GYSIN MAPS FOR ORIENTED $\mathbb{P}^1$-SPECTRA

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Categories always mean ∞-categories. Everything is derived. Let Ani be the category of animas. Let Ring be the category of animated rings. Let St be the category of Zariski stacks, i.e. accessible functors $\mathsf{Ring} \to \mathsf{Ani}$ that satisfy Zariski descent. Let $\mathsf{St}^{\mathrm{ex}} \subseteq \mathsf{St}$ be the full subcategory consisting of stacks that take blowup squares of derived regular immersions to pullback squares, so there is a localization functor $\mathsf{St} \to \mathsf{St}^{\mathrm{ex}}$ that takes all such squares

$$\begin{array}{ccc} D & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

to pushouts, where $Y \to X$ is a regular immersion; $\mathsf{St}^{\mathrm{ex}}$ is initial with respect to this property. Let $\mathsf{Sp}_{\mathbb{P}^1}(\mathsf{St}^{\mathrm{ex}})$ be the category of $\mathbb{P}^1$-motivic spectra as in [AI22, Definition 2.2.3]. Consider the initial symmetric monoidal category $\mathsf{Sp}_{\mathbb{P}^1}(\mathsf{St}^{\mathrm{ex}})[c]$ under $\mathsf{Sp}_{\mathbb{P}^1}(\mathsf{St}^{\mathrm{ex}})$ with a specified orientation, i.e. a left inverse $c\colon \mathrm{B}\mathbb{G}_m \to \mathbb{P}^1$ to $\mathcal{O}(1)\colon \mathbb{P}^1 \to \mathrm{B}\mathbb{G}_m$. Note that by [AI22, Lemma 3.1.7, Theorem 2.4.5], $\mathsf{Sp}_{\mathbb{P}^1}(\mathsf{St}^{\mathrm{ex}})[c]$ is stable. There is obviously a chain of symmetric monoidal left adjoints

$$\mathsf{St} \to \mathsf{St}^{\mathrm{ex}} \to \mathsf{Sp}_{\mathbb{P}^1}(\mathsf{St}^{\mathrm{ex}}) \to \mathsf{Sp}_{\mathbb{P}^1}(\mathsf{St}^{\mathrm{ex}})[c].$$

I offen tacitly view things on the left as things on the right along these arrows. Note that by this convention, $\mathbb{P}^1$ is not the same as $\Sigma_{\mathbb{P}^1}1$, which is the pointed $\mathbb{P}^1$.

In this writeup, I will associate to every regular immersion $Y \to X$ of codimension n a Gysin map $X/(X \setminus Y) \to \Sigma^n_{\mathbb{P}^1} Y$ in $\mathsf{Sp}_{\mathbb{P}^1}(\mathsf{St}^{\mathrm{ex}})[c]$.

Remark 1. Everything in St is canonically an $\mathbb{E}_\infty$-coring by the diagonal map, and for a map $X \to Y$ of stacks, X is canonically a coalgebra over Y. Therefore, the same happens in $\mathsf{Sp}_{\mathbb{P}^1}(\mathsf{St}^{\mathrm{ex}})[c]$ for images of stacks and their maps. This does not apply to $c\colon \mathrm{B}\mathbb{G}_m \to \mathbb{P}^1$, since it does not come from a stack map.

Remark 2. Let c_1 denote the composite map $\mathrm{B}\mathbb{G}_m \to \Sigma_{\mathbb{P}^1}1$ of $c\colon \mathrm{B}\mathbb{G}_m \to \mathbb{P}^1$ and the obvious map $\mathbb{P}^1 \to \Sigma_{\mathbb{P}^1}1$. For $n \in \mathbb{N}$, let c_1^n denote the map $\mathrm{B}\mathbb{G}_m \to \Sigma^n_{\mathbb{P}^1}1$ obtained by applying the Remark 1. By Yoneda, [AI22, Lemma 3.3.5] implies that, for a vector bundle $V \to X$ of rank n, the map

$$\sum_{i=0}^{n-1} c_1^i(\mathcal{O}(1))\colon \mathbb{P}(V) \to \bigoplus_{i=0}^{n-1} \Sigma^i_{\mathbb{P}^1} X$$

is an isomorphism of X-comodules in $\mathsf{Sp}_{\mathbb{P}^1}(\mathsf{St}^{\mathrm{ex}})[c]$. Similarly, [AI22, Proposition 4.1.5] implies that the map

$$\sum_{i=0}^{\infty} c_1^i(\mathcal{O}(1))\colon \mathrm{B}\mathbb{G}_m \to \bigoplus_{i=0}^{\infty} \Sigma^i_{\mathbb{P}^1} 1$$

is well-defined and is an isomorphism in $\mathsf{Sp}_{\mathbb{P}^1}(\mathsf{St}^{\mathrm{ex}})[c]$.

Definition 3 (Chern classes)**.** This is a refinement of [AI22, Definition 4.2.1]. Let $V \to X$ be a vector bundle of rank n. Then for $1 \le i \le n$, the i^{th} *Chern class* of V is the map $c_i(V)\colon X \to \Sigma^i_{\mathbb{P}^1} 1$ defined by twisting the composite

$$\Sigma^{n-i}_{\mathbb{P}^1} X \to \mathbb{P}(V) \to \mathrm{B}\mathbb{G}_m \to \Sigma^n_{\mathbb{P}^1} 1$$

back by $\Sigma^{i-n}_{\mathbb{P}^1}$, where the first arrow is the direct inclusion, the second arrow is $\mathcal{O}(1)$, and the third arrow is the direct projection. By convention the 0^{th} *Chern class* is the unique map $X \to 1$. The *total Chern class* of V is the map $c(V)\colon X \to \bigoplus_{i=0}^n \Sigma^i_{\mathbb{P}^1} 1$ obtained by summing all the c_i's.

By an argument of Marc Hoyois that Toni Annala taught me, for any vector bundle $V \to X$, the natural map $V/\mathbb{G}_m \to X \times \mathrm{B}\mathbb{G}_m$ is an isomorphism in the stabilization of $\mathsf{Sp}_{\mathbb{P}^1}(\mathsf{St}^{\mathrm{ex}})$, where $\mathbb{G}_m$ acts on V by multiplication. I will refer to this fact by *weighted homotopy invariance* in what follows.

Definition 4 (Weighted Thom map)**.** Let $V \to X$ be a vector bundle of rank n. By Remark 2, the stack map $\mathcal{O}(1)\colon \mathbb{P}(V) \to X \times \mathrm{B}\mathbb{G}_m$, after postcomposed with the direct projection

$$X \times \mathrm{B}\mathbb{G}_m = \bigoplus_{i=0}^{\infty} \Sigma^i_{\mathbb{P}^1} X \to \bigoplus_{i=0}^{n-1} \Sigma^i_{\mathbb{P}^1} X,$$

becomes an isomorphism of X-comodules. Therefore, the direct inclusion

$$\bigoplus_{i=n}^{\infty} \Sigma^i_{\mathbb{P}^1} X \to \bigoplus_{i=0}^{\infty} \Sigma^i_{\mathbb{P}^1} X = X \times \mathrm{B}\mathbb{G}_m$$

gives an isomorphism $\bigoplus_{i=n}^{\infty} \Sigma^i_{\mathbb{P}^1} X \cong (X \times \mathrm{B}\mathbb{G}_m)/\mathbb{P}(V)$. The *weighted Thom map* of V is a map in the stabilization of $\mathsf{Sp}_{\mathbb{P}^1}(\mathsf{St}^{\mathrm{ex}})$, defined as

$$(V/\mathbb{G}_m)/((V/\mathbb{G}_m) \setminus (0/\mathbb{G}_m)) \cong (X \times \mathrm{B}\mathbb{G}_m)/\mathbb{P}(V) \cong \bigoplus_{i=n}^{\infty} \Sigma^i_{\mathbb{P}^1} X \to \Sigma^n_{\mathbb{P}^1} X$$

of X-comodules, where the first isomorphism is the weighted homotopy invariance and the definition of $\mathbb{P}(V)$, the second isomorphism is as explained in the previous sentence, and the final map is the direct projection.

In what follows, let $Y \to X$ be a regular immersion of codimension n. Consider the weighted deformation to the normal cone $\mathfrak{Y} \to \mathfrak{X}$ as in [Tan22, Definition 5.11]. In short, $\mathfrak{X} = (\mathrm{Bl}_Y(\mathbb{A}^1_X) \setminus \mathrm{Bl}_Y(X))/\mathbb{G}_m$ thought of as over $\mathbb{A}^1_X/\mathbb{G}_m$, and $\mathfrak{Y} = \mathbb{A}^1_Y/\mathbb{G}_m$ is naturally closed immersed into $\mathfrak{X}$. (So the second part of [Tan22, Remark 5.13] is a mistake.) Let $\mathfrak{Y}_0 \to \mathfrak{X}_0$ be the fiber of $\mathfrak{Y} \to \mathfrak{X}$ over $0/\mathbb{G}_m \to \mathbb{A}^1/\mathbb{G}_m$. Then it can be identified with the inclusion $0/\mathbb{G}_m \to \mathcal{N}_{Y/X}/\mathbb{G}_m$ of the zero section of the weighted normal bundle of Y in X. Note that the fiber of $\mathfrak{Y} \to \mathfrak{X}$ over $\mathbb{G}_m/\mathbb{G}_m \to \mathbb{A}^1/\mathbb{G}_m$ is $Y \to X$.

Proposition 5. *Let $\mathcal{C}$ denote the stabilization of $\mathsf{Sp}_{\mathbb{P}^1}(\mathsf{St}^{\mathrm{ex}})$. Then the natural map $\mathfrak{X}_0/(\mathfrak{X}_0 \setminus \mathfrak{Y}_0) \to \mathfrak{X}/(\mathfrak{X} \setminus \mathfrak{Y})$ is an isomorphism in $\mathcal{C}$.*

Proof. Since $\mathrm{Bl}_Y(X) \cap \mathbb{A}^1_Y = \varnothing$ in $\mathrm{Bl}_Y(\mathbb{A}^1_X)$, by Zariski excision one can replace $\mathfrak{X}$ by $\mathrm{Bl}_Y(\mathbb{A}^1_X)/\mathbb{G}_m$. Consider the two blowup squares

$$\begin{array}{ccc} \mathbb{P}(\mathcal{N}_{Y/X}) & \longrightarrow & \mathrm{Bl}_Y(X) \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array} \qquad \begin{array}{ccc} \mathbb{P}(\mathcal{N}_{Y/X} \oplus \mathcal{O}) & \longrightarrow & \mathrm{Bl}_Y(\mathbb{A}^1_X) \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathbb{A}^1_X \end{array}$$

where the former has an obvious map to the latter. Note that after quotient by $\mathbb{G}_m$, the two lower rows become isomorphic; therefore by stability of $\mathcal{C}$, the cofibers of the two upper rows also become isomorphic. This means that

$$\begin{array}{ccc} \mathbb{P}(\mathcal{N}_{Y/X}) \times \mathrm{B}\mathbb{G}_m & \longrightarrow & \mathrm{Bl}_Y(X) \times \mathrm{B}\mathbb{G}_m \\ \downarrow & & \downarrow \\ \mathbb{P}(\mathcal{N}_{Y/X} \oplus \mathcal{O})/\mathbb{G}_m & \longrightarrow & \mathrm{Bl}_Y(\mathbb{A}^1_X)/\mathbb{G}_m \end{array}$$

is a fiber square in $\mathcal{C}$. Now note that $\mathrm{Bl}_Y(\mathbb{A}^1_X) \backslash \mathbb{A}^1_Y$ is the total space of $\mathcal{O}_{\mathrm{Bl}_Y(X)}(-1)$; this implies that the cofiber of the right column is $\mathfrak{X}/(\mathfrak{X} \setminus \mathfrak{Y})$. Similarly, since $\mathbb{P}(\mathcal{N}_{Y/X} \oplus \mathcal{O}) \setminus \mathbb{P}(\mathcal{O})$ is the total space of $\mathcal{O}_{\mathbb{P}(\mathcal{N}_{Y/X})}(-1)$, the cofiber of the left column is $\mathfrak{X}_0/(\mathfrak{X}_0 \setminus \mathfrak{Y}_0)$. This proves the proposition. □

Definition 6 (Gysin map)**.** The *Gysin map* of $Y \to X$ is defined as the composite

$$X/(X \setminus Y) \to \mathfrak{X}/(\mathfrak{X} \setminus \mathfrak{Y}) \cong \mathfrak{X}_0/(\mathfrak{X}_0 \setminus \mathfrak{Y}_0) \to \Sigma^n_{\mathbb{P}^1} Y$$

in $\mathsf{Sp}_{\mathbb{P}^1}(\mathsf{St}^{\mathrm{ex}})[c]$, where the first map is induced by the inclusion $X \to \mathfrak{X}$ and the final map is the weighted Thom map of the vector bundle $\mathcal{N}_{Y/X}$.

Question 7. Is there an unoriented Gysin map $X/(X \setminus Y) \to \Sigma^n_V Y$ in (the stabilization of) $\mathsf{Sp}_{\mathbb{P}^1}(\mathsf{St}^{\mathrm{ex}})$?

References

[AI22] Toni Annala and Ryomei Iwasa. *Motivic spectra and universality of K-theory.* 2022. URL: https://arxiv.org/abs/2204.03434.

[Tan22] Longke Tang. *Syntomic cycle classes and prismatic Poincaré duality.* 2022. URL: https://arxiv.org/abs/2210.14279.