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Abstract. Things with the adjective Lie, like Lie derivatives, Lie groups, and Lie algebras. Except Riemannian geometry on Lie groups is in the Riemannian geometry file. Theorems and lemmas from Lee Smooth Manifolds and Warner Lie groups without much proof or discussion.

1. Lie derivative and commuting vector fields

This actually doesn't have to involve Lie groups or Lie algebras. But it has the word Lie so it's going in this file.

This is essentially how to take a directional derivative of a vector field. It is easy in $\mathbb{R}^n$ to take a directional derivative in the direction v. But in general the problem is that for a vector field W we can't look at the difference quotient $W_{p+tv} - W_p$, since those vectors belong to different tangent spaces. So instead of looking at the derivative in the direction of a single vector v, we look at the derivative along a vector field $V \in \mathfrak{X}(M)$ so we can use the flow to push values back to p, and then differentiate.

Definition 1 (Lie derivative)**.** Let V be a smooth vector field on M and θ the flow of V. Define a (rough) vector field

$$(\mathcal{L}_V W)_p := \frac{d}{dt}\bigg|_{t=0} d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}).$$

Theorem 1 (how to compute the Lie derivative)**.** *If M is a smooth manifold and $V, W \in \mathfrak{X}(M)$, then*

$$\mathcal{L}_V W = [V, W].$$

Corollary 1 (properties of Lie derivative)**.** *Let $V, W, X \in \mathfrak{X}(M)$.*

(i) $\mathcal{L}_V W = -\mathcal{L}_W V$.
(ii) $\mathcal{L}_V [W, X] = [\mathcal{L}_V W, X] + [W, \mathcal{L}_V X]$.
(iii) $\mathcal{L}_{[V,W]} X = \mathcal{L}_V \mathcal{L}_W X - \mathcal{L}_W \mathcal{L}_V X$.
(iv) If $g \in C^\infty(M)$, then $\mathcal{L}_V(gW) = (Vg)W + g\mathcal{L}_V W$ (product rule, also get for $\mathcal{L}_{fV} W$ using (i))
(v) If $F : M \to N$ is a diffeomorphism, then $F_*(\mathcal{L}_V X) = \mathcal{L}_{F_* V} F_* X$.

Definition 2 (invariant under flow)**.** A vector field W is *invariant under a flow θ* if W is θ_t-related to itself for each t, i.e.

$$((\theta_t)_* W)_{\theta(p)} \equiv d(\theta_t)_p(W_p) = W_{\theta_t(p)}, \qquad \forall (t, p) \in \text{domain}(\theta).$$

Theorem 2 (commuting and invariant under flow)**.** *The following are equivalent:*

(i) V and W commute.
(ii) V is invariant under the flow of W.
(iii) W is invariant under the flow of V.

In particular, every smooth vector field is invariant under its own flow.

Theorem 3 (commute, flows commute)**.** *Smooth vector fields commute if and only if their flows commute.*

Proposition 1. *Lie derivative wrt vector field X can be defined similarly on forms and tensors in general. Let $X \in \mathfrak{X}(M)$.*

(i) $L_X f = X(f)$ for $f \in C^\infty(M)$.
(ii) $L_X Y = [X, Y]$ for $Y \in \mathfrak{X}(M)$.
(iii) L_X, on differential forms, is a derivation and commutes with d.
(iv) On differential forms, $L_X = i(X) \circ d + d \circ i(X)$ (Cartan magic formula). This can be taken to be the definition of L_X on differential forms.

2. Lie groups and algebras

Definition 3 (Lie group)**.** A *Lie group G* is a differentiable manifold with a group structure $G \times G \to G$ such that multiplication and inversion is C^∞, i.e. $(\sigma, \tau) \mapsto \sigma\tau^{-1}$ is C^∞.

Examples. $(\mathbb{R}^n, +)$, $(\mathbb{C}^*, \cdot)$, $(S^1 \subset \mathbb{C}, \cdot)$, T^n, $GL(n, \mathbb{R})$, affine motions of $\mathbb{R}^n$.

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Definition 4 (Lie algebra)**.** A *Lie algebra* $\mathfrak{g}$ over $\mathbb{R}$ is an $\mathbb{R}$-vector space with a bilinear *bracket* $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

(i) $[x,y] = -[y,x]$ (anti-commuattivity)
(ii) $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$ (Jacobi identity)

As we will see, every Lie group has an associated finite dimensional Lie algebra, and many of the Lie group properties carry over to its Lie algebra.

Examples. Smooth vector fields with $[X,Y] = XY - YX$, vector space with all brackets equal to 0 ("abelian Lie algebra"), $M_{n\times n}(\mathbb{R})$ with $[A,B] = AB - BA$, $\mathbb{R}^3$ with cross product.

3. Left-invariant vector fields

Definition 5 (left/right translation)**.** Let $g \in G$. *Left translation by* g and *right translation by* g are respectively the diffeomorphisms l_g and $r_g : G \to G$ defined by

$$l_g(\tau) = g\tau$$
$$r_g(\tau) = \tau g.$$

A vector field X (apriori not necessarily smooth) on G is *left-invariant* if for all $g \in G$, X is l_g-related to itself, i.e.

$$dl_g \circ X = X \circ l_g$$
$$(dl_g)_{g'}(X_{g'}) = X_{gg'},$$

or equivalently,

$$((l_g)_* X)_{gg'} = X_{gg'}.$$

Proposition 2 (Left-invariant vector fields as the Lie algebra)**.** *Let G be a Lie group and let* $\mathrm{Lie}(G)$ *be its set of left invariant vector fields. Then*

(i) $\mathrm{Lie}(G)$ *is a Lie algebra under the Lie bracket on vector fields.*
(ii) The map $\alpha : \mathrm{Lie}(G) \to T_eG$ that sends $X \mapsto X(e)$ is an isomorphism. Thus there is an isomorphism between the space of left invariant vector fields and the tangent space to G at the identity, and $\dim \mathrm{Lie}(G) = \dim T_eG = \dim G$.
(iii) Left invariant vector fields are smooth.
(iv) The Lie bracket of two left invariant vector fields is left invariant.

Proof sketch. (ii) Prove injectivity and surjectivity and left invariance. (i) comes from properties of Lie bracket of vector fields. □

As a result of (ii), we may associate the Lie algebra $\mathrm{Lie}(G)$ with the tangent space $T_e(G)$ with the Lie algebra structure acquired by demanding α be an isomorphism of Lie algebras. (need $\alpha([X,Y]) = [\alpha(X), \alpha(Y)]$)

Examples. $\mathbb{R}$ with $\mathrm{Lie}(\mathbb{R}) = \{\lambda(\partial_r) : \lambda \in \mathbb{R}\}$. $GL_n(\mathbb{R})$ with $\mathrm{Lie}(GL_n(\mathbb{R})) \simeq M_{n\times n}(\mathbb{R})$ via tangent space identification. Also $\mathrm{Lie}(GL_n(\mathbb{C})) \simeq M_{n\times n}(\mathbb{C})$.

Example. Let V be an n-dimensional $\mathbb{R}$-vector space. Let $\mathrm{End}(V)$ be the set of endomorphisms of V (linear maps $V \to V$), and $\mathrm{Aut}(V) \subset \mathrm{End}(V)$ the set of automorphisms of V (invertible endomorphisms). Then $\mathrm{End}(V)$ has dimension n^2 and is a Lie algebra with bracket $[f_1, f_2] := f_1 \circ f_2 - f_2 \circ f_1$. Also $\mathrm{Aut}(V)$ is a Lie group under composition (send to $GL_n(\mathbb{R})$). Under identification $T_e \mathrm{End}(V) = T_e \mathrm{Aut}(V)$ and using $T_e \mathrm{End}(V) \simeq \mathrm{End}(V)$, can identify $\mathrm{Lie}(\mathrm{Aut}(V)) = \mathrm{End}(V)$.

4. Exponential map

The exponential map links a Lie group with its Lie algebra. In particular, for matrices this coincides with the usual matrix exponentiation.

Theorem 4 (relation Lie group and algebra, differential)**.** *Let G, H be Lie groups with G simply connected. Let $\psi : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ be a homomorphism. Then there exists a unique homomorphism $\varphi : G \to H$ such that $d\varphi = \psi$.*

Remark. A corollary is that if G, H are both simply connected Lie groups and have isomorphic Lie algebras, then they are isomorphic.

Definition 6 (Exponential map)**.** Let $X \in \mathrm{Lie}(G)$. Then there is a homomorphism $\mathrm{Lie}(\mathbb{R}) \to \mathrm{Lie}(G)$ given by $\lambda\partial_r \mapsto \lambda X$. By the previous theorem, then there exists a unique map

$$\exp_X : \mathbb{R} \to G$$

such that

$$d\exp_X(\lambda\partial_r) = \lambda X.$$

The map $t \mapsto \exp_X(t)$ is the unique homomorphism $\mathbb{R} \to G$ with tangent vector $X(e)$ at 0. Define the *exponential map*

$$\begin{aligned} \exp : \mathrm{Lie}(G) &\to G \\ X &\mapsto \exp_X(1) \end{aligned}$$

Proposition 3 (Properties of exp)**.** *Let $X \in \mathrm{Lie}(G)$. Then*

(i) $\exp(tX) = \exp_X(t)$, $t \in \mathbb{R}$

(ii) $\exp(t_1 + t_2)X = (\exp t_1 X)(\exp t_2 X)$, $t_1, t_2 \in \mathbb{R}$

(iii) $\exp(-tX) = (\exp tX)^{-1}$, $t \in \mathbb{R}$.

(iv) $\exp : \mathrm{Lie}(G) \to G$ *is* C^∞ *and* $d\exp : T_0 \mathrm{Lie}(G) \to T_e G$ *is the identity map. Thus* exp *is a diffeomorphism of a neighborhood of* $0 \in \mathrm{Lie}(G)$ *to a neighborhood of* $e \in G$.

(v) $l_g \circ \exp_X$ *is the unique integral curve of* X *with value* g *at* 0. *Thus left-invariant vector fields are always complete (no blow up).*

Proof of (v). TODO □

Theorem 5 (why exp is important)**.** *Let $\varphi : H \to G$ be a homomorphsim of Lie groups, i.e. C^∞ and a group homomorphism. Then the following diagram commutes:*

$$\begin{array}{ccc} H & \xrightarrow{\varphi} & G \\ \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} \\ \mathrm{Lie}(H) & \xrightarrow{d\varphi} & \mathrm{Lie}(G) \end{array}$$

Proof idea. The map $t \mapsto \varphi(\exp tX)$ is a curve in G with tangent $d\varphi(X(e))$ at 0. Compare with uniqueness of $t \mapsto \exp t(d\varphi(X))$ to obtain

$$\varphi(\exp tX) = \exp t(d\varphi(X))$$

and take $t = 1$. □

Example. Matrix exponentiation: Show the smooth map $\mathbb{R} \to GL_n(\mathbb{C})$ given by $t \mapsto e^{tA}$ (where e^{tA} is matrix exponentiation) has tangent vector A at 0 by differentiating term by term. It is a homomorphism since $e^{tA+t'A} = e^{tA}e^{t'A}$. By uniqueness of $t \mapsto \exp_X(t)$ this implies

$$\exp(A) = e^A, \qquad A \in M_{n\times n}(\mathbb{C}).$$

Remark. Let $\varphi : G \to \mathrm{Aut}(V)$ be a representation and let $X \in \mathrm{Lie}(G)$. Then

$$d\varphi(X) = \lim_{t\to 0} \frac{\varphi(\exp tX) - 1}{t} = \left.\frac{d}{dt}\right|_{t=0} (\varphi(\exp tX)).$$

since the curve $t \mapsto \varphi(\exp tX)$ has velocity vector $d\varphi(X(e))$ at $t = 0$ and we identify X with $X(e)$.

5. The adjoint representation

Groups actions: smooth $G \times M \to M$, $(gg')m = g(g'm)$ and $em = m$.

A Lie group G acts on itself by inner automorphism, $a : (g, h) \mapsto ghg^{-1}$, or $a_g : h \mapsto ghg^{-1}$. Define the *adjoint representation* by

$$\begin{aligned} \mathrm{Ad} : G &\to \mathrm{Aut}(\mathrm{Lie}(G)) \simeq \mathrm{Aut}(T_e G) \\ g &\mapsto (da_g)_e. \end{aligned}$$

The differential we denote by

$$ad := d(Ad) : T_e G \to \mathrm{End}(\mathrm{Lie}(G)) \simeq T_e \mathrm{End}(\mathrm{Lie}(G))$$

Denote $\mathrm{Ad}(g) \in \mathrm{Aut}(\mathrm{Lie}(G))$ by Ad_g. By the Theorem that exp is important, the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{a_g} & G \\ \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} \\ \mathrm{Lie}(G) & \xrightarrow{\mathrm{Ad}_g} & \mathrm{Lie}(G) \end{array}$$

Thus

$$\exp t\,\mathrm{Ad}_g(X) = g(\exp tX)g^{-1} : \mathrm{Lie}(G) \to G.$$

Example. Let $G = \mathrm{Aut}(V)$. Let $B \in \mathrm{Aut}(V)$ and $C \in \mathrm{End}(V) = \mathrm{Lie}(G)$.Then $\mathrm{Ad}_B(C) = BCB^{-1}$, because

$$\begin{aligned}\mathrm{Ad}_B(C) &= \frac{d}{dt}\bigg|_{t=0} (a_B(\exp tC)) = \frac{d}{dt}\bigg| (Be^{tC}B^{-1}) \\ &= \frac{d}{dt}\bigg|_{t=0} e^{tBCB^{-1}} = BCB^{-1}.\end{aligned}$$

Proposition 4. *Let $X, Y \in \mathrm{Lie}(G)$. Then*

$$\mathrm{ad}_X Y = [X, Y].$$

6. Lie group and algebra homomorphisms

Definition 7 (homomorphism, representation)**.** A map $\varphi : G \to H$ is a Lie group homomorphism is φ is both C^∞ and a group homomorphism. If $H = \mathrm{Aut}(V)$ then $\varphi : G \to H$ is a *representation of the Lie group G.*

A map $\psi : \mathfrak{g} \to \mathfrak{h}$ is a *Lie algebra homomorphism* if it is linear and preserves brackets, $\psi[X, Y] = [\psi(X), \psi(Y)]$. If $\mathfrak{h} = \mathrm{End}(V)$ then it is a *representation of the Lie algebra* $\mathfrak{g}$.

Definition 8 (differential of homormorphism)**.** Let $\varphi : G \to H$ be a homomorphism. Since φ maps identity to identity, the differential $d\varphi$ takes $T_eG \to T_eH$. Identifying these tangent spaces with $\mathrm{Lie}(G), \mathrm{Lie}(H)$, we will overload the symbol $d\varphi$ to make a new map

$$\begin{aligned} d\varphi : \mathrm{Lie}(G) &\to \mathrm{Lie}(H) \\ X &\mapsto d\varphi(X), \end{aligned}$$

defined such that if $X \in \mathrm{Lie}(G)$, then $d\varphi(X)$ is the unique left invariant vector field on H such that

$$d\varphi(X)(e) = d\varphi(X(e)),$$

the right side being the usual differential on TG.

Theorem 6. *Let G, H be Lie groups and $\varphi : G \to H$ a homormophism. Then*

(i) X and $d\varphi(X)$ are φ-related for each $X \in \mathrm{Lie}(G)$, i.e.

$$d\varphi(X)(\varphi(g)) = d\varphi(X(g)).$$

(ii) $d\varphi : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is a Lie algebra homomorphism.

7. Left-invariant forms

Definition 9 (left invariant forms)**.** A form ω on G is *left invariant* if

$$(l_g)^*\omega = \omega, \qquad \forall g \in G.$$

Left-invariant 1-forms are called *Maurer-Cartan forms.*

Proposition 5. *(i) Left invariant forms are smooth.*

(ii) If ω is a Maurer-Cartan form and $X, Y \in \mathrm{Lie}(G)$, then

$$d\omega(X, Y) = -\omega([X, Y]).$$

(iii) Let $\{X_1, \dots, X_d\}$ be a basis of $\mathrm{Lie}(G)$ with dual basis $\{\omega_1, \dots, \omega_d\}$ for the Maurer-Cartan forms. Then there exist structural constants of G wrt $\{X_i\}$, *c_{ijk}, such that*

$$[X_i, X_j] = \sum_{k=1}^{d} c_{ijk} X_k.$$

They satisfy

$$\begin{aligned} c_{ijk} + c_{jik} &= 0 \\ \sum_r (c_{ijr}c_{rks} + c_{jkr}c_{ris} + c_{kir}c_{rjs}) &= 0. \end{aligned}$$

The exterior derivative of ω_i is given by the Maurer-Cartan equations

$$d\omega_i = \sum_{j<k} c_{jki}\omega_k \wedge \omega_j.$$