Four-coloring P_6-free graphs. II. Finding an excellent precoloring

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Abstract

This is the second paper in a series of two. The goal of the series is to give a polynomial time algorithm for the 4-COLORING PROBLEM and the 4-PRECOLORING EXTENSION PROBLEM restricted to the class of graphs with no induced six-vertex path, thus proving a conjecture of Huang. Combined with previously known results this completes the classification of the complexity of the 4-COLORING PROBLEM for graphs with a connected forbidden induced subgraph.

In this paper we give a polynomial time algorithm that starts with a 4-precoloring of a graph with no induced six-vertex path, and outputs a polynomial-size collection of so-called excellent precolorings. Excellent precolorings are easier to handle than general ones, and, in addition, in order to determine whether the initial precoloring can be extended to the whole graph, it is enough to answer the same question for each of the excellent precolorings in the collection. The first paper in the series deals with excellent precolorings, thus providing a complete solution to the problem.

1 Introduction

All graphs in this paper are finite and simple. We use $[k]$ to denote the set $\{1, \dots, k\}$. Let G be a graph. A *k-coloring* of G is a function $f : V(G) \to [k]$. A k-coloring is *proper* if for every edge $uv \in E(G)$, $f(u) \neq f(v)$, and G is *k-colorable* if G has a proper k-coloring. The k-COLORING PROBLEM is the problem of deciding, given a graph G, if G is k-colorable. This problem is well-known to be NP-hard for all $k \geq 3$.

Let G be a graph. For $X \subseteq V(G)$ we denote by $G|X$ the subgraph induced by G on X, and by $G \setminus X$ the graph $G|(V(G) \setminus X)$. We say that X is *connected* if $G|X$ is connected. If $X = \{x\}$ we write $G \setminus x$ to mean $G \setminus \{x\}$. For disjoint subsets $A, B \subset V(G)$ we say that A is *complete* to B if every vertex of A is adjacent to every vertex of B, and that A is *anticomplete* to B if every vertex of A is non-adjacent to every vertex of B. If $A = \{a\}$ we write a is complete (or anticomplete) to B to mean that $\{a\}$ is complete (or anticomplete) to B. If a is not complete and not anticomplete to B, we say that a is *mixed* on B. Finally, if H is an induced subgraph of G and $a \in V(G) \setminus V(H)$, we say that a is *complete to, anticomplete to* or *mixed on* H if a is complete to, anticomplete to or mixed on $V(H)$, respectively. For $X \subseteq V(G)$, we say that $e \in E(G)$ is *an edge of* X if both endpoints of e are in X. For $v \in V(G)$ we write $N_G(v)$ (or $N(v)$ when there is no danger of confusion) to mean the set of vertices of G that are adjacent to v. Observe that since G is simple, $v \notin N(v)$. For $X \subseteq V(G)$ we define $N(X) = (\bigcup_{v \in X} N(v)) \setminus X$.

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A function $L : V(G) \to 2^{[k]}$ that assigns a subset of $[k]$ to each vertex of a graph G is a *k-list assignment* for G. For a k-list assignment L, a function $f : V(G) \to [k]$ is an L-coloring if f is a k-coloring of G and $f(v) \in L(v)$ for all $v \in V(G)$. We say that G is *L-colorable*, or that (G, L) is *colorable*, if G has a proper L-coloring. The k-LIST COLORING PROBLEM is the problem of deciding, given a graph G and a k-list assignment L, if G is L-colorable. Since this generalizes the k-coloring problem, it is NP-hard for all $k \geq 3$.

A *k-precoloring* (G, X, f) of a graph G is a function $f : X \to [k]$ for a set $X \subseteq V(G)$ such that f is a proper k-coloring of $G|X$. Equivalently, a k-precoloring is a k-list assignment L in which $|L(v)| \in \{1, k\}$ for all $v \in V(G)$. A *k-precoloring extension* for (G, X, f) is a proper k-coloring g of G such that $g|_X = f$, and the k-PRECOLORING EXTENSION PROBLEM is the problem of deciding, given a graph G and a k-precoloring (G, X, f), if (G, X, f) has a k-precoloring extension.

We denote by P_t the path with t vertices. Given a path P, its *interior* is the set of vertices that have degree two in P; the interior of P is denoted by P^*. A *path in a graph* G is a sequence $v_1 - \ldots - v_t$ of pairwise distinct vertices where for $i, j \in [t]$, v_i is adjacent to v_j if and only if $|i - j| = 1$; the *length* of this path is t. We denote by $V(P)$ the set $\{v_1, \ldots, v_t\}$, and if $a, b \in V(P)$, say $a = v_i$ and $b = v_j$ and $i < j$, then $a - P - b$ is the path $v_i - v_{i+1} - \ldots - v_j$, and $b - P - a$ is the path $v_j - v_{j-1} - \ldots - v_i$. For $v \in V(P)$, the *neighbors of v in P* are the neighbors of v in $G|V(P)$. A *P_t in G* is a path of length t in G. A graph is *P_t-free* if there is no P_t in G.

Throughout the paper by "polynomial time" or "polynomial size" we mean running time, or size, that are polynomial in $|V(G)|$, where G is the input graph. Since the k-COLORING PROBLEM and the k-PRECOLORING EXTENSION PROBLEM are NP-hard for $k \geq 3$, their restrictions to graphs with a forbidden induced subgraph have been extensively studied; see [2, 8] for a survey of known results. In particular, the following is known (given a graph H, we say that a graph G is *H-free* if no induced subgraph of G is isomorphic to H):

Theorem 1 ([8]). *Suppose that $P \neq NP$. Let H be a (fixed) graph, and let $k > 2$. If the k-COLORING PROBLEM can be solved in polynomial time when restricted to the class of H-free graphs, then every connected component of H is a path.*

Thus if we assume that H is connected, then the question of determining the complexity of k-coloring H-free graphs is reduced to studying the complexity of coloring graphs with certain induced paths excluded, and a significant body of work has been produced on this topic. Below we list a few such results.

Theorem 2 ([1]). *The 3-COLORING PROBLEM can be solved in polynomial time for the class of P_7-free graphs.*

Theorem 3 ([6]). *For every $k \in \mathbb{N}$, the k-COLORING PROBLEM can be solved in polynomial time for the class of P_5-free graphs.*

Theorem 4 ([7]). *The 4-COLORING PROBLEM is NP-complete for the class of P_7-free graphs.*

Theorem 5 ([7]). *For all $k \geq 5$, the k-COLORING PROBLEM is NP-complete for the class of P_6-free graphs.*

The only cases for which the complexity of k-coloring P_t-free graphs is not known are $k = 4$, $t = 6$, and $k = 3$, $t \geq 8$. This is the second paper in a series of two. The main result of the series is the following:

Theorem 6. *The 4-PRECOLORING EXTENSION PROBLEM can be solved in polynomial time for the class of P_6-free graphs.*

Theorem 6 proves a conjecture of Huang [7], thus resolving the former open case above, and completes the classification of the complexity of the 4-COLORING PROBLEM for graphs with a connected forbidden induced subgraph.

A *starred precoloring* is a 7-tuple $P = (G, S, X_0, X, Y, Y^*, f)$ such that

(A) $f : S \cup X_0 \to \{1, 2, 3, 4\}$ is a proper coloring of $G|(S \cup X_0)$;

(B) $V(G) = S \cup X_0 \cup X \cup Y \cup Y^*$ and the sets S, X_0, X, Y, Y^* are disjoint;

(C) $G|S$ is connected and no vertex in $V(G) \setminus S$ is complete to S;

(D) every vertex in Y has a neighbor in S;

(E) for every vertex $x \in X$, $|f(N(x) \cap S)| \geq 2$;

(F) Y is anticomplete to Y^*;

(G) no vertex in X is mixed on a component of $G|Y^*$;

(H) for every component C of $G|Y^*$, there is a vertex in $S \cup X_0 \cup X$ complete to $V(C)$.

A starred precoloring is *excellent* if $Y = \emptyset$; we write it is as a 6-tuple (G, S, X_0, X, Y^*, f). The set S is called the *seed* of the starred precoloring. We define $L_P(v) = L_{S,f}(v) = \{1,2,3,4\} \setminus (f(N(v) \cap S))$. A *precoloring extension* of a starred precoloring is a function $f' : V(G) \setminus (S \cup X_0) \to \{1,2,3,4\}$ such that $f' \cup f$ is a proper coloring of G. The main result of the first paper of the series [4] is

Theorem 7. *For every fixed positive integer C, there exists a polynomial-time algorithm with the following specifications.*

Input: *An excellent starred precoloring $P = (G, S, X_0, X, Y^*, f)$ of a P_6-free graph G with $|S| \leq C$.*

Output: *A precoloring extension of P or a determination that none exists.*

In this paper, we reduce the 4-PRECOLORING EXTENSION PROBLEM for P_6-free graphs to the case handled by Theorem 7. Our main result is the following.

Theorem 8. *There exists an integer $C > 0$ and a polynomial-time algorithm with the following specifications.*

Input: *A 4-precoloring (G, X_0, f) of a P_6-free graph G.*

Output: *A collection $\mathcal{L}$ of excellent starred precolorings of G such that*

1. *$|\mathcal{L}| \leq |V(G)|^C$,*
2. *for every $(G', S', X'_0, X', Y^*, f') \in \mathcal{L}$*
   - *$|S'| \leq C$,*
   - *$X_0 \subseteq S' \cup X'_0$,*
   - *G' is an induced subgraph of G, and*
   - *$f'|_{X_0} = f|_{X_0}$.*
3. *if we know for every $P \in \mathcal{L}$ whether P has a precoloring extension, then we can decide in polynomial time if (G, X_0, f) has a 4-precoloring extension; and*
4. *given a precoloring extension for every $P \in \mathcal{L}$ such that P has a precoloring extension, we can compute a 4-precoloring extension for (G, X_0, f) in polynomial time, if one exists.*

Clearly Theorem 8 and Theorem 7 together imply Theorem 6, and, as an immediate corollary, we obtain that the 4-COLORING PROBLEM for P_6-free graphs is also solvable in polynomial time. In contrast, the 4-LIST COLORING PROBLEM restricted to P_6-free graphs is NP-hard as proved by Golovach, Paulusma, and Song [8].

The proof of Theorem 8 consists of several steps. At each step we replace the problem that we are trying to solve by a polynomial-sized collection of simpler problems, where by "simpler" we mean "closer to being an excellent starred precoloring". The strategy at every step is to "guess" (by exhaustively enumerating) a bounded number of vertices that have certain key properties, and their colors, add these vertices to the seed, and show that the resulting precoloring is better than the one we started with. In this process we make sure that the size of the seed remains bounded, so that we can apply Theorem 7. Since we will follow a gradual process to establish all the required properties, on of our main challenges will be to ensure that each step preserves the progress we have made in the previous step; and we will introduce a few tools specifically for this purpose (including "normal subcases", subproblems that are derived from the original in a restricted way that preserves some of its properties).

This paper is organized as follows. In Section 1.1, we introduce a few helpful definitions and lemmas. In Section 2, we transform an instance of the 4-PRECOLORING EXTENSION PROBLEM into a polynomial number of subproblems, each of which is a starred precoloring that has additional properties. In particular, we ensure that every component of $G|(Y \cup Y^*)$ that contains a vertex v with $L_P(v) = [4]$ is completely contained in Y^*. The axioms listed at the beginning of Section 2 will serve as our road map: we start with a precoloring extension, and we will obtain a starred precoloring. In Section 3, we start with a list of starred precolorings, and replace each of the subproblems produced in Section 2 by yet another polynomial-sized collection of problems and prove Theorem 8. Again, a set of axioms listed at the beginning of the section serves as a road map; and while Section 2 mainly deals with vertices with $L_P(v) = [4]$, Section 3 deals with vertices with 3 colors in their lists.

1.1 Definitions

For two functions f, f' with $f : X \to \{1, \ldots, k\}$ and $f' : Y \to \{1, \ldots, k\}$ such that $f|_{X \cap Y} \equiv f'|_{X \cap Y}$, we define their *union* $f \cup f' : X \cup Y \to \{1, \ldots, k\}$ as $(f \cup f')(z) = f(z)$ if $z \in X$, and $(f \cup f')(z) = f'(z)$ if $z \in Y$. For a set X with $X = \{x\}$, we do not distinguish between X and x (when there is no danger of confusion).

In Section 2, we will work with seeded precolorings: A *seeded precoloring* of a graph G is a 7-tuple

$$P = (G, S, X_0, X, Y_0, Y, f)$$

such that

- the function $f : (S \cup X_0) \to \{1, 2, 3, 4\}$ is a proper coloring of $G|(S \cup X_0)$;
- $V(G) = S \cup X_0 \cup X \cup Y_0 \cup Y$; and
- S, X_0, X, Y_0, Y are pairwise disjoint.

The set S is called the *seed* of the seeded precoloring. A *precoloring extension* of a seeded precoloring P is a 4-precoloring extension of $(G, S \cup X_0, f)$. For a seeded precoloring P and a collection $\mathcal{L}$ of seeded precolorings, we say that $\mathcal{L}$ is an *equivalent collection* for P if P has a precoloring extension if and only if at least one of the seeded precolorings in $\mathcal{L}$ does, and, given a precoloring extension of a member of $\mathcal{L}$, we can construct a precoloring of P in polynomial time.

Given a seeded precoloring $P = (G, S, X_0, X, Y_0, Y, f)$ and a seeded precoloring P', we say that P' is a *normal subcase* of P if $P' = (G \setminus Z, S', X_0', X', Y_0', Y', f')$ such that

- $Z \subseteq Y_0$;
- $G|S'$ connected and $S \subseteq S'$;
- every vertex in $X_0' \cap Y_0$ has a neighbor in S';
- $S \subseteq S' \subseteq S \cup X \cup Y_0 \cup Y$;
- $X_0 \subseteq X_0' \subseteq X_0 \cup X \cup Y_0 \cup Y$, and
- there is a function $g : (S' \cup X_0') \setminus (S \cup X_0) \to \{1, 2, 3, 4\}$ such that $f' = f \cup g$.

Normal subcases will be helpful in our arguments, because we will be able to deduce that if P' is a normal subcase of P, then P' retains some of the nice properties of P. The simplifications we make throughout this paper, such as adding vertices to the seed and separating connected components into difference instances, will usually result in normal subcases.

For a seeded precoloring $P = (G, S, X_0, X, Y_0, Y, f)$ of a graph G, we let $L_P(v) = L_{S,f}(v) = f(v)$ for $v \in S \cup X_0$, and $L_P(v) = L_{S,f}(v) = \{1, 2, 3, 4\} \setminus f(N(v) \cap S)$ otherwise. Note that either $|L_P(v)| = 1$, or $L_P(v)$ is determined by the neighbors of v in S, but not those in X_0. For a set $Z \subseteq V(G)$ and list $L \subseteq \{1, 2, 3, 4\}$, we let Z_L denote the set of vertices v in Z with $L_P(v) = L$. This notation will be used throughout the paper, for example in (vii).

We finish this section with two useful results.

Lemma 1. *Let G be a graph and let $X \subseteq V(G)$ be connected. If $v \in V(G) \setminus X$ is mixed on X, the there is an edge xy of X such that v is adjacent to x and not to y.*

Proof. Since v is mixed on X, both the sets $N(v) \cap X$ and $X \setminus N(v)$ are non-empty. Now since X is connected, there exist $x \in N(v) \cap X$ and $y \in X \setminus N(v)$ such that x is adjacent to y, as required. This proves Lemma 1. □

Theorem 9 ([5]). *There is a polynomial time algorithm that tests, for graph H and a list assignment L with $|L(v)| \leq 2$ for every $v \in V(H)$, if (H, L) is colorable, and finds a coloring if one exists.*

2 Establishing the Axioms on Y_0

Given a P_6-free graph G and a precoloring (G, A, f), our goal is to construct a polynomial number of seeded precolorings $P = (G, S, X_0, X, Y_0, Y, f)$ satisfying the following axioms, and such that if we can decide for each of them if it has a precoloring extension, then we can decide if (G, A, f) has a 4-precoloring extension, and construct one if it exists.

(i) $G \setminus X_0$ is connected.

(ii) S is connected and no vertex in $V(G) \setminus S$ is complete to S.

(iii) $Y_0 = V(G) \setminus (N(S) \cup X_0 \cup S)$.

(iv) No vertex in $V(G) \setminus (Y_0 \cup X_0)$ is mixed on an edge of Y_0.

(v) If $|L_{S,f}(v)| = 1$ and $v \notin S$, then $v \in X_0$; if $|L_{S,f}(v)| = 2$, then $v \in X$; if $|L_{S,f}(v)| = 3$, then $v \in Y$; and if $|L_{S,f}(v)| = 4$, then $v \in Y_0$.

(vi) There is a color $c \in \{1, 2, 3, 4\}$ such that for every vertex $y \in Y$ with a neighbor in Y_0, $f(N(y) \cap S) = \{c\}$. We let $L = \{1, 2, 3, 4\} \setminus \{c\}$.

(vii) With L as in (vi), we let Y_L^* be the subset of Y_L of vertices that are in connected components of $G|(Y_0 \cup Y_L)$ containing a vertex of Y_0. Then no vertex of $Y \setminus Y_L^*$ has a neighbor in $Y_0 \cup Y_L^*$, and no vertex in X is mixed on an edge of $Y_0 \cup Y_L^*$.

(viii) With Y_L^* as in (vii), for every component C of $G|(Y_0 \cup Y_L^*)$, there is a vertex v in X complete to C.

We begin by establishing the first axiom.

Lemma 2. *Given a 4-precoloring (G, X_0, f) of a P_6-free graph G, there is an algorithm with running time $O(|V(G)|^2)$ that outputs a collection $\mathcal{L}$ of seeded precolorings such that:*

- *$|\mathcal{L}| \leq |V(G)|$;*
- *every $P' \in \mathcal{L}$ is of the form $P' = (G|(V(C) \cup X_0), \emptyset, X_0, \emptyset, V(C), \emptyset, f)$ for a component C of $G \setminus X_0$;*
- *every $P' \in \mathcal{L}$ satisfies* (i)
- *(G, X_0, f) has a 4-precoloring extension if and only if each of the seeded precolorings $P' \in \mathcal{L}$ has a precoloring extension; and*
- *given a precoloring extension for each of the seeded precolorings $P' \in \mathcal{L}$, we can compute a 4-precoloring extension for (G, X_0, f) in polynomial time.*

Proof. For each connected component C of $G \setminus X_0$, the algorithm outputs the seeded precoloring $(G|(V(C) \cup X_0), \emptyset, X_0, \emptyset, V(C), \emptyset, f)$. Since the coloring is fixed on X_0, it follows that (G, X_0, f) has a 4-precoloring extension if and only if the 4-precoloring on X_0 can be extended to every connected component C of $G \setminus X_0$. This implies the statement of the lemma. □

The next lemma is used to arrange the following axioms, which we restate:

(ii) S is connected and no vertex in $V(G) \setminus S$ is complete to S.

(iii) $Y_0 = V(G) \setminus (N(S) \cup X_0 \cup S)$.

Lemma 3. *There is a constant C such that the following holds. Let $P = (G, \emptyset, X_0, \emptyset, Y_0, \emptyset, f)$ be a seeded precoloring of a P_6-free graph G with P satisfying* (i)*. Then there is an algorithm with running time $O(|V(G)|^C)$ that outputs an equivalent collection $\mathcal{L}$ for P such that*

- *$|\mathcal{L}| \leq |V(G)|^C$;*
- *every $P' \in \mathcal{L}$ is a normal subcase of G;*
- *every $P' = (G', S', X_0', X', Y_0', Y', f') \in \mathcal{L}$ with seed S' satisfies $|S'| \leq C$; and*
- *every $P' \in \mathcal{L}$ satisfies* (i), (ii) *and* (iii).

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension for P in polynomial time.

Proof. If $|V(G) \setminus X_0| \leq 5$, we enumerate all possible colorings. Now let $v \in V(G) \setminus X_0$, and let $S' = \{v\}$. While there is a vertex w in $V(G) \setminus S'$ complete to S', we add w to S'. Let S denote the set S' when this procedure terminates. If either $|S| \geq 5$ or $(G|(S \cup X_0), \emptyset, X_0, S, \emptyset, \emptyset, f)$ has no precoloring extension, then we output $\emptyset$. Otherwise, we construct $\mathcal{L}$ as follows. For every proper coloring f' of $G|S$ such that $f \cup f'$ is a proper coloring of $G|(S \cup X_0)$, we add

$$P' = (G, S, X_0 \setminus S, N(S) \setminus X_0, V(G) \setminus (X_0 \cup S \cup N(S)), \emptyset, f \cup f')$$

to $\mathcal{L}$. Since $|S| \leq 4$, it follows that the first three bullets hold, and (iii) holds for P' by the definition of P'. Since X_0 is unchanged, it follows that (i) holds. Since S is a maximal clique, we have that (ii) holds for P'. This concludes the proof. □

The next four lemmas are technical tools that we use several times in the course of the proof. They are used to show that if we start with a seeded precoloring that has certain properties, and then move to one of its normal subcases, then these properties are preserved (or at least they can be restored with a simple modification).

For a seeded precoloring $P = (G, S, X_0, X, Y_0, Y, f)$, a *type* is a subset of S. For $v \in V(G) \setminus (S \cup X_0)$, the *type of* v, denoted by $T_P(v) = T_S(v)$, is $N(v) \cap S$. For a type T and a set A, we let $A(T) = \{v \in A : T_P(v) = T\}$.

Lemma 4. *Let $P = (G, S, X_0, X, Y_0, Y, f)$ be a seeded precoloring of a P_6-free graph G satisfying* (ii) *and* (iii)*, and let Let $T, T' \subseteq S$ with $|f(T)| = |f(T')| = 1$ and such that $f(T) \neq f(T')$. Let $y, y' \in N(Y_0)$ such that $T_P(y) = T$ and $T_S(y') = T'$. Let $z, z' \in Y_0$ be such that yz and $y'z'$ are edges, and suppose that z is non-adjacent to z' and that y is non-adjacent to y'. Then either yz' or $y'z$ is an edge.*

Proof. Suppose both the pairs yz' and $y'z$ are non-adjacent. Since P satisfies (ii) and (iii), it follows that $G|S$ is connected and both y, y' have neighbors in S. Let Q be a shortest path from y to y' with interior in S. Since $|f(T)| = |f(T')| = 1$ and $f(T) \neq f(T')$, it follows that $T \cap T' = \emptyset$, and so $|Q^*| > 1$ (recall that Q^* denotes the interior of Q). But now $z - y - Q - y' - z'$ is a path with at least six vertices in G, a contradiction. This proves Lemma 4. □

Lemma 5. *Let $P = (G, S, X_0, X, Y_0, Y, f)$ be a seeded precoloring of a P_6-free graph satisfying* (ii), (iii) *and* (iv)*, and let $P' = (G', S', X_0', X', Y_0', Y', f')$ be a normal subcase of P satisfying* (iii)*. Then no $v \in Y_0 \setminus (S' \cup Y_0')$ has both a neighbor in S' and a neighbor in Y_0'.*

Proof. Suppose that such v exists. Let $y \in Y_0'$ be a neighbor of v. Since P' is a normal subcase of P, P' satisfies (ii). Since v has both a neighbor in Y_0' and a neighbor in S', and since P' satisfies (iii), it follows that $v \in X' \cup Y' \cup X_0'$. Since $v \in Y_0$, it follows that v is anticomplete to S. Since $v \notin Y_0'$, it follows that v has a neighbor in $S' \setminus S \subseteq X \cup Y \cup Y_0$. Since P' satisfies (ii), there is a path Q from v to a vertex s of S with $Q^* \subseteq S'$. Then $V(Q) \setminus \{v\}$ is anticomplete to Y_0'. Let R be the maximal subpath of $v - Q - s$, with $v \in V(R)$, such that $V(R) \subseteq Y_0$. Then $s \notin V(R)$, and there is a unique vertex $t \in V(Q) \setminus V(R)$ with a neighbor in $V(R)$. Since $t \in N(Y_0)$, it follows that $t \notin S \cup Y_0$, and so $t \in X \cup Y$. But t is mixed on $V(R) \cup \{y\} \subseteq Y_0$, contrary to the fact that P satisfies (iv). This proves Lemma 5. □

Lemma 6. *There is a constant C such that the following holds. Let $P = (G, S, X_0, X, Y_0, Y, f)$ be a seeded precoloring of a P_6-free graph G with P satisfying* (i)*, and let $P' = (G', S', X_0', X', Y_0', Y', f')$ be a normal subcase of P satisfying* (iii) *and* (iv)*. Then there is an algorithm with running time $O(|V(G)|^C)$ that outputs an equivalent collection $\mathcal{L}$ for P', such that $|\mathcal{L}| \leq 1$, and if $\mathcal{L} = \{P''\}$, then*

- *there is a $Z \subseteq Y_0'$ such that $P'' = (G' \setminus Z, S', X_0', Y_0' \setminus Z, Y', f)$ and P'' is a normal subcase of P';*
- *P'' satisfies* (i)—(iv)*;*
- *if P' satisfies* (v)*, then P'' satisfies* (v).

Moreover, given a precoloring extension of P'', we can compute a precoloring extension for P in polynomial time.

Proof. Since P' is a normal subcase of P, it follows that $S \subseteq S'$, and $G|S'$ is connected; therefore P' satisfies (ii). We may assume that P' does not satisfy (i), for otherwise we can set $\mathcal{L} = \{P'\}$. Now let C be a connected component of $G' \setminus X_0'$ with $S' \cap V(C) = \emptyset$. From (iii), it follows that $V(C) \subseteq Y_0'$ and C is a component of $G|Y_0'$.

Let $x \in N(V(C)) \cap (X_0' \setminus X_0)$. Since P satisfies (i), such a vertex x exists. By Lemma 5, $x \in X \cup Y$. Since P' satisfies (iv), it follows from Lemma 1 that x is complete to $V(C)$. Let $f'(x) = c$. Then in every precoloring extension d of P' we have $d(v) \neq c$ for every $v \in V(C)$.

Let $A = \{v \in X_0' : f'(v) \neq c\}$. By Theorem 2, and since G is P_6-free, we can decide in polynomial time if $(G'|(V(C) \cup A), A, f'|_A)$ has a precoloring extension with colors in $\{1, 2, 3, 4\} \setminus \{c\}$. If not, then P' has no precoloring extension, and we set $\mathcal{L} = \emptyset$. If $(G'|(V(C) \cup A), A, f'|_A)$ has a precoloring extension using only colors in $\{1, 2, 3, 4\} \setminus \{c\}$, then P' has a precoloring extension if and only if $(G' \setminus V(C), S', X_0', X', Y_0' \setminus V(C), Y', f')$ does.

We repeat this process a polynomial number of times until $G' \setminus X_0'$ is connected, and output the resulting seeded precoloring $P'' = (G'', S', X_0', X', Y_0'', Y', f')$ satisfying (i). Since $Y_0'' \subseteq Y_0'$, and the other sets of P'' remain the same as in P', it follows that the P'' satisfies (ii)–(iv), and if P' satisfies (v), then so does P''. This proves Lemma 6. □

Lemma 7. *Let $P = (G, S, X_0, X, Y_0, Y, f)$ be a seeded precoloring of a P_6-free graph satisfying* (ii)*,* (iii) *and* (iv)*, and let $P' = (G', S', X_0', X', Y_0', Y', f')$ be a normal subcase of P satisfying* (iii)*. Then P' satisfies* (iv)*. Moreover, if P satisfies* (vi)*, then P' satisfies* (vi).

Proof. Since P' is a normal subcase of P, P' satisfies (ii). First we show that P' satisfies (iv). Suppose not, then there exists $v \in V(G) \setminus X_0'$ mixed on an edge xy of Y_0', say v is adjacent to y and not to x. It follows that $v \in X' \cup Y'$, and since P satisfies (iv), $v \in Y_0$. By the definition of $X' \cup Y'$, v has a neighbor in S'. So $v \in Y_0 \setminus (Y_0' \cup S')$ has a neighbor in S' and Y_0', which contradicts Lemma 5. This proves that P' satisfies (iv).

Next assume that P satisfies (vi). We show that P' satisfies (vi). Let L be the set as in (vi) with respect to P. Suppose there exists $y \in N(Y_0')$ with $L_{P'}(y) \neq L$ and $|L_{P'}(y)| = 3$. Since P satisfies (vi), it follows that $y \in Y_0 \setminus Y_0'$, and y has a neighbor $s \in S'$, contrary to Lemma 5. This proves that P' satisfies (vi).

This completes the proof of Lemma 7. □

The next lemma is another technical tool, used to establish axioms (iv) and (vii). We want to show that no vertex in a certain set has exactly one neighbor among vertex set of an edge of another set; so we replace "edge" by "clique of size j" and establish this property by an iterative procedure. Since we may assume that there is no clique of size 5 (which would preclude being 4-colorable), this statement is true for $j = 5$; so we establish it next for $j = 4$, then $j = 3$, and finally $j = 2$, which yields the desired statement.

Lemma 8. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y_0, Y, f)$ be a seeded precoloring of a P_6-free graph G with P satisfying* (i)*,* (ii) *and* (iii)*. Let $L \subseteq [4]$ with $|L| = 3$, let c_4 be the unique element of $[4] \setminus L$. Let $R \subseteq Y_0 \cup Y_L$ such that $Y_0 \subseteq R$. Assume further that if $t \in (X \cup Y) \setminus R$ has a neighbor in R, then for every $z \in R$, $L_P(t) \neq L_P(z)$, and that there is no path $t - z_1 - z_2 - z_3$ with $t \in (X \cup Y) \setminus R$ and $z_1, z_2, z_3 \in R$. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs an equivalent collection $\mathcal{L}$ for P such that*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;
- *every* $P' \in \mathcal{L}$ *is a normal subcase of* P;
- *every* $P' = (G', S', X'_0, X', Y'_0, Y', f') \in \mathcal{L}$ *with seed* S' *satisfies* $|S'| \leq q(|S|)$;
- *every* $P' \in \mathcal{L}$ *satisfies* (ii) *and* (iii).
- *no vertex of* $(X' \cup Y') \setminus R$ *is mixed on an edge of* $(Y' \cup Y'_0) \cap R$.

Moreover, for every $P' \in \mathcal{L}$*, given a precoloring extension of* P'*, we can compute a precoloring extension of* P *in polynomial time.*

Proof. If G contains a K_5, then P has no precoloring extension; we output $\mathcal{L} = \emptyset$ and stop. Thus from now on we assume that G has no cliques of size five. Let $Y_0^5 = R$ and let $Z^5 = (X \cup Y) \setminus R$. Let $\mathcal{T}^5$ be the set of types (with respect to S) of vertices in Z^5, and set $j = 4$.

Let $\mathcal{Q}_j$ be the set of $|\mathcal{T}^{j+1}|$-tuples $(S^{j,T})_{T \in \mathcal{T}^{j+1}}$, where each $S^{j,T} \subseteq Z^{j+1}(T)$ and $S^{j,T}$ is constructed as follows (starting with $S^{j,T} = \emptyset$):

- If $R = Y_0$ or $c_4 \in f(T)$ proceed as follows. While there is a vertex $z \in Z^{j+1}(T)$ complete to $S^{j,T}$ and such that there is a clique $\{a_1, \dots, a_j\} \subseteq {Y_0}^{j+1}$ with $N(z) \cap \{a_1, \dots, a_j\} = \{a_1\}$, choose such z with $N(z) \cap R$ maximal and add it to $S^{j,T}$.
- If $R \neq Y_0$ and $c_4 \notin f(T)$, while there is $z \in Z^{j+1}(T)$ complete to $S^{j,T}$ such that there is clique $\{a_1, \dots, a_j\} \subseteq {Y_0}^{j+1}$ with $N(z) \cap \{a_1, \dots, a_j\} = \{a_1\}$, add z to $S^{j,T}$. Let $X_0(z)$ be the set of all $z' \in Z^{j+1}(T)$ such that
  - z' is complete to $S^{j,T} \setminus \{z\}$
  - there is a clique $\{b_1, \dots, b_j\} \subseteq {Y_0}^{j+1}$ such that $N(z') \cap \{b_1, \dots, b_j\} = \{b_1\}$, and
  - $N(z') \cap R$ is a proper subset of $N(z) \cap R$.

  When no such vertex z exists, let $X_0^{j,T} = \bigcup_{z \in S^{j,T}} X_0(z)$. Define $f'(z') = c_4$ for every $z' \in X_0^{j,T}$ (observe that since $c_4 \notin f(T)$, it follows that $c_4 \in L_P(z')$).

Since G has no clique of size five, it follows that $|S^{j,T}| \leq 4$ for all T. Let $Q \in \mathcal{Q}_j$; write $Q = (S^{j,T})_{T \in \mathcal{T}^{j+1}}$. Let $S^j = S^{j,Q} = \bigcup_{T \in \mathcal{T}^{j+1}} S^{j,T}$. Let $Y_0^j = Y_0^{j,Q} = Y_0^{j+1} \setminus N(S^j)$, $X_0^j = X_0^{j,Q} = \bigcup_{T \in \mathcal{T}^{j+1}} X_0^{j,T}$. $Z^j = Z^{j,Q} = (Z^{j+1} \setminus X_0^j) \cup (Y_0^{j+1} \setminus Y_0^j)$ and let $\mathcal{T}^j$ be the set of types of Z^j (in P). If $j > 2$, decrease j by 1 and repeat the construction above, to obtain a new set $\mathcal{Q}_{j-1}$; repeat this for each $Q \in \mathcal{Q}_j$.

Suppose $j = 2$. Then Q was constructed by fixing $Q_4 \in \mathcal{Q}_4$, constructing $\mathcal{Q}_3$ (with Q_4 fixed), fixing $Q_3 \in \mathcal{Q}_3$, constructing $\mathcal{Q}_2$ (with Q_3 fixed), and finally fixing $Q \in \mathcal{Q}_2$. Write $Q_2 = Q$. For consistency of notation we write $Q_5 = \emptyset$, $Z^5 = Z^{5,Q_5}$ and $Y_0^5 = Y_0^{5,Q_5}$. Let $S' = S \cup \bigcup_{j=2}^4 S^{j,Q_j}$. If $R \neq Y_0$, let $X'_0 = X_0 \cup \bigcup_{j=2}^4 X_0^{j,Q_j}$; if $R = Y_0$, let $X'_0 = X_0$.

For every function $f' : S' \setminus S \to \{1,2,3,4\}$ such that $f \cup f'$ is a proper coloring of $G|(S' \cup X'_0)$, let

$$P_{f',Q} = (G, S', X'_0, Z^{2,Q} \cap X, Y_0^{2,Q}, Z^{2,Q} \cap Y, f \cup f').$$

Let $\mathcal{L}$ be the set of all $P_{Q,f'}$ as above. Observe that S' is obtained from S by adding a clique of size at most four for each type in $\mathcal{T}^j$ at each of the three steps ($j = 4,3,2$), and since $|\mathcal{T}^j| \leq 2^{|S|}$ for every j, it follows that $|S \cup S'| \leq |S| + 12 \times 2^{|S|}$. Since $|S' \setminus S| \leq 12 \times 2^{|S|}$, it follows that $|\mathcal{L}| \leq (4|V(G)|)^{12 \times 2^{|S|}}$.

In the remainder of the proof we show that every $P_{Q,f'} \in \mathcal{L}$ satisfies the required properties.

(1) $S \cup \bigcup_{k=j}^4 S^k$ *is connected for every* $j \in \{2, \dots, 4\}$*. In particular* S' *is connected.*

Since for every j, we have that $S^{j,Q_j} \subseteq Z^{j+1}$, it follows that every vertex of S^{j,Q_j} has a neighbor in $S \cup \bigcup_{k=j+1}^4 S^{k,Q_k}$, and (1) follows.

(2) *Let* $j \in \{2, \dots, 5\}$*. There is no path* $z - a - b - c$ *with* $z \in Z^{j,Q_j}$ *and* $a, b, c \in Y_0^{j,Q_j}$.

Suppose for a contradiction that there exist j and z violating (2); we may assume z is chosen with j maximum. By assumption, $j \neq 5$ and $z \in Y_0^{j,Q_j} \setminus Y_0^{j+1,Q_{j+1}}$. It follows that z has a neighbor $z' \in S^{j,Q_j}$ and that z is anticomplete to $S \cup \bigcup_{k=j+1}^{4} S^{k,Q_k}$. Since $z' \in S^{j,Q_j} \subseteq Z^{j+1,Q_j}$, it follows that z' has a neighbor $s \in S \cup \bigcup_{k=j+1}^{4} S^{k,Q_k}$. But now $s - z' - z - a - b - c$ is a P_6 in G, a contradiction. This proves (2).

(3) *Let $j \in \{2, \ldots, 4\}$. No vertex $z \in Z^{j,Q_j}$ has exactly one neighbor in a clique $\{a_1, \ldots, a_j\} \subseteq Y_0^{j,Q_j}$.*

Suppose for a contradiction that there exist j and z violating (3); we may assume that z is chosen with j maximum. Write $Q_j = (S^{j,T})$. Let $\{a_1, \ldots, a_j\} \subseteq Y_0^{j,Q_j}$ be a clique with $N(z) \cap \{a_1, \ldots, a_j\} = \{a_1\}$.

Suppose first that $z \in R$. Let k be maximum such that $z \in Z^{k,Q_k}$. Then $z \notin Z^{k+1,Q_{k+1}}$, and thus $z \in Y_0^{k+1,Q_{k+1}}$, z has a neighbor $z' \in S^{k,Q_k}$, and z is anticomplete to $S \cup \bigcup_{l=k+1}^{4} S^{l,Q_l}$. It follows that $z' \in Z^{k+1,Q_{k+1}}$. But now $z' - z - a_1 - a_j$ is a path with $z, a_1, a_j \in Y_0^{k+1,Q_{k+1}}$ contrary to (2). This proves that $z \notin R$.

It follows that $z \in Z^{j+1,Q_{j+1}} \cap (X \cup Y)$, and in particular z has a neighbor in S. Let $T = T_P(z)$. It follows that $S^{j,T} \neq \emptyset$; let $z' \in S^{j,T}$ be the first vertex that was added to $S^{j,T}$ that is non-adjacent to z (such a vertex exists by the definition of $S^{j,T}$). Then $L_P(z) = L_P(z')$. Since $z' \in S^{j,Q_j}$, it follows that z' is anticomplete to Y_0^{j,Q_j}. Since $a_1 \in Y_0^{j,Q_j} \subseteq Y_0^{j+1,Q_{j+1}}$, it follows that z has a neighbor in $Y_0^{j+1,Q_{j+1}}$ non-adjacent to z', and hence (by the choice of z' if $Y_0 = R$, and since $z \notin X_0(z')$ if $Y_0 \neq R$), it follows that z' has a neighbor $a' \in Y_0^{j+1}$ that is non-adjacent to z.

Suppose first that a' is complete to $\{a_1, \ldots, a_j\}$. Since G contains no clique of size five, it follows that $j < 4$. But now $N(z') \cap \{a', a_1, \ldots, a_j\} = \{a'\}$, contrary to the maximality of j.

Suppose next that a' is mixed on $\{a_1, \ldots, a_j\}$. Let x be a neighbor and y be a non-neighbor of a' in $\{a_1, \ldots, a_j\}$. Then $z' - a' - x - y$ is a path, which contradicts an assumption of the theorem.

It follows that a' is anticomplete to $\{a_1, \ldots, a_j\}$. Since $z, z' \notin R$ and have neighbors in R, it follows that there is a vertex $t \in T$ that is anticomplete to R (this is immediate if $R = Y_0$, and follows from the fact that $L_P(z) \neq L$ if $R \neq Y_0$). Now $a' - z' - t - z - a_1 - a_j$ is a P_6 in G, a contradiction. This proves (3).

By (1), $P_{f',Q}$ satisfied (ii), and by construction (iii) holds. Now from (3) with $j = 2$ we deduce that no vertex of $(X' \cup Y') \setminus R$ is mixed on an edge of $(Y' \cup Y'_0) \cap R$.

It remains to show that $\mathcal{L}$ is equivalent to P. Clearly for every $P' \in \mathcal{L}$, a precoloring extension of P' is also a precoloring extension of P.

Let d be a precoloring extension of P. We show that some $P' \in \mathcal{L}$ has a precoloring extension. Let $j \in \{2, 3, 4\}$; define $S^{j,T}$ and f' as follows (starting with $S^{j,T} = \emptyset$):

- If $R = Y_0$ or $c_4 \in f(T)$ proceed as follows. While there is a vertex $z \in Z^{j+1}(T)$ complete to $S^{j,T}$ and such that there is clique $\{a_1, \ldots, a_j\} \subseteq {Y_0}^{j+1}$ with $N(z) \cap \{a_1, \ldots, a_j\} = \{a_1\}$, choose such z such that $N(z) \cap R$ maximal and add it to $S^{j,T}$; set $f'(z) = d(z)$.
- If $R \neq Y_0$ and $c_4 \notin f(T)$, while there is $z \in Z^{j+1}(T)$ complete to $S^{j,T}$ such that there is clique $\{a_1, \ldots, a_j\} \subseteq {Y_0}^{j+1}$ with $N(z) \cap \{a_1, \ldots, a_j\} = \{a_1\}$, choose such z with $d(z) \neq c_4$ and subject to that with $N(z) \cap R$ maximal; add z to $S^{j,T}$ and set $f'(z) = d(z)$. Let $X_0(z)$ be the set of all $z' \in Z^{j+1}(T)$ such that
  - z' is complete to $S^{j,T} \setminus \{z\}$,
  - there is a clique $\{b_1, \ldots, b_j\} \subseteq {Y_0}^{j+1}$ such that $N(z') \cap \{b_1, \ldots, b_j\} = \{b_1\}$, and
  - $N(z') \cap R$ is a proper subset of $N(z) \cap R$.

  It follows from the choice of z that $d(z') = c_4$ for every $z' \in X_0(z)$. When no such vertex z exists, let $X_0^{j,T} = \bigcup_{z \in S^{j,T}} X_0(z)$; thus $d(z') = c_4$ for every $z' \in X_0^{j,T}$. Define ${f'}_{j,T}(z') = c_4$ for every $z' \in X_0^{j,T}$, then ${f'}_{j,T}(z) = d(z)$ for every $z \in X_0^{j,T}$.

Let $Q_j = (S^{j,T})$ and let ${f'}_j = \bigcup_T {f'}_{j,T}$. It follows that $P_{f_2,Q_2} = (G, S', X'_0, X', Y'_0, Y', f \cup f')$ satisfies $d(v) = f_2(v)$ for every $v \in S' \cup X'_0$, and thus d is is a precoloring extension of P_{f_2,q_2}, as required. This proves Lemma 8. $\square$

The next lemma is used to arrange the following axiom, which we restate:

(iv) No vertex in $V(G) \setminus (Y_0 \cup X_0)$ is mixed on an edge of Y_0.

Lemma 9. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y_0, Y, f)$ be a seeded precoloring of a P_6-free graph G with P satisfying* (i), (ii) *and* (iii). *Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs an equivalent collection $\mathcal{L}$ for P such that*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$*;*
- *every $P' \in \mathcal{L}$ is a normal subcase of P;*
- *every $P' = (G', S', X'_0, X', Y'_0, Y', f') \in \mathcal{L}$ with seed S' satisfies $|S'| \leq q(|S|)$; and*
- *every $P' \in \mathcal{L}$ satisfies* (i), (ii), (iii) *and* (iv).

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension for P in polynomial time.

Proof. Let $S^5 = \emptyset$. Let $Z = X \cup Y$. Since P satisfies (iii), it follows that every vertex of Z has a neighbor in S. While there is a vertex $z \in Z$ complete to S^5 and a path $z - a - b - c$ with $a, b, c \in Y_0$, we add z to S^5. If $|S^5| \geq 5$, then G contains a K_5 and thus it has no precoloring extension; set $\mathcal{L} = \emptyset$ and stop. Thus we may assume that $|S^5| \leq 4$. Let $Y_0^5 = Y_0 \setminus N(S^5)$ and let $Z^5 = Z \cup (Y_0 \setminus Y_0^5)$. Since S is connected, and since every vertex of S^5 has a neighbor in S, it follows that $S \cup S^5$ is connected.

(4) *There is no path $z - a - b - c$ with $z \in Z^5$ and $a, b, c \in Y_0^5$.*

Suppose for a contradiction that such a path exists, and suppose first that $z \in Z$. By the choice of S^5, it follows that $S^5 \neq \emptyset$, and there exists a vertex $z' \in Z \cap S^5$ non-adjacent to z. Since $S \cup S^5$ is connected, there exists a path Q connecting z and z' with interior in $S \cup S^5$. Since P satisfies (iii) and by the construction of S^5, it follows that Q^* is anticomplete to $\{a, b, c\}$. But now $z' - Q - z - a - b - c$ is a path with at least six vertices in G, a contradiction.

It follows that $z \in N(S^5) \setminus Z$, and thus $z \in Y_0 \setminus Y_0^5$. Let $s' \in S^5 \cap N(z)$. Then s' is anticomplete to $\{a, b, c\}$. Moreover, $s' \in Z$, and so s' has a neighbor $s \in S$. Since P satisfies (iii), s is anticomplete to Y_0, and so s is anticomplete to $\{z, a, b, c\}$. But now $s - s' - z - a - b - c$ is a P_6 in G, a contradiction. This proves (4).

For every $f' : S^5 \to [4]$ such that $f \cup f'$ is a proper coloring of $G|(S \cup S^5)$, let $P_{f'} = (G, S \cup S^5, X_0, Z^5, Y_0^5, \emptyset, f \cup f')$. Then $P_{f'}$ is a normal subcase of P that satisfies (i)-(iii).

Let $\mathcal{M}_{f'}$ be the collection of seeded precolorings obtained by applying Lemma 8 to P'_f with $R = Y_0^5$, and let $\mathcal{M}$ be the union of all such $\mathcal{M}_{f'}$. By (4), every $P'' \in \mathcal{M}$ satisfies (ii)–(iv).

Finally let $\mathcal{L}$ be obtained from $\mathcal{M}$ by applying Lemma 6 to every member of $\mathcal{M}$. Then every $P' \in \mathcal{L}$ satisfies (i)–(iv), as required. This proves Lemma 9. ☐

The purpose of Lemma 10 is to organize vertices according to their lists (which, in turn, arise from the colors of their neighbors in the seed) to satisfy the following axiom:

(v) If $|L_{S,f}(v)| = 1$ and $v \notin S$, then $v \in X_0$; if $|L_{S,f}(v)| = 2$, then $v \in X$; if $|L_{S,f}(v)| = 3$, then $v \in Y$; and if $|L_{S,f}(v)| = 4$, then $v \in Y_0$.

Moreover, we will construct new seeded precolorings in controlled ways from seeded precolorings satisfying (i), (ii), (iii), and (iv), to arrange that these axioms as well as (v) still hold for the new instances.

Lemma 10. *There is a constant C such that the following holds. Let $P = (G, S, X_0, X, Y_0, Y, f)$ be a seeded precoloring of a P_6-free graph G with P satisfying* (i), (ii), (iii) *and* (iv), *and let $P' = (G', S', X'_0, X', Y'_0, Y', f')$ be a normal subcase of P. Then there is an algorithm with running time $O(|V(G)|^C)$ that outputs an equivalent collection $\mathcal{L}$ for $\{P\}$ of seeded precoloring with $|\mathcal{L}| \leq 1$, such that if $\mathcal{L} = \{P''\}$ then*

- *P'' is a normal subcase of P', and*

- *P'' satisfies* (i)*,* (ii)*,* (iii)*,* (iv) *and* (v)*.*
- *If P' satisfies* (vi)*, then P'' satisfies* (vi)*.*
- *If P' satisfies* (vii)*, then P'' satisfies* (vii)*.*

Moreover, given a precoloring extension of P'', we can compute a precoloring extension for P in polynomial time.

Proof. Since P' is a normal subcase of P, it follows that P' satisfies (ii). By replacing P' with

$$(G', S', X_0', X', (Y' \cup Y_0') \cap (N(S') \cup X_0'), (Y' \cup Y_0') \setminus (N(S') \cup X_0')),$$

we may assume that P' satisfies (iii). By Lemma 7, P' satisfies (iv).

Let $Z_i = \{v \in V(G) \setminus (S' \cup X_0') : |L_{P'}(v)| = i\}$. If $Z_0 \neq \emptyset$, then P' has no precoloring extension, and we output this and $\mathcal{L} = \emptyset$ and stop. Thus, we may assume that $Z_0 = \emptyset$. Let $f'' : Z_1 \to \{1, 2, 3, 4\}$ with $f''(v) = c$ if $L_{P'}(v) = \{c\}$. Since P' satisfies (iii), it follows that $Y_0' = Z_4$, and so the seeded precoloring $\tilde{P} = (G', S', X_0' \cup Z_1, Z_2, Z_4, Z_3, f' \cup f'')$ satisfies (iv). For the same reason, if P' satisfies (vi), then so does $\tilde{P}$, and if P' satisfies (vii), then so does $\tilde{P}$. Let P'' be obtained from the precoloring $\tilde{P}$ by applying Lemma 6. It follows that P'' satisfies (i)–(v), and P'' is a normal subcase of P'. Clearly if $\tilde{P}$ satisfies (vi), then so does P'', and if $\tilde{P}$ satisfies (vii), then so does P''. This proves Lemma 10. □

In the next lemma we establish (vi), which we restate:

(vi) There is a color $c \in \{1, 2, 3, 4\}$ such that for every vertex $y \in Y$ with a neighbor in Y_0, $f(N(y) \cap S) = \{c\}$. We let $L = \{1, 2, 3, 4\} \setminus \{c\}$.

The proof proceeds in two steps to address two separate cases; two vertices y, y' in Y with a neighbor in Y_0 and with different lists may receive colors that are in the intersection of their lists (second step), or they may not (first step).

Lemma 11. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y_0, Y, f)$ be a seeded precoloring of a P_6-free graph G with P satisfying* (i)*,* (ii)*,* (iii)*,* (iv) *and* (v)*. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs an equivalent collection $\mathcal{L}$ for P such that*

- *$|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;*
- *every $P' \in \mathcal{L}$ is a normal subcase of P;*
- *every $P' = (G', S', X_0', X', Y_0', Y', f') \in \mathcal{L}$ with seed S' satisfies $|S'| \leq q(|S|)$; and*
- *every $P' \in \mathcal{L}$ satisfies* (i)*,* (ii)*,* (iii)*,* (iv)*,* (v) *and* (vi)*.*

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension for P in polynomial time.

Proof. A seeded precoloring $P = (G, S, X_0, X, Y_0, Y, f)$ is *acceptable* if for every precoloring extension c of P and for every non-adjacent $y, y' \in Y \cap N(Y_0)$ with $L_P(y) \neq L_P(y')$, we have $\{c(y), c(y')\} \not\subseteq L_P(y) \cap L_P(y')$.

First we construct a collection $\mathcal{M}$ of seeded precolorings that is an equivalent collection for P, and such that every member of $\mathcal{M}$ is acceptable. We proceed as follows. Let $\mathcal{T}$ be the set of all pairs (T, T') with $T, T' \subseteq S$ and $|f(T)| = |f(T')| = 1$ and $f(T) \neq f(T')$. Write $\mathcal{T} = \{(T_1, T_1'), \dots, (T_t, T_t')\}$. Let $\mathcal{Q}$ be the set of all t-tuples $Q = (Q_{T_1,T_1'}, \dots, Q_{T_t,T_t'})$ such that $Q_{T_i,T_i'} = (P_{T_i,T_i'}, M_{T_i,T_i'}, N_{T_i,T_i'})$ where

- $|P_{T_i,T_i'}| = |M_{T_i,T_i'}| \leq |N_{T_i,T_i'}| \leq 1$.
- $P_{T_i,T_i'} \subseteq Y(T_i)$ and $N_{T_i,T_i'} \subseteq Y(T_i')$.
- $M_{T_i,T_i'} \subseteq Y_0$.
- $M_{T_i,T_i'}$ is complete to $P_{T_i,T_i'} \cup N_{T_i,T_i'}$.

- P_{T_i,T'_i} is anticomplete to N_{T_i,T'_i}.

Let $V(Q_{T_i,T'_i}) = P_{T_i,T'_i} \cup M_{T_i,T'_i} \cup N_{T_i,T'_i}$ and let $S(Q) = \bigcup_{i=1}^{t} V(Q_{T_i,T'_i})$. Let $(T_i, T'_i) \in \mathcal{T}$. Define $Z(T_i, T'_i)$ as follows.

- If $|P_{T_i,T'_i}| = |M_{T_i,T'_i}| = |N_{T_i,T'_i}| = 0$, then $Z(T_i, T'_i) = Y(T'{}_i) \cap N(Y_0)$.
- If $|P_{T_i,T'_i}| = |M_{T_i,T'_i}| = 0$ and $|N_{T_i,T'_i}| = 1$, then $Z(T_i, T'_i) = (Y(T'{}_i) \cap N(Y_0)) \setminus N(N_{T_i,T'_i})$.
- If $|P_{T_i,T'_i}| = |M_{T_i,T'_i}| = |N_{T_i,T'_i}| = 1$, then $Z(T_i, T'_i) = \emptyset$.

Let $Z(Q) = \bigcup_{(T_i,T'_i)\in\mathcal{T}} Z(T_i, T'_i)$. A function f' is said to be Q-*admissible* if $f' : S(Q) \cup Z(Q) \to \{1, \dots, 4\}$ and for every $i \in \{1, \dots, t\}$ it satisfies:

- $f'(P_{T_i,T'_i}), f'(N_{T_i,T'_i}) \in [4] \setminus (f(T_i) \cup f(T'_i))$.
- If $Z(T_i, T'_i) \subseteq Y(T'_i)$, then $f'(Z(T_i, T'_i)) = f(T_i)$.
- If $Z(T_i, T'_i) \subseteq Y(T_i)$, then $f'(Z(T_i, T'_i)) = f(T'_i)$.
- The coloring $f \cup f'$ of $G|(S \cup S(Q) \cup X_0 \cup Z(Q))$ is proper.

For every Q-admissible function f' with domain $S(Q) \cup Z(Q)$, let

$$P_{Q,f'} = (G, S \cup S(Q), X_0 \cup Z(Q), X, Y_0 \setminus (S(Q) \cup N(S(Q))), (Y \setminus (S(Q) \cup Z(Q))) \cup (N(S(Q)) \cap Y_0), f \cup f').$$

Then $P_{Q,f'}$ is a normal subcase of P.

Since every vertex in $X \cup Y$ has a neighbor in S, it follows that $P_{Q,f'}$ satisfies (ii); by construction, (iii) holds. By Lemma 7, $P_{Q,f'}$ satisfies (iv). Let $\mathcal{M}$ be the union of the collections obtained by applying Lemma 10, where the union is taken over all Q, f' as above. Then every member of $\mathcal{M}$ satisfies (i)–(v).

We show that there is a function $q_1 : \mathbb{N} \to \mathbb{N}$ such that $|S \cup S(Q)| \leq q_1(|S|)$ and $|\mathcal{M}| \leq |V(G)|^{q_1(|S|)}$. Since there are at most $2^{|S|}$ types, it follows that $t \leq 2^{2|S|}$. Now, since for every $(T_i, T'_i) \in \mathcal{T}$ we have that $|V(Q_{T_i,T'_i})| \leq 3$, it follows that for every $Q \in \mathcal{Q}$ we have $|S(Q)| \leq 3 \times 2^t$, and so $|S \cup S(Q)| \leq |S| + 3 \times 2^{2|S|}$ and $|\mathcal{Q}| \leq |V(G)|^{3\times 2^{2|S|}}$. Finally, for every Q, there are at most $4^{|S(Q)|} = 4^{3t}$ possible precoloring of $S(Q)$, since every precoloring of $S(Q)$ extends to an admissible function in a unique way, and we deduce that $|\mathcal{M}| \leq 4^{3t} \times |\mathcal{Q}| \leq 4^{3\times 2^{2|S|}} \times |V(G)|^{3\times 2^{2|S|}} \leq (4|V(G)|)^{3\times 2^{2|S|}}$ as required.

(5) *Let $P' \in \mathcal{M}$ with $P' = (G, S', X'_0, X', Y'_0, Y', f')$. If $y \in Y'$ has a neighbor $z \in Y'_0$, then $y \in Y$.*

Suppose that $y \notin Y$. Then $y \in Y_0 \cap Y'$ and there exist $s \in S' \setminus S$ such that y is adjacent to s, contrary to Lemma 5. This proves (5).

Next we show that every precoloring in $\mathcal{M}$ is acceptable. Let $P' = (G, S', X'_0, X', Y'_0, Y', f') \in \mathcal{M}$, and suppose there exist non-adjacent $y, y' \in N(Y'_0) \cap Y'$ with $L_{P'}(y) \neq L_{P'}(y')$ and such that there exists a precoloring extension c with $c(y), c(y') \in L_{P'}(y) \cap L_{P'}(y')$. Let $z \in N(y) \cap Y'_0$ and $z' \in N(y') \cap Y'_0$. Then $z, z' \in Y_0$, and so by Lemma 5, $y, y' \in Y$, $L_P(y) = L_{P'}(y)$ and $L_P(y') = L_{P'}(y')$. Let $T = T(y)$ and $T' = T(y')$ (in P). Then $T \cap T' = \emptyset$. By Lemma 4, we may assume that $z = z'$. Since $Y' \cap Y(T)$ and $Y' \cap Y(T')$ are both non-empty, it follows that $|V(Q_{T,T'})| > 1$. Let $P_{T,T'} = \{p\}$, $M_{T,T} = \{m\}$ and $N_{T,T'} = \{n\}$. Since $z \in Y'_0$, it follows that z is anticomplete to $V(Q_{T,T'})$. Since P' satisfies (v), $f'(p), f'(n) \in L_P(y) \cap L_P(y')$ and $|L_{P'}(y)| = |L_{P'}(y')| = 3$, it follows that $\{y, y', p, n\}$ is a stable set. By symmetry, we may assume that $f'(m) \in L_{P'}(y)$, and hence y is not adjacent to m. Let $s \in T \setminus T'$; then $z - y - s - p - m - n$ is a P_6 in G, a contradiction. This proves that every seeded precoloring in $\mathcal{M}$ is acceptable.

Next we show that $\mathcal{M}$ is equivalent to P. Clearly every precoloring extension of a member of $\mathcal{M}$ is a precoloring extension of P. For the converse, let c be a precoloring extension of P. For every pair of types $(T, T') \in \mathcal{T}$ for which there exist non-adjacent $y \in Y(T) \cap N(Y_0)$ and $y' \in Y(T') \cap N(Y_0)$, such that $c(y), c(y') \notin f(T) \cup f(T')$, choose such a pair y, y' and let z be a common neighbor of y, y' in Y_0 (such z exists by Lemma 4); set $P_{T,T'} = \{y\}$, $M_{T,T'} = \{z\}$ and $N_{T,T'} = \{y'\}$, and define $f'(y) = c(y)$, $f'(y') = c(y')$ and $f'(z) = c(z)$. Let $Z(T_i, T'_i) = \emptyset$.

Now let $(T,T') \in \mathcal{T}$ be such that no such y, y' exist. Suppose that there exists $y \in Y(T') \cap N(Y_0)$ with $c(y) \neq f(T)$, let $N_{T,T'} = \{y\}$, $P_{T,T'} = M_{T,T'} = \emptyset$, and let $f'(y) = c(y)$. Let $Z(T_i, T_i') = (Y(T) \cap N(Y_0)) \setminus N(y)$, and set $f'(v) = f(T')$ for every $v \in Z(T_i, T_i')$. Since (T,T') does not have the property described in the previous paragraph, it follows that $c((Y(T) \cap N(Y_0)) \setminus N(y)) = f(T')$, and so $c(v) = f'(v)$ for every $v \in Z(T_i, T_i')$. Finally, suppose that $c(Y(T') \cap N(Y_0)) = f(T)$. Then set $P_{T,T'} = M_{T,T'} = N_{T,T'} = \emptyset$ and $Z(T_i, T_i') = Y(T') \cap N(Y_0)$. Define $f'(v) = f(T)$ for every $v \in Z(T_i, T_i')$. Let Q consist of all the triples $Q_{T,T'} = (P_{T,T'}, M_{T,T'}, N_{T,T'})$ as above. Let $S(Q) = \bigcup_{(T,T') \in \mathcal{T}} V(Q_{T,T'})$, and $Z(Q) = \bigcup_{(T,T') \in \mathcal{T}} Z(T_i, T_i')$. Let

$$P_{Q,f'} = (G, S \cup S(Q), X_0 \cup Z(Q), X, Y_0 \setminus (S(Q) \cup N(S(Q))), (Y \setminus (S(Q) \cup Z(Q))) \cup (N(S(Q)) \cap Y_0), f \cup f').$$

Then c is a precoloring extension of $P_{Q,f'}$. Moreover, $P_{Q,f'}$ was one of the seeded precoloring we considered in the process of constructing $\mathcal{M}$, and so $\mathcal{M}$ contains the seeded precoloring obtained from $P_{Q,f'}$ by applying Lemma 10. It follows that $\mathcal{M}$ is an equivalent collection for P.

Let $P' = (G, S', X_0', X', Y_0', Y', f') \in \mathcal{M}$ be an acceptable seeded precoloring. For $c \in \{1,2,3,4\}$ and a precoloring extension d of P', we say that is *c is active for L and d* if there exists a vertex $v \in Y' \cap N(Y_0')$ with $L_{P'}(v) = L$ and $d(v) = c$.

Define $\mathcal{L}_1(P')$ as follows. For every function $g : Y' \cap N(Y_0') \to [4]$ such that

- $g(v) \in L_{P'}(v)$ for every $v \in Y \cap N(Y_0')$,
- $|g(Y_L' \cap N(Y_0')| = 1$ for every $L \in \binom{[4]}{3}$, and
- $f' \cup g$ is a proper coloring of $G|(S' \cup X_0' \cup (Y' \cap N(Y_0')))$,

let

$$P_g'' = (G, S', X_0' \cup (Y' \cap N(Y_0')), X', Y_0', Y' \setminus N(Y_0'), f' \cup g).$$

It is easy to check that P_g'' satisfies (ii)—(vi). Let P_g' be obtained from P_g'' by applying Lemma 10. Then P_g' satisfies (i)—(vi). Let $\mathcal{L}_1(P')$ be the collections of all such P_g'.

Next we construct $\mathcal{L}_2(P')$. For every $L \in \binom{[4]}{3}$, for every $y_1, y_2 \in Y'_L \cap N(Y'_0)$, and for every $c_1, c_2 \in L$, define a function g as follows. Let $g(y_i) = c_i$. For every $L' \in \binom{[4]}{3} \setminus L$, let $Z(L')$ be the set of vertices $v \in Y'_{L'}$ such that v has a non-neighbor $n \in \{y_1, y_2\}$ with $g(n) \in L'$. For every $v \in Z(L')$, let $g(v)$ be the unique element of $L' \setminus L$. Finally, let $Z = \bigcup_{L' \in \binom{[4]}{3} \setminus L} Z(L')$.

If $f' \cup g$ is a proper coloring of $G|(S \cup X_0 \cup \{y_1, y_2\})$, let

$$P''_{L,y_1,y_2,c_1,c_2} = (G, S \cup \{y_1, y_2\}, X_0 \cup Z, X, Y_0 \setminus N(\{y_1, y_2\}), Y \setminus (Z \cup \{y_1, y_2\}), f' \cup g).$$

It is easy to check that P''_{L,y_1,y_2,c_1,c_2} satisfies (i)—(vi). Let P'_{L,y_1,y_2,c_1,c_2} be obtained from P''_{L,y_1,y_2,c_1,c_2} by applying Lemma 10. Let $\mathcal{L}_2(P')$ be the collection of all P'_{L,y_1,y_2,c_1,c_2} constructed this way; then every member of $\mathcal{L}_2$ satisfies (i)—(vi).

We claim that $\mathcal{L}(P') = \mathcal{L}_1(P') \cup \mathcal{L}_2(P')$ is an equivalent collection for $\{P'\}$. Clearly a precoloring extension of an element of $\mathcal{L}(P')$ is a precoloring extension of P. Now let c be a precoloring extension of P. If for every $L \in \binom{[4]}{3}$ there is at most one active color for L and c, then c is a precoloring extension of a member of $\mathcal{L}_1(P)$, so we may assume that there is $L_0 \in \binom{[4]}{3}$ such that at least two colors are active for L and c. We may assume that $L = \{1,2,3\}$ and the colors $1, 2$ are active. Let $y_i \in Y'_{L_0}$ with $c(y) = i$. We claim that c is a precoloring extension of $P''_{L_0,y_1,y_2,1,2}$. Let $L \in \binom{[4]}{3} \setminus L_0$. Since P' is acceptable, for every $v \in Y_L'$ that has a non-neighbor $n \in \{y_1, y_2\}$ with $c(n) \in L'$, we have that $c(v) \in L' \setminus L_0$. It follows that $c(v) = g(v)$, and the claim holds. This proves that $\mathcal{L}(P')$ is an equivalent collection for $\{P'\}$.

Finally, setting

$$\mathcal{L} = \bigcup_{P' \in \mathcal{M}} \mathcal{L}(P'),$$

Lemma 11 follows. This completes the proof. □

The next lemma is used to arrange the following axiom, which we restate:

(vii) With L as in (vi), we let Y_L^* be the subset of Y_L of vertices that are in connected components of $G|(Y_0 \cup Y_L)$ containing a vertex of Y_0. Then no vertex of $Y \setminus Y_L^*$ has a neighbor in $Y_0 \cup Y_L^*$, and no vertex of X is mixed on $Y_0 \cup Y_L^*$.

Lemma 12. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y_0, Y, f)$ be a seeded precoloring of a P_6-free graph G with P satisfying* (i), (ii), (iii), (iv), (v) *and* (vi). *Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs an equivalent collection $\mathcal{L}$ for P such that*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$*;*
- *every $P' \in \mathcal{L}$ is a normal subcase of P;*
- *for every $P' = (G', S', X_0', X', Y_0', Y', f') \in \mathcal{L}$, $|S'| \leq q(|S|)$;*
- *every $P' \in \mathcal{L}$ satisfies* (i), (ii), (iii), (iv), (v), (vi) *and* (vii)*;*

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension for P in polynomial time.

Proof. We may assume that G contains no K_5, for otherwise, P does not have a precoloring extension and we output $\mathcal{L} = \emptyset$ and stop.

With L as in (vi) and Y_L^* as in (vii), let $Y^* = (X \cup (Y \setminus Y_L^*)) \cap N(Y_0 \cup Y_L^*)$. By the definition of Y_L^*, it follows that $L_P(y) \neq L$ for every $y \in Y^*$, and if $y \in Y^* \cap Y$, then y is anticomplete to Y_0. Let $\mathcal{T} = \{T_1, \ldots, T_t\}$ be the set of types of vertices in Y^*. Let $L = \{c_1, c_2, c_3\}$ and $\{c_4\} = \{1, 2, 3, 4\} \setminus L$. Let $\mathcal{Q}$ consist of all t-tuples $Q = ((S_{T_1}, R_{T_1}), \ldots, (S_{T_t}, R_{T_t}))$ such that

- $|R_{T_i}| \leq |S_{T_i}| \leq 1$.
- $S_{T_i} \cup R_{T_i} \subseteq Y^*(T_i)$.
- S_{T_i} is complete to R_{T_i}.

Let $V(Q) = \bigcup_{i=1}^{t} (S_{T_i} \cup R_{T_i})$. For every $Q \in \mathcal{Q}$ and for every $f' : V(Q) \to L$ with $f'(v) \in L_P(v) \setminus \{c_4\}$ for all $v \in V(Q)$, we proceed as follows. Let $\tilde{Y}^1_{Q,f'}$ be the set of all vertices v in Y^* such that $S_{T(v)} = \emptyset$. Let $\tilde{Y}^2_{Q,f'}$ be the set of all vertices v in Y^* such that $S_{T(v)} \neq \emptyset$, $R_{T(v)} = \emptyset$ and v is complete to $S_{T(v)}$. Let $\tilde{Y}_{Q,f'} = \tilde{Y}^1_{Q,f'} \cup \tilde{Y}^2_{Q,f'}$. Let $f'(v) = c_4$ for every $v \in \tilde{Y}_{Q,f'}$. Since $V(Q) \subseteq Y^*$, it follows that $G|(S \cup V(Q))$ is connected. Suppose that $f \cup f'$ is a proper coloring of $G|(S \cup X_0 \cup V(Q) \cup \tilde{Y}_{Q,f'})$. Let $\mathcal{L}'$ be obtained from the normal subcase

$$(G, S \cup V(Q), X_0 \cup \tilde{Y}_{Q,f'}, X \setminus (\tilde{Y}_{Q,f'} \cup V(Q)), Y_0, Y \setminus (\tilde{Y}_{Q,f'} \cup V(Q)), f \cup f' \cup g)$$

of P by applying Lemma 10. Suppose that $\mathcal{L}' = \{P_{Q,f'}\}$. Write $P_{Q,f'} = (G, S', X_0', X', Y_0', Y', f')$. Then $P_{Q,f'}$ satisfies (i)–(vi). Furthermore, $P_{Q,f'}$ has a precoloring extension if and only if P has a precoloring extension d such that $d(v) = f'(v)$ for every $v \in V(Q)$, and $d(v) = c_4$ for every $v \in Y^*$ such that either

- $S_{T(v)} = \emptyset$, or
- $S_{T(v)} \neq \emptyset$, $R_{T(v)} = \emptyset$, and v is complete to $S_{T(v)}$.

Moreover, $|V(Q)| \leq 2|\mathcal{T}| \leq 2^{|S|+1}$.

Let $\mathcal{L}_1$ be the set of all seeded precolorings $P_{Q,f'}$ as above (ranging over all $Q \in \mathcal{Q}$). Then $\mathcal{L}_1$ is an equivalent collection for P, and $|\mathcal{L}_1| \leq (3|V(G)|)^{2^{|S|+1}}$. Let $P' \in \mathcal{L}_1$ with $P' = (G, S', X_0', X', Y_0', Y', f')$. Since P' satisfies (vi), let L be as in (vi) and let ${Y'}^*_L$ be as in (vii).

(6) *There is no path $z - a - b - c$ with $z \in (X' \cup Y') \setminus {Y'}^*_L$ and $a, b, c \in {Y'}^*_L \cup Y_0'$.*

Suppose that such a path $z - a - b - c$ exists. First we show that $z \in X \cup Y$. Suppose not; then $z \in Y_0$ and z has a neighbor $s' \in S' \setminus S$. Since $S' \setminus S \subseteq Y^*$, it follows that $L_P(s) \neq L$. Since P satisfies (vi), it follows that $s' \notin Y$, and hence $s \in X$. Since $\{z, a, b, c\} \subseteq Y_0 \cup Y_L$, and since P satisfies (v), we deduce that there exists $s \in T(s')$ with $f(s) \in L$. Consequently, s is anticomplete to $\{z, a, b, c\}$. But now $s - s' - z - a - b - c$ is a P_6 in G, a contradiction. This proves that $z \in X \cup Y$.

Since $L_{S,f}(z) \neq L$, there exists $t \in T(z)$ with $f(t) \in L$. Since $z \notin X_0'$, it follows that $S_{T(v)} \neq \emptyset$, and either

- $R_{T(z)} \neq \emptyset$, or
- $R_{T(z)} = \emptyset$, and z is not complete to $S_{T(z)}$.

Let $S_{T(z)} = \{s\}$. Since $f'(s) \in L$, it follows that s is anticomplete to $\{a, b, c\}$. If z is non-adjacent to s, then $s - t - z - a - b - c$ is a P_6, a contradiction. It follows that z is adjacent to s, and therefore $R_{T(z)} \neq \emptyset$; say $R_{T(z)} = \{r\}$. Since s is adjacent to r, it follows that $f'(z) \neq f'(r)$. Since $z \notin X_0$, and since (v) holds, it follows that z is non-adjacent to r. Since $f'(r) \in L$, it follows that r is anticomplete to $\{a, b, c\}$. But now $r - s - z - a - b - c$ is a P_6 in G, a contradiction. This proves (6).

In view of (6), let $\mathcal{L}_2(P')$ be the collection of precolorings obtained from P' by applying Lemma 8 with $R = Y_0' \cup {Y'}^*_L$. Let $P'' \in \mathcal{L}_2(P')$; write $P'' = (G, S'', X_0'', X'', Y_0'', Y'', f'')$. Then P'' satisfies (ii) and (iii) and no vertex of $(X'' \cup Y'') \setminus R$ is mixed on $(Y'' \cup Y_0'') \cap R$. By Lemma 7, P'' satisfies (iv) and (vi).

Let $\mathcal{L}_3(P'')$ be obtained by applying Lemma 10 to P'', and let $\tilde{P} \in \mathcal{L}_3(P'')$. Write $\tilde{P} = (\tilde{G}, \tilde{S}, \tilde{X_0}, \tilde{X}, \tilde{Y_0}, \tilde{Y}, \tilde{f})$. By Lemma 10, $\tilde{P}$ satisfies (i)–(vi). Since P'' satisfies (iii), $\tilde{S} = S''$ and $\tilde{Y_0} = Y_0''$. Define $\tilde{Y}_L^*$ as in (vii), then $\tilde{Y}_L^* = R \cap \tilde{Y}$. Since no vertex of $(X'' \cup Y'') \setminus R$ is mixed on $(Y'' \cup Y_0'') \cap R$, it follows that no vertex of $(\tilde{X} \cup \tilde{Y}) \setminus \tilde{Y}_L^*$ is mixed on $Y_0'' \cup \tilde{Y}_L^*$, and since $\tilde{P}$ satisfies (vi), we deduce that $\tilde{P}$ satisfies (vii). Now setting

$$\mathcal{L} = \bigcup_{P_1 \in \mathcal{L}_1} \bigcup_{P_2 \in \mathcal{L}_2(P_1)} \mathcal{L}_3(P_2)$$

Lemma 12 follows. □

We are now ready to prove the final lemma of this section, used to prove the following axiom, which we restate:

(viii) With Y_L^* as in (vii), for every component C of $G|(Y_0 \cup Y_L^*)$, there is a vertex v in X complete to C.

The previous axiom (vii) states that vertices of X cannot be mixed on such components C; therefore it suffices to deal with components C with no neighbors in X. For these components, we will use their structure (and in particular, that we can use the color in $[4] \setminus L$ on the Y_0-side of them without impacting the Y_L-side) to decide if the coloring can be extended to them, and then remove them.

Lemma 13. *There is a constant c such that the following holds. Let $P = (G, S, X_0, X, Y_0, Y, f)$ be a seeded precoloring of a P_6-free graph G satisfying* (i), (ii), (iii), (iv), (v), (vi), *and* (vii). *Let L be as in* (vi) *and let Y_L^* as in* (vii). *There is an algorithm with running time $O(|V(G)|^c)$ that outputs an equivalent collection $\mathcal{L}$ of seeded precolorings, such that $|\mathcal{L}| \leq 1$, and if $\mathcal{L} = \{P'\}$, then*

- *there is $Z \subseteq Y_0 \cup Y_L^*$ such that $P' = (G \setminus Z, S, X_0, X, Y_0 \setminus Z, Y \setminus Z, f)$, and*
- *P' satisfies* (i)–(viii).

Proof. We may assume that P does not satisfy (viii) for otherwise we set $\mathcal{L} = \{P\}$. A component C of $G|(Y_0 \cup Y_L^*)$ is *deficient* if no vertex of X is complete to $V(C)$. Let C be a deficient component. It follows from (vii) that X is anticomplete to $V(C)$. Let $A = V(C) \cap Y_0$, $B = V(C) \setminus A = V(C) \cap Y_L^*$. For every vertex $v \in A \cup B$, let $L(v) = \{1, 2, 3, 4\} \setminus (f(N(v) \cap (S \cup X_0)))$. It follows that $L(v) \subseteq L$ for $v \in B$. Moreover, by (i), it follows that $B \neq \emptyset$. Let $L = \{c_1, c_2, c_3\}$ and let $\{c_4\} = \{1, 2, 3, 4\} \setminus L$.

For every component D of $G|A$, we proceed as follows.

Let $\mathcal{P}(D)$ be the set of lists $L^* \subseteq \{1, 2, 3, 4\}$ with $|L^*| \leq 3$ such that D can be colored with list assignment $L'(x) = L(x) \cap L^*$ for $x \in V(D)$. Since G is P_6-free, it follows from Theorem 2 that $\mathcal{P}(D)$ can be computed in polynomial time. Since C is connected, it follows from (iv) that some vertex of B is complete to D. Consequently, in any precoloring extension of P, at most three colors appear in D, and at least one color of L does not appear in D. Therefore, if $\mathcal{P}(D) = \emptyset$, or if $L \subseteq L'$ for every $L' \in \mathcal{P}(D)$, then P has no precoloring extension we set $\mathcal{L} = \emptyset$ and stop. Let $\mathcal{P}^*(D)$ be the set of $L' \subseteq \{1,2,3,4\}$ such that $L' \notin \mathcal{P}(D)$, but for every proper superset $L'' \subseteq \{1,2,3,4\}$ of L' with $|L''| \leq 3$, we have that $L'' \in \mathcal{P}(D)$. Let $d \in V(D)$. We now replace D by a stable set $R(D) = \{d(L^*)\}_{L^*}$ of copies of d (each with the same set of neighbors as d), one for each $L^* \in \mathcal{P}^*(D)$ with $c_4 \in L^*$, and set $L'(d(L^*)) = \{1,2,3,4\} \setminus L^*$. Then $L'(d(L^*)) \subseteq L$. Let C' denote the graph obtained by this process (repeated for every component of $C|Y_0$) from C. Let $L'(v) = L(v)$ for every $v \in V(C) \setminus Y_0$. Since C' is obtained from an induced subgraph of G by replacing vertices with stable sets, it follows that C' is P_6-free.

We claim that C has a proper L-coloring if and only if C' has a proper L'-coloring. Suppose that C has a proper L-coloring c. We need to show that $c|_{V(C)\setminus Y_0}$ can be extended to each $R(D)$. We can consider each D separately.

Let D be a component of $C|Y_0$. Let $L^* = c(D)$. Let $L^{**} = c(N(D))$. We claim that for every $r \in R(D)$, $L'(r) \setminus L^{**} \neq \emptyset$. Suppose $L'(r) \subseteq L^{**}$. Then $\{1,2,3,4\} \setminus L'(r) \in \mathcal{P}^*(D)$, but $L^* \subseteq \{1,2,3,4\} \setminus L^{**} \subseteq \{1,2,3,4\} \setminus L'(r)$, a contradiction. This proves that for every $r \in R(D)$, there exists $d(r) \in L'(r) \setminus L^{**}$, and setting $c(r) = d(r)$ we obtain a coloring of C'.

Next suppose that C' has a proper L'-coloring c. Let $L^* = \{1,2,3,4\} \setminus c(N(D))$. If $L^* \in \mathcal{P}(D)$, then we color D with an L-coloring using only those colors in L^*; this is possible by the definition of $\mathcal{P}(D)$. Thus we may assume that $L^* \notin \mathcal{P}(D)$. Since $L(x) \subseteq L$ for all $x \in N(D) \subseteq B$, it follows that $c_4 \in L^*$. From the definition of $\mathcal{P}^*(D)$, it follows that some superset L^{**} of L^* is in $\mathcal{P}^*(D)$. Then $L'(d'(L^{**}) = \{1,2,3,4\} \setminus L^{**} \subseteq \{1,2,3,4\} \setminus L^* = c(N(D))$. However, $c(d') \in L'(d')$, and thus $c(d) \in c(N(D)) = c(N(d))$, contrary to the fact that c is a proper coloring. This proves that C has a proper L-coloring if and only if C' has a proper L'-coloring.

We have so far proved the following:

- C' has a proper L'-coloring if and only if C has a proper L-coloring;
- C' is P_6-free; and
- for every $x \in V(C')$, we have that $L'(x) \subseteq L$.

By Theorem 2, we can decide in polynomial time if C' has a proper L'-coloring, and thus we can decide in polynomial time if C has a proper L-coloring. If not, then P has no precoloring extension; we set $\mathcal{L} = \emptyset$ and stop. If C has a proper L-coloring, then $(G \setminus V(C), S, X_0, X, Y_0 \setminus V(C), Y \setminus V(C), f)$ has a precoloring extension if and only if P does.

By repeatedly applying this algorithm to every deficient component C of $G|(Y_0 \cup Y_L^*)$, and setting $Z = \bigcup V(C)$ where the union is taken over all such components, we set $P' = (G \setminus Z, S, X_0, X, Y_0 \setminus Z, Y \setminus Z, f)$ and output $\mathcal{L} = \{P'\}$. Then P' satisfies (i)-(viii), and Lemma 13 follows. $\square$

We call a seeded precoloring *good* if it satisfies (i), (ii), (iii), (iv), (v), (vi), (vii), and (viii).

By applying Lemmas 2, 3, 9, 10, 11, 12 and 13, each to every seeded precoloring in the output of the previous one, we finally derive the main theorem of Section 2.

Theorem 10. *There is a constant C such that the following holds. Let G be a P_6-free graph, and let (G, A, f) be a 4-precoloring of G. Then there exists a polynomial-time algorithm that computes a collection $\mathcal{L}$ of seeded precolorings such that*

- *$\mathcal{L}$ is equivalent for P.*
- *for every $(G', S', X'_0, X', Y'_0, Y', f') \in \mathcal{L}$, G' is an induced subgraph of G, $A \subseteq X'_0 \cup S'$ and $f'|_A = f|_A$.*
- *every $P \in \mathcal{L}$ is good*
- *every seeded precoloring in $\mathcal{L}$ has a seed of size at most C;*
- *$|\mathcal{L}| \leq |V(G)|^C$.*

By Theorem 10, to solve the 4-precoloring extension problem in polynomial time, it is sufficient to solve the precoloring extension problem for good seeded precolorings of P_6-free graphs (with seed size bounded by a constant) in polynomial time.

3 Establishing the Axioms on Y

In the previous section, we arranged that components of $G|(Y_0 \cup Y)$ containing a vertex of Y_0 are well-behaved. In this section, we deal with components of $G|(Y_0 \cup Y)$ that do not contain a vertex of Y_0.

Let P be a starred precoloring. We say that a collection $\mathcal{L}$ of starred precolorings is an *equivalent collection* for P if P has a precoloring extension if and only if at least one of the starred precolorings in $\mathcal{L}$ does.

The following are the axioms we want to establish for starred precolorings.

(I) Every vertex y in Y satisfies $|L_P(y)| = 3$.

(II) Let $L_1, L_2 \subseteq \{1,2,3,4\}$ with $|L_1| = |L_2| = 3$ and $L_1 \neq L_2$. Then there is no path $a - b - c$ with $L_P(a) = L_1$, $L_P(b) = L_P(c) = L_2$ and $a, b, c \in Y$.

(III) Let $L_1, L_2, L_3 \subseteq \{1,2,3,4\}$ with $|L_1| = |L_2| = |L_3| = 3$ and such that L_1, L_2, L_3 are pairwise distinct. Then there is no path $a - b - c$ with $L_P(a) = L_1$, $L_P(b) = L_2$, $L_P(c) = L_3$ and $a, b, c \in Y$.

(IV) Let $L_1 \subseteq \{1,2,3,4\}$ with $|L_1| = 3$. Then there is no path $a - b - c$ with $L_P(b) = L_P(c) = L_1$ and $a \in X$, $b, c \in Y$.

(V) Let $L_1, L_2 \subseteq \{1,2,3,4\}$ with $|L_1| = |L_2| = 3$. Then there is no path $a-b-c$ with $L_P(b) = L_1, L_P(c) = L_2$ and $a \in X$ with $L_P(a) \neq L_1 \cap L_2$.

(VI) For every component C of $G|Y$, for which there is a vertex of X is mixed on C, there exist $L_1, L_2 \subseteq \{1,2,3,4\}$ with $|L_1| = |L_2| = 3$ such that C contains a vertex x_i with $L_P(x_i) = L_i$ for $i = 1,2$, every vertex x in C satisfies $L_P(x) \in \{L_1, L_2\}$, and every $x \in X$ mixed on C satisfies $L_P(x) = L_1 \cap L_2$.

(VII) For every component C of $G|Y$ such that some vertex of X is mixed on C, and for L_1, L_2 as in (VI), $L_P(v) = L_1 \cap L_2$ for every vertex $v \in X$ with a neighbor in C.

(VIII) $Y = \emptyset$.

We begin by showing that starred precolorings exist, and we establish axiom (I).

Lemma 14. *Let $P = (G, S, X_0, X, Y_0, Y, f)$ be a good seeded precoloring of a P_6-free graph G. Then*

$$P' = (G, S, X_0, X, Y \setminus Y_L^*, Y_L^* \cup Y_0, f)$$

(with Y_L^* as in (vii)*) is a starred precoloring satisfying* (I) *and P' has a precoloring extension if and only if P does, and every precoloring extension of P' is a precoloring extension of P.*

Proof. This is easily verified by checking the definition of a starred precoloring. □

Our next goal is to establish axiom (II), which we restate.

(II) Let $L_1, L_2 \subseteq \{1,2,3,4\}$ with $|L_1| = |L_2| = 3$ and $L_1 \neq L_2$. Then there is no path $a - b - c$ with $L_P(a) = L_1$, $L_P(b) = L_P(c) = L_2$ and $a, b, c \in Y$.

This lemma will also be useful for proving (IV). Our proof strategy, as is often the case for proofs that a certain configuration can be avoided, is to guess one or more occurrences of such a path, and add parts of them to the seed. We will also see an earlier strategy again; reducing the maximum size of a clique such that some vertex has exactly one neighbor in it.

Lemma 15. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $L_1 \subseteq \{1,2,3,4\}$ with $|L_1| = 3$, and let $P = (G, S, X_0, X, Y, Y^*, f)$ be a starred precoloring of a P_6-free graph G with P satisfying* (I)*. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs an equivalent collection $\mathcal{L}$ for P such that*

- *$|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;*
- *every $P' \in \mathcal{L}$ is a starred precoloring of G;*

- *every $P' \in \mathcal{L}$ with seed S' satisfies $|S'| \leq q(|S|)$;*
- *every $P' = (G, S', X'_0, X', Y', Y^*, f') \in \mathcal{L}$ satisfies* (I) *and $Y' \subseteq Y$;*
- *if there is no path $a - b - c$ with $L_P(a) \neq L'_1$, $L_P(b) = L_P(c) = L'_1$ and $a, b, c \in Y$ for some L'_1 with $|L'_1| = 3$, then there is no path $a - b - c$ with $L_{P'}(a) \neq L'_1$, $L_{P'}(b) = L_{P'}(c) = L'_1$ and $a, b, c \in Y'$;*
- *if P satisfies* (II)*, and if there is no path $a - b - c$ with $L_P(a) \neq L'_1$, $L_P(b) = L_P(c) = L'_1$ and $a, b, c \in X \cup Y$ for some L'_1 with $|L'_1| = 3$, then there is no path $a - b - c$ with $L_{P'}(a) \neq L'_1$, $L_{P'}(b) = L_{P'}(c) = L'_1$ and $a, b, c \in X' \cup Y'$; and*
- *there is no path $a - b - c$ with $L_{P'}(a) \neq L_1$, $L_{P'}(b) = L_{P'}(c) = L_1$ and $a, b, c \in X' \cup Y'$.*

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension for P in polynomial time, if one exists.

Proof. Let $L_1 \subseteq \{1, 2, 3, 4\}$ with $|L_1| = 3$, and let $P = (G, S, X_0, X, Y, Y^*, f)$ be a starred precoloring of a P_6-free graph G with P satisfying (I). We check in polynomial time if G contains a K_5. If so, then P does not have a precoloring extension and we output $\mathcal{L} = \emptyset$ as an equivalent collection. Therefore, for the remainder of the proof we may assume that G contains no K_5.

Let $\mathcal{L} = \emptyset$. Let $Y_1 = \{y \in Y : L_P(y) = L_1\}$. Let $\mathcal{T} = \{T_1, \ldots, T_r\}$ be the set of types $T \subseteq S$ with $f(T) \neq \{1, 2, 3, 4\} \setminus L_1$ and $|f(T)| \leq 2$, and if P satisfies (II), $|f(T)| = 2$. Let $\mathcal{Q}$ be the set of all r-tuples of quadruples $((Q_1, R_1, c_1, d_1), \ldots, (Q_r, R_r, c_r, d_r))$ such that for every $i \in \{1, \ldots, r\}$,

- $c_i, d_i \in L_1$ with $c_i \neq d_i$ if $Q_i, R_i \neq \emptyset$;
- $1 \geq |Q_i| \geq |R_i|$ and $Q_i \cup R_i$ is a clique; and
- $Q_i \cup R_i \subseteq (X \cup Y)(T_i)$.

For every $Q = ((Q_1, R_1, c_1, d_1), \ldots, (Q_r, R_r, c_r, d_r)) \in \mathcal{Q}$, we proceed as follows. Let $S'^Q = Q_1 \cup R_1 \cup \cdots \cup Q_r \cup R_r$, and let $f' : S' \to L_1$ be such that $f'(q_i) = c_i$ for all i for which $Q_i = \{q_i\}$, and $f'(r_i) = d_i$ for all i for which $R_i = \{r_i\}$. Let

$$\tilde{Y}^Q = \bigcup_{i : Q_i = \emptyset} (X \cup Y)(T_i),$$

and let $g^Q : \tilde{Y}^Q \to \{1, 2, 3, 4\} \setminus L_1$ be the constant function. Let

$$\tilde{Z}^Q = \bigcup_{i : R_i = \emptyset, Q_i \neq \emptyset} ((X \cup Y)(T_i) \cap N(Q_i)),$$

and let $g'^Q : \tilde{Z}^Q \to \{1, 2, 3, 4\} \setminus L_1$ be the constant function.

For $i \in \{1, \ldots, r\}$, let $\tilde{X}_i$ and g''^Q_i be defined as follows. If $|f(T_i)| = 1$, we let $\tilde{X}_i = X(T_i) \cap N(Q_i) \cap N(R_i)$. If $|f(T_i)| = 2$, we let $\tilde{X}_i = X(T_i) \cap N(Q_i)$. We define $g''^Q(\tilde{X}_i) = \{1, 2, 3, 4\} \setminus (f'(T_i) \cup \{c_i, d_i\})$. This is well-defined, since $f'(T_i) \cap \{c_i, d_i\} = \emptyset$, and $\tilde{X}_i$ is non-empty only if either $|f'(T_i)| = 2$ or $|\{c_i, d_i\}| = 2$ (since our definition for $\tilde{X}_i$ requires that both Q_i and R_i are non-empty in the case that $|f'(T_i)| = 1$). Let $\tilde{X}^Q = \tilde{X}_1 \cup \cdots \cup \tilde{X}_r$.

Then, if $f \cup f' \cup g^Q \cup g'^Q \cup g''^Q$ is a proper coloring of $G|(S \cup S'^Q \cup X_0 \cup \tilde{Y}^Q \cup \tilde{Z}^Q \cup \tilde{X}^Q)$, we add the starred precoloring

$$\begin{aligned} P'^Q = (&G, S \cup S'^Q, \\ &X_0 \cup \tilde{X}^Q \cup \tilde{Y}^Q \cup \tilde{Z}^Q, \\ &X \setminus (\tilde{X}^Q \cup \tilde{Y}^Q \cup \tilde{Z}^Q \cup S'^Q), \\ &Y \setminus (\tilde{X}^Q \cup \tilde{Y}^Q \cup \tilde{Z}^Q \cup S'^Q), \\ &Y^*, f \cup f' \cup g^Q \cup g'^Q \cup g''^Q) \end{aligned}$$

to $\mathcal{L}$.

This starred precoloring satisfies (I). Every precoloring extension of P'^Q is a precoloring extension of P. Moreover, suppose that c is a precoloring extension of P. Let $Q = ((Q_1, R_1, c_1, d_1), \ldots, (Q_r, R_r, c_r, d_r))$ be defined as follows:

- For every type $T_i \in \mathcal{T}$ such that $c((X \cup Y)(T_i)) \subseteq \{1,2,3,4\} \setminus L_1$, we let $Q_i = R_i = \emptyset$ and $c_i, d_i \in L_1$ arbitrary.
- For every type $T_i \in \mathcal{T}$ such that there exist $x, y \in (X \cup Y)(T_i)$ with $c(x), c(y) \in L_1$ and $xy \in E(G)$, we let $Q_i = \{x\}$, $R_i = \{y\}$ and $c_i = c(x), d_i = c(y)$.
- For every type $T_i \in \mathcal{T}$ such that do not there exist x, y as above, but there is a vertex $v \in (X \cup Y)(T_i)$ with $c(v) \in L_i$, we let $Q_i = \{v\}, R_i = \emptyset, c_i = c(v), d_i = c(v)$.

Note that if $|Q_i \cup R_i| < 2$, then every vertex v in $(X \cup Y)(T_i)$ complete to $Q_i \cup R_i$ satisfies $c(v) \notin L_1$, and so g and g' agree with c on $\tilde{Y}^Q$ and $\tilde{Z}^Q$, respectively. It follows that $P'^Q \in \mathcal{L}$ and c is a precoloring extension of P'^Q. Consequently, that $\mathcal{L}$ is an equivalent collection for P.

We now prove that every $P'^Q \in \mathcal{L}$ satisfies the claims of the lemma. Let

$$Q = ((Q_1, R_1, c_1, d_1), \ldots, (Q_r, R_r, c_r, d_r))$$

with $P'^Q \in \mathcal{L}$, and write $P' = P'^Q \in \mathcal{L}$ with $P' = (G, S', X_0', X', Y', Y^*, f')$. Let $Y_1' = \{y \in Y' : L_{P'}(y) = L_1\}$.

(7) *If there is no path $a - b - c$ with $L_P(a) \neq L_1'$, $L_P(b) = L_P(c) = L_1'$ and $a, b, c \in Y$ for some L_1' with $|L_1'| = 3$, then there is no path $a - b - c$ with $L_{P'}(a) \neq L_1'$, $L_{P'}(b) = L_{P'}(c) = L_1'$ and $a, b, c \in Y'$; and if P satisfies* (II)*, and if there is no path $a - b - c$ with $L_P(a) \neq L_1'$, $L_P(b) = L_P(c) = L_1'$ and $a, b, c \in X \cup Y$ for some L_1' with $|L_1'| = 3$, then there is no path $a - b - c$ with $L_{P'}(a) \neq L_1'$, $L_{P'}(b) = L_{P'}(c) = L_1'$ and $a, b, c \in X' \cup Y'$.*

Suppose not; and let $a - b - c$ be a path violating one of the statements of (7). Since $b, c \in Y' \subseteq Y$, it follows that $L_P(b) = L_P(c) = L_1'$. By the assumption of (7), it follows that $L_P(a) \neq L_{P'}(a)$, and so $a \in Y \cap X'$. This implies that $|L_{P'}(a)| = 2$. Since $a \notin Y'$, it follows that the first statement of (7) is proved.

Therefore, we may assume that (II) holds for P. Since P satisfies (II), it follows that $L_P(a) = L_1'$. Moreover, there is a vertex $s \in S' \setminus S$ with $f(s) \in L_1'$ and $as \in E(G)$. Since $b \in Y'$, it follows that $s - a - b$ is a path. But since P satisfies (II), it follows that $S' \setminus S \subseteq X$ by construction, and so $s \in X$. But then the path $s - a - b$ contradicts the assumption of (7). This implies (7).

(8) *There is no path $z - a - b - c$ with $z \in (X' \cup Y') \setminus Y_1'$ and $a, b, c \in Y_1'$.*

Suppose not; and let $z-a-b-c$ as in (8). It follows that $z \in X \cup Y$ and $a, b, c \in Y_1$. Let $T_i = N(z) \cap S \in \mathcal{T}$. Since $z \notin X_0'$, it follows that $z \notin \tilde{X}^Q \cup \tilde{Y}^Q \cup \tilde{Z}^Q$. Therefore, $Q_i \cup R_i$ contains a vertex y non-adjacent to z. Since $c_i, d_i \in L_1$, it follows that y is anticomplete to $\{z, a, b, c\}$. Let $s \in T_i$ with $f(s) \in L_1$; then s is a common neighbor of y and z. It follows that s is not adjacent to a, b, c. But then $y - s - z - a - b - c$ is a P_6 in G, a contradiction. This proves (8).

Let $\mathcal{L}_5 = \mathcal{L}$. We repeat the following procedure for $j = 4, 3, 2$. For every $P' = (G, S', X_0', X', Y', Y^*, f') \in \mathcal{L}_{j+1}$, we proceed as follows. We let $\mathcal{L}_j(P') = \emptyset$. Let $Y_1' = \{y \in Y' : L_{P'}(y) = L_1\}$. Let Y_1^* be the set of vertices y in $(X' \cup Y') \setminus Y_1'$ such that there is a clique $\{a_1, \ldots, a_j\} \subseteq Y_1'$ and $N(y) \cap \{a_1, \ldots, a_j\} = \{a_1\}$. Let $\mathcal{T}^j = \left\{T_1^j, \ldots, T_{r_j}^j\right\}$ be the set of all types $T \subseteq S'$ such that $f(T) \neq \{1,2,3,4\} \setminus L_1$ and $|f(T)| \leq 2$, and if P' satisfies (II), $|f(T)| = 2$. Let $\mathcal{Q}(P')$ be the set of all r_j-tuples of quadruples $((Q_1, R_1, c_1, d_1), \ldots, (Q_{r_j}, R_{r_j}, c_{r_j}, d_{r_j}))$ such that for every $i \in \{1, \ldots, r_j\}$,

- $c_i, d_i \in L_1$;
- $1 \geq |Q_i| \geq |R_i|$ and $Q_i \cup R_i$ is a clique; and
- $Q_i \cup R_i \subseteq (X \cup Y)(T_i)$.

For every $Q = ((Q_1, R_1, c_1, d_1), \ldots, (Q_{r_j}, R_{r_j}, c_{r_j}, d_{r_j})) \in \mathcal{Q}(P')$, we proceed as follows. Let $S'^Q = Q_1 \cup R_1 \cup \cdots \cup Q_{r_j} \cup R_{r_j}$, and let $g^Q : S' \to L_1$ such that $g^Q(q_i) = c_i$ for all i such that $Q_i = \{q_i\}$, and $g^Q(x_i) = d_i$ for all i such that $R_i = \{x_i\}$.

For $i \in \{1, \ldots, r_j\}$, we let Z_i be the set of vertices $z \in (X \cup Y)(T_i)$ such that one of the following holds:

- $Q_i = \emptyset$;
- $Q_i = \{q_i\}$, and $N(q_i) \cap Y'_1 \subsetneq N(z) \cap Y'_2$;
- $Q_i = \{q_i\}$, $R_i = \{x_i\}$, z is adjacent to q_i and $N(x_i) \cap Y'_2 \subsetneq N(z) \cap Y'_1$;

We let $\tilde{Z}^Q = Z_1 \cup \dots \cup Z_{r_j}$ and $g'^Q : \tilde{Z}^Q \to \{1,2,3,4\} \setminus L_1$. Let

$$\tilde{X}^Q = \bigcup_{i: R_i = \emptyset, Q_i \neq \emptyset} ((X \cup Y)(T_i) \cap N(S_i)),$$

and let $g''^Q : \tilde{X}^Q \to \{1,2,3,4\} \setminus L_1$ be the constant function. Let

$$\begin{aligned} P'^Q = &(G, S' \cup S^Q, X'_0 \cup \tilde{Z}^Q \cup \tilde{X}^Q, \\ &X' \setminus (S^Q \cup \tilde{Z}^Q \cup \tilde{X}^Q), \\ &Y' \setminus (S^Q \cup \tilde{Z}^Q \cup \tilde{X}^Q), Y^*, \\ &f' \cup g^Q \cup g'^Q \cup g''^Q). \end{aligned}$$

If $f' \cup g^Q \cup g'^Q \cup g''^Q$ is proper coloring of $G|(S' \cup S^Q \cup \tilde{Z}^Q \cup \tilde{X}^Q)$, then we add P'^Q to $\mathcal{L}_j(P')$.

It follows that for every $Q \in \mathcal{Q}(P')$, every precoloring extension of P'^Q is a precoloring extension of P'. Moreover, suppose that c is a precoloring extension of P'. We define $Q = ((Q_1, R_1, c_1, d_1), \dots, (Q_{r_j}, R_{r_j}, c_{r_j}, d_{r_j}))$ as follows:

- For every type $T_i \in \mathcal{T}$ such that $c((X \cup Y)(T_i)) \cap L_1 = \emptyset$, we let $Q_i = R_i = \emptyset$ and $c_i, d_i \in L_1$ arbitrary.
- For every type $T_i \in \mathcal{T}$ such that $c((X \cup Y)(T_i)) \cap L_1 \neq \emptyset$, we let v a vertex $v \in (X \cup Y)(T_i)$ with $c(v) \in L_1$ with $N(v) \cap Y_1$ maximal. We let $Q_i = \{v\}, c_i = c(v)$. If there is a vertex w in $N(v) \cap (X \cup Y)(T_i)$ with $c(w) \in L_1$, then we choose such a vertex with $N(w) \cap Y_1$ maximal and let $R_i = \{w\}, d_i = c(w)$; otherwise we let $R_i = \emptyset$ and $d_i \in L_1$ arbitrary.

The second bullet implies that $c(x) \notin L_1$ for every $x \in (X \cup Y)(T_i)$ such that $Q_i = \{q_i\}$ and $N(q_i) \cap Y'_1 \subsetneq N(v) \cap Y'_1$. Similarly, $c(x) \notin L_1$ for every $x \in (X \cup Y)(T_i) \cap N(Q_i)$ such that $R_i = \{r_i\}$ and $N(r_i) \cap Y'_1 \subsetneq N(v) \cap Y'_1$. It follows that $Q \in \mathcal{Q}(P')$, $P'^Q \in \mathcal{L}_j(P')$, and c is a precoloring extension of P'^Q. Thus $\mathcal{L}_j(P')$ is an equivalent collection for P'. By construction, P'^Q satisfies (I) for every $Q \in \mathcal{Q}(P')$.

Now let

$$\mathcal{L}_j = \bigcup_{P' \in \mathcal{L}_{j+1}} \mathcal{L}_j(P').$$

Since $\mathcal{L}_{j+1}$ is an equivalent collection for P and since $\mathcal{L}_j$ is the union of equivalent collections for every $P' \in \mathcal{L}_{j+1}$, it follows that $\mathcal{L}_j$ is an equivalent collection for P.

Let $P' \in \mathcal{L}_{j+1}$. Let $Q = ((Q_1, R_1, c_1, d_1), \dots, (Q_r, R_r, c_r, d_r)) \in \mathcal{Q}(P')$, and let

$$P'^Q = (G, S'', X''_0, X'', Y'', Y^*, f'') \in \mathcal{L}_j(P').$$

Let $Y''_1 = \{y \in Y'' : L_{P'^Q}(y) = L_1\}$. From the previous step $(j+1)$ of our argument, we may assume that (8) and (9) hold for P', that is, for $j+1$, the statements (8) and (9) hold with X'', Y'', Y''_1 replaced by X', Y', Y'_1, respectively, where X', Y' are the sets in $P' = (G, S', X'_0, X', Y', Y^*, f')$, and $Y'_1 = \{y \in Y' : L_{P'}(y) = L_1\}$. This is true when $j = 4$ as well, since G contains no K_5 and by (8).

(9) *There is no vertex $z \in (X'' \cup Y'') \setminus Y''_1$ with $N(z) \cap \{a_1, \dots, a_j\} = \{a_1\}$ for a clique $\{a_1, \dots, a_j\} \subseteq Y''_1$.*

Suppose for a contradiction that z is such a vertex. Write $P' = (G, S', X'_0, X', Y', Y^*, f')$. Let $Y'_i = \{y \in Y' : L_{P'}(y) = L_i\}$ for $i = 1, 2$. Suppose first that $z \in Y'_1$. Then z has a neighbor $s \in S'' \setminus S'$. It follows that $f''(s) \in L_1$ and $s \notin Y'_1$. Consequently, s is anticomplete to $\{a_1, \dots, a_j\}$. But then the path $s - z - a_1 - a_j$ contradicts the fact that (8) holds for P'.

It follows that $z \in (X' \cup Y') \setminus Y_1'$ and $\{a_1, \ldots, a_j\} \subseteq Y_1'$. Let i be such that $S' \cap N(z) = T_i$. Since $z \notin X_0''$, it follows $Q_i \neq \emptyset$; say $Q_i = \{q_i\}$. If z is non-adjacent to q_i, let $s = q_i$. Otherwise, it follows that $R_i = \{r_i\}$, say; let $s = r_i$. In both cases, it follows that s is non-adjacent to z.

Since $a_1, \ldots, a_j \notin X''$, it follows that s is non-adjacent to $a_1, \ldots, a_j$. The definition of Z_i implies that $N(s) \cap Y_2' \not\subseteq N(z) \cap Y_2'$. Since $a_1 \in (N(z) \setminus N(s)) \cap Y_1'$, we deduce that there exists a vertex $y \in (N(z) \setminus N(s)) \cap Y_1'$.

Let $s' \in T_i$ with $f(s') \in L_1$. Then, s' is non-adjacent to $a_1, \ldots, a_j$. But $y - s - s' - z - a_1 - a_j$ is not a P_6 in G, and thus y has a neighbor in $\{a_1, \ldots, a_j\}$. But y is not complete to $\{a_1, \ldots, a_j\}$, since P' satisfies (9) for $j+1$. It follows that y is mixed on $\{a_1, \ldots, a_j\}$, and thus by Lemma 1, there is a path $y - a - b$ with $a, b \in \{a_1, \ldots, a_j\}$. But then $s - y - a - b$ is a path, contrary to the fact that P' satisfies (8). This concludes the proof of (9).

(10) *There is no path $z - a - b - c$ with $z \in (X'' \cup Y'') \setminus Y_1''$ and $a, b, c \in Y_1''$.*

Suppose not; and let $z - a - b - c$ be such a path. Since $Y_1'' \subseteq Y_1'$, the fact that P' satisfies (8) implies that $z \notin X' \cup Y'$, and thus $z \in Y_1'$. Thus z has a neighbor $s \in S'' \setminus S'$ with $f(s) \in L_1$. It follows that $s \in X' \cup Y'$, and thus $s - z - a - b$ is a path, contrary to the fact that (8) holds for P'. This proves (10).

(11) *If there is no path $a - b - c$ with $L_{P'}(a) \neq L_1'$, $L_{P'}(b) = L_{P'}(c) = L_1'$ and $a, b, c \in Y'$ for some L_1' with $|L_1'| = 3$, then there is no path $a - b - c$ with $L_{P''}(a) \neq L_1'$, $L_{P''}(b) = L_{P''}(c) = L_1'$ and $a, b, c \in Y''$; and if P' satisfies* (II)*, and if there is no path $a - b - c$ with $L_{P'}(a) \neq L_1'$, $L_{P'}(b) = L_{P'}(c) = L_1'$ and $a, b, c \in X \cup Y$ for some L_1' with $|L_1'| = 3$, then there is no path $a - b - c$ with $L_{P'}(a) \neq L_1'$, $L_{P''}(b) = L_{P''}(c) = L_1'$ and $a, b, c \in X'' \cup Y''$.*

Suppose not; and let $a - b - c$ be a path with $L_{P''}(b) = L_{P''}(c) = L_1'$ with $b, c \in Y''$ that violates one of the statements of (11). Since $b, c \in Y'' \subseteq Y'$, it follows that $L_{P'}(b) = L_{P'}(c) = L_1'$. By the assumption of (11), it follows that $L_{P'}(a) \neq L_{P''}(a)$, and so $a \in Y' \cap X''$. This implies that $|L_{P''}(a)| = 2$. Since $a \notin Y''$, it follows that the first statement of (11) is proved.

Therefore, we may assume that (II) holds for P'. Since P' satisfies (II), it follows that $L_{P'}(a) = L_1'$. Moreover, there is a vertex $s \in S'' \setminus S'$ with $f'(s) \in L_1'$ and $as \in E(G)$. Since $b \in Y''$, it follows that $s - a - b$ is a path. But since P' satisfies (II), it follows that $S'' \setminus S' \subseteq X'$ by construction, and so $s \in X'$. But then the path $s - a - b$ contradicts the assumption of (11). This implies (11).

It follows that (8) and (11) holds for P'^Q for every $P' \in \mathcal{L}_{j+1}$ and $Q \in \mathcal{Q}(P')$. Moreover, by construction, $\mathcal{L}_j$ is an equivalent collection for P. If $j > 2$, we repeat the procedure for $j - 1$; otherwise, we stop.

At termination, we have constructed an equivalent collection $\mathcal{L}_2$ for P and every $P' = (G, S', X_0', X', Y', Y_0', f') \in \mathcal{L}_2$ satisfies (I) and (9) for $j = 2$, and thus the last bullet of the lemma. The third-to-last and second-to-last bullets of the lemma follow from (7) and (11). Thus, $\mathcal{L}_2$ satisfies the properties of the lemma, and hence, the lemma is proved. □

Lemma 16. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y, Y^*, f)$ be a starred precoloring of a P_6-free graph G with P satisfying* (I)*. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs an equivalent collection $\mathcal{L}$ for P such that*

- *$|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;*
- *every $P' \in \mathcal{L}$ is a starred precoloring of G;*
- *every $P' \in \mathcal{L}$ with seed S' satisfies $|S'| \leq q(|S|)$; and*
- *every $P' \in \mathcal{L}$ satisfies* (I) *and* (II).

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension for P in polynomial time, if one exists.

Proof. Let $\mathcal{L} = \{P\}$. We repeat the following for every pair L_1, L_2 of distinct lists of size three contained in $\{1, 2, 3, 4\}$. We apply Lemma 15 to every starred precoloring $P' \in \mathcal{L}$, and replace $\mathcal{L}$ by the union of the equivalent collections produced by Lemma 15. Then we move to the next pair of lists. □

The next lemma is a simple tool that we will use to establish further axioms.

Lemma 17. *Let G be a P_6-free graph with $u, v \in V(G)$ such that $V(G) = \{u, v\} \cup N(u) \cup N(v)$, $uv \notin E(G)$, $N(u) \cap N(v) = \emptyset$, and $N(u), N(v)$ stable. Then there is a partition $A_0, A_1, \ldots, A_k$ of $N(u)$ and a partition $B_0, B_1, \ldots, B_k$ of $N(v)$ with $k \geq 0$ such that*

- *A_0 is complete to $N(v)$;*
- *B_0 is complete to $N(u)$; and*
- *for $i = 1, \ldots, k$, $A_i, B_i \neq \emptyset$ and A_i is complete to $N(v) \setminus B_i$ and B_i is complete to $N(u) \setminus A_i$, and A_i is anticomplete to B_i.*

Proof. Let G, u, v be as in the lemma. The result holds if $N(u) = \emptyset$ or $N(v) = \emptyset$; thus we may assume that both sets are non-empty. Let $a \in N(u), b \in N(v)$. If $ab \in E(G)$, we let $A_0 = \{a\}, B_0 = \{b\}$; otherwise, we let $A_1 = \{a\}, B_1 = \{b\}$. Now let $A_0, A_1, \ldots, A_k, B_0, B_1, \ldots, B_k$ be chosen such that their union is maximal subject to satisfying the conditions of the lemma. If their union is $V(G) \setminus \{u, v\}$, then there is nothing to show; thus we may assume that there is a vertex $x \notin \{u, v\}$ not contained in their union. Without loss of generality, we may assume that $x \in N(v)$.

If x is complete to $A = A_0 \cup A_1 \cup \cdots \cup A_k$, we can add x to B_0, contrary to the maximality of our choice of sets. Suppose first that x is complete to $A_1 \cup \cdots \cup A_k$. Let $A_{k+1} = A_0 \setminus N(x)$. Then A_{k+1} is non-empty, since x has a non-neighbor in A. But then $A_0 \setminus A_{k+1}, A_1, \ldots, A_k, A_{k+1}, B_0, B_1, \ldots, B_k, \{x\}$ satisfies the conditions of the lemma and has strictly larger union; a contradiction.

It follows that x has a non-neighbor in $A \setminus A_0$; without loss of generality we may assume that there is $y \in A_1$ non-adjacent to x. Let $w \in B_1$. Suppose that x has a neighbor $z \in A_1$. Then $w - v - x - z - u - y$ is a P_6 in G, a contradiction. It follows that x has no neighbor in A_1. If x is complete to $A \setminus A_1$, we can add x to B_1 and enlarge the structure, a contradiction; hence x has a non-neighbor z in $A \setminus A_1$. It follows that z is adjacent to w. But then $x - v - w - z - u - y$ is a P_6 in G, a contradiction. This concludes the proof of the lemma. □

The purpose of the following lemmas is to establish the following axiom, which we restate:

(III) Let $L_1, L_2, L_3 \subseteq \{1, 2, 3, 4\}$ with $|L_1| = |L_2| = |L_3| = 3$ and such that L_1, L_2, L_3 are pairwise distinct. Then there is no path $a - b - c$ with $L_P(a) = L_1$, $L_P(b) = L_2$, $L_P(c) = L_3$ and $a, b, c \in Y$.

It may be helpful to begin by reading (12) in order to understand the construction for $\mathcal{Q}$.

Lemma 18. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $L_1, L_2, L_3 \subseteq \{1, 2, 3, 4\}$ with $|L_1| = |L_2| = |L_3| = 3$ and such that L_1, L_2, L_3 are pairwise distinct. Let $P = (G, S, X_0, X, Y, Y^*, f)$ be a starred precoloring of a P_6-free graph G with P satisfying* (I) *and* (II). *Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs an equivalent collection $\mathcal{L}$ for P such that*

- *$|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;*
- *every $P' \in \mathcal{L}$ is a starred precoloring of G;*
- *every $P' \in \mathcal{L}$ with seed S' satisfies $|S'| \leq q(|S|)$;*
- *every $P' \in \mathcal{L}$ satisfies* (I) *and* (II)*;*
- *every $P' = (G, S', X'_0, X', Y', Y^*, f') \in \mathcal{L}$ satisfies that there is no path $a - b - c - d$ with $L_{P'}(a) = L_1$, $L_{P'}(b) = L_{P'}(d) = L_2$, $L_{P'}(c) = L_3$ and $a, b, c, d \in Y'$; and*
- *if P satisfies the previous bullet for L_1, L_2, L_3 and for L_3, L_2, L_1, then every $P' = (G, S', X'_0, X', Y', Y^*, f') \in \mathcal{L}$ satisfies that there is no path $a - b - c$ with $L_{P'}(a) = L_1$, $L_{P'}(b) = L_2$, $L_{P'}(c) = L_3$ and $a, b, c \in Y'$.*

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension for P in polynomial time.

Proof. We say that *the conditions of the last bullet hold for* P if P satisfies the second-to-last bullet for L_1, L_2, L_3 and L_3, L_2, L_1.

Let $Y_i = \{y \in Y : L_P(y) = L_i\}$ for $i = 1, 2, 3$. Let $\mathcal{T} = \{T_1, \dots, T_r\}$ be the set of types $T \subseteq S$ with $f(T) = \{1, 2, 3, 4\} \setminus L_1$. We let $\mathcal{Q}$ be the set of all r-tuples $(Q_1, \dots, Q_r)$, where for each i, $Q_i = (S_i^1, S_i^2, R_i^1, R_i^2, c_i^1, c_i^2, c_i^3, c_i^4)$ such that the following hold:

1. $\{c_i^1, c_i^2\} \subseteq \{1, 2, 3, 4\}$.
2. $1 \geq |S_i^1| \geq |S_i^2| \geq |R_i^1| \geq |R_i^2|$.
3. $S_i^1 \neq \emptyset$ if and only if one of the following holds:
   - there is a path $a - b - c - d$ with $a \in Y_1, b, d \in Y_2, c \in Y_3$ and $N(a) \cap S = T_i$; or
   - the conditions of the last bullet hold for P and there is a path $a - b - c$ with $a \in Y_1, b \in Y_2, c \in Y_3$ and $N(a) \cap S = T_i$.
4. $S_i^1 \cup S_i^2$ is a stable set, and $S_i^1 \cup S_i^2 \subseteq Y_1(T_i)$.
5. If $S_i^1 = \{s_i^1\}$, then s_i^1 has a neighbor in Y_2.
6. If $S_i^2 = \{s_i^2\}$, then s_i^2 has a neighbor in Y_2.
7. If $S_i^2 \neq \emptyset$, then $\{c_i^1, c_i^2\} = L_1 \setminus (L_2 \cap L_3)$ and $c_i^1 \in L_3, c_i^2 \in L_2$.
8. $R_i^1 \subseteq (N(S_i^1) \setminus N(S_i^2)) \cap Y_2$.
9. $R_i^2 \subseteq (N(S_i^2) \setminus N(S_i^1)) \cap Y_3$.
10. $R_i^1 \cup R_i^2$ is a stable set.
11. $\{c_i^3, c_i^4\} \subseteq L_2 \cap L_3$.

We let $S'^Q = \bigcup_{i=1}^r (S_i^1 \cup S_i^2)$ and $T'^Q = \bigcup_{i=1}^r (R_i^1 \cup R_i^2)$. Define $f'^Q : S'^Q \cup T'^Q \to \{1, 2, 3, 4\}$ by setting $f'^Q(v) = c_i^j$ if $S_i^j = \{v\}$ for $j = 1, 2$ and $f'^Q(v) = c_i^{j+2}$ if $R_i^j = \{v\}$ for $j = 1, 2$. Let S_1' be the set of $v \in (T'^Q \cup S'^Q)$ such that $f'^Q(v) \in L_2 \cap L_3$. Let S_2' be the set of $v \in (T'^Q \cup S'^Q)$ such that $f'^Q(v) \in L_2 \setminus L_3$, and let S_3' be the set of $v \in (T'^Q \cup S'^Q)$ such that $f'^Q(v) \in L_3 \cap L_2$. Let

$$\tilde{X}^Q = (N(S_1') \cap (Y_1 \cup Y_2 \cup Y_3)) \cup (N(S_2') \cap (Y_1 \cup Y_2)) \cup (N(S_3') \cap (Y_1 \cup Y_2)) \cup (N(T'^Q) \cap (Y_2 \cup Y_3)).$$

For $i \in \{1, \dots, r\}$, we further define $\tilde{Z}_i = \emptyset$ if $|S_i^1 \cup S_i^2| < 2$ or $|R_i^1| > 0$, and $\tilde{Z}_i = (N(S_i^1) \setminus N(S_i^2)) \cap Y_2$ otherwise. We let $\tilde{Z}^Q = \bigcup_{i=1}^r \tilde{Z}_i$. Let $g^Q : \tilde{Z} \to L_2 \setminus L_3$ be the constant function. For $i \in \{1, \dots, r\}$, we let $\tilde{Y}_i = \emptyset$ if $|S_i^1 \cup S_i^2| < 2$ or $|R_i^1 \cup R_i^2| \neq 1$, and $\tilde{Y}_i = (N(S_i^2) \setminus (N(S_i^1) \cup N(R_i^1))) \cap Y_3$ otherwise. We let $\tilde{Y}^Q = \bigcup_{i=1}^r \tilde{Y}_i$. Let $g'^Q : \tilde{Y} \to L_3 \setminus L_2$ be the constant function. For $i \in \{1, \dots, r\}$, we let $\tilde{W}_i = \emptyset$ if $|S_i^1 \cup S_i^2| \neq 1$ or $c_i^1 \in L_1 \cap L_2 \cap L_3$, and $\tilde{W}_i = Y_1(T_i) \setminus S_i^1$ otherwise. We let $\tilde{W}^Q = \bigcup_{i=1}^r \tilde{W}_i$. We define $g''^Q : \tilde{W} \to L_1$ by setting $g''(\tilde{W}_i \setminus N(S_i^1)) = \{c_i^1\}$ and $g''(\tilde{W}_i \cap N(S_i^1)) = L_1 \setminus (\{c_i^1\} \cup (L_2 \cap L_3)$.

Let P'^Q be the starred precoloring

$$(G, S \cup S'^Q \cup T'^Q, X_0 \cup \tilde{W}^Q \cup \tilde{Y}^Q \cup \tilde{Z}^Q, X \cup \tilde{X}^Q, Y \setminus (S'^Q \cup T'^Q \cup \tilde{W}^Q \cup \tilde{X}^Q \cup \tilde{Y}^Q \cup \tilde{Z}^Q), Y^*, f \cup f'^Q \cup g^Q \cup g'^Q \cup g''^Q).$$

Since P satisfies (II), it follows that P' satisfies (II) as well. We let

$$\mathcal{L} = \{P'^Q : Q \in \mathcal{Q}, f \cup f'^Q \cup g^Q \cup g'^Q \cup g''^Q \text{ is a proper coloring}\}.$$

(12) *$\mathcal{L}$ is an equivalent collection for P.*

Let $L_1 = \{c^1, c^2, c^3\}, L_2 = \{c^1, c^2, c^4\}$ and $L_3 = \{c^1, c^3, c^4\}$. Let Y_1^* denote the set of vertices in Y_1 with a neighbor in Y_2. Every precoloring extension for $P'^Q \in \mathcal{L}$ is a precoloring extension for P. Now suppose that P has a precoloring extension $c : V(G) \to \{1, 2, 3, 4\}$. We define an r-tuple $(Q_1, \dots, Q_r)$, where for each i, $Q_i = (S_i^1, S_i^2, R_i^1, R_i^2, c_i^1, c_i^2, c_i^3, c_i^4)$. For $i \in \{1, \dots, r\}$, we define $Q_i = (S_i^1, S_i^2, R_i^1, R_i^2, c_i^1, c_i^2, c_i^3, c_i^4)$ as follows:

- If neither bullet of 3 is satisfied, we let $Q_i = (\emptyset, \emptyset, \emptyset, \emptyset, c^1, c^1, c^1, c^1)$.
- If $Y_1^*(T_i)$ contains a vertex v with $c(v) = c^1$, we let $Q_i = (\{v\}, \emptyset, \emptyset, \emptyset, c^1, c^1, c^1, c^1)$.
- If $Y_1^*(T_i)$ contains a vertex v with $c(v) = c^2$ such that $c(Y_1^*(T_i) \setminus N(v)) \subseteq \{c^3\}$, we let $Q_i = (\{v\}, \emptyset, \emptyset, \emptyset, c^2, c^1, c^1, c^1)$.
- If $Y_1^*(T_i)$ contains a vertex v with $c(v) = c^3$ such that $c(Y_1^*(T_i) \setminus N(v)) \subseteq \{c^2\}$, we let $Q_i = (\{v\}, \emptyset, \emptyset, \emptyset, c^3, c^1, c^1, c^1)$.
- Let $u, v \in Y_1^*(T_i)$ such that $c(u) = c^2, c(v) = c^3$ and $uv \notin E(G)$. We let $A = (N(u) \setminus N(v)) \cap Y_2$ and $B = (N(v) \setminus N(u)) \cap Y_3$. We proceed as follows:
  - If $c(A) \subseteq L_2 \setminus L_3$, we let $Q_i = (\{u\}, \{v\}, \emptyset, \emptyset, c^2, c^3, c^1, c^1)$.
  - If there is a vertex $x \in A$ such that $c(x) \in L_2 \cap L_3$ and $c(B \setminus N(x)) \subseteq L_3 \setminus L_2$, we let $Q_i = (\{u\}, \{v\}, \{x\}, \emptyset, c^2, c^3, c(x), c^1)$.
  - If there is $x \in A$ and $y \in B$ such that $c(x), c(y) \in L_2 \cap L_3$ and $xy \notin E(G)$, we let $Q_i = (\{u\}, \{v\}, \{x\}, \{y\}, c^2, c^3, c(x), c(y))$.

It follows from the definitions of $\tilde{Y}^Q, \tilde{Z}^Q, \tilde{W}^Q$ that $c|_{(\tilde{Y}^Q \cup \tilde{Z}^Q \cup \tilde{W}^Q)} = g^Q|_{\tilde{Z}^Q} \cup g'^Q|_{\tilde{Y}^Q} \cup g''^Q|_{\tilde{W}^Q}$. It follows that $Q \in \mathcal{Q}$ and c is a precoloring extension of P'^Q. Thus $\mathcal{L}$ is an equivalent collection for P, which proves (12).

Let $Q \in \mathcal{Q}$ and let $P'^Q \in \mathcal{L}$ with $P'^Q = (G, S', X_0', X', Y', Y^*, f')$, and let $Y_i' = \{y \in Y' : L_{P'}(y) = L_i\}$ for $i = 1, 2, 3$. We claim the following.

(13) *For every $i \in \{1, \dots, r\}$ such that $S_i^1 = \{u\}, S_i^2 = \{v\}$, we have that $N(u) \cap (Y_2' \cup Y_3')$ is anticomplete to $N(v) \cap (Y_2' \cup Y_3')$.*

From the properties of Q, we know that $f'(u) \in L_1 \cap L_3$ and $f'(v) \in L_1 \cap L_2$. Since $u, v \in S'$, it follows that $N(u) \cap Y_3' = \emptyset$, since $N(u) \cap Y_3 \subseteq \tilde{X}^Q$; similarly, $N(v) \cap Y_2' = \emptyset$. We let $A = (N(u) \setminus N(v)) \cap Y_2$ and $B = (N(v) \setminus N(u)) \cap Y_3$. It follows that v is anticomplete to A and u is anticomplete to B. Let $a_1, \dots, a_t$ be the components of $G|A$, and let $b_1, \dots, b_s$ be the components of $G|B$. Since P satisfies (II), it follows that for every $i \in [t]$ and $j \in [s]$, $V(a_i)$ is either complete or anticomplete to $V(b_j)$.

Let H be the graph with vertex set $\{u, v\} \cup \{a_1, \dots, a_t\} \cup \{b_1, \dots b_s\}$; where $N_H(u) = \{a_1, \dots, a_t\}$, $N_H(v) = \{b_1, \dots, b_s\}$, the sets $\{a_1, \dots, a_t\}$ and $\{b_1, \dots, b_s\}$ are stable, and a_i is adjacent to b_j if and only if $V(a_i)$ is complete to $V(b_j)$ in G. We apply Lemma 17 to H, u and v to obtain a partition $A_0', A_1', \dots, A_k'$ of $\{a_1, \dots, a_t\}$ and a partition $B_0', B_1', \dots, B_k'$ of $\{b_1, \dots, b_t\}$. For $i \in [k]$, let $A_i = \bigcup_{a_j \in A_i} V(a_j)$ and $B_i = \bigcup_{b_j \in B_i} V(b_j)$.

It follows from the definition of H that in G,

- A_0 is complete to B;
- B_0 is complete to A; and
- for $j = 1, \dots, k$, $A_j, B_j \neq \emptyset$ and A_j is complete to $B \setminus B_j$ and B_j is complete to $A \setminus A_j$, and A_j is anticomplete to B_j.

If $R_i^1 = \emptyset$, then $A \subseteq \tilde{Z}^Q$, and so $A \cap Y' = \emptyset$, and (13) follows. Thus $R_i^1 \neq \emptyset$. Suppose that $R_i^2 = \emptyset$. Then one of the following holds:

- $R_i^1 \subseteq A_0$, and so $B \subseteq \tilde{X}^Q$; or
- $R_i^1 \subseteq A_j$ for some $j > 0$, and so $B \setminus B_j \subseteq \tilde{X}^Q$ and $B_j \subseteq \tilde{Y}^Q$.

It follows that $N(v) \cap Y_2' = \emptyset$, and (13) follows. Thus we may assume that $R_i^2 \neq \emptyset$, then there exists a $j > 0$ such that $R_i^1 \subseteq A_j$ and $R_i^2 \subseteq B_j$, and so $(A \setminus A_j) \cup (B \setminus B_j) \subseteq \tilde{X}^Q$, and again, (13) holds.

(14) *There is no path $z - a - b - c$ with $z \in Y_1'$, $a, c \in Y_2'$ and $b \in Y_3'$.*

Suppose that $z - a - b - c$ is such a path. Let $i \in \{1, \dots, r\}$ such that $N(z) \cap S = T_i$. Since $z \notin X_0'$, it follows that $S_i^1 \neq \emptyset$. Write $S_i^1 = \{u\}$. Let $s \in T_i$; then $f'(s) \in L_2 \cup L_3$, and therefore s is anticomplete to $\{a, b, c\}$.

Suppose that $S_i^2 = \emptyset$. Then $f'(u) \in L_2 \cap L_3$, and thus u is non-adjacent to z, a, b, c. Now $u - s - z - a - b - c$ is a P_6 in G, a contradiction. Thus it follows that $S_i^2 = \{v\}$, and z is non-adjacent to u and v. By construction, it follows that $f'(u) \in L_2 \setminus L_3$, and $f'(v) \in L_3 \setminus L_2$. Since neither $u - s - z - a - b - c$ nor $v - s - z - a - b - c$ is a P_6 in G, it follows that u, v each have a neighbor in $\{a, b, c\}$. Since neighbors of u in Y_2 are in $\tilde{X}^Q$, it follows that u is non-adjacent to a and c, and hence u is adjacent to b. Since neighbors of v in Y_3 are in $\tilde{X}^Q$, it follows that v is non-adjacent to b, and v is adjacent to a or c. This contradicts (13), and thus (14) follows.

(15) *If the conditions of the last bullet hold for P, then there is no path $z - a - b$ with $z \in Y_1'$, $a \in Y_2'$ and $b \in Y_3'$.*

Suppose not, and let $z - a - b$ be such a path. Let $i \in \{1, \dots, r\}$ such that $N(z) \cap S = T_i$. Let $s \in T_i$. Then $f'(s) \in L_2 \cap L_3$, since $f'(s) \notin L_1$, and hence s is anticomplete to a, b. Since $z \notin X_0'$, it follows that $S_i^1 \neq \emptyset$, say $S_i^1 = \{u\}$. Suppose first that $S_i^2 = \emptyset$. Since $z \notin X_0'$, it follows that $f'(u) \in L_2 \cap L_3$, and thus u is non-adjacent to z, a, b. By construction, u has a neighbor y in Y_2, and since u is anticomplete to a, b, it follows that $y \neq a, b$. Since $y - u - s - z - a - b$ is not a P_6 in G, it follows that y has a neighbor in $\{z, a, b\}$. Since P satisfies (II), it follows that $u - y - a$ is not a path and so y is not adjacent to a. Since P satisfies the second-to-last bullet for L_1, L_2, L_3, it follows that $u - y - b - a$ is not a path, and so u is not adjacent to b. But then u is adjacent to z; and $b - a - z - u$ is a path contrary to the second-to-last bullet for L_3, L_2, L_1. This is a contradiction, and hence $S_i^2 \neq \emptyset$, say $S_i^2 = \{u\}$.

By construction, it follows that $f'(u) \in L_2 \setminus L_3$, and $f'(v) \in L_3 \setminus L_2$. If one of u, v has no neighbor in $\{a, b\}$, then we reach a contradiction as above. Since neighbors of u in Y_2 are in $\tilde{X}^Q$, it follows that u is adjacent to b, but not a. Since neighbors of v in Y_3 are in $\tilde{X}^Q$, it follows that v is adjacent to a, but not b. This contradicts (13), and proves (15).

We now replace every $P' \in \mathcal{L}$ by P'' satisfying (I) by moving vertices with lists of size less than three from Y' to X'. It follows that P'' still satisfies (II) and (14). This concludes the proof of the lemma. ☐

Lemma 19. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y, Y^*, f)$ be a starred precoloring of a P_6-free graph G with P satisfying* (I) *and* (II)*. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs an equivalent collection $\mathcal{L}$ for P such that*

- *$|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;*
- *every $P' \in \mathcal{L}$ is a starred precoloring of G;*
- *every $P' \in \mathcal{L}$ with seed S' satisfies $|S'| \leq q(|S|)$; and*
- *every $P' \in \mathcal{L}$ satisfies* (I), (II) *and* (III).

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension for P in polynomial time.

Proof. Let $\mathcal{L} = \{P\}$. For every triple (L_1, L_2, L_3) of distinct lists of size three included in $[4]$ we repeat the following. Apply Lemma 18 to every member of $\mathcal{L}$; replace $\mathcal{L}$ by the union of the collections thus obtained, and move to the next triple of lists. At the end of this process we have an equivalent collection $\mathcal{L}$ for P, in which every starred precoloring satisfies the second-to-last bullet of Lemma 18 for every (L_1, L_2, L_3).

Repeat the procedure described in the previous paragraph. Since the second-to-last bullet of the conclusion of Lemma 18 holds for each starred precoloring we input this time, it follows that the last bullet of Lemma 18 holds for the output for every (L_1, L_2, L_3). Thus (III) holds; this concludes the proof. ☐

Let $P = (G, S, X_0, X, Y, Y^*, f)$ be a starred precoloring. For $W \subseteq V(G)$ and $L \subseteq [4]$, we say that W *meets* L if $L_P(w) = L$ for some $w \in W$. We now have the following convenient property.

Lemma 20. *Let $P = (G, S, X_0, X, Y, Y^*, f)$ be a starred precoloring of a P_6-free graph G satisfying* (I)*,* (II) *and* (III)*. Let L_1, L_2, L_3, L_4 be the subsets of $[4]$ of size three. Let C be a component of $G|Y$ that meets at least three of the lists L_1, L_2, L_3, L_4. For $i \in [4]$, let $C_i = \{v \in V(C) : L_P(v) = L_i\}$. Then for every $i \neq j$, C_i is complete to C_j.*

Proof. Let $Q = p_1 - \ldots - p_k$ be a path such that for some $i \neq j$, $p_1 \in C_i$, $p_k \in C_j$, p_1 is non-adjacent to p_k, and subject to that with k minimum (and since Q is induced, it follows that $k \geq 3$). Since P satisfies (II), it follows that $p_2 \notin C_i$; say $p_2 \in C_l$. Since P satisfies (II) and (III), it follows that $p_3 \in C_i$. Similarly, $p_4 \notin C_i$. By the minimality of k, we deduce that $k = 4$. By (III) applied to $p_2 - p_3 - p_4$, we deduce that $l = j$. Let C' be a component of $C|(C_i \cup C_j)$ with $p_1, \ldots, p_4 \in V(C')$. Since C is connected, and since $V(C) \neq C_i \cup C_j$, there exists $c \in C_l$ with $l \neq i, j$ such that c has a neighbor in C'. Since P satisfies (II) and (III), it follows from Lemma 1 that c is complete to C'. But now $p_1 - c - p_4$ contradicts the fact that P satisfies (III). This proves Lemma 20. □

Our next goal is to establish axiom (IV), which we restate.

(IV) Let $L_1 \subseteq \{1, 2, 3, 4\}$ with $|L_1| = 3$. Then there is no path $a - b - c$ with $L_P(b) = L_P(c) = L_1$ and $a \in X$, $b, c \in Y$.

Lemma 21. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y, Y^*, f)$ be a starred precoloring of a P_6-free graph G with P satisfying* (I)*. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs an equivalent collection $\mathcal{L}$ for P such that*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$*;*
- *every $P' \in \mathcal{L}$ is a starred precoloring of G;*
- *every $P' \in \mathcal{L}$ with seed S' satisfies $|S'| \leq q(|S|)$; and*
- *every $P' \in \mathcal{L}$ satisfies* (I)*,* (II)*,* (III) *and* (IV)*.*

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension for P in polynomial time.

Proof. Let $\mathcal{L} = \{P\}$. For every list $L \subseteq \{1, 2, 3, 4\}$ of size three, apply Lemma 15 to every member of $\mathcal{L}$, replace $\mathcal{L}$ by the union of the equivalent collections thus obtained, and move to the next list. At the end of the process we obtained the required equivalent collection for $\{P\}$. □

We now begin to establish the following axiom, which we restate below.

(V) Let $L_1, L_2 \subseteq \{1, 2, 3, 4\}$ with $|L_1| = |L_2| = 3$. Then there is no path $a - b - c$ with $L_P(b) = L_1, L_P(c) = L_2$ and $a \in X$ with $L_3 = L_P(a) \neq L_1 \cap L_2$.

We define the following auxiliary statement:

(16) *Let $L_1, L_2 \subseteq \{1, 2, 3, 4\}$ with $|L_1| = |L_2| = 3$. Then there is no path $a - b - c - d$ with $L_P(b) = L_P(d) = L_1, L_P(c) = L_2$ and $a \in X$ with $L_3 = L_P(a) \neq L_1 \cap L_2$.*

Again, it may be helpful to begin by reading (18) in order to understand the construction better.

Lemma 22. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $L_1, L_2 \subseteq \{1, 2, 3, 4\}$ with $|L_1| = |L_2| = 3$ and $L_1 \neq L_2$, and let $L_3 \subseteq \{1, 2, 3, 4\}$ with $|L_3| = 2$ and $L_3 \neq L_1 \cap L_2$. Let $P = (G, S, X_0, X, Y, Y^*, f)$ be a starred precoloring of a P_6-free graph G with P satisfying* (I)*,* (II)*,* (III) *and* (IV)*. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs an equivalent collection $\mathcal{L}$ for P such that*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$*;*

- *every $P' \in \mathcal{L}$ is a starred precoloring of G;*
- *every $P' \in \mathcal{L}$ with seed S' satisfies $|S'| \leq q(|S|)$;*
- *every $P' \in \mathcal{L}$ satisfies* (I), (II), (III) *and* (IV)*;*
- *every $P' \in \mathcal{L}$ satisfies* (16) *for every three lists L'_1, L'_2, L'_3 such that P satisfies* (16) *for L'_1, L'_2, L'_3;*
- *if P satisfies* (16) *for every three lists, then every $P' \in \mathcal{L}$ satisfies* (V) *for every three lists L'_1, L'_2, L'_3 such that P satisfies* (V) *for L'_1, L'_2, L'_3;*
- *every $P' \in \mathcal{L}$ satisfies* (16) *for L_1, L_2, L_3.*
- *if P satisfies* (16) *for every three lists L'_1, L'_2, L'_3 such that $|L'_1| = |L'_2| = 3, L'_1 \neq L'_2, |L'_3| = 2, L'_3 \neq L'_1 \cap L'_2$, then every $P' = (G, S', X'_0, X', Y', Y^*, f') \in \mathcal{L}$ satisfies that there is no path $a - b - c$ with $L_{P'}(a) = L_3$, $L_{P'}(b) = L_1$, $L_{P'}(c) = L_2$ with $a \in X$, $b, c \in Y'$.*

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension for P in polynomial time.

Proof. Let $\mathcal{L} = \emptyset$. Let $Y_i = \{y \in Y : L_P(y) = L_i\}$ for $i = 1, 2$, and let X_3 be the set of vertices v in X with list L_3 such that v starts a path $v - b - c - d$ ($v - b - c$ if the condition of the last bullet holds for P) with $v \in X, b, d \in Y_1, c \in Y_2$. Let L_4, L_5 be the two three-element lists in $\{1, 2, 3, 4\}$ that are not L_1, L_2, and let $Y_i = \{y \in Y : L_P(y) = L_i\}$ for $i = 4, 5$. We call a component C of $G|Y$ *bad* if $V(C) \cap Y_1 \neq \emptyset, V(C) \cap Y_2 \neq \emptyset$ and $V(C) \cap Y_i \neq \emptyset$ for some $i \in \{4, 5\}$.

Let $\mathcal{T} = \{T_1, \ldots, T_r\}$ be the set of types $T \subseteq S$ with $f(T) = \{1, 2, 3, 4\} \setminus L_3$. We let $\mathcal{Q}$ be the set of all r-tuples $(Q_1, \ldots, Q_r)$, where for each i,

$$Q_i = (S_i^1, S_i^2, R_i^1, R_i^2, R_i^3, R_i^4, C_i^1, C_i^2, X_i^{1,1}, X_i^{1,2}, X_i^{2,1}, X_i^{2,2}, f_i, case_i)$$

such that the following hold:

1. $f_i : S_i^1 \cup S_i^2 \cup R_i^1 \cup R_i^2 \cup R_i^3 \cup R_i^4 \cup X_i^{1,1} \cup X_i^{1,2} \cup X_i^{2,1} \cup X_i^{2,2} \to \{1, 2, 3, 4\}$.
2. $f_i(S_i^1 \cup S_i^2) \subseteq L_3$.
3. $1 \geq |S_i^1| \geq |S_i^2| \geq |R_i^1| \geq |R_i^2| \geq |R_i^3| \geq |R_i^4|$.
4. $S_i^1 \cup S_i^2$ is a stable set and $S_i^1 \cup S_i^2 \subseteq X_3(T_i)$.
5. If $S_i^1 = \emptyset$, then $X_3(T_i) = \emptyset$.
6. If $S_i^2 \neq \emptyset$, then $f_i(S_i^1 \cup S_i^2) = L_3$ and $L_3 \cap L_1 \cap L_2 = \emptyset$.
7. For $j = 1, 2$, if $S_i^j = \left\{s_i^j\right\}$ and s_i^j is mixed on a bad component, then C_i^j is the vertex set of a bad component on which s_i^j is mixed; otherwise, $C_i^j = \emptyset$.
8. For $j, k = 1, 2$, $|X_i^{j,k}| \leq 1$, and $|X_i^{j,k}| = 1$ if and only if $C_i^j \neq \emptyset$.
9. For $j = 1, 2$, if $C_i^j \neq \emptyset$, then there exist $p \neq q$ such that $X_i^{j,1} \cap C_i^j \cap Y_p \neq \emptyset$ and $X_i^{j,2} \cap C_i^j \cap Y_q \neq \emptyset$.
10. For $j = 1, 2, 3, 4$, $f_i(R_i^j) \subseteq L_1 \cap L_2$.
11. $case_i \in \{\emptyset, (a), (b), (c), (d), (e), (f)\}$.
12. $case_i \in \{\emptyset, (a), (b)\}$ if and only if $R_i^j = \emptyset$ for all $j \in \{1, 2, 3, 4\}$.
13. $case_i \in \{(c), (d), (e)\}$ if and only if $R_i^3, R_i^4 = \emptyset$ and $R_i^1, R_i^2 \neq \emptyset$.
14. $case_i = (f)$ if and only if $R_i^j \neq \emptyset$ for all $j \in \{1, 2, 3, 4\}$.
15. If $S_i^2 = \emptyset$, then $case_i = \emptyset$.

16. If $case_i \neq \emptyset$, then let $\{u, v\} = S_i^1 \cup S_i^2$ such that $u \in S_i^1$ if and only if $f_i(u) \in L_1$; then $R_i^1, R_i^3 \subseteq N(u) \cap (Y_2 \setminus N(v))$ and $R_i^2, R_i^4 \subseteq N(v) \cap (Y_1 \setminus N(u))$.

17. If $case_i = (c)$, R_i^1 is anticomplete to R_i^2.

18. If $case_i \in \{(d), (e)\}$, R_i^1 is complete to R_i^2.

19. If $case_i = (f)$, then R_i^1 is complete to R_i^2 and anticomplete to R_i^4, and R_i^3 is anticomplete to R_i^2 and anticomplete to R_i^4.

We let

$$S'^Q = \bigcup_{i\in\{1,\dots,r\}} (S_i^1 \cup S_i^2 \cup R_i^3 \cup R_i^4 \cup X_i^{1,1} \cup X_i^{1,2} \cup X_i^{2,1} \cup X_i^{2,2}) \cup \bigcup_{i\in\{1,\dots,r\},case_i \neq (c)} (R_i^1 \cup R_i^2),$$

and let $f'^Q = f_1 \cup \dots \cup f_r$.

For every $i \in \{1, \dots, r\}$, we let $\tilde{Y}_i = \bigcup_{j,k\in\{1,2\}} \bigcup_{p\in\{1,2,4,5\}, X_i^{j,k} \cap C_i^j \cap Y_p \neq \emptyset} (C_i^j \cap Y_p)$, and we let $h_i(C_i^j \cap Y_p \cap \tilde{Y}_i) \subseteq f_i(X_i^{j,k})$. Let $\tilde{Z}_i = (C_i^1 \cup C_i^2) \setminus \tilde{Y}_i$. Let $\tilde{Y}^Q = \bigcup_{i\in\{1,\dots,r\}} \tilde{Y}_i$ and $\tilde{Z}^Q = \bigcup_{i\in\{1,\dots,r\}} \tilde{Z}_i$ and $h^Q = h_1 \cup \dots \cup h_r$.

Let S_1' be the set of $v \in S'^Q$ such that $f'(v) \in L_1 \cap L_2$; let S_2' be the set of $v \in S'^Q$ such that $f'(v) \in L_1 \setminus L_2$, and let S_3' be the set of $v \in S'^Q$ such that $f'(v) \in L_2 \setminus L_1$. Let

$$\tilde{X}^Q = (N(S_1') \cap (Y_1 \cup Y_2)) \cup (N(S_2') \cap (Y_1)) \cup (N(S_3') \cap (Y_2)).$$

Let $\tilde{W}_i = X_3(T_i)$ if $S_i^1 = \{v\}, S_i^2 = \emptyset$ and $f'(v) \notin L_1 \cap L_2 \cap L_3$, and $\tilde{W}_i = \emptyset$ otherwise. If $\tilde{W}_i \neq \emptyset$, we let $g_i'' : \tilde{W}_i \to L_3$ such that $g''(y) = f'(v)$ is y if non-adjacent to v, and $g''(y)$ is the unique color in $L_3 \setminus (\{f'(v)\})$ otherwise. Let $\tilde{W}^Q = \bigcup_{i\in\{1,\dots,r\}} \tilde{W}_i$ and let $g''^Q = g_1''^Q \cup \dots \cup g_r''^Q$.

Let $\tilde{V}^Q$ be the set of vertices v in X with list L_3 such that S'^Q contains a neighbor s of v, and let $h'^Q : \tilde{V} \to L_3$ such that $h'^Q(v) \in L_3 \setminus (f'(s))$.

Let $\tilde{U}_i$ be the set of all vertices $x \in X_3(T_i)$ such that $S_i^1 = \{v\}$ and such that $f'(v) \in L_1 \cap L_2$ and $N(v) \cap Y_1 \subsetneq N(x) \cap Y_1$, and let $g_i : \tilde{U}_i \to L_3 \setminus (L_1 \cap L_2)$. Let $\tilde{U}^Q = \bigcup_{i\in\{1,\dots,r\}} \tilde{U}_i$ and $g^Q = g_1 \cup \dots \cup g_r$.

Let $\tilde{U'}_i$ be the set of all vertices $x \in X_3(T_i)$ such that $S_i^1 = \{u\}, S_i^2 = \{v\}$ such that $xu \notin E(G)$, and $N(v) \cap Y_1 \subsetneq N(x) \cap Y_1$, and let $g_i' : \tilde{U}_i \to \{f'(u)\}$. Let $\tilde{U'}^Q = \bigcup_{i\in\{1,\dots,r\}} \tilde{U'}_i$ and $g'^Q = g_1' \cup \dots \cup g_r'$.

Finally, we define $\tilde{T}_i$ as follows: If $case_i = \emptyset$, then $\tilde{T}_i = \emptyset$. Otherwise, let $\{u, v\} = S_i^1 \cup S_i^2$ such that $f'(u) \in L_1$, and let $A = N(u) \cap (Y_2 \setminus N(v))$ and $B = N(v) \cap (Y_1 \setminus N(u))$. If $case_i =$

(a) then $\tilde{T}_i = A$;

(b) then $\tilde{T}_i = B$;

(c) then $\tilde{T}_i = (A \cap N(R_i^2)) \cup (B \cap N(R_i^1))$;

(d) then $\tilde{T}_i = B \setminus N(R_i^1)$;

(e) then $\tilde{T}_i = A \setminus N(R_i^2)$;

(f) then $\tilde{T}_i = \emptyset$.

We let $\tilde{T}^Q = \bigcup_{i\in\{1,\dots,r\}} \tilde{T}_i$ and let $h''^Q : \tilde{T}^Q \to (L_1 \setminus L_2) \cup (L_2 \setminus L_1)$ be the unique function such that $h''^Q(v) \in L_P(v)$ for all $v \in \tilde{T}^Q$.

The following statement could be proved using Lemma 17, but we give a shorter proof here:

(17) *Let i such that $\{u, v\} = S_i^1 \cup S_i^2$ and $f(u) \in L_1$. Let $R = R_i^1 \cup R_i^2 \cup R_i^3 \cup R_i^4$ if $case_i \neq (c)$ and $R = \emptyset$ otherwise. Then $(N(u) \cap Y_2) \setminus (N(v) \cup \tilde{T}_i \cup N(R))$ is anticomplete to $(N(v) \cap Y_1) \setminus (N(u) \cup \tilde{T}_i \cup N(R)))$.*

Let $A' = A \setminus (\tilde{T}_i \cup N(R))$, $B' = B \setminus (\tilde{T}_i \cup N(R))$; then it suffices to prove that A' is anticomplete to B'. If $case_i = (a), (b), (d), (e)$, this follows since A' or B' is empty in each of these cases. In case (f), we have that $G|(\{u,v\} \cup R)$ is a six-cycle. Since the graph arising from a six-cycle by adding a vertex with exactly one neighbor in the cycle contains a P_6, it follows that $A', B' = \emptyset$. In case (c), we let $x'y'$ be an edge from A' to B', and we let $x \in A_i^1, y \in A_i^2$. Then $x - u - x' - y' - v - y$ is a P_6 in G, a contradiction. Again it follows that A' is anticomplete to B', and (17) follows.

Let P'^Q be the starred precoloring obtained from

$$\begin{aligned}
&(G, S \cup S'^Q\\
&\quad X_0 \cup \tilde{Y}^Q \cup \tilde{W}^Q \cup \tilde{V}^Q \cup \tilde{U}^Q \cup \tilde{U'}^Q \cup \tilde{T}^Q\\
&\quad (X \setminus (\tilde{W}^Q \cup \tilde{V}^Q \cup \tilde{U}^Q \cup \tilde{U'}^Q)) \cup \tilde{Z}^Q \cup \tilde{X}^Q\\
&\quad Y \setminus (\tilde{Y}^Q \cup \tilde{Z}^Q \cup \tilde{X}^Q \cup \tilde{T}^Q)\\
&\quad Y^*, f \cup f'^Q \cup h^Q \cup h'^Q \cup h''^Q \cup g^Q \cup g_i'^Q \cup g''^Q)
\end{aligned}$$

by moving every vertex with a list of size at most two X, and every vertex with a list of size at most one to X_0. Since P satisfies (II) and (III), it follows that P'^Q satisfies (II) and (III) as well. Moreover, P'^Q satisfies (I).

We let

$$\mathcal{L} = \left\{ P'^Q : Q \in \mathcal{Q}, f \cup f'^Q \cup h^Q \cup h'^Q \cup h''^Q \cup g^Q \cup g_i'^Q \cup g''^Q \text{ is a proper coloring} \right\}.$$

(18) *$\mathcal{L}$ is an equivalent collection for P.*

For every $P'^Q \in \mathcal{L}$, every precoloring extension of P'^Q is a precoloring extension of P. Conversely, let c be a precoloring extension of P, and define $Q = (Q_1, \dots, Q_r)$, where for each i,

$$Q_i = (S_i^1, S_i^2, R_i^1, R_i^2, R_i^3, R_i^4, C_i^1, C_i^2, X_i^{1,1}, X_i^{1,2}, X_i^{2,1}, X_i^{2,2}, f_i', case_i)$$

is defined as follows:

- If $X_3(T_i) = \emptyset$, then $Q_i = (\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, f_i, \emptyset)$, where f_i is the empty function.
- If $X_3(T_i)$ contains a vertex v with $c(v) \in L_1 \cap L_2$, we choose v with $N(v) \cap Y_1$ maximal and let $S_i^1 = \{v\}$, $case = \emptyset$. In this case, we let $S_i^2 = \emptyset$.
- If $X_3(T_i)$ contains no vertex v with $c(v) \in L_1 \cap L_2$, we let $u \in X_3(T_i)$ with $N(u) \cap Y_1$ maximal, and set $S_i^1 = \{u\}$. If there is a vertex $v \in X_3(T_i)$ with $c(v) \neq c(u)$ and $uv \notin E(G)$, we choose v with $N(v) \cap Y_1$ maximal and set $S_i^2 = \{v\}$; otherwise we let $S_i^2 = \emptyset$.
- If $S_i^2 = \emptyset$, we let $case_i = \emptyset$ and $R_i^j = \emptyset$ for $j = 1,2,3,4$. Otherwise, we let $\{u,v\} = S_i^1 \cup S_i^2$ such that $c(u) \in L_1$. We let $A = N(u) \cap (Y_2 \setminus N(v))$ and $B = N(v) \cap (Y_1 \setminus N(u))$. Let $a_1, \dots, a_t$ be the components of $G|A$, and let $b_1, \dots, b_s$ be the components of $G|B$. Since P satisfies (II), it follows that for every $i \in [t]$ and $j \in [s]$, $V(a_i)$ is either complete or anticomplete to $V(b_j)$.

  Let H be the graph with vertex set $\{u,v\} \cup \{a_1, \dots, a_t\} \cup \{b_1, \dots b_s\}$; where $N_H(u) = \{a_1, \dots, a_t\}$, $N_H(v) = \{b_1, \dots, b_s\}$, the sets $\{a_1, \dots, a_t\}$ and $\{b_1, \dots, b_s\}$ are stable, and a_i is adjacent to b_j if and only if $V(a_i)$ is complete to $V(b_j)$ in G. Apply 17 to H, u and v to obtain a partition $A_0', A_1', \dots, A_k'$ of $\{a_1, \dots, a_t\}$ and a partition $B_0', B_1', \dots, B_k'$ of $\{b_1, \dots, b_t\}$. For $i \in [k]$, let $A_i = \bigcup_{a_j \in A_i} V(a_j)$ and $B_i = \bigcup_{b_j \in B_i} V(b_j)$.

  It follows from the definition of H that in G,

  - A_0 is complete to $N(v)$;
  - B_0 is complete to $N(u)$; and
  - for $j = 1, \dots, k$, $A_j, B_j \neq \emptyset$ and A_j is complete to $N(v) \setminus B_j$ and B_j is complete to $N(u) \setminus A_j$, and A_j is anticomplete to B_j.

If $A_0 = B_0 = \emptyset$ and $k = 1$, then A is anticomplete to B, and we let $case_i = \emptyset$. Otherwise, we consider the following cases, setting $case_i =$

(a) if $c(A) \subseteq L_2 \setminus L_1$;

(b) if $c(B) \subseteq L_1 \setminus L_2$;

(c) if there is an $i \in \{1, \dots, k\}$ such that $c(A \setminus A_i) \subseteq L_2 \setminus L_1$, and $c(B \setminus B_i) \subseteq L_1 \setminus L_2$;

(d) if there exist $x \in A$, $y \in B$ adjacent such that $c(x), c(y) \in L_2 \cap L_1$ and $c(B \setminus N(x)) \subseteq L_1 \setminus L_2$;

(e) if there exist $x \in A$, $y \in B$ adjacent such that $c(x), c(y) \in L_2 \cap L_1$ and $c(A \setminus N(y)) \subseteq L_2 \setminus L_1$;

(f) if there exist $x, x' \in A, y, y' \in B$, with x, y adjacent, x' non-adjacent to y, y' non-adjacent to x, and (consequently) x' adjacent to y', and $c(x), c(y), c(x'), c(y') \in L_2 \cap L_1$.

It is easy to verify that one of these cases occurs.

With the notation as above, if $case_i =$

(a) then we let $R_i^j = \emptyset$ for $j = 1, 2, 3, 4$;

(b) then we let $R_i^j = \emptyset$ for $j = 1, 2, 3, 4$;

(c) then we let $x \in A_i, y \in B_i$ and set $R_i^1 = \{x\}$, $R_i^2 = \{y\}$, $R_i^3 = R_i^4 = \emptyset$;

(d) then we let $R_i^1 = \{x\}$, $R_i^2 = \{y\}$, $R_i^3 = R_i^4 = \emptyset$;

(e) then we let $R_i^1 = \{x\}$, $R_i^2 = \{y\}$, $R_i^3 = R_i^4 = \emptyset$;

(f) then we let $R_i^1 = \{x\}$, $R_i^2 = \{y\}$, $R_i^3 = \{x'\}$, $R_i^4 = \{y'\}$.

- For $j = 1, 2$, we proceed as follows. If $S_i^j = \emptyset$ or the vertex $v \in S_i^j$ is not mixed on a bad component, then we let $X_i^{j,1} = X_i^{j,2} = C_i^j = \emptyset$. Otherwise, let $v \in S_i^j$ and let C be a bad component of $G|Y$ on which v is mixed. We set $C_i^j = V(C)$. By Lemma 20 applied to C, it follows that for $p \neq q$, $V(C) \cap Y_p$ is complete to $V(C) \cap Y_q$. Since $Y_p \cap V(C) \neq \emptyset$ for at least three different $p \in \{1, 2, 4, 5\}$, it follows that there exist $p, q \in \{1, 2, 4, 5\}$ with $p \neq q$ such that $|c(V(C) \cap Y_p)| = 1$ and $|c(V(C) \cap Y_q)| = 1$. Let $X_i^{j,1} \subseteq V(C) \cap Y_p$, $X_i^{j,2} \subseteq V(C) \cap Y_q$, such that $|X_i^{j,k}| = 1$ for $k = 1, 2$.

We let $f_i = c|_{S_i^1 \cup S_i^2 \cup R_i^1 \cup R_i^2 \cup R_i^3 \cup R_i^4 \cup X_i^{1,1} \cup X_i^{1,2} \cup X_i^{2,1} \cup X_i^{2,2}}$. It follows from the definition of Q that $Q \in \mathcal{Q}$. Moreover, c is a precoloring extension of P'^Q by the definition of Q and P'^Q. This proves (18).

Let $P' \in \mathcal{L}$ with $P' = (G, S', X_0', X', Y', Y^*, f')$ such that $P' = P'^Q$ for $Q = (Q_1, \dots, Q_r)$, where for each i,

$$Q_i = (S_i^1, S_i^2, R_i^1, R_i^2, R_i^3, R_i^4, C_i^1, C_i^2, X_i^{1,1}, X_i^{1,2}, X_i^{2,1}, X_i^{2,2}, f_i, case_i).$$

Let $Y_i' = \{y \in Y' : L_{P'}(y) = L_i\}$ for $i = 1, 2$. We claim the following.

(19) *P' satisfies* (IV).

Suppose not; and let $x - a - b$ be a path with $x \in X'$ and $a, b \in Y'$ with $L_{P'}(a) = L_{P'}(b) = L$. Since P satisfies (II) and (IV), it follows that $x \notin X$, and so $x \in Y$ and $L_P(x) = L$. Moreover, since $x \in X' \setminus X$, it follows that x has a neighbor $s' \in S' \setminus S$ with $f'(s') \in L$. Since P satisfies (II) and (IV), and since s' is adjacent to x but not a, it follows that $s' \in Y$ and $L_P(s') = L$. Since s' has a neighbor $x \in Y$ with a neighbor $a \in Y'$, it follows that $x \notin \tilde{Y}^Q \cup \tilde{Z}^Q$. Since $s' \notin X$, it follows that $s' \notin S_i^1 \cup S_i^2$, and hence there exists $i \in \{1, \dots, r\}$ such that $s' \in R_i^j$ for some $j \in \{1, 2, 3, 4\}$. Thus $L_P(s') \in \{L_1, L_2\}$. Let $\{u, v\} = S_i^1 \cup S_i^2$ such that $s' \in N(u) \setminus N(v)$. It follows that $case_i \in \{(d), (e), (f)\}$, and hence there is a vertex $t' \in R_i^1 \cup R_i^2 \cup R_i^3 \cup R_i^4$ such that t' is adjacent to s' and v, but t' is not adjacent to u, and $L_P(t') \in \{L_1, L_2\} \setminus \{L\}$, and $f'(t') \in L_1 \cap L_2$. But then $t' - s' - x$ or $t' - x - a$ is a path (since $a \in Y'$ it follows that a is not adjacent to t'); contrary to the fact that (II) holds for P. This is a contradiction, and (19) follows.

(20) *If P satisfies* (16) *for lists L_1', L_2', L_3', then P' satisfies* (16) *for L_1', L_2', L_3'.*

Suppose not; and let $x - a - b - c$ be a path such that $L_{P'}(x) = L'_3$ with $|L'_3| = 2$ and $L'_3 \neq L'_1 \cap L'_2$, $L_{P'}(a) = L'_1 = L_{P'}(c)$, $L_{P'}(b) = L'_2$. Since P satisfies (II), (16) for L'_1, L'_2, L'_3, and (III), it follows that $L_P(x) = L'_2$. Consequently, x has a neighbor s' in $S' \setminus S$ with $f'(s') \in L'_2$. Since $L'_3 \neq L'_1 \cap L'_2$, it follows that $f'(s') \in L'_1$. Thus $s' - x - a - b - c$ is a path. Suppose first that $s' \in Y$. It follows that $s' \notin S^1_i \cup S^2_i$. Since s' has a neighbor $x \in Y$ with a neighbor $a \in Y'$, it follows that $x \notin \tilde{Y}^Q \cup \tilde{Z}^Q$. This implies that there exist $i \in \{1, \dots, r\}$ and $j \in \{1, 2, 3, 4\}$ such that $s' \in R^j_i$. Since P satisfies (II) and (III), it follows that $L_P(s') = L'_1$. Let $\{u, v\} = S^1_i \cup S^2_i$ such that u is adjacent to s' and v is not. It follows that $case_i \in \{(d), (e), (f)\}$, and hence there is a vertex $t' \in R^1_i \cup R^2_i \cup R^3_i \cup R^4_i$ such that t' is adjacent to s' and v, but t' is not adjacent to u, and $f'(t') \in L'_1 = L_P(s')$. Since $t' - s' - x - a - b - c$ is not a P_6 in G, it follows that $f'(t') \notin L'_1 \cap L'_2$. Therefore, $L_P(t') \notin \{L'_1, L'_2\}$. Since P satisfies (III), it follows that t' is adjacent to x (since $t' - s' - x$ is not a path). Since $f'(t') \in L'_1$, it follows that t' is not adjacent to a. Now $t' - x - a$ is a path in G, contrary to the fact that P satisfies (III). Thus, $s' \in X$.

Suppose that $L_P(s') \neq L'_1 \cap L'_2$. Then s' has a neighbor s in S with $f'(s) \in L'_1 \cap L'_2$. Now $s - s' - x - a - b - c$ is a P_6 in G, a contradiction. It follows that $s' \in X$ and $L_P(s') = L'_1 \cap L'_2$. Since $s' \in S^1_i \cup S^2_i$, it follows that there is a path $s' - y - z$ with $y \in L_1, z \in L_2$, and $L_P(s') \neq L_1 \cap L_2$. It follows that either $L_1 \notin \{L'_1, L'_2\}$ or $L_2 \notin \{L'_1, L'_2\}$. Since $z - y - s' - x - a - b - c$ is not a P_7 in G, it follows that $G|\{z, y, x, a, b, c\}$ is connected. Let $w \in \{y, z\}$ such that $L_P(w) \notin \{L'_1, L'_2\}$. Since P satisfies (III), it follows that w is complete to x, a, b, c. But then $x - w - c$ is a path, contrary to the fact that (III) holds for P. This implies (20).

(21) *If P satisfies* (V) *for lists L'_1, L'_2, L'_3, and P satisfies* (16) *for all lists, then P' satisfies* (V) *for L'_1, L'_2, L'_3.*

Suppose not; and let $x - a - b$ be a path such that $L_{P'}(x) = L'_3$ with $|L'_3| = 2$ and $L'_3 \neq L'_1 \cap L'_2$, $L_{P'}(a) = L'_1$, $L_{P'}(b) = L'_2$. Since P satisfies (II), (V) for L'_1, L'_2, L'_3, and (III), it follows that $L_P(x) = L'_2$. Consequently, x has a neighbor s' in $S' \setminus S$ with $f'(s') \in L'_2$. Since $L'_3 \neq L'_1 \cap L'_2$, it follows that $f'(s') \in L'_1$. Thus $s' - x - a - b$ is a path. Suppose first that $s' \in X$. Then there exist $i \in \{1, \dots, r\}$ and $j \in \{1, 2\}$ such that $s' \in S^j_i$. It follows that $L_P(s') = L'_1 \cap L'_2$, since P satisfies (16) for all lists. By construction, it follows that there is a path $s' - y - z$ with $y \in L_1, z \in L_2$, and $L_P(s') \neq L_1 \cap L_2$. We choose such $y, z \in C^j_i$ if $C^j_i \neq \emptyset$. Since $z - y - s' - x - a - b$ is not a six-vertex path in G, it follows that $G|\{z, y, x, a, b\}$ is connected. Since $C^j_i \cap Y' = \emptyset$ by construction, it follows that $C^j_i = \emptyset$, and so s' is not mixed on a bad component. Since $L_1 \cap L_2 \neq L'_1 \cap L'_2$, it follows that either $L_1 \notin \{L'_1, L'_2\}$ or $L_2 \notin \{L'_1, L'_2\}$. Let $w \in \{y, z\}$ such that $L_P(w) \notin \{L'_1, L'_2\}$. Then $G|\{z, y, x, a, b\}$ is contained in a component of $G|Y$ containing vertices with lists L'_1, L'_2 and $L_P(w)$, hence a bad component. But since $s' - x - a - b$ is a path, s' is mixed on this bad component, a contradiction. It follows that $s' \in Y$.

Since P satisfies (II) and (III), it follows that $L_P(s') = L'_1$. Since s' has a neighbor $x \in Y$ with a neighbor $a \in Y'$, it follows that $s' \notin \tilde{Y}^Q \cup \tilde{Z}^Q$. Thus, there exist $i \in \{1, \dots, r\}$ and $j \in \{1, 2, 3, 4\}$ such that $s' \in R^j_i$. By construction, it follows that $L_P(s') \in \{L_1, L_2\}$. Let $\{u, v\} = S^1_i \cup S^2_i$ such that u is adjacent to s' and v is not. It follows that $case_i \in \{(d), (e), (f)\}$, and hence there is a vertex $t' \in R^1_i \cup R^2_i \cup R^3_i \cup R^4_i$ such that t' is adjacent to s' and v, but t' is not adjacent to u, and $f'(t') \in L_1 \cap L_2$.

Suppose first that $\{L'_1, L'_2\} = \{L_1, L_2\}$. Then $f'(s'), f'(t') \in L'_1 \cap L'_2$. Let $s \in S$ be a common neighbor of u, v with $f'(s) \in L_1 \cap L_2$. Since $s - u - s' - x - a - b$ is not a P_6 in G, it follows that u is adjacent to a. Since $t' - v - s - u - a - b$ is not a P_6 in G, it follows that v has a neighbor in $\{u, a, b\}$. Since $f'(v) \in L_1$, it follows that v is non-adjacent to a. Thus v is adjacent to b. Since $a, b \notin \tilde{T}^Q$, it follows that $case_i = (f)$. By symmetry, we may assume that $s' \in R^1_i, t' \in R^2_i$. Let $x' \in R^3_i, y' \in R^4_i$. Then x', y' are non-adjacent to a, b. But then $x' - u - a - b - v - y$ is a P_6 in G, a contradiction. It follows that $\{L'_1, L'_2\} \neq \{L_1, L_2\}$.

Consequently, $L_P(t') \notin \{L'_1, L'_2\}$. Since P satisfies (III), it follows that $t' - s' - x$ is not a path, and so t' is adjacent to x. Since $f'(t') \in L'_1$, it follows that t' is not adjacent to a. Now $t' - x - a$ is a path, contrary to the fact that (III) holds for P. This proves (21).

(22) *P' satisfies* (16) *for L_1, L_2, L_3.*

Suppose not; and let $z - a - b - c$ be a path with $L_{P'}(z) = L_3$, $L_{P'}(a) = L_{P'}(c) = L_1$, $L_{P'}(b) = L_2$. Suppose first that $z \in X$. Let i such that $T_i = N(z) \cap S$. Then $S^1_i \neq \emptyset$. Let $s' \in S^1_i \cup S^2_i$, and let s be a common neighbor of s' and z in S with $f(s) \in L_1 \cap L_2$. Since $s' - s - z - a - b - c$ is not a path, it follows that z, a, b, c contains a neighbor of s' for every $s' \in S^1_i \cup S^2_i$. But z is anticomplete to $S^1_i \cup S^2_i$, for otherwise,

$z \in \tilde{V}^Q$. If $S_i^2 = \emptyset$, then, since $z \notin X_0'$, it follows that $f(s') \in L_1 \cap L_2$ and so z is anticomplete to a, b, c, a contradiction. Therefore, $S_i^2 \neq \emptyset$. But then $S_i^1 \cup S_i^2 = \{u, v\}$ with $f'(u) \in L_2 \setminus L_1$, say. Since $a, b, c \in Y'$, it follows that u is adjacent to a or c, and v is adjacent to b; and no other edges between u, v and a, b, c exist. Now, Y' contains an edge between $N(u) \cap (Y_1 \setminus N(v))$ and $N(v) \cap (Y_2 \setminus N(u))$; but this contradicts (17).

Since P satisfies (II) and (III), it follows that $L_P(z) = L_2$. Then z has a neighbor $s' \in S' \setminus S$ with $f'(s') \in L_1 \cap L_2$ (for if $f'(s') \notin L_1$, then $L_{P'}(z) = L_1 \cap L_2 \neq L_3$), and $s' - z - a - b - c$ is a path. Suppose first that $s' \in Y$. Since P satisfies (II) and (III), it follows that $L_P(s') = L_1$. Moreover, by construction, s' has a neighbor $t' \in S'$ with $L_P(t') = L_2$ and $f'(t') \in L_1 \cap L_2$. But then $t' - s' - z - a - b - c$ is a P_6 in G, a contradiction. It follows that $s' \in X$.

Since $s' \in X$, it follows that $L(s') = L_3$, and so s' has a neighbor $s \in S$ with $f(s) \in L_1 \cap L_2$. But then $s - s' - z - a - b - c$ is a P_6 in G, a contradiction. This proves (22).

(23) *If P satisfies* (16) *for every three lists, then P' satisfies* (V) *for L_1, L_2, L_3.*

Suppose not; and let $z - a - b$ be a path with $L_{P'}(z) = L_3$, $L_{P'}(a) = L_1$, $L_{P'}(b) = L_2$.

Suppose first that $z \in X$. Let $i \in \{1, \dots, r\}$ such that $T_i = N(z) \cap S$. By construction, it follows that $S_i^1 \neq \emptyset$. Let $s' \in S_i^1 \cup S_i^2$, and let s be a common neighbor of s' and z in S with $f(s) \in L_1 \cap L_2$. Let c be a neighbor of s' in Y_1; by construction, we may choose c to be non-adjacent to z. Then $c \neq a, b$ (since $b \notin Y_1$). Since $c - s' - s - z - a - b$ is not a path, it follows that either s' or c has a neighbor in $\{a, b\}$. Since P satisfies (IV), it follows that $s' - c - a$ is not a path. Since P satisfies (16) for all lists, it follows that $z - a - b - c$ is not a path. Consequently, s' has a neighbor in $\{a, b\}$. It follows that $f'(s') \notin L_1 \cap L_2$. Therefore, $S_i^1 \cup S_i^2 = \{u, v\}$ and both u, v have a neighbor in $\{a, b\}$. Since $a, binY'$, it follows that both a, b have a non-neighbor in $\{u, v\}$. This is a contradiction by (17).

Since $z \in Y$ and P satisfies (II) and (III), it follows that $L_P(z) = L_2$. Consequently, z has a neighbor s' in $S' \setminus S$ with $f'(s') \in L_2$. Since $L_3 \neq L_1 \cap L_2$, it follows that $f'(s') \in L_1$. Thus $s' - z - a - b$ is a path. Since s' has a neighbor $z \in Y$ with a neighbor $a \in Y'$, it follows that $s' \notin \tilde{Y}^Q \cup \tilde{Z}^Q$. Suppose first that $s' \in X$. Then there exist $i \in \{1, \dots, r\}$ and $j \in \{1, 2\}$ such that $s' \in S_i^j$. It follows that $L_P(s') = L_1 \cap L_2$ since P satisfies (16) for all lists. But $S_i^j \subseteq X_3$ and so $L_P(s') \neq L_1 \cap L_2$, a contradiction. It follows that $s' \in Y$.

Since P satisfies (II) and (III), it follows that $L_P(s') = L_1$, and there exist $i \in \{1, \dots, r\}$ and $j \in \{1, 2, 3, 4\}$ such that $s' \in R_i^j$. Moreover, $S_i^1 \cup S_i^2 = \{u, v\}$. By symmetry, we may assume that u is adjacent to s' and v is not. It follows that $case_i \in \{(d), (e), (f)\}$, and hence there is a vertex $t' \in R_i^1 \cup R_i^2 \cup R_i^3 \cup R_i^4$ such that t' is adjacent to s' and v, but t' is not adjacent to u. By construction, it follows that $f'(s'), f'(t') \in L_1 \cap L_2$. Let $s \in T_i$ with $f'(s) \in L_1 \cap L_2$. Since $s - u - s' - z - a - b$ is not a P_6 in G, it follows that u is adjacent to a or to z. Note that if $uz \in E(G)$, then z is adjacent to both s' and u, both of which are in S' and $f(s', u) \subseteq L_1$. This implies that $z \in X_0'$. It follows that u is adjacent to a. Since $t' - v - s - u - a - b$ is not a P_6 in G, it follows that v is adjacent to b. This contradicts (17) and concludes the proof of (23).

The statement of the lemma follows; we have proved every claim in (19), (20), (21), (22) and (23). $\square$

Lemma 23. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y, Y^*, f)$ be a starred precoloring of a P_6-free graph G with P satisfying* (I), (II), (III) *and* (IV). *Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs an equivalent collection $\mathcal{L}$ for P such that*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;
- *every $P' \in \mathcal{L}$ is a starred precoloring of G;*
- *every $P' \in \mathcal{L}$ with seed S' satisfies $|S'| \leq q(|S|)$; and*
- *every $P' \in \mathcal{L}$ satisfies* (I), (II), (III), (IV) *and* (V).

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension for P in polynomial time, if one exists.

Proof. Let $\mathcal{L} = \{P\}$. For every triple (L_1, L_2, L_3) of lists of size three, we repeat the following. Apply Lemma 22 to every member of $\mathcal{L}$, replace $\mathcal{L}$ with the union of the equivalent collections thus obtained, and move to the next triple. At the end of thus process (16) holds for every $P' \in \mathcal{L}$.

Now repeat the procedure of the previous paragraph. Since at this stage all inputs satisfy (16) for every triple of lists, it follows that (V) holds for every starred precoloring of the output. $\square$

We now observe that the next axiom, which we restate, holds.

(VI) For every component C of $G|Y$, for which there is a vertex of X is mixed on C, there exist $L_1, L_2 \subseteq \{1,2,3,4\}$ with $|L_1| = |L_2| = 3$ such that C contains a vertex x_i with $L_P(x_i) = L_i$ for $i = 1, 2$, every vertex x in C satisfies $L_P(x) \in \{L_1, L_2\}$, and every $x \in X$ mixed on C satisfies $L_P(x) = L_1 \cap L_2$.

Lemma 24. *Let $P = (G, S, X_0, X, Y, Y^*, f)$ of a P_6-free graph G satisfying* (I)-(V)*, and let C be a component of $G|Y$ such that some vertex $x \in X$ is mixed on C. Then C meets exactly two lists L_1, L_2, and $L_P(x) = L_1 \cap L_2$.*

Proof. Since P satisfies (IV), Lemma 1 implies that C meets more than one list. By Lemma 1, there exist a, b in C such that $x - a - b$ is a path. By (IV), $L_P(a) \neq L_P(b)$, and by (V), $L_P(x) = L_P(a) \cap L_P(b)$. Let $c \in V(C)$ be such that $L_P(c) \neq L_P(a), L_P(b)$. By Lemma 20, c is complete to $\{a, b\}$. But then x is mixed on one of $\{a, c\}$, $\{b, c\}$, contrary to (V). This proves Lemma 24. ☐

The following lemma establishes that:

(VII) For every component C of $G|Y$ such that some vertex of X is mixed on C, and for L_1, L_2 as in (VI), $L_P(v) = L_1 \cap L_2$ for every vertex $v \in X$ with a neighbor in C.

Lemma 25. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y, Y^*, f)$ be a starred precoloring of a P_6-free graph G with P satisfying* (I), (II), (III), (IV), (V) *and* (VI). *Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs an equivalent collection $\mathcal{L}$ for P such that*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$*;*
- *every $P' \in \mathcal{L}$ is a starred precoloring of G;*
- *every $P' \in \mathcal{L}$ with seed S' satisfies $|S'| \leq q(|S|)$; and*
- *every $P' \in \mathcal{L}$ satisfies* (I), (II), (III), (IV), (V), (VI) *and* (VII).

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension for P in polynomial time, if one exists.

Proof. Let $\mathcal{R} = \{T_1, \ldots, T_r\}$ be the set of all $T \subseteq S$ with $|f(T)| = 2$, let $S = \{s_1, \ldots, s_s\}$, and let $\mathcal{T} = \{(L_1^1, L_2^1), \ldots, (L_1^t, L_2^t)\}$ be the set of all pairs (L_1, L_2) with $|L_1| = |L_2| = 3$ and $L_1 \neq L_2$. We let $\mathcal{Q}$ be the set of all $(rst+1)$-tuples $Q = (Q_{1,1,1}, \ldots, Q_{r,s,t}, f')$, where $i \in [r], j \in [s]$ and $k \in [t]$, and for each i, j, k the following statements hold:

- $Q_{i,j,k} \subseteq X(T_i)$ and $|Q_{i,j,k}| \leq 1$;
- $Q_{i,j,k} = \emptyset$ if $[4] \setminus f(T_i) = L_1^k \cap L_2^k$ or $f(s_j) \in f(T_i)$;
- if $Q_{i,j,k} = \{x\}$, then there is a component C of $G|Y$ such that
  - s_j has a neighbor in $V(C)$;
  - some vertex of X is mixed on C, and C meets L_1^k, L_2^k as in (VI);
  - x has neighbors in $V(C)$

  and x has the maximum number of such components C among all vertices in $X(T_i)$;
- if $Q_{i,j,k} = \emptyset$, then no vertex $x \in X(T_i)$ and component C as above exist,
- Let $\tilde{Q} = \bigcup_{i \in \{1,\ldots,r\}, j \in \{1,\ldots,s\}, k \in \{1,\ldots,t\}} Q_{i,j,k}$, then $f' : \tilde{Q} \to \{1,2,3,4\}$ satisfies that $f' \cup f$ is a proper coloring of $G|(S \cup X_0 \cup \tilde{Q})$.

For $Q \in \mathcal{Q}$, we construct a starred precoloring P^Q from P as follows. We let $\tilde{Z}^Q$ be the set of vertices z in $X \setminus \tilde{Q}$ such that $\tilde{Q}$ contains a neighbor x of z with $f'(x) \in L_P(z)$, and let $g^Q : \tilde{Z}^Q \to \{1,2,3,4\}$ be the unique function such that $g^Q(z) \in L_P(z) \setminus f'(N(z) \cap \tilde{Q})$. We let $\tilde{X}^Q$ be the set of vertices z in Y such that $\tilde{Q}$ contains a neighbor x of z with $f'(x) \in L_P(z)$.

We let

$$P^Q = (G, S \cup \tilde{Q}, X_0 \cup \tilde{Z}^Q, (X \setminus (\tilde{Z}^Q \cup \tilde{Q})) \cup \tilde{X}^Q, Y \setminus \tilde{X}^Q, Y^*, f \cup f' \cup g^Q),$$

and let $\mathcal{L} = \{P^Q : Q \in \mathcal{Q}, f \cup f' \cup g^Q \text{ is a proper coloring}\}$. It is easy to check that $\mathcal{L}$ is an equivalent collection for P.

Let $Q \in \mathcal{Q}$, and let $P^Q = (G', S', X'_0, X', Y', Y^*, f')$. By construction, P^Q satisfies (I). Since P satisfies (II), (III), so does P^Q. Since P satisfies (II), it follows that P^Q satisfies (IV).

(24) *P^Q satisfies* (V).

Suppose not; and let $a - b - c$ be a path with $a \in X'$, $b, c \in Y'$ such that $L_{P^Q}(a) = L_3, L_{P^Q}(b) = L_1, L_{P^Q}(c) = L_2$ and $L_1 \neq L_2$, $L_3 \neq L_1 \cap L_2$. Since P satisfies (V), it follows that $a \in Y$. Since P satisfies (II) and (III), it follows that $L_P(a) = L_2$, and there is a vertex $x \in \tilde{Q}$, say $x \in Q_{i,j,k}$ such that x is adjacent to a and $f'(x) \in L_P(a)$. Since $c \in Y'$, it follows that x is not adjacent to c. Since x is mixed on a component of $G|Y$ meeting L_1 and L_2, and since P satisfies (VI), it follows that $L_P(x) = L_1 \cap L_2$. Thus $x - a - b - c$ is a path, and there is a component C of $G|Y$ such that $V(C)$ meets L_1^k, L_2^k and x has a neighbor in C and $L_1^k \cap L_2^k \neq L_P(x) = L_1 \cap L_2$. It follows that $a, b, c \notin V(C)$, and so $V(C)$ is anticomplete to a, b, c. By symmetry, we may assume that $L_1^k \notin \{L_1, L_2\}$. Let $d \in V(C)$ with $L_P(d) = L_1^k$. Since P satisfies (V) and (IV), and since x has a neighbor in C, it follows that x is complete to C and thus adjacent to d. Since $L_P(d) \notin \{L_1, L_2\}$, it follows that there is a vertex $s \in S$ with $f(s) \in L_1 \cap L_2$ and s adjacent to d. But then $c - b - a - x - d - s$ is a P_6 in G, a contradiction. This proves (24).

Now by Lemma 24, P^Q satisfies (VI).

(25) *P^Q satisfies* (VII).

Suppose not. Let C be a component of $G'|Y'$ such that some vertex of X' is mixed on C, and with L_1, L_2 as in (VI), and let $v \in X'$ with $N(v) \cap C \neq \emptyset$ such that $L_{P^Q}(v) \neq L_1 \cap L_2$.

Since $L_{P^Q}(v) \neq L_1 \cap L_2$, we may assume that $[4] \setminus L_1 \subseteq L_P(v)$. Let $s \in S$ with $f(s) = [4] \setminus L_1$, such that s has a neighbor in C. Since P^Q satisfies (VI), it follows that v is complete to C.

We claim that every $x \in X' \cap Y$ is complete to C. Suppose that $x \in Y \cap X'$ is mixed on C. Since P^Q satisfies (VI), it follows that $L_{P^Q}(x) = L_1 \cap L_2$. By symmetry, we may assume that $L_P(x) = L_1$, and therefore, x has a neighbor s in $\tilde{Q} \cap X$ and $f(s) = L_1 \setminus L_2$. But then s is mixed on the component $\tilde{C}$ of $G|Y$ containing $V(C) \cup \{x\}$, $\tilde{C}$ meets L_1 and L_2, and $L_P(s) \neq L_1 \cap L_2$, contrary to the fact that P satisfies (VI). This proves the claim. Now since some vertex of X' is mixed on C, it follows that some vertex of X is mixed on C.

Next we claim that $v \in X$. Suppose $v \in Y$. Then there is a component $\tilde{C}$ of $G|Y$ such that $V(C) \cup \{v\} \subseteq V(\tilde{C})$. Since some $x \in X$ is mixed on C, and since P satisfies (VI), we deduce that $L_P(v) \in \{L_1, L_2\}$. Consequently, v has a neighbor s in $\tilde{Q}$. Therefore $q \in X$. Since v is complete to C, it follows that v has a neighbor n in C with $L_P(n) = L_P(v)$. But then x is mixed on the edge vn, contrary to the fact that P satisfies (IV). This proves that $v \in X$.

By construction, Q contains an entry $Q_{i,j,k}$ with $T_i = T(v)$, $s_j = s$ and $(L_1^k, L_2^k) = (L_1, L_2)$, and in view of the claims of the previous two paragraphs, $Q_{i,j,k} \neq \emptyset$. Write $Q_{i,j,k} = \{z\}$. Let C' be a component of $G|Y$ meeting both L_1 and L_2, such that some vertex of X is mixed on C', and both s and z have a neighbor in C'. Since $f'(z) \in L_1 \cup L_2$, it follows that z is not complete to C. Since $L_P(z) \neq L_1 \cap L_2$, it follows from the fact that P satisfies (VI) that z is not mixed on either of C, C'. Consequently, z is complete to C', and z is anticomplete to C. Now, by the maximality of z, we may assume that v is anticomplete to C'. Since $[4] \setminus L_1 \subseteq L_P(z) = L_P(v)$, it follows that s is anticomplete to $\{z, v\}$.

Let $a \in V(C) \cap N(s)$ and $a' \in V(C') \cap N(s)$. Since each of C, C' meets L_2, we can also choose $b \in V(C) \setminus N(s)$ and $b' \in V(C') \setminus N(s)$. $L_P(z) \neq L_1 \cap L_2$, there exists $t \in T_i$ with $f(t) \in L_1 \cap L_2$. Then t is anticomplete to $V(C) \cup V(C')$. It t is non-adjacent to s, then $s - a - v - t - z - a'$ is a P_6 in G, so t is adjacent to s. If a is non-adjacent to b, then $b - v - a - s - a' - z$ is a P_6, so a is adjacent to b. But now $b - a - s - t - z - a'$ is a P_6, a contradiction. Thus, (25) follows.

This concludes the proof of the Lemma 25. □

We are now ready to prove the final axiom.

(VIII) $Y = \emptyset$.

Lemma 26. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y, Y^*, f)$ be a starred precoloring of a P_6-free graph G with P satisfying* (I), (II), (III), (IV), (V), (VI), (VII). *Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs collection $\mathcal{L}$ of starred precolorings such that*

- *if we know for every $P' \in \mathcal{L}$ whether P' has a precoloring extension or not, then we can decide if P has a precoloring extension in polynomial time;*
- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$*;*
- *every $P' \in \mathcal{L}$ is a starred precoloring of G;*
- *every $P' \in \mathcal{L}$ with seed S' satisfies $|S'| \leq q(|S|)$; and*
- *every $P' \in \mathcal{L}$ satisfies* (VIII).

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension for P in polynomial time, if one exists.

Proof. Let $P = (G, S, X_0, X, Y, Y^*, f)$. For every component C of $G \setminus (S \cup X_0)$, Let P_C be the starred precoloring

$$(G|(V(C) \cup S \cup X_0), S, X_0, X \cap V(C), Y \cap V(C), Y^* \cap V(C), f).$$

Then P_C satisfies (I)–(VII). Let $\mathcal{L}_0$ be the collection of all such starred precolorings P_C. Clearly P has a precoloring extension if and only if every member of $\mathcal{L}_0$ does, so from now on we focus on constructing an equivalent collection for each P_C separately. To simplify notation, from now on we will simply assume that $G \setminus (X_0 \cup S)$ is connected.

In the remainder of the proof we either find that P has no precoloring extension, output $\mathcal{L} = \emptyset$ and stop, or construct two disjoint subsets U and W of Y, and a subset $\tilde{X}_0$ of X such that

- $U \cup W = Y$,
- No vertex of X is mixed on a component of $G|W$,
- For every component C of $G|W$, some vertex of $X \cup X_0 \cup S$ is complete to C.
- There is a set F with $|F| \leq 2^6$ of colorings of $G|\tilde{X}_0$ that contains every coloring of $G|\tilde{X}_0$ that extends to a precoloring extension of P, and F can be computed in polynomial time.
- P has a precoloring extension if and only if for some $f' \in F$

  $$(G \setminus U, S, X_0 \cup \tilde{X}_0, X \setminus \tilde{X}_0, W, Y^*, f \cup f')$$

  has a precoloring extension.

Having constructed such $U, W, \tilde{X}_0$ and F, for each $f' \in F$ we set

$$P_{f'} = (G \setminus U, S, X_0 \cup \tilde{X}_0, X \setminus \tilde{X}_0, \emptyset, Y^* \cup W, f \cup f')$$

and output the collection $\mathcal{L} = \{P_{f'}\}_{f' \in F}$, which has the desired properties.

Start with $U = W = \tilde{X}_0 = \emptyset$. For $v \in Y$, let $M(v) = L_P(v) \setminus f(N(v) \cap (S \cup X_0))$. For $L \subseteq [4]$, we denote by M_L the list assignment $M_L(v) = M(v) \cap L$. To construct U, W and $\tilde{X}_0$, we first examine each component of $G|Y$ separately. Every time we enlarge U, we will "restart" the algorithm with $(G, S, X_0, X, Y, Y^*, f)$ replaced by $(G \setminus U, S, X_0, X, Y \setminus U, Y^*, f)$. Since we only do this when U is enlarged, there will be at most $|V(G)|$ such iterations, and so it is enough to ensure that each iteration can be done in polynomial time.

Let C be a component of $G|Y$. If no vertex of X is mixed on C, and some vertex of $S \cup X_0 \cup X$ is complete to C, we add $V(C)$ to W. So we may assume that either some vertex of X is mixed on C, or no vertex of X is complete to C. Let $C_i = \{v \in V(C) \ : \ L_P(v) = [4] \setminus \{i\}\}$. Since P satisfies (I), it follows that $V(C) = \bigcup_{i=1}^{4} C_i$,

Suppose first that C meets exactly one list L. Since P satisfies (VI), it follows that no vertex of X is mixed on C, and so $N(V(C)) \subseteq S \cup X_0$. By Theorem 2, we can test in polynomial time if (C, M) is colorable. If not, then P has no precoloring extension, we set $\mathcal{L} = \emptyset$ and stop. If (C, M) is colorable, then deleting $V(C)$ does not change the existence of a precoloring extension for P, and we add $V(C)$ to U.

Now suppose that C meets at least three lists. By Lemma 20, C_i is complete to C_j for every $i \neq j$. Since P satisfies (VI), it follows that no vertex of X is mixed on C, and so $N(V(C)) \subseteq S \cup X_0$. Since C_i is non-empty for at least three values of i, it follows that in every proper coloring of C, at most two colors appear in C_i, and for $i \neq j$ the sets of colors that appear in C_i and C_j are disjoint. By Theorem 9, for every $L \subset [4]$ with $|L| \leq 2$ and for every i, we can test in polynomial time if $(C|C_i, M_L)$ is colorable. If there exist disjoint lists $L_1, \dots, L_4$ such that (G_i, M_{L_i}) is colorable for all i, then deleting $V(C)$ does not change the existence of a precoloring extension for P, and we add $V(C)$ to U. If no such $L_1, \dots, L_i$ exist, then P has no precoloring extension, we set $\mathcal{L} = \emptyset$ and stop.

Thus we may assume that C meets exactly two lists, say $V(C) = C_3 \cup C_4$. Let $A_1, \dots, A_k$ be the components of $C|C_3$ and $A_{k+1}, \dots, A_t$ be the components of $C|C_4$. Since P satisfies (II), for every $i \in [k]$ and $j \in \{k+1, \dots, t\}$, A_i is either complete or anticomplete to A_j, and since P satisfies (IV), for every $i \in [t]$ no vertex of X is mixed on A_i. Since P satisfies (VII), if $x \in X$ has a neighbor in C, then $L_P(x) = \{1, 2\}$. By Theorem 2, for every A_i and for every $L \subseteq [4]$ with $|L \cap \{1, 2\}| \leq 1$, we can test in polynomial time if (A_i, M_L) is colorable. If (A_i, M_L) is colorable, we say that the set $M_L \cap \{1, 2\}$ *works* for A_i. Suppose that $\emptyset$ works for i. We may assume $i = 1$. It follows that (A_1, M) can be colored with color 3. Since $N(V(A_1)) \subseteq S \cup X_0 \cup X_{\{1,2\}} \cup C_4$, it follows that deleting A_i does not change the existence of a precoloring extension for P, and so we add $V(A_i)$ to U. Thus we may assume that $\emptyset$ does not work for any i.

Since C is connected and both C_3, C_4 are non-empty, it follows that for every i there is j such that A_i is complete to A_j, and so in every proper coloring of C, at most one of the colors $1, 2$ appears in each $V(A_i)$. Since $\emptyset$ does not work for any i, it follows that in every precoloring extension of P, exactly one of the colors $1, 2$ appears in each $V(A_i)$, and both 1 and 2 appear in $V(C)$. If some $x \in X$ is complete to C, then $x \in X_{\{1,2\}}$, and so G has no precoloring extension; we set $\mathcal{L} = \emptyset$, and stop. Thus we may assume that no vertex of X is complete to $V(C)$.

Let X_C be the set of vertices of X that are mixed on $V(C)$. Then $X_C \subseteq X_{\{1,2\}}$, and $N(V(C)) \subseteq S \cup X_0 \cup X_C$. Let $A_C = \{a_1, \dots, a_t\}$. Let H_C be the graph with vertex set $X_C \cup A_C$, where

- $a_i a_j \in E(H_C)$ if and only if A_i is complete to A_j,
- for $x \in X_C$, $x a_i \in E(H_C)$ if and only if x is complete to A_i, and
- $H_C|(X_C) = G|(X_C)$.

Let $T_C(a_i)$ be the the union of all the sets that work for i. Suppose first that $X_C = \emptyset$. Then $N(V(C)) \subseteq S \cup X_0$. By Theorem 9, we can test in polynomial time if (H_C, T_C) is colorable. If (H_C, T_C) is not colorable, then P has no precoloring extension; we output $\mathcal{L} = \emptyset$ and stop. Thus we may assume that (H_C, T_C) is colorable. Since $N(V(C)) \subseteq S \cup X_0$, deleting $V(C)$ does not change the existence of a precoloring extension, and we add $V(C)$ to U. Thus we may assume that $X_C \neq \emptyset$.

Now let $C^1, \dots, C^l$ be all the components of $G|Y$ for which $V(C^i) = C_3^i \cup C_4^i$ and $X_C \neq \emptyset$. Let H be the graph with vertex set $\bigcup_{i=1}^{l} V(H_{C^i})$ and such that $uv \in E(H)$ if and only if either

- $uv \in E(H_{C^i})$ for some i, or
- $u, v \in X$ and $uv \in E(G)$.

Let $T(v) = T_C(v)$ if $v \in V(H) \setminus X$, and let $T(v) = M(v)$ if $v \in V(H) \cap X$. By Theorem 9, we can test in polynomial time if (H, T) is colorable. If (H, T) is not colorable, then P has no precoloring extension; we output $\mathcal{L} = \emptyset$ and stop. Thus we may assume that (H, T) is colorable. Note that $T(v) \subseteq \{1, 2\}$ for every $v \in V(H)$.

Next we will show H is connected, and therefore (H, T) has at most two proper colorings, and we can compute the set of all proper colorings of (H, T) in polynomial time. Suppose that H is not connected. Since each C^i is connected, it follows that $H|A_{C^i}$ is connected for all i, and since for every i, every vertex of X_{C^i} has a neighbor in A_{C^i}, it follows that $H|V(H_{C^i})$ is connected for every i. Let D_1, D_2 be distinct components of H. Since $G \setminus (S \cup X_0)$ is connected, there is exist $p, q \in [l]$ such that $V(H_{C^p}) \subseteq D_1$, $V(H_{C^q}) \subseteq D_2$, and there is a path $P = p_1 - \ldots - p_m$ in $G \setminus (S \cup X_0)$ with $p_1 \in V(C^p) \cup X_{C^p}$, $p_m \in V(C^q) \cup X_{C^q}$, and P^* is disjoint from $\bigcup_{i=1}^{l}(V(C^i) \cup X_{C^i})$. Since for every i, $N(V(C^i)) \subseteq S \cup X_0 \cup X_{C^i}$, it follows that $p_1 \in X_{C^p}$ and $p_m \in X_{C^q}$, and P^* is anticomplete to $V(C^p) \cup V(C^q)$. By Lemma 1, there exist $a_p, b_p \in V(C^p)$ such that $p_m - a_p - b_p$ is a path, and there exist $a_q, b_q \in V(C^q)$ such that $p_m - a_q - b_q$ is a path. But now $b_p - a_p - p_1 - P - p_m - a_q - b_q$ is a path of length at least six in G, a contradiction. This proves that H is connected.

Let $\tilde{X}_0^{3,4} = V(H) \cap X$, and let $F^{3,4}$ be the set of all proper colorings of $(G|\tilde{X}_0^{3,4}, M)$ that extend to a coloring of (H, T). Then $|F^{3,4}| \leq 2$, and we can compute $F^{3,4}$ in polynomial time. Let $U^{3,4} = \bigcup_{i=1}^{l} V(C^i)$. Since for each i, $N(C^i) \subseteq \tilde{X}_0^{3,4} \cup S \cup X_0$, it follows that

P has a precoloring extension if and only if for some $f' \in F^{3,4}$

(26) $$(G \setminus U^{3,4}, S, X_0 \cup \tilde{X}_0^{3,4}, X \setminus \tilde{X}_0^{3,4}, Y \setminus U^{3,4}, Y^*, f \cup f')$$

has a precoloring extension.

For every $i, j \in [4]$ with $i \neq j$ define $U^{i,j}$, $F^{i,j}$ and $\tilde{X}_0^{i,j}$ similarly. Let $\tilde{X}_0 = \bigcup \tilde{X}_0^{i,j}$. Let F be the set of all functions $f' : \tilde{X}_0 \to [4]$ such that $f'|_{\tilde{X}_0{}^{i,j}} \in F^{i,j}$. Then $|F| \leq 2^6$. Let $U' = \bigcup U^{i,j}$.

If follows from (26) that P has a precoloring extension if and only if

$$(G \setminus U', S, X_0 \cup \tilde{X}, X \setminus \tilde{X}_0, Y \setminus U', Y^*, f \cup f')$$

has a precoloring extension for some $f' \in F$. Now we add U' to U, and Lemma 26 follows. □

We are now ready to prove our the main result, which we restate:

Theorem 11. *There exists an integer $C > 0$ and a polynomial-time algorithm with the following specifications.*

Input: *A 4-precoloring (G, X_0, f) of a P_6-free graph G.*

Output: *A collection $\mathcal{L}$ of excellent starred precolorings of G such that*

1. *$|\mathcal{L}| \leq |V(G)|^C$,*
2. *for every $(G', S', X_0', X', \emptyset, Y^*, f') \in \mathcal{L}$*
   - *$|S'| \leq C$,*
   - *$X_0 \subseteq S' \cup X_0'$,*
   - *G' is an induced subgraph of G, and*
   - *$f'|_{X_0} = f|_{X_0}$.*
3. *if we know for every $P \in \mathcal{L}$ whether P has a precoloring extension, then we can decide in polynomial time if (G, X_0, f) has a 4-precoloring extension; and*
4. *given a precoloring extension for every $P \in \mathcal{L}$ such that P has a precoloring extension, we can compute a 4-precoloring extension for (G, X_0, f) in polynomial time, if one exists.*

Proof. Let (G, X_0, f) be a 4-precoloring of a P_6-free graph G. We apply Theorem 10 to (G, X_0, f) to obtain a collection $\mathcal{L}_0$ of good seeded precolorings with the desired properties. Then we apply Lemma 14 to each seeded precoloring in $\mathcal{L}_0$ to obtain a starred precoloring satisfying (I); let $\mathcal{L}_1$ be the collection thus obtained. Next, starting with $\mathcal{L}_1$, apply Lemma 16, Lemma 19, Lemma 21, Lemma 23, Lemma 24, Lemma 25 and Lemma 26 to each element in the output of the previous one, to finally obtain a collection $\mathcal{L}$. Then $\mathcal{L}$ is an equivalent collection for P, and every element of $\mathcal{L}$ satisfies (II), (III), (IV), (V), (VI), (VII) and (VIII). Finally, (VIII) implies that each starred precoloring in $\mathcal{L}$ is excellent, as claimed. □

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