Square-free graphs with no induced fork

M. Chudnovsky* Shenwei Huang† T. Karthick‡
Jenny Kaufmann§

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Abstract

The *claw* is the graph $K_{1,3}$, and the *fork* is the graph obtained from the claw $K_{1,3}$ by subdividing one of its edges once. In this paper, we prove a structure theorem for the class of (claw, C_4)-free graphs that are not quasi-line graphs, and a structure theorem for the class of (fork, C_4)-free graphs that uses the class of (claw, C_4)-free graphs as a basic class. Finally, we show that every (fork, C_4)-free graph G satisfies $\chi(G) \leq \lceil \frac{3\omega(G)}{2} \rceil$ via these structure theorems with some additional work on coloring basic classes.

1 Introduction

All graphs in this work are finite and simple. For a positive integer n, K_n will denote the complete graph on n vertices, and P_n will denote the path on n vertices. For integers $n > 2$, C_n will denote the cycle on n vertices; the graph C_4 is called a *square*. For positive integers m, n, $K_{m,n}$ will denote the complete bipartite graph with classes of size m and n. The *claw* is the graph $K_{1,3}$, and the *fork* is the tree obtained from the claw $K_{1,3}$ by subdividing one of its edges once. A *clique* (*stable set* or an *independent set*) is a set of vertices that are pairwise adjacent (nonadjacent). The *clique number* $\omega(G)$ (*independence number* $\alpha(G)$) of a graph G is the size of a largest clique (stable set) in G. A *triad* is a stable set of size 3. A *k-vertex coloring* of a graph G is a function $c : V(G) \to \{1, 2, \ldots k\}$ such that for any adjacent vertices v and w, we have $c(v) \neq c(w)$. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number k such that G admits a k-vertex coloring. A graph is $(G_1, G_2, \ldots G_k)$*-free* if it does not contain any graph in $\{G_1, G_2, \ldots, G_k\}$ as an induced subgraph.

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†College of Computer Science, Nankai University, Tianjin 300350, China. This research is partially supported by the National Natural Science Foundation of China (11801284).

‡Computer Science Unit, Indian Statistical Institute, Chennai Centre, Chennai 600029, India. This research is partially supported by DST-SERB, Government of India under MATRICS scheme.

§Harvard University, Cambridge, MA 02138. This work was performed when the author was at Princeton University.

Clearly, for every graph G, we have $\chi(G) \geq \omega(G)$. In 1955, Mycielski constructed an infinite sequence of graphs G_n with $\omega(G_n) = 2$ and $\chi(G) = n$ for every n [9]. Thus, in general, there is no function of $\omega(G)$ that gives an upper bound for $\chi(G)$; however, there do exist such upper bounding functions for some restricted classes of graphs. To be precise, if $\mathcal{G}$ is a class of graphs, and there exists a function f (called *χ-binding function*) such that $\chi(G) \leq f(\omega(G))$ for all $G \in \mathcal{G}$, then we say that $\mathcal{G}$ is *χ-bounded*; and is *linearly χ-bounded*, if f is linear. The field of χ-boundedness is primarily concerned with determining which forbidden induced subgraphs $G_1, G_2, \ldots, G_k$ give χ-bounded classes, and finding the smallest χ-binding functions for these classes. It is known that if none of $G_1, G_2 \ldots G_k$ is acyclic, then the class of $(G_1, G_2, \ldots, G_k)$-free graphs is not χ-bounded [11]. Gyárfás [6] and Sumner [12] both independently conjectured that for every tree T, the class of T-free graphs is χ-bounded. Gyárfás [6] showed that the class of $K_{1,t}$-free graphs is χ-bounded and its smallest χ-binding function f satisfies $\frac{R(t,\omega+1)-1}{t-1} \leq f(\omega) \leq R(t, \omega)$, where $R(m, n)$ denotes the classical Ramsey number. A famous result of Kim [8] show that the Ramsey number $R(3, t)$ has order of magnitude $O(t^2/\log t)$. Thus for any claw-free graph G, we have $\chi(G) \leq O(\omega(G)^2/\log \omega(G))$. Further, it is known that there exists no linear χ-binding function for the class of claw-free graphs; see [11]. More precisely, for the class of claw-free graphs the smallest χ-binding function f satisfies $f(\omega) \in O(\omega^2/\log \omega)$. The first author and Seymour [4] studied the structure of claw-free graphs in detail, and they obtained the tight χ-bound for claw-free graphs containing a triad [5]. That is, if G is connected and claw-free with $\alpha(G) \geq 3$, then $\chi(G) \leq 2\omega(G)$.

The class of fork-free graphs generalizes the class of claw-free graphs. The class of fork-free graphs is comparatively less studied. Kierstead and Penrice showed that fork-free graphs are χ-bounded [7]. However, the best χ-binding function for fork-free graphs is not known, and an interesting question of Randerath and Schiermeyer [11] asks for the existence of a polynomial χ-binding function for the class of fork-free graphs. Randerath, in his thesis, obtained tight χ-bounds for several subclasses of fork-free graphs [10]. Here we are interested in linearly χ-bounded fork-free graphs. Recently the first author with Cook and Seymour [2] studied the structure of (fork, anti-fork)-free graphs and showed linear χ-binding function for such class of graphs. Since the class of $(3K_1, 2K_2)$-free graphs does not admit a linear χ-bounding function [1], if $\mathcal{G}$ is a linearly χ-bounded class of (fork, H)-free graphs with $|V(H)| = 4$, then $H \in \{P_4, C_4, K_4, K_4 - e, \overline{K_{1,3}}, \text{paw}\}$. When $H = P_4$, then every (fork, P_4)-free graph G is again P_4-free, and it is well known that every such G satisfies $\chi(G) = \omega(G)$; when $H \in \{K_4, K_4 - e, \text{paw}\}$, it follows from the results of [10] that every (fork, H)-free graph G satisfies $\chi(G) \leq \omega(G) + 1$, and from a result of [2] that every (fork, $\overline{K_{1,3}}$)-free graph G satisfies $\chi(G) \leq 2\omega(G)$. Thus the problem of obtaining a (best) linear χ-binding function for the class of (fork, C_4)-free graphs is open.

In this paper, we show that every (fork, C_4)-free graph G satisfies $\chi(G) \leq \lceil \frac{3\omega(G)}{2} \rceil$. To do this, we need to achieve in three major steps:

- First, we obtain a structure theorem for the class of (fork, C_4)-free graphs that uses the class of (claw, C_4)-free graphs as a basic class (Section 3).

- Next, we prove a new structure theorem for the class of (claw, C_4)-free graphs that are not quasi-line graphs (Section 4).
- Finally, we prove our $\lceil \frac{3\omega}{2} \rceil$-bound for the chromatic number via these structure theorems with additional work on coloring basic classes (Section 5).

2 Notations and Terminology

Given a vertex $v \in V(G)$, we say the *neighborhood* of v, $N_G(v)$, is the set of neighbors of v, and the *non-neighborhood* of v, $M_G(v)$, is the set of non-neighbors of v; we may write $N(v)$ and $M(v)$ when the relevant graph is unambiguous. We say $N[v] = N(v) \cup \{v\}$, $M[v] = M(v) \cup \{v\}$, and $deg(v) = |N(v)|$. If $S \subseteq V(G)$, then $N(S) = \cup_{v \in S} N(v) \setminus S$ and $M(S) = \cup_{v \in S} M(v) \setminus S$.

Given $S \subseteq V(G)$, we define $\alpha(S)$ to be $\alpha(G[S])$. A vertex v is *important* if for all $w \in V(G)$, $\alpha(N(v)) \geq \alpha(N(w))$. A vertex v in G is a *root of a claw* if v has neighbors a, b, c in G such that v, a, b, c induces a claw in G. A vertex v in a graph G is *good* if $deg(v) \leq \lceil \frac{3\omega(G)}{2} \rceil - 1$.

Given disjoint vertex sets S, T, we say that S is *complete* to T if every vertex in S is adjacent to every vertex in T; we say S is *anticomplete* to T if every vertex in S is nonadjacent to every vertex in T; and we say S is *mixed* on T if S is not complete or anticomplete to T. When $S = \{v\}$ is a single vertex, we can instead say that v is complete to, anticomplete to, or mixed on T. A vertex v is called *universal* if it is complete to $V(G) \setminus \{v\}$. A vertex set S is *homogeneous* if $1 < |S| < |V(G)|$ and for every $v \notin S$, v is complete to S or anticomplete to S. A *homogeneous clique* is a homogeneous set that induces a clique.

We say that disjoint vertex sets Y, Z are *matched* (*antimatched*) if each vertex in Y has a unique neighbor (non-neighbor) in Z and vice versa. Note that if Y and Z are matched or antimatched, then $|Y| = |Z|$.

A graph H is called a *thin candelabrum* (with base Z) if its vertices can be partitioned into non trivial disjoint sets Y, Z such that Y is a stable set, Z is a clique, and Y and Z are matched. Candelabra, which were introduced later by Chudnovsky, Cook, and Seymour in [2], are a generalization of thin candelabra. In this work we deal only with thin candelabra, and henceforth use "candelabrum" to mean "thin candelabrum." One can add a candelabrum to a graph G via the following procedure: Let H be a candelabrum with base Z. Take the disjoint union of G and H, then add edges to make Z complete to $V(G)$. We refer to this construction procedure as *candling* the graph G. We say that a graph G is *candled* if it can be constructed by candling some induced subgraph $G_0 \subseteq G$.

An *anticandelabrum* with base Z is the complement of a candelabrum with base Z. We say that a graph G is *anticandled* if $\overline{G}$ is candled. We will refer to the analogous construction procedure as *anticandling.* Anticandling can also be thought of as adding an anticandelabrum H with base Z to a graph, so that Z is anticomplete to the graph and $V(H) \setminus Z$ is complete to the graph.

A graph G is a *quasi-line graph* if for every vertex v, the set of neighbors of v can be expressed as the union of two cliques.

A *blowup* of a graph H is any graph G such that $V(G)$ can be partitioned into $|V(H)|$ (not necessarily non-empty) cliques Q_v, $v \in V(H)$, such that Q_u is complete to Q_v if $uv \in E(H)$, and Q_u is anticomplete to Q_v if $uv \notin E(H)$.

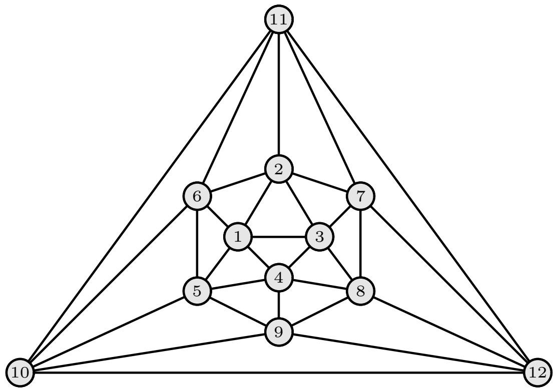

Figure 1: Icosahedron

The *icosahedron* is the unique planar graph with twelve vertices all of degree five. Consider the graph of the icosahedron, say I, with vertex labels as in Figure 1. A blowup of the icosahedron I is *special* if Q_i is non-empty for each $i \in \{1, \ldots, 6\}$.

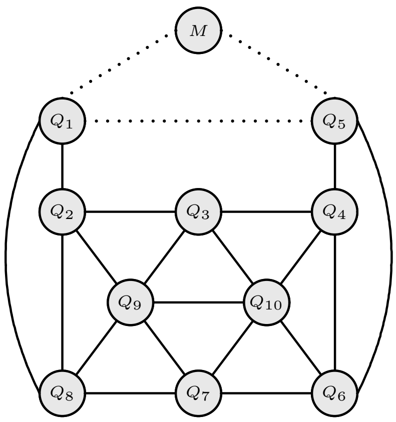

Figure 2: Schematic representation of a crown. Each circle represents a set. Each Q_i is a clique. A line between two sets means that the two sets are complete to each other, a dotted line between the two sets means that the edges between the two sets are arbitrary, and the absence of a line between two sets means that the two sets are anticomplete to each other.

We say that a connected (claw, C_4)-free graph G is a *crown* (see Figure 2), if $V(G)$ can be partitioned into eleven sets $Q_1, \ldots, Q_{10}$ and M such that the following hold.

- Each Q_i is a clique, and is non-empty for each $i \neq 5$. We write Q for the set $\cup_{i=1}^{10} Q_i$.

- The set M is anti-complete to $Q \setminus (Q_1 \cup Q_5)$.
- Q_1 is complete to $Q_2 \cup Q_8$ and is anticomplete to $Q \setminus (Q_2 \cup Q_5 \cup Q_8)$, Q_2 is complete to $Q_3 \cup Q_8 \cup Q_9$ and is anticomplete to $Q \setminus (Q_1 \cup Q_3 \cup Q_8 \cup Q_9)$, Q_3 is complete to $Q_4 \cup Q_9 \cup Q_{10}$ and is anticomplete to $Q \setminus (Q_2 \cup Q_4 \cup Q_9 \cup Q_{10})$, Q_4 is complete to $Q_5 \cup Q_6 \cup Q_{10}$ and is anticomplete to $Q \setminus (Q_3 \cup Q_5 \cup Q_6 \cup Q_{10})$, Q_5 is complete to Q_6 and is anticomplete to $Q \setminus (Q_1 \cup Q_4 \cup Q_6)$, Q_6 is complete to $Q_7 \cup Q_{10}$ and is anticomplete to $Q \setminus (Q_4 \cup Q_5 \cup Q_7 \cup Q_{10})$, Q_7 is complete to $Q_8 \cup Q_9 \cup Q_{10}$ and is anticomplete to $Q \setminus (Q_6 \cup Q_8 \cup Q_9 \cup Q_{10})$, Q_8 is complete to Q_9 and is anticomplete to $Q \setminus (Q_1 \cup Q_2 \cup Q_7 \cup Q_9)$, Q_9 is complete to Q_{10} and is anticomplete to $Q \setminus (Q_2 \cup Q_3 \cup Q_7 \cup Q_8 \cup Q_{10})$, and Q_{10} is anticomplete to $Q \setminus (Q_3 \cup Q_4 \cup Q_6 \cup Q_7 \cup Q_9)$.
- Adjacency between Q_1 and Q_5, and between $Q_1 \cup Q_5$ and M are not specified, but they are restricted by the fact that G is (claw, C_4)- free.

3 Structure of (fork, C_4)-free graphs

In this section, we obtain a structure theorem for the class of (fork, C_4)-free graphs that uses the class of (claw, C_4)-free graphs as a basic class.

Theorem 1 *Let G be a (fork, C_4)-free graph. Then at least one of the following hold:*

- *G is not connected.*
- *G contains a universal vertex.*
- *G contains a homogeneous clique.*
- *G is candled or anticandled.*
- *G is claw-free.*

Proof. Let G be a (fork, C_4)-free graph. Suppose that G is a connected graph which has no universal vertex, no homogeneous clique, and that G contains a claw. We show that G is either candled or anticandled. Let $v \in V(G)$ be an important vertex. Then since G is not claw-free, there is some claw rooted at v. Let $L(v) \subseteq N(v)$ be the leaves of claws rooted at v and let Q denote the set $N(v) \setminus L(v)$. So if S is a maximum stable set in $N(v)$, then $S \subseteq L(v)$. Let $M(v) := V(G) \setminus (N(v) \cup \{v\})$. Since v is not a universal vertex, $M(v)$ is not empty. Then we have the following:

(1) $L(v)$ is anticomplete to $M(v)$.

Proof of (1): Suppose $x \in M(v)$ has a neighbor a in a triad $\{a, b, c\} \subseteq L(v)$. Then since $\{v, a, b, c, x\}$ does not induce a C_4, x is not adjacent to b and c. But then $\{v, a, b, c, x\}$ induces a fork, a contradiction. So (1) holds. $\Diamond$

Let $Q_1(v)$ be the maximal subset of Q that is anticomplete to $M(v)$, and let $Q_2(v) := N(M(v)) \cap Q = Q \setminus Q_1(v)$.

(2) If $t \in Q$ is complete to $L(v)$, then $t \in Q_1(v)$.

Proof of (2): Suppose $t \in Q$ is complete to $L(v)$. If t has a neighbor $x \in M(v)$, then, by (1), $\alpha(G[N(t)]) > \alpha(G[N(v)])$, a contradiction to the fact that v is an important vertex. So (2) holds. $\Diamond$

(3) $Q_2(v)$ is a clique, and $Q_1(v)$ is complete to $Q_2(v)$.

Proof of (3): Suppose to the contrary that there are non adjacent vertices $t \in Q_2(v)$ and $t' \in Q_1(v) \cup Q_2(v)$. Let $x \in M(v)$ be a neighbor t. Then since $\{v, t, x, t'\}$ does not induce a C_4, t' is not adjacent to x. By (2), t has a non-neighbor $a \in L(v)$. By (1), a is not adjacent to x. Then since $\{x, t, v, t', a\}$ does not induce a fork, t' is adjacent to a. Let $b, c \in L(v)$ be such that $\{v, a, b, c\}$ induces a claw. Again by (1), x is anticomplete to $\{b, c\}$. Now since $t, t' \notin L(v)$, we see that t and t' are each adjacent to at least two vertices in $\{a, b, c\}$. Thus t is adjacent to b and c, and we may assume that t' is adjacent to b. Then since $\{t, b, t', c\}$ does not induce a C_4, t' is not adjacent to c. But then $\{t', b, t, c, x\}$ induces a fork which is a contradiction. So (3) holds. $\Diamond$

(4) Q is a clique.

Proof of (4): By (3), it is enough to show that $Q_1(v)$ is a clique. Suppose to the contrary that there are non adjacent vertices $Q_1(v)$, say t and t'. Since $M(v) \neq \emptyset$ and since G is connected, there exists a vertex $x \in M(v)$ which has a neighbor $w \in Q_2(v)$. By (3), w is complete to $\{t, t'\}$, and by the definition of $Q_1(v)$, x is anticomplete to $\{t, t'\}$. By (2), w has a non-neighbor $a \in L(v)$. Let $b, c \in L(v)$ be such that $\{v, a, b, c\}$ induces a claw. Then by (1), x is anticomplete to $\{a, b, c\}$. Now since $\{a, w, t, t', x\}$ does not induce a fork or a C_4, we see that a is anticomplete to $\{t, t'\}$. Then since $t, t' \notin L(v)$, we have $\{t, t'\}$ is complete to $\{b, c\}$. But then $\{t, b, t', c\}$ induces a C_4 which is a contradiction. So (4) holds. $\Diamond$

(5) If C is a connected component of $M(v)$, every $t \in N(v)$ is complete or anticomplete to C. In particular, C is a homogeneous set or a singleton.

Proof of (5): Suppose not. Then since G is connected, we may assume that there are adjacent vertices $x, y \in V(C)$, and there exists a vertex $t \in N(v)$ which is adjacent to x and not adjacent to y. By (1) and by our definition of $Q_1(v)$, $t \notin L(v) \cup Q_1(v)$. So $t \in Q_2(v)$. Then since G does not contain a fork, t is adjacent to at least two vertices in any given triad $\{a, b, c\} \subseteq L(v)$; we may assume $a, b \in N(t)$. Then $\{y, x, t, a, b\}$ induces a fork, which is a contradiction. $\Diamond$

(6) If C is a connected component of $M(v)$, then $V(C)$ is a clique.

Proof of (6): Since G is connected, there is some $t \in N(V(C))$. As in (5), $t \in Q_2(v)$. So, by (2), t has a non-neighbor $a \in L(v)$. Now if there are non adjacent vertices x and y in $V(C)$, then, by (5), we see that $\{a, v, t, x, y\}$ induces a fork. So any two vertices in $V(C)$ are adjacent, and hence $V(C)$ is a clique. $\Diamond$

(7) $M(v)$ is a stable set.

Proof of (7): Since G has no homogeneous cliques, the proof follows by (5) and (6). $\Diamond$

(8) Each vertex in $Q_2(v)$ has at most one neighbor in $M(v)$.

Proof of (8): Suppose to the contrary that $t \in Q_2(v)$ has two neighbors in C, say x and y. Then by (7), x and y are not adjacent. Since $t \in Q_2(v)$, by (2), t has a non-neighbor $a \in L(v)$. But then $\{a, v, t, x, y\}$ induces a fork which is a contradiction. So (8) holds. $\diamondsuit$

(9) Every vertex in Q has a non-neighbor in $L(v)$.

Proof of (9): Suppose to the contrary that there exists a vertex $t \in Q$ which is complete to $L(v)$. Then by (2), $t \in Q_1(v)$. But then by (3) and (4), and by the definition of $Q_1(v)$, $\{v, t\}$ is a homogeneous clique in G, a contradiction to our assumption that G has no homogeneous cliques. So (9) holds. $\diamondsuit$

We now prove the theorem in two cases. Suppose that $|M(v)| > 1$. Then we have the following.

Claim 1.1 *Any $a \in L(v)$ is either complete to $Q_2(v)$ or anticomplete to $Q_2(v)$.*

Proof of Claim 1.1. Suppose to the contrary that there exists a vertex $a \in L(v)$ which is mixed on $Q_2(v)$. Then by using (3), there are adjacent vertices t and t' in $Q_2(v)$ such that a is adjacent to t and a is not adjacent to t'. Let $x \in M(v)$ be a neighbor of t and let $x' \in M(v)$ be a neighbor of t'. If $x \neq x'$, then by using (7) and (8), we see that $\{x, t, t', x', a\}$ induces a fork. So we may assume that $x = x'$. Then since $|M(v)| > 1$, there exists a vertex $y \in M(v)$ (which is distinct from x and x'), and so there exists a vertex $t'' \in Q_2(v)$ which is adjacent to y. Then by using (7), (8) and (3), we see that either $\{x, t, t'', y, a\}$ or $\{x, t', t'', y, a\}$ induces a fork which is a contradiction. This proves Claim 1.1. $\diamondsuit$

By Claim 1.1, we partition $L(v)$ into two sets as follows: Let $L_1(v)$ denote the set $\{a \in L(v) \mid a \text{ is complete to } Q_2(v)\}$ and let $L_0(v)$ denote the set $L(v) \setminus L_1(v) := \{a \in L(v) \mid a \text{ is anticomplete to } Q_2(v)\}$. Then by (9), $L_0(v) \neq \emptyset$. Fix a vertex $x \in M(v)$, and let $t \in Q_2(v)$ be a neighbor of x. Then we have the following.

Claim 1.2 *$L_0(v)$ is anticomplete to $L_1(v)$.*

Proof of Claim 1.2. Suppose to the contrary that there are adjacent vertices $c \in L_1(v)$ and $d \in L_0(v)$. Then by definitions of $L_0(v)$ and $L_1(v)$, we have c is adjacent to t, and d is not adjacent to t. Let $\{a, b\} \subset L(v)$ be such that $\{a, b, c\}$ is a triad in $L(v)$. Then by (1), x is anticomplete to $\{a, b, c, d\}$. Then since $\{x, t, v, a, b\}$ does not a fork, we may assume that t is adjacent to a. Then since $\{a, t, c, d\}$ does not induce a C_4, a is not adjacent to d. But then $\{d, c, t, x, a\}$ induces a fork which is a contradiction. This proves Claim 1.2. $\diamondsuit$

Claim 1.3 *$L_0(v)$ is a clique.*

Proof of Claim 1.3. If there are non-adjacent vertices a and b in $L_0(v)$, then $\{a, v, t, x, b\}$ induces a fork. $\diamondsuit$

Consider a maximum stable set $S \subseteq N(v)$; then $S \subseteq L(v)$. We have $|S \cap L_0(v)| = 1$, because $L_0(v)$ is a clique component of $L(v)$ (by Claim 1.3). So

$|S \cap L_1(v)| = |S| - 1$. A maximum stable set in $N(t)$ is $(S \cap L_1(v)) \cup \{x\}$, which has size $|S| = \alpha(N(v))$. Therefore, $\alpha(N(t)) = \alpha(N(v))$, so t is also an important vertex. So $M(t)$ is a stable set, by (7). Since $L_0(v)$ is a nonempty component of $M(t)$, it is a singleton, say $L_0(v) := \{l\}$. Then we have the following claim.

Claim 1.4 *$L_0(v) = \{l\}$ is anticomplete to $Q_1(v)$.*

Proof of Claim 1.4. Suppose that there exists a vertex $q \in Q_1(v)$ which is adjacent to l. Then by (4), t and q are adjacent, and by the definition of $L_0(v)$, l and t are not adjacent. Now by (9), q has a non-neighbor, say $a \in L(v)$. Then $a \in L_1(v)$, and hence a is adjacent to t. Also by Claim 1.2 and (1), a is anticomplete to $\{l, x\}$. But then $\{l, q, t, x, a\}$ induces a fork. ♢

Claim 1.5 *No two vertices in $Q_2(v)$ share a common neighbor in $M(v)$.*

Proof of Claim 1.5. Suppose that there are vertices t' and t'' in $Q_2(v)$ which have a common neighbor $x' \in M(v)$. Then by above observations, since $\{t', t''\}$ is complete to $(Q \setminus \{t', t''\}) \cup L_1(v) \cup \{v\}$, and is anticomplete to $L_0(v) \cup (M(v) \cup \{x'\})$, we see that $\{t', t''\}$ is a homogenous clique, a contradiction to our assumption that G has no homogenous cliques. ♢

Now we partition the vertex set of G as follows: Let Z' denote the set $\{v\} \cup (N(M(v)) \cap Q_2(v))$, let Z'' denote the set $(\{v\} \cup Q_2(v)) \setminus Z'$, and let $V(G_0)$ denote the set $Q_1(v) \cup Z'' \cup L_1(v)$. Then by above observations, we see that Z' is a clique, $M(v) \cup \{l\}$ is a stable set, Z' and $M(v) \cup \{l\}$ are matched, $V(G_0)$ is complete to Z', and $V(G_0)$ is anticomplete to $M(v) \cup \{l\}$. Thus we conclude that G is candled.

So we may assume that every important vertex in G has exactly one non-neighbor. In this case, we claim that G is anticandled. Let $Y = Q_2(v) \cup \{v\}$. Then by (3), Y is a clique. Let m be the unique vertex in $M(v)$. Then there exists a vertex $t \in Q_2(v)$ such that t is adjacent to m. If S is a maximum stable set in $N(v)$, then by (1), $S \cup \{m\}$ is a stable set of size $\alpha(N(v)) + 1$. Since $t \notin L(v)$, t is adjacent to at least $|S| - 1$ of the vertices in S, so $\alpha(N(t)) = |S| = \alpha(N(v))$. So every vertex $t \in Q_2(v)$ is important and hence by assumption has a unique non-neighbor.

Since $\{t, t'\}$ are not a homogeneous clique, for any $t' \in Q_2(v)$, they do not share a non-neighbor. Therefore, each vertex in $M(Y)$ has a distinct non-neighbor in Y, so in particular $M(Y)$ and Y are antimatched.

Consider distinct $m, m' \in M(Y)$ with respective non-neighbors $t, t' \in Y$. Then since $\{m, m', t, t'\}$ does not induce a C_4, m and m' are nonadjacent. Thus $M(Y)$ is stable.

Suppose $m \in M(Y)$ has a neighbor u. Let $t \in Y$ be a non-neighbor of m; then since u is adjacent to the unique non-neighbor of t, we have by (1) that $u \in Q_2(t)$. Then u is important, so by assumption u has a unique non-neighbor. Thus $u \notin L(v)$, since it is not part of a triad. Moreover, by (9), every vertex in $Q_1(v)$ has a non-neighbor in $L(v)$, so has at least two-non-neighbors. Then $u \notin Q_1(v)$. Thus, $u \in Y$. So $M(Y)$ is anticomplete to $V(G) \setminus (Y \cup M(Y))$.

Hence we conclude that $Y \cup M(Y)$ induces an anticandelabrum with base $M(Y)$, with $G \setminus (Y \cup M(Y))$ complete to Y and anticomplete to $M(Y)$.

This completes the proof of the theorem. □

Corollary 1 *Let G be a connected (fork, C_4)-free graph. Then G is claw-free or G has a universal vertex or G has a clique cutset.*

Proof. Let G be a (fork, C_4)-free graph. Suppose that G has no universal vertex, and no clique cutset. We show that G is claw-free. Suppose to the contrary that G contains a claw. Let $v \in V(G)$ be an important vertex. Let $L(v) \subseteq N(v)$ be the leaves of claws rooted at v and let Q denote the set $N(v) \setminus L(v)$. So if S is a maximum stable set in $N(v)$, then $S \subseteq L(v)$. Let $M(v) := V(G) \setminus (N(v) \cup \{v\})$. Since v is not a universal vertex, $M(v)$ is not empty. Let Q_1 be the maximal subset of Q that is anticomplete to $M(v)$, and let $Q_2 := N(M(v)) \cap Q = Q \setminus Q_1$. Then it follows from Theorem 1 (See item (3) and note that items (1)–(3) hold regardless of whether G has a homogeneous-clique or not.) that Q_2 is a clique. But then we see that Q_2 is a clique cutset separating $\{v\}$ and $M(v)$ which is a contradiction. This completes the proof. □

4 Structure of (claw, C_4)-free graphs

In this section, we obtain a structure theorem for the class of (claw, C_4)-free graphs that is not a quasi-line graph.

Theorem 2 *Let G be a connected (claw, C_4)-free graph. Then at least one of the following hold:*

- *G is a quasi-line graph.*
- *G is a special blowup of the Icosahedron graph.*
- *G has a clique cutset.*
- *G has a good vertex.*
- *G is a crown.*

Proof. Let G be a connected (claw, C_4)-free graph. We may assume that G has no clique cutset. Let $v \in V(G)$. First suppose that $G[N(v)]$ is chordal. Then since G is claw-free, $G[N(v)]$ is a triad-free chordal graph. Since the complement graph of a triad-free chordal graph is a bipartite graph, we see that $N(v)$ can be expressed as the union of two cliques. Since v is arbitrary, G is a quasi-line graph. So we may assume that $G[N(v)]$ is not chordal and hence $G[N(v)]$ contains an induced C_k for some $k \geq 5$. Since $\alpha(C_k) \geq 3$ for $k \geq 6$, and since G is (claw, C_4)-free, we have $k = 5$. That is, $G[N(v)]$ contains an induced C_5, say $C := v_1\text{-}v_2\text{-}v_3\text{-}v_4\text{-}v_5\text{-}v_1$. Let $T = \{x \in V(G) \setminus V(C) \mid N(x) \cap V(C) = V(C)\}$, let $R = \{x \in V(G) \setminus V(C) \mid N(x) \cap V(C) = \emptyset\}$, and for each $i \in \{1, 2, \ldots, 5\}$, i mod 5, let:

$$\begin{aligned} A_i &= \{x \in V(G) \setminus V(C) \mid N(x) \cap V(C) = \{v_i, v_{i+1}\}\}, \\ B_i &= \{x \in V(G) \setminus V(C) \mid N(x) \cap V(C) = \{v_{i-1}, v_i, v_{i+1}\}\} \cup \{v_i\}. \end{aligned}$$

Let $A := A_1 \cup \cdots \cup A_5$ and $B := B_1 \cup \cdots \cup B_5$. Note that $v \in T$, and so $T \neq \emptyset$. Then the following properties hold for each $i \in \{1, 2, \ldots, 5\}$, i mod 5:

(1) $V(G) = A \cup B \cup T \cup R$.

Proof of (1): Suppose that there exists a vertex $x \in V(G) \setminus (A \cup B \cup T \cup R)$. Then for some i, either $N(x) \cap V(C) = \{v_i\}$ or $\{v_{i-1}, v_{i+1}\} \subseteq N(x) \cap V(C)$ with $v_i \notin N(x)$. But then $\{v_{i-1}, v_i, v_{i+1}, x\}$ induces either a claw or a C_4. $\diamondsuit$

(2) A_i and $B_i \cup T$ are cliques.

Proof of (2): Let $i = 1$ and suppose that there are non adjacent vertices x and y in one of the listed sets. If $x, y \in A_1$, then $\{x, y, v_1, v_5\}$ induces a claw, and if $x, y \in B_1 \cup T$, then $\{x, v_5, y, v_2\}$ induces a C_4. $\diamondsuit$

(3) A_i is anticomplete to T.

Proof of (3): If $a \in A_i$ and $t \in T$ are adjacent, then $\{v_{i-1}, t, v_{i+2}, a\}$ induces a claw. $\diamondsuit$

(4) A_i is complete to $A_{i-1} \cup A_{i+1} \cup B_i \cup B_{i+1}$.

Proof of (4): By symmetry, it suffices to show that A_i is complete to $A_{i+1} \cup B_{i+1}$. Suppose that there are non adjacent vertices $x \in A_i$ and $y \in A_{i+1} \cup B_{i+1}$. If $y \in A_{i+1}$, then $\{x, y\}$ is anti-complete to v (by (3)), and then $\{v, x, y, v_{i+1}\}$ induces a claw. So $y \in B_{i+1}$. Then $\{v_{i-1}, x, y, v_i\}$ induces a claw. $\diamondsuit$

(5) A_i is anticomplete to $A_{i+2} \cup A_{i-2} \cup B_{i+2} \cup B_{i-1} \cup B_{i-2}$.

Proof of (5): By symmetry, it suffices to show that A_i is anti-complete to $A_{i+2} \cup B_{i+2} \cup B_{i-2}$. Suppose that there are adjacent vertices $x \in A_i$ and $y \in A_{i+2} \cup B_{i+2} \cup B_{i-2}$. If $y \in A_{i+2} \cup B_{i-2}$, then $\{x, v_{i+1}, v_{i+2}, y\}$ induces a C_4. So $y \in B_{i+2}$. Now since x is not adjacent to v (by (3)), and y is adjacent to v (by (2)), we see that $\{x, v_i, v, y\}$ induces a C_4. $\diamondsuit$

(6) B_i is complete to $B_{i+1} \cup B_{i-1}$.

Proof of (6): By symmetry, it suffices to show that B_i is complete to B_{i+1}. If there are nonadjacent vertices $x \in B_i$ and $y \in B_{i+1}$, then $\{x, y\}$ is complete to v (by (2)), and then $\{v_{i-2}, v, x, y\}$ induces a claw. $\diamondsuit$

(7) B_i is anticomplete to $B_{i+2} \cup B_{i-2}$.

Proof of (7): If there are adjacent vertices $x \in B_i$ and $y \in B_{i+2} \cup B_{i-2}$, then either $\{x, v_{i-1}, v_{i-2}, y\}$ or $\{x, v_{i+1}, v_{i+2}, y\}$ induces a C_4. $\diamondsuit$

(8) If $r \in R$, then $N(r) \cap (B \cup T) = \emptyset$.

Proof of (8): If there is a vertex $x \in N(r) \cap (B \cup T)$, then for some i, $\{v_{i-1}, v_{i+1}\} \subset N(x) \cap V(C)$, and then $\{v_{i-1}, v_{i+1}, x, r\}$ induces a claw. $\diamondsuit$

(9) Any $r \in R$ which has a neighbor in A_i is complete to $A_{i+1} \cup A_{i-1}$.

Proof of (9): Let $r \in R$ be such that r has a neighbor $a \in A_i$. If r is not adjacent to a vertex $b \in A_{i+1} \cup A_{i-1}$, then since a is adjacent to b (by (4)), we see that either $\{r, a, v_i, b\}$ or $\{r, a, v_{i+1}, b\}$ induces a claw. $\diamondsuit$

(10) If A_i and A_{i+1} are not empty, for some i, then any $r \in R$ which has a neighbor in $A_i \cup A_{i+1} \cup A_{i-1}$ is complete to $A_i \cup A_{i+1} \cup A_{i-1}$.

Proof of (10): This follows by (4) and (9). $\Diamond$

First suppose that R is empty. Then since $B_i \cup T \neq \emptyset$, for each i, we set for each $i \in \{1, \ldots, 5\}$, $Q_i := B_i$, and $Q_6 = T$. So by above properties we see that G is a special blowup of the icosahedron graph (shown in Figure 1).

So we may assume that $R \neq \emptyset$. Then by (8), $A \neq \emptyset$. Since G has no clique cutset, using (10), we may assume that there exists an index i such that A_i and A_{i+2} are not empty, say $i = 1$. Now if $A_2 \neq \emptyset$, then by (10) and since G is claw-free, any $r \in R$ is complete to A. Moreover, since G is C_4-free, R is a clique. So G is a special blowup of the icosahedron graph. So we may assume that $A_2 = \emptyset$.

Next suppose that $A_4 \cup A_5 = \emptyset$. In this case, we show that one of the vertices v_2 or v_5 is good. Suppose not. Then since $T \cup B_1 \cup B_5 \cup \{v_1, v_5\}$ and $T \cup B_4 \cup B_5 \cup \{v_4, v_5\}$ are cliques, we see that $|B_1 \cup \{v_1\}| > \frac{\omega(G)}{2}$ and $|B_4 \cup \{v_4\}| > \frac{\omega(G)}{2}$. Since v_2 is not a good vertex and since $A_1 \cup B_1 \cup B_2 \cup \{v_1, v_2\}$ is a clique, we have $|T \cup B_3 \cup \{v_3\}| > \frac{\omega(G)}{2}$. Then we see that $T \cup B_3 \cup B_4 \cup \{v_3, v_4\}$ is a clique of size $> \omega(G)$ which is a contradiction. Thus one of the vertices v_2 or v_5 is good.

So we may assume that $A_5 \neq \emptyset$ and $A_4 = \emptyset$. Let R' be the set $\{r \in R \mid r \text{ has a neighbor in } A_1 \cup A_5\}$, and let R'' be the set $R \setminus R'$. Then by (10), R' is complete to $A_1 \cup A_5$. Also if there are nonadjacent vertices $r_1, r_2 \in R'$, then for any $a \in A_1$, $\{r_1, r_2, v_2, a\}$ induces a claw, and so R' is a clique. Now by above properties we see that G is a crown, where we set $B_1 := Q_{10}$, $B_2 := Q_7$, $B_3 := Q_8$, $B_4 := Q_2$, $B_5 := Q_3$, $T := Q_9$, $A_1 := Q_6$, $A_3 := Q_1$, $A_5 := Q_4$, $R' := Q_5$ and $R'' := M$.

This completes the proof of the theorem. $\Box$

5 Coloring (claw/fork, C_4)-free graphs

In this section, we show that every (fork, C_4)-free graph satisfies $\chi(G) \leq \lceil \frac{3\omega(G)}{2} \rceil$.

Theorem 3 ([3]) *If G is a quasi-line graph, then $\chi(G) \leq \lceil \frac{3}{2}\omega(G) \rceil$.*

Lemma 1 *Let I be the icosahedron graph with vertex labels as in Figure 1. Let G be a special blowup of the icosahedron I. Then $\chi(G) \leq \lceil \frac{4\omega(G)}{3} \rceil$.*

Proof. We prove by induction on $|V(G)|$, and let $q = \omega(G)$. Obviously the desired result holds if G is any induced subgraph of I. Let X be a subset of $V(G)$ obtained by taking $\min\{1, |Q_i|\}$ vertices from Q_i for each $i \in \{1, \ldots, 12\}$. Since Q_i is non-empty for each $i \in \{1, \ldots, 6\}$, we see that $\chi(G[X]) = 4$ and $\omega(G \setminus X) \leq q-3$. By the induction hypothesis we have $\chi(G \setminus X) \leq \lceil \frac{4}{3}\omega(G \setminus X) \rceil \leq \lceil \frac{4}{3}(q-3) \rceil = \lceil \frac{4}{3}q - 4 \rceil \leq \lceil \frac{4}{3}q \rceil - 4$. Since $\chi(G) \leq \chi(G \setminus X) + \chi(G[X])$, we have $\chi(G) \leq \lceil \frac{4\omega(G)}{3} \rceil$. $\Box$

Given a graph G and a set of colors C, a *list assignment* is a function $L : V \to \mathcal{P}(C)$, where $\mathcal{P}(C)$ is the power set of C. For a vertex v in G, the set $L(v)$ is called the *list*, and represents the set of possible colors for v. Given a graph G and a list assignment L, a *list coloring* (or *L-coloring*) is a vertex coloring c of G such that $c(v) \in L(v)$, for each $v \in V(G)$. We say that a graph G is L-colorable if G admits a L-coloring for some list assignment L. We say that a graph F with list assignment L is *L-degenerate* if there exists a vertex ordering $v_1, \dots, v_n$ of $V(F)$ such that each v_i has at most $|L(v)| - 1$ neighbors in $\{v_1, \dots, v_{i-1}\}$ for $i \in \{1, 2, \dots, n\}$. Clearly, if a graph is L-degenerate, then it is L-colorable.

Lemma 2 *Let G be a graph with the structure described in Figure 2 except that $M = \emptyset$ and Q_1 and Q_5 are possibly empty. Let L be a list assignment of G such that*

(1) For every $v \in V(G - (Q_1 \cup Q_5))$, $L(v) \subseteq \{1, 2, \dots, \lceil \frac{3}{2}\omega(G) \rceil\}$.

(2) For every vertex $v \in Q_1 \cup Q_5$, $|L(v)| = 1$.

(3) Every two vertices in $Q_2 \cup Q_8$ have the same list L_{28} such that $|L_{28}| \geq |Q_2| + |Q_8| + \lceil \frac{1}{2}\omega(G) \rceil$ and L_{28} is disjoint from the list of every vertex in Q_1.

Every two vertices in $Q_4 \cup Q_6$ have the same list L_{46} such that $|L_{46}| \geq |Q_4| + |Q_6| + \lceil \frac{1}{2}\omega(G) \rceil$ and L_{46} is disjoint from the list of every vertex in Q_5.

(4) For every $v \in Q_3 \cup Q_7 \cup Q_9 \cup Q_{10}$, $L(v) = \{1, 2, \dots, \lceil \frac{3}{2}\omega(G) \rceil\}$.

Then G is L-colorable.

Proof. Let $n := |V(G)|$ and $\omega := \omega(G)$. We prove this by induction on n. Since $|Q_i| \geq 1$ for each $i \in \{1, 2, \dots, 10\} \setminus \{1, 5\}$, $n \geq 8$. If $n = 8$, then $\omega = 3$ and clearly G is L-colorable. We now assume that $n > 8$ and the lemma is true for any graph of order smaller than n with a list assignment satisfying (1), (2), (3) and (4).

First, observe that one of the following inequality holds for otherwise we would obtain a contradiction.

- $|Q_2| + |Q_8| + \lceil \frac{\omega}{2} \rceil \geq |Q_2| + |Q_6|$.
- $|Q_2| + |Q_8| + \lceil \frac{\omega}{2} \rceil \geq |Q_4| + |Q_8|$.
- $|Q_4| + |Q_6| + \lceil \frac{\omega}{2} \rceil \geq |Q_4| + |Q_8|$.
- $|Q_4| + |Q_6| + \lceil \frac{\omega}{2} \rceil \geq |Q_2| + |Q_6|$.

By symmetry, we may assume that $|Q_2| + |Q_8| + \lceil \frac{\omega}{2} \rceil \geq |Q_2| + |Q_6|$. This implies that

$$|L_{28}| \geq |Q_2| + |Q_8| + \lceil \frac{\omega}{2} \rceil \geq |Q_2| + |Q_6|.$$

So we can choose a subset $W \subseteq L_{28}$ of size $|Q_2|$ and a subset $X \subseteq L_{46}$ of size $|Q_6|$ so that W and X are disjoint. We can also choose a subset $Y \subseteq L_{28} \setminus W$ of size $|Q_8|$ and a subset $Z \subseteq L_{46} \setminus X$ of size $|Q_4|$. Then choose colors $a \in W$,

$b \in X$, $c \in Y$ and $d \in Z$. We may choose these colors since Q_i is not empty for $i \in \{2,4,6,8\}$. Then $a \notin X \cup Y$, $b \notin W \cup Z$, $c \notin W$ (possibly $b = c$), and $d \notin X$ (possibly $d = c$, but $d \neq b$). Therefore $a \neq b$, $a \neq c$ and $b \neq d$. Choose a vertex $q_i \in Q_i$ for each $i \in \{1,2,\ldots,10\} \setminus \{1,5\}$, and choose $q_j \in Q_j$ for $j \in \{1,5\}$ if $Q_j \neq \emptyset$. Let T be the subgraph of G induced by these chosen vertices. Note that assigning colors a,b,c,d to q_2,q_6,q_8,q_4 in this order and coloring q_1 and q_5 (if exist) with colors $L(q_1)$ and $L(q_5)$ form a proper coloring of $T - \{q_3,q_7,q_9,q_{10}\}$. It can be readily checked that one can extend this coloring to a proper coloring of T using a set R of at most 4 colors containing a,b,c,d.

Let $G' := G - T$ and $Q'_i := Q_i \setminus \{q_i\}$ for $i \in \{1,2,\ldots,10\}$. For every $v \in V(G')$, let $L'(v) := L(v) \setminus R$. Denote by the resulting list assignment L'. Observe that $|L'(v)| \geq |L(v)| - 4$ for every $v \in V(G') \setminus (Q'_1 \cup Q'_5)$, $\omega(G') \leq \omega - 3$, and $|Q'_i| \leq |Q_i| - 1$ for each $i \in \{2,4,6,8\}$. Consequently, the list assignment L' of G' satisfies (1), (2), (3) and (4). If Q'_i is not empty for each $i \in \{1,2,\ldots,10\} \setminus \{1,5\}$, then G' is L'-colorable by the inductive hypothesis and so G is L-colorable. Therefore, we assume that $Q'_i = \emptyset$ for some $i \in \{1,2,\ldots,10\} \setminus \{1,5\}$.

We now consider the following three cases (up to symmetry) and in each case we show that G' is L'-colorable using the degeneracy of the graph. Let $\omega' := \omega(G')$ and $H = G' - (Q'_1 \cup Q'_5)$. Below we show that H is L'-degenerate. Note that every two vertices in Q'_i have the same degree in H.

Case 1. $Q'_9 = \emptyset$.

Since $|Q'_2| + |Q'_8| \leq \omega'$, one of Q'_2 and Q'_8 has size at most $\omega'/2$, say Q'_2 by symmetry. Then for any $v \in Q'_3$, it follows that $d_H(v) \leq (\omega'-1) + |Q'_2| \leq \frac{3}{2}\omega' - 1$. If $|Q'_8| \leq \omega'/2$, then $d_H(v) \leq \frac{3}{2}\omega' - 1$ for any $v \in Q'_7$. This implies that H is L'-degenerate: the ordering of the vertices in H is $\{Q'_2, Q'_4, Q'_6, Q'_8\}$, Q'_{10}, $\{Q'_3, Q'_7\}$. (It does not matter which vertex comes first among Q'_2, Q'_4, Q'_6, Q'_8 or between Q'_3 and Q'_7.) So $|Q'_8| > \omega'/2$. This implies that $|Q'_7| < \omega'/2$. Then for any $v \in Q'_8$, it follows that $d_H(v) = |Q'_2| + |Q'_8| - 1 + |Q'_7| < |Q'_2| + |Q'_8| - 1 + \omega'/2 < |L'(v)|$. So H is L'-degenerate with the ordering $Q'_2, Q'_4, Q'_6, Q'_{10}, Q'_7, Q'_8, Q'_3$.

Case 2. $Q'_3 = \emptyset$.

If $|Q'_7 \cup Q'_9| \leq \omega'/2$, then for any $v \in Q'_2 \cup Q'_8$, it follows that $d_H(v) \leq |Q'_2| + |Q'_8| - 1 + \omega'/2 < |L'(v)|$. Then H is L'-degenerate with the ordering $Q'_4, Q'_6, Q'_{10}, Q'_7, Q'_9, Q'_8, Q'_2$. So we assume that $|Q'_7 \cup Q'_9| > \omega'/2$. By symmetry, $|Q'_7 \cup Q'_{10}| > \omega'/2$. This implies that each of $Q'_6, Q'_8, Q'_9, Q'_{10}$ has size less than $\omega'/2$. Then every vertex in $Q'_9 \cup Q'_{10}$ has degree at most $\frac{3}{2}\omega' - 1$ in $G' - (Q'_2 \cup Q'_4)$ So H is L'-degenerate with the ordering $Q'_8, Q'_6, Q'_7, Q'_{10}, Q'_9, Q'_4, Q'_2$.

Case 3. $Q'_2 = \emptyset$.

If $|Q'_9| \leq \omega'/2$, then for every $v \in Q'_3$ it follows that $d_H(v) \leq \frac{3}{2}\omega' - 1$. By Case 2, $H - Q'_3$ is L'-degenerate and thus H is L'-degenerate. So $|Q'_9| > \omega'/2$. This implies that $|Q'_i \cup Q'_{10}| < \omega'/2$ for $i \in \{3,7\}$. This implies that $d_H(v) < |Q'_4| + |Q'_6| - 1 + \omega'/2 < |L'(v)|$ for any $v \in Q'_4 \cup Q'_6$. So H is L'-degenerate with the ordering $Q'_8, Q'_7, Q'_9, Q'_{10}, Q'_3, Q'_4, Q'_6$.

Since H is L'-degenerate, H is L'-colorable and so G' is L'-colorable. This completes the proof. □

Theorem 4 *Let G be a (claw, C_4)-free graph. Then* $\chi(G) \leq \lceil \frac{3}{2}\omega(G) \rceil$.

Proof. Let G be a (claw, C_4)-free graph. By Theorem 3, we may assume that G is not a quasi-line graph. We prove the theorem by induction on $|V(G)|$, and we apply Theorem 2.

If G is a special blow of the icosahedron graph, then the theorem follows from Lemma 1.

If G has a good vertex u, then by induction hypothesis, we have $\chi(G \setminus u) \leq \lceil \frac{3}{2}\omega(G \setminus u)\rceil \leq \lceil \frac{3}{2}\omega(G)\rceil$. So we can take any $\chi(G \setminus u)$-coloring of $G \setminus u$ and extend it to a $\chi(G)$-coloring of G, using for u a color that does not appear in its neighborhood.

If G has a clique cutset K, let A, B be a partition of $V(G)\setminus K$ such that both A, B are non-empty and $[A, B] = \emptyset$. Clearly $\chi(G) = \max\{\chi(G[K\cup A]), \chi(G[K\cup B])\}$, so the desired result follows from the induction hypothesis on $G[K \cup A]$ and $G[K \cup B]$.

So we can assume that G is a crown. Since $|Q_1 \cup Q_5 \cup M| < |V(G)|$, there exists a $\lceil \frac{3}{2}\omega(G)\rceil$-coloring ϕ of $G[Q_1 \cup Q_5 \cup M]$ by the inductive hypothesis. Let $G' = G - M$. Let L be the list assignment of G' such that

$$L(v) = \begin{cases} \phi(v) & \text{if } v \in Q_1 \cup Q_5, \\ \{1, 2, \ldots, \lceil \frac{3}{2}\omega(G')\rceil\} \setminus \phi(Q_1) & \text{if } v \in Q_2 \cup Q_8, \\ \{1, 2, \ldots, \lceil \frac{3}{2}\omega(G')\rceil\} \setminus \phi(Q_5) & \text{if } v \in Q_4 \cup Q_6, \\ \{1, 2, \ldots, \lceil \frac{3}{2}\omega(G')\rceil\} & \text{if } v \in Q_3 \cup Q_7 \cup Q_9 \cup Q_{10}. \end{cases}$$

Since $|Q_1| + |Q_2| + |Q_8| \leq \omega(G')$, it follows that $|L(v)| \geq \lceil \frac{3}{2}\omega(G')\rceil - |Q_1| \geq |Q_2|+|Q_8|+\lceil \frac{1}{2}\omega(G')\rceil$ for any $v \in Q_2\cup Q_8$. Similarly, $|L(v)| \geq \lceil \frac{3}{2}\omega(G')\rceil - |Q_5| \geq |Q_4|+|Q_6|+\lceil \frac{1}{2}\omega(G')\rceil$ for any $v \in Q_4\cup Q_6$. Then G' and L satisfy the conditions in Lemma 2, and therefore G' admits an L-coloring. This L-coloring and ϕ give a $\lceil \frac{3}{2}\omega(G)\rceil$-coloring of G. □

Theorem 5 *Let G be a (fork, C_4)-free graph. Then $\chi(G) \leq \lceil \frac{3}{2}\omega(G)\rceil$.*

Proof. Let G be any (fork, C_4)-free graph. We prove the theorem by induction on $|V(G)|$.

If G has a universal vertex u, then $\omega(G) = \omega(G\setminus u)+1$, and by the induction hypothesis we have $\chi(G) = \chi(G \setminus u) + 1 \leq \lceil \frac{3}{2}(\omega(G \setminus u)\rceil + 1$, which implies $\chi(G) \leq \lceil \frac{3}{2}(\omega(G)\rceil$.

If G has a clique cutset K, let A, B be a partition of $V(G)\setminus K$ such that both A, B are non-empty and $[A, B] = \emptyset$. Clearly $\chi(G) = \max\{\chi(G[K\cup A]), \chi(G[K\cup B])\}$, so the desired result follows from the induction hypothesis on $G[K \cup A]$ and $G[K \cup B]$.

Finally, if G has no universal vertex and no clique cutset, then the result follows from Corollary 1 and Theorem 4. □

We remark that we do not have any example of a (claw/fork, C_4)-free graph G such that $\chi(G) = \lceil \frac{3}{2}\omega(G)\rceil$ except C_5. However, for an integer $m \geq 1$, consider the blowup G of the icosahedron graph I where $|Q_v| = m$, for each vertex v in I. Then clearly $\omega(G) = 3m$, and since $\alpha(G) = 3$, we have $\chi(G) \geq \frac{|V(G)|}{\alpha(G)} = \frac{12m}{3} = 4m$.

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