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Abstract. Riemannian geometry the fast version. Theorems and lemmas from Lee Riemannian Manifolds [1] and do Carmo Riemannian Geometry [2], without much proof or discussion.

1. Background

Lemma 1 (Tensor characterization lemma). *A map*

$$\tau : \mathcal{T}^1(M) \times \cdots \times \mathcal{T}^1(M) \times \mathcal{T}^1 \times \cdots \times \mathcal{T}(M) \to C^\infty(M)$$

is induced by a (p,q) tensor field via iff it is multilinear over $C^\infty(M)$.

In other words, to check something is a tensor, it suffices to just show that it is linear in $C^\infty(M)$ in each term. This can also be used to easily show something is *not* a tensor.

We will also use this to identify "vector-valued tensors" with target space for example $\mathfrak{X}(M)$.

2. Riemannian metrics

Definition 1 (metric). Let M be a smooth manifold. A *Riemannian metric* is a 2-tensor field that is symmetric ($g(X,Y) = g(Y,X)$) and positive definite ($g(X,X) > 0$ if $X \neq 0$). In other words, it is a (smoothly-changing) inner product on each T_pM, $\langle X, Y\rangle := g(X,Y)$ for $X, Y \in T_pM$.

Theorem 1 (Nash, hard). *Any Riemannian metric on any manifold can be realized as the induced metric of some embedding in $\mathbb{R}^n$.*

Nevertheless we are interested in intrinsic properties of Riemannian manifolds and don't want to deal with embeddings and distinguishing which properties depend only on the metric.

Definition 2 (semi-Riemannian). A *peudo-Riemannian metric* or *semi-Riemannian metric* is a symmetric 2-tensor field g that is *nondegenerate* at each $p \in M$, i.e. only zero is orthogonal to everything: $g(X,Y) = 0$ for all $Y \in T_pM$ iff $X = 0$. This is equivalent to writing $g = g_{ij}\varphi^i\varphi^j$ and requiring (g_{ij}) to be invertible.

Definition 3 (isometry). Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be Riemannian manifolds. A diffeomorphism $\varphi : M \to \widetilde{M}$ is an *isometry* if $\varphi^*\widetilde{g} = g$.

In general we care about properties that are invariant under isometries.

Definition 4 (conformal). Two metrics g_1 and g_2 on a manifold M are *conformal* to each other if there is a positive $f \in C^\infty$ such that $g_2 = fg_1$. Two Riemannian manifolds (M_1, g_1) and (M_2, g_2) are *conformally equivalent* if there is a diffeomorphism $\varphi : M_1 \to M_2$ such that φ^*g_2 is conformal to g_1.

Remark. Two metrics are conformal iff they define the same angles. (but they may not preserve length.) A diffeomorphism is a conformal equivalence iff it preserves angles.

2.1. Coefficient matrix.

Let $(E_1, \ldots, E_n)$ be a local frame (n-tuple of smooth vector field defined on some U such that $(E_j|_p)$ is a basis for T_pM for each $p \in U$). Let (φ_j) be the dual coframe. Then we may write

$$g = g_{ij}\varphi^i \otimes \varphi^j, \qquad g_{ij} := \langle E_i, E_j\rangle.$$

Since g is symmetric, we may define the *symmetric product* of two 1-forms ω and η via $\omega\eta := \frac{1}{2}(\omega \otimes \eta + \eta \otimes \omega)$ and write

$$g = g_{ij}\, dx^i\, dx^j. \tag{2.1}$$

We also define $g^{ij} := (g_{ij})^{-1}$ as a matrix inverse.

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2.2. **Examples and constructing more metrics.**

(1) The *Euclidean metric* $\bar{g}$ is the usual inner product on $T_x\mathbb{R}^n \simeq \mathbb{R}^n$, written in standard Euclidean coordinates as
$$\bar{g} = \sum_i dx^i\, dx^i = \sum_i (dx^i)^2 = \delta_{ij} dx^i\, dx^j.$$

(2) Submanifolds: If $i : N \hookrightarrow M$ is an immersed submanifold of M, then the *induced metric* on N is $g_N := i^* g$, which is just the restirction of g to vectors tangent to N.

To do computations on N (a submanifold of dimension n), it is useful to use the chain rule with a local parametrization, i.e. an embedding $X : U_n \to M_m$ with $U_n \subset \mathbb{R}^n$ open and $X(U_n)$ an open subset of N.

(3) Products: If (M_1, g_1) and (M_2, g_2) are Riemannian manifolds, their product $M_1 \times M_2$ has the *product metric* $g := g_1 \oplus g_2$,
$$g(X_1 + X_2, Y_1 + Y_2) := g_1(X_1, Y_1) + g_2(X_2, Y_2), \qquad X_i, Y_i \in T_{p_i} M_i,$$
and usoing $T_{(p_1,p_2)}(M_1 \times M_2) \simeq T_{p_1} M_1 \oplus T_{p_2} M_2$. In coordinates, the product metric has the block diagonal form
$$(g_{ij}) = \begin{pmatrix} (g_1)_{ij} & 0 \\ 0 & (g_2)_{ij} \end{pmatrix}.$$

2.3. **Flat, sharp, and trace.** TODO

2.4. **Extension to tensors.**

2.5. **Volume element.**

Lemma 2. *Let (M, g) be an oriented Riemannian n-manifold. Then there is a unique n-form dV such that $dV(E_1, \dots, E_n) = 1$ for every oriented orthonormal basis $(E_1, \dots, E_n)$ on some tangent space T_pM. This form is called the* Riemannian volume element.

Remark. For any oriented local frame $\{E_j\}$,
$$dV = \sqrt{\det(g_{ij})}\, \varphi^1 \wedge \cdots \wedge \varphi^n.$$

2.6. **The spheres.** TODO

3. Connections

We want to find geodesics, the equivalent of straight lines in Euclidean space. We can charaterize a straight line as one having zero acceleration. But on a general manifold, the acceleration $\ddot{\gamma}$ is not coordinate-independent. (e.g. $(x(t), y(t)) = (\cos t, \sin t)$ and compare in polar coordinates.) The problem is that we cannot simply take a difference quotient of the tangent vectors $\dot{\gamma}(t)$ and $\dot{\gamma}(t_0)$ since they live in different tangent spaces. We will use a connection to connect the tangent spaces so we can define geodesics.

Definition 5. A *linear connection* on M is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying

(a) $\nabla_X Y$ is linear over $C^\infty(M)$ in X:
$$\nabla_{fX_1 + gX_2} Y = f \nabla_{X_1} Y + g \nabla_{X_2} Y, \qquad f, g \in C^\infty(M).$$

(b) $\nabla_X Y$ is linear over $\mathbb{R}$ in Y:
$$\nabla_X (aY_1 + bY_2) = a \nabla_X Y_1 + b \nabla_X Y_2, \qquad a, b \in \mathbb{R}.$$

(c) ∇ satisfies the product rule:
$$\nabla_X (fY) = f \nabla_X Y + (Xf) Y, \qquad f \in C^\infty(M).$$

It is a way to compute covariant derivatives of vector fields.

Remarks.

* This is not a tensor field because by the product rule it is not linear over $C^\infty(M)$ in Y.
* Connections are local operators, using bump functions one can prove that $\nabla_X Y|_p$ depends only on X, Y in an arbitrarily small neighborhood of p. In fact, one can improve this to only requiring Y in a neighborhood of p and the value of X at p.

3.1. **Existence of connections.** The *Euclidean connection* on $\mathbb{R}^n$ is

$$\bar\nabla_X(Y^j\partial_j) = (XY^j)\partial_j. \tag{3.1}$$

It is just the vector field with components the ordinary directional derivatives of the components of Y in the direction X. For this connection all Christoffel symbols are zero.

We can always construct a connection on a single coordinate chart, and then extend to the whole manifold using a partition of unity.

Lemma 3 (connection on single chart)**.** *Let M be a manifold covered by a single coordinate chart. Then there is a bijection between the linear connections on M and choices of n^3 smooth functions $\{\Gamma_{ij}^k\}$ on M, given by*

$$\nabla_X Y = (X^i\partial_i Y^k + X^i Y^j \Gamma_{ij}^k)\partial_k. \tag{3.2}$$

Proof. For any connection, (5.5) with $E_i = \partial_i$ becomes our desired equation. For the other direction, just check this defines a connection. ☐

Lemma 4 (existence of connection)**.** *Every manifold admits a linear connection.*

Proof. Get a partition of unity, a connection ∇^α on each coordinate patch, and define

$$\nabla_X Y = \sum_\alpha \varphi_\alpha \nabla_X^\alpha Y.$$

Check this is a connection. (Note in general that ∇^1 and ∇^2 connections does not imply $\nabla^1+\nabla^2$ or $\frac{1}{2}\nabla^1$ are connections, they will fail the product rule.) ☐

3.2. **Covariant derivatives of tensor fields.** A linear connection induces connections on all tensor fields.

Lemma 5 (induced connection on tensor fields)**.** *Let ∇ be a linear connection on M. There is a unique connection on each tensor bundle $T_\ell^k M$, also denoted ∇, such that*

(a) On TM, ∇ agrees with original connection.
(b) On T^0M, ∇ is ordinary differentiation of functions, $\nabla_X f = Xf$.
(c) Product rule for tensor products:

$$\nabla_X(F\otimes G) = (\nabla_X F)\otimes G + F\otimes(\nabla_X G). \tag{3.3}$$

(d) Product rule between covector field ω and vector field Y:

$$\nabla_X\langle\omega, Y\rangle = \langle\nabla_X\omega, Y\rangle + \langle\omega, \nabla_X Y\rangle.$$

Definition 6 (total covarinant derivative)**.** The *total covariant derivative* ∇ on (p,q) forms is a $(p,q+1)$ tensor defined by

$$\nabla F : \Omega^1(M)\times\cdots\Omega^1(M)\times\mathfrak{X}(M)\times\cdots\mathfrak{X}(M)\to T_{q+1}^p(M)$$
$$(\omega^1,\dots,\omega^p,Y_1,\dots,Y_k,X)\mapsto\nabla_X F(\omega^1,\dots,\omega^p,Y_1,\dots,Y_k).$$

3.3. **Vector fields along curves, covariant derivative along curves.** Let $\gamma : I\to M$ be a curve. The *velocity* $\dot\gamma(t)$ of γ is the push-forward $\gamma_*(d/dt)$,

$$\dot\gamma(t)f = \frac{d}{dt}(f\circ\gamma)(t).$$

This is the usual notion of velocity. In coordinates it is

$$\dot\gamma(t) = \dot\gamma^i(t)\partial_i.$$

Definition 7. A *vector field along a curve* $\gamma : I\to M$ is a smooth map $V : I\to TM$ such that $V(t)\in T_{\gamma(t)}M$ for every $t\in I$.

For example, the velocity vector $\dot\gamma(t)\in T_{\gamma(t)}M$. Check smoothness using the coordinate definition. If we have a vector field on M we can restrict it to the curve and check in coordinates that it is smooth. A vector field V along γ is *extendible* if it is the restriction of some vector field on M. (But for example if $\gamma(t_1)=\gamma(t_2)$ but $\dot\gamma(t_1)\neq\dot\gamma(t_2)$ then $\dot\gamma$ is not extendible.)

We define directional derivative of a vector field along a curve.

Lemma 6. *Let ∇ be a linear connection on M. For each curve $\gamma : I\to M$, ∇ determines a unique operator*

$$D_t : \mathfrak{X}(\gamma)\to\mathfrak{X}(\gamma)$$

satisfying

(a) Linearity over $\mathbb{R}$:

$$D_t(aV+bW) = aD_tV + bD_tW, \qquad a,b\in\mathbb{R}. \tag{3.4}$$

(b) *Product rule:*

$$D_t(fV) = \dot{f}V + fD_tV, \qquad f \in C^\infty(I). \tag{3.5}$$

(c) *If V is extendible, then for any extension $\widetilde{V}$ of V,*

$$D_tV(t) = \nabla_{\dot{\gamma}(t)}\widetilde{V}. \tag{3.6}$$

4. Geodesics

Definition 8 (acceleration, geodesic)**.** Let M be a manifold with a linear connection ∇ and let γ be a curve in M. The *acceleration* of γ is the vector field $D_t\dot{\gamma}$ along γ. A curve γ is a *geodesic* wrt ∇ if $D_t\dot{\nabla} \equiv 0$.

Theorem 2 (existence and uniqueness of geodesics)**.** *Let M be a manifold with a linear conection. For any $p \in M$, any $V \in T_pM$, and any $t_0 \in \mathbb{R}$, there exists an open interval $I \subset \mathbb{R}$ containing t_0 and a geodesic $\gamma : I \to M$ satisfying $\gamma(t_0) = p$, $\dot{\gamma}(t_0) = V$. Any two such geodesics agree on their common domain.*

The proof uses existence and uniqueness of ODE.

Remark. As a result, there is a unique maximal geodesic with initial point p and initial velocity V, denoted γ_V.

Definition 9 (parallel)**.** Let M be a manifold with a linear connection ∇. A vector field V along a curve γ is *parallel along* γ wrt ∇ if $D_tV \equiv 0$.

Thus a geodesic is a curve whose velocity vector field is parallel transported along the curve.

Any tangent vector at a single point on a curve can be uniquely extended to a parallel vector field along the entire curve.

Theorem 3 (parallel transport)**.** *Given a curve $\gamma : I \to M$, $t_0 \in I$, and a vector $V_0 \in T_{\gamma(t_0)}M$, there exists a unique parallel vector field V along γ such that $V(t_0) = V_0$. This is called the* parallel translate *or* parallel transport *of V_0 along γ.*

The proof uses existence and uniqueness of linear ODE, with solutions existing for all time.

Remark (doing exercises with parallel transport)**.**

* Intuitively, parallel transporting a vector along a curve means to keep the vector parallel to itself and constant length. Parallel transport also preserves the angle between starting vectors. Thus for a geodesic, parallel transport preserves the angle with the tangent vector.
* In the plane $\mathbb{R}^2$ this is easy to visualize, literally just translate the vector so the tail stays on the curve. For simple 2-surfaces like cones or spheres, sometimes it works to visualize "cuting" the surface flat and then doing parallel transport in the plane.
* For a manifold embedded in $\mathbb{R}^n$, we can compute the covariant derivative of a vector field, by computing the ordinary directional derivative in $\mathbb{R}^n$, then projecting it onto the tangent plane. This lets us check if the covariant derivative is nonzero or not.

  For example, this allows us to see that a non-great circle on S^2 is not a geodesic, as the directional derivative points inward and has nonzero projection.

5. Levi-Civita and Riemannian geodesics

We will define the Riemannian connection so we can talk about geodesics on Riemannian surfaces.

First we consider $M \subset \mathbb{R}^n$ an embedded submanifold. As on a homework sheet we can extend a vector field on M to one on $\mathbb{R}^n$.

Lemma 7 (tangential connection)**.** *Define a map*

$$\nabla^\top : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \qquad \nabla^\top_X Y := \pi^\top(\overline{\nabla}_X Y),$$

where X, Y are extended to $\mathbb{R}^n$, $\overline{\nabla}$ is the Euclidean connection on $\mathbb{R}^n$, and $\pi^\top : T_p\mathbb{R}^n \to T_pM$ the orthogonal projection. Then $\nabla^\top$ is well-defined and is a connection on M, called the tangential connection.

Definition 10 (metric/compatible)**.** Let g be a Riemannian or semi-Riemannian metric on M. A linear connection ∇ is *compatible with* g or *metric* if it satisfies the product rule,

$$\nabla_X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \qquad X, Y, Z \in \mathfrak{X}(M). \tag{5.1}$$

Lemma 8. *The following are equivalent for a linear connection ∇ on a Riemannian manifold.*

(a) *∇ is compatible with g.*

(b) *$\nabla g \equiv 0$.*

(c) If V, W are vector fields along a curve γ,

$$\frac{d}{dt}\langle V, W\rangle = \langle D_t V, W\rangle + \langle V, D_t W\rangle.$$

(d) If V, W are parallel vector fields along γ, then $\langle V, W\rangle$ is constant.
(e) Parallel translation $P_{t_0 t_1} : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M$ is an isometry for each t_0, t_1.

Definition 11 (torsion)**.** The *torsion tensor* $\tau : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is

$$\tau(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]. \tag{5.2}$$

Theorem 4 (fundamental lemma of Riemannian geometry)**.** *Let (M, g) be a Riemannian or semi-Riemannian manifold. There exists a unique linear connection ∇ on M that is metric (compatible with g) and torsion-free. This is called the* Riemannian connection *or the* Levi-Civita connection *of g.*

Proof sketch for uniqueness. First show as follows that a torsion-free metric connection must satisfy the Koszul equation. Use the metric (compatibility) condition and cyclically permute the three variables. Then use torsion-free on one term in each line. Add two equations and subtract the third, then solve for $\langle \nabla_X Y, Z\rangle$. The result is the *Koszul equation*,

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle + \langle X, [Z, Y]\rangle). \tag{5.3}$$

We obtain uniqueness of the Levi-Cevita by letting $\nabla, \widetilde{\nabla}$ be two such connections, and then noting the right side is independent of the connection. □

We use *Christoffel symbols* to express connections in coordinates. Let x^μ be local coordinates for TM on an open set U. The *Christoffel symbols* Γ^k_{ij} of ∇ are defined via the expansion

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k. \tag{5.4}$$

The n^3 Christoffel symbols determine the ∇ on U as follows.

Lemma 9 (Connection in local frame with Christoffel symbols)**.** *Let ∇ be a linear connection, and let $X = X^i\partial_i$ and $Y = Y^j\partial_j$. Then*

$$\nabla_X Y = (XY^k + X^i Y^j \Gamma^k_{ij})E_k. \tag{5.5}$$

The proof is by using the definition of a connection and computing.

Lemma 10 (Christoffel symbol formula)**.** *The Christoffel symbols can be computed by*

$$\Gamma^k_{ij} = \frac{1}{2}g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}). \tag{5.6}$$

This follows from the Koszul formula (5.3).

Remark. Compatibility with g and torsion-free are invariantly-defined properties that force the Levi-Civita connection to agree with the tangential connection when M is a submanifold of $\mathbb{R}^n$ with the induced metric.

Lemma 11 (naturality of Riemann connection)**.** *Let $\varphi : (M, g) \to (\widetilde{M}, \widetilde{g})$ be an isometry. Then*

(a) φ takes the Levi-Civita connection ∇ of g to the Levi-Civita connectoin $\widetilde{\nabla}$ of $\widetilde{g}$,

$$\varphi_*(\nabla_X Y) = \widetilde{\nabla}_{\varphi_* X}(\varphi_* Y).$$

(b) If γ is a curve in M and V is a vector field along γ, then

$$\varphi_* D_t V = \widetilde{D}_t(\varphi_* V).$$

(c) φ takes geodesics to geodesics. If $\gamma_{V,p}$ geodesic in M, then $\varphi \circ \gamma_{\varphi_* V, \varphi(p)}$ geodesic in $\widetilde{M}$.

6. Curvature

Definition 12 (curvature endomorphism)**.** If M is a Riemannian manifold, the *(Riemann) curvature endomorphism* is the map $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z.$$

Sometimes we just call this the curvature. It is a 3-tensor (show it is linear over $C^\infty(M)$ in X, Y, Z.).

Example (Euclidean curvature is zero)**.** Let $Z = Z^k \partial_k$. Then

$$\nabla_X \nabla_Y Z = \nabla_X (Y Z^k \partial_k) = XYZ^k \partial_k,$$

and the rest follows by equality of mixed partials. (for general manifolds we don't have equality of mixed partials)

Proposition 1 (properties of curvature)**.** *The curvature endomorphism has the following properties*

(i) R is bilinear in $\mathfrak{X}(M) \times \mathfrak{X}(M)$, i.e.

$$\begin{aligned} R(fX_1 + gX_2, Y_1) &= fR(X_1, Y_1) + gR(X_2, Y_2) \\ &= R(X_1, fY_1 + gY_2). \end{aligned}$$

(ii) $R(X, Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is linear, i.e.

$$R(X, Y)(fZ + W) = fR(X, Y)Z + R(X, Y)W.$$

(iii) Bianchi identity:

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0.$$

6.1. Flat manifolds.

Definition 13 (flat)**.** A Riemannian manifold is *flat* if it is locally isometric to Euclidean space.

Theorem 5 (flat)**.** *A Riemannian manifold is flat if and only if its curvature tensor vanishes identically, $R \equiv 0$.*

6.2. Symmetries of the curvature tensor.

Definition 14 (curvature 4-tensor)**.** The *(Riemann) curvature 4-tensor* is $Rm := R^\flat$. On vector fields,

$$Rm(X, Y, Z, W) = \langle R(X, Y)Z, W \rangle.$$

Lemma 12 (curvature is an isometry invariant)**.** *The Riemann curvature endomorphism and curvature 4-tensor are local isometry invariants. If $\varphi : (M, g) \to (\widetilde{M}, \widetilde{g})$ is a local isometry, then*

$$\varphi^* \widetilde{Rm} = Rm$$

$$\widetilde{R}(\varphi_* X, \varphi_* Y)\varphi_* Z = \varphi_*(R(X, Y)Z).$$

Proposition 2 (symmetries of curvature tensor)**.** *The curvature 4-tensor has the following symmetries for any vector fields W, X, Y, Z:*

(a) $Rm(W, X, Y, Z) = -Rm(X, W, Y, Z)$
(b) $Rm(W, X, Y, Z) = -Rm(W, X, Z, Y)$
(c) $Rm(W, X, Y, Z) = Rm(Y, Z, W, X)$
(d) $Rm(W, X, Y, Z) + Rm(X, Y, W, Z) + Rm(Y, W, X, Z) = 0$ (algebraic or first Bianchi identity)

Proof idea. (a) is immediate from the definition of curvature endomorphism. (b) comes from compatibility of the connection. (d) comes from torsion-free. (c) follows from (a), (b), (d). □

Proposition 3 (differential (second) Bianchi identity)**.** *The total covariant derivative of the curvature tensor satisfies*

$$\nabla Rm(X, Y, Z, V, W) + \nabla Rm(X, Y, V, W, Z) + \nabla Rm(X, Y, W, Z, V) = 0.$$

Equivalently,

$$(\nabla X R)(Y, Z)W + (\nabla_Y R)(Z, X)W + (\nabla_Z R)(Y, X)W = 0.$$

Proof by computation.

6.3. Sectional curvature.

Definition 15 (sectional curvature)**.** Let (M, g) be a Riemannian manifold and $X, Y \in T_pM$ linearly independent. Then the *sectional curvature* is

$$K(X, Y) := \frac{g(R(X, Y)Y, X)}{g(X, X)g(Y, Y) - (g(X, Y))^2}.$$

Proposition 4. *$K(X, Y)$ only depends on the plane spanned by X, Y, i.e. any choice of two linearly independent vectors in the plane leads to the same sectional curvature.*

Example (Sphere)**.** The sectional curvature of S^n is constant: The orthogonal group acts by isomotries on (S^n, g) and can thus move any 2-plane on TS^n to any other 2-plane.

6.4. **Ricci and scalar curvatures.** The curvature tensor is a 3 or 4-tensor which is complicated. To simplify things we take partial traces.

Definition 16 (Ricci and scalar curvature)**.** Let (M, g) be a (semi-)Riemannian manifold and ∇ the Levi-Civita connection. Fix an orthonormal basis E_i of T_pM. Then the Ricci and scalar curvatures at p are given by

$$\begin{aligned} \mathrm{Ric}(X, Y) &= \sum_i g(R(E_i, X)Y, E_i) \\ \mathrm{scal} &= \sum_j \mathrm{Ric}(E_j, E_j) = \sum_{i,j} g(R(E_i, E_j)E_j, E_i). \end{aligned}$$

Example (sphere)**.** Viewing the sphere S^n as a submanifold of $\mathbb{R}^{n+1}$, it turns out

$$R(X, Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y.$$

Then

$$\begin{aligned} \mathrm{Ric}(X, Y) = \mathrm{tr}\, R(\cdot, Y)Z &= \sum_{i=1}^{n} g(R(E_i, X)Y, E_i) \\ &= \sum_{i=1}^{n} (\langle X, Y\rangle\langle E_i, E_i\rangle - \langle X, E_i\rangle\langle E_i, Y\rangle) = (n-1)\langle X, Y\rangle \end{aligned}$$

Scalar curvature of S^n_R is $n(n-1)/R^2$.

Definition 17 (trace)**.** TODO

Definition 18 (alt, Ricci and scalar curvature)**.** The *Ricci curvature Ric* is a 2-tensor field defined as the trace of the curvature endomorphism on the first and last indices. The *scalar curvature S* is a function defined as the trace of the Ricci tensor.

Definition 19 (Einstein metric)**.** A Riemmanian metric is an *Einstein metric* if its Ricci curvature is a scalar multiple of the metric at each point, i.e. $Ric = \lambda g$ for some function λ.

In fact, we must have

$$Ric = \frac{1}{n}\mathrm{scal} \cdot g. \tag{6.1}$$

Proposition 5 (Einstein metric constant scalar curvature)**.** *If g is an Einstein metric on a connection manifold of dimension ≥ 3, then its scalar curvature is constant.*

7. Gradient, divergence

Definition 20 (divergence)**.** Let $X \in \mathfrak{X}(M)$ where M is a semi-Riemannian manifold of dimension n. Then $L_X \mathrm{vol}$ is an n-form, and we define the *divergence* $\mathrm{div}(X)$ by

$$\mathrm{div}(X)\,\mathrm{vol} = L_X \,\mathrm{vol} = d i_X \,\mathrm{vol}.$$

Proposition 6 (Properties of div)**.**

$$\mathrm{div}(X) = \sum_i g(\nabla_{E_i} X, E_i)$$

Let $\alpha(\cdot) := g(X, \cdot)$ the dual 1-form of X. Then

$$\mathrm{div}(X) = \delta\alpha.$$

To prove the first, use that vol is parallel so $\nabla_Z \mathrm{vol} = 0$ for all Z. To prove the second, use Stokes theorem and inner product.

Definition 21 (gradient)**.** The gradient $\mathrm{grad}\, f$ is the vector field ∇f such that for any $X \in \mathfrak{X}$,

$$g(\nabla f, X) = Xf \ (= \nabla_X f).$$

In coordinates,

$$\nabla f = g^{ik}\partial_k \partial_i.$$

Definition 22 (Hodge inner product)**.** The *Hodge inner product* for any two p-forms α, β on M is

$$(\alpha, \beta) := \int_M \alpha \wedge *\beta = \int_M \langle \alpha, \beta\rangle * (1).$$

It is symmetric, $(\alpha, \beta) = (\beta, \alpha)$.

In this sense, the codifferential δ is the adjoint of the exterior derivative d,

$$(\delta\omega, \eta) = (\omega, d\eta).$$

Definition 23 (Musical isomorphisms)**.** The *musical isomorphisms* give an isomorphism between TM and T^*M of a Riemannian manifold. For $X = X^i\partial_i \in \mathfrak{X}(M)$,

$$X^\flat := g_{ij}X^i dx^j =: X_j dx^j,$$

equivalent to

$$X^\flat(Y) = g(X, Y).$$

This is called "lowering an index".

For $\omega = \omega_i dx^i \in T^*M$,

$$\omega^\sharp := g^{ij}\omega_i\partial_j =: \omega^j\partial_j,$$

equivalent to

$$g(\omega^\sharp, Y\rangle = \omega(Y).$$

This is called "raising an index".

Definition 24 (Dual metric)**.** The *dual metric* $g : \Omega^1(M) \times \Omega^1(M) \to C^\infty(M)$ is defined by

$$g(\omega, \eta) := g(\omega^\sharp, \eta^\sharp).$$

8. Riemannian geometry on Lie groups

Definition 25 (left-invariant metric)**.** A metric on a Lie group G is *left-invariant* if left translation is always an isometry, i.e. for all $g \in G$,

$$\langle X, Y\rangle = \langle (l_g)_*X, (l_g)_*Y\rangle.$$

Right-invariant metric and bi-invariant defined analogously.

Lemma 13. *If X is a left-invariant vector field, then*

$$\nabla_X X = 0.$$

Corollary 1 (Levi-Civita formula)**.** *If X, Y are left-invariant vector fields, then*

$$\nabla_X Y = \frac{1}{2}[X, Y].$$

For the proof use $\nabla_{X+Y}(X + Y) = 0$ and torsion-free.

Example. Let g be a bi-invariant metric on a Lie group G. Using $\nabla_X Y = \frac{1}{2}[X, Y]$ for left-invariant vector fields, we can show

$$R(X, Y)Z = \frac{1}{4}[Z, [X, Y]]$$

and using invariance under the Adjoint map, that

$$g([Z, X], Y) + g(X, [Z, Y]) = 0,$$

for left-invariant vector fields X, Y, Z. Then the sectional curvature is

$$K(X, Y) = \frac{g([X, Y], [X, Y])}{4(g(X, X)g(Y, Y) - (g(X, Y))^2)}.$$

Theorem 6 (geodesics on Lie group)**.** *The geodesics on a Lie group are the left-translates of the 1-paramter subgroups (recall these were smooth homomorphisms $(R, +) \to G$). Thus geodesics are complete.*

References

[1] J. M. Lee, *Riemannian Manifolds: An Introduction to Curvature*, Graduate Texts in Mathematics 176 (1997).
[2] M. P. do Carmo, *Riemannian Geometry* (1992).