Introduction and definitions

Three-variable bracket

A state of a link diagram is a choice of smoothing at each crossing of the diagram. There are two choices of smoothing at a crossing. Given a state s of a diagram D, let a(s), b(s) denote the number of A- and B-smoothings in s, and j(s) denote the number of loops resulting in applying s to D. Then the three-variable bracket \( D(A, B, D) \) is given by

\[
\sum_s A^{a(s)}B^{b(s)}(-1)^{j(s)}.
\]

This polynomial is not a link invariant. However, it is invariant under a flype. Any two reduced alternating diagrams of a link are related by flypes, so the three-variable bracket is an invariant of reduced alternating links.

Tutte polynomial

Label the edges of a graph \( G = (V, E) \) as 1, 2, \ldots, |E| and denote the labeling of edge e by \( l(e) \). Let \( T \) be the set of spanning trees of \( G \). For \( T \in T \), let \( e \in e(G) \) with respect to \( T \):

- externally active if \( e \in T \) and \( l(e) < l(f) \) for all \( f \) on the walk in \( T \) between endpoints of \( e \),
- internally active if \( e \in T \) and \( l(e) < l(f) \) for all \( f \) with endpoints in the two components of \( T - e \),
- inactive otherwise.

Let \( i(T), j(T) \) denote the number of edges which are externally or internally active with respect to \( T \), respectively. Then the Tutte polynomial of \( G \) is

\[
T(G, x, y) = \sum_{T \in T} x^{i(T)}y^{j(T)}.
\]

Current work

Relationship between three-variable bracket and Tutte polynomial

Given a non-split alternating link diagram \( D \), form its B-graph \( G = (V, E) \):

- Perform a B-smoothing at each crossing.
- Form a graph with a vertex for each resulting loop, and an edge for each crossing between adjacent loops.

Associate a state s with the subgraph \( G_s \) of the B-graph which includes all vertices and edges at A-smoothings of s. Note |s| = i components of \( G_s \) and j bounded faces of \( G_s \).

Therefore, the coefficient of \( A^{i-1}B^{j}(-1)^{k} \) in \( D \) is the number of spanning subgraphs \( G_s \) with i edges which have j components and k − j + 1 bounded faces. Such a graph can be formed uniquely by removing j − 1 internally active edges and adding k − j + 1 externally active edges from/to a maximal subtree of \( G \), so \( j = (|V| + k - j + 1)/2 \).

Thus, the coefficient of \( A^{i-1}B^{j}(-1)^{k} \) in \( D \) is

\[
\sum_{T \in T} i(T) \binom{k}{j} j(T) = \frac{1}{(1 - j)(k - j + 1)} \binom{|T|}{k} |G(x, y)| \frac{1}{y^{k+1-j}x^{j-1}} (1, 1).
\]

That is,

\[
\langle D \rangle = \sum_{k \geq 0} \left( \sum_{|B| = k} \binom{|V| - 1}{B} \frac{1}{(1 - j)(k - j + 1)} \frac{1}{y^{k+1-j}x^{j-1}} (1, 1) \right) B^{k} E^{1-|V|} T[G](dA^{-1}B + 1, dAB^{-1} + 1).
\]

Future questions

Can we obtain signature/write from the three-variable bracket?

Given a knot K with alternating diagram \( D \), the signature \( \sigma(K) \) is given by \( \sigma(K) = \sigma(D) \), where \( G \) is the Goeritz matrix of \( D \) and \( \sigma \) a correction term. Since \( D \) is alternating, \( \sigma \) is given by \( |w(D)| = c(D)/2 \) or \( |w(D)| = c(D)/2 \) where \( w(D) \) is the writhe of \( D \), depending on the choice of coloring of \( D \).

Since \( D \) is alternating, \( G \) is definite. Therefore, if \( |w(D)| \geq 0 \) then \( G \) is positive definite and the one term in the three-variable bracket of \( D \) of the form \( A^{-1}B \) satisfies \( j = \sigma(G) \). Similarly, if \( |w(D)| < 0 \) then \( G \) is negative definite and the one term of the form \( A^{-1}B \) satisfies \( j = -\sigma(G) \).

Thus, to recover \( \sigma(K) \) from the three-variable bracket it is sufficient to recover \( w(D) \).

Questions

- Is writhe determined by the B-graph?
- Is there a method of determining writhe from the Tutte polynomial?

Hypothesis

- Can writhe be determined from the Tutte polynomial?

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