ESSENTIAL SPECTRUM (MQM WS2016-17)

Definition 1. The *essential spectrum* of A is

$$\sigma_{\text{ess}}(A) := \sigma(A) \setminus \sigma_d(A),$$

where $\lambda \in \sigma_d(A)$ the *discrete spectrum* if there exists a contour γ around λ that is contained in the resolvent $\rho(A)$ and

$$\dim \left(\frac{1}{2\pi i} \oint_\gamma \frac{1}{z - A} \, dx \right) < \infty.$$

For self-adjoint operators, $\lambda \in \sigma_{\text{ess}}(A)$ iff for all $\varepsilon > 0$, $\dim \chi_{(\lambda-\varepsilon, \lambda+\varepsilon)}(A) = \infty$.

1. One-particle operators

Warm-up: Consider the Laplacian $-\Delta : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Then via the Fourier transform, we can compute

$$\sigma_{\text{ess}}(-\Delta) = \sigma_{\text{ess}}(|\xi|^2) = [0, \infty).$$

Now consider the operator $-\Delta - \varphi$, where φ has some moderate singularities and tends to zero in some weak sense. It turns out we can expect $\sigma_{\text{ess}}(-\Delta - \varphi) = \sigma_{\text{ess}}(-\Delta)$.

TODO why can we expect

1.1. **Perturbations.**

Theorem 1 (Weyl)**.** *Suppose A, B are self-adjoint, and there exists $z \in \rho(A) \cap \rho(B)$ such that $\frac{1}{z-A} - \frac{1}{z-B} \in \mathfrak{S}^\infty(\mathcal{H})$ compact. Then $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B)$.*

We are not going to prove this.

Definition 2. Let A be self-adjoint. Then K is called *relatively compact* wrt A if

$$K \cdot \frac{1}{z - A}$$

is compact for some $z \in \rho(A)$.

Assume K is relatively A compact. Write

$$\frac{1}{i + A + K} = \frac{1}{i + A} - \frac{1}{i + A} \cdot K \cdot \frac{1}{i + A + K}.$$

If $A + K$ is self-adjoint, then $A + K$ has real spectrum so $A + K + i$ is invertible and $\frac{1}{i+A+C}$ is bounded. Compact operators are an ideal in the bounded linear operators, so we obtain

$$\frac{1}{i + A + K} - \frac{1}{i + A} \in \mathfrak{S}(\mathcal{H})$$

and so $\sigma_{\text{ess}}(A + K) = \sigma_{\text{ess}}(A)$ by Weyl.

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1.2. **Apply to Hamiltonians.** Let $A = -\Delta$ and $K = \varphi$. If we want to apply the above, we need $\frac{1}{i-\Delta}\varphi$ compact. Suppose $\varphi \in L^p(\mathbb{R}^3) + (L^\infty)_\varepsilon(\mathbb{R}^3)$, i.e.

$$\forall \varepsilon > 0 \, \exists \, \varphi_1 \in L^p(\mathbb{R}^3), \varphi_2 \in L^\infty(\mathbb{R}^3) \text{ with } \varphi = \varphi_1 + \varphi_2 \text{ and } \|\varphi_2\|_\infty \leq \varepsilon.$$

Then we split up

$$\frac{1}{-\Delta - z}\varphi = \frac{1}{-\Delta - z}\varphi_1 + \frac{1}{-\Delta - z}\varphi_2.$$

Pick $p = 2$, and compute

$$\left\| \frac{1}{-\Delta - z}\varphi_1 \right\|_2^2 \overset{z=-1}{=} \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy \left| \frac{e^{-|x-y|}}{4\pi|x-y|}\varphi_1(y) \right|^2$$
$$= \frac{1}{16\pi^2} \int dx \, \frac{e^{-2|x|}}{|x|^2} \int dy \, \varphi_1(y)^2 < \infty.$$

Also,

$$\left\| \frac{1}{-\Delta + 1}\varphi - \frac{1}{-\Delta + 1}\varphi_1 \right\|_{\mathcal{L}(\mathcal{H})} = \left\| \frac{1}{-\Delta + 1}\varphi_2 \right\|_{\mathcal{L}(\mathcal{H})} \leq C\|\varphi_2\|_\infty \leq C\varepsilon.$$

Thus we are approximating $\frac{1}{-\Delta+1}\varphi$ by a sequence of Hilbert-Schmidt $\mathfrak{S}^2$ functions with error arbitrarily small. Thus $\frac{1}{-\Delta+1}\varphi$ is compact. The fact $\varphi \in L^2 + (L^\infty)_\varepsilon$ implies φ relatively compact. Finally the operator $-\Delta - \varphi$ with the above assumptions is self-adjoint by Kato-Rellich.

Theorem 2. *Let $\varphi \in L^2(\mathbb{R}^3) + L^\infty_\varepsilon(\mathbb{R}^3)$. Then*

$$\sigma_{\text{ess}}(-\Delta - \varphi) = \sigma_{\text{ess}}(-\Delta) = [0, \infty).$$

Examples.

(1) Let $\varphi = \frac{Z}{|x|}$, with $\varphi_2(x) := \frac{Z}{|x|}\chi_{|x|>\frac{Z}{\varepsilon}}$. Then $\varphi_2(x)| \leq \frac{Z}{Z/\varepsilon} = \varepsilon$. Since $\varphi_1(x) = \frac{Z}{|x|}\chi_{B_{2/\varepsilon}}(x)$ then $\varphi_1 \in L^2$ so $\sigma_{\text{ess}}(-\Delta - \frac{Z}{|x|}) = [0, \infty)$.

(2) We cannot apply it to $\varphi(x) = \frac{C}{|x|^2}$ since $2^2 = 4 > 3$.

(3) Consider $H = \sqrt{1 + |p|^2} - \frac{Z}{|x|}$. Note $\frac{1}{|x|}$ is not relatively compact with respect to $\sqrt{p^2 + 1}$.

(4)

References

[1] Reed Simon.
[2] Teschl schroe2.pdf