DIRECT SUMMAND THEOREM BY ARC COHOMOLOGY

LONGKE TANG

The derived direct summand theorem is

Theorem 1. *Let R be a regular ring and $f\colon X \to \operatorname{Spec} R$ be proper surjective. Then the natural map $f^*\colon R \to \mathrm{R}\Gamma(X, \mathcal{O}_X)$ splits in $\mathsf{D}(R)$.*

In this note we give a quick proof of it with the theory developed in [BS22].

Proof. We first make some reductions. Let $C_f = \operatorname{cofib}(f^*) \in \mathsf{D}(R)$, then f determines a class $\alpha_f \in \operatorname{Ext}^1(C_f, R)$, and the theorem just says that $\alpha_f = 0$. From this we know that the theorem is fpqc-local, so we can assume R is a complete regular local ring with residue field k algebraically closed.

Now if $\operatorname{char} k = 0$, then by Cohen structure theorem $R = k[[x_1, \ldots, x_n]]$. By Artin–Popescu approximation, R is a colimit of smooth k-algebras; also k is a colimit of regular rings of finite type over $\mathbb{Z}$; so R is a colimit of regular rings of finite type over $\mathbb{Z}$. The theorem can be obviously passed to limit, so it is reduced to the case that R is finite type over $\mathbb{Z}$; but now all the maximal ideals of R has residue fields of positive characteristic, so we have reduced $\operatorname{char} k = 0$ case to $\operatorname{char} k = p$ case.

We also need some reductions about X. For $g\colon Y \to X$ with fg surjective, one easily sees that the class α_{fg} is carried to α_f under the natural map $C_f \to C_{fg}$, so the theorem about fg implies that about f. By this we can assume that X is integral and f is generically finite. We can further assume that f is generically étale: if $R = k[[x_1, \ldots, x_n]]$ is of equal characteristic p, just base change to $k[[x_1^{p^{-N}}, \ldots, x_n^{p^{-N}}]]$ and then take the reduced scheme structure, for a sufficiently large N.

Now we arrive at the following situation: R is a complete regular local ring with residue field k algebraically closed with characteristic p. X is an integral scheme and $f\colon X \to R$ is proper surjective and generically étale. We need to prove that the class $\alpha = \alpha_f \in \operatorname{Ext}^1(C_f, R) = 0$. By passage to limit one can take a nonzero $r \in R$ so that $X_{R_r} \to \operatorname{Spec} R_r$ is already finite étale. Let $X \to \operatorname{Spec} S \to \operatorname{Spec} R$ denote the Stein factorization of f; then on $\operatorname{Spec} R_r$, the first arrow is an isomorphism and the second is finite étale.

By Cohen structure theorem, $R = W(k)[[x_1, \ldots, x_n]]/E$ where $E - p \in (x_1, \ldots, x_n)$. Let $R_N = W(k)[[x_1^{p^{-N}}, \ldots, x_n^{p^{-N}}]]/E$ and $R_\infty = (\operatorname{colim}_N R_N)^\wedge_p$. Then R_∞ is a perfectoid ring p-completely faithfully flat over R. By [BS22, Theorem 7.14], there exists an absolutely integrally closed R' that is p-completely faithfully flat over R_∞. Denote by X', S', f', α' the base change to R' of X, S, f, α, e.g. f' is the composition $X' \to \operatorname{Spec} S' \to \operatorname{Spec} R'$. Take compatible p^N-th roots of r in R' denoted $r^{p^{-N}}$, and let $(r^{p^{-\infty}})$ denote the ideal generated by them. Then $(R', (r^{p^{-\infty}}))$ is an almost setting, denoted r-almost.

I claim that it suffices to prove that α' is r-almost zero. In fact, by the faithful flatness of $R \to R'$ and Grothendieck coherence which implies that $C_f \in \mathsf{D}(R)$ is bounded with finitely generated cohomology, we have

$$\operatorname{Ext}^1_{R'}(C_{f'}, R') = \operatorname{Ext}^1_R(C_f, R') = \operatorname{Ext}^1_R(C_f, R) \otimes_R R',$$

so $\operatorname{Ann}_{R'}(\alpha') = \operatorname{Ann}_R(\alpha)R'$. If α' is r-almost zero, then $\operatorname{Ann}_{R'}(\alpha') \supseteq (r^{p^{-\infty}})$, which means that $\operatorname{Ann}_{R'}(\alpha')^{p^N} \supseteq rR'$ and thus $r \in \operatorname{Ann}_R(\alpha)^{p^N}$ for every N. Since $r \neq 0$, the Krull's intersection theorem forces $\operatorname{Ann}_R(\alpha) = 1$, which means $\alpha = 0$.

Finally we prove that α' is r-almost zero, i.e. $R' \to \mathrm{R}\Gamma(X', \mathcal{O}_{X'})$ splits in the category of p-complete r-almost R'-complexes. We use the p-complete arc topology. Since it is obviously finer than Zariski topology, we have a natural map $\mathrm{R}\Gamma(X', \mathcal{O}_{X'}) \to \mathrm{R}\Gamma(X'_{\mathrm{arc}}, \mathcal{O}_{\mathrm{arc}})$. Now it suffices to prove that $R' \to \mathrm{R}\Gamma(X'_{\mathrm{arc}}, \mathcal{O}_{\mathrm{arc}})$ is p-completely r-almost split. Let I denotes the ideal $(r^{p^{-\infty}})S'$. At this stage, note that since $X' \to \operatorname{Spec} S'$ is proper and an isomorphism outside $V(I)$, the proof of [BS22, Corollary 8.11] actually shows that the diagram

$$\begin{array}{ccc} S'_{\mathrm{perfd}} & \longrightarrow & \mathrm{R}\Gamma(X'_{\mathrm{arc}}, \mathcal{O}_{\mathrm{arc}}) \\ \downarrow & & \downarrow \\ (S'/I)_{\mathrm{perfd}} & \longrightarrow & \mathrm{R}\Gamma((X' \times_{\operatorname{Spec} S'} \operatorname{Spec} S'/I)_{\mathrm{arc}}, \mathcal{O}_{\mathrm{arc}}) \end{array}$$

is a pullback. The two lower modules are over S'/I, thus r-almost zero, so the upper arrow is an r-almost isomorphism. Now it remains to prove that $R' \to S'_{\mathrm{perfd}}$ is p-completely r-almost split, but this follows from [BS22, Theorem 10.9]. $\square$

References

[BS22] Bhargav Bhatt and Peter Scholze. "Prisms and prismatic cohomology". In: *Annals of Mathematics* 196.3 (2022), pp. 1135–1275. URL: https://doi.org/10.4007/annals.2022.196.3.5.