TREE-INDEPENDENCE NUMBER VIII. 3PC-FREE GRAPHS HAVE LOGARITHMIC TREE-INDEPENDENCE NUMBER

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Abstract. A *three-path-configuration* is a graph consisting of three pairwise internally vertex-disjoint paths, the union of every two of which is an induced cycle of length at least four. A graph is *3PC-free* if no induced subgraph of it is a three-path-configuration. We show that 3PC-free graphs have logarithmic tree-independence number, which is asymptotically best possible.

1. Introduction

All graphs in this paper are finite and simple and all logarithms are base 2. We treat a graph and its vertex set interchangeably. Throughout this paper we use n to denote the number of vertices of a graph G. For graphs G and H, we say that G is *H-free* if no induced subgraph of G is isomorphic to H. For $X, Y \subseteq V(G)$, we say that X and Y are *anticomplete* if $X \cap Y = \emptyset$ and there is no edge in G with an end in X and an end in Y. The *closed neighborhood* of a set X, denoted by $N[X]$, is the set of vertices of G that are in X or have at least one neighbor in X. For standard graph-theoretic terminology, the reader is referred to [4].

For a graph G, a *tree decomposition* (T, χ) of G consists of a tree T and a map $\chi : V(T) \to 2^{V(G)}$ with the following properties:

(i) For every $v \in V(G)$, there exists $t \in V(T)$ such that $v \in \chi(t)$.
(ii) For every $uv \in E(G)$, there exists $t \in V(T)$ such that $u, v \in \chi(t)$.
(iii) For every $v \in V(G)$, the subgraph of T induced by $\{t \in V(T) \mid v \in \chi(t)\}$ is connected.

The *width* of a tree decomposition (T, χ) is $\max_{t \in V(T)} |\chi(t)| - 1$. The *independence number* of a tree decomposition (T, χ) is $\max_{t \in V(T)} \alpha(\chi(t))$. The *treewidth* of G, denoted by $\mathrm{tw}(G)$, is the minimum width of a tree decomposition of G. The *tree-independence number* of G, denoted by $\mathsf{tree}\text{-}\alpha(G)$, is the minimum independence number of a tree decomposition of G.

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A *path* in a graph is an induced subgraph that is a path. The *interior* of a path P is denoted by P^*. A *theta* is a graph consisting of two non-adjacent vertices a, b and three paths P_1, P_2, P_3 from a to b, such that P_i^* is anticomplete to P_j^* for $i \neq j$. A *three-path-configuration* is a graph consisting of three pairwise internally vertex-disjoint paths, the union of every two of which is an induced cycle of length at least four. A graph is *3PC-free* if no induced subgraph of it is a three-path-configuration.

It is known that 3PC-free graphs have polylogarithmic tree-independence number and if their clique number is bounded by a constant t, then they also have logarithmic treewidth.

Theorem 1.1 (Abrishami, Chudnovsky, Hajebi, Spirkl [1])**.** *For every $t \in \mathbb{N}$ there exists a constant c_t such that if a graph G is 3PC-free and K_t-free, then $\mathrm{tw}(G) \leq c_t \log n$.*

Theorem 1.2 (Chudnovsky, Hajebi, Lokshtanov, Spirkl [3])**.** *There exists a constant c such that if a graph G with $n > 1$ is 3PC-free, then $\mathsf{tree}\text{-}\alpha(G) \leq c(\log n)^2$.*

On the other hand, the best lower bound on the tree-independence number of 3PC-free graphs is $\log n$ (Sintiari and Trotignon [7]). This led to a question [2, 3] of whether 3PC-free graphs with superlogarithmic tree-independence number exist.

Recently, Hajebi and Spirkl [5] introduced a powerful new notion of loose separability. A direct application of their method allows us to close the gap between the upper and the lower bounds. We prove:

Theorem 1.3. *There exist constants c, d such that if a graph G with $n > 1$ is 3PC-free, then $\mathsf{tree}\text{-}\alpha(G) \leq c + d \log n$.*

Note that since the class of theta-free graphs is "χ-bounded" (see [6] for details), Theorem 1.3 yields another proof of Theorem 1.1.

2. Proof

We include some definitions from [3] and [5] that will be useful for us. If S is a set of sets, a *hitting set for S* is a set X such that $X \cap Y \neq \varnothing$ for every $Y \in S$. For $b \in \mathbb{N}$, a *b-system* $\mathcal{S}$ in G is a set of non-empty subsets of $V(G)$, each of cardinality at most b (thus, $\mathcal{S} = \varnothing$ is allowed, whereas $\mathcal{S} = \{\varnothing\}$ is not). We say that a b-system $\mathcal{S}$ in G is *anticomplete* if every two distinct sets $S_1, S_2 \in \mathcal{S}$ are anticomplete in G. We denote the set of all (X, Y)-paths in G by $\mathcal{P}_G(X, Y)$.

For $b, c, d \in \mathbb{N}$, a graph G is *(b, c)-loosely d-α-separable* if for every b-system $\mathcal{S}$ in G, either

- there is a hitting set $X \subseteq V(G)$ for $\mathcal{S}$ with $\alpha(X) \leq c$; or
- there exist $S_1, S_2 \in \mathcal{S}$ such that $\mathcal{P}_G(S_1, S_2)$ has a hitting set $Y \subseteq V(G) \setminus (S_1 \cup S_2)$ with $\alpha(Y) \leq d$; in particular, S_1 and S_2 are anticomplete in G.

For a graph G, a function $w : V(G) \to [0, 1]$ is a *normal weight function* on G if $w(V(G)) = 1$. Let $c \in [0, 1]$ and let w be a normal weight function on G. A set $X \subseteq V(G)$ is a *(w, c)-balanced separator* if $w(D) \leq c$ for every component D of $G \setminus X$. The set X is a *w-balanced separator* if X is a $(w, \frac{1}{2})$-balanced separator. For $b \in \mathbb{N}$, a graph G is *b-separator-dominated* if for every normal weight function w on G, there exists $X \subseteq V(G)$ with $|X| \leq b$ such that $N[X]$ is a w-balanced separator in G.

Hajebi and Spirkl [5] prove:

Lemma 2.1 (Hajebi, Spirkl [5]). *For all $a, b, q \in \mathbb{N}$, there is a constant $c_{2.1} = c_{2.1}(a, b, q) \in \mathbb{N}$ such that for every $K_{a,a}$-free graph G and every b-system $\mathcal{S}$ in G, either there is a hitting set $X \subseteq V(G)$ for $\mathcal{S}$ with $\alpha(X) < c_{2.1}$, or there is an anticomplete b-system $\mathcal{Q} \subseteq \mathcal{S}$ in G with $|\mathcal{Q}| \geq q$.*

Lemma 2.2 (Hajebi, Spirkl [5]). *Let $b, c, d \in \mathbb{N}$ and let G be a (b, c)-loosely d-α-separable graph such that every induced subgraph of G is b-separator-dominated. Then, $\mathsf{tree}\text{-}\alpha(G) \leq 10c + 5d$.*

The following was proved in [3]:

Theorem 2.3 (Chudnovsky, Hajebi, Lokshtanov, Spirkl [3]). *There is an integer b with the following property. Let G be a 3PC-free graph, and let w be a normal weight function on G. Then, there exists $Y \subseteq V(G)$ such that*

- *$|Y| \leq b$, and*
- *$N[Y]$ is a w-balanced separator in G.*

Theorem 2.4 (Chudnovsky, Hajebi, Lokshtanov, Spirkl [3]). *Let G be a 3PC-free graph and let $a, b \in V(G)$ be non-adjacent. Then, there is a hitting set $X \subseteq V(G) \setminus \{a, b\}$ for $\mathcal{P}_G(\{a\}, \{b\})$ with $\alpha(X) \leq 32 \log n$.*

Proof of Theorem 1.3:

By Theorem 2.3, there exists an integer b such that every 3PC-free graph is b-separator-dominated. In particular, every induced subgraph of G is b-separator-dominated. Let $\mathcal{S}$ be a b-system in G. Since G is theta-free, G is $K_{3,3}$-free. By Lemma 2.1, there exists an integer c such that either there is a hitting set $X \subseteq V(G)$ for $\mathcal{S}$ with $\alpha(G[X]) < c$, or there are two anticomplete sets $S_1, S_2 \in \mathcal{S}$. Let $s_1 \in S_1, s_2 \in S_2$, and let $G_{s_1 s_2} = (G \setminus (S_1 \cup S_2)) \cup \{s_1, s_2\}$. Theorem 2.4 implies that there exists a hitting set $Y_{s_1 s_2}$ for $\mathcal{P}_{G_{s_1 s_2}}(\{s_1\}, \{s_2\})$ in $G_{s_1 s_2}$ with $\alpha(Y_{s_1 s_2}) \leq 32 \log n$. Thus, $\mathcal{P}_G(S_1, S_2)$ has a hitting set $Y = \cup_{s_1 \in S_1, s_2 \in S_2} Y_{s_1 s_2} \subseteq V(G) \setminus (S_1 \cup S_2)$ with $\alpha(G[Y]) \leq 32 b^2 \log n$; consequently, G is (b, c)-loosely $(32 b^2 \log n)$-α-separable. By Lemma 2.2, $\mathsf{tree}\text{-}\alpha(G) \leq 10c + 160 b^2 \log n$. This completes the proof.

■

3. AI Disclosure

We used GPT-6 Astra to proofread the manuscript. No AI tools were used to create the proof.

References

[1] Tara Abrishami, Maria Chudnovsky, Sepehr Hajebi, and Sophie Spirkl. Induced subgraphs and tree decompositions III. Three-path-configurations and logarithmic treewidth. *Adv. Comb.*, pages Paper No. 6, 29, 2022.

[2] Maria Chudnovsky and Sepehr Hajebi. *Barbados Graph Theory Workshop*, 2024.

[3] Maria Chudnovsky, Sepehr Hajebi, Daniel Lokshtanov, and Sophie Spirkl. Tree independence number II. Three-path-configurations. *Journal of Combinatorial Theory, Series B*, 176:74–96, 2026.

[4] Reinhard Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Berlin, Fifth edition, 2017.

[5] Sepehr Hajebi and Sophie Spirkl. Tree-alpha and excluding finitely many graphs. 2026. arXiv:2605.01223.

[6] Alex Scott and Paul Seymour. A survey of χ-boundedness. *Journal of Graph Theory*, 95(3):473–504, 2020.

[7] Ni Luh Dewi Sintiari and Nicolas Trotignon. (Theta, triangle)-free and (even hole, K_4)-free graphs—Part 1: Layered wheels. *Journal of Graph Theory*, 97(4):475–509, 2021.