

INDUCED SUBGRAPHS AND TREE DECOMPOSITIONS

III. THREE-PATH-CONFIGURATIONS AND LOGARITHMIC TREEWIDTH

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ABSTRACT. A *theta* is a graph consisting of two non-adjacent vertices and three internally disjoint paths between them, each of length at least two. For a family \mathcal{H} of graphs, we say a graph G is \mathcal{H} -free if no induced subgraph of G is isomorphic to a member of \mathcal{H} . We prove a conjecture of Sintiari and Trotignon, that there exists an absolute constant c for which every (theta, triangle)-free graph G has treewidth at most $c \log(|V(G)|)$. A construction by Sintiari and Trotignon shows that this bound is asymptotically best possible, and (theta, triangle)-free graphs comprise the first known hereditary class of graphs with arbitrarily large yet logarithmic treewidth.

Our main result is in fact a generalization of the above conjecture, that treewidth is at most logarithmic in $|V(G)|$ for every graph G excluding the so-called *three-path-configurations* as well as a fixed complete graph. It follows that several NP-hard problems such as STABLE SET, VERTEX COVER, DOMINATING SET and COLORING admit polynomial time algorithms in graphs excluding the three-path-configurations and a fixed complete graph.

1. INTRODUCTION

All graphs in this paper are finite and simple. Let $G = (V(G), E(G))$ be a graph. For a set $X \subseteq V(G)$ we denote by $G[X]$ the subgraph of G induced by X . For $X \subseteq V(G)$, $G \setminus X$ denotes the subgraph induced by $V(G) \setminus X$. In this paper, we use induced subgraphs and their vertex sets interchangeably. Let $v \in V(G)$. The *open neighborhood* of v , denoted by $N(v)$, is the set of all vertices in $V(G)$ adjacent to v . The *closed neighborhood* of v , denoted by $N[v]$, is $N(v) \cup \{v\}$. Let $X \subseteq V(G)$. The *open neighborhood* of X , denoted by $N(X)$, is the set of all vertices in $V(G) \setminus X$ with at least one neighbor in X . The *closed neighborhood* of X , denoted by $N[X]$, is $N(X) \cup X$. If H is an induced subgraph of G and $X \subseteq V(G)$, then $N_H(X) = N(X) \cap H$ and $N_H[X] = N_H(X) \cup X$. Let $Y \subseteq V(G)$ be disjoint from X . We say X is *complete* to Y if all edges with an end in X and an end in Y are present in G , and X is *anticomplete* to Y if there are no edges between X and Y .

For a graph $G = (V(G), E(G))$, a *tree decomposition* (T, χ) of G consists of a tree T and a map $\chi : V(T) \rightarrow 2^{V(G)}$ with the following properties:

- (i) For every $v \in V(G)$, there exists $t \in V(T)$ such that $v \in \chi(t)$.
- (ii) For every $v_1 v_2 \in E(G)$, there exists $t \in V(T)$ such that $v_1, v_2 \in \chi(t)$.
- (iii) For every $v \in V(G)$, the subgraph of T induced by $\{t \in V(T) \mid v \in \chi(t)\}$ is connected.

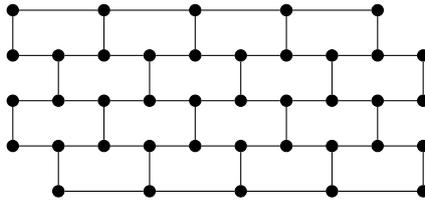
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FIGURE 1. $W_{5 \times 5}$

For each $t \in V(T)$, we refer to $\chi(t)$ as a *bag* of (T, χ) . The *width* of a tree decomposition (T, χ) , denoted by $\text{width}(T, \chi)$, is $\max_{t \in V(T)} |\chi(t)| - 1$. The *treewidth* of G , denoted by $\text{tw}(G)$, is the minimum width of a tree decomposition of G .

Treewidth, first introduced by Robertson and Seymour in their monumental work on graph minors, is an extensively studied graph parameter, mostly due to the fact that graphs of bounded treewidth exhibit interesting structural [15] and algorithmic [5] properties. Accordingly, one would naturally desire to understand the structure of graphs with large treewidth, and in particular the unavoidable substructures emerging in them. For instance, for each k , the $(k \times k)$ -wall, denoted by $W_{k \times k}$, is a planar graph with maximum degree three and with treewidth k (see Figure 1; a precise definition can be found in [2]). Every subdivision of $W_{k \times k}$ is also a graph of treewidth k . The unavoidable subgraphs of graphs with large treewidth are fully characterized by the Grid Theorem of Robertson and Seymour, the following.

Theorem 1.1 ([15]). *For every integer t there exists $c = c(t)$ such that every graph of treewidth at least c contains a subdivision of $W_{t \times t}$ as a subgraph.*

Following the same line of thought, our motivation is to study the unavoidable induced subgraphs of graphs with large treewidth. Together with subdivided walls mentioned above, complete graphs and complete bipartite graphs are easily observed to have arbitrarily large treewidth: the complete graph K_{t+1} and the complete bipartite graph $K_{t,t}$ both have treewidth t . Line graphs of subdivided walls form another family of graphs with unbounded treewidth, where the *line graph* $L(F)$ of a graph F is the graph with vertex set $E(F)$, such that two vertices of $L(F)$ are adjacent if the corresponding edges of F share an end. One may ask whether these graphs are all we have to exclude as induced subgraphs to obtain a constant bound on the treewidth:

Question 1.2. *Is it true that for all t , there exists $c = c(t)$ such that every graph with treewidth more than c contains as an induced subgraph either a K_t , or a $K_{t,t}$, or a subdivision of $W_{t \times t}$ or the line graph of a subdivision of $W_{t \times t}$?*

Sintiari and Trotignon [17] provided a negative answer to this question. To describe their result, we require a few more definitions. Let H be a graph. We say G *contains* H if G has an induced subgraph isomorphic to H . We say G is H -free if G does not contain H . For a family \mathcal{H} of graphs we say that G is \mathcal{H} -free if G is H -free for every $H \in \mathcal{H}$. Given a graph G , a *path* in G is an induced subgraph of G that is a path. If P is a path in G , we write $P = p_1 \cdots p_k$ to mean that $V(P) = \{p_1, \dots, p_k\}$, and p_i is adjacent to p_j if and only if $|i - j| = 1$. We call the vertices p_1 and p_k the *ends* of P , and say that P is *from* p_1 *to* p_k . The *interior* of P , denoted by P^* , is the set $V(P) \setminus \{p_1, p_k\}$. The *length* of a path P is the number of edges in P .

A *theta* is a graph consisting of two non-adjacent vertices a, b and three paths P_1, P_2, P_3 from a to b of length at least two, such that P_1^*, P_2^*, P_3^* are mutually disjoint and anticomplete to each other. If a graph G contains an induced subgraph H which is a theta, and a, b are the two vertices of degree three in H , then we say that G contains a theta *between* a *and* b . Note that the complete bipartite graph $K_{2,3}$ is a theta. Also, it is readily seen that for large enough k , all subdivisions of $W_{k \times k}$ contain thetas, and of course line graphs of subdivisions of $W_{k \times k}$ contain triangles. So the following theorem provides a negative answer to Question 1.2.

Theorem 1.3 ([17]). *For every integer $\ell \geq 1$, there exists a (theta, triangle)-free graph G_ℓ such that $\text{tw}(G_\ell) \geq \ell$.*

The authors of [17] observed that the number of vertices of the graphs G_ℓ from Theorem 1.3 is exponential in their treewidth, while for walls and their line graphs, the number of vertices is polynomial in the treewidth. This radical difference leads to the following conjecture.

Conjecture 1.4 ([17]). *There exists a constant c such that if G is a (theta, triangle)-free graph, then $\text{tw}(G) \leq c \log(|V(G)|)$.*

We prove this conjecture. Indeed, our main result is a substantial generalization of Conjecture 1.4 involving the so-called *three-path-configurations*, which we define next. A *hole* in a graph is an induced cycle of length at least four. (The *length* of a hole is the number of vertices in it.)

A *pyramid* is a graph consisting of a vertex a and a triangle $\{b_1, b_2, b_3\}$, and three paths P_i from a to b_i for $1 \leq i \leq 3$, all of length at least one, such that for $i \neq j$, the only edge between $P_i \setminus \{a\}$ and $P_j \setminus \{a\}$ is $b_i b_j$, and at most one of P_1, P_2, P_3 has length exactly one. We say a is the *apex* of the pyramid and $b_1 b_2 b_3$ is the *base* of the pyramid.

A *prism* is a graph consisting of two triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, and three paths P_i from a_i to b_i for $1 \leq i \leq 3$, all of length at least one, and such that for $i \neq j$ the only edges between P_i and P_j are $a_i a_j$ and $b_i b_j$. A *pinched prism* is a graph consisting of a hole H of length at least six, together with a vertex b_1 such that $N_H(b_1)$ is an induced two-edge matching. We call b_1 the *center* of the pinched prism. (This graph is often called a ‘line wheel’, but here we choose to emphasize its similarity to a prism with a “pinched” path). A *generalized prism* is a graph that is either a prism or a pinched prism.

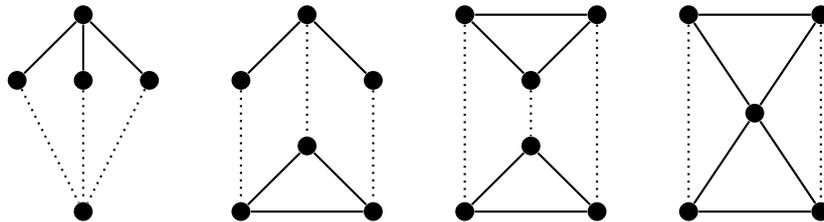


FIGURE 2. Theta, pyramid, prism, and pinched prism. The dotted lines represent paths of length at least one. In the pinched prism, the dotted lines represent paths of length at least two.

Finally, a graph is a *three-path-configuration* if it is a theta, a prism, or a pyramid. Let \mathcal{C} be the class of (theta, pyramid, generalized prism)-free graphs. Also, for every integer $t \geq 1$, let \mathcal{C}_t be the class of (theta, pyramid, generalized prism, K_t)-free graphs. Our main result is the following.

Theorem 1.5. *For every $t \geq 1$ there exists a constant c_t such that every $G \in \mathcal{C}_t$ has treewidth at most $c_t \log(|V(G)|)$.*

A *clique cutset* in a graph G is a clique K of G such that $G \setminus K$ is not connected. A special case of Lemma 3.1 from [7] shows that clique cutsets do not affect treewidth:

Theorem 1.6. *For every graph G there exists an induced subgraph G' of G such that G' has no clique cutset, and $\text{tw}(G) = \text{tw}(G')$.*

It follows that in order to prove Theorem 1.5, it is enough to prove the following:

Theorem 1.7. *For every $t \geq 1$ there exists a constant c_t such that every graph in $G \in \mathcal{C}_t$ with no clique cutset has treewidth at most $c_t \log(|V(G)|)$.*

Note that since \mathcal{C}_t contains all (theta, triangle)-free graphs for each $t \geq 3$, Theorem 1.5 settles Conjecture 1.4. As discussed above, the construction from Theorem 1.3 shows that the bound provided by Theorem 1.5 is asymptotically best possible. In fact, Theorem 1.5 is the first result establishing a logarithmic bound on the treewidth in a hereditary (that is, closed under isomorphism and taking induced subgraphs) class of graphs. This is remarkable, especially because a considerable number of algorithmic advantages of bounded treewidth are still accessible in graphs of logarithmic treewidth. We elaborate on this in Section 9.

The proof of Theorem 1.5 builds on a method developed in two previous papers of this series [2, 4] to bound the treewidth of graph classes with bounded maximum degree. Incidentally, unlike subdivided walls and their line graphs, the graphs G_ℓ from Theorem 1.3 contain vertices of arbitrarily large degree. So the following question is asked in [14]:

Question 1.8 ([14]). *Is it true that for every $\Delta > 0$, there exists $c = c(\Delta)$ such that for every (theta, triangle)-free graph G of maximum degree at most Δ , we have $\text{tw}(G) \leq c$?*

In [2], with Dibek, Rzażewski and Vušković, we gave an affirmative answer to this question. More generally, it is also conjectured in [1] that there is an affirmative answer to Question 1.2 restricted to graphs of bounded maximum degree.

Conjecture 1.9 ([1]). *For all $k, \Delta > 0$, there exists $c = c(k, \Delta)$ such that every graph with maximum degree at most Δ and treewidth more than c contains a subdivision of $W_{k \times k}$ or the line graph of a subdivision of $W_{k \times k}$ as an induced subgraph.*

This is still open, while several interesting special cases of it are proved in earlier papers of this series [2, 4]. In the same vein, the following may be true as far as we know (this is a variant of a conjecture of [17]):

Conjecture 1.10. *For all $t \geq 0$ there exists $c = c(t)$ such that if G is a graph with no K_t , no $K_{t,t}$, no subdivision of $W_{t \times t}$ and no line graph of a subdivision of $W_{t \times t}$ as induced subgraphs, then $\text{tw}(G) \leq c \log(|V(G)|)$.*

We conclude this section with the following result, which is an immediate consequence of the Helly property of subtrees of a tree:

Theorem 1.11 ([11]). *Let G be a graph, let K be a clique of G , and let (T, χ) be a tree decomposition of G . Then, there is $v \in V(T)$ such that $K \subseteq \chi(v)$.*

1.1. Proof outline and organization. Let us now discuss the main ideas of the proof of Theorem 1.5. We will give precise definitions of the concepts used below later in the paper; our goal here is to sketch a road map of where we are going. By Theorem 1.6 we may assume that the graph in question does not admit a clique cutset. By Theorem 3.1 we may restrict our attention to cube-free graphs (the ‘‘cube’’ is a certain eight vertex graph defined later). Obtaining a tree decomposition is usually closely related to producing a collection of ‘‘non-crossing decompositions,’’ which roughly means that the decompositions ‘‘cooperate’’ with each other, and the pieces that are obtained when the graph is simultaneously decomposed by all the decompositions in the collection ‘‘line up’’ to form a tree structure.

In the case of graphs in \mathcal{C}_t , there is a natural family of decompositions to turn to; they correspond to special vertices of the graph called ‘‘hubs,’’ and are discussed in Section 3. Unfortunately, these natural decompositions are very far from being non-crossing, and therefore we cannot use them in traditional ways to get tree decompositions. We were able to overcome this issue in [4] by using a bound on the maximum degree of the graph, but the same methods do not apply when no such bound exists. What we can do instead is use degeneracy (that is given by a result of [12]) to partition the set of all hubs (which yields a partition of all the natural decompositions) of an n -vertex graph G in \mathcal{C}_t into collections S_1, \dots, S_p , where each S_i is ‘‘non-crossing’’ (this property is captured in Lemma 6.4), $p \leq C(t) \log n$ (where $C(t)$ only

depends on t and works for all $G \in \mathcal{C}_t$), and vertices in S_i have a bounded (as a function of t) number of neighbors in $\bigcup_{j=i}^p S_j$. Our main result is that the treewidth of G is bounded by a linear function of $p + \log n$.

It follows immediately from Theorem 2.3 and Theorem 4.4 that the treewidth of a graph in \mathcal{C}_t is bounded (by a constant depending on t) if G has no hubs; thus we may assume that $p > 0$. It is sometimes the case that one of the hubs we get is not “useful” to us, and then we turn to the arguments in Section 5.

In the general case, for $p > 0$, we proceed as follows. We first decompose G , simultaneously, by all the decompositions corresponding to the hubs in S_1 . This allows us to define a natural induced subgraph $\beta(S_1)$ of G that we call the “central bag” for S_1 . The parameter p is smaller for $\beta(S_1)$ than it is for G , and so we can use induction to obtain a bound on the treewidth of $\beta(S_1)$. We then start with a special optimal tree decomposition of $\beta(S_1)$, where each bag is a “potential maximal clique” (see Section 2). Also inductively (this time on the number of vertices) we have tree decompositions for each component of $G \setminus \beta(S_1)$.

Now we use the special nature of our “natural decompositions” and properties of potential maximal cliques to combine the tree decompositions above into a tree decomposition of G , where the size of the bag only grows by an additive constant. This is possible because in the growing process all we need to do is add to each existing bag the neighbor sets of some non-hub vertices in that bag. As the number of such vertices in each bag is bounded by Theorem 4.4, and due to the “degeneracy” property of the partition S_1, \dots, S_p , we can ensure a bound on the growth.

The paper is organized as follows. In Section 2 we introduce potential maximal cliques and discuss their properties. In Section 3 we prove structural results guaranteeing the existence of useful decompositions. In Section 4 we discuss the bounds on the number of non-hub vertices in minimal separators and in potential maximal cliques. Section 5 contains lemmas that will allow us to deal with hubs that do not fall into the framework of Section 6. In Section 6 we discuss collections of non-crossing decompositions and properties of their central bags. In Section 7 we show how to construct the partition S_1, \dots, S_p . Section 8 puts together the results of all the previous sections to prove Theorem 1.5. Finally, Section 9 discusses algorithmic consequences of Theorem 1.5.

2. POTENTIAL MAXIMAL CLIQUES

In the proof of Theorem 1.5 we will use a special kind of tree decomposition that we explain now.

For a graph G and a set $F \subseteq \binom{V(G)}{2} \setminus E(G)$, we denote by $G + F$ the graph obtained from G by making the pairs in F adjacent. A set $F \subseteq \binom{V(G)}{2} \setminus E(G)$ is a *chordal completion* or *fill-in* of G if $G + F$ is chordal; a chordal completion is *minimal* if it is inclusion-wise minimal. Let $X \subseteq V(G)$. The set X is a *minimal separator* if there exist $u, v \in V(G)$ such that u and v are in different connected components of $G \setminus X$, and u and v are in the same connected component of $G \setminus Y$ for every $Y \subsetneq X$. A component D of $G \setminus X$ is a *full component* for X if $N(D) = X$. It is well-known that a set $X \subseteq V(G)$ is a minimal separator if and only if there are at least two distinct full components for X .

A *potential maximal clique* (PMC) of a graph G is a set $\Omega \subseteq V(G)$ such that Ω is a maximal clique of $G + F$ for some minimal chordal completion F of G . The following result of [8] characterizes PMCs:

Theorem 2.1. *A set $\Omega \subseteq V(G)$ is a PMC of G if and only if:*

- (1) *for every distinct $x, y \in \Omega$ with $xy \notin E(G)$, there exists a component D of $G \setminus \Omega$ such that $x, y \in N(D)$.*
- (2) *for every component D of $G \setminus \Omega$ it holds that $N(D) \subsetneq \Omega$.*

If $\Omega \subseteq V(G)$ and D is a component of $G \setminus \Omega$ with $x, y \in N(\Omega)$ non-adjacent (as in (1) above), we say that D covers the non-edge xy .

We also need the following result of [8] relating PMCs and minimal separators:

Theorem 2.2. *Let $\Omega \subseteq V(G)$ be a PMC of G . Then, for every component D of $(G \setminus \Omega)$, the set $N(D)$ is a minimal separator of G .*

Let us say that a tree decomposition (T, χ) of a graph G is *structured* if $\chi(v)$ is a PMC of G for every $v \in V(T)$. We denote by $\omega(G)$ the maximum size of a clique in G . The following is a striking but easy fact (this was observed by multiple authors in the past, but we include the proof here for completeness):

Theorem 2.3. *Every graph G has a structured tree decomposition of width $\text{tw}(G)$.*

Proof. Let (T', χ') be a tree decomposition of G of width $\text{tw}(G)$. It is easy to check that the graph G' obtained from G by adding all edges xy such that $x, y \in \chi'(v)$ for some $v \in V(T')$ is chordal. It follows that there exists a minimal chordal completion F of G such that $F \subseteq E(G') \setminus E(G)$; let $G'' = G + F$. In particular, every clique of G'' is a subset of a clique of G' . Since by Theorem 1.11 every clique of G' is contained in a bag $\chi'(v)$ for some $v \in V(T')$, it follows that $\omega(G'') \leq \omega(G') \leq \text{tw}(G) + 1$. Next, since G'' is chordal, there is a tree decomposition (T'', χ'') of G'' such that $\chi''(v)$ is a clique of G'' (and therefore a PMC of G) for every $v \in V(T'')$. Lastly, since G is a subgraph of G'' , it follows that (T'', χ'') is a tree decomposition of G . Since $\omega(G'') \leq \text{tw}(G) + 1$, it follows that $\text{width}(T'', \chi'') = \text{tw}(G)$, as required. This proves Theorem 2.3. ■

3. STRUCTURAL RESULTS

In this section we establish some useful structural properties of (theta, pyramid, generalized prism)-free graphs containing either a ‘cube’ or a ‘wheel’. Let us define these notions and state our theorems properly.

The *cube* is the graph with vertex set $\{a_1, \dots, a_6, b_1, b_2\}$ in which $\{a_1, \dots, a_6\}$ is a hole, b_1 is complete to $\{a_1, a_3, a_5\}$, b_2 is complete to $\{a_2, a_4, a_6\}$, and there are no other edges. Let G be a graph. We say a graph H is a *clique blow-up* of G if $V(H)$ is the disjoint union of $|V(G)|$ non-empty and pairwise disjoint cliques $(X_v; v \in G)$ such that for all distinct $u, v \in G$, if $uv \in E(G)$ then X_u is complete to X_v in H , and if $uv \notin E(G)$ then X_u is anticomplete to X_v in H . A partition (V_1, V_2) of the vertex set of a graph G is said to be a *cube partition* if V_1 is a clique blow-up of the cube, V_2 is a clique and V_1 is complete to V_2 .

According to our first result, the following, it turns out that even with only thetas and pyramids excluded, containing a cube results in a structurally simple class of graphs.

Theorem 3.1. *Let G be a (theta, pyramid)-free graph. If G contains a cube, then G admits either a clique cutset or a cube partition.*

We continue with more definitions. Let G be a graph. Let W be a hole in G and $v \in G \setminus W$. A *sector* of (W, v) is a path P of W of length at least one, such that both ends of P are adjacent to v and v is anticomplete to P^* . A sector P is *long* if $P^* \neq \emptyset$. A *wheel* in G is a pair (W, v) where W is a hole of length at least five, v has at least three neighbors in W and (W, v) has at least two long sectors. For $v \in V(G)$ a wheel (W, v) is *optimal* if for every wheel (W', v) in G we have $|N_W(v)| \leq |N_{W'}(v)|$. Let \mathcal{C}^* be the class of all the cube-free graphs in \mathcal{C} . Our second result is the following.

Theorem 3.2. *Let $G \in \mathcal{C}^*$ and let (W, v) be an optimal wheel in G . Then there is no component D of $G \setminus N[v]$ such that $W \subseteq N[D]$.*

Theorem 3.2 is used later to prove Theorem 6.1, which is a key step in our proof. It may also be of independent interest in the study of graphs in \mathcal{C} . We prove Theorems 3.1 and 3.2 in the upcoming two subsections. Let us conclude with a theorem concerning degeneracy. Recall that for an integer $\delta > 0$, a graph G is δ -degenerate if every subgraph of G contains a vertex of degree less than δ . The following is an easy consequence of the main theorem of [12]:

Theorem 3.3. *For every $t \geq 1$, there exists $\delta_t > 0$ such that every (θ, K_t) -free graph is δ_t -degenerate.*

To deduce Theorem 3.3 from [12] observe that theta-free graphs do not contain subdivisions of the complete bipartite graph $K_{2,3}$ as induced subgraphs.

3.1. Cube attachments. Here we prove Theorem 3.1.

Proof of Theorem 3.1. Suppose not. Let S be the largest subset of $V(G)$ admitting a cube partition. Since G contains a cube, say Q , it follows that $V(Q)$ admits a cube partition with $V_1 = V(Q)$ and $V_2 = \emptyset$. This shows that S exists. Since G does not admit a cube partition, it follows that $S \neq V(G)$. Let (V_1, V_2) be a cube partition of S . We may assume that V_1 admits a partition into eight non-empty cliques $A_1, \dots, A_6, B_1, B_2$, such that

- for each $i \in \{1, \dots, 6\}$, A_i is complete to A_{i+1} (where $A_7 = A_1$); and
- B_1 is complete to $A_1 \cup A_3 \cup A_5$ and B_2 is complete to $A_2 \cup A_4 \cup A_6$;

and there no more edges in V_1 .

(1) *Let $w \in G \setminus S$ have two non-adjacent neighbors $x, y \in S$. Then x is at distance two from y in S .*

Suppose not. By symmetry, we may assume that $x \in B_1$ and $y \in B_2$. Since $A_2 \cup A_3 \cup A_5 \cup A_6 \cup B_1 \cup B_2 \cup \{w\}$ contains no theta, w is complete to at least one of A_2, A_3, A_5, A_6 , say A_6 . Also, since $A_1 \cup A_5 \cup A_6 \cup B_1 \cup \{w\}$ contains no theta, w is complete to at least one of A_1, A_5 ; by symmetry, let w be complete to A_5 . Suppose that w has a non-neighbor in one of A_1 and A_4 , say the former. Since $A_1 \cup A_2 \cup A_6 \cup B_1 \cup \{w\}$ contains no theta, w is anticomplete to A_2 . As a result, w is complete to A_4 , as otherwise depending on whether w has a neighbor in A_3 or not, either $A_2 \cup A_3 \cup A_4 \cup B_2 \cup \{w\}$ or $A_2 \cup A_3 \cup A_4 \cup B_1 \cup B_2 \cup \{w\}$ contains a theta, which is impossible. Consequently, since $A_2 \cup A_3 \cup A_4 \cup B_1 \cup B_2 \cup \{w\}$ contains no pyramid, w is complete to A_3 . But then $A_1 \cup A_2 \cup A_3 \cup B_1 \cup B_2 \cup \{w\}$ contains a pyramid, a contradiction. This proves that w is complete to both A_1 and A_4 . Next suppose that w has a non-neighbor in one of A_2 or A_3 , say the former. Since $A_1 \cup A_2 \cup A_3 \cup B_2 \cup \{w\}$ contains no theta, w is anticomplete to A_3 . But then $A_2 \cup A_3 \cup A_4 \cup B_1 \cup B_2 \cup \{w\}$ contains a pyramid, a contradiction. Therefore, w is complete to both A_2 and A_3 . This restores the symmetry between B_1 and B_2 . Finally, if w has a non-neighbor $u \in B_1 \cup V_2$, then $A_1 \cup A_3 \cup A_5 \cup \{u, w\}$ contains a theta. So w is complete to both B_1 and V_2 , and symmetrically to B_2 . In conclusion, w is complete to $V_1 \cup V_2$. But then $(V_1, V_2 \cup \{w\})$ is a cube partition for $S \cup \{w\}$, a contradiction with the choice of S . This proves (1).

(2) *For every $w \in G \setminus S$, $N_S(w)$ is a clique.*

Suppose not. Then by (1) and symmetry, we may assume that w has a neighbor in A_1 and a neighbor in A_3 . It follows from (1) that w is anticomplete to A_4 and A_6 . Also, since $A_3 \cup A_4 \cup A_5 \cup B_2 \cup \{w\}$ contains no theta, w is anticomplete to at least one of A_5, B_2 ; by symmetry, let w be anticomplete to A_5 . Now, since $A_1 \cup \dots \cup A_6 \cup \{w\}$ contains no theta, w is complete to A_2 . In addition, since $A_1 \cup A_3 \cup A_4 \cup A_5 \cup A_6 \cup B_2 \cup \{w\}$ contains no theta, w is complete to B_2 . The latter, along with (1), implies that w is anticomplete to B_1 . Moreover, if w has a non-neighbor in one of A_1 and A_3 , say the former, then $A_1 \cup A_3 \cup A_4 \cup A_6 \cup B_1 \cup B_2 \cup \{w\}$ contains a theta, which is impossible. Thus, w is complete to $A_1 \cup A_3$. Finally, if w has a

non-neighbor $u \in V_2$, then $A_1 \cup A_3 \cup B_2 \cup \{u, w\}$ contains a theta. So w is complete to V_2 . In conclusion, w is complete to A_1, A_2, A_3, B_2 and V_2 and anticomplete to the rest of S . But then adding w to A_2 , $(V_1 \cup \{w\}, V_2)$ is a cube partition for $S \cup \{w\}$, a contradiction with the choice of S . This proves (2).

(3) For every component J of $G \setminus S$, $N_S(J)$ is a clique.

Suppose not. Then we may choose an induced path $R = r_1 \cdots r_l$ of smallest length such that $R \subseteq G \setminus S$, and (having chosen R) non-adjacent vertices $x \in N_S(r_1)$ and $y \in N_S(r_l)$ such that the distance between x and y in S is as small as possible. It follows from (2) that $l \geq 2$. Now, if the distance between x and y in S is three, we may assume, without loss of generality, that $x \in B_1$ and $y \in B_2$. From the choice of R , x and y , and the fact that every vertex in $V_1 \setminus (B_1 \cup B_2)$ is at distance two from either x or y , it follows that that R^* is anticomplete to V_1 , $N_S(r_1) \subseteq B_1$ and $N_S(r_l) \subseteq B_2$. But then $A_2 \cup A_3 \cup A_5 \cup A_6 \cup R \cup \{x, y\}$ contains a theta, which is impossible. Therefore, the distance between x and y in S is two, and so due to symmetry, we may assume that $x \in A_1$ and $y \in A_3$. From the choice of R , it immediately follows that R^* is anticomplete to $A_4 \cup A_5 \cup A_6 \cup B_2$, r_1 is anticomplete to $A_4 \cup A_5 \cup B_2$ and r_l is anticomplete to $A_5 \cup A_6 \cup B_2$. But then $A_1 \cup A_3 \cup A_4 \cup A_5 \cup A_6 \cup B_2 \cup R$ contains a theta, which is impossible. This proves (3).

Since $S \neq V(G)$, $G \setminus S$ has a component J . By (3), $N_S(J)$ is a clique. So $S \setminus N(J) \neq \emptyset$, as S is not a clique. But then $N_S(J)$ is a cutset in G separating J from $S \setminus N(J)$, a contradiction. This concludes the proof of Theorem 3.1. \blacksquare

3.2. Wheel attachments. The goal of this subsection is to prove Theorem 3.2, which falls into several steps. First, we need a couple of definitions. Let G be a graph and (W, v) be a wheel in G . Throughout, we fix a cyclic orientation of W and refer to it as *clockwise*. For all $x, y \in W$, we denote by $W[x, y]$ the subpath of W joining x to y in the clockwise orientation, with the convention that $W[x, x] = \{x\}$. Also, for every $z \in W$, let $z^- \in W$ (resp. $z^+ \in W$) be the vertex which appears immediately before (resp. after) z with respect to the clockwise orientation of W . For every vertex $w \in G \setminus (N[v] \cup W)$, we say w is (W, v) -local if $N_W(w)$ is contained in a sector of (W, v) . Similarly, a component J of $G \setminus N[v]$ is said to be (W, v) -local if $N_W[J]$ is contained in a sector of (W, v) (note that a local component may meet W).

A wheel (W, v) in a graph G is *stranded* if for some $k \geq 2$, there exists a (unique) enumeration a_1, \dots, a_k, b of the vertices in $N_W(v)$ such that the following hold.

- The vertices a_1, \dots, a_k, b appear in this order with respect to the clockwise orientation of W .
- For each $i \in \{1, \dots, k-1\}$, we have $a_{i+1} = a_i^+$.
- Both $W[b, a_1]$ and $W[a_k, b]$ are long sectors of (W, v) .

We refer to (a_1, \dots, a_k, b) as the *contour* of (W, v) . Note that if $k = 2$, then $W \cup \{v\}$ is a pyramid in G . Let us begin with a lemma.

Lemma 3.4. *Let G be a (theta, pyramid, prism)-free graph and (W, v) be an optimal wheel in G . Assume that (W, v) is a stranded wheel with contour (a_1, \dots, a_k, b) . Then for every $w \in G \setminus (N[v] \cup W)$, either w is (W, v) -local or $N_W(w) = \{b^-, b, b^+\}$.*

Proof. Suppose for a contradiction that w is not (W, v) -local and $N_W(w) \neq \{b^-, b, b^+\}$. In this proof, by $N(w)$ we mean $N_W(w)$. Note that since G has no pyramid, we have $k \geq 3$.

(4) The vertex w has a neighbor in $W \setminus \{a_1, \dots, a_k\}$.

Suppose not. Then since w is not (W, v) -local, it has a least two neighbors in $\{a_1, \dots, a_k\}$. Choose $i \in \{1, \dots, k\}$ minimum such that $a_i \in N(w)$, and $j \in \{1, \dots, k\}$ maximum such

that $a_j \in N(w)$. Again, since w is not (W, v) -local, we have $j \geq i + 2$. But then $W' = (W \setminus W[a_{i+1}, a_{j-1}]) \cup \{w\}$ is a hole in G , and (W', v) is a wheel in G with $|N_{W'}(v)| < |N_W(v)|$, a contradiction with the optimality of (W, v) . This proves (4).

(5) *Neither $N(w) \cap W[b^+, a_1^-]$ nor $N(w) \cap W[a_k^+, b^-]$ is empty.*

For otherwise by symmetry we may assume that $N(w) \cap W[b^+, a_1^-] = \emptyset$. If w has no neighbor in $\{a_1, \dots, a_{k-1}\}$, then w is (W, v) -local, a contradiction. So we may choose $i \in \{1, \dots, k-1\}$ minimum such that $a_i \in N(w)$. Also, by (4), w has a neighbor in $W[a_k^+, b]$. Traversing the path $W[a_k^+, b]$ from a_k^+ to b , let z be the last vertex in $N(w) \cap W[a_k^+, b]$. Now, if $i = 1$ (resp. $i = 2$), then $W[z, a_i] \cup \{v, w\}$ is a theta (resp. pyramid) in G , which is impossible. Therefore, we have $i \geq 3$, and so $k \geq 4$. But then $W' = (W \setminus W[a_{i+1}, z^-]) \cup \{w\}$ is a hole in G , and (W', v) is a wheel in G with $|N(v) \cap W'| < |N(v) \cap W|$, a contradiction with the optimality of (W, v) . This proves (5).

(6) *We have $b \in N(w)$.*

Suppose not. By (5), neither $N(w) \cap W[b^+, a_1^-]$ nor $N(w) \cap W[a_k^+, b^-]$ is empty. Traversing the path $W[b^+, a_1^-]$ from b^+ to a_1^- , let x be the first vertex in $N(w) \cap W[b^+, a_1^-]$. Also, traversing the path $W[a_k^+, b^-]$ from a_k^+ to b^- , let z be the last vertex in $N(w) \cap W[a_k^+, b^-]$. If $N(w) \cap W = \{x, z\}$, then $W \cup \{w\}$ is a theta in G , which is impossible. Also, if w is adjacent to a_i for some $i \in \{2, \dots, k-1\}$, then $W[z, x] \cup \{a_i, v, w\}$ is a theta in G , a contradiction. Therefore, w has a neighbor in either $W[a_k, z^-]$ or $W[x^+, a_1]$, say the former. Traversing the path $W[a_k, z^-]$ from a_k to z^- , let y be the first vertex in $N(w) \cap W[a_k, z^-]$. Then depending on whether y is adjacent to z or not, $W[z, x] \cup W[a_k, y] \cup \{v, w\}$ is either a pyramid or a theta in G , a contradiction. This proves (6).

By (5), neither $N(w) \cap W[b^+, a_1^-]$ nor $N(w) \cap W[a_k^+, b^-]$ is empty. Traversing the path $W[b^+, a_1^-]$ from b^+ to a_1^- , let x be the last vertex in $N(w) \cap W[b^+, a_1^-]$. Also, traversing the path $W[a_k, b^-]$ from a_k to b^- , let z be the first vertex in $N(w) \cap W[a_k, b^-]$. Due to symmetry, we may assume that $|W[b, x]| \geq |W[z, b]|$. If $|W[b, x]| \geq 3$, then depending on whether $|W[z, b]| \geq 3$ or not, $W[x, a_1] \cup W[a_k, z] \cup \{b, v, w\}$ is either a theta or a pyramid in G , a contradiction. We deduce that $|W[b, x]| = |W[z, b]| = 2$, from which we have $x = b^+$, $z = b^-$ and $N(w) \cap W[a_k^+, a_1^-] = \{b^-, b, b^+\}$. Furthermore, by the assumption, we have $N(w) \cap W \neq \{b^-, b, b^+\}$. So w has a neighbor in $\{a_1, \dots, a_k\}$. Now, if w is adjacent to both a_1 and a_k , then $\{a_1, a_k, b, v, w\}$ is a theta in G , which is impossible. Therefore, we may assume, without loss of generality, that w is not adjacent to a_1 , and so $a_i \in N(w)$ for some $i \in \{2, \dots, k\}$. But then depending on whether $i = 2$ or not, $W[b, a_1] \cup \{a_i, v, w, x\}$ is either a prism or a pyramid in G , a contradiction. This completes the proof of Lemma 3.4. \blacksquare

From Lemma 3.4, we deduce the following.

Lemma 3.5. *Let G be a (theta, pyramid, prism)-free graph and (W, v) be an optimal wheel in G . Assume that (W, v) is a stranded wheel with contour (a_1, \dots, a_k, b) . Then no component of $G \setminus (N[v] \setminus \{a_2, \dots, a_{k-1}\})$ contains $W \setminus \{a_1, a_k, b\}$.*

Proof. Suppose not. Note that since G has no pyramid, we have $k \geq 3$. Then we may choose an induced path $P = p_1 \cdots p_l$ in G such that the following hold.

(P1) We have $P \subseteq G \setminus (W \cup N[v])$.

(P2) The vertex p_1 has a neighbor in $\{a_2, \dots, a_{k-1}\}$ and p_l has a neighbor in $W[a_k^+, a_1^-] \setminus \{b\}$.

(P3) Subject to (P1) and (P2), $|P|$ is as small as possible.

It follows from Lemma 3.4 that $l \geq 2$, $N_W(p_1) \subseteq \{a_1, \dots, a_k\}$ and either p_l is (W, v) -local or $N_W(p_l) = \{b^-, b, b^+\}$. Also, we deduce:

(7) P^* is anticomplete to $(W \cup \{v\}) \setminus \{a_1, a_k\}$.

Suppose for a contradiction that p_i has a neighbor $q \in (W \cup \{v\}) \setminus \{a_1, a_k\}$ for some $i \in \{2, \dots, l-1\}$. Since $P \cap N[v] = \emptyset$, it follows that $q \neq v$. Then depending on whether $q \in \{a_2, \dots, a_{k-1}\}$ or $q \in W[a_k^+, a_1^-]$, either p_i - P - p_l or p_1 - P - p_i is an induced path in G with fewer vertices than P and satisfying (P1) and (P2), which violates (P3). This proves (7).

(8) We have $N_W(p_l) = \{b^-, b, b^+\}$.

Suppose not. Then p_l is (W, v) -local, and we may assume, without loss of generality, that $N_W(p_l) \subseteq W[a_k, b]$. Note that by (P2), p_l has a neighbor in $W[a_k^+, b]$. So traversing the path $W[a_k^+, b]$ from a_k^+ to b , let z be the last vertex in $N(p_l) \cap W[a_k^+, b]$. Also, note that again by (P2), p_1 has a neighbor in $\{a_2, \dots, a_{k-1}\}$, while p_l is anticomplete to $\{a_1, \dots, a_{k-1}\}$. So let $i \in \{1, \dots, l-1\}$ be maximum such that p_i has a neighbor in $\{a_1, \dots, a_{k-1}\}$, and choose $j \in \{1, \dots, k-1\}$ minimum such that p_i is adjacent to a_j . Now, if $j = 1$ (resp. $j = 2$), then $W[z, a_j] \cup \{p_i, \dots, p_l\} \cup \{v\}$ is a theta (resp. pyramid) in G , which is impossible. Therefore, we have $j \geq 3$, and so $k \geq 4$. But then $W' = (W \setminus W[a_{j+1}, z^-]) \cup \{p_i, \dots, p_l\}$ is a hole in G , and (W', v) is a wheel in G with $|N_{W'}(v)| < |N_W(v)|$, a contradiction with the optimality of (W, v) . This proves (8).

(9) The vertex p_1 is anticomplete to $\{a_1, a_k\}$ and P^* is anticomplete to $W \cup \{v\}$.

Suppose not. Then by (7), some vertex in $P \setminus \{p_l\}$ has a neighbor in $\{a_1, a_k\}$, while P^* is anticomplete to $(W \cup \{v\}) \setminus \{a_1, a_k\}$. Let us choose $i \in \{1, \dots, l-1\}$ maximum such that p_i is adjacent to either a_1 or a_k , say the latter. Also, by (8), we have $N(p_l) \cap W = \{b^-, b, b^+\}$. But then $W[a_k, b] \cup \{p_i, \dots, p_l\} \cup \{a_k, b^-, b, v\}$ is a pyramid in G , which is impossible. This proves (9).

Now, choose $i \in \{2, \dots, k-1\}$ minimum such that p_1 is adjacent to a_i . Then depending on whether $i = 2$ or not, $P \cup W[b, a_1] \cup \{a_i, v\}$ is either a prism or a pyramid in G , which is impossible. This concludes the proof of Lemma 3.5. \blacksquare

Lemma 3.6. *Let $G \in \mathcal{C}^*$ and (W, v) be an optimal wheel in G which is not stranded. Then for every $w \in G \setminus (N[v] \cup W)$, w is (W, v) -local.*

Proof. Suppose not. We deduce:

(10) We have $|N_W(w)| \geq 3$.

For otherwise either w is (W, v) -local or $W \cup \{w\}$ is a theta in G . This proves (10).

(11) The vertex w has a neighbor in $W \setminus N(v)$.

Suppose not. If $N_W(w)$ contains a stable set S of size three, then $S \cup \{v, w\}$ is a theta in G , which is impossible. So $N_W(w)$ has no stable set of size three. Now, if $N_W(w)$ is not connected, then by (10), $N_W(w)$ is either an edge plus an isolated vertex, or an induced two-edge matching. In the former case, $W \cup \{w\}$ is a pyramid, and in the latter case, $W \cup \{w\}$ is a pinched prism, violating the fact that $G \in \mathcal{C}^*$. So $N_W(w)$ is connected, and so since w is not (W, v) -local, $N_W(w)$ is a path on three or four vertices, say $w_1 \cdots w_l$ for $l \in \{3, 4\}$.

But then $W' = (W \setminus \{w_2, \dots, w_{l-1}\}) \cup \{w\}$ is hole in G , and (W', v) is a wheel in G with $|N_{W'}(w)| < |N_W(w)|$, a contradiction with the optimality of (W, v) . This proves (11).

(12) *There do not exist two consecutive sectors of (W, v) whose union contains $N_W(w)$.*

For otherwise there are two consecutive sectors of (W, v) , say $W[a, b]$ and $W[b, c]$, where $N_W(w) \subseteq W[a, c]$. Traversing $W[a, c]$ from a to c , let x be first vertex in $N(w) \cap W[a, c]$ and let z be last vertex in $N(w) \cap W[a, c]$. Note that since w is not (W, v) -local, we have $x \in W[a, b^-]$ and $z \in W[b^+, c]$. Also, since (W, v) is not stranded, $W[c, a]$ contains a long sector of (W, v) . But then $W' = (W \setminus W[x^+, z^-]) \cup \{w\}$ is a hole in G , and (W', v) is wheel in G with $|N_{W'}(v)| < |N_W(v)|$, a contradiction with the optimality of (W, v) . This proves (12).

(13) *Let $W[r, s]$ be a long sector of (W, v) such that w has a neighbor in the interior of $W[r, s]$. Then w has no neighbor in $W \setminus W[r^-, s^+]$.*

Suppose not. Note that v has a neighbor in $W \setminus W[r^-, s^+]$, as otherwise $W \cup \{v\}$ is a pyramid, a theta or a pinched prism, which violates the fact that $G \in \mathcal{C}^*$. Therefore, $(W \setminus W[r^-, s^+]) \cup \{v, w\}$ is connected, and we may choose an induced path L from v to w contained in $(W \setminus W[r^-, s^+]) \cup \{v, w\}$. Also, traversing $W[r, s]$ from r to s , let y be the first vertex in $N(w) \cap W[r, s]$ and z be the last vertex in $N(w) \cap W[r, s]$. Note that since w has a neighbor in $W[r^+, s^-]$, either y and z are distinct or $y = z \in W[r^+, s^-]$. Now, depending on whether y is adjacent to z or not, $W[r, y] \cup W[z, s] \cup L$ is either pyramid or a theta in G , which is impossible. This proves (13).

Note that (11), (11)(12) and (13) together with the fact that (W, v) is a wheel immediately imply the following.

(14) *Let $W[r, s]$ be a long sector of (W, v) such that w has a neighbor in the interior of $W[r, s]$. Then r^- and s^+ are distinct, w is adjacent to both r^- and s^+ , and w has no neighbor in $W \setminus W[r^-, s^+]$.*

Now, by (11), we may choose $a, b \in W$ such that $W[a, b]$ is a long sector of (W, v) and w has a neighbor in $W[a^+, b^-]$. Then by (14) applied to the sector $W[a, b]$, a^- and b^+ are distinct, w is adjacent to both a^- and b^+ , and w has no neighbor in $W \setminus W[a^-, b^+]$. Next we prove:

(15) *v is complete to $\{a^-, b^+\}$.*

Suppose that v is non-adjacent to a^- . Let $c \in W$ be such that $W[c, a]$ is a sector of (W, v) . Then $c \neq b$, for otherwise we get a theta. By (14) applied to $W[c, a]$ we deduce that w is complete to $\{a^+, c^-\}$ and has no other neighbor in $W[a^+, c^-]$. Since w is adjacent to b^+ , it follows that $c^- = b^+$, and $W[b, c]$ is a sector of (W, v) . Now it follows by symmetry that $a^+ = b^-$ and $c^+ = b^-$, and that $N_W(v) = \{a, b, c\}$ and $N_W(w) = W \setminus N(v)$. But now $W \cup \{v, w\}$ is a cube in G , which is impossible. This proves (15).

Let $W' = (W \setminus W[a, b]) \cup \{w\}$. Then W' is a hole and $|N_{W'}(v)| < |N_W(v)|$. By the optimality of (W, v) we deduce that the $N_{W'}(v)$ forms a path in W' . Since by (15) v is complete to $\{a^-, b^+\}$, it follows that v is complete to $W \setminus W[a, b]$. But then $N_W(v)$ forms a path in W , contrary to the fact that (W, v) is a wheel. This completes the proof of Lemma 3.6. ■

Lemma 3.7. *Let $G \in \mathcal{C}^*$ and (W, v) be an optimal wheel in G which is not stranded. Then every component of $G \setminus N[v]$ is (W, v) -local.*

Proof. Suppose for a contradiction that some component J of $G \setminus N[v]$ is not (W, v) -local. In other words, there are two distinct vertices $u_1, u_2 \in N_W(v)$ such that neither $N[J] \cap W[u_1^+, u_2^-]$ nor $N[J] \cap W[u_2^+, u_1^-]$ is empty. From this and Lemma 3.6, it is easily observed that, for fixed

v , the hole W in the non-stranded wheel (W, v) together with an induced path $Q = q_1 \cdots q_l$ in G can be chosen such that the following hold.

- (Q1) We have $l \geq 2$ and $Q \subseteq G \setminus (W \cup N[v])$.
- (Q2) Both q_1 and q_l have neighbors in W , and there are two distinct sectors S and S' of (W, v) such that $N_W(q_1) \subseteq S$ and $N_W(q_l) \subseteq S'$. Also, no sector of (W, v) contains $N_W(q_1) \cup N_W(q_l)$.
- (Q3) Subject to (Q1) and (Q2), $|W| + |Q|$ is as small as possible.
- (Q4) Subject to (Q1), (Q2) and (Q3), a component of $W \setminus (S^* \cup S'^*)$ with the smallest number of vertices has as few vertices as possible.

We deduce:

- (16) *If $S \cap S' \neq \emptyset$, then some vertex in Q^* has a neighbor in $W \setminus (S \cup S')$.*

Suppose not. Then Q is anticomplete to $W \setminus (S \cup S')$. Traversing $S \cup S'$ in the clockwise orientation of W , let y be the first vertex with a neighbor in Q and z be the last vertex with a neighbor in Q . Note that by (Q2), both y and z exist, one of them belongs to $S \setminus S'$ and the other one lies in $S' \setminus S$. Then $Q \cup \{y, z\}$ is connected, and hence contains an induced path R from y to z . Also, since (W, v) is not stranded, $W \setminus (S \cup S')^*$ contains a long sector of (W, v) . Now if $N(v) \setminus (S \cup S') = \emptyset$, then $W[z, y] \cup R \cup \{v\}$ is a theta in G , which is impossible. So v has a neighbor in $W \setminus (S \cup S')$. But then $W' = (W \setminus W[y, z]) \cup R$ is a hole in G , and (W', v) is a wheel in G with $|N_{W'}(v)| < |N_W(v)|$, a contradiction with the optimality of (W, v) . This proves (16).

- (17) *We have $S \cap S' = \emptyset$.*

Suppose not. Assume, without loss of generality, that $S = W[s, c]$ and $S' = W[c, s']$ for some $s, c, s' \in W$, appearing in this order with respect to the clockwise orientation of W . Then by (16), some vertex $q_i \in Q^*$ has a neighbor in $W \setminus (S \cup S')$. Now, if there exist $q_j \in Q$ with a neighbor in $(S \cup S')^*$ (in which case by Lemma 3.6, we have $i \neq j$), then assuming Q' to be the subpath of Q joining q_i to q_j , (W, v) and Q' satisfy both (Q1) and (Q2), yet $|Q'| < |Q|$, a contradiction with (Q3). So Q is anticomplete to $(S \cup S')^*$. As a result, we have $N(q_1) \cap W = \{s\}$ and $N(q_l) \cap W = \{s'\}$. It follows from (Q2) that $W \setminus (S \cup S')^*$ is not a sector of (W, v) , and so v has a neighbor in $W \setminus (S \cup S')$. Now, assume that there are two distinct vertices $x, y \in W \setminus (S \cup S')$ such that $x \in N_W(v)$ and $y \in N_W(Q^*)$, say $y \in W[s', x]$. Then defining Q' to be the subpath of Q from q_1 to q_i , (W, v) and Q' satisfy both (Q1) and (Q2), yet $|Q'| < |Q|$, a contradiction with (Q3). Consequently, there exists a vertex $z \in W \setminus (S \cup S')$ such that $N_{W \setminus (S \cup S')}(v) = N_W(Q^*) = \{z\}$. Now, if $W[z, s]$ is a long sector of (W, v) , then assuming R to be an induced path in $Q \cup \{s, z\}$ from s to z , $R \cup W[z, s] \cup \{v\}$ is a theta in G , which is impossible. So we have $|W[z, s]| = 2$, and similarly $|W[s', z]| = 2$. But then (W, v) is stranded, a contradiction. This proves (17).

- (18) *Both $N_W(q_1)$ and $N_W(q_l)$ are cliques of G , and so $|N_W(q_1)|, |N_W(q_l)| \in \{1, 2\}$.*

Suppose for a contradiction, and by symmetry that q_1 has two non-adjacent neighbors in W , which by (Q2) belong to S . Therefore, traversing S in the clockwise orientation of W , let y and z respectively be the first and the last vertex in $N_S(q_1) = N_W(q_1)$. Note that $W[y^+, z^-] \neq \emptyset$. Now $W' = (W \setminus W[y^+, z^-]) \cup \{q_1\}$ is a hole in G and so (W', v) is an optimal wheel in G , as $|N_{W'}(v)| = |N_W(v)|$. Also, (W, v) and (W', v) have the same number of long sectors. Thus, by Lemma 3.6, q_1 is (W', v) -local, and so $l \geq 3$. But then defining Q' to be the subpath of Q from q_2

to q_l , (W', v) and Q' satisfy both (Q1) and (Q2), yet $|W'| \leq |W|$ and $|Q'| < |Q|$, a contradiction with (Q3). This proves (18).

(19) Q^* is anticomplete to W .

For otherwise there exists a vertex $q_i \in Q^*$ with a neighbor $x \in W$. By (17), either $x \notin S$ or $x \notin S'$, say the former holds. Let $S = W[a, b]$ and $S' = W[c, d]$, with distinct vertices a, b, c, d appearing in this order with respect to the clockwise direction of W . If $x \in S'^*$, then defining Q' to be the subpath of Q from q_1 to q_i , (W, v) and Q' satisfy both (Q1) and (Q2), yet $|Q'| < |Q|$, a contradiction with (Q3). So we may assume, by symmetry, that $x \in W[b^+, c]$. Also, if for some neighbor $y \in W[a, b]$ of q_1 , v has a neighbor in $W[y^+, x^-]$, then again defining Q' to be the subpath of Q from q_1 to q_i , (W, v) and Q' satisfy both (Q1) and (Q2), yet $|Q'| < |Q|$, a contradiction with (Q3). Therefore, we have $N_W(q_1) = \{b\}$ and v has no neighbor in $W[b, x] \setminus \{b, x\}$. But then $x \neq c$, since otherwise we may replace S by the sector $W[b, c]$ of (W, v) , which violates (Q4). Similarly, if q_l has a neighbor in $z \in W[c, d]$ for which v has a neighbor in $W[x^+, z^-]$, then defining Q' to be the subpath of Q from q_i to q_l , (W, v) and Q' satisfy both (Q1) and (Q2), yet $|Q'| < |Q|$, a contradiction with (Q3). Consequently, we have $N_W(q_l) = \{c\}$ and v has no neighbor in $W[x, c] \setminus \{x, c\}$. On the hand, by (Q3), no sector of (W, v) contains $\{b, c\}$. Consequently, v is adjacent to x . But then we can replace S and S' respectively with the sectors $W[b, x]$ and $W[x, c]$ of (W, v) , which contradicts (Q4). This proves (19).

Now, (17) together with (18) and (19) immediately implies that $W \cup Q$ is either a theta, or a pyramid or a prism in G , which violates $G \in \mathcal{C}^*$. This concludes the proof of Lemma 3.7. ■

Now we are ready to prove Theorem 3.2, which we restate:

Theorem 3.2. *Let $G \in \mathcal{C}^*$ and let (W, v) be an optimal wheel in G . Then there is no component D of $G \setminus N[v]$ such that $W \subseteq N[D]$.*

Proof. Suppose not. Let D be a component of $G \setminus N[v]$ where $W \subseteq N[D]$. If (W, v) is stranded with contour (a_1, \dots, a_k, b) for some $k \geq 3$, then $W[b^+, a_1^-] \cup W[a_k^+, b^-] \subseteq D$ and $\{a_2, \dots, a_{k-1}\} \subseteq N(D)$. So there is a component of $G \setminus (N[v] \setminus \{a_2, \dots, a_{k-1}\})$ containing $D \cup \{a_2, \dots, a_{k-1}\}$, which in turn contains $W \setminus \{a_1, a_k, b\}$. But this violates Lemma 3.5. It follows that (W, v) is not stranded. Now, since $W \subseteq N[D]$ and (W, v) has at least three sectors, D is not (W, v) -local, a contradiction with Lemma 3.7. This completes the proof of Theorem 3.2. ■

4. HUB-NEIGHBORS AND SEPARATORS

A vertex v is a *hub* in G if there is a wheel (W, v) in G . If (W, v) is a wheel, we say v is a *hub for W* . We denote by $\text{Hub}(G)$ the set of all hubs of G . We write $\deg_{\text{Hub}(G)}(v) = |N_{\text{Hub}(G)}(v)|$. Let $v \in V(G)$ and let D be a component of $G \setminus N[v]$. The v -closure of D in G is the set $N[D] \cup \{v\}$. We denote the v -closure of D in G by $\text{cl}_{v,G}(D)$. Recall that the Ramsey number $R(t, s)$ is the minimum integer such that every graph on at least $R(t, s)$ vertices contains either a clique of size t or a stable set of size s .

The goal of this section is to show if G is a graph in \mathcal{C} and X is a minimal separator or PMC in G , then the size of the set $X \setminus \text{Hub}(G)$ is small. In later sections we develop tools to control the size of $X \cap \text{Hub}(G)$ for certain sets X .

We need the following result from [3].

Lemma 4.1. *Let x_1, x_2, x_3 be three distinct vertices of a graph G . Assume that H is a connected induced subgraph of $G \setminus \{x_1, x_2, x_3\}$ such that $V(H)$ contains at least one neighbor of each of x_1, x_2, x_3 , and that $V(H)$ is minimal subject to inclusion. Then, one of the following holds:*

- (i) *For some distinct $i, j, k \in \{1, 2, 3\}$, there exists P that is either a path from x_i to x_j or a hole containing the edge $x_i x_j$ such that*

- $V(H) = V(P) \setminus \{x_i, x_j\}$; and
 - either x_k has two non-adjacent neighbors in H or x_k has exactly two neighbors in H and its neighbors in H are adjacent.
- (ii) There exists a vertex $a \in V(H)$ and three paths P_1, P_2, P_3 , where P_i is from a to x_i , such that
- $V(H) = (V(P_1) \cup V(P_2) \cup V(P_3)) \setminus \{x_1, x_2, x_3\}$;
 - the sets $V(P_1) \setminus \{a\}$, $V(P_2) \setminus \{a\}$ and $V(P_3) \setminus \{a\}$ are pairwise disjoint; and
 - for distinct $i, j \in \{1, 2, 3\}$, there are no edges between $V(P_i) \setminus \{a\}$ and $V(P_j) \setminus \{a\}$, except possibly $x_i x_j$.
- (iii) There exists a triangle $a_1 a_2 a_3$ in H and three paths P_1, P_2, P_3 , where P_i is from a_i to x_i , such that
- $V(H) = (V(P_1) \cup V(P_2) \cup V(P_3)) \setminus \{x_1, x_2, x_3\}$;
 - the sets $V(P_1)$, $V(P_2)$ and $V(P_3)$ are pairwise disjoint; and
 - for distinct $i, j \in \{1, 2, 3\}$, there are no edges between $V(P_i)$ and $V(P_j)$, except $a_i a_j$ and possibly $x_i x_j$.

We use Lemma 4.1 in order to prove the following.

Theorem 4.2. *Let $G \in \mathcal{C}_t$, let X be a minimal separator of G , and let $Y \subseteq X \setminus \text{Hub}(G)$ be stable. Then $|Y| \leq 2$. Consequently, $|X \setminus \text{Hub}(G)| \leq R(t, 3)$.*

Proof. We first prove the first assertion of the theorem. Suppose $|Y| \geq 3$. Let $x_1, x_2, x_3 \in Y$. Let D_1, D_2 be distinct full components for X . For $i \in 1, 2$, let Z_i be a minimal connected subgraph of D_i containing neighbors of x_1, x_2, x_3 . We apply Lemma 4.1 to x_1, x_2, x_3 and Z_1, Z_2 . If outcome (ii) holds for both Z_1 and Z_2 , then G contains a theta, a contradiction. If outcome (ii) holds for Z_1 and outcome (iii) holds for Z_2 (or vice versa), then G contains a pyramid, a contradiction. If outcome (iii) holds for both Z_1 and Z_2 , then G contains a prism, a contradiction. Therefore, we may assume that outcome (i) holds for Z_1 , and that Z_1 is a path from x_1 to x_2 , where x_3 has at least two neighbors in Z_1 . Since x_3 is not a hub in G and G does not contain a pinched prism or a pyramid, it follows that:

- (20) *For every path P_2 from x_1 to x_2 with $P_2^* \subseteq D_2$, we have that x_3 is anticomplete to P_2 .*

Suppose outcome (ii) holds for Z_2 , so there exists a vertex $a \in Z_2$ and three paths P_1, P_2, P_3 in Z_2 from a to x_1, x_2, x_3 as in (ii) of Lemma 4.1. By (20), it follows that x_3 is not adjacent to a . If x_3 has two non-adjacent neighbors in Z_1 , then G contains a theta between x_3 and a , a contradiction. Thus, x_3 has exactly two adjacent neighbors in Z_1 , but now G contains a pyramid with apex a and base $x_3 \cup N_{Z_1}(x_3)$, a contradiction. Therefore, outcome (ii) does not hold for Z_2 .

Next, suppose outcome (iii) holds for Z_2 , so there exists a triangle $a_1 a_2 a_3$ in Z_2 and three paths P_1, P_2, P_3 as in (iii) of Lemma 4.1. If x_3 has two non-adjacent neighbors in Z_1 , then G contains a pyramid with apex x_3 and base $a_1 a_2 a_3$, a contradiction. Thus, x_3 has exactly two adjacent neighbors in Z_1 , but now G contains a prism with triangles $a_1 a_2 a_3$ and $x_3 \cup N_{Z_1}(x_3)$, a contradiction.

Consequently, outcome (i) holds for Z_2 . By (20), Z_2 is not a path from x_1 to x_2 . Thus, we may assume that Z_2 is a path from x_2 to x_3 , and that x_1 has at least two neighbors in Z_2 . If x_1 has two non-adjacent neighbors in Z_2 and x_3 has two non-adjacent neighbors in Z_1 , then G contains a theta between x_1 and x_3 , a contradiction. If x_1 has two non-adjacent neighbors in Z_2 and x_3 has exactly two adjacent neighbors in Z_1 , then G contains a pyramid with apex x_1 and base $x_3 \cup N_{Z_1}(x_3)$, a contradiction. If x_1 has exactly two adjacent neighbors in Z_2 and x_3 has exactly two adjacent neighbors in Z_1 , then G contains a prism with triangles $x_3 \cup N_{Z_1}(x_3)$ and $x_1 \cup N_{Z_2}(x_1)$, a contradiction. This proves the first assertion of the theorem. The second assertion follows immediately since G is K_t -free. This proves Theorem 4.2. \blacksquare

Theorem 4.2 has the following corollary that we will need later.

Corollary 4.3. *Let $G \in \mathcal{C}_t$, let $v \in V(G)$ and let D be a component of $G \setminus N[v]$. Let $Y \subseteq N(D) \setminus \text{Hub}(G)$ be stable. Then, $|Y| \leq 2$. Consequently, $|N(D) \setminus \text{Hub}(G)| \leq R(t, 3)$ and $|N(D)| < R(t, 3) + \deg_{\text{Hub}(\text{cl}_{v,G}(D))}(v) \leq R(t, 3) + \deg_{\text{Hub}(G)}(v)$.*

Proof. Observe that $N(D)$ is a minimal separator in $\text{cl}_{v,G}(D)$, so it follows from Theorem 4.2 that $|Y| \leq 2$ and $|N(D) \setminus \text{Hub}(\text{cl}_{v,G}(D))| < R(t, 3)$. We also have that $N(D) \cap \text{Hub}(\text{cl}_{v,G}(D)) \subseteq N_{\text{Hub}(\text{cl}_{v,G}(D))}(v) \subseteq \text{Hub}(G)$. Therefore, $|N(D) \setminus \text{Hub}(G)| < R(t, 3)$ and $|N(D)| < R(t, 3) + \deg_{\text{Hub}(\text{cl}_{v,G}(D))}(v) \leq R(t, 3) + \deg_{\text{Hub}(G)}(v)$. ■

We finish this section by proving an analogue of Theorem 4.2 for PMCs instead of minimal separators.

Theorem 4.4. *Let $G \in \mathcal{C}_t$, and let X be a PMC of G . Let $Y \subseteq X \setminus \text{Hub}(G)$ be stable. Then $|Y| \leq 3$. Consequently, $|X \setminus \text{Hub}(G)| \leq R(t, 4)$.*

Proof. Suppose $|Y| \geq 4$, let $y_1, y_2, y_3, y_4 \in Y$. By Theorem 2.1 for every $i \neq j \in \{1, \dots, 4\}$ there exists a component D_{ij} of $G \setminus X$ such that $y_i, y_j \in N(D_{ij})$; let P_{ij} be a path from y_i to y_j with interior in D_{ij} . Recall that by Theorem 2.2 all $N(D_{ij})$ are minimal separators, and therefore by Theorem 4.2 $N(D_{ij}) \cap Y = \{y_i, y_j\}$. But now $\{y_1, y_2, y_3, y_4\} \cup P_{12} \cup P_{23} \cup P_{13} \cup P_{14} \cup P_{24}$ is a theta in G , a contradiction. This proves the first assertion of the theorem. The second assertion follows from the fact that G is K_t -free. ■

5. EXTENDING TREE DECOMPOSITIONS OF NEIGHBORHOODS

Next we prove a result that allows us to extend a tree decomposition of the neighborhood of a vertex of G to a tree decomposition of G .

Let $v \in V(G)$, and let D_1, \dots, D_m be the components of $G \setminus N[v]$. By Corollary 4.3, for every $i \in \{1, \dots, m\}$, it holds that $|N(D_i) \setminus \text{Hub}(G)| \leq R(t, 3)$. Let H be the graph obtained from $N(v) \setminus \text{Hub}(G)$ by adding a stable set of new vertices $\{d_1, \dots, d_m\}$ where $N_H(d_i) = N(D_i) \setminus \text{Hub}(G)$ for every i . In other words, H is obtained from $G \setminus N_{\text{Hub}(G)}[v]$ by contracting each D_i to a corresponding kk vertex d_i .

Throughout this section, we fix G and H as above. We first prove a structural result about H .

Theorem 5.1. *If $G \in \mathcal{C}_t$, then $H \cup \{v\} \in \mathcal{C}_t$ and H is wheel-free.*

Proof. For $I \subseteq \{1, \dots, m\}$ let G_I be obtained from $G \setminus N_{\text{Hub}(G)}[v]$ by contracting each D_i with $i \in I$ to a vertex d_i . Then $H = G_{\{1, \dots, m\}}$ and $G \setminus \{v\} = G_\emptyset$.

We prove by induction on $|I|$ that $G_I \cup \{v\} \in \mathcal{C}_t$ and that $\text{Hub}(G_I \cup \{v\}) \cap (N(v) \cup \bigcup_{i \in I} \{d_i\}) = \emptyset$.

Since $N_{G_I}(d_i) \subseteq N_G(v)$ for every $i \in I$, it follows that $\omega(G_I \cup \{v\}) \leq w(G) < t$.

Now suppose $G_I \cup \{v\}$ contains a generalized prism, theta, or pyramid Q . We may assume that $1 \in I$ and $d_1 \in Q$. It follows that d_1 has at least two neighbors. Let $I' = I \setminus \{1\}$. Inductively, $G_{I'} \cup \{v\} \in \mathcal{C}_t$. If d_1 has exactly two neighbors, say a and b , in Q , then we can replace d_1 by a path from a to b with interior in D_1 to get a configuration of the same type as Q in $G_{I'} \cup \{v\}$, a contradiction. Thus, d_1 has at least three neighbors in Q . Note that inductively, $N(v) \cap \text{Hub}(G_{I'} \cup \{v\}) = \emptyset$, so by Corollary 4.3, $N(d_1)$ does not contain a stable set of size three. Consequently, one of the following holds (with notation as in the definitions of the corresponding graphs):

- (1) Q is a pyramid or a prism and $d_1 = b_1$, or Q is a pinched prism and $d_1 \in N(b_1) \cap Q$; or
- (2) Q is a pinched prism with center d_1 .

(21) *The first alternative does not hold.*

Suppose the first alternative holds, and so $N(d_1) \cap Q = \{x_1, x_2, x_3\}$, where x_2 is adjacent to x_3 , and there are no other edges with both ends in $\{x_1, x_2, x_3\}$. Let Z be a minimal connected subgraph of D_1 containing a neighbor of each of x_1, x_2, x_3 . Then $Q' = (Q \setminus \{d_1\}) \cup Z$ is an induced subgraph of $G_I \cup \{v\}$, and therefore Q' is not a generalized prism, a pyramid or a theta.

We apply Lemma 4.1 to x_1, x_2, x_3 and Z . Suppose outcome (i) of Lemma 4.1 holds. Suppose first that P is a hole containing the edge x_2x_3 . If x_1 has two non-adjacent neighbors in P , then $\{v, x_1\} \cup P$ contains a pyramid with apex x_1 and base vx_2x_3 , a contradiction. So x_1 has exactly two adjacent neighbors in P , and now $\{v, x_1\} \cup P$ is a prism with triangles $x_1 \cup (N(x_1) \cap P)$ and vx_2x_3 , a contradiction. Thus, we may assume that P is a path from x_1 to x_2 . Suppose first that Q is a pyramid with apex a' . Since x_3 has at least two neighbors in $P \setminus \{x_2\}$, it follows that x_3 has two non-adjacent neighbors in P . Therefore, x_3 has a neighbor in P non-adjacent to x_2 . If x_3 is non-adjacent to a' , then Q' is a theta from x_3 to a' , a contradiction, so x_3 is adjacent to a' . Now, x_3 is a hub for the hole given by $Q' \setminus x_3$, a contradiction. Therefore, Q is a prism or a pinched prism. Let $x'_1x'_2x'_3$ be the triangle of Q not containing d_1 , where possibly $x_3 = x'_3$ or $x_2 = x'_2$, and Q contains paths from x_i to x'_i for $1 \leq i \leq 3$. If $x_3 = x'_3$, then x_3 is a hub for the hole given by $Q' \setminus \{x_3\}$, a contradiction, so $x_3 \neq x'_3$. But now Q' contains a pyramid from x_3 to $x'_1x'_2x'_3$, a contradiction. Consequently, outcome (i) of Lemma 4.1 does not hold.

Next, suppose outcome (ii) of Lemma 4.1 holds, so there exists a vertex $a \in Z$ and three paths P_1, P_2, P_3 from a to x_1, x_2, x_3 , respectively, such that $Z = P_1 \cup P_2 \cup P_3 \setminus \{x_1, x_2, x_3\}$ and $P_1 \setminus \{a\}$, $P_2 \setminus \{a\}$, and $P_3 \setminus \{a\}$ are pairwise disjoint and anticomplete to each other, except for the edge x_2x_3 . We first consider the case in which both P_2 and P_3 have length one. Now Q' is isomorphic to the graph obtained from Q by subdividing the edge d_1x_1 , which is not in a triangle of Q . This implies that if Q is a pyramid, prism, or pinched prism, then so is Q' , a contradiction. It follows that at least one of P_2, P_3 has length more than one, so $Z \cup \{v, x_1, x_2, x_3\}$ is a pyramid, a contradiction. Consequently, outcome (ii) of Lemma 4.1 does not hold.

It follows that outcome (iii) of Lemma 4.1 holds, so there exists a triangle $a_1a_2a_3 \in Z$ and paths P_1, P_2, P_3 as in outcome (iii) of Lemma 4.1. But now $\{v, x_1, x_2, x_3\} \cup Z$ is a prism with triangles vx_1x_2 and $a_1a_2a_3$, a contradiction. Thus, the first alternative does not hold. This proves (21).

Next we prove:

(22) *The second alternative does not hold.*

Let Q be a pinched prism with $d_1 = b_1$. Let p, q, r, s be the neighbors of d_1 in Q , with p adjacent to q and r adjacent to s . Let $z \in D_1$. If z has two non-adjacent neighbors in $\{p, q, r, s\}$, then $(Q \setminus \{d_1\}) \cup \{z\}$ is a pinched prism, pyramid, or theta in $G_I \cup \{v\}$ (depending on whether z has four, three, or two neighbors in $\{p, q, r, s\}$), a contradiction. Therefore, $N(z) \cap \{p, q, r, s\}$ is either a subset of $\{p, q\}$ or a subset of $\{r, s\}$ for all $z \in D_1$.

Now let R be a shortest path in D_1 with ends u, w , say, such that u has a neighbor in $\{p, q\}$ and w has a neighbor in $\{r, s\}$. Then no vertex in R^* has a neighbor in $\{p, q, r, s\}$; consequently $(Q \setminus \{d_1\}) \cup R$ is a prism, pyramid, or theta in $G_I \cup \{v\}$ (depending on whether $|N(\{u, w\}) \cap \{p, q, r, s\}|$ equals four, three, or two), a contradiction. This proves (22).

Therefore, $G_I \cup \{v\}$ does not contain a generalized prism, theta, or pyramid. This proves that $G_I \cup \{v\} \in \mathcal{C}_t$. By Corollary 4.3, $N(d_i)$ does not contain a stable set of size three, so it follows that $d_i \notin \text{Hub}(G_I \cup \{v\})$ for all $i \in I$. It remains to show that $N(v) \cap \text{Hub}(G_I \cup \{v\}) = \emptyset$.

Suppose (W, x) is a wheel in $G_I \cup \{v\}$, where $x \in N(v)$. Inductively, $d_1 \in W$. Let r, s be the neighbors of d_1 in W , and let W' be the hole obtained from W by replacing d_1 by a path P from r to s with interior in D_1 . If d_1 is non-adjacent to x , then (W', x) is a wheel in $G_{I'}$, a contradiction. This proves that d_1 is adjacent to x . Since $\{x, r, s\}$ is not a stable set in $N(D_1)$, we may assume that x is adjacent to r . This implies that if two vertices of $W \setminus N(x)$ belong to different sectors of (W, x) , then they also belong to different sectors of (W', x) . Now it follows that (W', x) is a wheel in $G_{I'}$, a contradiction. This proves that $N(v) \cap \text{Hub}(G_I \cup \{v\}) = \emptyset$, and completes the proof of Theorem 5.1. \blacksquare

Let (T_0, χ_0) be a tree decomposition of H , and for $i \in \{1, \dots, m\}$, let (T_i, χ_i) be a tree decomposition of D_i . For $i \in \{1, \dots, m\}$ let $v(i) \in T_i$ be some vertex such that $d_i \in \chi_0(v(i))$. Let T be a tree obtained from the union of T_0, T_1, \dots, T_m by adding, for every $i > 0$, a unique edge from some vertex of T_i to $v(i)$. For $u \in T$, let $\chi(u)$ be defined as follows.

- If $u \in V(T_0)$, let

$$\chi(u) = (\chi_0(u) \setminus \{d_1, \dots, d_m\}) \cup N_{\text{Hub}(G)}(v) \cup \{v\} \cup \bigcup_{d_i \in \chi_0(u)} N_H(d_i).$$

- If $u \in V(T_i)$ for $i \in \{1, \dots, m\}$, let

$$\chi(u) = \chi_i(u) \cup N_H(d_i) \cup N_{\text{Hub}(G)}(v).$$

We prove the following:

Theorem 5.2. *With the notation as above, (T, χ) is a tree decomposition of G .*

Proof. Since T is obtained by adding a single edge from T_0 to each of the trees T_1, \dots, T_m , it follows that T is a tree. Clearly every vertex of G is in $\chi(u)$ for some $u \in V(T)$. Next we check that for every edge xy of G there exists $u \in V(T)$ such that $x, y \in \chi(u)$. This is clear if $x, y \in N[v]$, and if $x, y \in D_i$. Thus we may assume that $x \in N(v)$ and $y \in D_1$, say. Now $x \in N(D_1) \cup N_{\text{Hub}(G)}(v)$, and therefore $x \in \chi(u)$ for every $u \in V(T_1)$. Let $u \in V(T_1)$ such that $y \in \chi_i(u) \subseteq \chi(u)$; then $x, y \in \chi(u)$ as required.

Finally we show that $\chi^{-1}(x)$ is a tree for every $x \in V(G)$. Let $x \in V(G)$. For $i \in \{0, \dots, m\}$ define $F_i(x) = \{u \in V(T_i) \text{ such that } x \in \chi(u)\}$. Let $F(x) = F_0(x) \cup \bigcup_{i=1}^m F_i(x)$. We need to show that for every $x \in X$ we have that $T[F(x)]$ is connected. If $x \in N_{\text{Hub}(G)}(v)$, then $F(x) = V(T)$, and the claim holds. If $x = v$, then $F(x) = V(T_0)$, and again the claim holds. Thus we may assume that $x \in V(H) \cup \bigcup_{i=1}^m D_i$. If $F_0(x) = \emptyset$, then there exists a unique i such that $x \in D_i$; therefore $F(x) = F_i(x)$, and $T[F(x)]$ is connected because $T_i[F_i(x)]$ is connected. Thus we may assume that $x \in V(H) \cap N(v)$. Let $I \subseteq \{1, \dots, m\}$ be the set of all i such that $x \in N(D_i) = N(d_i)$. It follows that $F_i(x) = \emptyset$ for all $i \in \{1, \dots, m\} \setminus I$.

First we observe that $x \in \chi(v(i))$ for every $i \in I$, and so there is an edge between $T_i[F_i(x)] = T_i$ and $T_0[F_0(x)]$. Since $F_i(x) = V(T_i)$ is connected, it is enough to prove that $T_0[F_0(x)]$ is connected.

Observe that

$$F_0(x) = \chi_0^{-1}(x) \cup \bigcup_{i \in I} \chi_0^{-1}(d_i).$$

Since $xd_i \in E(H)$ for every $i \in I$, it follows that $\chi_0^{-1}(x) \cap \chi_0^{-1}(d_i) \neq \emptyset$ for every $i \in I$. Since $T_0[\chi_0^{-1}(u)]$ is a tree for every $u \in V(H)$, it follows that $T_0[F_0(x)]$ is connected, as required. This proves Theorem 5.2. \blacksquare

6. STABLE SETS OF UNBALANCED HUBS

Let G be a graph with $|V(G)| = n$. We say that $v \in V(G)$ is *balanced* if $|D| \leq \frac{n}{2}$ for every component D of $G \setminus N[v]$. If v is not balanced, then v is *unbalanced*.

In this section, we describe a way to decompose graphs in \mathcal{C}_t using stable sets of unbalanced hubs. Recall that \mathcal{C} is the class of (theta, pyramid, generalized prism)-free graphs. We begin with the following structural result.

Lemma 6.1. *Let G be a cube-free graph in \mathcal{C} , and suppose that v is a hub of G . Let D be a component of $G \setminus N[v]$. Then v is not a hub in $\text{cl}_{v,G}(D)$.*

Proof. Suppose v is a hub in $\text{cl}_{v,G}(D)$. Now, Theorem 3.2 applied to an optimal wheel with hub v in $\text{cl}_{v,G}(D)$ implies that either $\text{cl}_{v,G} \setminus N[v]$ is not connected, or some vertex of $N(v) \cap \text{cl}_{v,G}(v)$ has no neighbor in $\text{cl}_{v,G} \setminus N[v]$. But $\text{cl}_{v,G} \setminus N[v] = D$, and $N(v) \cap \text{cl}_{v,G}(v) = N(D)$ a contradiction. This proves Lemma 6.1. \blacksquare

Let G be an n -vertex graph, and let v be an unbalanced vertex of G . Then there exists a unique component D of $G \setminus N[v]$ with $|D| > \frac{n}{2}$. Write $B(v) = D$, $C(v) = \{v\} \cup N(D)$ and $A(v) = V(G) \setminus (B(v) \cup C(v))$. We call $(A(v), C(v), B(v))$ the *canonical star separation corresponding to v* . (Note that at this point we do not require that $A(v)$ is non-empty.)

Following [3], we say that two unbalanced vertices $u, v \in V(G)$ are *star twins* if $B(u) = B(v)$, $C(u) \setminus \{u\} = C(v) \setminus \{v\}$, and $A(u) \cup \{u\} = A(v) \cup \{v\}$. Note that every two of these conditions imply the third.

Let S be a stable set of unbalanced vertices. Let \mathcal{O} be a fixed total ordering of S . Let \leq_A be a relation on S defined as follows (we think of \leq_A as a ‘‘partial order by the A -sides’’):

$$x \leq_A y \quad \text{if} \quad \begin{cases} x = y, \text{ or} \\ x \text{ and } y \text{ are star twins and } \mathcal{O}(x) < \mathcal{O}(y), \text{ or} \\ x \text{ and } y \text{ are not star twins and } y \in A(x). \end{cases}$$

Note that if $x \leq_A y$, then either $x = y$, or $y \in A(x)$.

We start with the following three results:

Lemma 6.2. *If $y \in A(x)$, then $A(y) \cup \{y\} \subseteq A(x) \cup \{x\}$.*

Proof. Let $n = |V(G)|$. Since $C(y) \subseteq N[y]$ and y is anticomplete to $B(x)$, we have $B(x) \subseteq G \setminus N[y]$. Since $|B(x)|, |B(y)| > n/2$, it follows that $B(x) \cap B(y) \neq \emptyset$. Since $B(x)$ is connected and contains no vertex in $N(B(y)) \subseteq N(y)$, it follows that $B(x) \subseteq B(y)$. Let $a \in C(x) \setminus \{x\}$. Then a has a neighbor in $B(x)$ and thus in $B(y)$. If $a \in N[y]$, then $a \in C(y)$. If $a \notin N[y]$, then $a \in B(y)$. It follows that $C(x) \setminus \{x\} \subseteq C(y) \cup B(y)$. But now $A(y) \setminus \{x\} \subseteq A(x)$, as required. This proves Lemma 6.2. \blacksquare

Lemma 6.3. *\leq_A is a partial order on S .*

Proof. We first prove the following claim:

(23) *If x and y satisfy $A(y) \cup \{y\} = A(x) \cup \{x\}$, then x and y are star twins.*

Since $x \in A(y)$, it follows that $N[x] \subseteq A(y) \cup C(y)$; so $C(x) \setminus \{x\} = N(x) \cap C(x) = N(x) \setminus A(x) = N(x) \setminus A(y) = N(x) \cap C(y) \subseteq C(y) \setminus \{y\}$ (where we used that $y \notin N(x)$), which shows that $C(x) \setminus \{x\} \subseteq C(y) \setminus \{y\}$; by symmetry, it follows that $C(x) \setminus \{x\} = C(y) \setminus \{y\}$. This proves (23).

We show that \leq_A is reflexive, antisymmetric, and transitive. By definition, \leq_A is reflexive. Suppose $x \leq_A y$ and $y \leq_A x$ for some $x, y \in S$ with $x \neq y$. Since $x \leq_A y$, it follows that $y \in A(x)$, and since $y \leq_A x$, it follows that $x \in A(y)$. From Lemma 6.2, it follows that

$A(y) \cup \{y\} = A(x) \cup \{x\}$. Thus, by (23), x and y are star twins. But $x \leq_A y$ implies that $\mathcal{O}(x) < \mathcal{O}(y)$, and $y \leq_A x$ implies that $\mathcal{O}(y) < \mathcal{O}(x)$, a contradiction. Therefore, \leq_A is antisymmetric.

Next we prove transitivity. Suppose $x \leq_A y$ and $y \leq_A z$ for distinct $x, y, z \in V(G)$. We need to show that $x \leq_A z$. Since $y \leq_A z$, it follows that $z \in A(y)$. Since $x \leq_A y$, it follows that $y \in A(x)$. By Lemma 6.2, it follows that $A(z) \cup \{z\} \subseteq A(y) \cup \{y\} \subseteq A(x) \cup \{x\}$; so $z \in A(x)$. If x and z are not star twins, then $x \leq_A z$, so we may assume that x and z are star twins.

Since x and z are star twins, $A(x) \cup \{x\} = A(z) \cup \{z\}$. But now $A(z) \cup \{z\} = A(y) \cup \{y\} = A(x) \cup \{x\}$, so x, y, z are all pairwise star twins by (23). It follows that $\mathcal{O}(x) < \mathcal{O}(y) < \mathcal{O}(z)$, so $x \leq_A z$, as required. This proves Lemma 6.3. ■

Let G be a graph, and let $S \subseteq V(G)$ be a stable set of unbalanced vertices of G . Let $\text{Core}(S)$ be the set of all \leq_A -minimal elements of S .

Lemma 6.4. *Let S be as above, and let $u, v \in \text{Core}(S)$. Then $A(u) \cap C(v) = C(u) \cap A(v) = \emptyset$.*

Proof. Suppose $x \in A(v) \cap C(u)$. Since $x \in C(u)$, it follows that $u \in A(v) \cup C(v)$. Since u is non-adjacent to v , we have that $u \notin C(v)$, and therefore $u \in A(v)$. Since $u, v \in \text{Core}(S)$, we have that $v \not\leq_A u$, and therefore u and v are star twins. But then again u and v are comparable in \leq_A since they are comparable in \mathcal{O} , a contradiction. This proves Lemma 6.4. ■

Define $\beta(S) = \bigcap_{v \in \text{Core}(S)} (B(v) \cup C(v))$. We call $\beta(S)$ the *central bag* for S . Next, we prove several properties of central bags.

Theorem 6.5. *Let G, S be as above with $G \in \mathcal{C}_t$. The following hold:*

- (1) *For every $v \in \text{Core}(S)$ we have $C(v) \subseteq \beta(S)$.*
- (2) *For every $v \in \text{Core}(S)$ we have that $\deg_{\beta(S)}(v) < R(t, 3) + \deg_{\text{Hub}(G)}(v)$.*
- (3) *For every component D of $G \setminus \beta(S)$, there exists $v \in \text{Core}(S)$ such that $D \subseteq A(v)$. Further, if D is a component of $G \setminus \beta(S)$ and $v \in \text{Core}(S)$ such that $D \subseteq A(v)$, then $N(D) \subseteq C(v)$.*
- (4) *If G is cube-free, then $S \cap \text{Hub}(\beta(S)) = \emptyset$.*

Proof. (1) is immediate from Lemma 6.4; and (2) follows immediately from Corollary 4.3.

Next we prove (3). Let D be a component of $G \setminus \beta(S)$. Since $G \setminus \beta(S) = \bigcup_{v \in \text{Core}(S)} A(v)$, there exists $v \in \text{Core}(S)$ such that $D \cap A(v) \neq \emptyset$. If $D \setminus A(v) \neq \emptyset$, then, since D is connected, it follows that $D \cap N(A(v)) \neq \emptyset$; but then $D \cap C(v) \neq \emptyset$, contrary to (1). Since $N(D) \subseteq \beta(S)$ and $N(D) \subseteq A(v) \cup C(v)$, it follows that $N(D) \subseteq C(v)$. This proves (3).

To prove (4), let $u \in S \cap \text{Hub}(\beta(S))$. Then, by Lemma 6.1, it follows that $\beta(S) \not\subseteq \text{cl}_{u,G}(B(u))$. Since $\text{cl}_{u,G}(B(u)) = B(u) \cup C(u)$, it follows that $u \notin \text{Core}(S)$. But then $u \in A(v)$ for some $v \in \text{Core}(S)$, and so $u \notin \beta(S)$, a contradiction. This proves (4) and completes the proof of Theorem 6.5. ■

We finish this section with a theorem that allows us to transform a tree decomposition of $\beta(S)$ into a tree decomposition of G .

Let G, S be as above with $G \in \mathcal{C}_t$ connected and cube-free, and let D_1, \dots, D_m be the components of $G \setminus \beta(S)$. For $i \in \{1, \dots, m\}$ let $r(D_i)$ be the \mathcal{O} -minimal vertex of S such that $D_i \subseteq A(v)$ (such v exists by Theorem 6.5(3)).

Let (T_β, χ_β) be a tree decomposition of $\beta(S)$, and for $i \in \{1, \dots, m\}$ let (T_i, χ_i) be a tree decomposition of D_i . Let T be the tree obtained from the union of T_β, T_1, \dots, T_m by adding, for each $i \in \{1, \dots, m\}$ a unique edge from some vertex of T_i to some vertex $v \in V(T_\beta)$ such that $r(D_i) \in \chi_\beta(v)$. For $u \in T$, let $\chi(u)$ be defined as follows.

- If $u \in V(T_\beta)$, let

$$\chi(u) = \chi_\beta(u) \cup \bigcup_{v \in \text{Core}(S) \cap \chi_\beta(u)} C(v).$$

- If $u \in V(T_i)$ for $i \in \{1, \dots, m\}$, let

$$\chi(u) = \chi_i(u) \cup C(r(D_i)).$$

Theorem 6.6. *With the notation as above, (T, χ) is a tree decomposition of G .*

Proof. Since T is obtained by adding a single edge from T_β to each of the trees T_1, \dots, T_m , it follows that T is a tree. Clearly, every vertex of G is in $\chi(v)$ for some $v \in V(T)$. Next we check that for every edge xy of G there exists $v \in V(T)$ such that $x, y \in \chi(v)$. This is clear if $x, y \in \beta(S)$ or if $x, y \in D_i$ for some i ; thus we may assume that $x \in \beta(S)$ and $y \in D_1$, say. Then by Theorem 6.5(3), $x \in C(r(D_1))$, and therefore $x \in \chi(v)$ for every $v \in V(T_i)$. Let $v \in V(T_i)$ such that $y \in \chi_i(v) \subseteq \chi(v)$; then $x, y \in \chi(v)$ as required.

Let $x \in V(G)$. Define $F_\beta(x) = \{v \in V(T_\beta) \text{ such that } x \in \chi(v)\}$, and for $i \in \{1, \dots, m\}$ define $F_i(x) = \{v \in V(T_i) \text{ such that } x \in \chi(v)\}$. Let $F(x) = F_\beta(x) \cup \bigcup_{i=1}^m F_i(x)$. We need to show that for every $x \in X$ we have that $T[F(x)]$ is connected. If $x \notin \beta(S)$, then $F_\beta(x) = \emptyset$, and there exists a unique i such that $x \in D_i$; therefore $F(x) = F_i(x)$, and $T[F(x)]$ is connected because $T_i[F_i(x)]$ is connected. Thus we may assume that $x \in \beta(S)$. Let $I \subseteq \{1, \dots, m\}$ be the set of all i such that $x \in C(r(D_i))$. It follows that $F_i(x) = \emptyset$ for all $i \in \{1, \dots, m\} \setminus I$.

First we show that for every $i \in I$, there is an edge between $T_i[F_i(x)] = T_i$ and $T_\beta[F_\beta(x)]$. Let $i \in I$; and let $v \in T_\beta$ be such that v is adjacent to a vertex of T_i . Then $r(D_i) \in \chi_\beta(v)$, so $C(r(D_i)) \subseteq \chi(v)$, and therefore, $x \in \chi(v)$. Thus there is an edge between $T_i[F_i(x)]$ and $T_\beta[F_\beta(x)]$, as required.

Now to show that $T[F(x)]$ is connected, it is enough to prove that $T_\beta[F_\beta(x)]$ is connected. Let $J(x) = \{v \in \text{Core}(S) \text{ such that } x \in C(v)\}$. Observe that

$$F_\beta(x) = \chi_\beta^{-1}(x) \cup \bigcup_{v \in J(x)} \chi_\beta^{-1}(v).$$

By Theorem 6.5(1), $xv \in E(\beta(S))$ for every $v \in J(x) \setminus \{x\}$, so it follows that $\chi_\beta^{-1}(x) \cap \chi_\beta^{-1}(v) \neq \emptyset$ for every $v \in J(x)$. Since $T_\beta[\chi_\beta^{-1}(u)]$ is a tree for every $u \in V(G)$, it follows that $T_\beta[F_\beta(x)]$ is connected, as required. This proves Theorem 6.6. ■

7. A USEFUL VERTEX PARTITION

For the remainder of the paper, all logarithms are taken in base 2. The goal of this section is to prove the following.

Theorem 7.1. *Let $t \in \mathbb{N}$, and let G be (theta, K_t) -free with $|V(G)| = n$. Let δ_t be as in Theorem 3.3. Then, there is a partition (S_1, \dots, S_k) of $V(G)$ with the following properties:*

- (1) $k \leq \delta_t \log n$.
- (2) S_i is a stable set for every $i \in \{1, \dots, k\}$.
- (3) For every $i \in \{1, \dots, k\}$ and $v \in S_i$ we have $\deg_{G \setminus \bigcup_{j < i} S_j}(v) \leq 4\delta_t$.

Proof. We start with the following.

(24) *At least $\frac{n}{2}$ vertices of G have degree at most $4\delta_t$.*

Since by Theorem 3.3 G is δ_t -degenerate, it follows that $|E(G)| \leq \delta_t n$. Let X be the set of vertices of G with degree at least $4\delta_t$. Then the number of edges incident with vertices of X is at least

$$\frac{1}{2}|X|4\delta_t \leq |E(G)| \leq \delta_t n$$

and therefore $|X| \leq \frac{n}{2}$. This proves (24).

(25) *There is a partition T_1, \dots, T_m of $V(G)$ with the following properties:*

(1) $m \leq \log n$.

(2) *For every $i \in \{1, \dots, m\}$ and $v \in T_i$ we have $\deg_{G \setminus \bigcup_{j < i} T_j}(v) \leq 4\delta_t$.*

The proof is by induction on n . By (24) there is a set $T_1 \subseteq V(G)$ with $|T_1| = \frac{n}{2}$ such that every vertex of T_1 has degree at most $4\delta_t$ in G . Inductively, $V(G \setminus T_1)$ has a partition T_2, \dots, T_m satisfying the requirement of (25) with $m - 1 \leq \log \frac{n}{2} = \log n - 1$. Now T_1, \dots, T_m is a required partition of $V(G)$ and $m \leq \log n$. This proves (25).

Since by Theorem 3.3 G is δ_t -degenerate, it follows that for every $i \in \{1, \dots, m\}$ the graph $G[T_i]$ can be δ_t -colored; let $S_1^i, \dots, S_{\delta_t}^i$ be such a coloring of T_i . Now

$$S_1^1, \dots, S_{\delta_t}^1, \dots, S_1^m, \dots, S_{\delta_t}^m$$

is a required partition of $V(G)$. This proves Theorem 7.1. ■

8. PUTTING EVERYTHING TOGETHER

Let G be a graph. A *hub-partition* of G is a partition S_1, \dots, S_k of $G[\text{Hub}(G)]$ as in Theorem 7.1; we call k the *order* of the partition. We call the *hub-dimension* of G (denoting it by $\text{hdim}(G)$) the smallest k such that G as a hub partition of order k .

We can now state a strengthening of Theorem 1.5 that we will prove by induction on $|V(G)|$ and $\text{hdim}(G)$.

Theorem 8.1. *Let $G \in \mathcal{C}_t$ be a graph with $|V(G)| = n$. Then $\text{tw}(G) \leq R(t, 4) + R(t, 4)(4\delta_t + R(t, 3))(\log n + \text{hdim}(G))$.*

By Theorem 7.1, $\text{hdim}(G) \leq \delta_t \log n$ for every $G \in \mathcal{C}_t$, so Theorem 8.1 immediately implies Theorem 1.5. We now prove Theorem 8.1.

Proof. The proof is by induction on $\text{hdim}(G)$, and for fixed $\text{hdim}(G)$ by induction on $|V(G)|$. Let $G \in \mathcal{C}_t$. By Theorem 1.6 we may assume that G has no clique cutset (and thus in particular, G is connected). If G contains the cube, then by Theorem 3.1 $|V(G)| < 9t$, so we may assume that G does not contain a cube. Let S_1, \dots, S_k be a hub-partition of G with $k = \text{hdim}(G)$. Let $\beta_0 = G$. Having defined β_i , if β_{final} is not yet defined, proceed as follows. If $\text{Hub}(\beta_i) = \emptyset$, let $\beta_{\text{final}} = \beta_i$. Now assume that $\text{Hub}(\beta_i) \neq \emptyset$. Let $S'_{i+1} = S_{i+1} \cap \text{Hub}(\beta_i)$. (Recall that every vertex of S_{i+1} is a hub in G , whereas S'_{i+1} is the set of all the vertices of S_{i+1} that remain hubs in β_i , and so results of Section 6 can be applied to S'_{i+1} in β_i .) If some vertex in S'_{i+1} is balanced (in β_i), let $\beta_{\text{final}} = \beta_i$. In both cases define $S_{\text{final}} = S'_{i+1}$, $\text{depth}_{\text{final}} = i$ and $\mathcal{P}_{\text{final}} = \bigcup_{j=1}^i S_j$. Now we may assume that every vertex of S'_{i+1} is unbalanced (in β_i). Applying the construction described in Section 6 to S'_{i+1} and β_i , let $\beta_{i+1} = \beta(S'_{i+1})$. Notice that, since we are working in β_i , canonical star separations for S'_{i+1} are defined in β_i , and the relation \leq_A and the set

$\text{Core}(S'_{i+1})$ in β_i rely on the canonical star separations defined in β_i . By repeated applications of Theorem 6.5(4), we deduce:

(26) *We have $S_j \cap \text{Hub}(\beta_i) = \emptyset$ for every $j \leq i$. In particular, $\mathcal{P}_{final} \cap \text{Hub}(\beta_{final}) = \emptyset$.*

By (26) and assertion (3) of Theorem 7.1, we have the following useful fact.

(27) *For all $1 \leq i < k$ and for all $v \in S_{i+1}$, we have $\deg_{\text{Hub}(\beta_i)}(v) \leq 4\delta_t$.*

Next we show:

(28) *We have $\text{tw}(\beta_{final}) \leq R(t, 4) + R(t, 4)(4\delta_t + R(t, 3))(\text{hdim}(G) - \text{depth}_{final} + \log(|\beta_{final}|))$.*

Suppose first that $\text{Hub}(\beta_{final}) \neq \emptyset$. Then, there is a balanced vertex $v \in S_{final}$. By (26), it follows that $\text{Hub}(\beta_{final}) \subseteq \text{Hub}(G) \setminus \mathcal{P}_{final}$, and so $\text{hdim}(\beta_{final}) \leq k - \text{depth}_{final}$. Let D_1, \dots, D_m be the components of $\beta_{final} \setminus N[v]$. Since v is balanced, it holds that $|D_i| \leq \frac{|\beta_{final}|}{2}$ for every $1 \leq i \leq m$. Let H be the graph obtained from $\beta_{final} \setminus \{v\}$ by contracting each D_i to a vertex d_i . By Theorem 5.1, we have $H \in \mathcal{C}_t$ and $\text{Hub}(H) = \emptyset$. Therefore, $\text{hdim}(H) = 0$, and it follows inductively that

$$\text{tw}(H) \leq R(t, 4) + R(t, 4)(4\delta_t + R(t, 3)) \log(|V(H)|).$$

By Theorem 2.3, H has a tree decomposition (T_0, χ_0) of width $\text{tw}(H)$, where $\chi_0(v)$ is a PMC of H for all $v \in V(T_0)$. Let T be a tree decomposition of β_{final} as in Theorem 5.2 (with $G = \beta_{final}$). We will show that (T, χ) has the required width.

Let $u \in V(T)$; we will show an upper bound on $|\chi(u)|$. Suppose first that $u \in T_0$. By the construction in Section 5, we have $\chi(u) = (\chi_0(u) \setminus \{d_1, \dots, d_m\}) \cup N_{\text{Hub}(\beta_{final})}(v) \cup \{v\} \cup \bigcup_{d_i \in \chi_0(u)} N_H(d_i)$. Since $\{d_1, \dots, d_m\} \cap \text{Hub}(H) = \emptyset$, it follows from Theorem 4.4 that $|\chi_0(u) \cap \{d_1, \dots, d_m\}| \leq R(t, 4)$. Next, by Corollary 4.3, for every $i \in \{1, \dots, m\}$ we have that $|N_H(d_i)| \leq R(t, 3)$. Finally, by (27), it follows that $\deg_{\text{Hub}(\beta_{final})}(v) \leq 4\delta_t$, so

$$|\chi(u)| \leq |\chi_0(u)| + R(t, 4)(R(t, 3) + 4\delta_t).$$

By the upper bound on $\text{tw}(H)$, it holds that $|\chi_0(u)| \leq R(t, 4) + R(t, 4)(4\delta_t + R(t, 3)) \log(|V(H)|)$, and so

$$|\chi(u)| \leq R(t, 4) + R(t, 4)(4\delta_t + R(t, 3))(\log(|V(H)|) + 1).$$

Since $|V(H)| \leq |\beta_{final}|$ and $\text{hdim}(G) - \text{depth}_{final} \geq \text{hdim}(\beta_{final}) \geq 1$, the required upper bound holds. Thus, we may assume that $u \in V(T_i)$. By the construction in Section 5, we have $|\chi(u)| = |\chi_i(u)| + |N(d_i)| + |N_{\text{Hub}(\beta_{final})}(v)|$. By Corollary 4.3, it holds that $|N(d_i)| \leq R(t, 3)$, and by (27), it holds that $|N_{\text{Hub}(\beta_{final})}(v)| \leq 4\delta_t$. Therefore, $|\chi(u)| \leq |\chi_i(u)| + 4\delta_t + R(t, 3)$. Since $|D_i| \leq \frac{|\beta_{final}|}{2}$, it follows inductively that

$$\begin{aligned} \text{tw}(D_i) &\leq R(t, 4) + R(t, 4)(4\delta_t + R(t, 3))(\log(|D_i|) + \text{hdim}(\beta_{final})) \leq \\ &R(t, 4) + R(t, 4)(4\delta_t + R(t, 3))(\log(|\beta_{final}|) - 1 + \text{hdim}(\beta_{final})). \end{aligned}$$

Since $R(t, 4)(4\delta_t + R(t, 3)) \geq 4\delta_t + R(t, 3)$, the required upper bound follows.

Now we may assume that $\text{Hub}(\beta_{final}) = \emptyset$. By Theorem 4.4, it follows that every PMC of β_{final} has size at most $R(t, 4)$. Now, by Theorem 2.3, $\text{tw}(\beta_{final}) \leq R(t, 4)$, as required. This proves (28).

Now we prove by induction on s that for every $s \leq \text{depth}_{final}$, it holds that

$$\text{tw}(\beta_s) \leq R(t, 4) + R(t, 4)(4\delta_t + R(t, 3))(k - s + \log(|\beta_s|)).$$

The assertion follows from (28) for $s = \text{depth}_{final}$. Now assume that we know the assertion for β_s ; we prove it for β_{s-1} .

By Theorem 2.3, β_s has a tree decomposition (T_0, χ_0) of width $\text{tw}(\beta_s)$, where $\chi_0(v)$ is a PMC of β_s for every $v \in V(T_0)$. Let T be a tree decomposition of β_{s-1} as in Theorem 6.6 (with $G = \beta_{s-1}$ and $\beta(S) = \beta_s$, where we use (T_0, χ_0) as (T_β, χ_β)). We will show that (T, χ) has the required width. Let D_1, \dots, D_m be the components of $\beta_{s-1} \setminus \beta_s$.

Let $u \in V(T)$; we will show an upper bound on $|\chi(u)|$. Let $j \in \{0, \dots, m\}$ be such that $u \in T_j$. We show:

(29) *We have*

$$|\chi_j(u)| \leq R(t, 4) + R(t, 4)(4\delta_t + R(t, 3))(k - s + \log(|\beta_{s-1}|)).$$

If $j = 0$, (29) is true inductively since $|\beta_s| \leq |\beta_{s-1}|$. Thus we may assume that $j \geq 1$. By (26), $\text{hdim}(\beta_{s-1}) \leq k - s + 1$. By Theorem 6.5(3), it holds that $|D_j| \leq \frac{|\beta_{s-1}|}{2}$, so we deduce inductively that

$$\begin{aligned} |\chi_j(u)| &\leq \text{tw}(D_j) \leq R(t, 4) + R(t, 4)(4\delta_t + R(t, 3))(\log(|D_j|) + \text{hdim}(\beta_{s-1})) \leq \\ &R(t, 4) + R(t, 4)(4\delta_t + R(t, 3))(\log(|\beta_{s-1}|) - 1 + (k - s + 1)). \end{aligned}$$

This proves (29).

Next, by (27) and by Theorem 6.5(2), we have the following:

$$(30) \text{ For every } v \in \text{Core}(S_s) \text{ we have that } \deg_{\beta_s}(v) < 4\delta_t + R(t, 3).$$

Now we show:

$$(31) |\chi(u)| \leq |\chi_j(u)| + R(t, 4)(4\delta_t + R(t, 3)).$$

Suppose first that $j > 0$. Then, by the construction in Section 6, we have $\chi(u) = \chi_j(u) \cup C(r(D_j))$. Now by (30), it holds that $|\chi(u)| \leq |\chi_j(u)| + 4\delta_t + R(t, 3)$. Since $R(t, 4)(4\delta_t + R(t, 3)) \geq 4\delta_t + R(t, 3)$, the required upper bound follows. Thus we may assume that $j = 0$. By the construction in Section 6, we have $\chi(u) = \chi_0(u) \cup \bigcup_{v \in \text{Core}(S_s) \cap \chi_0(u)} C(v)$. By (26), it holds that $S_s \cap \text{Hub}(\beta_s) = \emptyset$, so by Theorem 4.4 it follows that $\text{Core}(S_s) \cap \chi_0(u) \leq R(t, 4)$. It follows from (30) that $|\chi(u)| \leq |\chi_0(u)| + R(t, 4)(4\delta_t + R(t, 3))$. This proves (31).

Now the result follows immediately from (29) and (31). This completes the proof of Theorem 8.1. ■

9. ALGORITHMIC CONSEQUENCES

Finally, let us shed some light on the algorithmic significance of Theorem 1.5. It is well-known that a wide range of NP-hard problems are polynomial-time solvable on graphs of bounded treewidth, provided that a tree decomposition of bounded width is given. In particular, according to a celebrated theorem of Courcelle [9], for every graph property P which is definable in the monadic second-order logic, there exists a computable function f such that given a graph G and a tree decomposition of G of width at most k , one can check in time $\mathcal{O}(f(k)|V(G)|)$ whether G satisfies P . The function f can (and in fact ‘should’, in some sense) be quite huge for some properties. But for a broad category of ‘local’ problems, the standard dynamic programming on a tree decomposition of bounded width [5] yields $f(k) = 2^{\mathcal{O}(k)}$. Roughly speaking, *local* problems, such as STABLE SET, VERTEX COVER, DOMINATING SET and k -COLORING (with fixed k), are those for which, while applying the dynamic programming to a tree decomposition of bounded width, for each bag it suffices to keep track of those subsets of the bag that belong

to a partial solution (again, see [5] for more details). See also [6, 10] for more information about problems attaining $f(k) = 2^{\mathcal{O}(k)}$.

Note that while Courcelle’s theorem guarantees a polynomial-time algorithm in graph classes of bounded treewidth, aligning with Theorem 1.5, the major advantage of f being singly exponential (with linear exponent) in k is that the corresponding problem becomes polynomial-time solvable in much richer classes of graphs, namely those with logarithmic treewidth.

This leaves as the only concern to compute efficiently a tree decomposition of logarithmic width. The following 4-approximation for computing a tree decomposition of minimum width due to Robertson and Seymour [16] is strong enough for our purposes.

Theorem 9.1 ([16], see also [13]). *There exists an algorithm which runs in time $\mathcal{O}(27^k \cdot k^2 \cdot n^2)$ and given a graph G and an integer k , the algorithm either correctly outputs $\text{tw}(G) > k$ or computes a tree decomposition of G of width at most $4k$.*

From this, we immediately deduce:

Theorem 9.2. *Let P be a problem which admits an algorithm running in time $\mathcal{O}(2^{\mathcal{O}(k)}|V(G)|)$ on graphs G with a given tree-decomposition of width at most k . Also, let \mathcal{G} be a class of graphs for which there exists a constant $c = c(\mathcal{G})$ such that $\text{tw}(G) \leq c \log(|V(G)|)$ for all $G \in \mathcal{G}$. Then P is polynomial-time solvable in \mathcal{G} .*

In view of Theorems 1.5 and 9.2, we conclude the following.

Theorem 9.3. *Let $k \geq 1$ be fixed and P be a problem which admits an algorithm running in time $\mathcal{O}(2^{\mathcal{O}(t)}|V(G)|)$ on graphs G with a given tree-decomposition of width at most k . Then P is polynomial-time solvable in \mathcal{C}_t . In particular, STABLE SET, VERTEX COVER, DOMINATING SET and q -COLORING (with fixed q) are all polynomial-time solvable in \mathcal{C}_t .*

There is still one important problem to look into, and that is COLORING. By Theorem 3.3, for every fixed t , all graphs in \mathcal{C}_t have chromatic number at most δ_t . Also, for each fixed k , by Theorem 9.3, k -COLORING is polynomial-time solvable in \mathcal{C}_t . Now by solving k -COLORING for every $k \in \{1, \dots, \delta_t\}$, we obtain a polynomial-time algorithm for COLORING in \mathcal{C}_t .

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