COARSE BALANCED SEPARATORS IN BICLIQUE-INDUCED-MINOR-FREE GRAPHS

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Abstract. It is a classical theorem of Robertson and Seymour (1986) that the treewidth of a graph is linearly related to its separation number: the smallest integer k such that, for every weight function on the vertices, the graph admits a balanced separator of size at most k. Motivated by recent progress on coarse treewidth, Abrishami, Czyżewska, Kluk, Pilipczuk, Pilipczuk, and Rzażewski (2025) conjectured the following coarse analogue: for every $r \in \mathbb{N}$ there exists an $r' \in \mathbb{N}$ such that every graph that admits balanced separators that can be covered by a bounded number of balls of bounded radius r admits a tree decomposition where every bag can be covered by a bounded number of balls of radius r'. We verify a stronger variant of this conjecture for all $r \in \mathbb{N}$ for the hereditary class of $K_{t,t}$-induced-minor-free graphs of bounded clique number. A key step in the proof is the following result, which we expect to be of independent interest. In $K_{t,t}$-induced-minor-free graphs with clique number bounded by s, given a large subset of vertices $Y \subseteq V(G)$, there is a set Z whose size is bounded by a function polynomial in s, such that no ball of radius r in $G - Z$ covers a large proportion of Y.

1. Introduction

All graphs in this paper are finite and simple. We refer the reader to Section 1.1 for notation and definitions. *Treewidth* is a graph parameter which, informally, captures "how close" a graph is to a tree, and graphs with small treewidth are necessarily sparse. Formally, a *tree decomposition* $\mathcal{T} = (T, \beta)$ of a graph G consists of a tree T and a map $\beta : V(T) \to 2^{V(G)}$ with the following properties:

(i) For every $v \in V(G)$, there exists $t \in V(T)$ such that $v \in \beta(t)$.
(ii) For every $uv \in E(G)$, there exists $t \in V(T)$ such that $u, v \in \beta(t)$.
(iii) For every $v \in V(G)$, the subgraph of T induced by $\{t \in V(T) \mid v \in \beta(t)\}$ is connected.

For each $t \in V(T)$, we refer to $\beta(t)$ as a *bag of* (T, β). The *width* of a tree decomposition (T, β), denoted by $width(T, \beta)$, is $\max_{t \in V(T)} |\beta(t)| - 1$. The *treewidth* of G, denoted by $tw(G)$, is the minimum width of a tree decomposition of G. Graphs of bounded treewidth are well-understood both structurally [10] and algorithmically [2].

Let $w : V(G) \to \mathbb{R}_{\geq 0}$ be a weight function on the vertices of G. For a subset $C \subseteq V(G)$, we set $w(C) = \sum_{v \in C} w(v)$. A set $X \subseteq V(G)$ is a *w-balanced separator* if $w(C) \leq \frac{1}{2} w(V(G))$ for every connected component C of $G \setminus X$. The *separation number* of G, denoted by $\text{sep}(G)$, is the minimum $k \in \mathbb{N}$ such that, for every weight function $w : V(G) \to \mathbb{R}_{\geq 0}$ there is a w-balanced separator X with size at most k. Given a subset $Y \subseteq V(G)$, we say that X is a *balanced separator for Y* if X is a w-balanced separator for the weight function w given by

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$w(v) = 1$ for $v \in Y$ and $w(v) = 0$ otherwise. It is a classical result of Robertson and Seymour [10] that there is a linear correspondence between treewidth and separation number.

Theorem 1.1 ([10])**.** *For every graph G,* $\operatorname{sep}(G) - 1 \leq \operatorname{tw}(G) \leq 4 \operatorname{sep}(G)$.

This paper will investigate a "coarse analogue" of this theorem. For a set $S \subseteq V(G)$, we say that $\hat{S}$ is an *r-cover* of S if $S \subseteq N_G^r[\hat{S}]$, and $\hat{S}$ is a *minimum r-cover* of S if there is no r-cover $\hat{T}$ of S with $|\hat{T}| < |\hat{S}|$. The set S is *(k,r)-coverable* if there exists an r-cover $\hat{S}$ of S with $|\hat{S}| \leq k$, and S is *internally (k,r)-coverable* if $G[S]$ is (k,r)-coverable. We say that G *admits (k,r)-balanced separators* if, for every weight function $w : V(G) \to \mathbb{R}_{\geq 0}$, there is a (k,r)-coverable w-balanced separator. If every induced subgraph of G admits (k,r)-coverable balanced separators, we say that G is *(k,r)-separable*. A class of graphs is *(k,r)-separable* if every graph in it is (k,r)-separable. We say that a tree decomposition is *(internally) (k,r)-coverable* if every bag is (internally) (k,r)-coverable.

Abrishami, Czyżewska, Kluk, Pilipczuk, Pilipczuk, and Rzażewski proposed the following coarse counterpart to Theorem 1.1.

Conjecture 1.2 (Conjecture 1.1 in [1])**.** *For every $k, r \in \mathbb{N}$, there exist $k', r' \in \mathbb{N}$ such that if G admits (k,r)-balanced separators, then G admits a (k',r')-coverable tree decomposition.*

Conversely, it is straightforward to show that every graph that admits a tree decomposition where every bag is (k,r)-coverable also admits (k,r)-balanced separators using a standard argument. Therefore, if proven true, Conjecture 1.2 would establish that the correspondence between balanced separators and treewidth is preserved under quasi-isometry. The following strengthening of Conjecture 1.2 is also of interest.

Conjecture 1.3. *For every $k, r \in \mathbb{N}$, there exists $k' \in \mathbb{N}$ such that if G admits (k,r)-balanced separators, then G admits a (k',r)-coverable tree decomposition.*

Abrishami et al. reduced Conjecture 1.2 to the case $r = 1$ [1], and proved a weakening of Conjecture 1.2, allowing logarithmic dependence on the number of vertices in either the number of balls or the radius. Abrishami et.al. [1] also confirmed Conjecture 1.2 in graph classes with bounded doubling dimension (which is the requirement that, for every $r \in \mathbb{N}$, every ball of radius $2r$ can be covered by a bounded number of balls of radius r). Chudnovsky and Hickingbotham proved Conjecture 1.2 for hereditary $K_{t,t}$-induced-subgraph-free graph classes when $r = 1$.

Theorem 1.4 ([5])**.** *For all $k, t \in \mathbb{N}$, there exist k' such that the following holds: Let G be $K_{t,t}$-induced-subgraph-free and $(k,1)$-separable. Then G admits an internally $(k',2)$-coverable tree decomposition.*

For a positive integer t, we denote by $\mathcal{C}_t$ the class of $K_{t,t}$-induced-minor-free graphs. Our results strengthen Theorem 1.4 in multiple ways at the cost of bounding the clique number. We prove that Conjecture 1.3 holds in separable subclasses of $\mathcal{C}_t$ with bounded clique number.

Theorem 1.5. *For every $r, t \in \mathbb{N}$, there exists a polynomial $p = p_{r,t}$ such that the following holds. For every $k, s \in \mathbb{N}$, if $G \in \mathcal{C}_t$ such that $\omega(G) < s$ and G is (k,r)-separable then G admits an internally $(p(ks), r)$-coverable tree decomposition.*

This shows that Conjecture 1.3 holds for graphs in $\mathcal{C}_t$ with bounded clique number, while also providing additional control over the dependence of k' on k and the clique number.

The proof of Theorem 1.4 relies on a result concerning the behavior of large neighborhoods within a given set in $K_{t,t}$-induced-subgraph-free graphs (Theorem 5.1 in [4]). A substantial part of the present paper is devoted to establishing a distance-r generalization of this result for graphs in $\mathcal{C}_t$:

Theorem 1.6. *For all $t, r \in \mathbb{N}$, there exists a polynomial $f = f_{t,r}$ such that the following holds. Let $C, s \in \mathbb{N}$ such that $C \geq 2$, let $G \in \mathcal{C}_t$ with $\omega(G) < s$, and let $Y \subseteq V(G)$. Then there is a subset $X \subseteq V(G)$ with $|X| \leq f(Cs)$ such that for every $v \in G \setminus X$ we have that*

$$|N^r_{G\setminus X}[v] \cap Y| \leq \frac{|Y|}{C}.$$

Theorem 1.6 will be used in a forthcoming paper by the first two authors, Lokshtanov, and E S.

1.1. **Preliminaries and notation.**

We take standard concepts and notation not explained here from [6]. For a positive integer k, we use $[k]$ to denote the set $\{1, 2, \ldots, k\}$. Given a graph G, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. An *independent set*, or *stable set*, is a subset of vertices of $V(G)$ which are pairwise non-adjacent. A *clique* in a graph G is a set of pairwise adjacent vertices. A *complete bipartite graph with bipartition (X, Y)* is a graph G with vertex set $X \cup Y$ such that $xy \in E(G)$ if and only if $x \in X$ and $y \in Y$. For a positive integer t, let K_t denote a clique on t vertices and let $K_{t,t}$ denote a complete bipartite graph where X and Y contain exactly t vertices each. The *independence number* of a graph G, denoted by $\alpha(G)$, is the maximum size of an independent set in G. The *clique number* of G, denoted by $\omega(G)$, is the maximum size of a clique in G. For a set $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X, and by $G \setminus X$ the subgraph of G induced by $V(G) \setminus X$. In this paper, we use induced subgraphs and their vertex sets interchangeably. For a graph H, we say that H is an *induced subgraph of G* if there exists $X \subseteq V(G)$ such that H is isomorphic to $G[X]$, and we say that H is an *induced minor of G* if H can be obtained from G by a sequence of vertex deletions and edge contractions (and by deleting parallel edges generated in the process).

A *path* in a graph is an induced subgraph that is a path. We denote by $P = p_1\text{-}\ldots\text{-}p_k$ a path in G, where $p_ip_j \in E(G)$ if $|j - i| = 1$. We say that p_1 and p_k are the *ends* of P. The *interior* of P, denoted by P^*, is the set $P \setminus \{p_1, p_k\}$. For $i, j \in \{1, \ldots, k\}$ we denote by $p_i\text{-}P\text{-}p_j$ the subpath of P with ends p_i, p_j. The *length* of a path is the number of edges in it. A *(u, v)-path* is a path whose set of endpoints is equal to $\{u, v\}$. Given a graph G and sets $X, Y \subseteq V(G)$ of vertices, an *(X, Y)-path* is an (x, y)-path in G with $x \in X$, $y \in Y$, and no internal vertex in $X \cup Y$. The *distance between two vertices* $u, v \in V(G)$, denoted by $\text{dist}_G(u, v)$, is the length of the shortest (u, v)-path in G. The *distance between two sets* X and Y, denoted by $\text{dist}_G(X, Y)$, is the minimum of the distance $\text{dist}_G(x, y)$ among all $x \in X$ and $y \in Y$. For a vertex $v \in V(G)$, we use $N^r_G(v)$ and $N^r_G[v]$ to denote the sets of vertices at distance exactly r and at most r from v in G, respectively. For a set $X \subseteq V(G)$, we let $N^r_G[X] = \bigcup_{v\in X} N^r_G[v]$ and $N^r_G(X) = N^r_G[X] \setminus N^{r-1}_G[X]$. We say that a vertex u is an *r-neighbor* of v if $u \in N^r_G(v)$. If $r = 1$ we may omit the superscript or the prefix. Similarly, if the ambient graph G is clear from the context, we may omit it in the notation of both distances and neighborhoods.

2. Large r-neighborhoods: the proof of Theorem 1.6

The following result due to Chudnovsky, Codsi, Lokshtanov, Milanič, and Sivashankar [4] is an important step in the proof of Theorem 1.4.

Lemma 2.1 (Theorem 5.1 in [4]). *Let $C, t \in \mathbb{N}$ with $C \geq 2$, and let G be $K_{t,t}$-induced-minor-free. Fix $Y \subseteq V(G)$ and define*

$$Z = \left\{ z \in V(G) \colon \alpha(N(z) \cap Y) \geq \frac{\alpha(Y)}{C} \right\}.$$

Then $\min(\alpha(Y), \alpha(Z)) \leq (512C)^{2t^{2t}}$.

The objective of this section is to prove Theorem 1.6, which can be seen as a generalization of Lemma 2.1 to distance-r neighborhoods. This will be a crucial step in the proof of Theorem 1.5, and is likely to be of independent interest.

We start with the following:

Theorem 2.2. *There exists a function $M(C, s, t, r)$ that is polynomial in the parameters s and C such that the following holds: Let $C, s, t, r \in \mathbb{N}$ and let $G \in \mathcal{C}_t$ such that $\omega(G) < s$ and $C \geq 2$. Let $Y \subseteq V(G)$. Then there is a subset $X \subseteq V(G)$ with $\alpha(X) \leq M(C, s, t, r)$ such that if*

$$Z = \left\{ z \in V(G) \setminus X \ : \alpha(N^r_{G \setminus X}[z] \cap Y) \geq \frac{\alpha(Y)}{C} \right\}$$

then $\min(\alpha(Y), \alpha(Z)) \leq M(C, s, t, r)$.

To prove this, the following results will be required. For positive integers a, b, let $R(a, b)$ be the smallest integer, often referred to as the *Ramsey number*, such that every graph on $R(a, b)$ vertices contains either a clique of size a or stable set of size b.

Theorem 2.3 ([9]). *For all $a, b \in \mathbb{N}$, $R(a, b) \leq a^{b-1}$.*

A *hypergraph* $H = (V(H), E(H))$ consists of a non-empty set of vertices $V(H)$ and a non-empty set of *hyperedges* $E(H) \subseteq 2^{V(H)}$. Given a hypergraph H, we define the following parameters. We denote by $\tau(H)$ the minimum size of a *hitting set*, which is a set of vertices that meets all the hyperedges. We denote by $\nu(H)$ the maximum size of a *matching*, which is a set of pairwise disjoint hyperedges. We denote by $\lambda(H)$ the size of the largest set E of hyperedges such that for every $e, f \in E$, there exists $v \in V(H)$ such that

$$v \in (e \cap f) \setminus \bigcup_{\substack{g \in E \\ g \neq e, f}} g.$$

We will require the following result due to Ding, Seymour, and Winkler.

Theorem 2.4 (Theorem 1.1 in [7]). *For every hypergraph H with no empty hyperedge,*

$$\tau(H) \leq 11\lambda(H)^2(\lambda(H) + \nu(H) + 3)\binom{\lambda(H) + \nu(H)}{\nu(H)}^2.$$

Let G be a graph. For $k \in \mathbb{N}$, a *proper k-coloring* of G is a mapping $c : V(G) \to [k]$ such that $c(u) \neq c(v)$ whenever $uv \in E(G)$. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum k such that G admits a proper k-coloring. Since every color class forms an independent set, we deduce the following simple observation.

Observation 2.5. *For every graph G, $|V(G)| \leq \alpha(G) \cdot \chi(G)$.*

A class of graphs $\mathcal{C}$ is *(polynomially) χ-bounded* if there exists a (polynomial) function $p : \mathbb{N} \to \mathbb{N}$ such that $\chi(G) \leq p(\omega(G))$ for every $G \in \mathcal{C}$. Bourneuf, Bucić, Cook, and Davies [3] proved the following result.

Lemma 2.6 (Theorem 1.2 [3])**.** *For every $t \in \mathbb{N}$, $\mathcal{C}_t$ is polynomially χ-bounded.*

Lastly, we require a Ramsey-type result of Lozin and Razgon [8].

Lemma 2.7 (Lemma 2 in [8])**.** *For all $d, t \in \mathbb{N}$, there is a positive integer $K = K(d,t)$ such that if a graph G contains a collection of K pairwise disjoint subsets of vertices, each of size at most d and with at least one edge between every two of them, then G contains a (not necessarily induced) $K_{t,t}$-subgraph.*

Proof of Theorem 2.2. Let $K(\cdot,\cdot)$ be the constant given by Lemma 2.7, let $R(\cdot,\cdot)$ denote the Ramsey number, and let $c_t(\cdot)$ be the polynomial given by Lemma 2.6. We define

$$R(y) = R(K(yt,t),t), \qquad \nu(x,y) = 2xc_t(s)^{y+1}(R(y)+1)$$

$$\tau(x,y) = 11(2t)^2(2t+\nu(x,y)+3)\binom{2t+\nu(x,y)}{\nu(x,y)}^2, \qquad \iota(x,y) = c_t(s)^{y+1}t \cdot \tau(x,y)^{R(y)}$$

and

$$M(C,s,t,r) = \begin{cases} (512C)^{2t^{2t}} & \text{if } r = 1 \\ 2M(2C\iota(C,r)(R(r)+1), s, t, r-1) & \text{otherwise.} \end{cases}$$

(1) *The function $M(C,s,t,r)$ is a polynomial in the parameters s and C.*

By construction, $M(C,s,t,1)$ is a polynomial the parameters C and s. Suppose inductively that $M(C,s,t,r-1)$ is a polynomial in s and C. Since $c_t(\cdot)$ is a polynomial, the function $\nu(C,r)$ is a polynomial in s and C. Note that the binomial term in $\tau(C,r)$ is bounded by $\binom{2t+\nu(C,r)}{\nu(C,r)} \leq \frac{(2t+\nu(C,r))^{2t}}{(2t)!}$. Since $\tau(C,r)$ is a polynomial in s and C, we conclude that $\iota(C,r)$ is a polynomial in s and C. Since the composition of polynomials is a polynomial, we conclude that $2M(2C\iota(C,r)(R(r)+1), s, t, r-1)$ is a polynomial in the parameters s and C. This proves (1).

We may assume that $\alpha(Y) > M(C,s,t,r)$ as otherwise there is nothing to show. We proceed by induction on r. When $r = 1$, the result follows from Lemma 2.1 with $X = \emptyset$ and the observation that $\alpha(N[z] \cap Y) = \alpha(N(z) \cap Y)$, unless $N[z] \cap Y = z$ in which case $z \notin Z$ since $\frac{\alpha(Y)}{C} > 1$.

Now let $r \in \mathbb{N}$, and assume that the result holds for $1, \ldots, r-1$. By definition of M, we have $\alpha(Y) > M(2C\iota(C,r)(R(r)+1), s, t, r-1)$. By the inductive hypothesis, there is a set X_{r-1} with $\alpha(X_{r-1}) \leq M(2C\iota(C,r)(R(r)+1), s, t, r-1)$ for which

$$Z_{r-1} = \left\{ z \in V(G) \setminus X_{r-1} : \alpha(N^{r-1}_{G \setminus X_{r-1}}[z] \cap Y) \geq \frac{\alpha(Y)}{2\iota(C,r)C(R(r)+1)} \right\}$$

has $\alpha(Z_{r-1}) \leq M(2C\iota(C,r)(R(r)+1), s, t, r-1)$. Let $X = Z_{r-1} \cup X_{r-1}$. Then, since $r \geq 2$,

$$\alpha(X) \leq 2M(2C\iota(C,r)(R(r)+1), s, t, r-1) = M(C,s,t,r).$$

In what follows, we abuse the notation by setting $\iota(C, r) = \iota$, $R = R(r)$, and $c = c_t(s)$.

(2) *For every $v \in G \setminus X$, we have that $\alpha(N^{r-1}_{G\setminus X}[v] \cap Y) < \frac{\alpha(Y)}{2\iota C(R+1)}$*

(2) is a direct consequence of the definition of X.

Let

$$Z = \left\{ z \in V(G) \setminus X : \alpha(N^r_{G\setminus X}[z] \cap Y) \geq \frac{\alpha(Y)}{C} \right\},$$

we will show that $\alpha(Z) \leq M(C, s, t, r)$. For the sake of a contradiction, suppose not. Let I be an independent subset of Z of size ι which exists since $M(C, s, t, r) \geq \iota$. Let $Y_{r-1} = N^{r-1}_{G\setminus X}[I] \cap Y$. Set $G' = G \setminus X$ and $Y' = Y \setminus (X \cup Y_{r-1})$. Then in G' there is no path of length less than r from $I \cap G'$ to Y'.

(3) *For every $z \in I$, we have that $\alpha(N^r_{G\setminus X}(z) \cap Y') \geq \frac{\alpha(Y)}{2C}$*

It follows from (2) that

$$\begin{aligned} \alpha(N^r_{G'}(z) \cap Y') &\geq \alpha(N^r_{G\setminus X}[z] \cap Y) - \alpha(Y_{r-1}) \\ &> \frac{\alpha(Y)}{C} - |I| \cdot \frac{\alpha(Y)}{2C\iota(R+1)} \\ &\geq \frac{\alpha(Y)}{2C}. \end{aligned}$$

In particular, for every $z \in I$, there exist paths from z to Y' of length r. This proves (3).

Since $\mathcal{C}_t$ is hereditary and by Lemma 2.6, $\chi(G') \leq c$. Fix a proper c-coloring of G'. For a given path p_0-p_1-$\ldots$-p_r of length r in G', we call the sequence of colors $(c(p_0), c(p_1), \ldots, c(p_r))$ the *type* of the path. For every $z \in I$ and every $y \in Y' \cap N^r_{G'}(z)$, let $P_{z,y}$ be a (z, y)-path of length r. For every $z \in I$, let the *type of z* be the most common type among the paths in the set $\{P_{z,y} : y \in Y' \cap N^r_{G'}(z)\}$ (breaking ties arbitrarily). Let T be the most common type among the vertices in I (again breaking ties arbitrarily). Let $I' \subseteq I$ be the set of vertices of type T, and let Y'' be the set of vertices in Y' that can be reached by paths of type T from vertices in I'. Let G'' be the graph induced by the union of the (I', Y'')-paths of type T. We define the *0th level* as $L_0 = I'$ and the *ith level* as $L_i = N^i_{G''}(I')$ for $i \in [r]$. For $v \in G''$, let $l(v)$ be the integer i such that $v \in L_i$.

(4) *$|I'| \geq \frac{1}{c^{r+1}}|I| \geq t\tau^R$, and for every $z \in I'$, $\alpha(N^r_{G''}(z) \cap Y'') \geq \frac{1}{c^{r+1}} \cdot \frac{\alpha(Y)}{2C}$.*

The first assertion is immediate from the definition of I' and since there are at most c^{r+1} distinct possible types. The second assertion follows from (3) and the fact that z is of type T. This proves (4).

(5) *G'' is bipartite and, therefore, G'' is $K_{t,t}$-subgraph-free.*

By construction, the levels are disjoint, and every edge has its ends in two consecutive levels (since otherwise we would be able to find a path of length less than r between I' and Y''). It is sufficient to show that each level L_i is an independent set, as in this case, the set of even levels and the set of odd levels form a bipartition of G''. Since $\mathrm{dist}(I', Y'') = r$ and every vertex in G'' is in an (I', Y'')-path of length r of type T, it follows that for every i and for every vertex $v \in L_i$, there exists an (I', v)-path of length i that is an initial segment of a path of type T. Therefore, all the vertices of L_i belong to the same color class, which proves the first assertion of (5). The second assertion follows since every bipartite graph in $\mathcal{C}_t$ is $K_{t,t}$-subgraph-free.

For $D \geq 0$, let $\nu_D = \frac{2Cc^{r+1}(R+1)}{D+1}$ for any $D \geq 0$. Note that $\nu = \nu_0$.

Claim 2.8. *For every D such that $0 \leq D \leq R$, the following holds: Let $G^\star$ be an induced subgraph of G'', and define $I^\star = G^\star \cap I'$ and $Y^\star = G^\star \cap Y''$. Assume that $|I^\star| \geq t\tau^D$ and that for each $z \in I^\star$ we have $\alpha(N^r_{G^\star}(z) \cap Y^\star) \geq \frac{\alpha(Y)}{\nu_D}$. Then there exist $S \subseteq I^\star$ with $|S| = t$, and disjoint induced connected subgraphs $J_1, \ldots, J_D \subseteq G^\star$, such that $|J_i| \leq tr$ and $S \subseteq N_{G''}(J_i)$ for all $i \in [D]$.*

Proof of Claim. We proceed by induction on D. When $D = 0$, the statement is vacuously true. Now, assume inductively that the result holds for $1, \ldots, D-1$.

By taking a subset if necessary, we may assume that $|I^\star| = t\tau^D$. Let H be a hypergraph with vertex set $Y^\star$ and hyperedge set $I^\star$, where for every $z \in I^\star$, the hyperedge e_z is defined as $\{y \in Y^\star : y \in N^r_{G^\star}(z)\}$. See Fig. 1 for a visualization. The proof proceeds by using

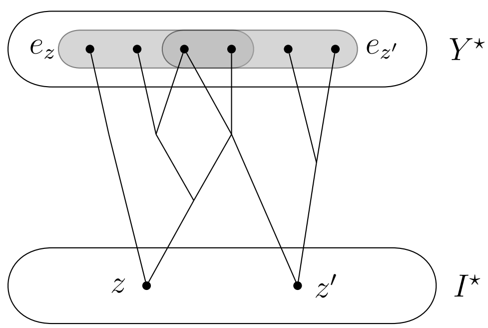

FIGURE 1. Construction of the hypergraph H.

Theorem 2.4 to find a vertex in $Y^\star$ that is contained in the r-neighborhood of many vertices of $I^\star$. We first show that $\nu(H)$ and $\lambda(H)$ are bounded by functions of t and ν_D.

(6) *It holds that $\nu(H) \leq \nu_D$.*

Suppose not, then there are more than ν_D pairwise disjoint subsets of $Y^\star$, each with at least $\frac{\alpha(Y)}{\nu_D}$ vertices. Therefore

$$|Y^\star| > \frac{\alpha(Y)}{\nu_D} \cdot \nu_D = \alpha(Y).$$

Since $Y^\star$ is an independent set contained in Y, this is a contradiction. This proves (6).

(7) *It holds that $\lambda(H) < 2t$.*

Suppose that $\lambda(H) \geq 2t$. We will obtain a contradiction by constructing a 1-subdivision of K_{2t}, from which we obtain a $K_{t,t}$ induced minor. Let $z_1, \ldots, z_{2t}$ be vertices in I' such that the hyperedges $e_{z_1}, \ldots, e_{z_{2t}}$ exhibit $\lambda(H) \geq 2t$. For every $i < j \in [2t]$, let $y_{i,j}$ be a vertex in $(e_{z_i} \cap e_{z_j}) \setminus \bigcup_{k \neq i,j} e_{z_k}$. We call $y_{i,j}$ the *private r-neighbor* of z_i and z_j. Let $P_{i,j}$ and $P_{j,i}$ be paths of length r from z_i to $y_{i,j}$ and from z_j to $y_{i,j}$, respectively, such that the union $|P_{i,j} \cup P_{j,i}|$ is minimal. Let $s_{i,j}$ be the vertex in $P_{i,j} \cap P_{j,i}$ with $l(s_{i,j})$ minimum. Such a vertex exists because $y_{i,j} \in P_{i,j} \cap P_{j,i}$. We define $P'_{i,j} = z_i\text{-}P_{i,j}\text{-}s_{i,j} \setminus s_{i,j}$ and $R_i = \bigcup_{j \neq i} P'_{i,j}$.

(8) *If $i \neq j$, then R_i and R_j are disjoint and anticomplete.*

Suppose not, then since $|R_i \cup R_j| \geq 2$ and since R_i and R_j are each connected subgraphs, there is an edge $e = u_i u_j$ with $u_i \in P'_{i,k} \subseteq R_i$ and $u_j \in P'_{j,l} \subseteq R_j$ for some k, l. See Figure 2 for a visualization.

Since for every edge e the ends of e belong to two consecutive levels, without loss of generality, $u_i \in L_m$ and $u_j \in L_{m-1}$ for some $m \in [r]$. Now, $z_j\text{-}P_{j,l}\text{-}u_j\text{-}u_i\text{-}P_{i,k}\text{-}y_{i,k}$ is a path of length r from z_j to $y_{i,k}$. Since $y_{i,k}$ is the private r-neighbor of z_i and z_k, it follows that $k = j$. Then u_i is a vertex in the intersection of a $(z_j, y_{i,j})$-path and $P_{i,j}$. Since $u_i \in P'_{i,j}$, $l(u_i) < l(s_{i,j})$, contradicting the minimality of $s_{i,j}$. This proves (8).

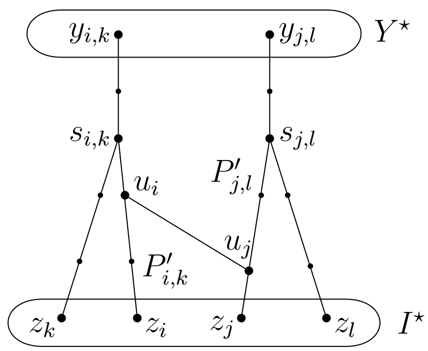

FIGURE 2. Visualization of the proof of (8).

(9) *If $k \notin \{i, j\}$, then for every l, $P_{k,l}$ is anticomplete to $s_{i,j}$.*

Suppose that $s_{i,j}$ is adjacent to a vertex $u \in P_{k,l}$. Then $|l(u) - l(s_{i,j})| = 1$. First, suppose that $s_{i,j} \in L_{m-1}$ and $u \in L_m$ for some $m \in [r]$. See the left-hand side of Figure 3. Then $z_i\text{-}P_{i,j}\text{-}s_{i,j}$ and $z_j\text{-}P_{j,i}\text{-}s_{i,j}$ are paths of length $m-1$ and $u\text{-}P_{k,l}\text{-}y_{k,l}$ is a path of length $r-m$. Since $s_{i,j}$ and u are adjacent, we have found a $(z_i, y_{k,l})$-path and a $(z_j, y_{k,l})$-path of length r. Since $y_{k,l}$ is the private r-neighbor of z_k and z_l, we conclude that $k \in \{i, j\}$.

Now suppose $s_{i,j} \in L_m$ and $u \in L_{m-1}$. See the right-hand side of Figure 3. Then $z_k\text{-}P_{k,l}\text{-}u$ is a path of length $m-1$ and $s_{i,j}\text{-}P_{i,j}\text{-}y_{i,j}$ is a path of length $r-m$. Since u and $s_{i,j}$ are adjacent, $z_k\text{-}P_{k,l}\text{-}u\text{-}s_{i,j}\text{-}P_{i,j}\text{-}y_{i,j}$ is a path of length r from z_k to $y_{i,j}$, which implies that $y_{i,j} \in e_{z_k}$. However, $y_{i,j}$ is the private r-neighbor of z_i and z_j, so we conclude that $k \in \{i, j\}$. This proves (9).

We now complete the proof of (7). Consider the graph Γ obtained by deleting every vertex that is not in $(\bigcup_{i \in [2t]} R_i) \cup (\bigcup_{i<j \in [2t]} s_{i,j})$ and contracting each R_i to a single vertex r_i. Then Γ is an induced minor of G. By (8), the set $\{r_i\}_{i \leq 2t}$ is stable. In addition, (9) implies that if $k \notin \{i, j\}$, then $s_{i,j}$ is not adjacent to any vertex in R_k. Therefore, in Γ, $s_{i,j}$ is adjacent to r_k if and only if $k \in \{i, j\}$. It also follows from (9) that if $\{k, l\} \neq \{i, j\}$, then $s_{i,j}$ and $s_{k,l}$ are not adjacent, so the set $\bigcup_{i<j \in [2t]} s_{i,j}$ is stable. Therefore, Γ is a 1-subdivision of a K_{2t},

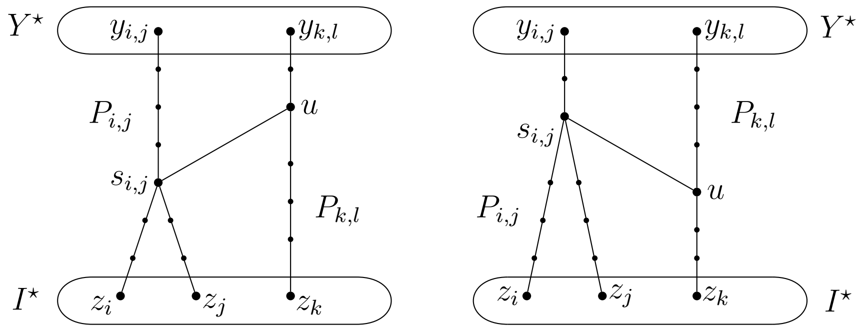

FIGURE 3. Visualization of the proof of (9).

which contains a $K_{t,t}$ induced minor, a contradiction. This completes the proof of (7).

By (6), $\nu(H) \leq \nu_D$ and by (7), $\lambda(H) \leq 2t$. Note that $\nu_D \leq \nu$. By Theorem 2.4, we have

$$\tau(H) \leq 11(2t)^2\,(2t+\nu_D+3)\binom{2t+\nu_D}{\nu_D}^2 \leq \tau.$$

Let $\mathcal{T} \subseteq Y^\star$ be a hitting set of minimum size. Since $|\mathcal{T}| \leq \tau$, there exists a vertex $x \in \mathcal{T}$ such that the set $\tilde{I} = \{z \in I^\star : x \in e_z\}$ has size $|\tilde{I}| \geq \frac{1}{\tau}|I^\star|$. For each $z \in \tilde{I}$, fix a (z,x)-path $Q_{z,x}$ of length r. Then set $\tilde{J}_D = \bigcup_{z\in\tilde{I}} (Q_{z,x} \setminus z)$.

Let $\tilde{G} = \left(G^\star \setminus \left(\tilde{J}_D \cup I^\star\right)\right) \cup \tilde{I}$. We now check that, after deleting $\tilde{J}_D$, every vertex in $\tilde{I}$ still reaches a sufficiently large proportion of $Y^\star$ via paths of length r, which will allow us to apply induction.

(10) *For every vertex $z \in \tilde{I}$, we have $|N^r_{\tilde{G}}(z) \cap Y^\star| \geq \frac{\alpha(Y)}{\nu_{D-1}}$.*

Let $z \in \tilde{I}$ and define $\tilde{Y}_z = (N^r_{G^\star}(z) \cap Y^\star) \setminus (N^{r-1}_{G^\star}(\tilde{J}_D \cap L_1) \cap Y^\star)$. We show that $\tilde{Y}_z \subseteq N^r_{\tilde{G}}(z) \cap Y^\star$. Let $y \in \tilde{Y}_z$, then there is a (z,y)-path $P_{z,y}$ in $G^\star$ of length r. We show that every such path is vertex-disjoint from $\tilde{J}_D$. Suppose, for contradiction, that there exists $u \in P_{z,y} \cap \tilde{J}_D$ with

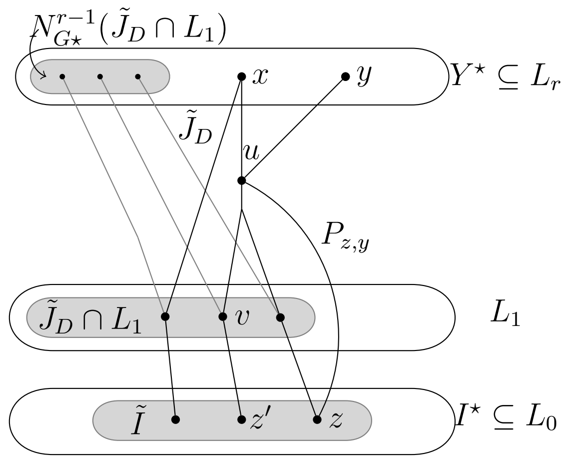

FIGURE 4. Visualization of the proof of (10).

$l(u) = m$ for some m. Since $u \in \tilde{J}_D$, there exist $z' \in \tilde{I}$ (not necessarily distinct from z), and a path $P_{z',x} \subseteq \tilde{J}_D$ such that $u \in P_{z',x}$. Let v denote the vertex in $P_{z',x} \cap L_1$. See Figure 4 for a visualization. Then v-$P_{z',x}$-u-$P_{z,y}$-y is a path of length $r-1$ from $v \in \tilde{J}_D \cap L_1$ to y. This implies $y \in N^{r-1}_{G^\star}(\tilde{J}_D \cap L_1) \cap Y^\star$, contrary to the assumption.

Therefore, the vertices in $\tilde{Y}_z$ can only be reached from z via paths of length r that are vertex-disjoint from $\tilde{J}_D$, which means that these paths are entirely contained in $\tilde{G}$. We conclude that $\tilde{Y}_z \subseteq N^r_{\tilde{G}}(z) \cap Y^\star$ and

$$\begin{aligned} |N^r_{\tilde{G}}(z) \cap Y^\star| &\geq |N^r_{G^\star}(z) \cap Y^\star| - |N^{r-1}_{G^\star}(\tilde{J}_D \cap L_1) \cap Y^\star| \\ &\geq \frac{\alpha(Y)}{\nu_D} - |I^\star| \cdot \frac{\alpha(Y)}{2\iota C(R+1)} \\ &\geq \frac{\alpha(Y)(D+1)}{2c^{r+1}C(R+1)} - \frac{\alpha(Y)}{2c^{r+1}C(R+1)} \\ &\geq \frac{\alpha(Y)}{\nu_{D-1}}, \end{aligned}$$

where the second inequality follows from (2) and the third inequality uses the fact $\frac{|I^\star|}{\iota} \leq c^{-(r+1)}$. This proves (10).

Observe that $\tilde{G}$ is an induced subgraph of $G^\star$ where $\tilde{G} \cap I' = \tilde{I}$ and $\tilde{G} \cap Y'' = Y^\star \setminus \{x\}$. By construction $|\tilde{I}| \geq \frac{1}{\tau}|I^\star| \geq t\tau^{D-1}$ and by (10) for each $z \in \tilde{I}$ we have $\alpha(N^r_{\tilde{G}}(z) \cap (Y^\star \setminus \{x\})) \geq \frac{\alpha(Y)}{\nu_{D-1}}$. Hence, by the inductive hypothesis, there exists a set $S \subseteq \tilde{I}$ and disjoint induced subgraphs $J_1, \dots, J_{D-1}$ of $\tilde{G}$ such that

- $|S| = t$,
- $|J_i| \leq rt$ for all $i \in [D-1]$,
- and $S \subseteq N_{G''}(J_i)$.

Recalling that $S \subseteq I^\star$, we let J_D be the subgraph of $\tilde{J}_D$ induced by $\bigcup_{z \in S} (Q_{z,x} \setminus \{z\})$. Now $|J_D| \leq rt$ and $S \subseteq N_{G''}(J_D)$. Moreover, for each $i \leq D-1$, J_i is vertex-disjoint from J_D because it is an induced subgraph of $\tilde{G}$. We conclude that the set S and $J_1, \dots, J_D$ satisfy the required properties and Claim 2.8 is proven. ■

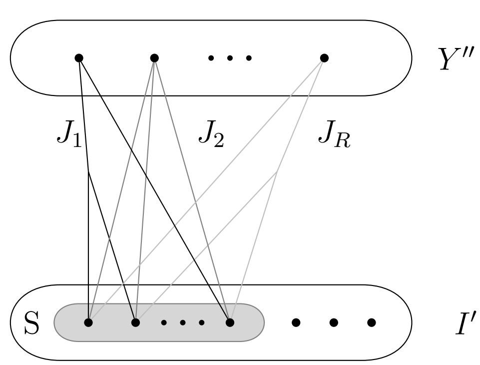

FIGURE 5. Visualization of the induced subgraphs $J_1, \dots, J_R$ given by Claim 2.8.

Recall from (4) that

$$|I'| \geq t\tau^R \quad \text{and} \quad \alpha(N^r_{G''}(z) \cap Y'') \geq \frac{\alpha(Y)(R+1)}{2c^{r+1}C(R+1)} \geq \frac{\alpha(Y)}{\nu_R} \text{ for every } z \in I'.$$

Therefore, we can apply Claim 2.8 to G'' to obtain a set $S \subseteq I'$ and disjoint induced subgraphs $J_1, \dots, J_R$ such that $|S| = t, |J_i| \leq rt$, and $S \subseteq N_{G''}(J_i)$ for all $i \in [R]$. See Figure 5 for a visualization. Let Λ be a graph with vertex set $J_1, \dots, J_R$, where two distinct vertices J_i and J_j are adjacent if and only if J_i and J_j are not anticomplete in G''.

(11) *Λ has no stable set of size t.*

For the sake of contradiction, suppose that Λ has a stable set of size t. Then, reindexing if necessary, there are t disjoint pairwise anticomplete induced subgraphs $J_1, \dots, J_t$ of G''. Let Γ be the graph obtained by deleting all vertices except $\left(\bigcup_{i=1}^t J_i\right) \cup S$ and contracting all the edges in each J_i to a vertex j_i. Since J_i and J_j are anticomplete for $i \neq j$, the set $\bigcup_{i \in [t]}\{j_i\}$ is stable, and since $S \subseteq N_{G''}(J_i)$, each j_i is complete to S. Therefore, Γ is an induced minor of G isomorphic to $K_{t,t}$, a contradiction. This proves (11).

By Theorem 2.3, it follows that Λ contains a clique of size K. Reindexing if necessary, there are K pairwise disjoint subsets $J_1, \dots, J_K$ in G'' such that there is at least one edge between any two sets. By Lemma 2.7, G'' contains a $K_{t,t}$ subgraph, which contradicts (5). This completes the proof of Theorem 2.2. ∎

Next, we provide an alternative formulation of Theorem 2.2 in terms of cardinalities rather than independence numbers.

Theorem 2.9. *There exists a function $M(C, s, t, r)$ that is polynomial in the parameters s and C such that the following holds: Let $r, s, t \in \mathbb{N}$, let $G \in \mathcal{C}_t$ with $\omega(G) < s$ and $C \geq 2$. For any $Y \subseteq V(G)$, there is a subset $X \subseteq V(G)$ with $|X| \leq M(C, s, t, r)$ such that if*

$$Z = \left\{ z \in V(G) \setminus X \, : |N^r_{G \setminus X}[z] \cap Y| \geq \frac{|Y|}{C} \right\}$$

then $\min(|Y|, |Z|) \leq M(C, s, t, r)$.

Proof. Let $M'(C, s, t, r)$ be the function given by Theorem 2.2, which is polynomial in the parameters s and C. By Lemma 2.6, there is a polynomial p_t such that $\chi(G) \leq p_t(s)$. Set $M(C, s, t, r) = p_t(s) \cdot M'(Cp_t(s), s, t, r)$, which is polynomial in s and C. We may assume that $|Y| > M(C, s, t, r)$, as otherwise there is nothing to show.

By Theorem 2.2, there is a set $X \subseteq V(G)$ with $\alpha(X) \leq M'(Cp_t(s), s, t, r)$ such that if we define

$$Z_\alpha = \left\{ z \in V(G) \setminus X \colon \alpha(N^r_{G \setminus X}[z] \cap Y) \geq \frac{\alpha(Y)}{p_t(s)C} \right\}$$

then $\min(\alpha(Z_\alpha), \alpha(Y)) \leq M'(Cp_t(s), s, t, r)$. Since $\alpha(Y) > M'(Cp_t(s), s, t, r)$, it follows that $\alpha(Z_\alpha) \leq M'(Cp_t(s), s, t, r)$.

Let

$$Z = \left\{ z \in V(G) \setminus X \colon |N^r_{G \setminus X}[z] \cap Y| \geq \frac{|Y|}{C} \right\}.$$

We show that X and Z satisfy the conclusion of Theorem 2.9. Let $z \in Z$ then, by Observation 2.5, it holds that

$$\chi(N^r_{G\setminus X}[z] \cap Y) \cdot \alpha(N^r_{G\setminus X}[z] \cap Y) \geq |N^r_{G\setminus X}[z] \cap Y| \geq \frac{|Y|}{C} \geq \frac{\alpha(Y)}{C}.$$

Since $N^r_{G\setminus X}[z] \cap Y$ is an induced subgraph of G, it has chromatic number at most $p_t(s)$. Rearranging the inequality yields

$$\alpha(N^r_{G\setminus X}[z] \cap Y) \geq \frac{\alpha(Y)}{p_t(s)C},$$

and we conclude that $Z \subseteq Z_\alpha$.

By Observation 2.5, we conclude

$$|Z| \leq |Z_\alpha| \leq \chi(Z_\alpha) \cdot \alpha(Z_\alpha) \leq p_t(s) \cdot M'(Cp_t(s), s, t, r) \leq M(C, s, t, r)$$

and

$$|X| \leq \chi(X) \cdot \alpha(X) \leq p_t(s) \cdot M'(Cp_t(s), s, t, r) \leq M(C, s, t, r),$$

as required. □

Finally, we restate and prove Theorem 1.6, which is another reformulation of Theorem 2.9.

Theorem 1.6. *For all $t, r \in \mathbb{N}$, there exists a polynomial $f = f_{t,r}$ such that the following holds. Let $C, s \in \mathbb{N}$ such that $C \geq 2$, let $G \in \mathcal{C}_t$ with $\omega(G) < s$, and let $Y \subseteq V(G)$. Then there is a subset $X \subseteq V(G)$ with $|X| \leq f(Cs)$ such that for every $v \in G \setminus X$ we have that*

$$|N^r_{G\setminus X}[v] \cap Y| \leq \frac{|Y|}{C}.$$

Proof. Let $f(Cs) = 2M(Cs, Cs, t, r)$ where M is defined as in Theorem 2.9, let X' be obtained by Theorem 2.9 and let Z be the corresponding set, and let $K \in \{Y, Z\}$ such that $|K| \leq M(C, s, t, r)$. Let $X = X' \cup K$; then $|X| \leq f(Cs)$. If $K = Y$ then $|N^r_{G\setminus X}[v] \cap Y| = 0$, otherwise, $K = Z$ and therefore $|N^r_{G\setminus X}[v] \cap Y| \leq \frac{|Y|}{C}$ for every $v \in G \setminus X$. □

3. Coarse Balanced Separators and Tree Decompositions

In this section we prove Theorem 1.5, which we restate for convenience.

Theorem 1.5. *For every $r, t \in \mathbb{N}$, there exists a polynomial $p = p_{r,t}$ such that the following holds. For every $k, s \in \mathbb{N}$, if $G \in \mathcal{C}_t$ such that $\omega(G) < s$ and G is (k, r)-separable then G admits an internally $(p(ks), r)$-coverable tree decomposition.*

The following lemma immediately implies Theorem 1.5 by setting $\hat{Y} = \emptyset$. Its proof is similar to that of Lemma 9 in [5].

Lemma 3.1. *Let $k, s, t, r \in \mathbb{N}$ and let $f = f_{2r,t}$ be the function given by Theorem 1.6. Let $G \in \mathcal{C}_t$ be (k, r)-separable such that $\omega(G) < s$. Then for every $\hat{Y} \subseteq V(G)$, where $|\hat{Y}| \leq 12f(8ks)$, there exists an internally $(24f(8ks), r)$-coverable tree decomposition of G with a bag containing $N^r_G[\hat{Y}]$.*

Proof. We proceed by induction on $|V(G)|$. Let $\hat{G}$ be a minimum r-cover of G. If G is $(24f(8ks), r)$-coverable, then the tree decomposition of G where $V(G)$ is in a single bag satisfies the statement. Therefore, we may assume that $|\hat{G}| > 24f(8ks)$. Note that if $|V(G)| \leq 24f(8ks)$, then we immediately have that G is $(24f(8ks), r)$-coverable, which proves the base case of the induction.

By adding vertices to $\hat{Y}$ if necessary, we may assume that $|\hat{Y}| = 12f(8ks)$. Note that $|\hat{G}| > \frac{|\hat{Y}|}{2}$. Let Z_G and Z_Y be the sets obtained by applying Theorem 1.6 to both $\hat{Y}$ and $\hat{G}$ with the parameters $C = 8k$ and $r' = 2r$. Let $Z = Z_G \cup Z_Y$, $G' = G \setminus Z$, $\hat{G}' = \hat{G} \setminus Z$, and $\hat{Y}' = \hat{Y} \setminus Z$. Then, $|Z| \leq 2f(8ks)$ and for every $v \in V(G')$, we have that

$$\left|N^{2r}_{G'}[v] \cap \hat{Y}'\right| \leq \frac{|\hat{Y}|}{8k} \quad \text{and} \quad \left|N^{2r}_{G'}[v] \cap \hat{G}'\right| \leq \frac{|\hat{G}|}{8k}.$$

Since G' is an induced subgraph of G, G' admits (k, r)-balanced separators. Therefore, by taking the union of two such separators, there exists $\hat{S}$ such that $|\hat{S}| \leq 2k$ and $S = N^r_{G'}[\hat{S}]$ is a balanced separator of both $\hat{G}'$ and $\hat{Y}'$ in G'. Let $\hat{Y}_S = \hat{Y}' \cap N^{2r}_{G'}[\hat{S}]$ and $\hat{G}_S = \hat{G}' \cap N^{2r}_{G'}[\hat{S}]$. Then

$$|\hat{Y}_S| \leq 2k \cdot \frac{|\hat{Y}|}{8k} = \frac{|\hat{Y}|}{4} \qquad \text{and} \qquad |\hat{G}_S| \leq 2k \cdot \frac{|\hat{G}|}{8k} = \frac{|\hat{G}|}{4}.$$

Let C be a connected component of $G' - S = G - (Z \cup S)$. Define

$$C^\star = G\big[C \cup S \cup Z \cup N^r_{G'}[\hat{Y}_S]\big] \qquad \text{and} \qquad \hat{Y}^\star = (C \cap \hat{Y}) \cup \hat{S} \cup Z \cup \hat{Y}_S.$$

Observe that $\hat{Y}^\star \subseteq V(C^\star)$. Next, we verify that $C^\star$ and $\hat{Y}^\star$ are suitable for the induction hypothesis.

(12) *It holds that $|V(C^\star)| < |V(G)|$.*

Since $\hat{G}$ was chosen to be a minimum r-cover of G, in order to prove (12), it suffices to prove that $C^\star$ can be covered by fewer than $|\hat{G}|$ balls of radius r. We check that balls centered on

$$\hat{C}^\star = (C \cap \hat{G}) \cup \hat{S} \cup Z \cup \hat{G}_S \cup \hat{Y}_S$$

have the required property. Since S is a balanced separator for $\hat{G}$, we have

$$|\hat{C}^\star| \leq |V(C) \cap \hat{G}| + |\hat{S}| + |Z| + |\hat{G}_S| + |\hat{Y}_S| \leq \frac{1}{2}|\hat{G}| + 2k + 2f(8ks) + \frac{|\hat{G}|}{4} + \frac{|\hat{Y}|}{4} < |\hat{G}|.$$

In the last inequality, we used that $f(x) \geq x$.

Now it remains to check that $V(C^\star) \subseteq N^r_G[\hat{C}^\star]$. See Figure 6 for a visualization. Suppose for a contradiction that $v \in C^\star$ but $v \notin N^r_G[\hat{C}^\star]$. If $v \in S \cup Z \cup N^r_{G'}[\hat{Y}_S]$, then evidently $v \in N^r_G[C^\star]$, and so we may assume $v \in C$. Since $G \subseteq N^r_G[\hat{G}]$, there exists $u \in \hat{G}$ such that $\mathrm{dist}_G(u, v) \leq r$. If $u \in C$, then we are done, so we may assume $u \notin C$. Since C is a component of $G - (Z \cup S)$, there is a (v, u)-path $P_{v,u}$ of length at most r intersecting $S \cup Z$. If there exists a vertex $x \in P_{v,u} \cap Z$, then $\mathrm{dist}(v, x) \leq r$ and so $v \in N^r_G[Z]$, which is a contradiction. Therefore, $P_{v,u} \cap S$ is non-empty. This implies that $\mathrm{dist}_{G'}(u, \hat{S}) \leq \mathrm{dist}_{G'}(u, S) + r \leq 2r$ and we deduce that $u \in \hat{G}_S$, and so $v \in N^r_G[\hat{G}_S]$. This contradiction proves (12).

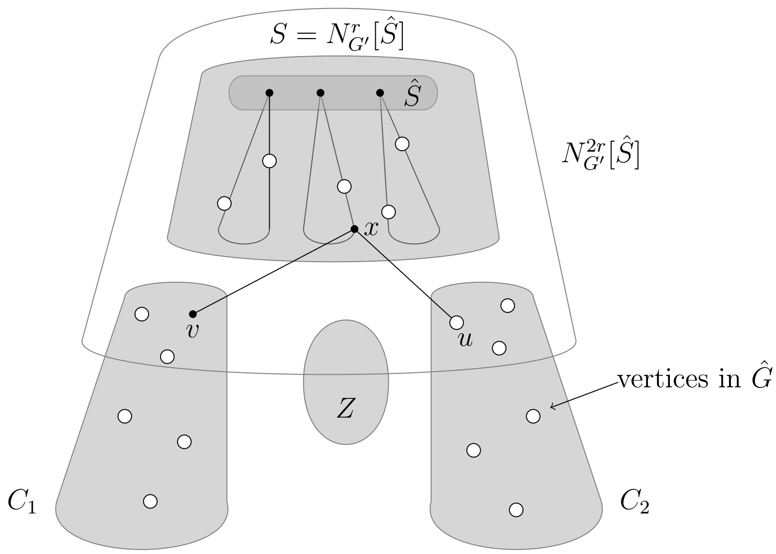

FIGURE 6. Visualization of the proof of (12). The vertices in $\hat{G}$ are distinguished by white fill and black outline.

Since S is a balanced separator for $\hat{Y}$, we can bound $|\hat{Y}^\star|$ as follows:

$$\begin{aligned}|\hat{Y}^\star| &\leq |V(C)\cap\hat{Y}| + |\hat{S}| + |Z| + |\hat{Y}_S|\\ &\leq \frac{1}{2}|\hat{Y}| + 2k + 2f(8ks) + \frac{|\hat{Y}|}{4}\\ &< \frac{3|\hat{Y}|}{4} + 3f(8ks) \leq 12f(8ks) = |\hat{Y}|.\end{aligned}$$

Now suppose that $C_1,\dots,C_m$ are the components of $G-(Z\cup S)$. By the inductive hypothesis, for every $i\in[m]$, the graph $C_i^\star$ admits a tree-decomposition $\mathcal{T}_i = (T_i,\beta_i)$ such that every bag is internally $(24f(8ks), r)$-coverable and there exists a bag $\beta_i(t_i)$ that contains $N^r_{C_i^\star}[\hat{Y}_i^\star]$.

Let T be obtained from the disjoint union of $T_1,\dots,T_m$ by adding a new vertex t_0 that is adjacent to $t_1,\dots,t_m$. Define $\beta(t)=\beta_i(t)$ for all $t\in T_i$ and $i\in[m]$, and set $\beta(t_0) = N^r_G[\hat{Y}]\cup S\cup Z$. Then $\mathcal{T}=(T,\beta)$ is a tree decomposition of G. To see this, note that for any $i<j\in[m]$ it holds that $C_i^\star\cap C_j^\star \subseteq N^r_G[\hat{Y}_S]\cup S\cup Z$, which is contained in the bag $\beta(t_0)$.

We observe that the set $N^r_G[\hat{Y}]\cup S\cup Z$ can be r-covered by the set $\hat{Y}\cup\hat{S}\cup Z$, which has at most $12f(8ks)+2k+4f(8ks)\leq 24f(8ks)$ vertices. Since the other bags do not change, every bag of $\mathcal{T}$ is internally $(24f(8ks), r)$-coverable and $\mathcal{T}$ has a bag containing $N^r_G[\hat{Y}]$, as required. □

4. ACKNOWLEDGMENTS

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