TENSOR PRODUCTS OVER NONUNITAL RINGS DEPEND ON BASE CATEGORIES

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For $R \in \mathsf{Alg}(\mathsf{Sp})$ and its right module M and left module N, one can define

$$M \otimes_R N = \operatorname{colim}\left(\cdots \; M \otimes R \otimes R \otimes N \; M \otimes R \otimes N \; M \otimes N \right)$$

as the simplicial colimit of the bar construction. Note that it can be computed as the semisimplicial colimit

$$M \otimes_R N = \operatorname{colim}\left(\cdots \; M \otimes R \otimes R \otimes N \; M \otimes R \otimes N \; M \otimes N \right)$$

which only depends on the nonunital ring structure of R, so it seems that one can define tensor products over nonunital rings by this semisimplicial colimit. This note aims to show that this definition is not well-behaved for general nonunital rings. More precisely, we will show that for a nonunital ring I, its modules M and N, and a ring A such that I is a nonunital A-algebra, the simplicial colimit

$$\operatorname{colim}\left(\cdots \; M \otimes_A I \otimes_A I \otimes_A N \; M \otimes_A I \otimes_A N \; M \otimes_A N \right)$$

actually depends on A, which will certainly not happen for unital algebras. Our example will be commutative.

Fix a commutative classical ring k. Let $A = k[x]$, $B = k[x, x^{-1}]$, $I = \mathrm{fib}(A \to B) = (x^{-1}k[x^{-1}])[-1]$, $M = N = I$. Then since $B \otimes_A B = B$, it is easy to see that $I \otimes_A I = I$, so the bar construction computing $I \otimes_I I$ over A gives the constant semisimplicial object with value I, whose colimit is thus I. I now claim that the bar construction over k gives something in degree -2, which is different from I that lives in degree -1. As $I[1]$ is a free k-module and hence so is $I^{\otimes_k i}[i]$, the claim follows from the following:

Proposition 1. *Let $X = (X_i)$ be a semisimplicial object in $\mathsf{D}(k)$ such that $X_i[i]$ is a free k-module for all $i \in \mathbb{N}$. Then $\operatorname{colim} X$ is a free k-module.*

Proof. Let $X_{\leq n}$ be the n-truncation of X and let $C_n = \operatorname{colim} X_{\leq n}$. It suffices to prove that C_n is free and the natural map $C_{n-1} \to C_n$ is a split injection. We do induction on n. For $n = 0$ it is obvious. Assume $n > 0$ and this is true for $n - 1$. Map $X_{\leq n}$ to the n-truncated semisimplicial object with X_n on the n^{th} place and 0 on every other place. The fiber of this map is $X_{\leq n-1}$ with a 0 appended on the n^{th} place. Taking colimits, we get a fiber sequence

$$C_{n-1} \to C_n \to X_n[n]$$

which has free k-modules on its first and third place. Therefore its second place C_n is also free and $C_{n-1} \to C_n$ is a split injection. $\square$