GYIN MAPS FOR ORIENTED $\mathbb{P}^1$-SPECTRA

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Categories always mean $\infty$-categories. Everything is derived. Let $\text{Ani}$ be the category of animas. Let $\text{Ring}$ be the category of animated rings. Let $\text{St}$ be the category of Zariski stacks, i.e. accessible functors $\text{Ring} \to \text{Ani}$ that satisfy Zariski descent. Let $\text{St}^{\text{ex}} \subseteq \text{St}$ be the full subcategory consisting of stacks that take blowup squares of derived regular immersions to pullback squares, so there is a localization functor $\text{St} \to \text{St}^{\text{ex}}$ that takes all such squares

$$
\begin{array}{ccc}
D & \longrightarrow & \hat{X} \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X
\end{array}
$$

to pushouts, where $Y \to X$ is a regular immersion; $\text{St}^{\text{ex}}$ is initial with respect to this property. Let $\text{Sp}_{\mathbb{P}^1}(\text{St}^{\text{ex}})$ be the category of $\mathbb{P}^1$-motivic spectra as in [AI22, Definition 2.2.3]. Consider the initial symmetric monoidal category $\text{Sp}_{\mathbb{P}^1}(\text{St}^{\text{ex}})[c]$ under $\text{Sp}_{\mathbb{P}^1}(\text{St}^{\text{ex}})$ with a specified orientation, i.e. a left inverse $c: B\mathbb{G}_m \to \mathbb{P}^1$ to $\mathcal{O}(1): \mathbb{P}^1 \to B\mathbb{G}_m$. Note that by [AI22, Lemma 3.1.7, Theorem 2.4.5], $\text{Sp}_{\mathbb{P}^1}(\text{St}^{\text{ex}})[c]$ is stable. There is obviously a chain of symmetric monoidal left adjoints

$$
\text{St} \to \text{St}^{\text{ex}} \to \text{Sp}_{\mathbb{P}^1}(\text{St}^{\text{ex}}) \to \text{Sp}_{\mathbb{P}^1}(\text{St}^{\text{ex}})[c].
$$

I often tacitly view things on the left as things on the right along these arrows. Note that by this convention, $\mathbb{P}^1$ is not the same as $\Sigma_{\mathbb{P}^1}1$, which is the pointed $\mathbb{P}^1$.

In this writeup, I will associate to every regular immersion $Y \to X$ of codimension $n$ a Gysin map $X/(X \setminus Y) \to \Sigma^n_{\mathbb{P}^1} Y$ in $\text{Sp}_{\mathbb{P}^1}(\text{St}^{\text{ex}})[c]$.

**Remark 1.** Everything in $\text{St}$ is canonically an $E_\infty$-coring by the diagonal map, and for a map $X \to Y$ of stacks, $X$ is canonically a coalgebra over $Y$. Therefore, the same happens in $\text{Sp}_{\mathbb{P}^1}(\text{St}^{\text{ex}})[c]$ for images of stacks and their maps. This does not apply to $c: B\mathbb{G}_m \to \mathbb{P}^1$, since it does not come from a stack map.

**Remark 2.** Let $c_1$ denote the composite map $B\mathbb{G}_m \to \Sigma_{\mathbb{P}^1}1$ of $c: B\mathbb{G}_m \to \mathbb{P}^1$ and the obvious map $\mathbb{P}^1 \to \Sigma_{\mathbb{P}^1}1$. For $n \in \mathbb{N}$, let $c_1^n$ denote the map $B\mathbb{G}_m \to \Sigma_{\mathbb{P}^1}^n$ obtained by applying the Remark 1. By Yoneda, [AI22, Lemma 3.3.5] implies that, for a vector bundle $V \to X$ of rank $n$, the map

$$
\sum_{i=0}^{n-1} c_1^i(\mathcal{O}(1)): \mathbb{P}(V) \to \bigoplus_{i=0}^{n-1} \Sigma_{\mathbb{P}^1}^i X
$$

is an isomorphism of $X$-comodules in $\text{Sp}_{\mathbb{P}^1}(\text{St}^{\text{ex}})[c]$. Similarly, [AI22, Proposition 4.1.5] implies that the map

$$
\sum_{i=0}^{\infty} c_1^i(\mathcal{O}(1)): B\mathbb{G}_m \to \bigoplus_{i=0}^{\infty} \Sigma_{\mathbb{P}^1}^i 1
$$

is well-defined and is an isomorphism in $\text{Sp}_{\mathbb{P}^1}(\text{St}^{\text{ex}})[c]$.
Definition 3 (Chern classes). This is a refinement of [AI22, Definition 4.2.1]. Let $V \to X$ be a vector bundle of rank $n$. Then for $1 \leq i \leq n$, the $i^{th}$ Chern class of $V$ is the map $c_i(V) : X \to \Sigma_{p_1}^i 1$ defined by twisting the composite
\[ \Sigma_{p_1}^{n-i} X \to \mathbb{P}(V) \to B\mathbb{G}_m \to \Sigma_{p_1}^n 1 \]
back by $\Sigma_{p_1}^{-n}$, where the first arrow is the direct inclusion, the second arrow is $O(1)$, and the third arrow is the direct projection. By convention the $0^{th}$ Chern class is the unique map $X \to 1$. The total Chern class of $V$ is the map $c(V): X \to \bigoplus_{i=0}^n \Sigma_{p_1}^i 1$ obtained by summing all the $c_i$’s.

By an argument of Marc Hoyois that Toni Annala taught me, for any vector bundle $V \to X$, the natural map $V/\mathbb{G}_m \to X \times B\mathbb{G}_m$ is an isomorphism in the stabilization of $\text{Sp}_{p_1}(\mathbb{S}^{\text{ex}})$, where $\mathbb{G}_m$ acts on $V$ by multiplication. I will refer to this fact by weighted homotopy invariance in what follows.

Definition 4 (Weighted Thom map). Let $V \to X$ be a vector bundle of rank $n$. By Remark 2, the stack map $O(1): \mathbb{P}(V) \to X \times B\mathbb{G}_m$, after postcomposed with the direct projection
\[ X \times B\mathbb{G}_m = \bigoplus_{i=0}^\infty \Sigma_{p_1}^i X \to \bigoplus_{i=0}^{n-1} \Sigma_{p_1}^i X, \]
becomes an isomorphism of $X$-comodules. Therefore, the direct inclusion
\[ \bigoplus_{i=n}^\infty \Sigma_{p_1}^i X \to \bigoplus_{i=0}^\infty \Sigma_{p_1}^i X = X \times B\mathbb{G}_m \]
gives an isomorphism $\bigoplus_{i=n}^\infty \Sigma_{p_1}^i X \cong (X \times B\mathbb{G}_m)/\mathbb{P}(V)$. The weighted Thom map of $V$ is a map in the stabilization of $\text{Sp}_{p_1}(\mathbb{S}^{\text{ex}})$, defined as
\[ (V/\mathbb{G}_m)/(\mathbb{P}(V) \setminus (0/\mathbb{G}_m)) \cong (X \times B\mathbb{G}_m)/\mathbb{P}(V) \cong \bigoplus_{i=n}^\infty \Sigma_{p_1}^i X \to \Sigma_{p_1}^n X \]
of $X$-comodules, where the first isomorphism is the weighted homotopy invariance and the definition of $\mathbb{P}(V)$, the second isomorphism is as explained in the previous sentence, and the final map is the direct projection.

In what follows, let $Y \to X$ be a regular immersion of codimension $n$. Consider the weighted deformation to the normal cone $\mathcal{Y} \to \mathcal{X}$ as in [Tan22, Definition 5.11]. In short, $\mathcal{X} = (\text{Bl}_Y(A^1_X) \setminus \text{Bl}_Y(X))/\mathbb{G}_m$ thought of as over $A^1_X/\mathbb{G}_m$, and $\mathcal{Y} = A^1_Y/\mathbb{G}_m$ is naturally closed immersed into $\mathcal{X}$. (So the second part of [Tan22, Remark 5.13] is a mistake.) Let $\mathcal{Y}_0 \to X_0$ be the fiber of $\mathcal{Y} \to \mathcal{X}$ over $0/\mathbb{G}_m \to A^1/\mathbb{G}_m$. Then it can be identified with the inclusion $0/\mathbb{G}_m \to \mathcal{N}/X/\mathbb{G}_m$ of the zero section of the weighted normal bundle of $Y$ in $X$. Note that the fiber of $\mathcal{Y} \to \mathcal{X}$ over $\mathbb{G}_m/\mathbb{G}_m \to A^1/\mathbb{G}_m$ is $Y \to X$.

Proposition 5. Let $\mathcal{C}$ denote the stabilization of $\text{Sp}_{p_1}(\mathbb{S}^{\text{ex}})$. Then the natural map $\mathcal{X}_0/(\mathcal{X}_0 \setminus \mathcal{Y}_0) \to \mathcal{X}/(\mathcal{X} \setminus \mathcal{Y})$ is an isomorphism in $\mathcal{C}$.
Proof. Since $Bl_Y(X) \cap \mathbb{A}^1_X = \emptyset$ in $Bl_Y(A^1_X)$, by Zariski excision one can replace $X$ by $Bl_Y(A^1_X)/\mathbb{G}_m$. Consider the two blowup squares

$$
\begin{array}{ccc}
\mathbb{P}(\mathcal{N}_{Y/X}) & \longrightarrow & Bl_Y(X) \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X
\end{array}
\quad
\begin{array}{ccc}
\mathbb{P}(\mathcal{N}_{Y/X} \oplus \mathcal{O}) & \longrightarrow & Bl_Y(A^1_X) \\
\downarrow & & \downarrow \\
Y & \longrightarrow & A^1_X
\end{array}
$$

where the former has an obvious map to the latter. Note that after quotient by $\mathbb{G}_m$, the two lower rows become isomorphic; therefore by stability of $\mathcal{C}$, the cofibers of the two upper rows also become isomorphic. This means that

$$
\begin{array}{ccc}
\mathbb{P}(\mathcal{N}_{Y/X} \times \mathbb{B}\mathbb{G}_m) & \longrightarrow & Bl_Y(X) \times \mathbb{B}\mathbb{G}_m \\
\downarrow & & \downarrow \\
\mathbb{P}(\mathcal{N}_{Y/X} \oplus \mathcal{O})/\mathbb{G}_m & \longrightarrow & Bl_Y(A^1_X)/\mathbb{G}_m
\end{array}
$$

is a fiber square in $\mathcal{C}$. Now note that $Bl_Y(A^1_X) \setminus A^1_Y$ is the total space of $\mathcal{O}_{Bl_Y(X)}(-1)$; this implies that the cofiber of the right column is $X/(X \setminus \mathcal{Y})$. Similarly, since $\mathbb{P}(\mathcal{N}_{Y/X} \oplus \mathcal{O}) \setminus \mathbb{P}(\mathcal{O})$ is the total space of $\mathcal{O}_{\mathbb{P}(\mathcal{N}_{Y/X})}(-1)$, the cofiber of the left column is $X_0/(X_0 \setminus \mathcal{Y}_0)$. This proves the proposition. □

Definition 6 (Gysin map). The Gysin map of $Y \to X$ is defined as the composite

$$
X/(X \setminus Y) \to X/(X \setminus \mathcal{Y}) \cong X_0/(X_0 \setminus \mathcal{Y}_0) \to \Sigma^n_{Sp} Y
$$

in $Sp_{pt}(St^{ex})[c]$, where the first map is induced by the inclusion $X \to X$ and the final map is the weighted Thom map of the vector bundle $\mathcal{N}_{Y/X}$.

Question 7. Is there an unoriented Gysin map $X/(X \setminus Y) \to \Sigma^n_{Sp} Y$ in (the stabilization of) $Sp_{pt}(St^{ex})$?

References
