

# Towards Erdős-Hajnal for graphs with no 5-hole

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### Abstract

The Erdős-Hajnal conjecture says that for every graph  $H$  there exists  $c > 0$  such that

$$\max(\alpha(G), \omega(G)) \geq n^c$$

for every  $H$ -free graph  $G$  with  $n$  vertices, and this is still open when  $H = C_5$ . Until now the best bound known on  $\max(\alpha(G), \omega(G))$  for  $C_5$ -free graphs was the general bound of Erdős and Hajnal, that for all  $H$ ,

$$\max(\alpha(G), \omega(G)) \geq 2^{\Omega(\sqrt{\log n})}$$

if  $G$  is  $H$ -free. We improve this when  $H = C_5$  to

$$\max(\alpha(G), \omega(G)) \geq 2^{\Omega(\sqrt{\log n \log \log n})}.$$

# 1 Introduction

All graphs in this paper are finite and have no loops or parallel edges, and the cardinalities of the largest stable sets and cliques in a graph  $G$  are denoted by  $\alpha(G), \omega(G)$  respectively. If  $G, H$  are graphs, we say that  $G$  *contains*  $H$  if some induced subgraph of  $G$  is isomorphic to  $H$ , and  $G$  is  *$H$ -free* otherwise.

The Erdős-Hajnal conjecture [5, 6] asserts:

**1.1 Conjecture:** *For every graph  $H$ , there exists  $\epsilon > 0$  such that every  $H$ -free graph  $G$  satisfies*

$$\max(\alpha(G), \omega(G)) \geq |V(G)|^\epsilon.$$

This is true for all  $H$  with at most four vertices, but is open when  $H = C_5$  ( $C_5$  denotes the cycle of length five). The problem for  $C_5$  has attracted a good deal of unsuccessful attention, for several reasons; not only is  $C_5$  arguably the smallest open case of 1.1, but also it has a good amount of symmetry, and more importantly, by excluding  $C_5$  we exclude its complement as well. (Excluding both a graph and its complement is an approach that has been quite fruitful lately, for instance [1, 2].) So we are happy to report some progress at last.

The best general bound for the Erdős-Hajnal conjecture to date was proved by Erdős and Hajnal in [6], namely:

**1.2** *For every graph  $H$ , there exists  $c > 0$  such that*

$$\max(\alpha(G), \omega(G)) \geq 2^{c\sqrt{\log n}}$$

*for every  $H$ -free graph  $G$  with  $n > 0$  vertices.*

(Logarithms are to base two, throughout the paper.) Until now, this was also the best bound known when  $H = C_5$ , but in this paper we will improve it to:

**1.3** *There exists  $c > 0$  such that*

$$\max(\alpha(G), \omega(G)) \geq 2^{c\sqrt{\log n \log \log n}}$$

*for every  $C_5$ -free graph  $G$  with  $n > 1$  vertices.*

If  $A, B \subseteq V(G)$  are disjoint and nonempty, the *edge-density* between them means the number of edges joining  $A, B$ , divided by  $|A| \cdot |B|$ . The proof of 1.3 is via the following conjecture of Conlon, Fox and Sudakov [4]:

**1.4 Conjecture:** *For every graph  $H$  there exist  $\epsilon, \sigma > 0$  such that for every  $H$ -free graph  $G$  on  $n > 1$  vertices, and all  $c$  with  $0 \leq c \leq 1/2$ ,  $V(G)$  contains two disjoint subsets  $A, B$  with  $|A| \geq \epsilon c^\sigma n$  and  $|B| \geq \epsilon n$ , such that the edge-density between  $A, B$  is either at most  $c$  or at least  $1 - c$ .*

This has not been proved so far for any graph  $H$  with more than four vertices, but in this paper we prove it for  $H = C_5$  (with  $\sigma = 1$ ), and this is the key to proving 1.3. We first prove it for sparse graphs  $G$ , and then use a theorem of Rödl to deduce it in general (both in the next section). The proof of 1.3 is completed in section 3.

We remark that 1.4 (for all  $H$ ) is equivalent to the same statement for sparse graphs (for all  $H$ ), because of the theorem of Rödl discussed in the next section; but for sparse graphs we can prove 1.4 for many more graphs  $H$  than just  $C_5$  (for instance, for all bipartite  $H$ , and all cycles of length at least four). These results will appear in a later paper [3]. But  $C_5$  is still the largest graph  $H$  for which we can show that both  $H$  and its complement satisfy 1.4 in sparse graphs, and so the largest for which we can prove 1.4.

## 2 Sparse graphs

In this section we prove 1.4 for  $H = C_5$ , and first we prove it when  $G$  is sufficiently sparse. Let us say the *closed degree* of a vertex is one more than its degree. (Counting cardinalities of subsets works out more conveniently using closed degree.) For disjoint  $A, B \subseteq V(G)$ , we say  $A$  is *anticomplete* to  $B$  if there are no edges between  $A$  and  $B$ . We will prove:

**2.1** *For all  $c$  with  $0 < c \leq 1/2$ , and every graph  $G$  with  $n > 0$  vertices, if  $G$  satisfies:*

- *every vertex has closed degree at most  $n/16$ , and*
- *for every two disjoint subsets  $A, B \subseteq V(G)$  with  $|A| \geq cn/2$  and  $|B| \geq n/16$ , the edge-density between  $A, B$  is at least  $c$ ,*

*then  $G$  contains  $C_5$ .*

**Proof.** Let  $0 < c \leq 1/2$ , and let  $G, n$  be as in the theorem. Since every vertex has closed degree at most  $n/16$ , it follows that  $n \geq 16$  and in particular,  $\lfloor n/2 \rfloor \geq n/4$ . Choose a set  $N_0 \subseteq V(G)$  of cardinality  $\lfloor n/2 \rfloor$ . It follows that  $|N_0| \geq n/4 \geq cn/2$ , and so the edge-density between  $N_0$  and its complement is at least  $c$ . In particular, some vertex in  $N_0$  has at least  $cn/2$  neighbours.

Let  $v_1$  be a vertex of degree at least  $cn/2$ , let  $N_1$  be the set of all neighbours of  $v_1$ , and let  $Z_2 = V(G) \setminus (N_1 \cup \{v_1\})$ . Since  $|N_1| + 1 \leq n/16$ , it follows that  $|Z_2| \geq 15n/16$ . But  $|N_1| \geq cn/2$ , and so fewer than  $n/16$  vertices in  $Z_2$  have no neighbour in  $N_1$ , since  $c > 0$ . Hence at least  $7n/8$  vertices in  $Z_2$  do have such a neighbour. Choose  $B_1 \subseteq N_1$  minimal such that  $B_1$  covers at least  $5n/16$  vertices in  $Z_2$ . Let  $B_2$  be the set of vertices in  $Z_2$  covered by  $B_1$ . Thus  $5n/16 \leq |B_2| \leq 3n/8$  from the minimality of  $B_1$ . Let  $A_2 = Z_2 \setminus B_2$ . Thus  $A_2$  is anticomplete to  $B_1$ , and  $|A_2| = |Z_2| - |B_2| \geq (15n/16 - 3n/8) = 9n/16$ .

Let  $A_1 = N_1 \setminus B_1$ . Since  $|N_1| \geq cn/2$ , the edge-density between  $N_1, A_2$  is at least  $c$ . In particular there is a vertex  $v_2 \in A_1$  with at least  $c|A_2| \geq 9cn/16 \geq cn/2$  neighbours in  $A_2$ . (Note that  $v_2 \notin B_1$  since  $B_1$  is anticomplete to  $A_2$ .) Let  $N_2$  be the set of neighbours of  $v_2$  in  $A_2$ . Let  $C_1$  be the set of vertices in  $B_1$  adjacent to  $v_2$ , and let  $D_2$  be the set of vertices in  $B_2$  that have a neighbour in  $B_1 \setminus C_1$ .

(1) *If  $|D_2| \geq n/8$  then  $G$  contains  $C_5$ .*

Assume that  $|D_2| \geq n/8$ . It follows that there is a set  $D'_2 \subseteq D_2$  of at least  $n/16$  vertices that are nonadjacent to  $v_2$ . The edge-density between  $N_2$  and  $D'_2$  is at least  $c$ , since  $|N_2| \geq cn/2$ , and in particular some vertex  $d_2 \in D'_2$  has a neighbour  $w \in N_2$ . Since  $d_2 \in D'_2 \subseteq D_2$ , it is adjacent to some vertex  $d_1 \in B_1$  that is nonadjacent to  $v_2$ ; but then

$$d_1-v_1-v_2-w-d_2-d_1$$

is an induced cycle of length 5. (Note that  $d_1$  is nonadjacent to  $w$  since  $B_1$  is anticomplete to  $A_2$ .) This proves (1).

Let  $Y_2 = A_2 \setminus N_2$ ; it follows that  $|Y_2| \geq |A_2| - n/16 \geq n/2$ . Since  $|N_2| \geq cn/2$  the edge-density between  $N_2, Y_2$  is at least  $c$ , and so some vertex  $v_3 \in N_2$  has at least  $c|Y_2| \geq cn/2$  neighbours in  $Y_2$ . Let  $N_3$  be the set of neighbours of  $v_3$  in  $Y_2$ . Let  $C_2$  be the set of vertices in  $B_2$  with a neighbour in  $C_1$ .

(2) If  $|C_2| \geq 3n/16$  then  $G$  contains  $C_5$ .

Assume that  $|C_2| \geq 3n/16$ . It follows that there is a set  $C'_2 \subseteq C_2$  of at least  $n/16$  vertices that are nonadjacent to both  $v_2, v_3$ . The edge-density between  $N_3$  and  $C'_2$  is at least  $c$ , since  $|N_3| \geq cn/2$ , and in particular some vertex  $c_2 \in C'_2$  has a neighbour  $w \in N_3$ . Since  $c_2 \in C'_2 \subseteq C_2$ , it is adjacent to some vertex  $c_1 \in C_1$ ; but then

$$c_1-v_2-v_3-w-c_2-c_1$$

is an induced cycle of length 5. (Note that  $c_1$  is nonadjacent to  $v_3, w$  since  $B_1$  is anticomplete to  $A_2$ .) This proves (2).

Since  $B_1$  covers  $B_2$ , it follows that  $C_2 \cup D_2 = B_2$ , and since  $|B_2| \geq 5n/16$ , the result follows from (1) and (2). This proves 2.1. ■

Next we apply a theorem of Rödl [8], the following. ( $\overline{G}$  denotes the complement graph of  $G$ .)

**2.2** For every graph  $H$  and all  $d > 0$  there exists  $\delta > 0$  such that for every  $H$ -free graph  $G$ , there exists  $X \subseteq V(G)$  with  $|X| \geq \delta|V(G)|$  such that in one of  $G[X], \overline{G}[X]$ , every vertex in  $X$  has degree at most  $d|X|$ .

We deduce:

**2.3** There exists  $\epsilon > 0$  such that for all  $c$  with  $0 \leq c \leq 1/2$ , if  $G$  is  $C_5$ -free with  $n > 1$  vertices, then there exist disjoint  $A, B \subseteq V(G)$  with  $|A| \geq \epsilon cn$  and  $|B| \geq \epsilon n$ , such that the edge-density between  $A, B$  is either less than  $c$  or more than  $1 - c$ .

**Proof.** Let  $\delta$  satisfy 2.2, taking  $d = 1/20$  and  $H = C_5$ . Now let  $\epsilon = \delta/16$ , and let  $G$  be  $C_5$ -free with  $n > 1$  vertices. Let  $v$  be a vertex; then it has either at least  $(n-1)/2$  neighbours or at least  $(n-1)/2$  non-neighbours; and since  $(n-1)/2 \geq \epsilon n$ , we may assume that  $1 < \epsilon cn$ , for otherwise the theorem holds taking  $A = \{v\}$ . In particular  $n > 2\epsilon^{-1} \geq 32\delta^{-1}$ .

By 2.2, there exists  $X \subseteq V(G)$  with  $|X| \geq \delta n$  such that every vertex of  $J$  has degree at most  $|V(J)|/20$ , where  $J$  is one of  $G[X], \overline{G}[X]$ . Since  $|V(J)| \geq \delta n \geq 32$ , it follows that every vertex of  $J$  has closed degree at most  $|V(J)|/16$ . Since  $C_5$  is isomorphic to its complement,  $J$  is  $C_5$ -free, and so from 2.1, there are two disjoint subsets  $A, B \subseteq V(J)$  with  $|A| \geq c|V(J)|/2$  and  $|B| \geq |V(J)|/16$ , such that the edge-density between  $A, B$  in  $J$  is at most  $c$ . Thus  $|A| \geq c\delta n/2 \geq \epsilon cn$  and  $|B| \geq \delta n/16 = \epsilon n$ , and the edge-density between  $A, B$  in  $G$  is either at most  $c$  or at least  $1 - c$ . This proves 2.3. ■

It is possible to deduce versions of 1.2 from versions of Rödl's theorem 2.2 directly, as follows. If we have  $d, \delta$  satisfying 2.2, then for any  $n$ , if we choose  $k \leq \min(\frac{1}{2d}, \frac{\delta n}{2})$  then we can use Turán's theorem to obtain a stable set or clique on  $k$  vertices from the set of at least  $2k$  vertices with density at most  $\frac{1}{2k}$  or at least  $1 - \frac{1}{2k}$  that 2.2 gives us. This motivates trying to improve the bound in 2.2.

- Rödl's original proof of 2.2 uses Szemerédi's regularity lemma and gives a tower-type bound for  $1/\delta$  in terms of  $1/d$ , which yields something worse than 1.2.
- In [7], a better bound of  $\delta = 2^{-15|V(H)|(\log(1/d))^2}$  in 2.2 is proved, which implies the bound of 1.2.
- It is conjectured that a polynomial dependence of  $\delta$  on  $d$  holds, and this would imply the Erdős-Hajnal conjecture itself.
- For  $H = C_5$  we can get mid-way between, and that provides a different route to proving 1.3, as follows. One can prove that for  $H = C_5$  we may take

$$\delta = 2^{-O(\log(1/d)^2 / \log \log(1/d))}$$

in 2.2 by appropriately adapting the proof of 2.2 in [7] using that we now know 1.4 for  $H = C_5$ . This would imply 1.3. But the details of the proof of this improved bound for 2.2 for  $C_5$  are involved and similar to that of the proof of 1.3 given in the next section, and we omit them for the sake of brevity.

### 3 The proof of 1.3.

Now we use 2.3 to prove 1.3. Since the argument to come is rather heavy, and works just as well for any graph  $H$  satisfying 1.4 instead of  $C_5$ , it might be wise to present it in full generality. Thus, let us say a class of graphs  $\mathcal{I}$  is *hereditary* if every graph isomorphic to an induced subgraph of a member of the class also belongs to the class. Let  $\epsilon$  be as in 2.3, and let  $\sigma > \log(\epsilon^{-1})$ . Then for  $c \leq 1/2$ ,  $c^\sigma \leq \epsilon$ , and so by 2.3, if  $G$  is  $C_5$ -free with  $n \geq 2$  vertices, and  $0 \leq c \leq 1/2$ , then there exist disjoint  $A, B \subseteq V(G)$  with  $|A| \geq c^\sigma n$  and  $|B| \geq \epsilon n$ , such that the edge-density between them is either at most  $c$  or at least  $1 - c$ . Then 1.3 follows from 2.3 and the following, applied to the hereditary class of all  $C_5$ -free graphs:

**3.1** *Let  $\mathcal{I}$  be a hereditary class of graphs, and let  $\sigma \geq 0$  and  $0 \leq \epsilon \leq 1$  with the following property: for every graph  $G \in \mathcal{I}$  with at least two vertices, and all  $c$  with  $0 \leq c \leq 1/2$ , there are disjoint subsets  $A, B \subseteq V(G)$  with  $|A| \geq c^\sigma n$  and  $|B| \geq \epsilon n$ , such that the edge-density between  $A, B$  is either at most  $c$  or at least  $1 - c$ , where  $n = |V(G)|$ . Then there exists  $\kappa > 0$  such that*

$$\max(\alpha(G), \omega(G)) \geq 2^{\kappa \sqrt{\log n \log \log n}}$$

for every  $G \in \mathcal{I}$ , where  $n = |V(G)| \geq 2$ .

**Proof.** Let us define  $r(n) = \sqrt{\log n \log \log n}$  for  $n \geq 2$ , for typographical convenience.

A *cograph* is a graph not containing a 4-vertex path. Thus the disjoint union of two cographs is a cograph, and so is the complement of a cograph. We prove 3.1 by showing that  $G$  contains a

cograph with at least  $2^{2\kappa r(n)}$  vertices. As cographs are perfect, there is a clique or independent set with  $2^{\kappa r(n)}$  vertices (and so of the desired cardinality).

For a graph  $G$ , let  $\phi(G)$  denote the maximum of  $|V(H)|$  over all cographs  $H$  contained in  $G$ . For each real number  $x \geq 0$ , let  $f(x)$  be the minimum of  $\phi(G)$ , over all graphs  $G \in \mathcal{I}$  with  $|V(G)| = \lceil x \rceil$  (we may assume there is some such graph  $G$ , or else the result is trivially true). It is easy to see that  $f(x)$  is non-decreasing with  $x$ .

We may assume that  $\sigma \geq 1$  (by increasing  $\sigma$  if necessary). Let  $\mu = (32\sigma)^{-1/2}$ . Choose  $n_0$  such that

$$\left\lfloor \frac{\sigma 2\mu r(n) - 1}{\log(2/\epsilon)} \right\rfloor \geq \sqrt{\log n}$$

for all  $n \geq n_0$ , and also such that  $\mu r(n_0) \geq 2$ , and  $\log n_0 \geq 4\sigma\mu r(n_0)$ . Choose  $\kappa \leq \mu/2$  such that  $2\kappa r(n_0) \leq 1$ . We will show that  $\kappa$  satisfies the theorem.

(1) For all  $n \geq 2$  and all  $c$  with  $0 \leq c \leq 1/2$ , either  $f(n) \geq 1/(4c)$  or  $f(n) \geq f(c^\sigma n/2) + f(\epsilon n/2)$ .

Let  $G \in \mathcal{I}$  with  $n \geq 2$  vertices, such that  $\phi(G) = f(n)$ . Since  $G \in \mathcal{I}$ , the hypothesis implies that there are disjoint sets  $A, B \subseteq V(G)$  with  $|A| \geq c^\sigma n$  and  $|B| \geq \epsilon n$  such that the edge-density between  $A$  and  $B$  is either at most  $c$  or at least  $1 - c$ . We suppose without loss of generality that this density is at most  $c$  (in the other case, we apply the same argument to  $\overline{G}$ ).

Let  $A''$  be the set of vertices in  $A$  with at least  $2c|B|$  neighbours in  $B$ . As the number of edges between  $A, B$  is at least  $2c|B||A''|$  and at most  $c|A||B|$ , it follows that  $|A''| \leq |A|/2$ . Let  $A' = A \setminus A''$ ; so  $|A'| = |A| - |A''| \geq |A|/2$  and every vertex in  $A'$  has at most  $2c|B|$  neighbours in  $B$ . Since  $G[A'] \in \mathcal{I}$ , it follows from the definition of  $f$  that  $\phi(G[A']) \geq f(|A'|)$ . Let  $A_0 \subseteq A'$  induce a cograph, with  $|A_0| = f(|A'|)$ .

If  $|A_0| > 1/(4c)$ , then  $f(n) = \phi(G) \geq |A_0| \geq 1/(4c)$  as required, so we may assume that  $|A_0| \leq 1/(4c)$ . Let  $B'$  be those vertices in  $B$  with no neighbours in  $A_0$ ; so  $|B'| \geq |B| - 2c|B||A_0| \geq |B|/2$ . Again from the definition of  $f$ ,  $\phi(G[B']) \geq f(|B'|) \geq f(\epsilon n/2)$ . Since  $A_0$  is anticomplete to  $B'$ , it follows that

$$f(n) = \phi(G) \geq |A_0| + \phi(G[B']) \geq f(c^\sigma n/2) + f(\epsilon n/2).$$

This proves (1).

(2) For all  $n \geq 2$  and all  $c$  with  $0 \leq c \leq 1/2$ , if  $\log n \geq \sigma \log(1/c)$  then either  $f(n) \geq 1/(4c)$  or  $f(n) \geq kf(c^{2\sigma}n)$ , where

$$k = \left\lfloor \frac{\sigma \log(1/c) - 1}{\log(2/\epsilon)} \right\rfloor.$$

We may assume that  $f(n) < 1/(4c)$ , and hence  $f(n') < 1/(4c)$  for all  $n' \leq n$ . From the definition of  $k$ ,  $k \log(2/\epsilon) \leq \sigma \log(1/c) - 1 \leq \log n - 1$ , and so  $n(\epsilon/2)^k \geq 2$ . Hence we may recursively apply (1)  $k$  times without violating the condition “ $n \geq 2$ ” in (1); and we obtain

$$f(n) \geq f(c^\sigma n/2) + f(c^\sigma(\epsilon/2)n/2) + f(c^\sigma(\epsilon/2)^2 n/2) + \cdots + f(c^\sigma(\epsilon/2)^k n/2).$$

Each of the  $k + 1$  terms on the right side is at least  $f(c^{2\sigma}n)$ , from the definition of  $k$ , and so  $f(n) \geq kf(c^{2\sigma}n)$ . This proves (2).

(3) For all  $n \geq 2$  and all  $c$  with  $0 \leq c \leq 1/2$ , if  $\log n \geq 2\sigma \log(1/c)$  and with  $k$  as in (2), either  $f(n) \geq 1/(4c)$  or  $f(n) \geq k^j$ , where

$$j = \left\lfloor \frac{\log n}{4\sigma \log(1/c)} \right\rfloor.$$

Again, we may assume that  $f(n) < 1/(4c)$ , and hence  $f(n') < 1/(4c)$  for all  $n' \leq n$ . From the definition of  $j$ ,  $c^{2\sigma j} n \geq n^{1/2}$ , and so  $\log(c^{2\sigma j} n) \geq \frac{1}{2} \log n \geq \sigma \log(1/c)$ . Moreover,  $c^{2\sigma(j-1)} n \geq n^{1/2} c^{-2\sigma} \geq 2$  since  $\sigma \geq 1$ . Hence we may apply (2) recursively  $j$  times, and deduce that  $f(n) \geq k^j f(c^{2\sigma j} n) \geq k^j$ . This proves (3).

(4) Let  $n \geq n_0$ , and  $c = 2^{-2\mu r(n)}$ . Then

- $c \leq 1/2$ ;
- $\log n \geq 4\sigma \mu r(n)$ ;
- $k \geq \sqrt{\log n}$ , where  $k$  is as defined in (2); and
- $1/(4c) \geq 2^{\mu r(n)}$ .

We observe first that  $c \leq 1/2$  if  $n \geq n_0$ , since  $\mu r(n_0) \geq 1$ . Also,  $\log n_0 \geq 4\sigma \mu r(n_0)$  from the choice of  $n_0$ , and since  $\frac{\log n}{r(n)}$  increases with  $n$ , it follows that  $\log n \geq 4\sigma \mu r(n)$  for  $n \geq n_0$ . But  $4\sigma \mu r(n) = 2\sigma \log(1/c)$ , and so  $\log n \geq 2\sigma \log(1/c)$ . This proves the second statement. The third statement follows from the choice of  $n_0$ . For the final statement, we must check that  $\log(1/c) - 2 \geq \mu r(n_0)$ , that is,  $2\mu r(n) \geq \mu r(n_0) + 2$ ; but  $\mu r(n) \geq \mu r(n_0)$  since  $n \geq n_0$ , and  $\mu r(n) \geq 2$  from the definition of  $n_0$ . This proves (4).

(5) If  $n \geq n_0$  then  $f(n) \geq 2^{\mu r(n)}$ .

Let  $c$  be as in (4) and let  $n \geq n_0$ . By the first two statements of (4); we may apply (3), and so either  $f(n) \geq 1/(4c)$  or  $f(n) \geq (\log n)^{j/2}$ , by the third statement of (4). In the first case, the claim follows from the final statement of (4), so we may assume that

$$f(n) \geq (\log n)^{j/2} \geq (\log n)^{(\log n)/(16\sigma \log(1/c))} = 2^{(16\sigma \cdot 2\mu)^{-1} r(n)}.$$

As  $\mu = (16\sigma \cdot 2\mu)^{-1}$  from the definition of  $\mu$ , this proves (5).

We recall that  $\kappa \leq \mu/2$  and  $2\kappa r(n_0) \leq 1$ . We claim that  $f(n) \geq 2^{2\kappa r(n)}$  for all  $n \geq 2$ . This is true if  $n \leq n_0$ , because then  $f(n) \geq 2 \geq 2^{2\kappa r(n)}$ ; and if  $n > n_0$  then it follows from (5). This proves 3.1. ■



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