FPPF COHOMOLOGY OF SMOOTH GROUPS

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This note aims to explain the proof of the following theorem of Grothendieck:

Theorem 1. *Let A be a ring and G be a smooth commutative group over A. Then $\mathrm{R}\Gamma_{\text{ét}}(A, G) = \mathrm{R}\Gamma_{\text{fppf}}(A, G)$. In fact, let a denote the canonical map from the big fppf site of A to the small étale site of A. Then $Ra_*G = G$.*

I got the rough idea from Milne's exposition [Mil80, Theorem 3.9], but I found it difficult to read its details. Here I will write the proof in a way that is for me much easier to understand. I will allow all my algebraic groups to be group algebraic spaces locally of finite presentation, and use a little modern technology that was not available at Grothendieck's and Milne's time.

Proof. Since the fppf topology is generated by étale maps and finitely presented finite flat maps, we are reduced to proving that for every finitely presented finite flat A-algebra B, the Čech complex

$$0 \to G(-) \to G(- \otimes_A B) \to G(- \otimes_A B \otimes_A B) \to \cdots$$

is an exact sequence of étale sheaves. For simplicity, let $G^n(-) = G(- \otimes_A B^{\otimes_A n})$. By [Ji+22, Theorem 5.2], G^n is representable by an algebraic space locally of finite presentation over A, being the Weil restriction of an algebraic space locally of finite presentation along a finitely presented finite flat map $A \to B^{\otimes_A n}$. Also, it is easy to see that G^n is smooth by checking formal smoothness. So the Čech complex is a complex

$$0 \to G^0 \to G^1 \to G^2 \to \cdots$$

of smooth groups over A, which is exact fppf locally as it splits over B, and we want to prove that it is exact étale locally. Taking kernels and doing induction, we are reduced to showing that for an fppf short exact sequence

$$0 \to K \to G \to H \to 0$$

with K and G smooth, both the map $G \to H$ and the group H are smooth, as then $G \to H$, being a smooth map which is an fppf surjection, will be an étale surjection. Now because $G \to H$ is an fppf K-torsor and smoothness can be tested fppf locally, $G \to H$ is smooth; because smoothness can be descended along quasicompact surjective flat maps, H is also smooth. □

References

[Ji+22] Lena Ji, Shizhang Li, Patrick McFaddin, Drew Moore, and Matthew Stevenson. "Weil restriction for schemes and beyond". In: *Stacks Project Expository Collection*. Ed. by Pieter Belmans, Wei Ho, and Aise Johan de Jong. London Mathematical Society Lecture Note Series. Cambridge University Press, 2022, pp. 194–221.

[Mil80] J. S. Milne. *Étale Cohomology (PMS-33)*. Princeton University Press, 1980. URL: http://www.jstor.org/stable/j.ctt1bpmbk1.