F. If a is anticomplete to F, the result follows from Theorem 7.1. Thus suppose that a has a neighbor in F. By Theorem 7.1 there exists a leaf t of J such that $N(F) \subseteq M_t \cup \{a, b\}$. Since $G \setminus N[a]$ is connected, it follows that t is a b-leaf and $N(F) \cap M_t \neq \emptyset$. Let P be a path from a to a vertex of $x \in M_t$ with $P^* \subseteq F$. Let Q be a path of J from t to an a-leaf t'. Concatenating rungs corresponding to edges of Q in M, we obtain a path R from x to a vertex $y \in M'_t$. Let H be the hole x-R-y-a-P-x. Since $H \cup b$ is not a theta, it follows that b has a neighbor in P^*. Since (H, b) is not a proper wheel and $N(a) \cap N(b) = \emptyset$, it follows that b has a unique neighbor $x' \in P^*$ and x' is adjacent to x and non-adjacent to a. Let t'' be a b-leaf in J such that $t'' \neq t$, and let S be a shortest path in J from t'' to Q. Since t is a leaf, it follows that $t \notin S$. Concatenating rungs corresponding to edges of S in M, we obtain a path T from $x'' \in M_{t''}$ to a vertex with two (consecutive) neighbors in R. But now $H \cup T \cup \{b\}$ is a loaded pyramid with loaded corner b and apex a, contrary to the fact that $b \notin \mathrm{Hub}(G)$. This proves Theorem 7.2. ■

8 From local to global separators

In this section we prove a theorem that is the heart of the proof of our main result. Qualitatively, the content of this theorem is the following. We have a graph G whose vertex set is partitioned into two subsets D and X where D is connected, and $X = N(D)$. We also have a dstinguished vertex $b \in D$. We are given a collection of clique cutsets separating individual vertices of X from b (and with additional properties). The theorem asserts that there is one cutset, whose clique cover number is bounded from above by an absolute constant, that separates a positive proportion of the vertices of X from b.

Let us now delve into the details. Let D be graph and let $b \in D$. Our first goal is to associate to each clique of D a canonical separation. First we handle cliques that do not contain b; in this case the definition is similar in spirit to other papers in the series. For every clique $K \subseteq D \setminus \{b\}$ let $B(K)$ be the component of $D \setminus K$ with $b \in B(K)$. Let $C(K) = N(B(K))$ and $A(K) = D \setminus (C(K) \cup B(K))$. We call $(A(K), C(K), B(K))$ the *b-canonical separation for K.*

Now we extend the definition of a b-canonical separation to all cliques of D. Thus let $K \subseteq D$ be a clique such that $b \in K$. Let $B(K)$ be the union of the components D' of $D \setminus K$ such that $b \in N[D']$. Let $C(K) = N(B(K))$ and let $A(K) = D \setminus (B(K) \cup C(K))$. Note that $(A(K), C(K), B(K)) = (A(K \setminus b), C(K \setminus b) \cup \{b\}, B(K \setminus b) \setminus \{b\})$.

Let $\mathcal{K}$ be the set of all cliques of D with $A(K)$ maximal. Let $\beta(D, b) = \bigcap_{K \in \mathcal{K}} (B(K) \cup C(K))$. (Some readers may recognize $\beta(D, b)$ as the "central bag" for a collection of separations defined in [7].) For every $v \in D \setminus \beta(D, b)$, let $F(v)$ be the component of $D \setminus \beta(D, b)$ such that $v \in F(v)$. The next two lemmas describe some of the properties of $\beta(D, b)$.

Lemma 8.1. *We have $b \in \beta(D, b)$. For every component F of $D \setminus \beta(D, b)$ there exists $K \in \mathcal{K}$ such that $F \subseteq A(K)$. In particular, $N(F(v))$ is a clique for every $v \in D \setminus \beta(D, b)$.*

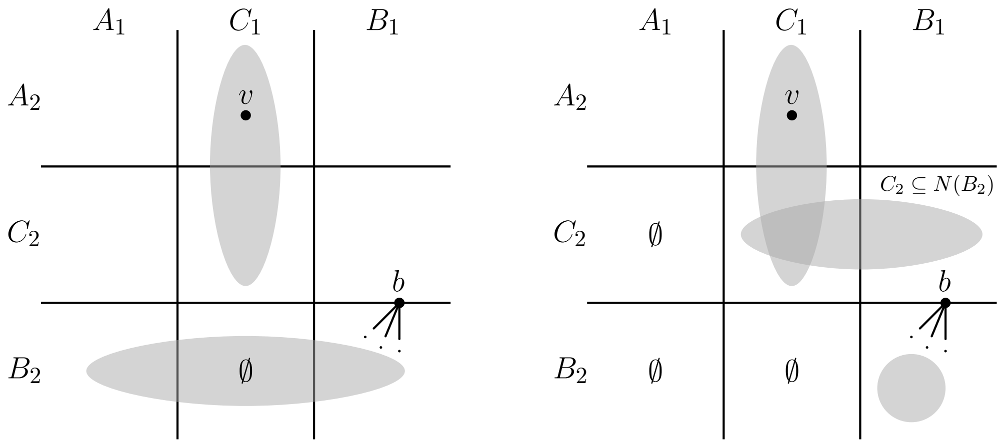

Figure 1: Proof of Lemma 8.1.

Proof. We prove the first assertion of the lemma first.

(28) $b \in \beta(D, b)$.

Let $K \in \mathcal{K}$. Then $b \in C(K) \cup B(K)$, and therefore (28) follows.

(29) *Let $K_1, K_2 \in \mathcal{K}$. Then $C(K_1) \cap A(K_2) = C(K_2) \cap A(K_1) = \emptyset$.*

Let (A_i, C_i, B_i) be the canonical separation associated with K_i. Suppose that there exists $v \in C_1 \cap A_2$. Then $C_1 \cap B_2 = \emptyset$.

First we show that $b \notin C_1 \cap C_2$. Suppose it is; then $b \in K_2$, and b has a neighbor in A_2, namely v, contrary to the definition of a canonical separation. This proves that $b \notin C_1 \cap C_2$. Since $C_1 \subseteq C_2 \cup A_2$, and $b \in \beta(D, b)$ by (28), it follows that $b \notin C_1$. By (28), $b \in B_1$; see Figure 1 (left).

Since $b \in B_1$, and $\{b\} \cup B_2$ is connected, and $(\{b\} \cup B_2) \cap C_1 = \emptyset$, it follows that $B_2 \subseteq B_1$; see Figure 1 (right). But $C_2 = N(B_2) \subseteq B_1 \cup C_1$, and so $A_1 \subseteq A_2 \setminus \{v\}$, contrary to the definition of $\mathcal{K}$. This proves (29).

Now let F be a component of $D \setminus \beta(D, b)$. Let $K \in \mathcal{K}$ be such that $F \cap A(K) \neq \emptyset$. By (29), $F \cap C(K) = \emptyset$. But now $F \subseteq A(K)$, as required. Since $N(F) \subseteq N(A(K)) \subseteq K$, the second assertion follows. ∎

Lemma 8.2. *If K is a clique cutset in the graph $\beta(D, b)$, then $b \in K$.*

Proof. Suppose that K is a clique cutset of $\beta(D, b)$ and $b \notin K$. Let D_1 be the component of $\beta(D, b) \setminus K$, with $b \in D_1$, and let $D_2 = \beta(D, b) \setminus (K \cup D_1)$. Since by Lemma 8.1, $N(F)$ is a clique for every component F of $D \setminus \beta(D, b)$, it follows that K is a clique cutset in D and no component of $D \setminus K$ meets both D_1 and D_2. But then $D_2 \cap A(K) \neq \emptyset$. It follows that there exists $K' \in \mathcal{K}$ such that $D_2 \cap A(K') \neq \emptyset$, contrary to the definition of $\beta(D, b)$. ∎

Next we define a *breaker* in a graph. Let G be a graph. Let $X \subseteq V(G)$ and let X_1, X_2, X_3 be subsets of X such that $|X_1| = |X_2| = |X_3|$ and X_1, X_2, X_3 are pairwise disjoint and anticomplete to each other. Write $D = G \setminus X$ and let $b \in D \setminus N[X]$. Assume that $N(D) = X$.

For $x_1, x_2, x_3 \in X$ let us say that $x_1x_2x_3$ is *partitioned* if $x_i \in X_i$ for every $i \in \{1, 2, 3\}$. For a partitioned triple $x_1x_2x_3$ we say that it is *b-separated* and *(i, j)-active* if b is non-adjacent to x_1, x_2, x_3 and there is a clique $K \subseteq D$ such that one of the following holds (For $i \in \{1, 2, 3\}$, let D_i be the union of components of $D \setminus K$ such that $N(x_i) \cap D_i \neq \emptyset$.)

- $b \notin K \cup D_i \cup D_j$; or
- We have $b \notin N[D_i]$. Moreover, there is a set $D'_j = D'_j(x_1x_2x_3)$ of vertices such that:
  - D'_j is not a clique;
  - D'_j is complete to K;
  - There is a vertex $q = q(x_1x_2x_3)$ with the following property. Either $b \in K$ and $q = b$; or there exists $k \in \{1, 2, 3\} \setminus \{i\}$ such that x_k is complete to $K \cup D'_j$ and $q = x_k$;
  - For every $v \in D'_j$, there is a path P in $D \cup \{x_j\}$ from b to x_j such that $P \cap N(q) \neq \emptyset$ and the (P, x_j)-last vertex in $P \cap N(q)$ is v; and
  - For every path P in $D \cup \{x_j\}$ from b to x_j, we have $P \cap N(q) \neq \emptyset$, and the (P, x_j)-last vertex of $P \cap N(q)$ is complete to K.

Under these circumstances we say that K is a *witness* for $x_1x_2x_3$. We say that $x_1x_2x_3$ is of *type 1* if the first bullet of (8) holds; otherwise, we say that $x_1x_2x_3$ is of *type 2*. We say an (i, j)-active triple $x_1x_2x_3$ is of *type 2a* if $q(x_1x_2x_3) = b$; it is of *type 2b* if $q(x_1x_2x_3) = x_j$; and it is of *type 2c* if $q(x_1x_2x_3) \neq b, x_j$. A triple is *b-separated* if it is b-separated and (i, j)-active for some distinct $i, j \in \{1, 2, 3\}$. Let $\delta \in (0, 1]$. We say that X is a *(δ, b)-breaker* in G if there exist at least $\delta|X|^3$ partitioned b-separated triples.

The main result of this section is the following:

Theorem 8.3. *Let $\delta \in (0, 1]$ and let $\epsilon \leq \frac{\delta^2}{48 \times 192}$. Let G be a C_4-free graph, and let X be a (δ, b)-breaker in G. Then there exists $S \subseteq D \setminus \{b\}$ with $\kappa(S) \leq (96/\delta)^2$ such that the component $D(b)$ of $D \setminus S$ with $b \in D(b)$ is disjoint from $N(x)$ for at least $\epsilon|X|$ vertices $x \in X$.*

Proof. We first show:

$$\begin{aligned}
\binom{\left\lceil \frac{96}{\delta} \right\rceil + 1}{2} &= \left(\left\lceil \frac{96}{\delta} \right\rceil\right)\left(\left\lceil \frac{96}{\delta} \right\rceil + 1\right)/2 \\
&\leq (96/\delta + 1)(96/\delta + 2)/2 \\
&= (96/\delta)^2 + 3 \cdot 96/(2\delta) + 1 - (96/\delta)^2/2 \\
&\leq (96/\delta)^2 + 3 \cdot 96/(2\delta) + 1 - 48 \cdot 96/\delta \\
&\leq (96/\delta)^2.
\end{aligned}$$

Now suppose that the statement is false. For every clique $K \subseteq D$, let $(A(K), C(K), B(K))$ be the b-canonical separation for K.

Let $x_1x_2x_3$ be a b-separated triple. We fix a witness $K(x_1, x_2, x_3)$ for $x_1x_2x_3$, where $K(x_1, x_2, x_3)$ is chosen such that $x_1x_2x_3$ is of type 1 if possible. Let $\mathcal{F}$ be the set of components of $D \setminus \beta(D, b)$. For every $x \in X$, let $\mathcal{F}(x)$ be the set of elements of $\mathcal{F}$ for which $N(x) \cap F \neq \emptyset$.

(30) *Let $x_1x_2x_3$ be a b-separated triple; write $K = K(x_1, x_2, x_3)$. Assume that the component $D(b)$ of $D \setminus (K \setminus b)$ with $b \in D(b)$ is anticomplete to x_1. Then x_1 is anticomplete to $\beta(D, b) \setminus K$. Moreover, let $F \in \mathcal{F}(x_1)$. Then either $K \cap F \neq \emptyset$, or $N(F) \subseteq K$.*

Since $\beta(D, b) \subseteq D(b) \cup K$, it follows that x_1 is anticomplete to $\beta(D, b) \setminus K$. Next, suppose that $F \cap K = \emptyset$ and that there is a vertex $p \in N(F) \setminus K$. By Lemma 8.2, $b \in \beta(D, b)$ and $\beta(D, b) \setminus (K \setminus b)$ is connected. Let P be a path from p to b with $P^* \subseteq \beta(D, b)$. Let Q be a path from $N(x_1)$ to p with interior in F. Then $R = Q\text{-}p\text{-}P\text{-}p$ is a path from $N(x_1)$ to b with $R^* \cap K = \emptyset$, a contradiction. This proves (30).

It follows from (30) that:

(31) *If $x_1x_2x_3$ is a $(1,2)$-active triple, then x_1 is anticomplete to $\beta(D,b) \setminus K(x_1,x_2,x_3)$. Moreover, if $x_1x_2x_3$ is of type 1, then x_2 is anticomplete to $\beta(D,b) \setminus K(x_1,x_2,x_3)$.*

Let $x \in X$. We define the *projection* of x, denoted by $\mathrm{Proj}(x)$, to be $N_{\beta(D,b)}(x) \cup \bigcup_{F \in \mathcal{F}(x)} N(F)$. Note that since b is non-adjacent to x_1, x_2, x_3 for b-seperated triples $x_1x_2x_3$, and since b is anticomplete to $A(K)$ for every $K \in \mathcal{K}$, it follows that $b \notin \mathrm{Proj}(x)$ whenever x is in a b-separated triple.

(32) *Let $x_1x_2x_3$ be a $(1,2)$-active triple; write $K = K(x_1,x_2,x_3)$. If $x_1x_2x_3$ is of type 1, then $\mathrm{Proj}(x_1) \cup \mathrm{Proj}(x_2)$ is a clique.*

For $i \in \{1,2\}$, let $F_i = \bigcup_{F \in \mathcal{F}(x_i)} F$. Suppose first that $K \cap F_i = \emptyset$ for all $i \in \{1,2\}$; then by (30) $N(F_i) \subseteq K$ for $i \in \{1,2\}$. Since by (31), $N_{\beta(D,b)}(x_i) \subseteq K$ for $i \in \{1,2\}$, (32) holds. Thus we may assume that there exists $F \in \mathcal{F}(x_1)$ such that $K \cap F \neq \emptyset$. Let $F' \in (F_1 \cup F_2) \setminus F$. Since K is a clique, it follows that $F' \cap K = \emptyset$, and by (30), $N(F') \subseteq K$.

Since $K \cap F \neq \emptyset$, we have that $K \cap \beta(D,b) \subseteq N(F)$, and so $N(F') \subseteq N(F)$. Moreover, by (31), for $i \in \{1,2\}$, we have $N_{\beta(D,b)}(x_i) \subseteq K$, and so $N_{\beta(D,b)}(x_i) \subseteq N(F)$. $N(F)$ is a clique by Lemma 8.1, and (32) follows.

(33) *Let $x_1x_2x_3$ be a $(1,2)$-active triple which is of type 2; write $K = K(x_1,x_2,x_3)$. Then $\mathrm{Proj}(x_1) \subseteq K$.*

By (31), we have $N_{\beta(D,b)}(x_1) \subseteq K$. By (30), we either have $N(F) \subseteq K$ or $K \cap F \neq \emptyset$ for every $F \in \mathcal{F}(x_1)$. If the former holds for all $F \in \mathcal{F}(x_1)$, then (33) holds; so we may assume that there exists $F \in \mathcal{F}(x_1)$ with $K \cap F \neq \emptyset$. By Lemma 8.1, it follows that $N(F)$ is a clique K'. From the definition of $\beta(D,b)$, it follows that $b \notin K'$. We claim that K' is a witness for $x_1x_2x_3$ that makes $x_1x_2x_3$ be of type 1 (and therefore contradicts the choice of $K = K(x_1x_2x_3)$). Suppose not; let P be a path from b to x_i for some $i \in \{1,2\}$ such that $P^* \cap K' = \emptyset$. From the definition of a b-separated $(1,2)$-active triple, it follows that $P^* \cap (K \cup D_2'(x_1x_2x_3)) \neq \emptyset$. Since $D_2'(x_1x_2x_3)$ is complete to K, it follows that $(K \cup D_2'(x_1x_2x_3)) \setminus K' \subseteq F$. Therefore, $P^* \cap F \neq \emptyset$. Since $P^* \cap N(F) = \emptyset$, it follows that $P \subseteq N[F] \cup \{x_i\}$, contrary to the fact that $b \notin N[F] \cup \{x_i\}$. This is a contradiction, and proves (33).

By permuting the indices if necessary, we may assume that $\frac{\delta}{6}|X|^3$ separated triples are $(1,2)$-active. Now, for one of the four possible types (1, 2a, 2b, 2c), there exist $\frac{\delta}{24}|X|^3$ distinct $(1,2)$-active triples $x_1x_2x_3$ of this type with respect to $K(x_1x_2x_3)$. Let l be the first entry of the list (1, 2a, 2b, 2c) for which this is the case; let us say that a triple $x_1x_2x_3$ is *manageable* if it is $(1,2)$-active of type l.

Let $Z_1 \subseteq X_1$ be the set of all vertices $y_1 \in X_1$ such that

$$|\{(y_2,y_3) \in X_2 \times X_3 : y_1y_2y_3 \text{ is a manageable triple}\}| \geq \frac{\delta}{48}|X|^2.$$

(34) $|Z_1| \geq \frac{\delta}{48}|X|$.

Suppose not. Each vertex $y_1 \in X_1 \setminus Z_1$ is in fewer than $\frac{\delta}{48}|X|^2$ manageable triples. Therefore, the total number of manageable triples is less than

$$|Z_1||X|^2 + |X_1 \setminus Z_1|\frac{\delta}{48}|X|^2 < \frac{\delta}{48}|X|^3 + \frac{\delta}{48}|X|^3 = \frac{\delta}{24}|X|^3,$$

a contradiction. This proves (34).

Recall that by (32), $\mathrm{Proj}(w)$ is a clique for every $w \in Z_1$. Since $|Z_1| > \epsilon|X|$, we have $\alpha(\bigcup_{w\in Z_1}\mathrm{Proj}(w)) > \frac{96}{\delta}$.

Let W_1 be a minimal subset of Z_1 such that $\alpha(\bigcup_{w\in W_1}\mathrm{Proj}(w)) \geq \frac{96}{\delta}$. Then $\alpha(\bigcup_{w\in W_1}\mathrm{Proj}(w)) = \left\lceil\frac{96}{\delta}\right\rceil$. Let $S = \bigcup_{w\in W_1}\mathrm{Proj}(w)$. By a theorem of [47] (using that the complements of even-hole-free graphs contain no induced two-edge matching), we have $\kappa(S) \leq \binom{\alpha(S)+1}{2} \leq \binom{\lceil\frac{96}{\delta}\rceil+1}{2}$.

Let J be a stable set of size $\left\lceil\frac{96}{\delta}\right\rceil$ in S. For every $j \in J$, let $x(j) \in W_1$ be such that $j \in \mathrm{Proj}(x(j))$. Since J is a stable set, the elements $x(j)$ are pairwise distinct.

Let H be the bipartite graph with bipartition $(\{x(j)\}_{j\in J}, X_2 \times X_3)$, such that $x(j)$ is adjacent to $(y_2, y_3) \in X_2 \times X_3$ if $x(j)y_2y_3$ is a manageable triple.

(35) *There exist distinct $j_1, j_2 \in J$ such that $|N_H(x(j_1)) \cap N_H(x(j_2))| \geq \frac{\delta^2}{48\times 192}|X|^2$.*

Suppose not. Let $j \in J$. Since $x(j) \in Z_1$, it follows that $x(j)$ has at least $\frac{\delta}{48}|X|^2$ neighbors in H. For every $j \in J$, let $M(j)$ be the set of vertices $y \in Y_2$ such that y is adjacent to $x(j)$ in H, and y is not adjacent in H to any other vertex $x(j')$ for $j' \neq j$. Since $|N_H(x(j_1)) \cap N_H(x(j_2))| < \frac{\delta^2}{48\times 192}|X|^2$ for all distinct $j_1, j_2 \in J$, and since each $x(j)$ has at least $\frac{\delta}{48}|X|^2$ neighbors in H, it follows that

$$|M(j)| > \frac{\delta}{48}|X|^2 - \left(\left\lceil\frac{96}{\delta}\right\rceil - 1\right)\frac{\delta^2}{48\times 192}|X|^2 > \frac{\delta}{96}|X|^2$$

for each $j \in J$. But now $\bigcup_{j\in J}|M(j)| > |J|\frac{\delta}{96}|X|^2 \geq |X|^2$, a contradiction. This proves (35).

Suppose that $l = 1$. Let j_1, j_2 be as in (35), and let

$$Z_2 = \{y_2 : (y_2, y_3) \in N_H(x(j_1)) \cap N_H(x(j_2)) \text{ for some } y_3 \in X_3\}.$$

Then $|Z_2| \geq \frac{\delta^2}{48\times 192}|X|$. Since $|Z_2| \geq \epsilon|X|$, we deduce that $\kappa(\bigcup_{y\in Z_2}\mathrm{Proj}(y)) > \binom{\lceil\frac{96}{\delta}\rceil+1}{2} > 4$. Therefore we can choose non-adjacent $k_1, k_2 \in (\bigcup_{y\in Z_2}\mathrm{Proj}(y)) \setminus (\mathrm{Proj}(x(j_1)) \cup \mathrm{Proj}(x(j_2)))$. For $i \in \{1,2\}$, let $y(k_i) \in Z_2$ be such $k_i \in \mathrm{Proj}(y_i)$. It follows that for every $p, q \in \{1,2\}$ there exists $y_3(p,q) \in Y_3$ such that $x(j_p), y(k_q), y_3(p,q)$ is a manageable triple. Now applying (32) to $x(j_p)y(k_q)y_3(p,q)$, we deduce that j_p is adjacent to k_q; consequently $\{j_1, j_2\}$ is complete to $\{k_1, k_2\}$. But then j_1-k_1-j_2-k_2-j_1 is a C_4 in G, a contradiction.

This proves that l is one of 2a, 2b, 2c. Let j_1, j_2 be as in (35) and let $(y_2, y_3) \in N_H(x(j_1)) \cap N_H(x(j_2))$. Note that since $x(j_1)y_2y_3$ and $x(j_2)y_2y_3$ are both $(1,2)$-active and of the same type in 2a, 2b, 2c, we have $q(x(j_1)y_2y_3) = q(x(j_2)y_2y_3)$. Let us define $q = q(x(j_1)y_2y_3)$. For $i \in \{1,2\}$, let $K_i = K(x(j_i)y_2y_3)$. By (33) we have $\mathrm{Proj}(x(j_1)) \subseteq K_1$ and so $j_1 \in K_1$; likewise, $j_2 \in K_2$.

Let $p_1, p_2 \in D_2'(x(j_1)y_2y_3)$ be non-adjacent. For $i \in \{1,2\}$, let P_i be a path from b to y_2 in $D \cup \{y_2\}$ such that the (P_i, y_2)-last vertex in $P_i \cap N(q)$ is p_i. It follows that p_1, p_2 are complete to K_1, K_2. But now j_1-p_1-j_2-p_2 is a C_4 in G, a contradiction. ■

9 Handling dangerous triples

Let G be a graph, let $a \in G$ and write $X = N(a)$ and $D = G \setminus N[a]$. Let X be partitioned into three equal-size subsets X_1, X_2, X_3 pairwise anticomplete to each other. We say that the triple $x_1x_2x_3$ with $x_i \in X_i$ is *dangerous with center x_2* if the edge ax_2 is a cross-edge of an extended near-prism in the graph $D \cup \{x_1, x_2, x_3, a\}$.

The goal of this section is to prove that if G contains many dangerous triples with a fixed center, then we can bypass the main argument of Section 10 and obtain the desired conclusion directly. The details of this are explained in Section 10.

We need the following definition: Let $G = D \cup X \cup \{a\}$ be a graph where D is connected, a is complete to X and anticomplete to D, and $N(D) = X$. Let $b \in D$. Let us say that $X' \subseteq X$ is *pure* if there does not exist a hole $H \subseteq D \cup X' \cup \{a\}$ such that $a, b \in H$.

Theorem 9.1. *Let $\delta \in (0, 1]$ and let $\epsilon \leq \frac{1}{8}\delta$. Let $G = D \cup X \cup \{a\}$ be a graph where D is connected, a is complete to X and anticomplete to D, and $N(D) = X$. Assume that $X \cap \mathrm{Hub}(G) = \emptyset$. Assume that X is partitioned into three pairwise anticomplete sets X_1, X_2, X_3 of equal size. Suppose that some $x_2 \in X_2$ is a center of $\delta|X|^2$ dangerous triples. Assume also that there is no clique of size $\epsilon|X|$ in X. Let $b \in D$. Then one of the following holds:*

- *there exists $S \subseteq D \setminus \{b\}$ with $\kappa(S) \leq 4$ such that the component $D(b)$ of $D \setminus S$ with $b \in D(b)$ is disjoint from $N(x)$ for at least $\epsilon|X|$ vertices $x \in X$; or*
- *X is not pure and there exists $X' \subseteq X$ with $|X'| \geq \frac{1-4\epsilon}{2}|X|$ such that X' is pure.*

Proof. Suppose not. Let $x_2 \in X_2$. For $i = 1, 3$ let $Y_i \subseteq X_i$ be the set of all $y \in X_i$ such that there exist at least $\frac{1}{2}\delta|X|$ elements $z \in X_{4-i}$ for which yx_2z is a dangerous triple with center x_2.

(36) *For $i = 1, 3$, $|Y_i| \geq \frac{1}{2}\delta|X|$.*

Suppose that $|Y_1| < \frac{1}{2}\delta|X|$. The number of dangerous triples with center x_2 and using an element of Y_1 is at most $|Y_1||X| \leq \frac{1}{2}\delta|X|^2$. The number of dangerous triples with center x_2 and not using an element of Y_1 is less than $|X_1 \setminus Y_1| \times \frac{1}{2}\delta|X| \leq \frac{1}{2}\delta|X|^2$. It follows that the number of dangerous triples with center x_2 is less than $\delta|X|^2$, a contradiction. This proves (36).

Let $G' = D \cup X_1 \cup X_3 \cup \{a, x_2\}$. Then ax_2 is a cross-edge of an extended near-prism in G' and no vertex of G' is adjacent to both a and x_2. Since $N(D) = X$, it follows from Theorem 6.5 that $a \not\in \mathrm{Hub}(G)$. Applying Theorem 7.2 to ax_2 and G', we obtain a J-strip system M with cross-edge ax_2 such that for every connected induced subgraph F of $G' \setminus (M \cup \{a, x_2\})$, we have that a is anticomplete to F, and the set of vertices in $M \cup \{x_2\}$ with a neighbor in F is local.

(37) *For every $x \in X_1 \cup X_3$, there exists an a-leaf t such that $x \in M_t$.*

Since $X_1 \cup X_3 \subseteq N(a)$, Theorem 7.2 implies that $X_1 \cup X_3 \subseteq V(M)$. Now (37) follows from the fact that ax_2 is a cross-edge for M.

Let $v \in V(J)$. Let e be an edge of J incident with v. We say that e is *special for* v if either v is a leaf, or the set $M_v \cap M_e$ is not a clique. Since G' is C_4-free and $M_v \cap M_e$ is complete to $M_v \cap M_{e'}$ for distinct edges e, e' incident with v, it follows that for every $v \in V(J)$, there is at most one special edge for v.

For $v \in V(J)$, let F_v be, the union of components F of $G' \setminus (M \cup \{a, x_2\})$ such that $N(F) \cap M \subseteq M_v$. For $e \in E(J)$ with ends u, v let F_e be the union of components F of $G' \setminus (M \cup \{a, x_2\})$ such that $N(F) \cap M \subseteq M_e$, and such that $N(F) \not\subseteq M_u$ and $N(F) \not\subseteq M_v$. Since D is connected and disjoint from $N(a)$, it follows that $F_t = \emptyset$ for every a-leaf t.

Let $e = uv$ be an edge of J. Define $\mu(e)$ as follows. If e is not special for either u or v, let $\mu(e) = F_e \cup M_e \setminus (M_u \cup M_v)$. If e is special for u and not for v, let $\mu(e) = F_e \cup (M_e \setminus M_v) \cup F_u$. If e is special for v and not for u, let $\mu(e) = F_e \cup (M_e \setminus M_u) \cup F_v$. If e is special for both u and v, let $\mu(e) = M_e \cup F_e \cup F_u \cup F_v$. In all cases let $\nu(e) = \emptyset$. Next, let $v \in V(J)$. If there is a special edge e for v, let $\mu(v) = M_v \setminus M_e$ and $\nu(v) = \emptyset$. If no edge is special for v, let $\mu(v) = M_v$ and $\nu(v) = F_v$. It follows that $\mu(v)$ is a clique for every $v \in V(J)$. Note that each vertex of $V(G') \setminus \{x_2, a\}$ is in at least one set in $\{\mu(x), \nu(x)\}_{x \in V(J) \cup E(J)}$; the only vertices of $V(G') \setminus \{x_2, a\}$ that are in two such sets are vertices in $M_u \cap M_v$ where uv is not special for u and not special for v; they are in both $\mu(v)$ and $\mu(u)$.

(38) *If e is an edge of J with ends u, v and $b \in \mu(e)$, then* $|(X_1 \cup X_3) \setminus (\mu(e) \cup \mu(v) \cup \mu(u))| \leq \epsilon|X|$.

Suppose $x \in (X_1 \cup X_3) \setminus (\mu(e) \cup \mu(v) \cup \mu(u))$. Then $N(x) \cap \mu(e) = \emptyset$. Since $L = \mu(v) \cup \mu(u)$ separates $\mu(e)$ from $D \setminus (\mu(e) \cup L)$ in D, it follows that the component of $D \setminus L$ that contains b is disjoint from $N(x)$. Since $\kappa(L) \leq 2$, there are at most $\epsilon|X|$ such vertices x, and (38) follows.

(39) *Suppose that $v \in V(J)$ and $b \in \mu(v)$. Let $e = vu \in E(J)$ such that $b \in M_e$. Moreover:*

- *Define L_1 and C_1 as follows. Let $f = vw$ be a special edge at v if one exists; in this case, let $L_1 = \mu(w) \cup \mu(f)$ and $C_1 = \mu(w)$; otherwise, $L_1 = C_1 = \emptyset$.*
- *Define L_2 and C_2 as follows. If $b \in M_u \cap M_v$ and there is a special edge uz at u with $z \neq v$, let $L_2 = \mu(uz) \cup \mu(z)$ and $C_2 = \mu(z)$. If $b \in M_u \cap M_v$ and there is no special edge at u except possibly uv, we let $L_2 = \nu(u)$ and $C_2 = \emptyset$. Finally, if $b \notin M_u$, let $L_2 = C_2 = \emptyset$.*

Then $|(X_1 \cup X_3) \setminus (\mu(e) \cup \mu(v) \cup L_1 \cup L_2 \cup \mu(u) \cup \nu(v))| \leq \epsilon|X|$.

Suppose that $x \in (X_1 \cup X_3) \setminus (\mu(e) \cup \mu(v) \cup L_1 \cup L_2 \cup \mu(u) \cup \nu(v))$. Let $L = ((\mu(v) \cup \mu(u) \cup C_1 \cup C_2) \setminus \{b\}) \cap D$. Then L separates $(\{b\} \cup \mu(e) \cup \nu(v) \cup (L_1 \setminus C_1) \cup (L_2 \setminus C_2)) \cap D$ from the rest of D, and in particular, from $D \cap (N(x) \setminus L)$. Since $\kappa(L) \leq 4$, there are at most $\epsilon|X|$ such vertices x, and (39) follows.

(40) *Suppose that $v \in V(J)$ and $b \in \nu(v)$. Then* $|(X_1 \cup X_3) \setminus (\mu(v) \cup \nu(v))| \leq \epsilon|X|$.

Suppose that $x \in (X_1 \cup X_3) \setminus (\mu(v) \cup \nu(v))$. Then $N(x) \cap \nu(v) = \emptyset$. Let $L = \mu(v)$. Then L separates $\nu(v)$ from $D \setminus (\nu(v) \cup \mu(v))$. We deduce that the component of $D \setminus L$ that contains b is disjoint from $N(x)$. Since $\kappa(L) = 1$, there are at most $\epsilon|X|$ such vertices x, and (40) follows.

It follows from (37), (38), (39) and (40) that there is an a-leaf t with $N_J(t) = \{t'\}$ such that $b \in \mu(t't) \cup M_{t'} \cup \nu(t')$. If $b \in \nu(t')$, then by (37) and (40), it follows that $|X \cap \mu(t')| \geq (1-\epsilon)|X|$. But $\mu(t')$ is a clique, and $\epsilon \leq 1/8$, and X contains no clique of size $\epsilon|X|$, a contradiction. It follows that $b \in \mu(t't) \cup M_{t'}$.

From Theorem 7.2, it follows that:

(41) *If $x \in N(x_2) \cap M$, then there is a b-leaf q such that $x \in M_q \cup F_q$.*

Next, we show:

(42) *Let t be an a-leaf, and let t' be the unique neighbor of t in J. Let $Z = \mu(tt') \cup \nu(t')$. Suppose that $y_1 \in X_1 \cap Z$ and $y_3 \in X_3 \cap Z$. Then $y_1x_2y_3$ is not a dangerous triple.*

Suppose not. Then, by the definition of a dangerous triple, there exists a path R from y_1 to y_3 with $x_2 \in R$ and such that $R \setminus \{y_1, x_2, y_3\} \subseteq D$. By (41), it follows that there exist b-leaves q_1, q_3 such that the neighbor of x_2 on the subpath of R from x_2 to y_i is in $M_{q_i} \cup F_{q_i}$ for $i \in \{1, 3\}$. Since $\mu(t')$ separates Z from $F_r \cup M_r$ for all $r \in V(J) \setminus \{t, t'\}$, it follows that the interiors of both the paths y_1-R-x_2 and x_2-R-y_3 meet $\mu(t')$, contrary to the fact that $\mu(t')$ is a clique. This proves (42).

(43) *Let t be an a-leaf, and let t' be the unique neighbor of t in J. Let t'' be a neighbor of t with $t'' \neq t$. Let $Z = \mu(tt') \cup \mu(t't'') \cup \mu(t') \cup \mu(t'') \cup \nu(t') \cup \nu(t'')$. Then $|X \cap Z| < (1-\epsilon)|X|$.*

From the definition of a J-strip structure with cross-edge ax_2, it follows that t'' is not an a-leaf.

Now, since X contains no clique of size $\epsilon|X|$, it follows that $|X \cap \mu(t') \cup \mu(t'')| \leq 2\epsilon|X|$. Furthermore, (37) implies that $X \cap \nu(t') = \emptyset$, $X \cap \nu(t'') = \emptyset$ and $X \cap \mu(t't'') = \emptyset$ (as t'', t' are not a-leaves). Therefore, $|(X_1 \cup X_3) \cap \mu(tt')| \geq (1-3\epsilon)|X|$. Since $\epsilon < \frac{1}{8}\delta$, it follows from (36) that there exist $y_1 \in Y_1 \cap (Z \setminus \mu(t'))$ and $y_3 \in X_3 \cap (Z \setminus \mu(t'))$ such that $y_1 x_2 y_3$ is a dangerous triple, contrary to (42). This proves (43).

To finish the proof of Theorem 9.1, we consider three cases. Suppose first that either:

- $b \in \mu(tt')$; or
- $b \in M_{tt'} \cap M_{t'}$ and no edge is special at t' (except possibly tt').

Let $Z = \mu(tt') \cup \mu(t')$ if $b \in \mu(tt')$, and let $Z = \mu(tt') \cup \mu(t') \cup \nu(t')$ otherwise. By (38) and (39) and since $\mu(t) = \nu(t) = \emptyset$, it follows that $|X \cap Z| \geq (1-\epsilon)|X|$. This contradicts (43) (choosing t'' arbitrarily).

Now suppose that $b \in (M_{t'} \cap M_{tt'}) \setminus \mu(tt')$. We may assume that we are not in the first case, and so it follows that there is an edge $t't''$ with $t'' \neq t$ which is special at t'. It follows that t'' is not an a-leaf. Let $Z = \mu(tt') \cup \mu(t') \cup \mu(t't'') \cup \mu(t'') \cup \nu(t')$. By (39) and since $\nu(t) = \mu(t) = \emptyset$, it follows that $|X \cap Z| \geq (1-\epsilon)|X|$. This contradicts (43).

It follows that $b \in M_{t'} \cap M_{t't''}$ for some neighbor t'' of t' with $t'' \neq t$. Then t'' is not an a-leaf. Suppose first that either:

- $b \notin M_{t''}$; or
- $b \in M_{t''} \cap M_{t'}$ and there is no special edge at t'' except possibly $t''t'$.

Let $Z = \mu(tt') \cup \mu(t't'') \cup \mu(t') \cup \mu(t'') \cup \nu(t') \cup \nu(t'')$. Using (39) (with $t' = v$; $t = w$; $t'' = u$), we conclude that $|X \cap Z| \geq (1-\epsilon)|X|$. Again, this contradicts (43).

It follows that $b \in M_{t'} \cap M_{t''}$, and there is a special edge $t''t'''$ at t'' with $t''' \neq t'$. Let $Z = \mu(tt') \cup \mu(t't'') \cup \mu(t') \cup \mu(t'') \cup \nu(t') \cup \nu(t'') \cup \mu(t''t''') \cup \mu(t''')$. Using (39) (with $t' = v$; $t = w$; $t'' = u$; $t''' = z$), we conclude that $|X \cap Z| \geq (1-\epsilon)|X|$. From (37), it follows that $X \cap (\nu(t') \cup \nu(t'')) = \emptyset$. Since X contains no clique of size at least $\epsilon|X|$, it follows that $|X \cap (\mu(t') \cup \mu(t'') \cup \mu(t'''))| < 3\epsilon|X|$. Neither t' nor t'' is a leaf, and so $X \cap \mu(t't'') = \emptyset$ by (37).

It follows that $|X \cap (\mu(tt') \cup \mu(t''t'''))| > (1-4\epsilon)|X|$. If $X \cap \mu(t''t''') = \emptyset$, then as before, there exist $y_1 \in \mu(tt') \cap Y_1$ and $y_3 \in \mu(tt') \cap Y_3$ such that $y_1 x_2 y_3$ is a dangerous triple, contrary to (42). So $X \cap \mu(t''t''') \neq \emptyset$. It follows that t''' is an a-leaf. There is symmetry (switching t, t', t'', t''' with t''', t'', t', t), and so $X \cap \mu(tt') \neq \emptyset$.

Let $x \in X \cap \mu(tt')$ and let $x' \in X \cap \mu(t''t''')$. Then $x \in M_t$ and $x' \in M_{t'''}$ by (37). Let R be a tt'-rung containing x, and let R' be a $t''t'''$-rung containing x'. Then x-R-b-R'-x'-a-x is a hole in $D \cup X \cup \{a\}$ containing a and b, and so X is not pure.

By symmetry, we may assume that $|X \cap \mu(tt')| \geq \frac{1-4\epsilon}{2}|X|$. Write $X' = X \cap \mu(tt')$. We claim that X' is pure (and so the second outcome of the theorem holds). Suppose not; let H be a hole containing a and b with $H \setminus a \subseteq X' \cup D$. Then H contains two internally disjoint paths from b to a, say P_1 and P_2. Since $Y = M_{t'} \cap M_{tt'}$ separates b from $X' \setminus Y$ in $D \cup X'$, it follows that P_1^* and P_2^* each contain a vertex in Y (and in particular, tt' is special at t'). Since $\mu(t')$ is complete to Y, it follows that $H \cap \mu(t') = \{b\}$. Consequently, $H \setminus b \subseteq \mu(tt')$. It follows that $R' \setminus b$ is anticomplete to $H \setminus b$. But now $H \cup R'$ is a theta in G with ends a, b, a contradiction. This concludes the proof. ■

10 Separating a pair of vertices: the hub-free case

In this section we set $\epsilon = \frac{1}{4\times 17^6 \times 48 \times 192}$, $\gamma = \frac{\epsilon(1-4\epsilon)}{2}$ and $C = 96^2 \times 4 \times 17^6 + 4$. The goal of this section is to prove the following:

Theorem 10.1. *Let $G \in \mathcal{C}$ with $|V(G)| = n$, and let $a, b \in V(G)$ be non-adjacent. Assume that $N(a) \cap \text{Hub}(G) = \emptyset$. Then there is a set $Z \subseteq V(G) \setminus \{a, b\}$ with $\kappa(Z) \leq -C\frac{1}{\log(1-\gamma)} \log n$ and such that every component of $G \setminus Z$ contains at most one of a, b.*

We need the following result from [5].

Lemma 10.2 (Abrishami, Chudnovsky, Dibek, Vušković [5])**.** *Let x_1, x_2, x_3 be three distinct vertices of a graph G. Assume that H is a connected induced subgraph of $G \setminus \{x_1, x_2, x_3\}$ such that $V(H)$ contains at least one neighbor of each of x_1, x_2, x_3, and that $V(H)$ is minimal subject to inclusion. Then, one of the following holds:*

(i) For some distinct $i, j, k \in \{1, 2, 3\}$, there exists P that is either a path from x_i to x_j or a hole containing the edge x_ix_j such that

- *$V(H) = V(P) \setminus \{x_i, x_j\}$; and*
- *either x_k has two non-adjacent neighbors in H or x_k has exactly two neighbors in H and its neighbors in H are adjacent.*

(ii) There exists a vertex $a \in V(H)$ and three paths P_1, P_2, P_3, where P_i is from a to x_i, such that

- *$V(H) = (V(P_1) \cup V(P_2) \cup V(P_3)) \setminus \{x_1, x_2, x_3\}$;*
- *the sets $V(P_1) \setminus \{a\}$, $V(P_2) \setminus \{a\}$ and $V(P_3) \setminus \{a\}$ are pairwise disjoint; and*
- *for distinct $i, j \in \{1, 2, 3\}$, there are no edges between $V(P_i) \setminus \{a\}$ and $V(P_j) \setminus \{a\}$, except possibly x_ix_j.*

(iii) There exists a triangle $a_1a_2a_3$ in H and three paths P_1, P_2, P_3, where P_i is from a_i to x_i, such that

- *$V(H) = (V(P_1) \cup V(P_2) \cup V(P_3)) \setminus \{x_1, x_2, x_3\}$;*
- *the sets $V(P_1)$, $V(P_2)$ and $V(P_3)$ are pairwise disjoint; and*
- *for distinct $i, j \in \{1, 2, 3\}$, there are no edges between $V(P_i)$ and $V(P_j)$, except a_ia_j and possibly x_ix_j.*

We also need the following; for a proof see, for example, [3]:

Theorem 10.3. *Let (T, χ) be a tree decomposition of a graph G. Then there exist a vertex $t_0 \in T$ such that $|D| \leq \frac{1}{2}|V(G)|$ for every component D of $G \setminus \chi(t_0)$.*

We start with a lemma.

Lemma 10.4. *Let n be an integer. Let G be a chordal graph with $n - \epsilon n$ vertices, and assume that G has no clique of size ϵn. Then there is $Z \subseteq V(G)$ such that*

- *$\kappa(Z) \leq 2$, and*
- *there exist subsets X_1, X_2, X_3 of $V(G) \setminus Z$, pairwise disjoint and anticomplete to each other, and such that $|X_i| = \left\lceil \frac{1}{17}n \right\rceil$ for every $i \in \{1, 2, 3\}$.*

Proof. Since G is chordal, there is a tree decomposition (T, χ) of G such that $\chi(t)$ is a clique for every $t \in T$ [33]. By Theorem 10.3, there exists a vertex $t_0 \in T$ such that $|D| \leq \frac{n}{2}$ for every component D of $G \setminus \chi(t_0)$. Let X be a minimal set of components of $G \setminus \chi(t_0)$ such that $|\bigcup_{D \in X} D| \geq (\frac{1}{4} - 2\epsilon)n$.

(44) $|G \setminus (\bigcup_{D \in X} D \cup \chi(t_0))| \geq \frac{1}{4}n$.

Suppose not. Then $|\bigcup_{D \in X} D \cup \chi(t_0)| > (\frac{3}{4} - \epsilon)n$. Since $\chi(t_0)$ is a clique, it follows that $|\chi(t_0)| < \epsilon n$, and so $|\bigcup_{D \in X} D| \geq (\frac{3}{4} - 2\epsilon)n$. Let $D_0 \in X$. Then $|D_0| \leq \frac{1}{2}n$, and so $|\bigcup_{D \in X \setminus \{D_0\}} D| \geq (\frac{1}{4} - 2\epsilon)n$, contrary to the minimality of X. This proves (44).

Let $X_1 = \bigcup_{D \in X} D$ and let $Z_1 = \chi(t_0)$. Let Y be a subset of $G \setminus (X_1 \cup Z_1)$ with $|Y| = \frac{1}{4}n$, and let $G' = G[Y]$. By Theorem 10.3, there exists a vertex $t_0' \in T$ such that $|D| \leq \frac{n}{8}$ for every component D of $G' \setminus \chi(t_0')$. Let X' be a minimal set of components of $G' \setminus \chi(t_0')$ such that $|\bigcup_{D \in X'} D| \geq (\frac{1}{16} - 2\epsilon)n$. Write $X_2 = \bigcup_{D \in X'} D$ and $Z_2 = \chi(t_0')$. Let $X_3 = G' \setminus (X_2 \cup Z_2)$. By (44), $|X_3| \geq \frac{1}{16}n$. Let $Z = Z_1 \cup Z_2$. Then $\kappa(Z) = 2$, the sets X_1, X_2, X_3 are pairwise disjoint and anticomplete to each other, and $|X_i| \geq (\frac{1}{16} - 2\epsilon)n = \frac{1}{17}n$. Now the conclusion of the lemma follows. ■

Next we prove the following, which immediately implies Theorem 10.1.

Theorem 10.5. *Let $G \in \mathcal{C}$ with $|V(G)| = n$, and let $a, b \in V(G)$ be non-adjacent. Assume that $N(a) \cap \text{Hub}(G) = \emptyset$ and $N(a) \neq \emptyset$. Then there is a set $Z \subseteq V(G) \setminus \{a, b\}$ with $\kappa(Z) \leq -C\frac{1}{\log(1-\gamma)}(\max(1, \log|N(a)|))$ and such that every component of $G \setminus Z$ contains at most one of a, b.*

Proof. We may assume that $|N(a)| > -C\frac{1}{\log(1-\gamma)}$. Let D be the component of $G \setminus N[a]$ such that $b \in D$. We may assume that $G = D \cup N[a]$. Write $X = N(a)$. Our first goal is to show the following:

(45) *Assume that either X is pure, or there does not exist $X' \subseteq X$ with $|X'| \geq \frac{1-4\epsilon}{2}|X|$ such that X' is pure. Then there exists $S \subseteq X \cup (D \setminus b)$ with $\kappa(S) \leq C - 2$ such the component $D(b)$ of $D \setminus S$ with $b \in D(b)$ meets $N(x)$ for at most $(1-\epsilon)|X|$ vertices $x \in X \setminus S$.*

The proof proceeds in several steps.

(46) *X is chordal.*

Suppose that there is a hole $H \subseteq X$. Then (H, a) is a wheel. But D is connected and $H \subseteq N(D)$, contrary to Theorem 6.5. This proves (46).

(47) *If there is a clique K of size $\epsilon|X|$ in X, then (45) holds.* .

Suppose such a clique K exists. Now setting $S = K$, it follows that (45) holds. This proves (47).

Let $Z' = N(b) \cap X'$. Since G is C_4-free, it follows that Z' is a clique. By (47), we can apply Lemma 10.4 to $X \setminus Z'$; let $Z, X_1, X_2, X_3 \subseteq X \setminus Z'$ as in the conclusion of the lemma.

Let $\delta = \frac{1}{6 \times 17^2}$. A triple $x_1x_2x_3$ is *partitioned* if $x_i \in X_i$. We remind the reader that dangerous triples were defined in Section 9.

(48) *Under the assumptions in* (45)*, if for some* $i \in \{1,2,3\}$*, some* $x_i \in X_i$ *is a center of* $\delta|X|^2$ *dangerous triples, then* (45) *holds.*

Since $\epsilon < \frac{1}{8}\delta$, it follows from (47) and Theorem 9.1 that there exists $S' \subseteq D \backslash b$ with $\kappa(X) \leq 4$ such that the component $D(b)$ of $D \setminus S'$ with $b \in D(b)$ is disjoint from $N(x)$ for at least $\epsilon|X|$ vertices $x \in X$. Setting $S = S' \cup Z'$, (45) holds. This proves (48).

In view of (48), in order to prove (45) we may assume that for every $i \in \{1,2,3\}$, every $x_i \in X_i$ is a center fewer that $\delta|X|^2$ dangerous triples.

(49) *At least* $\frac{1}{2}\left\lceil \frac{1}{17}|X| \right\rceil^3$ *of the partitioned triples are not dangerous.*

Since there are $\left\lceil \frac{1}{17}|X| \right\rceil^3$ partitioned triples, it is enough to prove that at most $\left\lceil \frac{1}{17}|X| \right\rceil^3/2$ of the partitioned triples are dangerous. Let $i \in \{1,2,3\}$. By (48), the number of dangerous triples with center in X_i is at most

$$|X_i| \times \delta|X|^2 \leq= \left\lceil \frac{1}{17}|X| \right\rceil \cdot \frac{1}{6 \times 17^2}|X| \leq \frac{1}{6}\left\lceil \frac{1}{17}|X| \right\rceil^3.$$

Since every dangerous triple has a center in one of the sets X_1, X_2, X_3, it follows that the total number of dangerous triples is at most

$$3 \times \frac{1}{6}\left\lceil \frac{1}{17}|X| \right\rceil^3 \leq \frac{1}{2}\left\lceil \frac{1}{17}|X| \right\rceil^3,$$

as required. This proves (49).

(50) *Every partitioned triple that is not dangerous is* b*-separated.*

Let $x_1x_2x_3$ be a partitioned triple that is not dangerous. Let $H = D \cup \{x_1, x_2, x_3, a\}$. Let F be a minimal connected subgraph of D such that each of x_1, x_2, x_3 has a neighbor in F. Since $X \cap \text{Hub}(G) = \emptyset$, it follows from Lemma 10.2 that $\Sigma = F \cup \{x_1, x_2, x_3, a\}$ is a pyramid with apex a. For $i \in \{1,2,3\}$, let $Q_i = a\text{-}x_i$. Since $x_1x_2x_3$ is not a dangerous triple, we deduce that a is not contained in a cross-edge of an extended near-prism in H. Now by Theorem 4.3 applied to Σ, Q_1, Q_2, Q_3, and H, it follows that the triple $x_1x_2x_3$ is b-separated, and (50) follows.

(51) X *is a* $(\frac{1}{2 \times 17^3}, b)$*-breaker in* G*.*

By (49), at least $\frac{1}{2}\left\lceil \frac{1}{17}|X| \right\rceil^3$ of the partitioned triples are not dangerous. Now by (50) at least $\frac{1}{2}\left\lceil \frac{1}{17}|X| \right\rceil^3 \geq \frac{1}{2 \times 17^3}|X|^3$ of the partitioned triples $x_1x_2x_3$ are b-separated, and (51) follows.

By Theorem 8.3, there exist $S' \subseteq D \setminus b$ with $\kappa(S) \leq C - 4$ such that the component $D(b)$ of $D \setminus S$ with $b \in D(b)$ is disjoint from $N(x)$ for at least $\epsilon|X|$ vertices $x \in X$. Setting $S = Z' \cup S$, (45) follows.

We complete the proof of Theorem 10.5 by induction on $|N_G(a)|$. If X is pure, let $X_0 = X$. If X is not pure and there does not exist $X' \subseteq X$ with $|X'| \geq \frac{1-4\epsilon}{2}|X|$ such that X' is pure, let $X_0 = X$. If X is not pure and there exists $X' \subseteq X$ with $|X'| \geq \frac{1-4\epsilon}{2}|X|$ such that X' is pure, let X_0 be a pure subset of X with $|X_0| \geq \frac{1-4\epsilon}{2}|X|$. Let $G_0 = G \setminus (X \setminus X_0)$. Note that

$X_0 = N_{G_0}(a) = N_{G_0}(D)$. Apply (45) in G_0 (and with $X = X_0$). Let S be as in (45). Let D_1 be the component of $D \setminus S$ with $b \in D_1$, let $X_1 = N_{G_0}(D_1)$ and let $G_1 = D_1 \cup X_1 \cup \{a\}$. By (45) $|N_{G_1}(a)| \leq (1-\epsilon)|N_{G_0}(a)| \leq (1-\epsilon)|X_0|$. Let $G_2 = G[V(G_1) \cup (X \setminus X_0)]$. Since $N_{G_2}(a) = (X \setminus X_0) \cup N_{G_1}(a)$, it follows that

$$|N_{G_2}(a)| \leq (1-\epsilon)|X_0| + |X| - |X_0| \leq |X| - \epsilon|X_0| \leq (1-\gamma)|X|.$$

If $|N_{G_2}(a)| \leq 1$, let $Z_1 = N_{G_2}(a)$. Otherwise, $\log |N_{G_2}(a)| \geq 1$ and inductively, there is a set $Z_1 \subseteq V(G_2) \setminus \{a, b\}$ with $\kappa(Z) \leq -C\frac{1}{\log(1-\gamma)}(\log |N_{G_2}(a)|)$ and such that every component of $G_2 \setminus Z_1$ contains at most one of a, b. In the latter case, $\kappa(Z_1) \leq -C\frac{1}{\log(1-\gamma)}\log(|N_G(a)|) - C$. Since $\kappa(S) \leq C$, the set $Z_1 \cup S$ satisfies the conclusion of the theorem. In the former case, $|Z_1 \cup S| \leq C$ and $Z_1 \cup S$ satisfies this conclusion of the theorem. ■

11 Stable sets of safe hubs

As we discussed in Section 3, in the course of the proof of Theorem 13.1, we will repeatedly decompose the graph by star cutsets arising from a stable set of appropriately chosen hubs (using Theorem 6.5). In this section we prepare the tools for handling one such step: one stable set of safe hubs.

Let d be an integer. In this section we again set $\epsilon = \frac{1}{4 \times 17^6 \times 48 \times 192}$, $\gamma = \frac{\epsilon(1-4\epsilon)}{2}$ and $C = 96^2 \times 4 \times 17^6 + 4$. Let $a, b \in V(G)$ be non-adjacent and such that no subset Z of G with $\kappa(Z) \leq -C\frac{1}{\log(1-\gamma)}\log n + d$ separates a from b. Following [13], we say that a vertex v is *d-safe* if $|N(v) \cap \text{Hub}(G)| \leq d$. As in [13], the goal of the next lemma is to classify d-safe vertices into "good ones" and "bad ones", and show that the bad ones are rare. A vertex $v \in G$ is *ab-cooperative* if there exists a component D of $G \setminus N[v]$ such that $a, b \in N[D]$.

Lemma 11.1. *If $v \in G$ is d-safe and not ab-cooperative, then v is adjacent to both a and b. In particular, the set of vertices that are not ab-cooperative is a clique.*

Proof. Suppose v is non-adjacent to b and v is not ab-cooperative. Let D be the component of $G \setminus N[v]$ such that $b \in D$. Let $X = N(D) \setminus \text{Hub}(G)$. Let $G' = D \cup X \cup \{v\}$. Then $a \notin G'$. We apply Theorem 10.1 to the vertices v, b in G' to obtain a subset $Z \subseteq G' \setminus \{v, b\}$ with $\kappa(Z) \leq -C\frac{1}{\log(1-\gamma)}\log n$ and such that every component of $G' \setminus Z$ contains at most one of v, b. We claim that $Z' = Z \cup (N(v) \cap \text{Hub}(G))$ separates a from b in G. Suppose that P is a path from a to b with $P^* \cap Z' = \emptyset$. Since $b \in D$ and $a \notin D$, there is a vertex $x \in P$ such that b-P-$x \subseteq D$ and $x \notin D$. Then $x \in X$. But now b-P-x-v is a path from b to v in $G' \setminus Z$, a contradiction. This proves the claim that Z' separates a from b in G. But $\kappa(Z') \leq -C\frac{1}{\log(1-\gamma)}\log n + d$, a contradiction. This proves Lemma 11.1. ■

Let S' be a stable set of hubs of G with $S' \cap \{a, b\} = \emptyset$, and assume that every $s \in S'$ is d-safe. Let $S'_{bad} = S' \cap N(a) \cap N(b)$. Since G is C_4-free, it follows that $|S'_{bad}| \leq 1$. Let $S = S' \setminus S'_{bad}$. By Lemma 11.1, every vertex in S is ab-cooperative.

A *separation* of G is a triple (X, Y, Z) of pairwise disjoint subsets of G with $X \cup Y \cup Z = G$ such that X is anticomplete to Z. We are now ready to move on to star cutsets. As in other papers on the subject, we associate a certain unique star separation to every vertex of S. The choice of the separation is the same as in [13].

Let $v \in S$. Since v is ab-cooperative, there is a component D of $G \setminus N[v]$ with $a, b \in N[D]$. Since $v \notin S'_{bad}$, it follows that v is not complete to $\{a, b\}$; consequently $D \cap \{a, b\} \neq \emptyset$, and so the component D is unique. Let $B(v) = D$, let $C(v) = N(B(v)) \cup \{v\}$, and let $A(v) = G \setminus (B(v) \cup C(v))$. Then $(A(v), C(v), B(v))$ is the *canonical star separation of G corresponding to v*.

As in [13], we observe:

Lemma 11.2. *The vertex v is not a hub of $G \setminus A(v)$.*

Proof. Suppose that (H, v) is a proper wheel or a loaded pyramid in $G \setminus A(v)$. Then $H \subseteq N[B(v)]$, contrary to Theorem 6.5. ■

Let $\mathcal{O}$ be a linear order on $S \cap \text{Hub}(G)$. Following [4], we say that two vertices of $S \cap \text{Hub}(G)$ are *star twins* if $B(u) = B(v)$, $C(u) \setminus \{u\} = C(v) \setminus \{v\}$, and $A(u) \cup \{u\} = A(v) \cup \{v\}$.

Let $\leq_A$ be a relation on $S \cap \text{Hub}(G)$ defined as follows:

$$x \leq_A y \quad \text{if} \quad \begin{cases} x = y, \text{ or} \\ x \text{ and } y \text{ are star twins and } \mathcal{O}(x) < \mathcal{O}(y), \text{ or} \\ x \text{ and } y \text{ are not star twins and } y \in A(x). \end{cases}$$

Note that if $x \leq_A y$, then either $x = y$, or $y \in A(x)$.

The following two results were proved in [13] with a slightly different setup: The set of hubs is defined differently. However, the proofs do not use the definition of the set of hubs.

Lemma 11.3 (Chudnovsky, Gartland, Hajebi, Lokshtanov, Spirkl [13], Lemma 4.8)**.** *$\leq_A$ is a partial order on $S \cap \text{Hub}(G)$.*

Let $\text{Core}(S')$ be a the set of all $\leq_A$-minimal elements of $S \cap \text{Hub}(G)$.

Lemma 11.4 (Chudnovsky, Gartland, Hajebi, Lokshtanov, Spirkl [13], Lemma 4.9)**.** *Let $u, v \in \text{Core}(S')$. Then $A(u) \cap C(v) = C(u) \cap A(v) = \emptyset$.*

As in [13], we define the *central bag*

$$\beta^A(S') = \left(\bigcap_{v \in \text{Core}(S')} (B(v) \cup C(v)) \right) \setminus S'_{bad}.$$

The next result describes important properties of $\beta^A(S')$.

Theorem 11.5. *The following hold:*

1. *For every $v \in \text{Core}(S')$, we have $C(v) \subseteq \beta^A(S')$.*

2. *For every component D of $G \setminus (\beta^A(S') \cup S'_{bad})$, there exists $v \in \text{Core}(S')$ such that $D \subseteq A(v)$. Further, if D is a component of $G \setminus (\beta^A(S') \cup S'_{bad})$ and $v \in \text{Core}(S')$ such that $D \subseteq A(v)$, then $N(D) \subseteq C(v) \cup S'_{bad}$.*

3. *$S' \cap \text{Hub}(\beta^A(S')) = \emptyset$.*

Proof. (1) is immediate from Lemma 11.4.

Next we prove (2). Let D be a component of $G \setminus (\beta^A(S') \cup S'_{bad})$. Since $G \setminus (\beta^A(S') \cup S'_{bad}) = \bigcup_{v \in \text{Core}(S')} A(v)$, there exists $v \in \text{Core}(S')$ such that $D \cap A(v) \neq \emptyset$. If $D \setminus A(v) \neq \emptyset$, then, since D is connected, it follows that $D \cap N(A(v)) \neq \emptyset$; but then $D \cap C(v) \neq \emptyset$, contrary to (1). Since $N(D) \subseteq \beta^A(S') \cup S'_{bad}$ and $N(D) \subseteq A(v) \cup C(v) \cup S'_{bad}$, it follows that $N(D) \subseteq C(v) \cup S'_{bad}$. This proves (2).

To prove (3), let $u \in S' \cap \text{Hub}(\beta^A(S'))$. Since $\beta^A(S') \cap S'_{bad} = \emptyset$, we deduce that $u \notin S'_{bad}$, and so $u \in S \cap \text{Hub}(G)$. By Lemma 11.2, it follows that $\beta^A(S') \not\subseteq B(u) \cup C(u)$, and therefore $u \notin \text{Core}(S')$. But then $u \in A(v)$ for some $v \in \text{Core}(S')$, and so $u \notin \beta^A(S')$, a contradiction. This proves (3) and completes the proof of Theorem 11.5. ■

In the course of the proof of Theorem 13.1, we will inductively obtain a small cutset separating a from b in $\beta^A(S')$, using that the vertices in S' are not hubs in $\beta^A(S')$. The next theorem lets us lift this cutset into a cutest that separates a from b in G.

Theorem 11.6. *Let (X, Y, Z) be a separation of $\beta^A(S')$ such that $a \in X$ and $b \in Z$. Then there exists a set $Y' \subseteq V(G)$ such that*

1. *Y' separates a from b in G, and*
2. *$\kappa(Y') \leq \kappa(Y) + |Y \cap \text{Core}(S')|(-C\frac{1}{\log(1-\gamma)} \log n + d) + 1$.*

Proof. Let $s \in Y \cap \text{Core}(S')$. Let $X = C(s) \setminus \text{Hub}(G)$. Let $G' = B(s) \cup X \cup \{s\}$. Suppose first that s is non-adjacent to b. We apply Theorem 10.1 to the vertices s, b in G' to obtain a subset $Z(s) \subseteq G' \setminus \{s, b\}$ with $\kappa(Z) \leq -C\frac{1}{\log(1-\gamma)} \log n$ and such that every component of $G' \setminus Z(s)$ contains at most one of s, b. Now suppose that s is adjacent to b. Since s is cooperative, it follows that s is non-adjacent to a. Now we apply Theorem 10.1 to the vertices s, a in G' to obtain a subset $Z(s) \subseteq G' \setminus \{s, a\}$ with $\kappa(Z(s)) \leq -C\frac{1}{\log(1-\gamma)} \log n$ and such that every component of $G' \setminus Z(s)$ contains at most one of s, a. We deduce:

(52) *For every $s \in \text{Core}(S')$, $Z(s)$ separates s from at least one of a, b.*

Now let
$$Y' = Y \cup \bigcup_{s \in Y \cap \text{Core}(S')} Z(s) \cup \bigcup_{s \in Y \cap \text{Core}(S')} (N(s) \cap \text{Hub}(G)) \cup S'_{bad}.$$
Since every vertex of S' is safe, we have that $\kappa(Y') \leq \kappa(Y) + |Y \cap \text{Core}(S')|(-C\frac{1}{\log(1-\gamma)} \log n + d)$.

We show that Y' separates a from b in G. Suppose not. Let D_b be the component of $\beta^A(S') \setminus Y$ such that $b \in D_b$, an let D_a be the component of $\beta^A(S') \setminus Y$ such that $a \in D_a$. Let $D'_b = D_b \cup \bigcup_{s \in D_b \cap \text{Core}(S')} A(s)$, and let $D'_a = D_a \cup \bigcup_{s \in D_a \cap \text{Core}(S')} A(s)$. Let P be a path from b to a in $G \setminus Y'$. Since $b \in D'_b$ and $a \notin D'_b$, there is $x \in P$ such that b-P-$x \subseteq D'_b$ and the neighbor y of x in the path x-P-a does not belong to D'_b. Since for every $s \in D_b \cap \text{Core}(S')$, $N_G(A(s)) \cap \beta^A(S') \subseteq C(s) \subseteq D_b \cup Y$, and since $Y \subseteq Y'$, it follows that $y \notin \beta^A(S')$. Let D' be the component of $G \setminus (\beta^A(S') \cup S'_{bad})$ such that $y \in D'$. By Theorem 11.5(2) there is an $s \in \text{Core}(S')$ such that $D' \subseteq A(s)$ and $N(D') \subseteq C \cup S'_{bad}$; consequently by Theorem 11.5(1), $N(D') \subseteq N_{\beta^A(S')}(s) \cup S'_{bad}$. In particular, $x \in N_{\beta^A(S')}(s)$. Since $y \notin D'_b$, it follows that $s \notin D_b$. Since $N_{\beta^A(S')}(s) \cap D_b \neq \emptyset$ and $s \notin D_b$, it follows that $s \in Y$. Consequently, $P \cap Z(s) = \emptyset$. Since $y \in A(s)$ and $a \in \beta^A(S') \subseteq B(s) \cup C(s)$, there exists $x' \in y$-P-a such that $x' \in C(s)$ and s has no other neighbors in the path x'-P-a. Now b-P-x-s and a-P-x'-s are paths from b to s and from a to s, respectively, and both are disjoint from $Z(s)$, contrary to (52). ■

12 Bounding the number of non-hubs

For $X \subseteq V(G)$, a component D of $G \setminus X$ is *full for* X if $N(D) = X$. $X \subseteq V(G)$ is a *minimal separator* in G if there exist two distinct full components for X. In this section we again set $\epsilon = \frac{1}{4 \times 17^6 \times 48 \times 192}$, $\gamma = \frac{\epsilon(1-4\epsilon)}{2}$ and $C = 96^2 \times 4 \times 17^6 + 4$, and let $D = -C\frac{1}{\log(1-\gamma)}$. Let $a, b \in V(G)$ be non-adjacent. The goal of this section is to start with a minimal separator in G separating a from b, and turn it into a separator that interfaces well with Theorem 11.6. Let d be an integer and let $S_1 \subseteq V(G) \setminus \text{Hub}(G)$ be a stable set of d-safe vertices. For a set $U \subseteq V(G)$ we denote by $\mu_d(U)$ the set $U \cap S_1$. We will prove the following:

Theorem 12.1. *Let d be an integer. Let G be an even-hole-free graph and let $a, b \in V(G)$ be non-adjacent. Let $|V(G)| = n$. Let Y be a minimal separator in G such that a and b belong to different components of $G \setminus Y$. Then there exists a set $Y' \subseteq V(G) \setminus \{a, b\}$ such that*

- *Y' separates a from b;*

- $|\kappa(Y' \setminus Y)| \leq D(D\log n + d)\log n$; *and*
- $|\mu_d(Y')| \leq D(D\log n + d)\log n$.

The main ingredient of the proof is the following:

Lemma 12.2. *Let d be an integer and let G be an even-hole-free graph where $G = D_1 \cup D_2 \cup X$, X is a minimal separator in G, and D_1, D_2 are full components for X. Let $a \in D_1$ and $b \in D_2$. Assume that X is a stable set, $X \cap \mathrm{Hub}(G) = \emptyset$, and that every vertex of X is d-safe. Then there exists $Z \subseteq V(G) \setminus \{a, b\}$ with $\kappa(Z) \leq D\log n + d$ such that either*

- *the component $D(b)$ of $D_2 \setminus Z$ with $b \in D(b)$ contains a neighbor of x for at most $(1-\epsilon)|X|$ vertices $x \in X \setminus Z$, or*
- *the component $D(a)$ of $D_1 \setminus Z$ with $a \in D(a)$ contains a neighbor of x for at most $(1-\epsilon)|X|$ vertices $x \in X \setminus Z$.*

Proof. We may assume that $|X| > D\log n + d$.

(53) *If some $v \in D_1$ has $\epsilon|X|$ neighbors in X, then the theorem holds.*

Suppose such v exists. Let $G' = D_2 \cup N_G[v]$. Apply Theorem 10.1 to v, b in G' (where v plays the role of a) to obtain a set Z as in the conclusion of the theorem. Let $D(b)$ be the component of $D_2 \setminus Z$ with $b \in D(b)$. Then $N_{G'}[v] \setminus Z$ is anticomplete to $D(b)$ in G' and therefore in G. Since $|N_{G'}(v)| \geq \epsilon|X|$, (53) follows.

In view of (53) (using the symmetry between a and b) from now on we assume that no $v \in D_1 \cup D_2$ has $\epsilon|X|$ neighbors in X.

(54) *If for some $x \in X \setminus N(a)$ at least $\epsilon|X|$ vertices of X are anticomplete to the component $D(a)$ of $D_1 \setminus N(x)$ with $a \in D(a)$, then the theorem holds.*

Suppose that such a vertex x exists. Let $D' = D(a)$, let $X' = (N(x) \cap N(D(a)) \setminus \mathrm{Hub}(G)$ and let $G' = D' \cup X' \cup \{x\}$. Apply Theorem 10.1 to x, a in G' (where x plays the role of a, and a plays the role of b) to obtain a set Z as in the conclusion of the theorem. Let $Z' = Z \cup (N(x) \cap \mathrm{Hub}(G))$. Since x is d-safe, it follows that $\kappa(Z') \leq D\log n + d$. Let $D'(a)$ be the component of $D_1 \setminus Z'$ with $a \in D'(a)$. Since Z separates x from a in G', it follows that $N[x] \setminus Z'$ is anticomplete to $D(a) \setminus Z'$, and therefore $D'(a) \subseteq D(a)$.

Consequently, if $x' \in X$ is anticomplete to $D(a)$, then x' is anticomplete to $D'(a)$. Since least $\epsilon|X|$ vertices of X are anticomplete to $D(a)$, (54) follows.

In view of (54), from now on we assume that for every $x \in X \setminus N(a)$ fewer than $\epsilon|X|$ vertices of X are anticomplete to the component $D(a)$ of $D_1 \setminus N(x)$ with $a \in D(a)$, and for every $x \in X \setminus N(b)$ fewer than $\epsilon|X|$ vertices of X are anticomplete to the component $D(b)$ of $D_2 \setminus N(x)$ with $b \in D(b)$. Let $X' = X \setminus (N(a) \cup N(b))$. Then $|X'| \geq (1 - 2\epsilon)|X|$. Since X is a stable set, we can choose disjoint and anticomplete subsets X_1, X_2, X_3 of X' such that $|X_1| = |X_2| = |X_3| = \left\lceil \frac{1}{4}|X| \right\rceil$.

Let us say that a partitioned triple $x_1x_2x_3$ is *b-triangular* if there is a minimal connected subgraph H of D_2 containing neighbors of x_1, x_2, x_3 such that either

- H satisfies the third outcome to Lemma 10.2, or
- (possibly with the roles of x_1, x_2, x_3 exchanged) H is the interior of a path from x_1 to x_3, and x_2 has exactly two neighbors in H and they are adjacent.

We define *a-triangular* triples similarly.

(55) *Every partitioned triple is either a-triangular or b-triangular.*

Let $x_1x_2x_3$ be a partitioned triple, and suppose that it is neither b-triangular nor a-triangular. For $i \in \{1,2\}$, let H_i be a minimal connected induced subgraph of D_i such that each of x_1, x_2, x_3 has a neighbor in H_i. We apply Lemma 10.2 to H_1 and H_2. If the second outcome of the lemma holds for both H_1 and H_2, then $H_1 \cup H_2 \cup \{x_1, x_2, x_3\}$ is a theta, a contradiction. Thus we may assume that H_1 is the interior of a path from x_1 to x_3 and x_2 has at least two non-adjacent neighbors in H_1. If H_2 is the interior of a path from x_1 to x_3 and x_2 has a neighbor in H_2, then $(H_1 \cup H_2, x_2)$ is a wheel in G, contrary to the fact that $x_2 \notin \mathrm{Hub}(G)$. Thus H_2 is not the interior of a path from x_1 to x_3 such that x_2 has a neighbor in H_2. Next suppose that the second outcome of Lemma 10.2 holds for H_2. Let P_1, P_2, P_3 be as in the the second outcome of Lemma 10.2, and let $a_2 = P_1 \cap P_2 \cap P_3$. Then x_2 is non-adjacent to a_2, and so $H_1 \cup H_2 \cup \{x_1, x_2, x_3\}$ contains a theta with ends a_2, x_2, a contradiction. It follows that the first outcome of Lemma 10.2 holds for H_2, and, by symmetry between x_1 and x_3, we may assume that H_2 is the interior of a path from x_1 to x_2, and x_3 has two non-adjacent neighbors in H_2. Now we get a theta with ends x_2, x_3 and paths x_2-H_1-x_3, x_2-H_2-x_3, and x_2-H_1-x_1-H_2-x_3, again a contradiction. This proves (55).

In view of (55), by switching the roles of D_1 and D_2 if necessary, we may assume that at least $\frac{1}{2}\left\lceil \frac{1}{4}|X| \right\rceil^3$ of the partitioned triples are b-triangular.

Let us say that the triple $x_1x_2x_3$ is *acceptable* if it is partitioned and for every $\{i,j,k\} = \{1,2,3\}$ there is path P_{ij} from x_i to x_j with interior in $D_1 \setminus N(x_k)$.

(56) *At most* $6\epsilon \left\lceil \frac{1}{4}|X| \right\rceil^2 |X|$ *partitioned triples are not acceptable.*

Let $x_1 \in X_1$ and let $D(a)$ be the component of $D_1 \setminus N(x_1)$ such that $a \in D(a)$. If $x_1x_2x_3$ is a partitioned triple such that there is no path from x_2 to x_3 in $D_1 \setminus N(x_1)$, then at least one of x_2, x_3 is anticomplete to $D(a)$. By the assumption following (54), there are fewer than $2\epsilon|X| \times \left\lceil \frac{1}{4}|X| \right\rceil$ such pairs x_2x_3, and therefore at most

$$\left(2\epsilon|X| \times \left\lceil \frac{1}{4}|X| \right\rceil\right) \times \left\lceil \frac{1}{4}|X| \right\rceil \leq 2\epsilon \left\lceil \frac{1}{4}|X| \right\rceil^2 |X|$$

such triples. Repeating this argument with x_2 and x_3 playing the role of x_1, we get that there are at most $6\epsilon \left\lceil \frac{1}{4}|X| \right\rceil^2 |X|$ partitioned triples that are not acceptable, and (56) follows.

Let $G' = D_2 \cup X$. Our next goal is to show the following:

(57) *X is a $(\frac{1}{256}, b)$-breaker in G'.*

By (56), there are fewer than $6\epsilon \left\lceil \frac{1}{4}|X| \right\rceil^2 |X|$ partitioned triples that are not acceptable. Since the total number of b-triangular triples at least $\frac{1}{2}\left\lceil \frac{1}{4}|X| \right\rceil^3$, we deduce that there are at least $\left(\frac{1}{2} - 24\epsilon\right)\left\lceil \frac{1}{4}|X| \right\rceil^3$ triples that are both acceptable and b-triangular.

Let $x_1x_2x_3$ be a b-triangular acceptable triple. Let H be a minimal connected subgraph of D_2 as in the definition of b-triangular. Then there exists a triangle $h_1h_2h_3$ in $H \cup \{x_1, x_2, x_3\}$ and three paths P_1, P_2, P_3, where P_i is a path from h_i to x_i (possibly of length zero), such that:

- $V(H) = (V(P_1) \cup V(P_2) \cup V(P_3)) \setminus \{x_1, x_2, x_3\}$;
- the sets $V(P_1)$, $V(P_2)$ and $V(P_3)$ are pairwise disjoint; and
- for distinct $i, j \in \{1, 2, 3\}$, there are no edges between $V(P_i)$ and $V(P_j)$, except $h_i h_j$.

If the path P_i has even length, let $q_i = 2$, and if the path P_i has odd length, let $q_i = 3$. Let G'' be the graph obtained from $D_2 \cup \{x_1, x_2, x_3\}$ by adding a new vertex v and paths Q_i from v to x_i such that

- Q_i has length q_i.
- $(Q_1 \cup Q_2 \cup Q_3) \setminus \{x_1, x_2, x_3\}$ is anticomplete to D_2.
- $Q_1 \setminus v$, $Q_2 \setminus v$, $Q_3 \setminus v$ are pairwise disjoint and anticomplete to each other.

(58) *G'' is even-hole-free.*

Suppose not, and let H be an even hole in G''. Since H is not an even hole in G, and since all internal vertices of each Q_i have degree two, we may assume that $Q_1 \cup Q_2 \subseteq H$. Then $H \setminus (Q_1^* \cup Q_2^*)$ is a path from x_1 to x_2 with interior in $D_2 \cup \{x_3\}$ and whose length has the same parity as $q_1 + q_2$. Since the triple $x_1x_2x_3$ is acceptable, there is a path R from x_1 to x_2 with $R^* \subseteq D_1 \setminus N(x_3)$. Since $(H \setminus (Q_1 \cup Q_2)) \cup R$ is not an even hole in G, it follows that the length of R has the same parity as $q_1 + q_2 + 1$. But now x_1-P_1-h_1-h_2-P_2-x_2-R-x_1 is an even hole in G, a contradiction. This proves (58).

By (58) and since $X \cap \text{Hub}(G) = \emptyset$, the assumptions of Theorem 4.3 are satisfied. Now Theorem 4.3 applied in G'' implies that the triple $x_1x_2x_3$ is b-separated in G'.

Since there are at least $\left(\frac{1}{2} - 24\epsilon\right) \left\lceil \frac{1}{4}|X| \right\rceil^3$ acceptable b-triangular triples, and since $\epsilon \leq 1/96$, (57) follows.

By Theorem 8.3 applied in G', there exists $Z \subseteq D_2 \setminus b$ with $\kappa(Z) \leq 96^2 \cdot 256^2 \leq C - 4$ such the component $D(b)$ of $D \setminus S$ with $b \in D(b)$ is disjoint from $N(x)$ for at least $\epsilon|X|$ vertices $x \in X$. This completes the proof. ■

Next we prove the following, which immediately implies Theorem 12.1.

Theorem 12.3. *Let d be an integer. Let G be an even-hole-free graph and let $a, b \in V(G)$ be non-adjacent. Let $|V(G) = n$. Let Y be a minimal separator in G such that a and b belong to different full components D_1 and D_2 of $G \setminus Y$. Assume that $\mu_d(Y) \neq \emptyset$. Then there exists a set $Y' \subseteq V(G) \setminus \{a, b\}$ such that*

- *Y' separates a from b;*
- *$\kappa(Y' \setminus Y) \leq D(D \log n + d) \max(1, \log |\mu_d(Y)|)$; and*
- *$|\mu_d(Y')| \leq D(D \log n + d) \max(1, \log |\mu_d(Y)|)$.*

Proof. We may assume that $|\mu_d(Y)| \geq D(D \log n + d)$. By Lemma 12.2 applied to the graph $D_1 \cup D_2 \cup \mu_d(Y)$ with $X = \mu_d(Y)$, we may assume that there exists $Z \subseteq V(G) \setminus \{a, b\}$ with $\kappa(Z) \leq D \log n + d$ such that the component $D(b)$ of $D_2 \setminus Z$ with $b \in D(b)$ meets $N(x)$ for at most $(1 - \epsilon)|X|$ vertices $x \in X \setminus Z$. Let $G' = G \setminus Z$. Let $Y_1 = N_{G'}(D(b))$. Then $Y_1 \subseteq Y \setminus Z$, and $\mu_d(Y_1) \subseteq X \setminus Z$; consequently $|\mu_d(Y_1)| \leq (1 - \epsilon)\mu_d(Y)$. Let $D(a)$ be the component of $G' \setminus Y_1$ such that $a \in D(a)$. Let $Y_2 = N_{G'}(D(a))$. Then $Y_2 \subseteq Y_1$, and therefore $|\mu_d(Y_2)| \leq (1 - \epsilon)|\mu_d(Y)|$. Let $G'' = D(a) \cup D(b) \cup Y_2$. Then Y_2 is a minimal separator in G'' where $D(a)$ and $D(b)$ are full components for Y_2. If $|Y_2| \leq 1$, let $Y'' = Y_2$. Now assume that $|Y_2| > 1$. Inductively, there is a set $Y'' \subseteq V(G'') \setminus \{a, b\}$ such that

- Y'' separates a from b;
- $|\kappa(Y'' \setminus Y_2)| \leq D(D \log n + d) \log |\mu_d(Y_2)|$; and
- $|\mu_d(Y'')| \leq D(D \log n + d) \log |\mu_d(Y_2)|$

In both cases, let $Y' = Y'' \cup Z$. Then $Y' \setminus Y \subseteq (Y'' \setminus Y_2) \cup Z$ and $\mu_d(Y') \subseteq \mu_d(Y'') \cup Z$. Since $|\mu_d(Y_2)| \leq (1-\epsilon)|\mu_d(Y)|$ and $\kappa(Z) \leq D \log n + d$, it follows that $|Y'|$ satisfies the conclusion of the theorem. ■

13 Separating a pair of vertices: a bound using clique size

We can now prove our first main result. We follow the outline of the proof of Theorem 1.3 from [3], using bounds from earlier sections of the present paper. As in the earlier sections, let $\epsilon = \frac{1}{4 \times 17^6 \times 48 \times 192}$, $\gamma = \frac{\epsilon(1-4\epsilon)}{2}$ and $C = 96^2 \times 4 \times 17^6 + 4$, and let $D = -C\frac{1}{\log(1-\gamma)}$.

Theorem 13.1. *Let t be an integer. Let G be an even-hole-free graph with $|V(G)| = n$ and with no clique of size $t+1$, and let $a, b \in V(G)$ be non-adjacent. Then there is a set $Z \subseteq V(G) \setminus \{a, b\}$ with*

$$\kappa(Z) \leq D \log n + 2D(D \log n + 8t)^2 2t \log^2 n$$

and such that every component of $G \setminus Z$ contains at most one of a, b.

We will need the main result of [17].

Theorem 13.2 (Chudnovsky, Seymour [17])**.** *Every even-hole-free graph has a vertex v such that $\kappa(N(v)) \leq 2$.*

Following the proof of Theorem 7.1 of [6], using Theorem 13.2 we deduce:

Theorem 13.3. *Let $t \in \mathbb{N}$, and let G be an even-hole-free graph with no clique of size $t+1$ and with $|V(G)| = n$. There exist a partition $(S_1, \dots, S_k)$ of $V(G)$ with the following properties:*

1. *$k \leq 2t \log n$.*
2. *S_i is a stable set for every $i \in \{1, \dots, k\}$.*
3. *For every $i \in \{1, \dots, k\}$ and $v \in S_i$ we have $\deg_{G \setminus \bigcup_{j<i} S_j}(v) \leq 8t$.*

For the remainder of this section, let us fix $t \in \mathbb{N}$. Let G be an even-hole-free graph with no clique of size $t+1$, and let $a, b \in V(G)$. A *hub-partition with respect to ab* of G is a partition $S_1, \dots, S_k$ of $\text{Hub}(G) \setminus \{a, b\}$ as in Theorem 13.3; we call k the *order* of the partition. We call the *hub-dimension* of (G, ab) (denoting it by $\text{hdim}(G, ab)$) the smallest k such that G has a hub-partition of order k with respect to ab.

Since, in view of Theorem 13.3, we have $\text{hdim}(G, ab) \leq 2t \log n$ for every $a, b \in V(G)$, Theorem 13.1 follows immediately from the next result:

Theorem 13.4. *Let G be an even-hole-free graph with $|V(G)| = n$ and with no clique of size $t+1$, and let $a, b \in V(G)$ be non-adjacent. Then there is a set $Z \subseteq V(G) \setminus \{a, b\}$ with*

$$\kappa(Z) \leq D \log n + 2D(D \log n + 8t)^2 \,\text{hdim}(G, ab) \log n$$

and such that every component of $G \setminus Z$ contains at most one of a, b.

Proof. Let $a, b \in G$ be non-adjacent and suppose that no such set Z exists. We will get a contradiction by induction on $\text{hdim}(G, ab)$. Suppose that $\text{hdim}(G, ab) = 0$. Then $\text{Hub}(G) \subseteq \{a, b\}$ and by Theorem 10.1, there is a set $Z \subseteq V(G) \setminus \{a, b\}$ with $\kappa(Z) \leq D \log n$ and such that every component of $G \setminus Z$ contains at most one of a, b. Thus we may assume $\text{hdim}(G, ab) > 0$.

Let $S_1, \ldots, S_k$ be a hub-partition of G with respect to ab and with $k = \text{hdim}(G, ab)$. We now use notation and terminology from Section 11. Write $d = 8t$. It follows from the definition of S_1 that every vertex in S_1 is d-safe. Let $\beta^A(S_1)$ be as in Section 11; then $a, b \in \beta^A(S_1)$. By Theorem 11.5(3), we have that $S_1 \cap \text{Hub}(\beta^A(S_1)) = \emptyset$ and $S_2 \cap \text{Hub}(\beta^A(S_1)), \ldots, S_k \cap \text{Hub}(\beta^A(S_1))$ is a hub-partition of $\beta^A(S_1)$ with respect to ab. It follows that $\text{hdim}(\beta^A(S_1), ab) \leq k-1$. Inductively there exists a set $Y_1 \subseteq \beta^A(S_1) \setminus \{a, b\}$ with

$$\kappa(Y_1) \leq D \log n + 2D(D \log n + d)^2(k-1) \log n$$

and such that every component of $\beta^A(S_1) \setminus Y_1$ contains at most one of a, b. Let $D(b)$ be the component of $G \setminus Y_1$ such that $b \in D$, and let $D(a)$ be the component of $\beta^A(S_1) \setminus N(D(b))$ with $a \in D(a)$. Write $N_{\beta^A(S_1)}(D(a)) = Y_2$. Then $Y_2 \subseteq Y_1$ and Y_2 is a minimal separator in $\beta^A(S_1)$ where $D(a)$ and the component of $\beta^A(S_1) \setminus Y_2$ containing $D(b)$ are two distinct full components for Y_2. By Theorem 12.1 applied in $\beta^A(S_1)$ and using S_1 to define $\mu_d(Y_2)$, there exists a set $Y \subseteq \beta^A(S_1) \setminus \{a, b\}$ such that

- Y separates a from b in $\beta^A(S_1)$, and
- $|\kappa(Y \setminus Y_2)| \leq D(D \log n + d) \log n$, and
- $|\mu_d(Y)| \leq D(D \log n + d) \log n$.

It follows that

$$\begin{aligned}\kappa(Y) &\leq D \log n + 2D(D \log n + d)^2(k-1) \log n + D(D \log n + d) \log n \\ &\leq D(\log n)(1 + 2(D \log n + d)^2(k-1) + D \log n + d).\end{aligned}$$

Since $\text{Core}(S_1) \cap Y \subseteq \mu_d(Y)$, we deduce that $|\text{Core}(S_1) \cap Y| \leq D(D \log n + d) \log n$. Now applying Theorem 11.6 to Y we obtain a set Y' such that

- Y' separates a from b in G; and
- $\kappa(Y') \leq \kappa(Y) + |Y \cap \text{Core}(S')|(D \log n + d) + 1$.

Consequently,

$$\begin{aligned}\kappa(Y') &\leq \kappa(Y) + |Y \cap \text{Core}(S')|(D \log n + d) + 1 \\ &\leq D(\log n)(1 + 2(D \log n + d)^2(k-1) + D \log n + d) + |Y \cap \text{Core}(S')|(D \log n + d) + 1 \\ &\leq D(\log n)(1 + 2(D \log n + d)^2(k-1/2) + D \log n + d) + 1 \\ &\leq D(\log n)(1 + 2(D \log n + d)^2 k).\end{aligned}$$

as required. ■

14 The proof of Theorem 3.1

We can finally prove Theorem 3.1. We will need a theorem from [37].

Theorem 14.1 (Korchemna, Lokshtanov, Saurabh, Surianarayanan, Xue [37])**.** *Let G be a graph with $|V(G)| = n$, $A, B \subseteq V(G)$, $\mathcal{F}$ a family of cliques of G, and f an integer. Then one of the following holds:*

- *There exists $S \subseteq V(G)$ such that S separates A from B and $\kappa(S) \leq f \log^2 n$.*
- *There exist an integer $t \leq 2\log(|\mathcal{F}|)$ and $t \times f$ paths $P_1, \dots, P_{t\times f}$ from A to B such that for every $K \in \mathcal{F}$, $K \cap P_i \neq \emptyset$ for fewer than $4t$ values of i.*

We also need the following result of [9] and [30]:

Theorem 14.2 (Alekseev [9], Farber [30]). *An n-vertex C_4-free graphs has at most n^2 maximal cliques.*

We now prove Theorem 3.1.

Proof. Let D be as in Section 12 and let $c = 16 \times 256D^3$. Let $a, b \in G$ and let $G' = G \setminus \{a, b\}$. Let $A = N_G(a)$ and $B = N_G(b)$. Let $\mathcal{F}$ be the set of all maximal cliques in G'. By Theorem 14.2, we have $|\mathcal{F}| \leq n^2$. Let $f = c \log^6 n$. We apply Theorem 14.1 to $G', A, B, \mathcal{F}$ and f. We may assume that the statement of the first bullet does not hold, and so there exists an integer $t \leq 2\log(|\mathcal{F}|)$ and $t \times f$ paths $P_1, \dots, P_{t\times f}$ from A to B such that for every $K \in \mathcal{F}$, $K \cap P_i \neq \emptyset$ for at most $4t$ values of i. We may assume that the paths $P_1, \dots, P_{t\times f}$ are induced. Let $G'' = \bigcup_{i=1}^{t\times f} P_i \cup \{a, b\}$. Since every clique of $G'' \setminus \{a, b\}$ is contained in an element of $\mathcal{F}$, we deduce that G'' does not have a clique of size $8t + 1 \leq 1 + 16 \log n$. By Theorem 13.1 there exists a set $Z \subseteq V(G'') \setminus \{a, b\}$ with

$$\begin{aligned}\kappa(Z) &\leq D \log n + 2D(D \log n + 8 \times 16 \log n)^2 2 \times 16 \log n \log^2 n \\ &< 256D^3 \log^5 n.\end{aligned}$$

and such that every component of $G'' \setminus Z$ contains at most one of a, b. Since for every $K \in \mathcal{F}$, $K \cap P_i \neq \emptyset$ for at most $4t$ values of i, it follows that the number of values of i for which $Z \cap P_i \neq \emptyset$ is less than $4t\kappa(Z) \leq c \log^6 n$. But this contradicts the fact that Z separates a from b in G''. ■

15 From pairs of vertices to tree decompositions.

In this section we prove our main result, following the outline of the last few sections of [14]. We need a theorem from [13]:

Theorem 15.1 (Chudnovsky, Gartland, Hajebi, Lokshtanov, Spirkl [13]). *There is an integer d with the following property. Let $G \in \mathcal{C}$ and let w be a normal weight function on G. Then there exists $Y \subseteq V(G)$ such that*

- *$|Y| \leq d$, and*
- *$N[Y]$ is a w-balanced separator in G.*

We also need some terminology and two results from [14]. Let L, d, r be integers. We say that an n-vertex graph G is (L, d, r)-*breakable* if

1. for every two disjoint and anticomplete cliques H_1, H_2 of G with $|H_1| \leq r$ and $|H_2| \leq r$, there is a set $X \subseteq G \setminus (H_1 \cup H_2)$ with $\alpha(X) \leq L$ separating H_1 from H_2, and
2. for every normal weight function w on G and for every induced subgraph G' of G there exists a set $Y \subseteq V(G')$ with $|Y| \leq d$ such that for every component D of $G' \setminus N[Y]$, $w(D) \leq \frac{1}{2}$.

Theorem 15.2 (Chudnovsky, Hajebi, Lokshtanov, Spirkl [14]). *For every integer $d > 0$ there is an integer $C(d)$ with the following property. Let $L, n, r > 0$ be integers such that $r \leq d(2+\log n)$ and let G be an n-vertex (L, d, r)-breakable theta-free graph. Then there exists a w-balanced separator Y in G such that $\alpha(Y) \leq C(d)\lceil \frac{d(2+\log n)}{r} \rceil (2 + \log n)L$.*

Lemma 15.3 (Chudnovsky, Hajebi, Lokshtanov, Spirkl [14]). *Let G be a graph, let $c \in [\frac{1}{2}, 1)$, and let d be a positive integer. If for every normal weight function w on G, there is a (c, w)-balanced separator X_w with $\alpha(X_w) \leq d$, then the tree independence number of G is at most $\frac{3-c}{1-c}d$.*

We now prove:

Theorem 15.4. *There exists an integer M with the following property. Let G be an even-hole-free graph with n vertices and let w be a normal function on G. Then there exists a w-balanced separator Y in G such that $\alpha(Y) \leq M \log^{10} n$.*

Proof. Let d be as in Theorem 15.1 and let c be as in Theorem 3.1. By Theorem 15.1 and Theorem 3.1, it follows that G is $(c \log^8 n, d, 1)$-breakable. Now the result follows from Theorem 15.2. ■

Theorem 1.1 now follows immediately from Theorem 15.4 and Lemma 15.3.

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