A SIMPLE LAYERED-WHEEL-LIKE CONSTRUCTION

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Abstract. In recent years, there has been significant interest in characterizing the induced subgraph obstructions to bounded treewidth and pathwidth. While this has recently been resolved for pathwidth, the case of treewidth remains open, and prior work has reduced the problem to understanding the *layered-wheel-like* obstructions – graphs that contain large complete minor models with each branching set inducing a path, exclude large walls as induced minors, exclude large complete bipartite graphs as induced minors, and exclude large complete subgraphs.

There are various constructions of such graphs, but they are all rather involved. In this paper, we present a simple construction of layered-wheel-like graphs with arbitrarily large treewidth. Three notable features of our construction are: (a) the vertices of degree at least four can be made arbitrarily far apart; (b) the girth can be made arbitrarily large; and (c) every outerstring induced subgraph of the graphs from our construction has treewidth bounded by an absolute constant. In contrast, among several previously known constructions of layered wheels, none achieves (a); at most one satisfies either (b) or (c); and none satisfies both (b) and (c) simultaneously.

In particular, this is related to a former conjecture of Trotignon, that every graph with large enough treewidth, excluding large walls and large complete bipartite graphs as induced minors, and large complete subgraphs, must contain an outerstring induced subgraph of large treewidth. Our construction provides the first counterexample to this conjecture that can also be made to have arbitrarily large girth.

1. Introduction

1.1. **Background and the main result.** Graphs in this paper have finite vertex sets, no loops and no parallel edges. Let $G = (V(G), E(G))$ be a graph. A graph H is a *minor of* G if H is isomorphic to a graph that can be obtained from G by a series of vertex deletions, edge deletions, and edge contractions, and H is an *induced minor of* G if H is isomorphic to a graph that can be obtained from G by a series of vertex deletions and edge contractions

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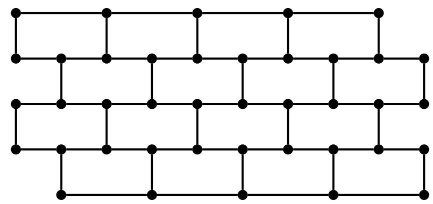

FIGURE 1. $W_{5\times 5}$

(and deleting the loops and the parallel edges produced in the contraction process). A *tree decomposition* (T, χ) of G consists of a tree T and a map $\chi\colon V(T) \to 2^{V(G)}$ such that the following hold.

- For every vertex $v \in V(G)$, there exists $t \in V(T)$ such that $v \in \chi(t)$.
- For every edge $v_1v_2 \in E(G)$, there exists $t \in V(T)$ such that $v_1, v_2 \in \chi(t)$.
- For every $v \in V(G)$, the subgraph of T induced by $\{t \in V(T)\colon v \in \chi(t)\}$ is connected.

For a tree decomposition (T, χ) of G with $V(T) = \{t_1, \dots, t_n\}$, the sets $\chi(t_1), \dots, \chi(t_n)$ are called the *bags of* (T, χ). The *width of* (T, χ) is defined as $\max_{t\in V(T)} |\chi(t)| - 1$. The *treewidth* of G, denoted by $\mathrm{tw}(G)$, is the minimum width of a tree decomposition of G.

The Grid Theorem of Robertson and Seymour, Theorem 1.1 below, fully describes the unavoidable subgraphs of graphs with large treewidth. For every $k \in \mathbb{N}$, the $(k \times k)$-*wall*, denoted by $W_{k\times k}$, is a planar graph with maximum degree three and with treewidth k (see Figure 1; a precise definition can be found in [2]). Every subdivision of $W_{k\times k}$ is also a graph of treewidth k.

Theorem 1.1 (Robertson and Seymour [21])**.** *There is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that every graph of treewidth at least $f(k)$ contains a subdivision of $W_{k\times k}$ as a subgraph.*

Theorem 1.1 also holds if "subgraph" is replaced by "minor" (in that case "subdivision" will not be necessary anymore). Recently, there has been growing interest in understanding the unavoidable *induced* subgraphs of graphs with large treewidth. For instance, subdivided walls, complete graphs, and complete bipartite graphs are easily observed to have arbitrarily large treewidth. Line graphs of subdivided walls form another family of graphs with unbounded treewidth (recall that the *line graph* $L(F)$ of a graph F is the graph with vertex set $E(F)$, such that two vertices of $L(F)$ are adjacent if the corresponding edges of G share an end). Since these four types of graphs do not contain each other as induced subgraphs, they must all be included in any potential family of induced subgraph obstructions to bounded treewidth; hence, they are often referred to as the "basic obstructions".

A full characterization of the induced subgraph obstructions to bounded treewidth remains unknown. Specifically, there are various constructions [5, 7, 8, 9, 10, 12, 14, 19, 22] showing that forbidding the basic obstructions as induced subgraphs does not guarantee bounded treewidth. Also, a recent result of Alecu, Bonnet, Villafana and Trotignon [5] shows that the only hereditary set of obstructions that can be forbidden (as induced subgraphs) from a graph class in order to guarantee bounded treewidth is the family of all graphs (a graph class is *hereditary* if it is closed under taking induced subgraphs).

Theorem 1.2 (Alecu, Bonnet, Villafana, Trotignon [5])**.** *For every $t \in \mathbb{N}$, there is a hereditary graph class $\mathcal{C}_t$ of unbounded treewidth and a constant c_t such that, for any graph H of treewidth at most t, every graph $G \in \mathcal{C}_t$ not containing H as an induced subgraph has treewidth at most c_t.*

On the other hand, we now know that every non-basic obstruction falls in one of only two categories: those that are "complete bipartite induced minor models" and those that are "linear complete minor models". Let us define these terms precisely. Given a graph G, a *model in G* is a graph H with vertex set $\{X_1, \dots, X_n\}$ such that the following hold.

(i) For all i with $1 \leq i \leq n$, we have $X_i \subseteq V(G)$ and $G[X_i]$ is connected.
(ii) For all i and j with $1 \leq i < j \leq n$, we have $X_i \cap X_j = \varnothing$.
(iii) For all i and j with $1 \leq i < j \leq n$, if X_i and X_j are adjacent in H, then there exist $v_i \in X_i$ and $v_j \in X_j$ such that $v_i v_j \in E(G)$.

We say that the model H in G is *linear* if $G[X_i]$ is path for all i with $1 \leq i \leq n$, and that H is *induced* if the converse to (iii) is also true; that is, for all i and j with $1 \leq i < j \leq n$, if X_i and X_j are not adjacent in H, then X_i and X_j are anticomplete in G. Note that for any choice of the subsets $\{X_1, \dots, X_n\}$ satisfying (i) and (ii), there is exactly one induced model in G with $\{X_1, \dots, X_n\}$ as its vertex set, which we call the *model in G induced by* $\{X_1, \dots, X_n\}$. It is easy to observe that a graph F is a minor of G if and only if F is isomorphic to a model in G, and F is an induced minor of G if and only if F is isomorphic to an induced model in G.

In [10], three of us proved the following.

Theorem 1.3 (Chudnovsky, Hajebi, Spirkl [10])**.** *For all $r, s, t \in \mathbb{N}$, there is a constant $c = c(r, s, t) \in \mathbb{N}$ such that for every graph G with $\operatorname{tw}(G) > c$, one of the following holds.*

- *There is an induced subgraph of G isomorphic to one of K_{r+1}, $K_{r,r}$, some subdivision of $W_{r\times r}$ or the line graph of some subdivision of $W_{r\times r}$.*
- *There is an induced model in G isomorphic to $K_{s,s}$.*
- *There is a linear model in G isomorphic to K_t.*

Moreover, the main result of [10] characterizes the unavoidable induced subgraphs of graphs with a large complete bipartite induced minors. Therefore, in order establish a full analog of Theorem 1.1 for induced subgraphs, it remains to understand those obstructions that are linear complete models which do not contain basic obstructions of large treewidth as induced subgraphs, and do not contain large complete bipartite graphs as induced minors.

We call these obstructions *layered-wheel-like*, and the naming is explained by the fact that the only known examples of such obstructions are (variations of) the so-called *layered wheels*. Indeed, there are several constructions of layered wheels, by Sintiari and Trotignon [22], by Chudnosvky and Trotignon [12] and by Alecu, Bonnet, Villafana and Trotignon [5]. But they are all quite intricate. In this paper, we present a new construction of layered-wheel-like graphs with arbitrarily large treewidth. Our construction is substantially simpler than all previous ones and simultaneously achieves several key properties, no two of which are attained individually by any earlier construction. Explicitly, our main result is as follows.

Theorem 1.4. *There exist $L, t_0 \in \mathbb{N}$ such that for all $g, k \in \mathbb{N}$, there is a graph G_k^g with the following properties.*

(i) *There is a linear model in* G^g_k *isomorphic to* K_k*. In particular,* $\mathrm{tw}(G^g_k) \geq k-1$.
(ii) G^g_k *does not contain* $W_{t_0 \times t_0}$ *or* K_{t_0,t_0} *as an induced minor.*
(iii) *If* $u, v \in V(G^g_k)$ *both have degree at least four in* G^g_k *and* $u \neq v$*, then* $\mathrm{dist}_{G^g_k}(u,v) \geq 2^g$.
(iv) G^g_k *has girth at least* g.
(v) *If* H *is an induced subgraph of* G^g_k *and* H *is an outerstring graph, then* $\mathrm{tw}(H) \leq L$.

No preexisting construction of layered wheels satisfies (iii) (and in fact they all contain every tree as induced subgraphs). Moreover, only one construction, namely the "theta-free" one [22], allows for arbitrarily large girth, and the most recent construction of layered wheels [5] (which is also the most complicated one) is the only one that achieves (v) – in particular, no other construction satisfies (iv) and (v) at once.

Property (v) is closely related to the following conjecture of Trotignon [23]. A *string representation* of a graph G is an assignment of the vertices of G to curves in the plane such that the curves corresponding to two vertices u, v intersect if and only if $uv \in E(G)$. An *outerstring representation* of G is a string representation of G in which all curves lie in the upper half of the plane and each curve has exactly one endpoint on the x-axis. A graph G is a *string graph* if it has a string representation, and it is an *outerstring graph* if it has an outerstring representation.

Conjecture 1.5 (Trotignon [23])**.** *For all* $r, t \in \mathbb{N}$*, there exists* $c = c(r,t) \in \mathbb{N}$ *such that for every graph* G*, if* G *does not contain* $W_{t\times t}$ *or* $K_{t,t}$ *as an induced minor, and every induced subgraph* H *of* G *that is an outerstring graph satisfies* $\mathrm{tw}(H) \leq r$*, then* $\mathrm{tw}(G) \leq c$.

This conjecture is already refuted by the construction from [5]. Our main construction provides a different counterexample to Conjecture 1.5, which has the advantage of being much simpler in structure, having arbitrarily large girth, and not containing all trees as induced subgraphs. In fact, our proof gives a slightly stronger property than (v); see Theorem 6.2.

1.2. **Definitions and notation.** We write $\mathbb{N}$ for the set of positive integers. Let G be a graph. An *induced subgraph of* G is a graph H obtained from G by deleting vertices. For $X \subseteq V(G)$, we let $G[X]$ denote the induced subgraph of G with vertex set X, and $G \setminus X$ denotes $G[V(G) \setminus X]$. We often use X to denote both the set X of vertices and the induced subgraph $G[X]$. We say that two subsets $X, Y \subseteq V(G)$ are *anticomplete* if $X \cap Y = \varnothing$ and there is no edge of G with an end in X and an end in Y. For $v \in V(G)$, we let $N_G(v)$ denote the set of all vertices in G adjacent to v, and we write $N_G[v] = N_G(v) \cup \{v\}$. The *degree of* v *in* G, denoted by $\deg_G(v)$, is $|N_G(v)|$. The *maximum degree of* G is $\max_{v\in V(G)} \deg_G(v)$, and the *minimum degree of* G is $\min_{v \in V(G)} \deg_G(v)$. For $X \subseteq V(G)$, we let $N_G(X)$ denote the set of all vertices in $G \setminus X$ with at least one neighbor in X, and we define $N_G[X] = N_G(X) \cup X$.

A *path* P is a graph with vertex set $\{v_1, \dots, v_n\}$ and edge set $\{v_iv_{i+1} \colon 1 \leq i \leq n-1\}$. Such a path is denoted by v_1-v_2-$\cdots$-v_{n-1}-v_n. We say that v_1 and v_n are the *ends* of P, and that P is a *path from* v_1 *to* v_n. The *interior of* P, denoted by P^*, is $P \setminus \{v_1, v_n\}$. The *length* of P is given by $|E(P)| = n-1$. A *path in* G *from* u *to* v is an induced subgraph of G that is a path from u to v. If G is connected, the *distance from* u *to* v *in* G, denoted by $\mathrm{dist}_G(u,v)$, is the length of the shortest path from u to v in G. For $u, v \in P$, the *subpath of* P *from* u *to* v is the (unique) path in P with ends u and v. Two paths P_1 and P_2 in G are *internally anticomplete* if P_1^* and P_2^* are anticomplete.

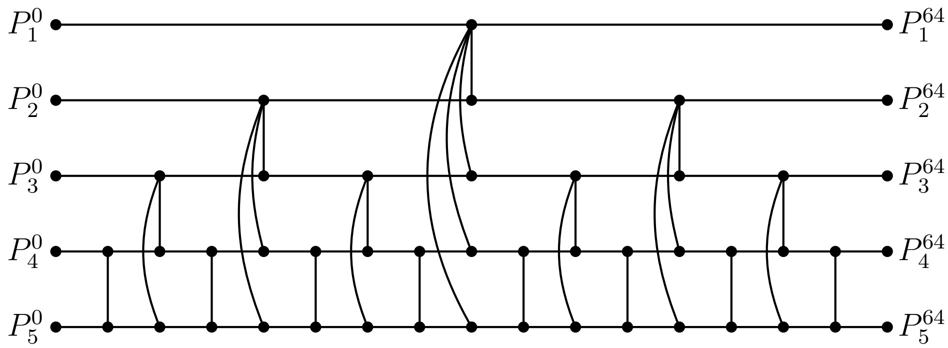

FIGURE 2. G_5^1 (internal small vertices are not drawn)

The k-vertex cycle C_k is obtained from a path on k vertices by adding an edge between its two ends. The *girth* of a graph G is the smallest k such that G contains C_k as an (induced) subgraph (and is infinite if G does not contain any cycles). A *hole in* G is an induced subgraph of G isomorphic to C_k for some $k \geq 4$.

2. The main construction

In this section, we give the description of the graph family that proves Theorem 1.4.

Construction 2.1. *For $g, k \in \mathbb{N}$ with $g \geq 1$, let G_k^g be the graph constructed as follows.*

(i) *$V(G_k^g)$ is partitioned into k paths $P_1, \dots, P_k$ of length 2^{k+g}, where $P_i = P_i^0\text{-}\cdots\text{-}P_i^{2^{k+g}}$ for each i with $1 \leq i \leq k$.*

(ii) *For all i and j with $1 \leq i < j \leq k$, the vertices P_i^x and P_j^y are adjacent in G_k^g if $x = y = b \cdot 2^{k-i+g}$ for some odd integer b; namely, $b \in \{1, 3, \dots, 2^i - 1\}$.*

See Figure 2. We say that the vertex $P_\ell^x \in V(G_k^g)$ has *layer* ℓ and *index* x. For $1 \leq \ell \leq k$ and $0 \leq x_1, x_2 \leq 2^{k+g}$, we let $P_\ell[x_1 : x_2]$ denote the subpath of P_ℓ with ends $P_\ell^{x_1}$ and $P_\ell^{x_2}$. For the purposes of analysis, we partition $V(G_k^g)$ into three sets: *big*, *medium*, and *small vertices*; where a vertex $P_i^x \in V(G_k^g)$ is *big* if $x = b \cdot 2^{k-i+g}$ for some odd b, *medium* if $x = b \cdot 2^{k-j+g}$ for some odd b and some $1 \leq j < i$, and *small* otherwise. The sets of big, medium, and small vertices of G_k^g are denoted by $B(G_k^g)$, $M(G_k^g)$, and $S(G_k^g)$, respectively. We note that all big vertices in $V(G_k^g) \setminus V(P_k)$ have degree at least 3, all medium vertices have degree exactly 3, and all small vertices have degree at most 2.

Some basic properties of the graph G_k^g are listed below (the proofs are easy and we leave the details to the reader to check).

Lemma 2.2. *For all $g, k \in \mathbb{N}$, the graph G_k^g has the following properties.*

(i) *There is a linear model in G_k^g isomorphic to K_k. In particular,* $\mathrm{tw}(G_k^g) \geq k - 1$.
(ii) *G_k^g is triangle-free.*
(iii) *G_k^g has girth at least g.*
(iv) *If $u, v \in V(G_k^g)$ are adjacent, then $\deg_{G_k^g}(u) \leq 3$ or $\deg_{G_k^g}(v) \leq 3$, and $\{u, v\} \not\subseteq B(G_k^g)$.*
(v) *If $u, v \in B(G_k^g)$ with $u \neq v$, then $\mathrm{dist}_{G_k^g}(u, v) \geq 2^g$. In particular, if $u, v \in V(G_k^g)$ with $u \neq v$ are non-adjacent and both have degree at least 4, then $\mathrm{dist}_{G_k^g}(u, v) \geq 2^g$.*

(vi) *If* $u \in V(G_k^g)$ *with* $\deg_{G_k^g}(u) = 3$*, then there is at most one* $v \in N_{G_k^g}(u)$ *such that* $\deg_{G_k^g}(v) \geq 3$*.*

Therefore, in order to prove Theorem 1.4, it remains to show that for all $g, k \in \mathbb{N}$, there is no large wall in G_k^g as an induced minor, there is no large complete bipartite graph in G_k^g as an induced minor, and all outerstring induced subgraphs of G_k^g have small treewidth. We will prove the latter in the next three sections, and then we will complete the proof of Theorem 1.4 in the final section.

3. No large wall as an induced minor

The main result of this section is the following lemma.

Lemma 3.1. *There exists* $h_0 \in \mathbb{N}$ *such that for all* $g, k \in \mathbb{N}$*, the graph* G_k^g *does not contain* $W_{h_0 \times h_0}$ *as an induced minor.*

The proof involves series-parallel graphs, which are exactly the graphs with treewidth at most 2. The definition given here is adapted from [16]. A *two-terminal graph* is a graph G where two distinct vertices $s, t \in V(G)$ are designated as the *terminals* of G, where s is the *source* and t is the *sink*. If G is a two-terminal graph with source s and sink t, and H is a two-terminal graph with source s' and sink t', then the *series-composition of* G *and* H is the two-terminal graph obtained by combining G and H via identifying t and s' into a single vertex (that is, replacing them by a single vertex with neighbor set $N_G(t) \cup N_H(s')$), and declaring s as the source and t' as the sink of the resulting graph. The *parallel-composition of* G *and* H is the two-terminal graph obtained by combining G and H via identifying s and s' into a single vertex and declaring that vertex as the source of the resulting graph, and identifying t and t' into a single vertex and declaring that vertex as the sink of the resulting graph. A two-terminal graph G with source s and sink t is (s,t)*-series-parallel* if G can be obtained from copies of K_2 via a sequence of series-compositions and parallel-compositions, where each copy of K_2 begins with one of its vertices as the source and the other as the sink. A graph G is *series-parallel* if it is a subgraph of some (s,t)-series-parallel graph.

We now return to the proof of Lemma 3.1. It is a consequence of Lemma 3.6 in [1] that, for every constant $t > 0$, there is a constant $h_t > 0$ such that every graph containing $W_{h_t \times h_t}$ as an induced minor contains a subdivision of $W_{t \times t}$ or the line graph of a subdivision of $W_{t \times t}$ as an induced subgraph. Recall that for all $g, k \in \mathbb{N}$, the graph G_k^g is triangle-free, and so it does not contain the line graph of a subdivision of any wall. Thus, to show the existence of the desired h_0, it suffices to show that G_k^g does not contain a subdivision of a large wall as an induced subgraph. This will be accomplished in Lemma 3.4, but we first need two more lemmas. Our goal is to show that for every induced subgraph of G_k^g, if we contract all edges not contained in $P_1, \ldots, P_k$ (the vertical edges in Figure 2), then the resulting graph is series-parallel.

Lemma 3.2. *Let* F *be a two-terminal graph with source* s *and sink* t*, and the following specifications.*

(i) *There are paths* $P_1, \ldots, P_k$ *from* s *to* t *such that* $V(F) = V(P_1) \cup \ldots \cup V(P_k)$ *and* $V(P_i^*) \cap V(P_j^*) = \varnothing$ *for* $i \neq j$*.*

(ii) *For $1 \leq i \leq k$ and every $v \in P_i^*$, exactly one of the following holds.*
- $N_F(v) \setminus V(P_i) = \varnothing$*; we let $S'(F)$ denote the set of all vertices in F for which this outcome holds.*
- $i < k$*, the vertex v has exactly one neighbor in P_j^* for every j with $i < j \leq k$, and v has no other neighbors apart from its neighbors in P_i; in this case, we write $N'_F[v] = (N_F(v) \setminus V(P_i)) \cup \{v\}$. We let $B'(F)$ denote the set of all vertices in F for which this outcome holds.*
- $i > 1$*, and v has exactly one neighbor in P_j^* for some $j < i$, and v has no other neighbors outside P_i.*

(iii) *Let v_1 and v_2 be distinct vertices in $B'(F) \cap V(P_i^*)$ for some i with $1 \leq i < k$ such that* $\text{dist}_{P_i}(s, v_1) < \text{dist}_{P_i}(s, v_2)$*. Then, for every j with $i < j \leq k$, we have* $\text{dist}_{P_j}(s, u_1) < \text{dist}_{P_j}(s, u_2)$*, where u_1 is the unique neighbor of v_1 in P_j^* and u_2 is the unique neighbor of v_2 in P_j^*.*

Then the model in F induced by the set

$$\{N'_F[v] \colon v \in B'(F)\} \cup \{\{w\} \colon w \in S'(F)\} \cup \{\{s\}, \{t\}\},$$

denoted by $c(F)$, is an (s,t)-series-parallel graph.

Proof. We proceed by induction on $|B'(F)|$. If $|B'(F)| = 0$, then $c(F)$ is isomorphic to F and consists of k pairwise internally anticomplete paths from s to t, and so it is (s,t)-series-parallel. We now assume that $|B'(F)| > 0$. We may further assume that P_1 contains a vertex in $B'(F)$; if this is not the case, then we may apply the argument on $F[V(P_t) \cup \ldots \cup V(P_k)]$, where t is minimal such that P_t contains a vertex in $B'(F)$, and then take the parallel-composition of this graph with $P_1, \ldots, P_{t-1}$ to show that F is (s,t)-series-parallel. Now, let $b \in B'(F) \cap V(P_1^*)$ with $\text{dist}_{P_1}(s, b)$ minimal. Since b has a neighbor in each of $P_2^*, \ldots, P_k^*$ in F, it follows that $F \setminus N'[b]$ has two components, one containing s and the other containing t. Let C_s and C_t, respectively, denote these components.

We now observe that the model F_s in F induced by $\{\{v\} \colon v \in V(C_s)\} \cup \{N'[b]\}$ satisfies the requirements of the lemma, with terminals s and $N'[b]$ and the paths from s to $N'[b]$ being the truncated versions of the paths $P_1, \ldots, P_k$. Clearly $B'(F_s) \subseteq B'(F)$; furthermore, this containment is strict, since $b \in B'(F)$, and $b \notin B'(F_s)$ as it is part of the terminal vertex $N'_F[b]$ in F_s. By induction, we find that $c(F_s)$ is $(s, N'_F[b])$-series-parallel. Similarly, $c(F_t)$ is $(N'_F[b], t)$-series-parallel, where F_t is the model in F induced by $\{\{v\} \colon v \in V(C_t)\} \cup \{N'_F[b]\}$. Since $c(F)$ is the series-composition of $c(F_s)$ and $c(F_t)$, it follows that $c(F)$ is series-parallel. □

Lemma 3.3. *Let $g, k \in \mathbb{N}$, and let H be an induced subgraph of G_k^g. For every vertex $b \in B(G_k^g) \cap V(H)$, let $N_H^M(b) = N_H(b) \cap M(G_k^g)$ and $N_H^M[b] = N_H^M(b) \cup \{b\}$. Let $B' = \{N_H^M[b] \colon b \in B(G_k^g) \cap V(H)\}$, and let H' be the model in H induced by $B' \cup \{\{v\} \colon v \in V(H),\, v \notin \bigcup_{Y \in B'} Y\}$; equivalently, H' is obtained from H by contracting every edge $e \in E(H)$ that has both ends in $N_H^M[b]$ for some $b \in B(G_k^g) \cap V(H)$. Then H' is series-parallel (in particular,* $\text{tw}(H') \leq 2$*).*

Proof. Let $X \subseteq V(G_k^g)$ be such that $H = G_k^g \setminus X$. Put $X_B = X \cap B(G_k^g)$, $X_M = X \cap M(G_k^g)$, and $X_S = X \cap S(G_k^g)$. Let F be obtained from G_k^g by removing, for each $b \in X_B$, every edge incident with b whose other end is in $M(G_k^g)$. Next, we add auxiliary terminal vertices s and t to F, where s is incident to P_ℓ^0 for $1 \leq \ell \leq k$ and t is incident to $P_\ell^{2^{k+g}}$ for $1 \leq \ell \leq k$.

Applying Lemma 3.2 gives that $c(F)$ is (s,t)-series-parallel, where $c(F)$ is as defined there. We now show that H' is a subgraph of $c(F)$. We observe that H' can be obtained from $c(F)$ as follows.

(i) Delete s and t.
(ii) For $v \in X_S$, delete the vertex $\{v\}$ from $c(F)$.
(iii) For $m \in X_M$, let b be the unique neighbor (in G_k^g) of m in $B(G_k^g)$, and let s_1 and s_2 be the two neighbors of m in $S(G_k^g)$.
  - If $b \in X_B$, delete $\{m\}$ from $c(F)$ (since $b \in X_B$, there is no edge between b and m in F, and so $\{m\}$ is a vertex of $c(F)$).
  - If $b \notin X_B$, there is a vertex S in $c(F)$ such that $b, m \in S$. If $s_1 \notin X_S$, delete the edge between v and $\{s_1\}$ in $c(F)$. Similarly, if $s_2 \notin X_S$, delete the edge between v and $\{s_2\}$ in $c(F)$.

(iv) For $b \in X_B$, delete $\{b\}$ from $c(F)$.

Thus H' is a subgraph of the series-parallel graph $c(F)$, and so H' is series-parallel. □

Lemma 3.4. *For all $g, k \in \mathbb{N}$, the graph G_k^g has no induced subgraph isomorphic to a subdivision of $W_{5\times5}$.*

Proof. Suppose that H is an induced subgraph of G_k^g that is isomorphic to a subdivision of $W_{5\times5}$. Say a vertex $v \in V(H)$ is a *branch vertex* of H if v is also a vertex of $W_{5\times5}$; that is, v is not a vertex that was created in the subdivision process. Let H' be as defined in Lemma 3.3, and for each vertex u of H, let $p(u)$ be the unique vertex of H' such that $u \in p(u)$. We show that H' contains a subdivision of $W_{3\times3}$. To see this, let J be a subdivision of $W_{3\times3}$ in H such that every two branch vertices of J have distance at least 3 in J (see Figure 1). Since $p(u) = p(v)$ can only hold if u and v have distance at most two in H, it follows that no two distinct branch vertices of J are mapped to the same vertex in H'; thus H' contains a subdivision of $W_{3\times3}$. But now $\mathrm{tw}(H') > 2$, which contradicts Lemma 3.3. □

4. No large complete bipartite induced minor

In this section we prove the following result.

Lemma 4.1. *There exists $r_0 \in \mathbb{N}$ such that for all $g, k \in \mathbb{N}$, the graph G_k^g does not contain K_{r_0,r_0} as an induced minor.*

We need to prepare for the proof. A graph T is a *wide theta of width m* if for some distinct vertices $a, b \in V(T)$ (called the *ends of T*), there are m pairwise internally anticomplete paths $P_1, \dots, P_m$ from a to b in T, each of length at least 2, and T has no other vertices or edges. The following is an immediate corollary of 1.3 in [10].

Lemma 4.2 (Chudnovsky, Hajebi, Spirkl [10])**.** *For all $h \in \mathbb{N}$, there exists $r = r(h) \in \mathbb{N}$ such that if G does not contain a wide theta of width 8 as an induced subgraph or $W_{h\times h}$ as an induced minor, then G does not contain $K_{r,r}$ as an induced minor.*

Accordingly, in what follows, we will show that for all $g, k \in \mathbb{N}$ and every pair of distinct vertices $u, v \in G_k^g$, there can be at most seven pairwise internally anticomplete paths between u and v in G_k^g. This will be achieved in Corollary 4.5, which, combined with Lemmas 3.1 and

4.2, gives a proof of Lemma 4.1. We remark that through further casework the proof shown here can be extended to obtain a bound of at most three paths, which is tight.

Note that for all $g, k \in \mathbb{N}$ and every pair of distinct vertices $u, v \in V(G_k^g)$, there can be at most $\min\{\deg_{G_k^g}(u), \deg_{G_k^g}(v)\}$ pairwise internally anticomplete paths between u and v in G_k^g. In particular, if u and v are not both in $B(G_k^g)$, there can be no more than three pairwise internally anticomplete paths between the two. Thus, we need only consider paths between two big vertices in G_k^g. We recall the notation used in defining G_k^g. Suppose $b_1 = P_{\ell_1}^{x_1} \in B(G_k^g)$ and $b_2 = P_{\ell_2}^{x_2} \in B(G_k^g)$ are two distinct vertices of G_k^g; by the symmetry of G_k^g, we may assume that $\ell_1 \leq \ell_2$ and $x_1 < x_2$. If R is a path from b_1 to b_2 in G_k^g, we say that R *switches layers at x, from ℓ to ℓ'*, if $E(R)$ contains an edge with ends P_ℓ^x and $P_{\ell'}^x$. We emphasize that if a path switches layers at x then it contains a big vertex with index x.

Suppose $\mathcal{R}$ is a set of paths from b_1 to b_2 that are pairwise internally anticomplete. For $R \in \mathcal{R}$, let R^- denote the unique neighbor of b_1 in R, and let R^+ denote the unique neighbor of b_2 in R. We say that a path $R \in \mathcal{R}$ is *standard* if $R^- = P_{\ell_1'}^{x_1}$ for some $\ell_1' > \ell_1$ and $R^+ = P_{\ell_2'}^{x_2}$ for some $\ell_2' > \ell_2$. Otherwise, we say that R is *nonstandard*. Note that $\mathcal{R}$ can include at most four nonstandard paths, since every nonstandard path uses either an edge of P_{ℓ_1} incident with $P_{\ell_1}^{x_1}$ or an edge of P_{ℓ_2} incident with $P_{\ell_2}^{x_2}$, and there are only four such edges. In what follows, we show that there may be at most three standard paths in $\mathcal{R}$.

Let $\mathcal{R}' = \{R^* \colon R \text{ is a standard path in } \mathcal{R}\}$. For $R^* \in \mathcal{R}'$, say that R is an *overpass* if, for some ℓ with $1 \leq \ell \leq k$, we have $V(P_\ell[x_1 : x_2]) \subseteq V(R^*)$ and $P_\ell[x_1 : x_2] \cap B(G_k^g) = \varnothing$. For ease of notation, we will also say that R^* is (or is not) an overpass to mean that R is (or is not) an overpass.

Lemma 4.3. *There is at most one $R^* \in \mathcal{R}'$ that is not an overpass.*

Proof. For $R^* \in \mathcal{R}'$, say $v \in V(R^*)$ is an *internal big vertex of* R^* if $v \in B(G_k^g)$ and b has index greater than x_1 and smaller than x_2.

We first show that if $R^* \in \mathcal{R}'$ is not an overpass, then contains some internal big vertex $b = P_\ell^x$. Indeed, for each x' with $x_1 \leq x' \leq x_2$, the path R^* includes a vertex $P_{\ell'}^{x'}$ for some ℓ' with $1 \leq \ell' \leq k$. If, for some ℓ', R^* contains all of $P_{\ell'}[x_1 : x_2]$, then one of the vertices of $P_{\ell'}[x_1 : x_2]$ is a big vertex as otherwise R^* would be an overpass. On the other hand, if there is no ℓ' such that R^* contains $P_{\ell'}^{x'}$ for all x' with $x_1 \leq x' \leq x_2$, then R^* switches layers at some index x that is strictly between x_1 and x_2, and so R^* contains a big vertex with index x.

Now assume that there is at least one element of $\mathcal{R}'$ that is not an overpass; let $b = P_{\ell_0}^{x_0}$ be a big vertex contained in some non-overpass of $\mathcal{R}'$ such that $x_1 < x_0 < x_2$ and ℓ_0 is minimal. Let R_0^* be the element of $\mathcal{R}'$ containing b, and suppose there is some other $R^* \in \mathcal{R}'$ that is not an overpass. The path R^* contains some vertex with index x_0, say $v = P_\ell^{x_0}$. It follows that $\ell < \ell_0$, as otherwise b and v would be adjacent or equal. Note that v is a small vertex as b is a big vertex and v has strictly smaller layer than b. Thus, there is some index x' such that R^* switches layers to ℓ at x' and such that $P_\ell[x', x_0]$ is contained in the subpath of R^* from R^- to v. It follows that R^* includes a big vertex with layer at most ℓ and index x'. By the choice of b and the fact that $\ell < \ell_0$, we deduce that either $x' < x_1$ or $x' > x_2$. The two cases are analogous; we show in detail how to handle the case where $x' < x_1$.

We now show that R^* contains $P_\ell[x_0 : x_2]$ (and thus all of $P_\ell[x_1 : x_2]$). Indeed, suppose this is not the case, and let x'' be maximal such that $P_\ell[x_0 : x'']$ is contained in R^*, where

$x_0 \leq x'' < x_2$. Since R^* does not contain $P_\ell^{x''+1}$, R^* switches layers at index x'' from ℓ to some other layer, and thus R^* contains a big vertex of index at most ℓ. But $\ell < \ell_0$ and $x_1 < x'' < x_2$, so this contradicts the choice of b. □

Lemma 4.4. *$\mathcal{R}'$ contains at most two overpasses.*

Proof. For every overpass $R^* \in \mathcal{R}'$, there is some $\ell \leq \ell_1$ such that $P_\ell^x \in V(R^*)$ for all x with $x_1 \leq x \leq x_2$, and none of these vertices are big. Furthermore, by the construction of G_k^g and the fact that R^* is a standard path, we have $\ell < \ell_1$. It follows that R^* contains a big vertex $b = P_{\ell'}^{x_0}$ for some $\ell' < \ell$, where $x_0 < x_1$.

Now suppose that R_α^* and R_β^* are two distinct elements of $\mathcal{R}'$. Let $b_\alpha = P_{\ell_\alpha}^{x_\alpha}$ be the big vertex in R_α^* with minimal distance to R_α^- in R_α^*, subject to the condition $\ell_\alpha < \ell_1$. Similarly, let $b_\beta = P_{\ell_\beta}^{x_\beta}$ be the big vertex in R_β^* with minimal distance to R_β^- in R_β^*, subject to the condition $\ell_\beta < \ell_1$.

We now show that either $x_\alpha < x_1 < x_\beta$ or $x_\beta < x_1 < x_\alpha$ holds. Suppose instead that both $x_\alpha < x_1$ and $x_\beta < x_1$ are true (the case where both are greater than x_1 is analogous). Without loss of generality, we assume that $x_\alpha > x_\beta$. Then the subpath of R_β^* from R_β^- to b_β contains some vertex v with index x_α and layer strictly larger than ℓ_α. But then b_α and v are adjacent, which is a contradiction.

Now suppose that $\mathcal{R}'$ contains three or more overpasses. Then there exist $R_\alpha^*, R_\beta^* \in \mathcal{R}'$ such that either $x_\alpha < x_1$ and $x_\beta < x_1$ or $x_\alpha > x_1$ and $x_\beta > x_1$, contrary to the claim of the previous paragraph. Thus, $\mathcal{R}'$ contains at most two overpasses. □

Corollary 4.5. *$|\mathcal{R}| \leq 7$.*

Proof. $\mathcal{R}$ contains at most four nonstandard paths. Of the standard paths in $\mathcal{R}$, there are at most two overpasses by Lemma 4.4, and at most one non-overpass by Lemma 4. Thus, there are at most seven paths in $\mathcal{R}$. □

5. No outerstring induced subgraph of large treewidth

In this section, we prove the following.

Lemma 5.1. *There exists $L \in \mathbb{N}$ such that for all $g, k \in \mathbb{N}$, every induced subgraph H of G_k^g that is an outerstring graph satisfies* $\operatorname{tw}(H) \leq L$.

We need several results from the literature. Let G be a graph, and let $w\colon V(G) \to [0,1]$. For $X \subseteq V(G)$, we write $w(X) = \sum_{x \in X} w(x)$, and for a subgraph H of G (not necessarily induced), we write $w(H)$ for $\sum_{x \in V(H)} w(x)$. We say that w is a *weight function on G* if $w(G) = 1$, and a *weak weight function on G* if $w(G) \leq 1$ (so all weight functions are weak weight functions). A set $X \subseteq V(G)$ is a *w-balanced separator in G* if $w(D) \leq \frac{1}{2}$ for every component D of $G \setminus X$. Treewidth and balanced separators are closely related through the following lemmas.

Lemma 5.2 ([2, 4, 17, 20])**.** *Let $m \in \mathbb{N}$, and let G be a graph such that for every weight function w on G, there is a w-balanced separator X_w in G with $|X_w| \leq m$. Then* $\operatorname{tw}(G) \leq 2m$.

Lemma 5.3 ([4, 13, 20])**.** *For every graph G and every weak weight function w on G, there is a w-balanced separator in G of size at most* $\operatorname{tw}(G) + 1$.

We also need the following.

Theorem 5.4 (Korhonen [18])**.** *For all $d, t \in \mathbb{N}$, there exists $L \in \mathbb{N}$ such that if G is graph of maximum degree at most d that does not contain any subdivision of $W_{t\times t}$ or the line graph of any subdivision of $W_{t\times t}$ as an induced subgraph, then $\operatorname{tw}(G) \leq L$.*

A *theta* is a graph T consisting of two non-adjacent vertices a and b and three internally anticomplete paths P_1, P_2, and P_3 from a to b, each of length at least 2, and no other vertices or edges. We call a and b the *ends of* T, and the *length of* T is $\operatorname{dist}_T(a, b)$. We say that T is an *ℓ-long theta* if its length is at least ℓ.

Next, we recall a result implicit in [15], that ℓ-long thetas are not outerstring graphs for $\ell \geq 4$. Since the class of outerstring graphs is hereditary, it follows that every graph containing an ℓ-long theta for $\ell \geq 4$ as an induced subgraph is not an outerstring graph either. This will be the main tool in the proof that every induced subgraph of our construction either has small treewidth or is not an outerstring graph.

Let G be a graph, and let $\prec$ be a linear order on $V(G)$. For $X \subseteq V(G)$, we let $\prec_X$ denote the restriction of $\prec$ to the set X. We say that the outerstring representation of G is *$\prec$-constrained* if for all $u, v \in V(G)$, we have $u \prec v$ if and only if the point at which the curve corresponding to u intersects the x-axis is to the left of the point at which the curve corresponding to v intersects the x-axis. It follows that, for every $X \subseteq V(G)$, the set of all curves in the representation corresponding to the vertices in X forms a $\prec_X$-constrained outerstring representation of $G[X]$. In particular, we have the following.

Lemma 5.5. *Let G be a graph. Assume that for every linear order $\prec$ on $V(G)$, there exists $X \subseteq V(G)$ such that $G[X]$ admits no $\prec_X$-constrained outerstring representation. Then G is not an outerstring graph.*

The following is implicit in [15].

Lemma 5.6. *For all $\ell \geq 4$, ℓ-long thetas are not outerstring graphs.*

Proof. Let T be an ℓ-long theta for some $\ell \geq 4$. In Proposition 6.2 of [15], it is shown that for every linear order $\prec$ on $V(T)$, there exists a 4-subset $X = \{x_1, x_2, x_3, x_4\}$ of vertices such that $x_1 \prec x_2 \prec x_3 \prec x_4$ and $E(T[W]) = \{x_1x_3, x_2x_4\}$. Clearly, this means there is no $\prec_X$-constrained outerstring representation of $T[X]$. Hence, by Lemma 5.5, the graph T is not an outerstring graph. □

In view of Lemmas 5.2 and 5.6, in order to prove Lemma 5.1, it suffices to show that, for every induced subgraph H of G_k^g with no induced long theta and every weight function w, there is a small w-balanced separator in H. We do this by finding small balanced separators with respect to certain weight functions on certain induced minors of H, which can then be translated back into a small w-balanced separator in H.

To this end, we fix $g, k \in \mathbb{N}$ and put $G = G_k^g$, $B = B(G)$, $M = M(G)$, and $S = S(G)$. Let H be an induced subgraph of G, and let w be a weight function on H. Let $B_H = B \cap V(H)$, $M_H = M \cap V(H)$, and $S_H = S \cap V(H)$. The graph H inherits from G the properties (iii), (iv), (v), and (vi) from Lemma 2.2, which are restated here.

Lemma 5.7. *The following hold.*

(i) *H is triangle-free.*
(ii) *If $u, v \in V(H)$ are adjacent, then $\deg_H(u) \leq 3$ or $\deg_H(v) \leq 3$, and $\{u, v\} \nsubseteq B_H$.*
(iii) *If $u, v \in B_H$ with $u \neq v$, then $\mathrm{dist}_{G_k^g}(u, v) \geq 2^g$.*
(iv) *If $u \in V(G_k^g)$ and $\deg_{G_k^g}(u) = 3$, then there is at most one $v \in N_{G_k^g}(u)$ such that $\deg_{G_k^g}(v) \geq 3$.*

For $b \in B$, let $N_H^M(b) = N_H(b) \cap M$ and $N_H^M[b] = N_H^M(b) \cup \{b\}$. Put $B_{H'} = \{N_H^M[b] : b \in B_H\}$, $M_{H'} = \{\{m\} : m \in M_H,\ m \notin \bigcup_{X \in B_{H'}} X\}$, and $S_{H'} = \{\{s\} : s \in S_H\}$. Let H' be the model in H induced by $B_{H'} \cup M_{H'} \cup S_{H'}$, and define $w' : V(H') \to [0, 1]$ by $w'(S) = \sum_{v \in S} w(v)$ for $S \in V(H')$. It is easy to see that w' is a weight function on H'. We note also that every element of $B_{H'}$ is uniquely identified by a vertex of B_H, and every element of $M_{H'}$ or $S_{H'}$ is uniquely identified by a vertex of M_H or S_H, respectively.

Lemma 5.8. *The following hold.*
(i) *$\deg_{H'}(v) \leq 2$ for every $v \in M_{H'} \cup S_{H'}$.*
(ii) *$\mathrm{dist}_{H'}(u_1, u_2) \geq 2^g/3 - 2$ for all pairs of distinct $u_1, u_2 \in B_{H'}$.*

Proof. If $v = \{s\} \in S_{H'}$, then $\deg_G(s) \leq 2$, so $\deg_H(s) \leq 2$ and thus $\deg_{H'}(\{s\}) \leq 2$. Next suppose that $v = \{m\} \in M_{H'}$, then $\deg_G(m) = 3$ and $N_G(m) = \{b, s_1, s_2\}$ for some $b \in B$ and $s_1, s_2 \in S$. Since m and b are adjacent in G, b cannot be a vertex of H, as then v would be an element of $N_H^M[b]$. Thus, $\deg_{H'}(\{m\}) \leq 2$. This proves (i). Statement (ii) follows from Lemma 5.7 (iii) and the observation that a path of length d from u_1 to u_2 in H' gives rise to a path of length at least $3(d + 2)$ in H. □

Note that H' here is defined analogously to Section 3, and so by Lemma 3.3, we have $\mathrm{tw}(H') \leq 2$. It follows from Lemma 5.3 that there is a w'-balanced separator $K' \subseteq V(H')$ for H' of size at most 3. It is straightforward to see that taking $K = \bigcup_{X \in K'} X$ gives a w-balanced separator in H. However, there is no bound on the size of the sets X, as each big vertex of G can have up to k medium neighbors, and our desired bound needs to be independent of k.

To remove this dependence on k, we now define a new induced minor of H related to H' and K'. First, we partition K' by defining the following sets:

$$
\begin{aligned}
Y'_{B,>3} &= \{b \in B_H : N_H^M[b] \in K',\ \deg_H(b) > 3\},\\
Y'_{B,\leq 3} &= \{b \in B_H : N_H^M[b] \in K',\ \deg_H(b) \leq 3\},\\
K'_{B,>3} &= \{N_H^M[b] : b \in Y'_{B,>3}\},\\
K'_{B,\leq 3} &= \{N_H^M[b] : b \in Y'_{B,\leq 3}\},\\
K'_M &= K' \cap M_{H'},\\
K'_S &= K' \cap S_{H'}.
\end{aligned}
$$

Note that $K' = K'_{B,>3} \cup K'_{B,\leq 3} \cup K'_M \cup K'_S$ and all of these subsets are pairwise disjoint. The set $K'_{B,>3}$ comprises "troublesome" vertices of K' in the sense that they are the vertices preventing $\bigcup_{S \in K'} S$ from having bounded size.

Let $\mathcal{N} = \bigcup_{b \in Y'_{B,>3}} \{\{n\} : n \in N_H^M(b)\}$ and $\mathcal{D} = \{\bigcup_{X \in V(D)} X : D \text{ is a component of } H' \setminus K'\}$. Let H'' be the model in H induced by $\mathcal{N} \cup \mathcal{D}$; it is straightforward to see that H'' is a bipartite graph with bipartition $(\mathcal{N}, \mathcal{D})$. Define $w'' : V(H'') \to [0, 1]$ by $w''(S) = \sum_{v \in S} w(v)$

for $S \in V(H'')$. Since $\bigcup_{S \in V(H'')} S \subseteq V(H)$, w'' is a weak weight function on H''. To bound the treewidth of H'', we need the following lemma.

Lemma 5.9 ([2, 3]). *Let T be a graph that does not contain K_3 as a subgraph. Let v_1, v_2, v_3 be distinct vertices of T, and assume that F is a connected induced subgraph of $T \setminus \{v_1, v_2, v_3\}$ such that $V(F)$ contains at least one neighbor of each of x_1, x_2, x_3, and that $V(F)$ is minimal subject to inclusion. Then, one of the following holds.*

(i) *For some distinct $\{i, j, k\} = \{1, 2, 3\}$, there exists P that is either a path from x_i to x_j or a hole containing the edge $x_i x_j$ such that*
  - *$V(F) = V(P) \setminus \{x_i, x_j\}$, and*
  - *x_k has at least two non-adjacent neighbors in F.*

(ii) *There is a vertex $a \in V(F)$ and three paths P_1, P_2, P_3, where P_i is a path from a to x_i, such that*
  - *$V(F) = (V(P_1) \cup V(P_2) \cup V(P_3)) \setminus \{x_1, x_2, x_3\}$, and*
  - *the sets $V(P_1) \setminus \{a\}$, $V(P_2) \setminus \{a\}$, and $V(P_3) \setminus \{a\}$ are pairwise disjoint, and*
  - *for distinct $i, j \in \{1, 2, 3\}$, there are no edges between $V(P_i)$ and $V(P_j)$ except possibly $x_i x_j$.*

Lemma 5.10. *At least one of the following holds.*

(i) *H'' has maximum degree less than 9.*

(ii) *H contains an ℓ-long theta as an induced subgraph, for some $\ell \geq 2^g - 1$.*

In particular, if H is an outerstring graph, then statement (i) holds.

Proof. Suppose that statement (i) does not hold. For $\{n\} \in \mathcal{N}$, we have $n \in M_H$ by construction, so $\deg_H(n) \leq 3$ and thus $\deg_{H''}(\{n\}) \leq 3$. This means that there is some $D \in \mathcal{D}$ with $\deg_{H''}(D) \geq 9$. Since $|K'_{B,>3}| \leq |K'| = 3$, there is $b \in Y'_{B,>3}$ such that there are at least three edges of H'' with one end D and the other end in $\{\{n\} \colon n \in N^M_H(b)\}$. It follows that there exist distinct $n_1, n_2, n_3 \in N^M_H(b)$ such that $N_H(n_i) \cap D \neq \varnothing$ for every $i \in \{1, 2, 3\}$.

Let $X \subseteq D$ be minimal (with respect to inclusion) such that $H[X]$ is connected and contains at least one neighbor of each of n_1, n_2, n_3. We now apply Lemma 5.9 with n_1, n_2, n_3 and $H[X]$.

Suppose that case (i) applies; let $\{i, j, k\} = \{1, 2, 3\}$ and P be such that P is a path in H from n_i to n_j, n_k has at least two non-adjacent neighbors in P, and $V(H[X]) = V(P) \setminus \{n_i, n_j\}$. Note that P is not a hole, since n_i and n_j are both adjacent to b, thus they are not adjacent to each other. Since n_k is adjacent to b in H, n_k is a medium vertex, thus $\deg_H(n_k) = 3$. Furthermore, every neighbor of n_k in P (of which there are at least 2) has degree at least 3 in H. This contradicts Lemma 5.7 (iv).

Suppose instead that case (ii) applies; let $a \in X$ and P_1, P_2, P_3 be such that P_i is a path from a to n_i for each $i \in \{1, 2, 3\}$, and $X = (V(P_1) \cup V(P_2) \cup V(P_3)) \setminus \{n_1, n_2, n_3\}$, and the sets $V(P_1) \setminus \{a\}$, $V(P_2) \setminus \{a\}$, and $V(P_3) \setminus \{a\}$ are pairwise anticomplete. Note that $\mathrm{dist}_H(a, b) \geq 2^g - 1$; this follows from Lemma 5.7 (iii) as $b \in B(G)$ and $a \in V(D) \subseteq G \setminus N[b]$ and a has degree at least 3 in G; so either $a \in B(G)$ and (iii) applies, or a is adjacent to a vertex a' in $B(G) \setminus \{b\}$ and the statement follows from (iii) applied to a' and b.

We now have that $H[V(P_1) \cup V(P_2) \cup V(P_3) \cup \{a, b\}]$ is an induced subgraph of H that is a theta of length at least $2^g - 1$, so statement (ii) holds.

□

From here onwards, we assume that H is an outerstring graph, so that we may proceed to bound the treewidth of H'' via Lemma 5.10 (i).

Corollary 5.11. *There exists $L'' \in \mathbb{N}$ (independent of k and H) such that* $\mathrm{tw}(H'') \leq L''$.

Proof. Let h_0 be as in Lemma 3.1. Since H'' is an induced minor of G and G does not contain $W_{h_0 \times h_0}$ as an induced minor, it follows that H'' does not contain $W_{h_0 \times h_0}$ as an induced minor, and so in particular H'' does not contain any subdivision of $W_{h_0 \times h_0}$ or $L(W_{h_0 \times h_0})$ as an induced subgraph. By Lemma 5.10, H'' has maximum degree less than 9. The result now follows from Theorem 5.4. □

By Lemma 5.3, H'' has a w''-balanced separator K'' of size at most $L''+1$. Let $K''_{\mathcal{N}} = K'' \cap \mathcal{N}$ and $K''_{\mathcal{D}} = K'' \cap \mathcal{D}$.

Lemma 5.12. *Let L'' be as in Corollary 5.11. Then*
 (i) $K''_{\mathcal{N}} \cup N_{H''}(K''_{\mathcal{D}})$ *is a w''-balanced separator in H'', and*
 (ii) $|K''_{\mathcal{N}} \cup N_{H''}(K''_{\mathcal{D}})| \leq 9(L''+1)$.

Proof. Let C be a component of $H'' \setminus (K''_{\mathcal{N}} \cup N_{H''}(K''_{\mathcal{D}}))$. First suppose that $V(C) \cap K''_{\mathcal{D}} \neq \varnothing$, and let $D \in V(C) \cap K''_{\mathcal{D}}$. Since $N_{H''}(D) \subseteq K''_{\mathcal{N}} \cup N_{H''}(K''_{\mathcal{D}})$, we have $\deg_{H'' \setminus (K''_{\mathcal{N}} \cup N_{H''}(K''_{\mathcal{D}}))}(D) = 0$, and so $V(C) = \{D\}$. As $H'[D]$ is a component of $H' \setminus K'$, it follows that $w''(D) = w'(H'[D]) \leq \frac{1}{2}$. Now suppose that $V(C) \cap K''_{\mathcal{D}} = \varnothing$. Then C is a connected induced subgraph of $H'' \setminus K''$, so in particular, there exists a component C^* of $H'' \setminus K''$ such that $V(C) \subseteq V(C^*)$. It follows that $w''(C) \leq w''(C^*) \leq \frac{1}{2}$. This proves (i).

To see that (ii) holds, we observe that $K''_{\mathcal{N}} \cup N_{H''}(K''_{\mathcal{D}})$ is obtained from K'' by removing a subset of its elements and replacing each removed element by at most nine new elements. Since $|K''| \leq L''+1$, the bound follows. □

We are ready to translate the balanced separators for H' and H'' back into a w-balanced separator in H.

Lemma 5.13. *Let $K^* = K'_S \cup K'_M \cup K'_{B,\leq 3} \cup K''_{\mathcal{N}} \cup N_{H''}(K''_{\mathcal{D}})$, and let $K = (\bigcup_{X \in K^*} X) \cup Y_{B,\geq 3}$. Then the following hold.*
 (i) *K is a w-balanced separator in H.*
 (ii) $|K| \leq 21 + 9(L''+1)$.

Proof. The induced subgraph $H \setminus ((\bigcup_{X \in K'_S \cup K'_M \cup K'_{B,\leq 3}} X) \cup Y_{B,\geq 3})$ of H, which will be denoted by F, has vertex set $\{n \colon \{n\} \in \mathcal{N}\} \cup (\bigcup_{D \in \mathcal{D}} V(D))$. Thus, for every component C of $F \setminus (\{n \colon \{n\} \in K''_{\mathcal{N}} \cup N_{H''}(K''_{\mathcal{D}})\})$, there is a corresponding component C'' of H'' such that $V(C) = \bigcup_{X \in C''} X$, and so it follows that $w(C) = w''(C'') \leq \frac{1}{2}$. This proves (i).

Since K'_S, K'_M, and $K'_{B,\leq 3}$ are all subsets of K', they each have size at most 3. Every element of K'_S and K'_M is a singleton set, and so $|\bigcup_{\{s\} \in K'_S} \{s\}| \leq 3$ and $|\bigcup_{\{m\} \in K'_M} \{m\}| \leq 3$. Every $N^M_H[b] \in K_{B,\leq 3}$ has size at most 4, and so

$$\left| \bigcup_{N^M_H[b] \in K'_{B,\leq 3}} N^M_H[b] \right| \leq 12.$$

Since $|K''_{\mathcal{N}} \cup N_{H''}(K''_{\mathcal{D}})| \leq 9(L''+1)$ by Lemma 5.12 (ii) and every element of $K''_{\mathcal{N}} \cup N_{H''}(K''_{\mathcal{D}})$ is a singleton, we have

$$\left| \bigcup_{\{n\} \in K''_{\mathcal{N}} \cup N_{H''}(K''_{\mathcal{D}})} \{n\} \right| \leq 9(L''+1).$$

Moreover, the set $Y_{B,\geq 3}$ has the same size as $K_{B,\geq 3}$, and thus it has size at most 3. Combining these bounds, we have $|K| \leq 3+3+12+9(L''+1)+3 = 21+9(L''+1)$. This proves (ii). ☐

We are now ready to prove Lemma 5.1.

Proof of Lemma 5.1. Let $g, k \in \mathbb{N}$, and let H be an induced subgraph of G^g_k that is an outerstring graph. Let L'' be as in Corollary 5.11. Then for every weight function w on H, by Lemma 5.13, there is a w-balanced separator X_w with $|X_w| \leq 21 + 9(L''+1)$. The claim now follows from Lemma 5.2, with $L = 42 + 18(L''+1)$. ☐

6. Completing the proof

We are now ready to complete the proof of our main result. As discussed at the end of Section 2, we only need to show the following.

Theorem 6.1. *There exist $t_0, L \in \mathbb{N}$ such that for all $g, k \in \mathbb{N}$, the following hold.*

(i) *G^g_k is $W_{t_0 \times t_0}$-induced-minor-free and K_{t_0,t_0}-induced-minor-free; and*
(ii) *if H is an induced subgraph of G and H is an outerstring graph, then* $\text{tw}(H) \leq L$.

Proof. Let h_0 and r_0 be as in Lemmas 3.1 and 4.1, and let $t_0 = \max\{h_0, r_0\}$. Let L be as in Lemma 5.1. Then, by Lemmas 3.1 and 4.1, G^g_k does not contain $W_{t_0 \times t_0}$ or K_{t_0,t_0} as an induced minor, and by Lemma 5.1, if an induced subgraph H of G^g_k is an outerstring graph, then $\text{tw}(H) \leq L$. ☐

We remark that, due to the more general setup of Lemma 5.10, our proof in fact gives the following stronger statement.

Theorem 6.2. *There exist $t_0, L \in \mathbb{N}$ such that for all $g, k \in \mathbb{N}$, the following hold.*

(i) *G^g_k is $W_{t_0 \times t_0}$-induced-minor-free and K_{t_0,t_0}-induced-minor-free; and*
(ii) *if H is an induced subgraph of G^g_k and* $\text{tw}(H) > L$, *then H contains an ℓ-long theta as an induced subgraph for some $\ell \geq 2^g - 1$.*

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