

2024 Gene Golub SIAM Summer School

Orbit Recovery from Moments

Liu Zhang

July 29, 2024

Problem Statement

Problem (Orbit Recovery)

Let $\theta_* \in V = \mathbb{R}^d$ be the unknown signal of interest. Let G be a compact Lie group that acts linearly, continuously, and orthogonally on $\mathbb{R}^d$.

- 1 G acts on V via a linear representation $\rho : G \rightarrow O(V)$,
- 2 the action map $(g, v) \mapsto \rho(g)(v)$ is continuous (equivalently, ρ is continuous),
- 3 $O(V)$ denotes the space of real orthogonal $d \times d$ matrices. G is a subgroup of $O(d)$.

- Given n samples:

$$Y_i = g_i \cdot \theta_* + \sigma \xi_i \in \mathbb{R}^d, \quad i \in 1, \dots, n, \quad (1)$$

- $g_1, \dots, g_n \stackrel{\text{iid}}{\sim} \Lambda$, Λ is the unique Haar probability measure on G ,
- $\xi_1, \dots, \xi_n \stackrel{\text{iid}}{\sim} \mathcal{N}(0, I_{d \times d})$, isotropic Gaussian noise.
- Goal: to estimate the orbit of θ_* , $O_{\theta_*} = \{g \cdot \theta_* : g \in G\}$.

Problem Statement

Problem (Unprojected cryo-EM)

- $f : \mathbb{R}^3 \rightarrow \mathbb{R}$: the target function.
- $G = SO(3)$. The action of $SO(3)$ on $\mathbb{R}^3$ is rotation about the origin. Denote the rotated function by $f_g(x) = f(g^{-1} \cdot x)$.
- Given n samples:

$$Y_i = f_g(x) + \sigma dW(x), \quad (2)$$

where $dW(x)$ is the standard Gaussian white noise process on $\mathbb{R}^3$.

- $\theta_* \in \mathbb{R}^d$: expansion coefficients of $f(x)$ in real spherical harmonic basis.
- Goal: to estimate the orbit of θ_* , $O_{\theta_*} = \{g \cdot \theta_* : g \in G\}$.

Method-of-Moment Estimation

- k -th order moment tensor:

$$\mathcal{T}_k(\theta) = \mathbb{E}_{\mathbf{g} \sim \Lambda}[(\mathbf{g} \cdot \theta)^{\otimes k}] = \int_G (\mathbf{g} \cdot \theta)^{\otimes k} d\Lambda(\mathbf{g}) \in \mathbb{R}^{d \times \dots \times d}, \quad (3)$$

$\mathcal{T}_k(\theta)$ is a tensor of order k . The entries of $\mathcal{T}_k(\theta)$ are the k -th order mixed moments of the distribution of $\mathbf{g} \cdot \theta$.

- k -th order moment variety (defined by fixing the true signal $\theta_* \in \mathbb{R}^d$):

$$\mathcal{V}_k(\theta_*) = \{\theta \in \mathbb{R}^d : M_k(\theta) = M_k(\theta_*)\}, \quad \mathcal{V}^{(0)}(\theta_*) = \mathbb{R}^d, \quad (4)$$

where $M_k(\theta) = \text{vec}(\mathcal{T}_1(\theta), \dots, \mathcal{T}_k(\theta)) \in \mathbb{R}^{d+d^2+\dots+d^k}$.

- Method-of-moment estimator:

$$\hat{\theta} = \underset{\theta \in \mathcal{V}_{k-1}(\theta_*)}{\text{argmin}} \quad \|\mathcal{T}_k(\theta) - \mathcal{T}_k(\theta_*)\|^2, \quad k = 1, 2, \dots \quad (5)$$

where $\|\cdot\|$ denotes the squared Euclidean norm between tensors.

- **Known:** (informal) $\hat{\theta}$ is close enough to θ_* given enough k .
- **Question/proposed idea:** Is $\hat{\theta}$ still close enough to θ_* if we subsample the entries of $\mathcal{T}_k(\theta)$? A subsampling procedure informed by group structure?