TORSION VANISHING FOR SOME SHIMURA VARIETIES

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Abstract. We generalize the torsion vanishing results of [CS17; CS19; Kos21; San23]. Our results apply to the cohomology of general Shimura varieties $(\mathbf{G}, X)$ of PEL type A or C, localized at a suitable maximal ideal $\mathfrak{m}$ in the spherical Hecke algebra at primes p such that $\mathbf{G}_{\mathbb{Q}_p}$ is a group for which we know the Fargues-Scholze local Langlands correspondence is the semi-simplification of a suitably nice local Langlands correspondence, as shown in [FS21; Ham21; HKW22; BHN22]. This is accomplished by combining Koshikawa's technique [Kos21], the theory of geometric Eisenstein series over the Fargues-Fontaine curve [Ham22], the work of Santos [San23] describing the structure of the fibers of the minimally and toroidally compactified Hodge-Tate period morphism for general PEL type Shimura varieties of type A or C, and ideas developed by Zhang [Zha23] on comparing Hecke correspondences on the moduli stack of G-bundles with the cohomology of Shimura varieties. In the process, we also establish a description of the generic part of the cohomology that bears resemblance to the work of Xiao-Zhu [XZ17]. Moreover, we also construct a filtration on the compactly supported cohomology that differs from Mantovan's filtration in the case that the Shimura variety is non-compact, allowing us to circumvent some of the circumlocutions taken in [CS19; Kos21]. Our method showcases a very general strategy for proving such torsion vanishing results, and should bear even more fruit once the inputs are generalized. Motivated by this, we formulate an even more general torsion vanishing conjecture (Conjecture 6.6).

Contents

1. Introduction

Let $\mathbf{G}$ be a connected reductive group over $\mathbb{Q}$ admitting a Shimura datum $(\mathbf{G}, X)$, and let $\mathbb{A}$ (resp. $\mathbb{A}_f$) denote the adeles (resp. finite adeles) of $\mathbb{Q}$. Fix a prime number $p > 0$ and let $G := \mathbf{G}_{\mathbb{Q}_p}$ be the base-change to $\mathbb{Q}_p$. We will assume that G is unramified so that there exists a hyperspecial subgroup $K_p^{\mathrm{hs}} \subset G(\mathbb{Q}_p)$ and a Borel B surjecting onto a maximal torus T which we now fix. We consider the open compact subgroup $K := K^p K_p^{\mathrm{hs}} \subset \mathbf{G}(\mathbb{A}_f)$, where $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$ denotes a sufficiently small level away from p. Let $\mathrm{Sh}(\mathbf{G}, X)_K$ denote the corresponding Shimura variety defined over the reflex field E. Given a prime $p \neq \ell$, we will be interested in understanding the ℓ-torsion cohomology groups

$$R\Gamma_c(\mathrm{Sh}(\mathbf{G}, X)_{K,\overline{E}}, \overline{\mathbb{F}}_\ell)$$

and

$$R\Gamma(\mathrm{Sh}(\mathbf{G}, X)_{K,\overline{E}}, \overline{\mathbb{F}}_\ell).$$

In particular, since the level at p is hyperspecial, the unramified Hecke algebra

$$H_{K_p^{\mathrm{hs}}} := \overline{\mathbb{F}}_\ell[K_p^{\mathrm{hs}} \backslash G(\mathbb{Q}_p)/K_p^{\mathrm{hs}}]$$

will act on these complexes via the right action. Given a maximal ideal $\mathfrak{m} \subset H_{K_p^{\mathrm{hs}}}$, we can localize both of these cohomology groups at $\mathfrak{m}$. We will be interested in describing this localization. To do this, we recall that, given such a maximal ideal $\mathfrak{m} \subset H_{K_p^{\mathrm{hs}}}$, this defines an unramified L-parameter

$$\phi_{\mathfrak{m}} : W_{\mathbb{Q}_p} \to {}^L G(\overline{\mathbb{F}}_\ell)$$

specified by a semisimple element $\phi_{\mathfrak{m}}(\mathrm{Frob}_{\mathbb{Q}_p})$. In particular, if T denotes the maximal torus of G then $\phi_{\mathfrak{m}}$ is induced from a parameter $\phi_{\mathfrak{m}}^T : W_{\mathbb{Q}_p} \to {}^L T(\overline{\mathbb{F}}_\ell) \subset {}^L G(\overline{\mathbb{F}}_\ell)$ factoring through the L-group of the maximal torus. Now, recall that the irreducible representations of ${}^L T$ correspond to the Γ-orbits $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$ of geometric dominant cocharacters of G. We have the following definition.

Definition 1.1. [Ham22, Definition 1.4] Given a toral L-parameter $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\overline{\mathbb{F}}_\ell)$, we say that ϕ_T is generic if, for all $\alpha \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$ corresponding to a Γ-orbit of coroots, we have that the complex $R\Gamma(W_{\mathbb{Q}_p}, \alpha \circ \phi_T)$ is trivial. Similarly, we say that $\mathfrak{m}$ is generic if $\phi_{\mathfrak{m}}^T$ is a generic toral parameter.

If $G = \mathrm{GL}_n$ then this coincides with the notion of decomposed generic considered in [CS17, Definition I.9]. We set $d = \dim(\mathrm{Sh}(\mathbf{G}, X)_K)$. Motivated by [CS17, Theorem 1.1] and [CS19, Theorem 1.1], we make the following conjecture.

Conjecture 1.2. *Let $(\mathbf{G}, X)$ be a Shimura datum such that $G = \mathbf{G}_{\mathbb{Q}_p}$ is unramified and $K = K_p K^p$ is a sufficiently small level with $K_p = K_p^{\mathrm{hs}}$ hyperspecial. If $\mathfrak{m} \subset H_{K_p^{\mathrm{hs}}}$ is a generic maximal ideal then the cohomology of $R\Gamma(\mathrm{Sh}(\mathbf{G}, X)_{K,\overline{E}}, \overline{\mathbb{F}}_\ell)_{\mathfrak{m}}$ (resp. $R\Gamma_c(\mathrm{Sh}(\mathbf{G}, X)_{K,\overline{E}}, \overline{\mathbb{F}}_\ell)_{\mathfrak{m}}$) is concentrated in degrees $d \leq i \leq 2d$ (resp. $0 \leq i \leq d$).*

We first recall the motivating situation of Caraiani-Scholze [CS17; CS19]. Let $F/\mathbb{Q}$ be a CM field, and let $(B, *, V, \langle\cdot,\cdot\rangle)$ be a PEL datum with B a central simple F-algebra and V a non-zero finite type left B-module. Let $(\mathbf{G}, X)$ denote the Shimura datum attached to it, where $\mathbf{G}$ is a connected reductive group over $\mathbb{Q}$ defined by the B-linear automorphisms of V preserving the choice of pairing $\langle\cdot,\cdot\rangle$. We have the following result.

Theorem 1.3. *[CS17; CS19; Kos21; San23] Assume that $(\mathbf{G}, X)$ is a PEL type Shimura datum of type A. If the prime p splits completely in F then Conjecture 1.2 is true.*

Remark 1.4. Koshikawa proved this under the assumption that $B = F$ and $V = F^{2n}$, and the global unitary group $\mathbf{G}$ is quasi-split, as well as in the case when p is split in F and the Shimura variety is compact. These additional assumptions were removed in the PhD thesis of Santos [San23].

Remark 1.5. Caraiani-Scholze actually proved a slightly different result. More precisely, let S be a set of finite places not containing p such that $\mathbf{G}$ is unramified and K^p is hyperspecial away from S. Consider a maximal ideal $\mathfrak{m} \subset \mathbb{T}^S$ in the spherical Hecke algebra such that $\mathfrak{m}$ is generic at p. Caraiani-Scholze show that the localization at $\mathfrak{m}^p \subset \mathbb{T}^{S\cup\{p\}}$ is concentrated in the relevant degrees.

Remark 1.6. In the case of Harris-Taylor Shimura varieties, there is also work of Boyer [Boy19], which describes the localization at non-generic maximal ideals.

Remark 1.7. We believe that Conjecture 1.2 is true under the weaker hypothesis that $H^2(W_{\mathbb{Q}_p}, \alpha \circ \phi_T)$ is trivial for all Γ-orbits of coroots α, as is shown in [CS19; San23; Kos21] in their particular case. However, the theory of geometric Eisenstein series which we will invoke in this paper becomes more complicated in this case (See the discussion around [Ham22, Conjecture 1.29]), and so a proof of this Theorem using our methods would require more deeply understanding geometric Eisenstein series when this assumption is dropped (cf. Remark 6.8).

Caraiani-Scholze [CS17; CS19] proved their results under some small restrictions, which Koshikawa [Kos21] was able to remove by using compatibility of the Fargues-Scholze local Langlands correspondence with the semi-simplification of the Harris-Taylor correspondence for GL_n. In the process, Koshikawa exhibited a much more flexible method for proving Theorem 1.3. The goal of the current paper is to expand the scope of Koshikawa's technique, motivated by work of the first author on geometric Eisenstein series in the Fargues-Fontaine setting [Ham22]. We then carry the strategy out in some particular cases using work on local-global compatibility of the Fargues-Scholze local Langlands correspondence beyond the case of GL_n.

One of the basic ingredients is the perspective on Mantovan's product formula provided by the Hodge-Tate period morphism. To explain this, we let $\mu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ denote the minuscule geometric dominant cocharacter of G determined by the Hodge cocharacter of X and an isomorphism $j : \mathbb{C} \simeq \overline{\mathbb{Q}}_p$ which we fix from now on. We consider the Kottwitz set $B(G)$ and with it the subset $B(G, \mu) \subset B(G)$ of μ-admissible elements. Let $\mathfrak{p}|p$ be the prime dividing p in the reflex field E, induced by the embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ given by the isomorphism j. We let $E_{\mathfrak{p}}$ be the completion at $\mathfrak{p}$, $C := \hat{\overline{E}}_{\mathfrak{p}}$ be the completion of the algebraic closure, and $\breve{E}_{\mathfrak{p}}$ be the compositum of $E_{\mathfrak{p}}$ with the completion of the maximal unramified extension of $\mathbb{Q}_p$. We recall that, attached to each element $b \in B(G, \mu)$, there exists a diamond

$$\mathrm{Sht}(G, b, \mu)_\infty \to \mathrm{Spd}(\breve{E}_{\mathfrak{p}})$$

parametrizing modifications

$$\mathcal{E}_b \dashrightarrow \mathcal{E}_0$$

of meromorphy μ between the G-bundle $\mathcal{E}_b$ corresponding to b and the trivial G-bundle. This space has an action by $G(\mathbb{Q}_p) = \mathrm{Aut}(\mathcal{E}_0)$ and $J_b(\mathbb{Q}_p) \subset \mathrm{Aut}(\mathcal{E}_b)$, where J_b is the σ-centralizer of b. This allows us to consider the quotients

$$\mathrm{Sht}(G, b, \mu)_\infty/\underline{K_p} \to \mathrm{Spd}(\breve{E}_{\mathfrak{p}})$$

for varying compact open subgroups $K_p \subset G(\mathbb{Q}_p)$. In certain cases, these quotients are representable by rigid analytic varieties called local Shimura varieties, but they are always representable as diamonds. We can consider the compactly supported cohomology

$$R\Gamma_c(\mathrm{Sht}(G, b, \mu)_{\infty,C}/\underline{K_p^{\mathrm{hs}}}, \overline{\mathbb{F}}_\ell)$$

at hyperspecial level with torsion coefficients. This has an action of $W_{E_{\mathfrak{p}}} \times J_b(\mathbb{Q}_p) \times H_{K_p^{\mathrm{hs}}}$. Now, the Mantovan product formula tells us that if we look at $R\Gamma(\mathrm{Sh}(\mathbf{G}, X)_{K,\overline{E}}, \overline{\mathbb{F}}_\ell)$ then this should always admit a filtration in the derived category whose graded pieces are

$$(R\Gamma_c(\mathrm{Sht}(G, b, \mu)_{\infty,C}/\underline{K_p^{\mathrm{hs}}}, \overline{\mathbb{F}}_\ell(d_b))[2d_b] \otimes_{\mathcal{H}(J_b)} R\Gamma(\mathrm{Ig}^b, \overline{\mathbb{F}}_\ell)$$

for varying $b \in B(G, \mu)$, where the objects are as follows.

(1) Ig^b is the perfect Igusa variety attached to an element $b \in B(G, \mu)$ in the μ-admissible locus inside $B(G)$ and $d_b := \dim(\mathrm{Ig}^b) = \langle 2\rho_G, \nu_b \rangle$, where ρ_G is the half sum of all positive roots and ν_b is the slope cocharacter of b.
(2) $\mathcal{H}(J_b) := C_c^{\infty}(J_b(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)$ is the usual smooth Hecke algebra.
(3) $\overline{\mathbb{F}}_\ell(d_b)$ is the sheaf on $\mathrm{Sht}(G, b, \mu)_{\infty, C}/\underline{K_p^{\mathrm{hs}}}$ with trivial Weil group action and $J_b(\mathbb{Q}_p)$ action as defined in [Kos21, Lemma 7.4].

Such a filtration should always exist, but is not currently proven in general. In the case that the Shimura datum $(\mathbf{G}, X)$ is PEL of type A or C, a modern proof of this result can be found in [Kos21, Theorem 7.1].

This filtration on the complex $R\Gamma(\mathrm{Sh}(\mathbf{G}, X)_{K,\overline{E}}, \overline{\mathbb{F}}_\ell)$ allows us to roughly split the verification of Conjecture 1.2 into two parts.

(1) Controlling the cohomology of the shtuka spaces $R\Gamma_c(\mathrm{Sht}(G, b, \mu)_{\infty, C}/\underline{K_p^{\mathrm{hs}}}, \overline{\mathbb{F}}_\ell(d_b))_{\mathfrak{m}}$.
(2) Controlling the cohomology of the Igusa varieties $R\Gamma(\mathrm{Ig}^b, \overline{\mathbb{F}}_\ell)$.

We first discuss point (1). One of the key observations underlying Koshikawa's method was that the cohomology of the space $\mathrm{Sht}(G, b, \mu)_\infty$ computes the action of a Hecke operator T_μ corresponding to μ on Bun_G the moduli stack of G-bundles of the Fargues-Fontaine curve. The Hecke operators commute with the action of the excursion algebra on Bun_G, and the action of the excursion algebra on a smooth irreducible representation ρ, viewed as a sheaf on Bun_G, determines the Fargues-Scholze parameter of ρ. It follows that $R\Gamma_c(\mathrm{Sht}(G, b, \mu)_{\infty, C}/\underline{K_p^{\mathrm{hs}}}, \overline{\mathbb{F}}_\ell(d_b))_{\mathfrak{m}}$ as a complex of smooth $J_b(\mathbb{Q}_p)$-modules will have irreducible constituents ρ with Fargues-Scholze parameter ϕ_ρ^{FS} equal to $\phi_{\mathfrak{m}}$ as conjugacy classes of parameters. When $\mathbf{G}_{\mathbb{Q}_p} = G$ is a product of GL_ns as in Theorem 1.3 (by the assumption that p splits in F), it follows from the work of Hansen-Kaletha-Weinstein [HKW22, Theorem 1.0.3] that the Fargues-Scholze correspondence for $J_b(\mathbb{Q}_p)$ with rational coefficients agrees with the semi-simplification of the Harris-Taylor correspondence, where we recall that J_b is a product of inner forms of GL_ns in this case. In particular, using that $\mathfrak{m}$ is generic, it follows that $\phi_\rho^{\mathrm{FS}} = \phi_{\mathfrak{m}}$ must lift to a $\overline{\mathbb{Z}}_\ell$ parameter which is also generic in the analogous sense, and the condition of generic implies that the lift cannot come from the semi-simplification of a parameter with non-trivial monodromy. Using this, one can deduce that such a ρ only exists if the group J_b is quasi-split. In this particular case (G is a product of GL_ns), this can only happen if $b \in B(G, \mu)$ corresponds to the ordinary element.

This argument of Koshikawa was formalized and generalized further in work of the first author [Ham22]. In particular, it was noted that, for a general quasi-split G and $\mathfrak{m}$ generic, the cohomology $R\Gamma_c(\mathrm{Sht}(G, b, \mu)_{\infty, C}/\underline{K_p^{\mathrm{hs}}}, \overline{\mathbb{F}}_\ell(d_b))_{\mathfrak{m}}$ will only be non-trivial if $b \in B(G, \mu)_{\mathrm{un}} := B(G)_{\mathrm{un}} \cap B(G, \mu)$, where $B(G)_{\mathrm{un}}$ is the set of elements lying in the image of the map $B(T) \to B(G)$, assuming that the Fargues-Scholze local Langlands correspondence has certain expected properties (Assumption 4.4). These unramified elements will be precisely the elements for which J_b is quasi-split. The set $B(G, \mu)_{\mathrm{un}}$ corresponds to Weyl group orbits of weights in the representation V_μ of $\hat{G}$ restricted to $\hat{G}^\Gamma$. In particular, if G is split then, since μ is minuscule, $B(G, \mu)_{\mathrm{un}}$ consists of only one element, corresponding to the unique Weyl group orbit of the highest weight. This is the situation occurring in the previous paragraph. Moreover, the contribution of the cohomology of this shtuka space is easily understood, and the problem completely reduces to controlling the cohomology of Ig^b when $b \in B(G, \mu)_{\mathrm{un}}$ is the μ-ordinary element. However, if G is not split then the restriction of V_μ to $\hat{G}^\Gamma$ may have multiple Weyl group orbits of weights. In particular, one needs to control the cohomology groups

$$R\Gamma_c(\mathrm{Sht}(G, b, \mu)_{\infty, C}/\underline{K_p^{\mathrm{hs}}}, \overline{\mathbb{F}}_\ell(d_b))_{\mathfrak{m}}$$

for all possible $b \in B(G,\mu)_{\mathrm{un}}$. This makes the situation much more complicated; in fact, for non-split G, the basic element could be unramified, and in this case the Igusa variety is just a profinite set, hence the problem of torsion vanishing for the contribution of the basic locus is completely reduced to controlling the generic part of the torsion cohomology of the local shtuka space attached to the basic element.

Such control of the cohomology of shtuka spaces with torsion coefficients for these more general situations was attained in [Ham22]. In order to understand this, it is helpful to move away from the language of isotypic parts of shtuka spaces and consider the action of Hecke operators on $\mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$, the derived category of étale $\overline{\mathbb{F}}_\ell$-sheaves on Bun_G. Since we are interested in cohomology localized at a generic maximal ideal $\mathfrak{m}$, we construct in appendix A a full-subcategory $\mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}} \subset \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ together with an idempotent localization map $(-)_{\phi_{\mathfrak{m}}} : \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell) \to \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}}$ such that, on smooth irreducible representations, the localization map is either an isomorphism or 0 depending on if the representation has Fargues-Scholze parameter conjugate to $\phi_{\mathfrak{m}}$ or not (Lemma 4.2 (1)). We let $\mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ denote the full subcategory of ULA objects, where we recall by [FS21, Theorem V.7.1], that this is equivalent to insisting that the restrictions to all the HN-strata indexed by $b \in B(G)$ are valued in the full subcategories $\mathrm{D}^{\mathrm{adm}}(J_b(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)$ of admissible complexes (i.e the invariants under all open compacts $K \subset J_b(\mathbb{Q}_p)$ is a perfect complex). Using the results of [Ham22], we show, under various technical hypothesis including the genericity of $\mathfrak{m}$, that one has a direct sum decomposition:

$$\mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}} \simeq \bigoplus_{b \in B(G)_{\mathrm{un}}} \mathrm{D}^{\mathrm{adm}}(J_b(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}}.$$

More precisely, we show that the ! and $*$ push-forwards with respect to the inclusion of HN-strata agree on this sub-category, and so the excision semi-orthogonal decomposition splits on $\mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}}$. This decomposition is a refinement of the fact mentioned above that only the shtuka spaces corresponding to the unramified elements $b \in B(G,\mu)_{\mathrm{un}}$ can contribute to the generic localization of the cohomology of the Shimura variety. The desired control of the shtuka spaces is now in turn encoded in understanding how Hecke operators interact with a perverse t-structure on Bun_G after restricting to the localized category $\mathrm{D}^{\mathrm{ULA}}\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}}$.

We recall $\mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ has an action by Hecke operators. In particular, for each geometric dominant cocharacter μ, we have a correspondence

$$\begin{tikzcd} & \mathrm{Hck}_{G,\leq\mu} \arrow[dl, "h_\mu^{\leftarrow}"'] \arrow[dr, "h_\mu^{\rightarrow}"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times \mathrm{Spd}(C) \end{tikzcd}$$

where $\mathrm{Hck}_{G,\leq\mu}$ is the stack parametrizing modifications $\mathcal{E}_1 \to \mathcal{E}_2$ of a pair of G-bundles with meromorphy bounded by μ at the closed Cartier divisor defined by the fixed untilt given by C, and $h_\mu^{\rightarrow}$ (resp. $h_\mu^{\leftarrow}$) remembers $\mathcal{E}_1$ (resp. $\mathcal{E}_2$). We define

$$T_\mu : \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell) \to \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)^{BW_{E_\mu}}$$
$$A \mapsto h^{\rightarrow}_{\mu*}(h_\mu^{\leftarrow*}(A) \otimes^{\mathbb{L}} \mathcal{S}_\mu)$$

where E_μ denotes the reflex field of μ and $\mathcal{S}_\mu$ is the sheaf on $\mathrm{Hck}_{G,\leq\mu}$ attached to the highest weight tilting module $\mathcal{T}_\mu \in \mathrm{Rep}_{\overline{\mathbb{F}}_\ell}(\hat{G})$ of highest weight μ by geometric Satake. The action of Hecke operators commutes with the action of excursion operators and therefore the action of the spectral Bernstein center. Moreover, it preserves the subcategory of ULA objects. It follows that we have an induced map

$$T_\mu : \mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}} \to \mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}}^{BW_{E_\mu}}$$

on the localized category (See Lemma 4.2 (2)).

We are almost ready to state the result on Hecke operators we will need. To do this, we recall that $\mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ has a natural perverse t-structure, which can be defined as follows. The v-stack Bun_G is cohomologically smooth of ℓ-dimension 0. Moreover, each one of the HN-strata Bun_G^b is isomorphic to $[*/\mathcal{J}_b]$, which is cohomologically smooth of ℓ-dimension $-d_b = -\dim(\mathrm{Ig}^b)$. Therefore, we can define a perverse t-structure ${}^{\mathrm{p}}\mathrm{D}^{\geq 0}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ (resp. ${}^{\mathrm{p}}\mathrm{D}^{\leq 0}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$) on $\mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ given by insisting that the ! (resp. $*$) restrictions to Bun_G^b are concentrated in degrees $\geq \langle 2\rho_G, \nu_b\rangle$ (resp. $\leq \langle 2\rho_G, \nu_b\rangle$). The key result that follows from the work of [Ham22] and various compatibility results for the Fargues-Scholze correspondence is as follows.

Theorem 1.8. *(Corollary 4.24) Let μ be a minuscule geometric dominant cocharacter and G a product of groups satisfying the conditions of Table (1) with p and ℓ satisfying the corresponding conditions. Then if $\mathfrak{m}$ is generic the restriction of the Hecke operator*

$$j_1^* T_\mu : \mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_{\phi_\mathfrak{m}} \to \mathrm{D}^{\mathrm{adm}}(G(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)_{\phi_\mathfrak{m}}^{BW_{E_\mu}}$$

is perverse t-exact. In particular, it induces maps

$$j_1^* T_\mu : {}^{\mathrm{p}}\mathrm{D}^{\mathrm{ULA},\geq 0}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_{\phi_\mathfrak{m}} \to \mathrm{D}^{\mathrm{adm},\geq 0}(G(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)_{\phi_\mathfrak{m}}^{BW_{E_\mu}}$$

and

$$j_1^* T_\mu : {}^{\mathrm{p}}\mathrm{D}^{\mathrm{ULA},\leq 0}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_{\phi_\mathfrak{m}} \to \mathrm{D}^{\mathrm{adm},\leq 0}(G(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)_{\phi_\mathfrak{m}}^{BW_{E_\mu}}$$

on the halves of the perverse t-structure, where we note that the perverse t-structure on $\mathrm{D}(\mathrm{Bun}_G^1, \overline{\mathbb{F}}_\ell) \simeq \mathrm{D}(G(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)$ coincides with the usual t-structure.

Here is the table summarizing our local constraints:

(1)

G	Constraint on G	ℓ	p
$\mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GL}_n)$	$L/\mathbb{Q}_p$ unramified	$(\ell, [L : \mathbb{Q}_p]) = 1$	
$\mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GSp}_4)$	$L = \mathbb{Q}_p$ $L/\mathbb{Q}_p$ unramified	$(\ell, 2(p^4 - 1)) = 1$ $(\ell, 2[L : \mathbb{Q}_p](p^{4[L:\mathbb{Q}_p]} - 1)) = 1$	 $p \neq 2$
$\mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GU}_2)$	$L/\mathbb{Q}_p$ unramified	$(\ell, [L : \mathbb{Q}_p]) = 1$	
$G = \mathrm{U}_n(L/\mathbb{Q}_p)$	n odd L unramified	$\ell \neq 2$	
$G = \mathrm{GU}_n(L/\mathbb{Q}_p)$	n odd L unramified	$\ell \neq 2$	
$G(\mathrm{SL}_{2,L})$	$L/\mathbb{Q}_p$ unramified	$(\ell, [L : \mathbb{Q}_p]) = 1$	
$G(\mathrm{Sp}_{4,L})$	$L/\mathbb{Q}_p$ unramified, $L \neq \mathbb{Q}_p$	$(\ell, 2[L : \mathbb{Q}_p](p^{4[L:\mathbb{Q}_p]} - 1)) = 1$	$p \neq 2$

The groups $G(\mathrm{SL}_{2,L})$ and $G(\mathrm{Sp}_{4,L})$ are the similitude subgroup of $\mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GL}_2)$ (resp. $\mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GSp}_4)$), i.e. the subgroup of elements such that the similitude factor lies in $\mathbb{Q}_p$. We will recall the definition of these groups in §4.3.

Remark 1.9. Assuming the Fargues-Scholze correspondence for G behaves as expected with rational coefficients, the analysis in [Ham22] allows one to verify this for any μ after imposing some additional conditions on the toral parameter $\phi_T^\mathfrak{m}$ attached to the maximal ideal $\mathfrak{m}$ ([Ham22, Condition/Definition 3.6]). However, for the groups considered, we will see that these additional conditions are superfluous and all one needs is generic, except for the case where $G = \mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GSp}_4)$ or $G = G(\mathrm{Sp}_{4,L})$ with $L/\mathbb{Q}_p$ non-trivial, where we need an extra banality assumption on the prime ℓ. It is conjectured ([Ham22, Conjecture 1.27]) that the results used to establish this theorem should always be true just under the condition that $\mathfrak{m}$ is generic.

Remark 1.10. We should warn the reader that some of the results of [Ham22] and in particular this consequence, are currently contingent on the proof of some ULA properties of sheaves on the moduli stack of B-structures [Ham22, Assumption 8.1]. However, this will appear in forthcoming work on geometric Eisenstein series [HHS].

These local torsion vanishing results would allow us to prove Conjecture 1.2 in several new cases if one could get control over the Igusa varieties Ig^b. In Koshikawa's argument, this is done by using a semi-perversity result proven by Caraiani-Scholze [CS19, Theorem 4.6.1], which was further generalized in work of Santos [San23]. Roughly speaking, we want to show that $R\Gamma(\mathrm{Ig}^b, \overline{\mathbb{F}}_\ell)$ is concentrated in degrees $\geq d_b$, so that the complex of $J_b(\mathbb{Q}_p)$-representations $R\Gamma(\mathrm{Ig}^b, \overline{\mathbb{F}}_\ell)$ defines the stalk of a semi-perverse sheaf on Bun_G at $b \in B(G)$, to which we can apply the previous result. In the case that the Shimura vareities $\mathrm{Sh}(\mathbf{G}, X)_K$ are compact, there is a simpler way of seeing this. In particular, Ig^b is known to be a perfect affine scheme in this case, and so the desired semi-perversity just follows by applying Artin vanishing and then using Poincaré duality on the global Shimura variety. It turns out that this style of argument can be made to work even in the non-compact case. In [CS17; CS19; Kos21; San23], the non-compactly supported cohomology $R\Gamma(\mathrm{Sh}(\mathbf{G}, X)_K, \overline{\mathbb{F}}_\ell)_{\mathfrak{m}}$ is studied together with its filtration involving $R\Gamma(\mathrm{Ig}^b, \overline{\mathbb{F}}_\ell)$ coming from Mantovan's formula, and shown to be concentrated in degrees $\geq d$. However, one could also study the compactly supported cohomology $R\Gamma_c(\mathrm{Sh}(\mathbf{G}, X)_K, \overline{\mathbb{F}}_\ell)_{\mathfrak{m}}$ and show that it is concentrated in degrees $\leq d$, à la Poincaré duality. To do this, we recall [CS19, Section 3.3] that, in the non-compact case, the perfect scheme Ig^b is not affine, but it admits a partial minimal compactification $g_b : \mathrm{Ig}^b \hookrightarrow \mathrm{Ig}^{b,*}$ which is affine, as proven in this more general setting of PEL type A or C by Santos [San23]. We define

$$V_b := R\Gamma_{c-\partial}(\mathrm{Ig}^b, \overline{\mathbb{F}}_\ell) := R\Gamma(\mathrm{Ig}^{b,*}, g_{b!}(\overline{\mathbb{F}}_\ell))$$

the partially compactly supported cohomology, which is supported in degrees $\leq d_b$ by Artin-vanishing (Proposition 3.7). Now, for $K \subset \mathbf{G}(\mathbb{A}_f)$ a sufficiently small open compact, we define $\mathcal{S}(\mathbf{G}, X)_K := (\mathrm{Sh}(\mathbf{G}, X)_K \otimes_E E_{\mathfrak{p}})^{\mathrm{ad}}$ to be the adic space over $\mathrm{Spa}(E_{\mathfrak{p}})$ attached to the Shimura variety. We can define the infinite level perfectoid Shimura varieties $\mathcal{S}(\mathbf{G}, X)_{K^p}$ by taking the inverse limit of $\mathcal{S}(\mathbf{G}, X)_{K^pK_p}$ as $K_p \to \{1\}$. The base-change $\mathcal{S}(\mathbf{G}, X)_{K^p,C}$ is representable by a perfectoid space if $(\mathbf{G}, X)$ is of pre-abelian type, and in general it is diamond. By the results of [Sch15; Han16], we have a Hodge-Tate period map

$$\pi_{\mathrm{HT}} : [\mathcal{S}(\mathbf{G}, X)_{K^p,C}/\underline{G(\mathbb{Q}_p)}] \to [\mathcal{F}\ell_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}]$$

recording the Hodge-Tate filtration on the abelian varieties with additional structure that $\mathcal{S}(\mathbf{G}, X)_{K^p,C}$ parametrizes. Here $\mathcal{F}\ell_{G,\mu^{-1}} := (G_C/P_{\mu^{-1}})^{\mathrm{ad}}$ is the adic flag variety attached to the parabolic in G_C given by a dominant inverse of μ and the dynamical method. We recall that the flag variety $[\mathcal{F}\ell_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}]$ admits a locally closed stratification $i_b : [\mathcal{F}\ell^b_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}] \hookrightarrow [\mathcal{F}\ell_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}]$ indexed by $b \in B(G, \mu)$, given by pulling the HN-stratification along the natural map $h^{\leftarrow} : [\mathcal{F}\ell_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}] \to \mathrm{Bun}_G$. We will now impose the following very mild assumption in what follows.

Assumption 1.11. *Write* $\partial\mathrm{Ig}^{b,*} \subset \mathrm{Ig}^{b,*}$ *for the closed complement of* Ig^b *in* $\mathrm{Ig}^{b,*}$*. We assume that* $(\mathbf{G}, X)$ *is a PEL datum of type A or C such that, for all* $b \in B(G, \mu)$*, the perfect scheme* $\partial\mathrm{Ig}^{b,*}$ *is empty or has codimension in* $\mathrm{Ig}^{b,*}$ *greater than* 2.

Remark 1.12. If $\mathbf{G}$ is simple then it is easy to show that this assumption will be satisfied if $\dim(\mathcal{S}(\mathbf{G}, X)_{K^p,C}) \geq 2$, by using that the boundary of the partially minimally compactified Igusa varieties is expressible as the Igusa varieties of Shimura varieties attached to Levis of $\mathbb{Q}$-rational parabolics of $\mathbf{G}$, as we will explain in §2.2.1. Moreover, if $\mathcal{S}(\mathbf{G}, X)_{K^p,C}$ is compact then it is automatic that $\partial\mathrm{Ig}^{b,*}$ is empty. Therefore, if $\mathbf{G}$ is simple, this is excluding the cases where $\dim(\mathcal{S}(\mathbf{G}, X)_{K^p,C}) = 1$ and $\mathcal{S}(\mathbf{G}, X)_{K^p,C}$ is non-compact. There are two possibilities; either $(\mathbf{G}, X)$ is the Shimura datum attached to the modular curve, or it is the Shimura datum attached to the unitary Shimura curve (See [Zha23, Proposition 1.9]). In the latter case, we have that the connected components are given by modular curves. In these cases, the results of [Kos21] are sufficient to prove Conjecture 1.2.

Now, assuming this, one can show that the stalk of $R\pi_{\mathrm{HT}!}(\overline{\mathbb{F}}_\ell)$ at a geometric point $x : \mathrm{Spa}(C, C^+) \to \mathcal{F}\ell_{G,\mu^{-1}}$ which lies in the adic Newton strata $\mathcal{F}\ell^b_{G,\mu^{-1}}$ is given by V_b. Moreover, if we write $h^{\leftarrow}_b : [\mathcal{F}\ell^b_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}] \to [\mathrm{Spd}(C)/\mathcal{J}_b] \simeq \mathrm{Bun}_G^b$ for the pullback of $h^{\leftarrow}$ to Bun_G^b then one can deduce that the complex $i_{b!}i_b^*R\pi_{\mathrm{HT}!}(\overline{\mathbb{F}}_\ell)$ is isomorphic to $h^{\leftarrow *}j_{b!}(V_b)$. Therefore, by excision, we deduce that the complex of $G(\mathbb{Q}_p) \times W_{E_\mathfrak{p}}$-representations

$$h^{\rightarrow}_* R\pi_{\mathrm{HT}!}(\overline{\mathbb{F}}_\ell) \simeq R\Gamma_c(\mathcal{S}_{K^p,C}, \overline{\mathbb{F}}_\ell) \simeq \mathrm{colim}_{K_p \to \{1\}} R\Gamma_c(\mathcal{S}_{K^pK_p,C}, \overline{\mathbb{F}}_\ell) \simeq \mathrm{colim}_{K_p \to \{1\}} R\Gamma_c(\mathrm{Sh}(\mathbf{G}, X)_{K^pK_p,C})$$

has a filtration with graded pieces isomorphic to $h^{\rightarrow}_* h^{\leftarrow *}(j_{b!}(V_b))$ for varying $b \in B(G,\mu)$, where $h^{\rightarrow} : [\mathcal{F}\ell_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}] \to [\mathrm{Spd}(C)/\underline{G(\mathbb{Q}_p)}]$ is the structure map quotiented by $G(\mathbb{Q}_p)$. Here the second isomorphism follows since taking compactly supported cohomology respects taking limits of spaces, and the third isomorphism is a standard comparison result due to Huber [Hub96, Theorem 3.5.13].

Now, via the Bialynicki-Birula isomorphism, the flag variety $[\mathcal{F}\ell_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}]$ identifies with an open substack of $\mathrm{Hck}_{G,\leq\mu}$ for the fixed minuscule μ. In particular, under this relationship the maps $h^{\rightarrow}_\mu$ and $h^{\leftarrow}_\mu$ identify with $h^{\rightarrow}$ and $h^{\leftarrow}$, and therefore we can relate the graded pieces of the excision filtration to Hecke operators. We write

$$R\Gamma_c(G, b, \mu) := \mathrm{colim}_{K_p \to \{1\}} R\Gamma_c(\mathrm{Sht}(G, b, \mu)/\underline{K_p}, \overline{\mathbb{F}}_\ell(d_b))$$

for the complex of $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_{E_\mathfrak{p}}$-modules defined by the compactly supported cohomology of this tower. Here $\overline{\mathbb{F}}_\ell(d_b)$ is the sheaf with $J_b(\mathbb{Q}_p)$-action defined as in [Kos21, Lemma 7.4].

We deduce the following variant of the Mantovan product formula for the compactly supported cohomology.

Theorem 1.13. *The complex $R\Gamma_c(\mathcal{S}_{K^p,C}, \overline{\mathbb{F}}_\ell)$ has a filtration as a complex of $G(\mathbb{Q}_p) \times W_{E_\mathfrak{p}}$-representations with graded pieces isomorphic to $j_1^*(T_\mu j_{b!}(V_b))[-d](-\frac{d}{2})$. More specifically, the graded pieces are isomorphic to*

$$(R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} V_b)[2d_b].$$

as $G(\mathbb{Q}_p) \times W_{E_\mathfrak{p}}$-modules.

Remark 1.14. When the Shimura variety is compact, we have that $R\Gamma_{c-\partial}(\mathrm{Ig}^b, \overline{\mathbb{F}}_\ell) \simeq R\Gamma(\mathrm{Ig}^b, \overline{\mathbb{F}}_\ell)$, and this recovers precisely [Kos21, Theorem 7.1].

We now apply our localization functor $(-)_{\phi_\mathfrak{m}} : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_G)_{\phi_\mathfrak{m}}$ for a generic maximal ideal $\mathfrak{m}$ to get a complex $R\Gamma_c(\mathcal{S}_{K^p,C}, \overline{\mathbb{F}}_\ell)_{\phi_\mathfrak{m}} \in \mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_{\phi_\mathfrak{m}}$, which we view as a sheaf on Bun_G by ! extending along the neutral strata. After applying $R\Gamma(K_p^{\mathrm{hs}}, -)$, this agrees with $R\Gamma_c(\mathcal{S}_{K^pK_p^{\mathrm{hs}},C}, \overline{\mathbb{F}}_\ell)_\mathfrak{m}$, the usual localization under the unramified Hecke algebra, which is the object we want to study. This in turn admits a filtration by $R\Gamma(K_p^{\mathrm{hs}}, (j_1^*T_\mu j_{b!}(V_b)))_{\phi_\mathfrak{m}}[-d](\frac{-d}{2})$. However, now we know, by the direct sum decomposition of $\mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_{\phi_\mathfrak{m}}$ described above, that the natural map $j_{b!}(V_b) \to j_{b*}(V_b)$ is an isomorphism after applying $(-)_{\phi_\mathfrak{m}}$. Moreover, one only has interesting contributions coming from the unramified elements $B(G,\mu)_{\mathrm{un}}$. In particular, we can deduce the following Corollary.

Theorem 1.15. *Suppose $(\mathbf{G}, X)$ is a PEL datum of type A or C such that $\mathbf{G}_{\mathbb{Q}_p}$ is a product of simple groups as in Table (1) with p and ℓ satisfying the corresponding conditions, the complex $R\Gamma_c(\mathcal{S}_{K,C}, \overline{\mathbb{F}}_\ell)_\mathfrak{m} \simeq R\Gamma_c(\mathrm{Sh}(\mathbf{G}, X)_{K,C}, \overline{\mathbb{F}}_\ell)_\mathfrak{m}$ breaks up as a direct sum*

$$\bigoplus_{b \in B(G,\mu)_{\mathrm{un}}} (R\Gamma_c(\mathrm{Sht}(G, b, \mu)_{\infty,C}/\underline{K_p^{\mathrm{hs}}}, \overline{\mathbb{F}}_\ell(d_b))_\mathfrak{m} \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} R\Gamma_{c-\partial}(\mathrm{Ig}^b, \overline{\mathbb{F}}_\ell))[2d_b]$$

of $H_{K_p^{\mathrm{hs}}} \times W_{E_\mathfrak{p}}$-modules.

Remark 1.16. As we will explain more in §6.1, in the case that the unique basic element $b \in B(G,\mu)_{\mathrm{un}}$ is unramified, the contribution of the corresponding summand to middle degree cohomology serves as a generic fiber analogue of the description of the middle degree cohomology on the special fiber of the integral model at hyperspecial level, as provided in [XZ17, Theorem 1.1.4].

As a consequence, we deduce our main Theorem, by combining Theorem 1.7 with the fact that $R\Gamma_{c-\partial}(\mathrm{Ig}^b, \overline{\mathbb{F}}_\ell) \in \mathrm{D}^{\leq d_b}(J_b(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)$, by Artin vanishing.

Theorem 1.17. *Suppose* $(\mathbf{G}, X)$ *is a PEL datum of type A or C such that* $\mathbf{G}_{\mathbb{Q}_p}$ *is a product of simple groups as in Table (1) with* p *and* ℓ *satisfying the corresponding conditions then Conjecture 1.2 is true.*

Remark 1.18. This notably allows one to relax the assumption in [CS17; CS19; Kos21; San23] that the prime p splits in F, answering a question of Caraiani.

We can also easily deduce the result for some abelian type Shimura varieties, such as Hilbert modular varieties, from the above result, which recovers work of Caraiani-Tamiozzo [CT21] (See Corollary 5.3).

Corollary 1.19. *Suppose* $(\mathbf{G}, X)$ *is an abelian-type Shimura datum which has an associated PEL-type datum* $(\mathbf{G}_1, X_1)$ *of type A or C such that* $\mathbf{G}_{1,\mathbb{Q}_p}$ *is a product of simple groups as in Table (1) with* p *and* ℓ *satisfying the corresponding conditions. Then Conjecture 1.2 is true.*

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Notation

- Fix distinct primes $\ell \neq p$.
- We write $\mathbb{Q}_p$ for the p-adic numbers, and $\breve{\mathbb{Q}}_p$ for the completion of the maximal unramified extension with Frobenius σ.
- We let $\overline{\mathbb{F}}_\ell$ denote the algebraic closure of the finite field $\mathbb{F}_\ell$. We fix a choice of square root of p in $\overline{\mathbb{F}}_\ell$ and define all half Tate twists and square roots of the norm character with respect to this choice.
- For $L/\mathbb{Q}_p$ a finite extension, we write $\breve{L} := L\breve{\mathbb{Q}}_p$ for the compositum of L with the maximal unramified extension and W_L for the Weil group of L. We let $\mathrm{WD}_L := W_L \times \mathrm{SL}(2, \overline{\mathbb{Q}}_\ell)$ denote the Weil-Deligne group of L.
- We let $\mathbb{A}$ (resp. $\mathbb{A}_f$) denote the adeles (resp. finite adeles) over $\mathbb{Q}$.
- A pair $(\mathbf{G}, X)$ will denote a Shimura datum. We will use E to denote the reflex field. For $K \subset \mathbf{G}(\mathbb{A}_f)$ a sufficiently small open compact, we write $\mathrm{Sh}(\mathbf{G}, X)_K \to \operatorname{Spec} E$ for the attached Shimura variety of level K.
- We fix an isomorphism $j : \overline{\mathbb{Q}}_p \xrightarrow{\simeq} \mathbb{C}$. Consider the induced embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ this gives a finite place $\mathfrak{p}$ of E. We write $E_{\mathfrak{p}}$ for the completion at $\mathfrak{p}$.
- We let $C := \hat{\overline{E}}_{\mathfrak{p}}$ be the completed algebraic closure of $E_{\mathfrak{p}}$.

- We use the symbol G to always denote a connected reductive group over $\mathbb{Q}_p$, usually taken to be $\mathbf{G}_{\mathbb{Q}_p}$. We will always assume that G is quasi-split with a fixed choice $T \subset B \subset G$ of maximal torus and Borel, respectively.
- If G is unramified then we let $K_p^{\mathrm{hs}} \subset G(\mathbb{Q}_p)$ be a choice of hyperspecial subgroup. We set $H_{K_p^{\mathrm{hs}}} := \overline{\mathbb{F}}_\ell[K_p^{\mathrm{hs}} \backslash G(\mathbb{Q}_p)/K_p^{\mathrm{hs}}]$ to be the unramified Hecke algebra with $\overline{\mathbb{F}}_\ell$-coefficients.
- We let $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ denote the set of geometric dominant cocharacters of G and let $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+/\Gamma$ denote the set of Galois orbits, where $\Gamma := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.
- Let $B(G) := G(\breve{\mathbb{Q}}_p)/(g \sim hg\sigma(h)^{-1})$ denote the Kottwitz set of G.
- For $b \in B(G)$, we write J_b for the σ-centralizer of b.
- For $\mu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$, we let $B(G, \mu)$ be the μ-admissible locus (as defined in [RV14, Definition 2.3]).
- Let Perf denote the category of affinoid perfectoid spaces in characteristic p over $* := \mathrm{Spd}(\overline{\mathbb{F}}_p)$ endowed with the v-topology. For a perfectoid space S, let Perf_S denote the category of affinoid perfectoid spaces over the tilt $S^\flat$.
- For $S \in \mathrm{Perf}$, let X_S denote the relative schematic Fargues-Fontaine curve over S.
- For $\mathrm{Spa}\,(F, \mathcal{O}_F) \in \mathrm{Perf}$ a geometric point, we will often drop the subscript on X_F and just write X for the associated Fargues-Fontaine curve.
- For $b \in B(G)$, we write $\mathcal{E}_b$ for the associated G-bundle on X.
- For $S \in \mathrm{Perf}$, we let $\mathcal{E}_0$ denote the trivial G-bundle on X_S.
- To a diamond or v-stack X over $*$, we write $\mathrm{D}(X, \overline{\mathbb{F}}_\ell)$ for the category of étale $\overline{\mathbb{F}}_\ell$-sheaves, as defined in [Sch18]. We let $\mathrm{D}^{\mathrm{ULA}}(X, \overline{\mathbb{F}}_\ell)$ denote the full subcategory of ULA sheaves over $*$.
- For an Artin v-stack X and $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$, we write $\mathrm{D}_{\blacksquare}(X, \Lambda)$ for the condensed ∞-category of solid $\overline{\mathbb{F}}_\ell$-sheaves on X, and write $\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \subset \mathrm{D}_{\blacksquare}(X, \Lambda)$ for the full sub-category of Λ-lisse-étale sheaves, as defined in [FS21, Chapter VII].
- If X is an Artin v-stack ([FS21, Definition IV.V.1]) admitting a separated cohomologically smooth surjection $U \to X$ from a locally spatial diamond U such that the etale site has a basis with bounded ℓ-cohomological dimension (which will always be the case for our applications) then we will regard it as a condensed ∞-category via the identification $\mathrm{D}_{\mathrm{lis}}(X, \overline{\mathbb{F}}_\ell) \simeq \mathrm{D}(X, \overline{\mathbb{F}}_\ell)$ when viewed as objects in $\mathrm{D}_{\blacksquare}(X, \overline{\mathbb{F}}_\ell)$ [FS21, Proposition VII.6.6].
- We let $\hat{G}$ denote the Langlands dual group of G with fixed splitting $(\hat{T}, \hat{B}, \{X_\alpha\})$.
- If E denotes the splitting field of G then the action of $W_{\mathbb{Q}_p}$on $\hat{G}$ factors through $Q := W_{\mathbb{Q}_p}/W_E$. We let ${}^LG := \hat{G} \rtimes Q$ denote the L-group.
- For I a finite index set, we let $\mathrm{Rep}_{\overline{\mathbb{F}}_\ell}({}^LG^I)$ (resp. $\mathrm{Rep}_{\overline{\mathbb{F}}_\ell}(\hat{G}^I)$) denote the category of finite-dimensional algebraic representations of ${}^LG^I$ (resp. $\hat{G}^I$) over $\overline{\mathbb{F}}_\ell$.
- For $\mu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$, we write $V_\mu \in \mathrm{Rep}_{\overline{\mathbb{F}}_\ell}(\hat{G})$ (resp. $\mathcal{T}_\mu \in \mathrm{Rep}_{\overline{\mathbb{F}}_\ell}(\hat{G})$) for the usual highest weight representation (resp. highest weight tilting module, as in [Don93]) of highest weight μ.
- To any condensed ∞-category $\mathcal{C}$, we write $\mathcal{C}^{BW^I_{\mathbb{Q}_p}}$ for the category of objects with continuous $W^I_{\mathbb{Q}_p}$-action, as defined in [FS21, Section IX.1].
- For any separated v-stack, $X \to \mathrm{Spa}(K, \mathcal{O}_K)$ where $\mathrm{Spa}(K, \mathcal{O}_K)$ is a non-archimedean field, we write $\overline{X}$ for the canonical compactification of X with respect to the structure map ([Sch18, Proposition 18.6], [Hub96, Theorem 5.15]).
- For a reductive group $H/\mathbb{Q}_p$, we write $\mathrm{D}(H(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)$ for the unbounded derived category of smooth $\overline{\mathbb{F}}_\ell$-representations.
- For an analytic adic space X, we will often abuse notation and use X to also denote the diamond $X^\diamond$ attached to it (as defined in [SW20, Lecture X]).

2. Preliminaries on Shimura Varieties

In this section we will recall some facts about Shimura varieties which we will need later in this paper.

2.1. **Shimura Varieties.**

We will mainly work with the following two types of Shimura varieties.

2.1.1. *PEL type A and C.* Let $(\mathcal{O}_B, *, L, \langle\cdot,\cdot\rangle)$ be an integral PEL datum, where B is a finite-dimensional semisimple $\mathbb{Q}$-algebra, $*$ is a $\mathbb{Q}$-linear involution of B, with fixed field F, $\mathcal{O}_B$ is a $*$-stable $\mathbb{Z}$-order of B, L is a lattice with $\mathcal{O}_B$-actions, and $\langle\cdot,\cdot\rangle : L \times L \to \mathbb{Z}(1)$ is a non-degenerate alternating form such that $\langle bv, v'\rangle = \langle v, b^* v'\rangle$, for all $b \in \mathcal{O}_B$ and $v, v' \in L$.

To our integral PEL datum, we have the following group scheme $\mathbf{G}$ over $\mathbb{Z}$ whose R-points, for each $\mathbb{Z}$-algebra R, are given by

$$\mathbf{G}(R) := \{(g, r) \in \mathrm{End}_{\mathcal{O}_B \otimes_{\mathbb{Z}} R}(L \otimes_{\mathbb{Z}} R) \times R | \langle gv, gw\rangle = r\langle v, w\rangle \text{ for all } v, w \in L \otimes_{\mathbb{Z}} R\}.$$

The PEL data is of type A if $(B \otimes_F \overline{F}, *)$ is isomorphic to $\mathrm{End}(W) \times \mathrm{End}(W)^{\mathrm{op}}$ with $(a, b)^* = (b, a)$ for some vector space W. The PEL data is of type C if $(B \otimes_F \overline{F}, *)$ is isomorphic to $\mathrm{End}(W)$ with $*$ being the adjoint map with respect to a symmetric bilinear form on W. We will assume from now on that we are in one of these cases.

We now further assume the data is unramified at p; namely, that each term in the decomposition $B_{\mathbb{Q}_p} = \prod_{\mathfrak{p}|p} B \otimes_F F_{\mathfrak{p}}$ is a matrix algebra over an unramified extension of $\mathbb{Q}_p$. We will thus moreover assume that $\mathcal{O}_B \otimes \mathbb{Z}_p$ is a maximal order in $B_{\mathbb{Q}_p}$, and L is self-dual after localization at p. This can be arranged following [Lan13, Remark 1.3.4.8]. Note that these conditions equivalently ensure that the group G is unramified.

We will now briefly discuss what conditions we may need to impose on the prime p so that the form of the local group G satisfies the conditions in Table (1). Firstly, suppose we are in type A. Then the center $Z(B) = F_c$ is a quadratic imaginary extension of F. Let n be the $\mathcal{O}_B$ rank of L. Observe that we will need to assume that the prime p satisfies for all primes $\mathfrak{p}$ of F above p,

(1) $\mathfrak{p}$ is split in F_c; or
(2) $F_{\mathfrak{p}} = \mathbb{Q}_p$, and n is odd.

These conditions imply that G will be a similitude subgroup of $\prod_{\mathfrak{p}} G_{\mathfrak{p}}$ where $G_{\mathfrak{p}}$ is either $\mathrm{Res}_{F_{\mathfrak{p}}/\mathbb{Q}_p}(\mathbb{G}_m \times \mathrm{GL}_n)$ or GU_n for an odd unitary group over $\mathbb{Q}_{p^2}$.

Now suppose we are in type C. Since the PEL data is unramified at p, we see that $B \otimes_F F_{\mathfrak{p}}$ is indefinite for all primes $\mathfrak{p}$ of F above p, and thus G will be a similitude subgroup of

$$\prod_{\mathfrak{p}} \mathrm{Res}_{F_{\mathfrak{p}}/\mathbb{Q}_p}(\mathrm{GSp}_{2n}).$$

Here, we will need to assume that the rank n of L as an $\mathcal{O}_B$ lattice is either 1 or 2 to satisfy the conditions in Table (1).

Both types of Shimura varieties will be moduli spaces of abelian varieties with extra structures, which we will briefly describe. Let $K^p \subset G(\mathbb{A}_f^p)$ be an open compact subgroup. To any PEL data, the Shimura variety $\mathfrak{S}(\mathbf{G}, X)_K$ over $\mathcal{O}_{E_{\mathfrak{p}}}$ is the scheme which represents the functor that associates to each locally Noetherian scheme S over $\mathcal{O}_{E_{\mathfrak{p}}}$ the set of isomorphism classes of tuples $(A, \lambda, \iota, \eta^p)$ consisting of

(1) An abelian scheme A/S of dimension $n[F : \mathbb{Q}]$ up to prime to p-isogeny,
(2) A prime-to-p polarization $\lambda : A \to A^\vee$,
(3) An embedding $\iota : \mathcal{O}_B \otimes \mathbb{Z}_{(p)} \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ of $\mathbb{Z}_{(p)}$-algebras such that

$$\lambda \circ \iota(b^*) = \iota(b)^\vee \circ \lambda,$$

(4) A section $\eta^p \in \Gamma(S, \mathrm{Isom}_B(L \otimes_{\mathbb{Z}} \mathbb{A}_f^p, \hat{V}(A)^p)/K^p)$, where $\mathrm{Isom}_B(L \otimes_{\mathbb{Z}} \mathbb{A}_f^p, \hat{V}(A)^p)$ is the étale torsor of isomorphisms that maps $\langle , \rangle$ to a $\mathbb{A}_f^{p\times}$ multiple of the pairing on $\hat{V}(A)^p$ defined by the Weil pairing,

satisfying the Kottwitz determinant condition that $\det(b|\mathrm{Lie}(A)) = \det(b|V^{-1,0})$ as polynomial functions on $\mathcal{O}_B$, where $V = L \otimes \mathbb{Q}$ and $V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1}$ is the Hodge decomposition.

2.1.2. *Compactifications.* We will now recall some constructions from the theory of toroidal compactifications of PEL type Shimura varieties from [Lan13]. To match the setting in [Lan13], we will moreover assume from now on that the level structure K is a principal congruence subgroup for some $N \geq 3$, namely

$$K = K(N) = \{g \in \mathbf{G}(\hat{\mathbb{Z}}) | g \equiv 1 \pmod N\}.$$

We first recall the definition of a split, symplectic and admissible filtration from [Lan13, §5.2.1]. Let R be a commutative ring.

Definition 2.1. A split, symplectic and admissible filtration on $L \otimes_{\mathbb{Z}} R$ is a two-step filtration on $L \otimes_{\mathbb{Z}} R$ by $(\mathcal{O}_B \otimes_{\mathbb{Z}} R)$-submodules, i.e. we have

$$0 = Z_{-3} \subset Z_{-2} \subset Z_{-1} \subset L \otimes_{\mathbb{Z}} R,$$

such that if we put $\mathrm{Gr}^Z_{-i} = Z_{-i}/Z_{-i-1}$ for $0 \leq i \leq 2$, and $\mathrm{Gr}^Z = \oplus_{0 \leq i \leq 2} \mathrm{Gr}^Z_{-i}$, we have

(1) Gr^Z_{-i} is isomorphic to $M \otimes_{\mathbb{Z}} R$ for some $\mathcal{O}_B$-lattice M
(2) There is some isomorphism of $\mathcal{O}_B$-lattices

$$L \otimes_{\mathbb{Z}} R \simeq \mathrm{Gr}^Z$$

(3) Z_{-2} and Z_{-1} are annihilators of each other under the pairing $\langle , \rangle$ induced from L.

Definition 2.2. Let M be a finite B-module. Since $B \simeq \prod_i B_i$ where each B_i is simple, we have a decomposition $M \simeq M_i^{m_i}$ where M_i is the unique simple left B_i-module. We call the tuple (m_i) the B-multi-rank of M.

Let $R = \hat{\mathbb{Z}}$ and suppose that we have a split symplectic admissible filtration $Z = Z_\bullet$ as above.

Definition 2.3. A torus argument Φ for Z is a tuple $\Phi = (X, Y, \phi, \varphi_{-2}, \varphi_0)$, where

(1) X and Y are $\mathcal{O}_B$-lattices of the same B-multi-rank, and $\phi : Y \hookrightarrow X$ is an $\mathcal{O}_B$-linear embedding
(2) We have isomorphisms $\varphi_{-2} : \mathrm{Gr}^Z_{-2} \simeq \mathrm{Hom}_R(X \otimes_{\mathbb{Z}} R, R(1))$ and $\varphi_0 : \mathrm{Gr}^Z_0 \simeq Y \otimes_{\mathbb{Z}} R$ such that the pairing $\langle , \rangle_{20} : \mathrm{Gr}^Z_{-2} \times \mathrm{Gr}^Z_0 \to R(1)$ is the pullback under these isomorphisms of the pairing

$$\langle \cdot, \cdot \rangle^\phi : \mathrm{Hom}_R(X \otimes R, R(1)) \times (Y \otimes R) \xrightarrow{\mathrm{id} \times \phi} \mathrm{Hom}_R(X \otimes R, R(1)) \times (X \otimes R) \to R(1),$$

where the last arrow is the tautological pairing.

We thus define a cusp label as a pair (Z, Φ), where Z is a split symplectic admissible filtration on $L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$, and Φ is a torus argument for Z. Note that this is the generalization of the cusp labels (Z, X) considered in [CS19, §2.5.2], as for the PEL type A Shimura data they considered, the assumption of principal polarization means we can set $X = Y$, and the torus argument Φ is determined by the $\mathcal{O}_F$-isomorphism.

There is an action of $\mathbf{G}(\mathbb{A}_f)$ on pairs (Z, Φ), as defined in [Lan13, §5.4.3], and we define a cusp label at level K to be a K-orbit of pairs (Z, Φ).

To each cusp label (Z, Φ), we can associate a split torus E_Φ over $\mathbb{Z}$, as constructed by Lan in [Lan13, §6.4]. Let $S_\Phi = \mathbb{X}^*(E_\Phi)$. Let $S_\Phi^\vee := \mathrm{Hom}_{\mathbb{Z}}(S_\Phi, \mathbb{Z})$ be the $\mathbb{Z}$-dual of S_Φ, and let $(S_\Phi)_{\mathbb{R}}^\vee := S_\Phi^\vee \otimes_{\mathbb{Z}} \mathbb{R}$. The $\mathbb{R}$-vector space $(S_\Phi)_{\mathbb{R}}^\vee$ is isomorphic to the space of Hermitian pairings $|\cdot, \cdot| : (Y \otimes \mathbb{R}) \times$

$(Y \otimes \mathbb{R}) \to \mathcal{O}_B \otimes \mathbb{R}$ by sending a Hermitian pairing $|\cdot,\cdot|$ to the function $y \otimes \phi(y') \mapsto \mathrm{Tr}_{B/\mathbb{Q}}(|y,y'|)$ in $\mathrm{Hom}_{\mathbb{Z}}(S_\Phi, \mathbb{R})$ (c.f. [Lan13, §6.2.5]).

Thus, we have an $\mathbb{R}$-vector space $(S_\Phi)^\vee_{\mathbb{R}}$ of Hermitian pairings, and we define P_Φ to be the subset of $(S_\Phi)^\vee_{\mathbb{R}}$ corresponding to positive semi-definite Hermitian pairings with admissible radicals (see [Lan13, Definition 6.2.5.4] and subsequent discussion for the precise definition of admissible radical). P_Φ will a rational polyhedral cone in $(S_\Phi)^\vee_{\mathbb{R}}$. Moreover, to every torus argument Φ we can also associate a stabilizer group Γ_Φ. We thus let Σ_Φ be a Γ_Φ-admissible rational polyhedral cone decomposition of P_Φ, as in [Lan13, Definition 6.1.1.14].

From now on, we will assume that we have fixed a compatible choice of admissible smooth rational polyhedral cone decomposition data (rpcd) Σ for K; namely, we have

(1) A complete set of representatives (Z, Φ) of cusp labels at level K,
(2) A Γ_Φ -admissible smooth rational polyhedral cone decomposition Σ_Φ for each cusp (Z, Φ) so that the cone decompositions are pairwise compatible.

The precise definition and proof of existence of such smooth admissible rpcd is [Lan13, §6.3.3.2, §6.6.3.3]. Associated to this admissible smooth rpcd, we have a toroidal compactification of $\mathfrak{S}(G,X)_K$, as in the following theorem of Lan [Lan13, Theorem 6.4.1.1].

Theorem 2.4. *To each compatible choice $\Sigma = \{\Sigma_\Phi\}$ of admissible smooth rational polyhedral cone decomposition data, there is an associated proper smooth algebraic scheme $\mathfrak{S}(G,X)^{\mathrm{tor}}_K$ over $\mathcal{O}_{E,\mathfrak{p}}$ containing $\mathfrak{S}(G,X)_K$ as an open dense subscheme, together with a semiabelian family $\mathcal{A}$ over $\mathfrak{S}(G,X)^{\mathrm{tor}}_K$. Moreover, we have a stratification*

$$\mathfrak{S}(G,X)^{\mathrm{tor}}_K = \coprod_{(\Phi,\sigma)} Z_{(\Phi,\sigma)},$$

where σ is a face of P_Φ.

Finally, observe that there is a cusp label corresponding to taking the filtration $Z_{-2} = 0, Z_{-1} = L \otimes R$, and the torus argument $X = Y = 0$. This trivial cusp label will correspond to the original Shimura variety $\mathfrak{S}(G,X)_K$ in the stratification above.

2.2. **Igusa Varieties.** We fix now some $b \in B(G,\mu)$, and consider a geometric point $x \in \mathfrak{S}(\mathbf{G},X)_K(\overline{\mathbb{F}}_p)$ lying in the Newton strata for b. This corresponds to some abelian variety $\mathcal{A}_x$, which has p-divisible group with G-structure $\mathbb{X} := \mathcal{A}_x[p^\infty]$ given by b. Up to replacing x by another element in its isogeny class, we can assume $\mathbb{X}$ is completely slope divisible. Thus, we can write $\mathbb{X} = \oplus_{i=1}^r \mathbb{X}_i$, where the $\mathbb{X}_i$ are isoclinic p-divisible groups of strictly decreasing slopes.

We consider the following subset

$$\mathscr{C}_{\mathbb{X}} := \{x \in \mathfrak{S}(\mathbf{G},X)_{K,\overline{\mathbb{F}}_p} : \exists \text{ isomorphism } \rho : \mathcal{A}_x[p^\infty] \times k(\overline{x}) \simeq \mathbb{X} \times k(\overline{x}) \text{ preserving } G\text{-structure}\},$$

where we denote by $\mathfrak{S}(\mathbf{G},X)_{K,\overline{\mathbb{F}}_p}$ the (geometric) special fiber of $\mathfrak{S}(\mathbf{G},X)_K$. This turns out to be a closed subset of $\mathfrak{S}(\mathbf{G},X)_{K,\overline{\mathbb{F}}_p}$, and thus we can give this subset the induced reduced scheme structure. We will continue to denote the associated scheme by $\mathscr{C}_{\mathbb{X}}$, and it turns out this scheme is smooth.

Let $\mathscr{G}$ be the p-divisible group of the restriction to $\mathscr{C}_{\mathbb{X}}$ of the universal abelian variety over $\mathfrak{S}(\mathbf{G},X)_K$. We further define Ig^b as the scheme over $\mathscr{C}_{\mathbb{X}}$ parametrizing, for any perfect $\mathscr{C}_{\mathbb{X}}$-scheme $\mathscr{T}$, isomorphisms $\mathscr{G} \times_{\mathscr{C}_{\mathbb{X}}} \mathscr{T} \simeq \mathbb{X} \times_{\overline{\mathbb{F}}_p} \mathscr{T}$ which preserve G-structure. Equivalently, we can define Ig^b as the functor sending an $\overline{\mathbb{F}}_p$-algebra R to the set

$$\mathrm{Ig}^b(R) = \{(\rho, x) : x \in \mathfrak{S}(\mathbf{G},X)_K(R), \rho : \mathcal{A}_x[p^\infty] \xrightarrow{\sim} \mathbb{X}_R \text{ preserving } G\text{-structure}\}. \tag{2}$$

By [CS17, Corollary 4.3.5], we know that this scheme is perfect, and hence it lifts uniquely to a flat p-adic formal scheme, which we denote by $\mathfrak{Ig}^b$ over $\mathrm{Spf}(W(\overline{\mathbb{F}}_p))$.

We write $\mathfrak{Ig}^b_C$ for the perfectoid space attached to the adic generic fiber of $\mathfrak{Ig}^b$ over C. These spaces are supposed to model the fibers of the Hodge-Tate period morphism, a connection we will elaborate upon in §3.

2.2.1. *Compactifications.* In order to understand (partial) minimal and toroidal compactifications of Ig^b, we must first consider compactifications of the central leaf $\mathscr{C}_{\mathbb{X}}$. As in the discussion in [CS19, §3.1], the central leaf $\mathscr{C}_{\mathbb{X}}$ is a well-positioned subset of $\mathfrak{S}(\mathbf{G}, X)_{K,\overline{\mathbb{F}}_p}$, and thus admits partial toroidal and minimal compactifications, which we will denote by $\mathscr{C}^{\mathrm{tor}}_{\mathbb{X}}$ and $\mathscr{C}^{*}_{\mathbb{X}}$ respectively. Moreover, let Z be a cusp label at level $K(N)$. This determines a locally closed boundary stratum $\mathscr{C}_{\mathbb{X},Z} \subset \mathscr{C}^{\mathrm{tor}}_{\mathbb{X}}$.

The Igusa variety Ig^b over $\mathscr{C}_{\mathbb{X}}$ extends to a perfect scheme $\mathrm{Ig}^{b,\mathrm{tor}}$ over $\mathscr{C}^{\mathrm{tor}}_{\mathbb{X}}$. More precisely, we can define $\mathrm{Ig}^{b,\mathrm{tor}}$ as follows. Let $\mathcal{A}$ denote the universal semi-abelian scheme over $\mathscr{C}^{\mathrm{tor}}_{\mathbb{X}}$. This is the restriction to $\mathscr{C}^{\mathrm{tor}}_{\mathbb{X}}$ of the universal semi-abelian scheme over $\mathfrak{S}(\mathbf{G}, X)_{K,\overline{\mathbb{F}}_p}$. Then, we know from [CS19, Proposition 3.2.1] that the connected part $\mathcal{A}[p^\infty]^\circ$ of $\mathcal{A}[p^\infty]$ is a p-divisible group. Moreover, if we denote by $\mathcal{A}[p^\infty]^\mu$ the multiplicative part, then this is also a p-divisible group. We thus let $\mathcal{A}[p^\infty]^{(0,1)} = \mathcal{A}[p^\infty]^\circ/\mathcal{A}[p^\infty]^\mu$ be the biconnected part. We can similarly define $\mathbb{X}^\circ, \mathbb{X}^{(0,1)}$ as the connnted and biconnected parts of $\mathbb{X}$.

Thus, we can define $\mathrm{Ig}^{b,\mathrm{tor}}$ to be the scheme which, for a perfect $\mathscr{C}^{\mathrm{tor}}_{\mathbb{X}}$-scheme $\mathscr{T}$, parametrizes $\mathcal{O}_B$ -linear isomorphisms $\rho : \mathcal{A}[p^\infty]^\circ \times_{\mathscr{C}^{\mathrm{tor}}_{\mathbb{X}}} \mathscr{T} \xrightarrow{\sim} \mathbb{X}^\circ_{\mathscr{T}}$ and a scalar in $\mathbb{Z}_p^\times(\mathscr{T})$ such that the induced isomorphism $\rho^{(0,1)} : \mathcal{A}[p^\infty]^{(0,1)} \times_{\mathscr{C}^{\mathrm{tor}}_{\mathbb{X}}} \mathscr{T} \xrightarrow{\sim} \mathbb{X}^{(0,1)}_{\mathscr{T}}$ obtained by quotienting by the multiplicative parts commutes with the polarizations up to the given element of $\mathbb{Z}_p^\times(\mathscr{T})$. Here, $\mathbb{Z}_p^\times(\mathscr{T})$ is the set of $\mathscr{T}$-points of the group scheme $\mathbb{Z}_p^\times \times \mathscr{C}^{\mathrm{tor}}_{\mathbb{X}}$.

Finally, we define the partial minimal compactification $\mathrm{Ig}^{b,*}$ as the normalization of $\mathscr{C}^*_{\mathbb{X}}$ in Ig^b. Since we have $\mathrm{Ig}^b \subset \mathrm{Ig}^{b,*}, \mathrm{Ig}^{b,\mathrm{tor}}$, we will denote the boundaries by $\partial\mathrm{Ig}^{b,*}$ and $\partial\mathrm{Ig}^{b,\mathrm{tor}}$, respectively. These schemes are all perfect, and we can lift them to p-adic formal schemes over $\mathrm{Spf}(W(\overline{\mathbb{F}}_p))$. We similarly denote by $\mathfrak{Ig}^{b,*}_C$, $\mathfrak{Ig}^{b,\mathrm{tor}}_C$, $\partial\mathfrak{Ig}^{b,*}_C$, and $\partial\mathfrak{Ig}^{b,\mathrm{tor}}_C$ the associated perfectoid spaces over C.

2.2.2. *Igusa Cusp Labels.* In order to understand the boundary components $\partial\mathrm{Ig}^{b,*}$ and $\partial\mathrm{Ig}^{b,\mathrm{tor}}$, we will recall the notion of Igusa cusp labels, as in [San23, Definition 3.2.19] (which we have slightly modified to match the definition of cusp labels previously introduced). We let $\mathbb{X}_b := \mathbb{X}$ be the completely slope divisible p-divisible group attached to b defined above. We reintroduce b in the notation to emphasise that all constructions here depend on b. Finally, observe that, since from the moduli problem the polarization on $\mathcal{A}_x$ is prime-to-p, the p-divisible group $\mathbb{X}_b = \mathcal{A}_x[p^\infty]$ is principally polarized.

Definition 2.5. We define an Igusa cusp label as a tuple $(Z_b, Z^p, X, Y, \phi, \varphi_0, \varphi_{-2}, \tilde{\varphi}_0, \delta_b)$ where

(1) Z_b is an $\mathcal{O}_B$ -stable filtration of $\mathbb{X}_b$ by p-divisible subgroups of the form
$$0 = Z_{b,-3} \subset Z_{b,-2} \subset Z_{b,-1} \subset \mathbb{X}_b,$$
where $\mathrm{Gr}^{Z_b}_{-2} = Z_{b,-2}$ is multiplicative, and $\mathrm{Gr}^{Z_b}_0 = \mathbb{X}_b/Z_{b,-1}$ is étale, and $Z_{b,-1}$, $Z_{b,-2}$ are Cartier dual to each other under the principal polarization on $\mathbb{X}_b$.

(2) δ_b is an $\mathcal{O}_B$-linear isomorphism
$$\delta_b : \mathrm{Gr}^{Z_b} \simeq \mathbb{X}_b$$

(3) Z^p is an $\mathcal{O}_B$ -stable split, symplectic and admissible filtration
$$0 = Z^p_{-3} \subset Z^p_{-2} \subset Z^p_{-1} \subset L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p$$

(4) X, Y are $\mathcal{O}_B$-lattices of the same B-multirank, together with an $\mathcal{O}_B$-linear embedding $\phi : Y \hookrightarrow X$, and we have isomorphisms
$$\varphi_0 : Y \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p \simeq \mathrm{Gr}^{Z^p}_0$$

$$\varphi_{-2} : \mathrm{Hom}(X \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p, \hat{\mathbb{Z}}^p(1)) \simeq \mathrm{Gr}_{-2}^{Z^p}$$

$$\tilde{\varphi}_0 : Y \otimes (\mathbb{Q}_p/\mathbb{Z}_p) \simeq \mathrm{Gr}_0^{Z_b}$$

such that the pairing $\langle , \rangle_{20} : \mathrm{Gr}_{-2}^{Z^p} \times \mathrm{Gr}_0^{Z^p} \to \hat{\mathbb{Z}}^p(1)$ induced from the one on L is the pullback via φ_{-2}, φ_0 of the one defined on X, Y.

There is an action of $J_b(\mathbb{Q}_p) \times \mathbf{G}(\mathbb{A}_f^p)$ on Igusa cusp labels. If $K \subset J_b(\mathbb{Q}_p) \times \mathbf{G}(\mathbb{A}_f^p)$ is a compact open subgroup then an Igusa cusp label at level K is a K-orbit of Igusa cusp labels. For a general closed subgroup $H \subset J_b(\mathbb{Q}_p) \times \mathbf{G}(\mathbb{A}_f^p)$, an Igusa cusp label at level H is a compatible family of Igusa cusp labels at level K for all $K \supset H$.

2.2.3. *Boundary components.* We can decompose the boundary $\partial \mathrm{Ig}^{b,\mathrm{tor}}$ according to Igusa cusp labels of prime-to-p level $K^p(N)$, in the following way. For every positive integer m, there is a level p^m-Igusa variety Ig_m^b, defined as in [CS17, Definition 4.3.6], and we let $\Gamma_{m,b}$-denote the Galois group of the finite étale cover $\mathrm{Ig}_m^b \to \mathscr{C}_{\mathbb{X}}$. We also have a toroidal extension of Ig_m^b to a level p^m-Igusa variety $\mathrm{Ig}_m^{b,\mathrm{tor}}$, as defined in [CS19, Definition 3.2.5]. We let $\Gamma_b(p^m) := \ker(\mathrm{Aut}(\mathbb{X}) \to \Gamma_{m,b})$, and note that if we let $K = \Gamma_b(p^m)K^p(N)$ then such Igusa cusp labels at level K have the same data as triples $(Z_{m,b}, Z, \Phi)$, where $Z = (Z, \Phi)$ is a cusp label as level $K(N)$, and $Z_{m,b}$ is an $\mathcal{O}_B$-filtration on $\mathbb{X}_b[p^m]$ together with an isomorphism $X/p^m \simeq \mathrm{Gr}_0^{Z_{m,b}}$. In particular, we can consider the locally closed boundary stratum $\mathscr{C}_{\mathbb{X},Z}$, and the p^m-Igusa variety $\mathrm{Ig}_{m,Z}^{b,\mathrm{tor}}$ which is the preimage of $\mathscr{C}_{\mathbb{X},Z}$. Moreover, from [San23, Theorem 3.2.22], we see that we have a decomposition

$$\mathrm{Ig}_{m,Z}^{b,\mathrm{tor}} = \coprod_{\tilde{Z}_m} \mathrm{Ig}_{m,\tilde{Z}_m}^{b,\mathrm{tor}},$$

where $\tilde{Z}_m$ denotes Igusa cusp labels of level $\Gamma_b(p^m)K^p(N)$ lying over Z.

Now, let $\tilde{Z}$ denote an Igusa cusp label of level $K^p(N)$. This is by definition a compatible system $\{\tilde{Z}_m\}$ of Igusa cusp labels for $\Gamma_b(p^m)K^p(N)$, for all positive integers m. We can thus define

$$\mathrm{Ig}_{\tilde{Z}}^{b,\mathrm{tor}} = \varprojlim_m \mathrm{Ig}_{m,\tilde{Z}_m}^{b,\mathrm{tor}}.$$

For later use, we will want to have a moduli description of points in $\mathrm{Ig}_{\tilde{Z}}^{b,\mathrm{tor}}$. We first recall some facts about degenerations of abelian schemes, from [Lan13, §3.4] and [CS19, §2.5.1]. Let C' be a complete algebraically closed nonarchimedean field with ring of integers $\mathcal{O}_{C'}$. Consider a polarized abelian variety (A, λ) over C' with $\mathcal{O}_B$-structure, and a degeneration $\mathcal{A}$ of A over $\mathcal{O}_{C'}$. Then this uniquely determines a short exact sequence

$$0 \to T \to \mathcal{G} \to \mathcal{B} \to 0$$

where T is a torus, $\mathcal{B}$ is an abelian scheme over $\mathcal{O}_{C'}$, and $\mathcal{G}$ is the Raynaud extension. Let $X = \mathbb{X}_*(T)$, which is a free abelian group over $\mathcal{O}_C$. The lattice X has an action of $\mathcal{O}_B$, and so does $\mathcal{B}$. Then, $\mathcal{G}$ determines, and is uniquely determined by, an $\mathcal{O}_B$-linear map $c : X \to \mathcal{B}^\vee$. Similarly, we can consider a degeneration $\mathcal{A}^\vee$ over $\mathcal{O}_{C'}$ of the dual $A^\vee/C'$, which gives us a short exact sequence

$$0 \to T^\vee \to \mathcal{G}^\vee \to \mathcal{B}^\vee \to 0,$$

and if we similarly let $Y = \mathbb{X}_*(T^\vee) = \mathbb{X}^*(T)$, the extension $\mathcal{G}$ determines, and is uniquely determined by an $\mathcal{O}_B$-linear map $c^\vee : Y \to \mathcal{B}$.

Moreover, the polarization $\mathcal{G} \to \mathcal{G}^\vee$ determines and is uniquely determined by the data of the polarization on $\mathcal{B}$, and an injective $\mathcal{O}_B$-linear map $\phi : Y \to X$.

Given $\tilde{Z}$, an Igusa cusp label of level $K^p(N)$, we want to understand the $(C, \mathcal{O}_C)$-valued points of $\mathfrak{Ig}^{b,\mathrm{tor}}$ which specialize to points in $\mathrm{Ig}_{\tilde{Z}}^{b,\mathrm{tor}}$. In particular, note that the data of the Igusa cusp label means we have fixed an $\mathcal{O}_B$-stable filtration

$$0 = Z_{b,-3} \subset Z_{b,-2} \subset Z_{b,-1} \subset \mathbb{X}_b,$$

as well as a $\mathcal{O}_B$ -linear splitting δ_b of this filtration. From the proof of [San23, Theorem 4.3.10], we see that $(C', \mathcal{O}_{C'})$-valued points are hence given by the following data:

(1) A polarized abelian variety $\mathcal{B}$ over $\mathcal{O}_{C'}$ with $\mathcal{O}_B$-action,
(2) An $\mathcal{O}_B$ -linear extension
$$0 \to T \to \mathcal{G} \to \mathcal{B} \to 0$$
where $\mathbb{X}_*(T) = X$; equivalently, an $\mathcal{O}_B$ -linear map $c : X \to \mathcal{B}^\vee$.
(3) An $\mathcal{O}_B$-linear isomorphism
$$\rho : \mathcal{G}[p^\infty] \simeq Z_{b,-1}$$
that is compatible with the identification $T[p^\infty] = Z_{b,-2}$. By the splitting δ_b, this induces a splitting of
$$0 \to T[p^\infty] \to \mathcal{G}[p^\infty] \to \mathcal{B}[p^\infty] \to 0,$$
and in particular c extends to a map $c : X[1/p] \to \mathcal{B}^\vee$.
(4) An $\mathcal{O}_B$ -linear extension
$$0 \to T^\vee \to \mathcal{G}^\vee \to \mathcal{B}^\vee \to 0$$
where $\mathbb{X}^*(T) = Y$. Equivalently, an $\mathcal{O}_B$ -linear map $c^\vee : Y \to \mathcal{B}$. By the splitting δ_b, as well as by duality (using that $\mathcal{G}[p^\infty]$ and $\mathcal{B}[p^\infty]$ will be principally polarized), we have a splitting of
$$0 \to T^\vee[p^\infty] \to \mathcal{G}^\vee[p^\infty] \to \mathcal{B}^\vee[p^\infty] \to 0$$
and in particular we extend $c^\vee$ to a map $c^\vee : Y[1/p] \to \mathcal{B}$.
(5) An $\mathcal{O}_{C'}$ -point of $\mathcal{P}'_{\Sigma_\Phi}$ whose special fibre lies in the boundary. Here, we note that away from the boundary we have a torsor $\mathcal{P}'$ over $\mathcal{O}_{C'}$ for the torus E_Φ with character group S_Φ, parametrizing lifts of $c^\vee$ to $\iota : Y[1/p] \to \mathcal{G}$. $\mathcal{P}'_{\Sigma_\Phi} \supset \mathcal{P}'$ is the torus embedding defined by the admissible rational polyhedral cone decomposition Σ_Φ for the cusp (Z, Φ).

3. Mantovan's Formula and the Hodge-Tate Period Morphism

For $K \subset \mathbf{G}(\mathbb{A}_f)$ a sufficiently small open compact, we define $\mathcal{S}_K := (\mathrm{Sh}(\mathbf{G}, X)_K \otimes_E E_\mathfrak{p})^{\mathrm{ad}}$ to be the adic space over $\mathrm{Spa}(E_\mathfrak{p})$ attached to the Shimura variety. When $K = K_p^{\mathrm{hs}} K^p$ with K_p^{hs} a hyperspecial subgroup, the space $\mathcal{S}_K$ has a canonical integral model $\mathfrak{S}_K$ over $\mathcal{O}_{E,\mathfrak{p}}$. Let $\mathcal{S}_K^\circ \subset \mathcal{S}_K$ be the good reduction locus, i.e. the open subspace of $\mathcal{S}_K$ obtained from the adic generic fiber of the p-adic completion $\mathfrak{S}_K^\wedge$ of the scheme $\mathfrak{S}_K$. We define $\mathcal{S}_{K'}^\circ \subset \mathcal{S}_{K'}$ for $K' \subset K$ by taking the preimage under the natural map from $\mathcal{S}_{K'}$ to $\mathcal{S}_K$. We also consider the adic spaces $\mathcal{S}_K^*$ and $\mathcal{S}_K^{\mathrm{tor}}$ attatched to the minimal and toroidal compactification of the Shimura variety $\mathcal{S}_K^*$.

Associated to the $G(\mathbb{R})$-conjugacy class X, we have a minuscule cocharacter μ of G_C which is defined over $E_\mathfrak{p}$. Let $\mathcal{F}\ell_{G,\mu^{-1}}$ be the flag variety over $\mathrm{Spa}(C)$ associated to μ^{-1} the dominant inverse of μ. Since μ is minuscule, via the Bialynicki-Birula isomorphism, when viewed as a diamond the flag variety $\mathcal{F}\ell_{G,\mu^{-1}}$ represents the following functor on Perf_C. Given any $S \in \mathrm{Perf}_C$, $\mathcal{F}\ell_{G,\mu^{-1}}(S)$ is the set of modifications of vector bundles $\mathcal{E} \dashrightarrow \mathcal{E}_0$ of meromorphy μ on X_S, the relative Fargues-Fontaine curve over S, such that the modification occurs over the untilt of S corresponding to the map $S \to \mathrm{Spd}(C)$.

Let

$$\mathcal{S}_{K^p}^\circ := \varprojlim_{K_p} \mathcal{S}_{K^pK_p}^\circ \subset \mathcal{S}_{K^p} := \varprojlim_{K_p} \mathcal{S}_{K^pK_p} \subset \mathcal{S}_{K^p}^{\mathrm{tor}} := \varprojlim_{K_p} \mathcal{S}_{K^pK_p}^{\mathrm{tor}} \to \mathcal{S}_{K^p}^* := \varprojlim_{K_p} \mathcal{S}_{K^pK_p}^*$$

be the associated perfectoid Shimura varieties. We also consider $\overline{\mathcal{S}}^{\circ}_{K^p,C}$, the canonical compactification of the good reduction locus. This will be a subspace of $\mathcal{S}_{K^p,C}$, since $\mathcal{S}_{K^p,C}$ is partially proper. Caraiani-Scholze [CS17, §2.1] consider the Hodge-Tate period morphism on $\mathcal{S}_{K^p}$

$$\pi_{\mathrm{HT}} : \mathcal{S}_{K^p,C} \to \mathcal{F}\ell_{G,\mu^{-1}},$$

which records the relative position of the Hodge-Tate filtration associated with the p-divisible group. This extends [CS19, §4.1] to a Hodge-Tate period morphism on the minimal compactification

$$\pi^*_{\mathrm{HT}} : \mathcal{S}^*_{K^p,C} \to \mathcal{F}\ell_{G,\mu^{-1}}$$

and toroidal compactification

$$\pi^{\mathrm{tor}}_{\mathrm{HT}} : \mathcal{S}^{\mathrm{tor}}_{K^p,C} \to \mathcal{F}\ell_{G,\mu^{-1}}.$$

We write $\pi^{\circ}_{\mathrm{HT}}$ for the restriction to the good reduction locus, and $\overline{\pi}^{\circ}_{\mathrm{HT}}$ for the canonical compactification of $\pi^{\circ}_{\mathrm{HT}}$, where we note that this again maps to $\mathcal{F}\ell_{G,\mu^{-1}}$ as this is proper over $\mathrm{Spa}(C)$.

These maps have the following properties:

(1) π^*_{HT} and $\pi^{\mathrm{tor}}_{\mathrm{HT}}$ are partially proper and qcqs; hence, proper.
(2) π_{HT} and $\overline{\pi}^{\circ}_{\mathrm{HT}}$ are partially proper, but not always qcqs.
(3) $\pi^{\circ}_{\mathrm{HT}}$ is qcqs, but not partially proper.

With these properties in mind, let us study the fibers of these maps. For our purposes, we will focus on the compactly supported cohomology of $\mathcal{S}_{K^p,C}$ and in turn the sheaf $R\pi_{\mathrm{HT}!}(\overline{\mathbb{F}}_\ell)$ on $\mathcal{F}\ell_{G,\mu^{-1}}$.

Remark 3.1. We note that it is always true that the compactly supported cohomology at infinite level is the colimit of the compactly supported cohomology at finite levels, but, for usual cohomology one needs to assume the spaces are qcqs for this to be true (e.g the tower defined by the good reduction locus or the minimal/toroidal compactifications).

Our goal is to describe the stalks of $R\pi_{\mathrm{HT}!}(\overline{\mathbb{F}}_\ell)$ at a geometric point $x : \mathrm{Spa}(C, C^+) \to \mathcal{F}\ell_{G,\mu^{-1}}$. We assume the geometric point x factors through the adic Newton strata $\mathcal{F}\ell^b_{G,\mu^{-1}}$ for $b \in B(G, \mu)$, and choose a completely slope divisible p-divisible group $\mathbb{X}_b$ over $\overline{\mathbb{F}}_p$ corresponding to $b \in B(G, \mu)$. Let Ig^b be the associated perfect Igusa variety as defined in §2.2, with toroidal compactification $\mathrm{Ig}^{b,\mathrm{tor}}$ and minimal compactification $\mathrm{Ig}^{b,*}$. Recall that we have associated perfectoid Igusa varieties, $\mathfrak{Ig}^b_C, \mathfrak{Ig}^{b,\mathrm{tor}}_C, \mathfrak{Ig}^{b,*}_C$, which should model the fibers of $\overline{\pi}^{\circ}_{\mathrm{HT}}$, $\pi^{\mathrm{tor}}_{\mathrm{HT}}$, and π^*_{HT}, respectively. We let $\partial\mathfrak{Ig}^{b,*}_C$ and $\partial\mathfrak{Ig}^{b,\mathrm{tor}}_C$ be the Zariski closed subspaces attached to the boundaries $\partial\mathrm{Ig}^{b,*}$ and $\partial\mathrm{Ig}^{b,\mathrm{tor}}$, respectively.

Let $g_b : \mathrm{Ig}^b \hookrightarrow \mathrm{Ig}^{b,*}$ be the natural open immersion of $\overline{\mathbb{F}}_p$-schemes. We define the partially compactly supported cohomology

$$R\Gamma_{c-\partial}(\mathrm{Ig}^b, \overline{\mathbb{F}}_\ell) := R\Gamma(\mathrm{Ig}^{b,*}, g_{b!}(\overline{\mathbb{F}}_\ell)).$$

Our goal is to show that this computes the fibers of $R\pi_{\mathrm{HT}!}(\overline{\mathbb{F}}_\ell)$ for geometric points in $\mathcal{F}\ell^b_{G,\mu^{-1}}$.

To get a clearer picture of how these spaces interact with each other, we have the following theorem.

Theorem 3.2. *[CS19; San23] There exists a diagram of spaces of the form*

$$\begin{tikzcd}
(\pi^{\circ}_{\mathrm{HT}})^{-1}(x) \arrow[r, hook] & \pi^{-1}_{\mathrm{HT}}(x) \arrow[r, hook] & (\pi^{\mathrm{tor}}_{\mathrm{HT}})^{-1}(x) \arrow[r] & (\pi^{*}_{\mathrm{HT}})^{-1}(x) \\
\mathfrak{Ig}^b_C \arrow[u, hook, "i"] \arrow[rr, hook] & & \mathfrak{Ig}^{b,\mathrm{tor}}_C, \arrow[u, hook, "{}^{\mathrm{tor}}i"] \arrow[r] & \mathfrak{Ig}^{b,*}_C \arrow[u, hook, "{}^{*}i"]
\end{tikzcd}$$

where the maps i, ${}^*i$, and ${}^{\mathrm{tor}}i$ are open immersions whose image contains all rank 1 points. Moreover, the fibers $(\pi^*_{\mathrm{HT}})^{-1}(x)$ and $(\pi^{\mathrm{tor}}_{\mathrm{HT}})^{-1}(x)$ are partially proper, so in particular ${}^*i$ and ${}^{\mathrm{tor}}i$ are canonical compactifications in the sense of [Sch18, Proposition 18.6].

Proof. This theorem in the case where $(\mathbf{G}, X)$ is of PEL type A attached to a globally quasi-split unitary group of even dimension is [CS19, Theorems 2.7.2, Theorem 4.5.1], and the general case of PEL type A or C is proven in [San23, Theorems 4.3.10, 4.3.12]. □

We will also combine this with the following result.

Theorem 3.3. *[CS19; San23] The partially minimally compactified Igusa variety $\mathrm{Ig}^{b,*}$ is affine; in particular, the attatched adic space $\mathfrak{Ig}^{b,*}_C$ is affinoid perfectoid. Moreover, there exists a proper map*

$$\mathfrak{Ig}^{b,\mathrm{tor}}_C \to \mathfrak{Ig}^{b,*}_C$$

which induces an isomorphism on global sections.

Proof. For the affineness, the case of PEL type A attached to a global quasi-split unitary group of even dimension is covered by [CS19, Theorem 1.7] and [CS19, Lemma 4.5.2]. The general case of PEL type A or C is covered in [San23, Lemma 3.3.7]. To see the properness, we note that the map

$$\mathfrak{Ig}^{b,\mathrm{tor}}_C \to \mathfrak{Ig}^{b,*}_C$$

is the one appearing in the Stein factorization described in [CS19, Proposition 3.3.4] and [San23, Proposition 3.3.5]. This also shows that one has an isomorphism on global sections. □

We will also need the following Corollary.

Corollary 3.4. *The boundary $\partial\mathfrak{Ig}^{b,\mathrm{tor}}_C$ is quasi-compact and $\partial\mathfrak{Ig}^{b,*}_C$ is affinoid perfectoid (In particular, quasi-compact).*

Proof. The fact that $\mathfrak{Ig}^{b,*}_C$ is affinoid perfectoid follows from the previous Theorem. Moreover, $\partial\mathfrak{Ig}^{b,*}_C$ is a Zariski closed subspace, since it came from considering the adic generic fiber of a formal model of the perfect closed subscheme $\partial\mathrm{Ig}^{b,*} \subset \mathrm{Ig}^{b,*}$. The claim for the toroidal compactification follows since the map $\mathfrak{Ig}^{b,\mathrm{tor}}_C \to \mathfrak{Ig}^{b,*}_C$ is proper, and maps the boundary $\partial\mathfrak{Ig}^{b,\mathrm{tor}}_C$ to $\partial\mathfrak{Ig}^{b,*}_C$. □

It is natural to wonder how one could describe the fiber of π_{HT} in terms of the spaces described above. In particular, we now deduce the following Corollary.

Corollary 3.5. *For $x : \mathrm{Spa}(C, C^+) \to \mathcal{F}\ell^b_{G,\mu^{-1}}$ a geometric point, we have isomorphisms*

$$\pi^{-1}_{\mathrm{HT}}(x) \simeq \overline{\mathfrak{Ig}}^{b,*}_C \setminus \overline{\partial\mathfrak{Ig}}^{b,*}_C \simeq \overline{\mathfrak{Ig}}^{b,\mathrm{tor}}_C \setminus \overline{\partial\mathfrak{Ig}}^{b,\mathrm{tor}}_C$$

induced by the natural open immersions $\pi^{-1}_{\mathrm{HT}}(x) \hookrightarrow (\pi^*_{\mathrm{HT}})^{-1}(x) \simeq \overline{\mathfrak{Ig}}^{b,*}_C$ (resp. $\pi^{-1}_{\mathrm{HT}}(x) \hookrightarrow (\pi^{\mathrm{tor}}_{\mathrm{HT}})^{-1}(x) \simeq \overline{\mathfrak{Ig}}^{b,\mathrm{tor}}_C$), as given by Theorem 3.2. Here $\overline{\partial\mathfrak{Ig}}^{b,*}_C$ (resp. $\overline{\partial\mathfrak{Ig}}^{b,\mathrm{tor}}_C$) is the Zarisiki closed subset of $\overline{\mathfrak{Ig}}^{b,*}_C$ (resp. $\overline{\mathfrak{Ig}}^{b,\mathrm{tor}}_C$) defined by the canonical compactification of the boundaries $\partial\mathfrak{Ig}^b_C \subset \mathfrak{Ig}^b_C$ (resp. $\partial\mathfrak{Ig}^{b,\mathrm{tor}}_C \subset \mathfrak{Ig}^{b,\mathrm{tor}}_C$).

Proof. We first establish the claim for the toroidal compactification. We consider the closed immersion

$$\overline{\mathfrak{Ig}}^{b,\mathrm{tor}}_C \times_{\mathcal{S}^{\mathrm{tor}}_{K^p,C}} \partial\mathcal{S}^{\mathrm{tor}}_{K^p,C} \hookrightarrow \overline{\partial\mathfrak{Ig}}^{b,\mathrm{tor}}_C$$

obtained by base-changing the closed immersion $\partial\mathcal{S}^{\mathrm{tor}}_{K^p,C} \hookrightarrow \mathcal{S}^{\mathrm{tor}}_{K^p,C}$ to the fiber $(\pi^{\mathrm{tor}}_{\mathrm{HT}})^{-1}(x)$ and applying Theorem 3.2. To show the desired claim, it suffices to show this is an isomorphism. Note that both the LHS and RHS are partially proper; therefore, to show this is an isomorphism, it suffices to show it induces an isomorphism on rank 1 points. In particular, given C'/C a complete algebraically closed non-archimedean field, we claim that there exists a Cartesian diagram of the form

$$\begin{tikzcd} \partial\mathfrak{Ig}^{b,\mathrm{tor}}_{C}(C',\mathcal{O}_{C'}) \arrow[r] \arrow[d] & \mathfrak{Ig}^{b,\mathrm{tor}}_{C}(C',\mathcal{O}_{C'}) \arrow[d] \\ \partial\mathcal{S}^{\mathrm{tor}}_{K^p,C}(C',\mathcal{O}_{C'}) \arrow[r] & \mathcal{S}^{\mathrm{tor}}_{K^p,C}(C',\mathcal{O}_{C'}) \end{tikzcd}.$$

This can be checked using the moduli interpretation, as in the proof of [CS19, Theorem 4.4.1]. In particular, given a $\mathrm{Spa}(C',\mathcal{O}_{C'})$ point of $\mathfrak{Ig}^{b,\mathrm{tor}}_{C}$ specializing to a boundary component indexed by an Igusa cusp label $\tilde{Z}$ (See §2.2.2 for the definition of Igusa cusp label), then from the discussion in §2.2.3 this corresponds to the datum of $(\mathcal{B},\mathcal{G},\mathcal{G}^\vee,\rho,y)$, where

(1) $\mathcal{B}$ is an abelian scheme over $\mathcal{O}_{C'}$ with polarization, and $\mathcal{O}_B$-structure,
(2) $\mathcal{G}$ is a Raynaud extension
$$0\to T\to\mathcal{G}\to\mathcal{B}\to 0$$
of $\mathcal{B}$ by a torus T with cocharacter group X,
(3) $\mathcal{G}^\vee$ is a Raynaud extension
$$0\to T^\vee\to\mathcal{G}^\vee\to\mathcal{B}^\vee\to 0$$
of $\mathcal{B}^\vee$ by a torus $T^\vee$ with cocharacter group Y,
(4) ρ is an $\mathcal{O}_B$-linear isomorphism $\rho:\mathcal{G}[p^\infty]\simeq Z_{b,-1}$, extending the isomorphism $T[p^\infty]\simeq Z_{b,-2}$,
(5) $y\in\mathcal{P}_{\Sigma_\Phi}(\mathcal{O}_{C'})$, where $\mathcal{P}_{\Sigma_\Phi}$ is the toroidal compactification (determined by an admissible rpcd Σ_Φ, as in §2.1.2) of a torsor under a torus $E_\Phi/\mathcal{O}_{C'}$, whose character group is given by S_Φ, and y is a point whose special fiber lies in the boundary.

The natural map $\mathfrak{Ig}^{b,\mathrm{tor}}(C',\mathcal{O}_{C'})\hookrightarrow\mathcal{S}^{\mathrm{tor}}_{K^p,C}(C',\mathcal{O}_{C'})$ is determined by forgetting the trivialization $\rho: Z_{b,-1}\simeq\mathcal{G}[p^\infty]$. The image under this map of a point lying in the boundary $\partial\mathfrak{Ig}^{b,\mathrm{tor}}_{C}$ is equivalent to insisting that y is a point whose special fiber lies in the boundary of $\mathcal{P}_{\Sigma_\Phi}$ for some Igusa cusp label $\tilde{Z}$, which is not the trivial one. Let $Z=(Z,\Phi)$ be the cusp label which $\tilde{Z}$ lives over. This is then equivalent to the condition guaranteeing that the image lies in the component of $\mathcal{S}^{\mathrm{tor}}_{K^p,C}$ indexed by $Z=(Z,\Phi)$, where Z is not the trivial cusp label, and thus lies in the boundary $\partial\mathcal{S}^{\mathrm{tor}}_{K^p,C}$. The claim follows.

It remains to see the analogous claim for the minimal compactification. This follows easily using Theorem 3.2 and the fact that the proper surjective map $\mathfrak{Ig}^{b,\mathrm{tor}}_{C}\to\mathfrak{Ig}^{b,*}_{C}$ sends $\partial\mathfrak{Ig}^{b,\mathrm{tor}}_{C}$ to $\partial\mathfrak{Ig}^{b,*}_{C}$ by construction. $\square$

We have the following Corollary.

Corollary 3.6. *For a geometric point $x:\mathrm{Spa}(C,C^+)\to\mathcal{F}\ell^b_{G,\mu^{-1}}$, we have an identification:*

$$R\Gamma_{c-\partial}(\mathrm{Ig}^b,\overline{\mathbb{F}}_\ell)\simeq R\pi_{\mathrm{HT}!}(\overline{\mathbb{F}}_\ell)_x.$$

Proof. We have an identification $\pi_{\mathrm{HT}}^{-1}(x)\simeq\overline{\mathfrak{Ig}}^{b,*}_{C}\setminus\overline{\partial\mathfrak{Ig}}^{b,*}_{C}$ by the previous Corollary, so, by proper base-change, we are tasked with computing the compactly supported cohomology of this space. We note, by Theorem 3.3, the adic spaces $\mathfrak{Ig}^{b,*}_{C}$ are affinoid perfectoid. It follows that the canonical compactification $\overline{\mathfrak{Ig}}^{b,*}_{C}\simeq(\pi^*_{\mathrm{HT}})^{-1}(x)$ is also affinoid perfectoid, by [Sch18, Proposition 18.7 (iv)]. In particular, it is quasi-compact and partially proper, so in particular proper. It therefore follows by excision[1] that we have a distinguished triangle

$$R\Gamma_c(\pi_{\mathrm{HT}}^{-1}(x),\overline{\mathbb{F}}_\ell)\to R\Gamma(\overline{\mathfrak{Ig}}^{b,*}_{C},\overline{\mathbb{F}}_\ell)\to R\Gamma(\overline{\partial\mathfrak{Ig}}^{b,*}_{C},\overline{\mathbb{F}}_\ell)\xrightarrow{+1}. \tag{3}$$

[1]One easily checks that the excision sequence is exact on points, and this is sufficient by [Sch18, Proposition 14.3].

Applying Theorem 3.2 again, we know that $k : \mathfrak{Ig}^{b,*} \hookrightarrow \overline{\mathfrak{Ig}}_C^{b,*}$ is a qcqs open immersion of perfectoid spaces inducing an isomorphism on rank 1 points, and it follows that the same is true for the induced map on the Zariski closed subspaces ${}^{\partial}k : \partial\mathfrak{Ig}^{b,*} \hookrightarrow \overline{\partial\mathfrak{Ig}}_C^{b,*}$. Therefore, we can apply [CS17, Lemma 4.4.2], this tells us that the natural maps

$$\overline{\mathbb{F}}_\ell \to k_*(\overline{\mathbb{F}}_\ell)$$

$$\overline{\mathbb{F}}_\ell \to {}^{\partial}k_*(\overline{\mathbb{F}}_\ell)$$

are isomorphisms, giving identifications $R\Gamma(\overline{\mathfrak{Ig}}_C^{b,*}, \overline{\mathbb{F}}_\ell) \simeq R\Gamma(\mathfrak{Ig}_C^{b,*}, \overline{\mathbb{F}}_\ell)$ and $R\Gamma(\overline{\partial\mathfrak{Ig}}_C^{b,*}, \overline{\mathbb{F}}_\ell) \simeq R\Gamma(\partial\mathfrak{Ig}_C^{b,*}, \overline{\mathbb{F}}_\ell)$. Now, by [CS17, Lemma 4.4.3], we have further identifications of $R\Gamma(\partial\mathfrak{Ig}_C^{b,*}, \overline{\mathbb{F}}_\ell)$ and $R\Gamma(\mathfrak{Ig}_C^{b,*}, \overline{\mathbb{F}}_\ell)$ with the cohomology of the perfect schemes $\partial\mathrm{Ig}^{b,*}$ and $\mathrm{Ig}^{b,*}$, respectively. Substituting this into the triangle (3), we get a distinguished triangle

$$R\Gamma_c(\pi_{\mathrm{HT}}^{-1}(x), \overline{\mathbb{F}}_\ell) \to R\Gamma(\mathrm{Ig}^{b,*}, \overline{\mathbb{F}}_\ell) \to R\Gamma(\partial\mathrm{Ig}^{b,*}, \overline{\mathbb{F}}_\ell) \xrightarrow{+1} .$$

By applying quasi-compact base-change [Sch18, Proposition 17.6] and then using that the inclusion $\partial\mathfrak{Ig}_C^{b,*} \subset \mathfrak{Ig}_C^{b,*}$ is induced from taking the rigid generic fiber over C of Witt vectors applied to $\partial\mathrm{Ig}^{b,*} \subset \mathrm{Ig}^{b,*}$, we identify the last map with the natural restriction map on the cohomology. However, this identifies the first term with precisely the partially compactly supported cohomology, as desired. □

We will combine this with the following proposition, which already hints at our expectation that $R\pi_{\mathrm{HT}!}(\overline{\mathbb{F}}_\ell)$ is connective in some suitable perverse t-structure.

Proposition 3.7. *If $d_b := \langle 2\rho_G, \nu_b\rangle = \dim(\mathrm{Ig}^{b,*}) = \dim(\mathrm{Ig}^b)$ then the cohomology of the complex*

$$R\Gamma_{c-\partial}(\mathrm{Ig}^b, \overline{\mathbb{F}}_\ell) \simeq R\pi_{\mathrm{HT}!}(\overline{\mathbb{F}}_\ell)_x$$

is concentrated in degrees $\leq d_b$.

Proof. We saw in Theorem 3.3 that $\mathrm{Ig}^{b,*}$ is an affine scheme. So we would like to apply Artin vanishing; however, $\mathrm{Ig}^{b,*}$ is also a perfect scheme so in particular not of finite type. To remedy this, consider the pro-étale cover

$$\mathrm{Ig}^b \to \mathcal{C}_{\mathbb{X}_b}^{\mathrm{perf}},$$

with Galois group $\mathrm{Aut}(\mathbb{X})(\overline{\mathbb{F}}_p)$ over the perfection of the central leaf attached to $\mathbb{X}_b$. This is obtained as the perfection of the limit of the finite étale covers

$$\mathrm{Ig}_m^b \to \mathcal{C}_{\mathbb{X}_b},$$

described in 2.2.3, as shown in [CS17, Proposition 4.3.8]. These spaces are of finite type over $\overline{\mathbb{F}}_p$. We now define $\mathrm{Ig}_m^{b,*}$ to be the normalization of $\mathcal{C}_{\mathbb{X}}^*$ in Ig_m^b of the finite étale cover $\mathrm{Ig}_m^b \to \mathcal{C}_{\mathbb{X}_b}$. By [San23, Theorem 3.33], $\mathcal{C}_{\mathbb{X}}^*$ is affine; therefore, it follows that $\mathrm{Ig}_m^{b,*}$ is a normal and affine scheme which will be of finite type, since Ig_m^b is. It follows, by definition of $\mathrm{Ig}^{b,*}$, that it is the perfection of $\lim_{m\geq 1}\mathrm{Ig}_m^{b,*}$. Therefore, since passing to perfections doesn't change the étale cohomology, we can conclude by combining Artin vanishing with an application of [Sta23, Tag 09QY] to the system of sheaves $g_{b,m!}(\overline{\mathbb{F}}_\ell)$, where $g_{b,m} : \mathrm{Ig}_m^b \to \mathrm{Ig}_m^{b,*}$ is the natural open inclusion at finite level. □

Now we would like to link this analysis with the semi-perversity of certain sheaves on Bun_G. We consider the Hodge-Tate period morphism

$$\pi_{\mathrm{HT}} : [\mathcal{S}_{K^p}/\underline{G(\mathbb{Q}_p)}] \to [\mathcal{F}\ell_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}]$$

quotiented out by $G(\mathbb{Q}_p)$. We let $h^{\to} : [\mathcal{F}\ell_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}] \to [\mathrm{Spd}(C)/\underline{G(\mathbb{Q}_p)}] \simeq \mathrm{Bun}_G^1$ be the structure map quotiented out by $G(\mathbb{Q}_p)$. Note this is a proper map, since $\mathcal{F}\ell_{G,\mu^{-1}}$ is proper over $\mathrm{Spd}(C)$.

Then we have an identification

$$R\Gamma_c(\mathcal{S}_{K^p,C}, \overline{\mathbb{F}}_l) \simeq h_*^{\rightarrow} R\pi_{\mathrm{HT}!}(\overline{\mathbb{F}}_\ell) \tag{4}$$

of $G(\mathbb{Q}_p)$-representations, and this computes the compactly supported torsion cohomology of the Shimura variety.

Similarly, we have a map

$$h^{\leftarrow} : [\mathcal{F}\ell_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}] \to \mathrm{Bun}_G$$

remembering the isomorphism class of the bundle $\mathcal{E}_1$ in the moduli interpretation of $\mathcal{F}\ell_{G,\mu^{-1}}$ as a diamond. This defines a cohomologically smooth map by [FS21, Theorem IV.1.19], and the image identifies with the open subset $B(G,\mu) \subset B(G)$ under the identification $|\mathrm{Bun}_G| \simeq B(G)$ of topological spaces, where $|\mathrm{Bun}_G|$ denotes the underlying topological space of Bun_G and $B(G)$ has the topology given by its natural partial ordering [Vie21]. For each b, we have a locally closed Harder-Narasimhan stratum $j_b : \mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$, and we can define the locally closed subset $[\mathcal{F}\ell^b_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}]$, by pulling back this HN-strata along $h^{\leftarrow}$. This defines a locally closed stratification of $[\mathcal{F}\ell_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}]$. Let $i_b : [\mathcal{F}\ell^b_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}] \hookrightarrow [\mathcal{F}\ell_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}]$ denote the associated locally closed immersion. We write $\pi^b_{\mathrm{HT}} : [\mathcal{S}^b_{K^p,C}/\underline{G(\mathbb{Q}_p)}] \to [\mathcal{F}\ell^b_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}]$ (resp. $\pi^{b,*}_{\mathrm{HT}} : [\mathcal{S}^{b,*}_{K_p,C}/\underline{G(\mathbb{Q}_p)}] \to [\mathcal{F}\ell^b_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}]$) for the pullbacks of π_{HT} (resp. π^*_{HT}) along i_b. On the good reduction locus, we also have an additional stratification coming from pulling back the Newton stratification on the special fiber along the specialization map. There is a rather subtle point that this does not agree with the pullback of the locally closed strata $\mathcal{F}\ell^b_{G,\mu^{-1}}$ (namely, the closure relationships are opposite with respect to the partial ordering on $B(G)$). We write $\mathcal{S}^{b,\circ,\mathrm{rd}}_{K^p,C}$ for these Newton strata coming from the special fiber. There exists a natural map

$$\mathcal{S}^{b,\circ,\mathrm{rd}}_{K^p,C} \times_{\mathcal{F}\ell_{G,\mu^{-1}}} \mathcal{F}\ell^b_{G,\mu^{-1}} \hookrightarrow \mathcal{S}^{b,\circ}_{K^p,C}$$

which is a qcqs open immersion containing all rank 1 points ([CS17, Page 68] [Kos21, Page 8]). We write $\pi^{b,\circ}_{\mathrm{HT}} : [(\mathcal{S}^{b,\circ,\mathrm{rd}}_{K^p,C} \times_{\mathcal{F}\ell_{G,\mu^{-1}}} \mathcal{F}\ell^b_{G,\mu^{-1}})/\underline{G(\mathbb{Q}_p)}] \to [\mathcal{F}\ell^b_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}]$ for the Hodge-Tate period map on this locus, and similarly we write $\overline{\pi}^{b,\circ}_{\mathrm{HT}} : [\overline{\mathcal{S}}^{b,\circ}_{K^p,C}/\underline{G(\mathbb{Q}_p)}] \to [\mathcal{F}\ell^b_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}]$ for the induced map on the canonical compactification, where we note that this agrees with the canonical compactification of $[(\mathcal{S}^{b,\circ,\mathrm{rd}}_{K^p,C} \times_{\mathcal{F}\ell_{G,\mu^{-1}}} \mathcal{F}\ell^b_{G,\mu^{-1}})/\underline{G(\mathbb{Q}_p)}]$ by the previous remark on rank 1 points and the fact that $\mathcal{F}\ell^b_{G,\mu^{-1}}$ is partially proper[2].

Define the group diamond $\mathcal{J}_b := \mathrm{Aut}(\mathcal{E}_b)$, as in [FS21, Proposition III.5.1]. We have an isomorphism $j_b : \mathrm{Bun}^b_G \simeq [\mathrm{Spd}(C)/\mathcal{J}_b] \hookrightarrow \mathrm{Bun}_G$ with the locally closed HN-strata in Bun_G defined by b. There is a $\mathcal{J}_b$-torsor over the adic Newton strata $\mathcal{F}\ell^b_{G,\mu^{-1}}$ given by rigidifying the bundle $\mathcal{E}$ to be isomorphic to $\mathcal{E}_b$. This gives a map

$$h^{\leftarrow}_b : [\mathcal{F}\ell^b_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}] \to [\mathrm{Spd}(C)/\mathcal{J}_b] \simeq \mathrm{Bun}^b_{G,C}$$

such that $h^{\leftarrow} \circ i_b = j_b \circ h^{\leftarrow}_b$.

The perfectoid Igusa variety $\mathfrak{Ig}^b_C$ comes equipped with an action of $\mathcal{J}_b$. Namely, using [CS17, Proposition 4.2.11], we get an action on the trivialization of $\mathbb{X}_b$, as in the moduli description in equation (2). This action extends to the formal model, giving rise to the action of $\mathcal{J}_b$ the generic fiber. This allows us to form the map of v-stacks:

$$\pi^b_{\mathfrak{Ig}} : [\mathfrak{Ig}^b_C/\mathcal{J}_b] \to [\mathrm{Spd}(C)/\mathcal{J}_b].$$

We would like to say $\pi^b_{\mathfrak{Ig}}$ pulls back to the map π^b_{HT}. However, as seen in Corollary 3.6, we need to account for the additional points in the fiber of the Hodge-Tate period morphism that are not seen

[2] The partial properness of these strata follows directly from the moduli interpretation, since the category of vector bundles on the Fargues-Fontaine curve is insensitive to the ring of definition.

by the perfectoid Igusa varieties $\mathfrak{Ig}^b_C$. To capture this, we need to show that $\pi^b_{\mathfrak{Ig}}$ also extends to the partial minimal compactification. We have the following.

Proposition 3.8. *[Zha23, Corollary 9.43] Assuming 1.11, the action of $\mathcal{J}_b$ on the perfectoid Igusa variety $\mathfrak{Ig}^b_C$ extends uniquely to an action on $\mathfrak{Ig}^{b,*}_C$. In particular, by functorality of the formation of the canonical compactification ([Sch18, Proposition 18.6]), we have a map*

$$\pi^{b,*}_{\mathfrak{Ig}} : [\overline{\mathfrak{Ig}}^{b,*}_C/\mathcal{J}_b] \to [\mathrm{Spd}(C)/\mathcal{J}_b]$$

extending $\pi^b_{\mathfrak{Ig}}$. This action preserves the boundary $\partial\mathfrak{Ig}^{b,*}$, so, in particular, we also get a map

$$\pi^{b,\partial}_{\mathfrak{Ig}} : [(\overline{\mathfrak{Ig}}^{b,*}_C \setminus \overline{\partial\mathfrak{Ig}}^{b,*}_C)/\mathcal{J}_b] \to [\mathrm{Spd}(C)/\mathcal{J}_b]$$

by restriction.

Proof. We consider the open immersion

$$g_b : \mathrm{Ig}^b \to \mathrm{Ig}^{b,*}$$

of perfect schemes, which we claim induces an isomorphism on global sections. To show this, we write g_b as the perfection of the limit of the corresponding maps at finite level

$$g_{b,m} : \mathrm{Ig}^b_m \hookrightarrow \mathrm{Ig}^{b,*}_m,$$

as explained in the proof of Proposition 3.7. Under Assumption 1.11, we can apply the algebraic form of Hartogs' principle (See for example [Sch15, Proposition III.2.9]) to the open inclusion $g_{b,m}$ to conclude an isomorphism of global sections via restriction. This gives the corresponding claim for the map g_b of perfect schemes. In particular, we have an isomorphism

$$\mathcal{O}(\mathrm{Ig}^b) \simeq \mathcal{O}(\mathrm{Ig}^{b,*}) \tag{5}$$

on global sections. Taking generic fibers of the corresponding integral models, we claim that we obtain an isomorphism

$$\mathcal{O}(\mathfrak{Ig}^b_C) \simeq \mathcal{O}(\mathfrak{Ig}^{b,*}_C), \tag{6}$$

of global sections. We need to be a bit careful here with analytic sheafification. In particular, for an index set I, we let $\{U_i\}_{i\in I}$ be an affine covering of Ig^b, and compute global sections via the Čech complex

$$0 \to \mathcal{O}(\mathrm{Ig}^b) \to \prod_{i\in I} \mathcal{O}(U_i) \to \prod_{i,j\in I} \mathcal{O}(U_i \cap U_j) \to \cdots . \tag{7}$$

We let $\mathfrak{U}_i$ be the formal schemes obtained by taking Witt vectors of this affine covering, with adic generic fibers $\mathfrak{U}_{i,C}$ over C. These form an affinoid perfectoid covering of the adic space $\mathfrak{Ig}^b_C$, and, by the acyclicity of affinoid perfectoids [Sch14, Theorem 1.8 (iv)], we have that

$$0 \to \mathcal{O}(\mathfrak{Ig}^b_C) \to \prod_{i\in I} \mathcal{O}(\mathfrak{U}_{i,C}) \to \prod_{i,j\in I} \mathcal{O}(\mathfrak{U}_{i,C} \cap \mathfrak{U}_{j,C}) \to \cdots . \tag{8}$$

It follows that the Čech complex (8) is obtained from Čech complex (7) by taking Witt vectors followed by taking the completed tensor product with C. Using this, we deduce that the identification (6) follows from the identification (5), as desired.

Now $\mathcal{J}_b$ acts on the LHS of (6), as discussed above. Using that $\mathfrak{Ig}^{b,*}$ is affinoid, this will give the desired action on $\mathfrak{Ig}^{b,*}$, and one can see that it preserves the boundary by using the moduli interpretation of the toroidal compactifications, and the description of the $J_b(\mathbb{Q}_p)$ action on cusp labels, as discussed in §2.2.2. $\square$

Lastly, we will consider the map $\overline{\pi}^b_{\mathfrak{Ig}} : [\overline{\mathfrak{Ig}}^b_C/\mathcal{J}_b] \to [*/\mathcal{J}_b]$, given by taking the canonical compactification of $\pi^b_{\mathfrak{Ig}}$, where we note, by [CS17, Proposition 4.2.22], the v-stack $[\mathrm{Spd}(C)/\mathcal{J}_b]$ is partially proper over $\mathrm{Spd}(C)$. We now have the following Proposition.

Proposition 3.9. *The maps constructed above fit into the following Cartesian squares*[3]

$$\begin{tikzcd} \left[(\mathcal{S}^{b,\circ,\mathrm{rd}}_{K^p,C} \times_{\mathcal{F}\ell_{G,\mu^{-1}}} \mathcal{F}\ell^b_{G,\mu^{-1}})/\underline{G(\mathbb{Q}_p)}\right] \arrow[r, "\pi^{b,\circ}_{\mathrm{HT}}"] \arrow[d, "\tilde{h}^{\leftarrow}_b"] & \left[\mathcal{F}\ell^b_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}\right] \arrow[d, "h^{\leftarrow}_b"] \\ \left[\mathfrak{Ig}^b_C/\mathcal{J}_b\right] \arrow[r, "\pi^b_{\mathfrak{Ig}}"] & \left[\mathrm{Spd}(C)/\mathcal{J}_b\right] \end{tikzcd} \tag{9}$$

and

$$\begin{tikzcd} \left[\overline{\mathcal{S}}^{b,\circ}_{K^p,C}/\underline{G(\mathbb{Q}_p)}\right] \arrow[r, "\overline{\pi}^{b,\circ}_{\mathrm{HT}}"] \arrow[d, "\tilde{h}^{\leftarrow}_b"] & \left[\mathcal{F}\ell^b_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}\right] \arrow[d, "h^{\leftarrow}_b"] \\ \left[\overline{\mathfrak{Ig}}^b_C/\mathcal{J}_b\right] \arrow[r, "\overline{\pi}^b_{\mathfrak{Ig}}"] & \left[\mathrm{Spd}(C)/\mathcal{J}_b\right]. \end{tikzcd} \tag{10}$$

Proof. Consider the moduli space of local shtukas $\mathrm{Sht}(G,b,\mu)_{\infty,C}$, as defined in [SW20, §23]. This represents the functor sending $S \in \mathrm{Perf}_C$ to the set of all pairs $(S^\#, \alpha)$ where $S^\#$ is the untilt of S coming from the map $S \to \mathrm{Spd}(C)$, and α is a modification from $\mathcal{E}_b \dashrightarrow \mathcal{E}_0$ with meromorphy along $S^\#$ and bounded by μ. We have a local Hodge-Tate period morphism

$$\mathrm{Sht}(G,b,\mu)_{\infty,C} \to \mathcal{F}\ell^b_{G,\mu^{-1}},$$

which fits into the following Cartesian diagram coming from the definition of $\mathrm{Sht}(G,b,\mu)_{\infty,C}$.

$$\begin{tikzcd} \mathrm{Sht}(G,b,\mu)_{\infty,C} \arrow[r] \arrow[d] & \mathrm{Spd}(C) \arrow[d] \\ \mathcal{F}\ell^b_{G,\mu^{-1}} \arrow[r] & \left[\mathrm{Spd}(C)/\mathcal{J}_b\right] \end{tikzcd} \tag{11}$$

Let $\mathrm{Sht}(G,b,\mu)_{\infty,C} \times^{\mathcal{J}_b} \mathfrak{Ig}^b_C$ denote the quotient of $\mathrm{Sht}(G,b,\mu)_{\infty,C} \times_C \mathfrak{Ig}^b_C$ by $\{(jx, j^{-1}y) : j \in \mathcal{J}_b, x \in \mathrm{Sht}(G,b,\mu)_{\infty,C}, y \in \mathfrak{Ig}^b_C\}$. To see that the diagram (9) above is Cartesian, observe that (11) implies we have an isomorphism

$$\mathcal{F}\ell^b_{G,\mu^{-1}} \times_{[\mathrm{Spd}(C)/\mathcal{J}_b]} [\mathfrak{Ig}^b_C/\mathcal{J}_b] \simeq \mathrm{Sht}(G,b,\mu)_{\infty,C} \times^{\mathcal{J}_b} \mathfrak{Ig}^b_C.$$

Moreover, we see, by [CS17, Corollary 4.3.19, Lemma 4.3.20], that we have an isomorphism

$$\mathrm{Sht}(G,b,\mu)_{\infty,C} \times_C \mathfrak{Ig}^b_C \simeq \mathcal{S}^{b,\circ,\mathrm{rd}}_{K^p} \times_{\mathcal{F}\ell_{G,\mu^{-1}}} \mathrm{Sht}(G,b,\mu)_{\infty,C}, \tag{12}$$

and again applying (11) implies that $\mathcal{S}^{b,\circ,\mathrm{rd}}_{K^p} \times_{\mathcal{F}\ell_{G,\mu^{-1}}} \mathcal{F}\ell^b_{G,\mu^{-1}}$ is isomorphic to the quotient of $\mathcal{S}^{b,\circ,\mathrm{rd}}_{K^p} \times_{\mathcal{F}\ell_{G,\mu^{-1}}} \mathrm{Sht}(G,b,\mu)_{\infty,C}$ by the action of $\mathcal{J}_b$ (here $\mathcal{J}_b$ acts via the action on the second factor). Thus, we have an isomorphism:

$$\mathrm{Sht}(G,b,\mu)_{\infty,C} \times^{\mathcal{J}_b} \mathfrak{Ig}^b_C \simeq \mathcal{S}^{b,\circ,\mathrm{rd}}_{K^p} \times_{\mathcal{F}\ell_{G,\mu^{-1}}} \mathcal{F}\ell^b_{G,\mu^{-1}}.$$

This gives us the Cartesian diagram (10). Now, the natural map,

$$\mathcal{S}^{b,\circ,\mathrm{rd}}_{K^p,C} \times_{\mathcal{F}\ell_{G,\mu^{-1}}} \mathcal{F}\ell^b_{G,\mu^{-1}} \hookrightarrow \mathcal{S}^{b,\circ}_{K^p,C}$$

[3] We emphasize that these are really diagrams of v-stacks and that all fiber products are formed in this category.

is a qcqs open immersion, which is an isomorphism on rank 1 points. In turn, it induces an isomorphism of canonical compactifications over the partially proper strata $\mathcal{F}\ell^b_{G,\mu^{-1}}$. Therefore, by passing to canonical compactifications over $\mathcal{F}\ell^b_{G,\mu^{-1}}$, we deduce that diagram (10) is also Cartesian. $\square$

We now invoke a result of Zhang [Zha23].

Theorem 3.10. *[Zha23, Theorem 1.3] Assuming 1.11, for all $b \in B(G,\mu)$ the Cartesian diagram (10) extends to a Cartesian diagram*

$$\begin{tikzcd} \left[\mathcal{S}^{b,*}_{K^p,C}/\underline{G(\mathbb{Q}_p)}\right] \arrow[r, "\pi^{b,*}_{\mathrm{HT}}"] \arrow[d, "{}^*\tilde{h}^{\leftarrow}_b"] & \left[\mathcal{F}\ell^b_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}\right] \arrow[d, "h^{\leftarrow}_b"] \\ \left[\overline{\mathfrak{Ig}}^{b,*}_C/\mathcal{J}_b\right] \arrow[r, "\pi^{b,*}_{\mathfrak{Ig}}"] & [\mathrm{Spd}(C)/\mathcal{J}_b] \end{tikzcd} \tag{13}$$

of v-stacks.

Remark 3.11. In fact Zhang shows a much stronger claim, that there exists a series of larger Cartesian diagrams living over $\mathrm{Bun}_{G.C}$ such that the diagrams (10) and (13), are the base-change along the inclusions $j_b : \mathrm{Bun}^b_{G,C} \hookrightarrow \mathrm{Bun}_{G,C}$ of HN-strata for $b \in B(G,\mu)$ varying.

Remark 3.12. The rough idea behind proving this is to apply a relative Spa construction in the category of diamonds to the horizontal maps of the diagram (13) by invoking Hartogs' principle, as in Proposition 3.8.

We now state the key Corollary that we will need.

Corollary 3.13. *Assuming 1.11, for all $b \in B(G,\mu)$ we have a Cartesian diagram*

$$\begin{tikzcd} \left[\mathcal{S}^{b}_{K^p,C}/\underline{G(\mathbb{Q}_p)}\right] \arrow[r, "\pi^{b}_{\mathrm{HT}}"] \arrow[d, "{}^{\partial}\tilde{h}^{\leftarrow}_b"] & \mathcal{F}\ell^b_{G,\mu^{-1}} \arrow[d, "h^{\leftarrow}_b"] \\ \left[(\overline{\mathfrak{Ig}}^{b,*}_C \setminus \overline{\partial\mathfrak{Ig}}^{b,*}_C)/\mathcal{J}_b\right] \arrow[r, "\pi^{b,\partial}_{\mathfrak{Ig}}"] & [\mathrm{Spd}(C)/\mathcal{J}_b]\,. \end{tikzcd} \tag{14}$$

Proof. This follows from the Cartesian diagram (13) and Corollary 3.5. $\square$

By the Cartesian diagram (14), if we look at the sheaf

$$i_b^* R\pi_{\mathrm{HT}!}(\overline{\mathbb{F}}_\ell)$$

we can see that this is canonically identified with

$$h^{\leftarrow *}_b R\pi^{b,\partial}_{\mathfrak{Ig}!}(\overline{\mathbb{F}}_\ell)$$

via proper base change. Moreover, we can identify $R\pi^{b,\partial}_{\mathfrak{Ig}!}(\overline{\mathbb{F}}_\ell)$ simply with the complex $V_b := R\Gamma_{c-\partial}(\mathrm{Ig}^b, \overline{\mathbb{F}}_\ell)$ of $J_b(\mathbb{Q}_p)$-modules under the identification $\mathrm{D}(\mathrm{Bun}^b_G, \overline{\mathbb{F}}_\ell) \simeq \mathrm{D}(J_b(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)$, as in Corollary 3.6. We can further refine this using the following lemma.

Lemma 3.14. *We have isomorphisms*

$$i_{b!} h^{\leftarrow *}_b(V_b) \simeq h^{\leftarrow *} j_{b!}(V_b)$$

and

$$i_{b*} h^{\leftarrow *}_b(V_b) \simeq h^{\leftarrow *} j_{b*}(V_b)$$

of sheaves on $\mathcal{F}\ell_{G,\mu^{-1}}$.

Proof. The first isomorphism follows from proper base-change. For the second isomorphism, we note that

$$h^{\leftarrow} : [\mathcal{F}\ell_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}] \to \mathrm{Bun}_G$$

is cohomologically smooth separated and representable in locally spatial diamonds; therefore, the result follows by smooth base-change [Sch18, Proposition 23.16 (2)]. □

Remark 3.15. In particular, the graded pieces of the filtration

$$R\Gamma([\mathcal{F}\ell_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}], i_{b!}i_b^*(R\pi_{\mathrm{HT}!}(\overline{\mathbb{F}}_\ell)))$$

on the cohomology of the Shimura variety are identified with $h_*^{\rightarrow}h^{\leftarrow *}j_{b!}(V_b) \in \mathrm{D}(G(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)$, and similarly for $R\Gamma([\mathcal{F}\ell_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}], i_{b*}i_b^*(R\pi_{\mathrm{HT}!}(\overline{\mathbb{F}}_\ell)))$ and $h_*^{\rightarrow}h^{\leftarrow *}j_{b*}(V_b)$.

All in all, we get the following.

Proposition 3.16. *Assuming 1.11, we have a filtration on $R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p,C}, \overline{\mathbb{F}}_\ell)$ by complexes of smooth representations of $G(\mathbb{Q}_p)$, with graded pieces isomorphic to $h_*^{\rightarrow}h^{\leftarrow *}j_{b!}(V_b)$, where $V_b \simeq R\Gamma_{c-\partial}(\mathrm{Ig}^b, \overline{\mathbb{F}}_\ell)$.*

The functor $h_*^{\rightarrow}h^{\leftarrow *}(-)$ appearing on the graded pieces is manifestly related to the action on $\mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ by Hecke operators. In particular, for each geometric dominant cocharacter $\mu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$, we have a correspondence

$$\begin{tikzcd} & \mathrm{Hck}_{G,\leq\mu} \arrow[dl, "h_\mu^{\leftarrow}"'] \arrow[dr, "h_\mu^{\rightarrow}"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times \mathrm{Spd}(C) \end{tikzcd}$$

where $\mathrm{Hck}_{G,\leq\mu}$ is the stack parametrizing modifications $\mathcal{E}_1 \to \mathcal{E}_2$ of a pair of G-bundles with meromorphy bounded by μ over the fixed untilt defined by C. We define the Hecke operator [FS21, Section IX.2]

$$T_\mu : \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell) \to \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)^{BW_{E_\mu}}$$
$$A \mapsto h_{\mu*}^{\rightarrow}(h_\mu^{\leftarrow *}(A) \otimes^{\mathbb{L}} \mathcal{S}_\mu)$$

where E_μ is the reflex field of μ and $\mathcal{S}_\mu$ is a sheaf on $\mathrm{Hck}_{G,\leq\mu}$ attached to the highest weight tilting module $\mathcal{T}_\mu$ by geometric Satake[4]. Here E_μ denotes the reflex field of μ.

If we now let μ be the minuscule cocharacter appearing above then the Bialynicki-Birula map gives an isomorphism of diamonds between the open locus of $\mathrm{Hck}_{G,\leq\mu}$ where $\mathcal{E}_1$ is isomorphic to the trivial bundle and $[\mathcal{F}\ell_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}]$, which identifies $h_\mu^{\rightarrow}$ (resp. $h_\mu^{\leftarrow}$)) with $h^{\rightarrow}$ (resp. $h^{\leftarrow}$). Moreover, this is a cohomologically smooth space of dimension $d := \langle 2\rho_G, \mu\rangle$, and we have an isomorphism $\mathcal{S}_\mu \simeq \overline{\mathbb{F}}_\ell[d](\frac{d}{2})$[5]. It follows, by proper base-change, that we have an isomorphism

$$h_*^{\rightarrow}h^{\leftarrow *}j_{b!}(V_b) \simeq j_1^*T_\mu(j_{b!}(V_b))[-d](-\frac{d}{2})$$

of $G(\mathbb{Q}_p) \times W_{E_\mathfrak{p}}$-modules, where $1 \in B(G)$ is the trivial element. The $W_{E_\mathfrak{p}}$-equivariance follows since the above Cartesian diagrams all descend to $\breve{E}_\mathfrak{p}$ and are also compatible with the Frobenius descent datum on $\mathrm{Sht}(G, b, \mu)_{\infty,C} \to \mathrm{Spd}(\breve{E}_\mathfrak{p})$ (This is true for (9) by the results of [CS17] and all the other diagrams are constructed from this one). This gives us Theorem 1.13.

[4]We note that, using [FS21, Proposition VII.5.2], we can replace the natural push-forward in the category of solid sheaves with the * push-forward in the usual category of étale $\overline{\mathbb{F}}_\ell$-sheaves when defining the Hecke operator.

[5]This is true for the highest weight module V_μ and this agrees with the highest weight tilting module $\mathcal{T}_\mu$, since μ is minuscule.

Corollary 3.17. *Assuming 1.11, the complex* $R\Gamma_c(\mathcal{S}(\mathbf{G},X)_{K^p,C},\overline{\mathbb{F}}_\ell)$ *has a* $G(\mathbb{Q}_p)\times W_{E_\mathfrak{p}}$*-equivariant filtration with graded pieces given by* $j_1^*T_\mu(j_{b!}(V_b))[-d](-\frac{d}{2})$ *for varying* $b\in B(G,\mu)$*, where* $V_b\simeq R\Gamma_{c-\partial}(\mathrm{Ig}^b,\overline{\mathbb{F}}_\ell)$*.*

Moreover, we obtain that each graded piece is isomorphic to

$$(R\Gamma_c(G,b,\mu)\otimes^{\mathbb{L}}_{\mathcal{H}(J_b)}V_b)[2d_b]$$

as $G(\mathbb{Q}_p)\times W_{E_\mathfrak{p}}$*-modules. Here*

$$R\Gamma_c(G,b,\mu):=\mathrm{colim}_{K_p\to\{1\}}R\Gamma_c(\mathrm{Sht}(G,b,\mu)_{\infty,C}/\underline{K_p},\overline{\mathbb{F}}_\ell(d_b))$$

is a complex of $G(\mathbb{Q}_p)\times J_b(\mathbb{Q}_p)\times W_{E_\mathfrak{p}}$*-modules, where* $\mathrm{Sht}(G,b,\mu)_{\infty,C}$ *is as defined above, and* $\overline{\mathbb{F}}_\ell(d_b)$ *is the sheaf with trivial Weil group action and* $J_b(\mathbb{Q}_p)$*-action given as in [Kos21, Lemma 7.4].*

Proof. It remains to explain the description of $j_1^*T_\mu(j_{b!}(V_b))[-d](-\frac{d}{2})$. By applying the second part of [Ham22, Proposition 11.12] and noting that $\mathcal{S}_\mu\simeq\overline{\mathbb{F}}_\ell[d](\frac{d}{2})$ since μ is minuscule (where we recall that, since μ is minuscule, the representation $\mathcal{T}_\mu$ agrees with the usual highest weight representation), we obtain that the graded pieces are isomorphic to

$$\mathrm{colim}_{K_p\to\{1\}}(R\Gamma_c(\mathrm{Sht}(G,b,\mu)_{\infty,C}/\underline{K_p},\overline{\mathbb{F}}_\ell)\otimes^{\mathbb{L}}_{\mathcal{H}(J_b)}V_b\otimes\kappa^{-1})[2d_b]$$

as desired, where κ is the character of $J_b(\mathbb{Q}_p)$ defined by the action of $J_b(\mathbb{Q}_p)$ on the compactly supported cohomology of the ℓ-adically contractible group diamond $\mathcal{J}_b^{>0}$, where $\mathcal{J}_b\simeq\mathcal{J}_b^{>0}\ltimes J_b(\mathbb{Q}_p)$ is the semi-direct product structure given by allowing $\mathrm{Aut}(\mathcal{E}_b)$ to act on its canonical reduction. However, by combining this with [Kos21, Lemma 7.6] and its proof, we can rewrite this as

$$(\mathrm{colim}_{K_p\to\{1\}}R\Gamma_c(\mathrm{Sht}(G,b,\mu)_{\infty,C}/\underline{K_p},\overline{\mathbb{F}}_\ell(d_b))\otimes^{\mathbb{L}}_{\mathcal{H}(J_b)}V_b)[2d_b],$$

as desired. $\square$

4. The Local Results

4.1. **The Spectral Action.** Let $G/\mathbb{Q}_p$ be a quasi-split connected reductive group with a choice of Borel B and maximal torus T as before. We will work with $\mathrm{D}(\mathrm{Bun}_G,\overline{\mathbb{F}}_\ell)$, the derived category of étale $\overline{\mathbb{F}}_\ell$-sheaves on the moduli stack of G-bundles. Our goal in this section will be to describe a localization $\mathrm{D}(\mathrm{Bun}_G,\overline{\mathbb{F}}_\ell)_{\phi_\mathfrak{m}}\subset\mathrm{D}(\mathrm{Bun}_G,\overline{\mathbb{F}}_\ell)$ for $\mathfrak{m}\subset H_{K_p^{\mathrm{hs}}}$ a generic maximal ideal with associated semi-simple L-parameter $\phi_\mathfrak{m}$ in the case that G is unramified. We will do this in slightly more generality using the spectral action [FS21, Section X.2]. We assume that ℓ is very good as in [FS21, Page 33], and consider the moduli stack $\mathfrak{X}_{\hat{G}}/\operatorname{Spec}\overline{\mathbb{F}}_\ell$ of $\overline{\mathbb{F}}_\ell$-valued Langlands parameters, as defined in [Dat+20; Zhu20]. We let $\mathrm{Perf}(\mathfrak{X}_{\hat{G}})$ denote the derived category of perfect complexes on this stack, and we write $\mathrm{Perf}(\mathfrak{X}_{\hat{G}})^{BW^I_{\mathbb{Q}_p}}$ for the derived category of objects with a continuous $W^I_{\mathbb{Q}_p}$ action for a finite index set I, and $\mathrm{D}(\mathrm{Bun}_G,\overline{\mathbb{F}}_\ell)^\omega$ for the triangulated sub-category of compact objects in $\mathrm{D}(\mathrm{Bun}_G,\overline{\mathbb{F}}_\ell)$. By [FS21, Corollary X.I.3], there exists a $\overline{\mathbb{F}}_\ell$-linear action

$$\mathrm{Perf}(\mathfrak{X}_{\hat{G}})\to\mathrm{End}(\mathrm{D}(\mathrm{Bun}_G,\overline{\mathbb{F}}_\ell)^\omega)$$

$$C\mapsto\{A\mapsto C\star A\}$$

which, extending by colimits, gives

$$\mathrm{Ind}(\mathrm{Perf}(\mathfrak{X}_{\hat{G}}))\to\mathrm{End}(\mathrm{D}(\mathrm{Bun}_G,\overline{\mathbb{F}}_\ell))$$

We recall the following basic properties of this action.

(1) For $V = \boxtimes_{i \in I} V_i \in \mathrm{Rep}_{\overline{\mathbb{F}}_\ell}({}^L G^I)$, there is an attached vector bundle $C_V \in \mathrm{Perf}(\mathfrak{X}_{\hat{G}})^{BW^I_{\mathbb{Q}_p}}$ whose evaluation at a $\overline{\mathbb{F}}_\ell$-point of $\mathfrak{X}_{\hat{G}}$ corresponding to a (not necessarily semi-simple) L-parameter $\tilde{\phi}: W_{\mathbb{Q}_p} \to {}^L G(\overline{\mathbb{F}}_\ell)$ is the vector space V with $W^I_{\mathbb{Q}_p}$-action given by $\boxtimes_{i \in I} r_{V_i} \circ \tilde{\phi}$. The endomorphism

$$C_V \star (-) : \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell) \to \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)^{BW^I_{\mathbb{Q}_p}}$$

is the Hecke operator T_V attached to V.

(2) The action is monoidal in the sense that, given $C_1, C_2 \in \mathrm{Perf}(\mathfrak{X}_{\hat{G}})$, we have a natural equivalence of endofunctors

$$(C_1 \otimes^{\mathbb{L}} C_2) \star (-) \simeq C_1 \star (C_2 \star (-)).$$

If $\phi : W_{\mathbb{Q}_p} \to {}^L G(\overline{\mathbb{F}}_\ell)$ is a semi-simple L-parameter then this defines a closed $\overline{\mathbb{F}}_\ell$-point x in the moduli stack of Langlands parameters, which maps to a closed point in the coarse moduli space. We let $\mathfrak{m}_\phi \subset \mathcal{O}_{\mathfrak{X}_{\hat{G}}}(\mathfrak{X}_{\hat{G}})$ denote the corresponding maximal ideal. We recall that, for all $f \in \mathcal{O}_{\mathfrak{X}_{\hat{G}}}(\mathfrak{X}_{\hat{G}})$ and $A \in \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$, one obtains an endomorphism

$$A \simeq \mathcal{O}_{\mathfrak{X}_{\hat{G}}} \star A \to \mathcal{O}_{\mathfrak{X}_{\hat{G}}} \star A \simeq A$$

induced by multiplication by f. Under the description of $\mathcal{O}_{\mathfrak{X}_{\hat{G}}}(\mathfrak{X}_{\hat{G}})$ in terms of the excursion algebra, this encodes the action of the excursion algebra on $\mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ [Zou22, Theorem 5.2.1]. More precisely, we recall that, since ℓ is very good [FS21, Page 33], to any Schur-irreducible $A \in \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ we can, by [FS21, Proposition I.9.3], attach a conjugacy class of semi-simple L-parameters

$$\phi^{\mathrm{FS}}_A : W_{\mathbb{Q}_p} \to {}^L G(\overline{\mathbb{F}}_\ell)$$

called the Fargues-Scholze parameter of A. By [FS21, Theorem VIII.3.6], we have an identification between the ring of global functions $\mathcal{O}_{\mathfrak{X}_{\hat{G}}}(\mathfrak{X}_{\hat{G}})$ and excursion operators. Since A is Schur irreducible the endomorphisms corresponding to $f \in \mathcal{O}_{\mathfrak{X}_{\hat{G}}}(\mathfrak{X}_{\hat{G}})$ determine a non-zero scalar in $\overline{\mathbb{F}}_\ell$ which will be determined by the excursion datum evaluated at the Fargues-Scholze parameter ϕ^{FS}_A.

With this in hand, we can make our key definition.

Definition 4.1. We define $\iota_\phi : \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_\phi \hookrightarrow \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ to be the full-subcategory of objects A for which the endomorphisms $A \to A$ induced by $f \in \mathcal{O}_{\mathfrak{X}_{\hat{G}}} \setminus \mathfrak{m}_\phi$ are isomorphisms.

It is easy to check that the subcategory $\mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_\phi \subset \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ is preserved under colimits and limits, and therefore, by the ∞-categorical adjoint functor theorem [Lur09, Corollary 5.5.2.9], there exists a left adjoint to the inclusion ι_ϕ denoted by $\mathcal{L}_\phi$. We define $(-)_\phi := \iota_\phi \mathcal{L}_\phi(-)$. This, by the fully faithfulness of ι_ϕ, will define an idempotent functor (See Appendix A for details).

We now have the following key lemma.

Lemma 4.2. *The following is true.*

(1) *Any Schur irreducible object $A \in \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_\phi$ has Fargues-Scholze parameter equal to ϕ as conjugacy classes of parameters.*

(2) *Given $V \in \mathrm{Rep}_{\overline{\mathbb{F}}_\ell}({}^L G^I)$, the Hecke operator $T_V : \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell) \to \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)^{BW^I_{\mathbb{Q}_p}}$ takes the subcategory $\mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_\phi$ to $\mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_\phi^{BW^I_{\mathbb{Q}_p}}$, and there is a natural isomorphism $T_V((-)_\phi) \simeq (T_V(-))_\phi$.*

(3) *Given $A \in \mathrm{D}(G(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell) \subset \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$, we have an isomorphism*

$$R\Gamma(K^{\mathrm{hs}}_p, A)_{\mathfrak{m}} \simeq R\Gamma(K^{\mathrm{hs}}_p, A_{\phi_{\mathfrak{m}}}),$$

where the LHS is the usual localization under the smooth Hecke algebra.

(4) *If $A \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ is ULA then one has a direct sum decomposition*
$$A \simeq \bigoplus_{\phi} A_\phi$$
ranging over all semi-simple L-parameters.

Proof. Claims (2) and (4) follow from Proposition A.2 and Proposition A.5, respectively, where for claim (2) we use the relationship between Hecke operators and the spectral action described above.

For (1), this follows since the action of $\mathcal{O}_{\mathfrak{X}_{\hat{G}}}(\mathfrak{X}_{\hat{G}})$ on A will factor through the maximal ideal $\mathfrak{m}_A$ defined by the semi-simple L-parameter ϕ_A^{FS} attached to A by the above discussion, and therefore $A \in \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_\phi$ forces an equality of maximal ideals: $\mathfrak{m}_A = \mathfrak{m}_\phi$.

For (3), we use the arguments in Koshikawa [Kos21, Page 6]. Consider the map
$$\mathcal{O}_{\mathfrak{X}_{\hat{G}}}(\mathfrak{X}_{\hat{G}}) \to \mathrm{End}_{G(\mathbb{Q}_p)}(\mathrm{cInd}_{K_p^{\mathrm{hs}}}^{G(\mathbb{Q}_p)}(\overline{\mathbb{F}}_\ell)) \simeq H_{K_p^{\mathrm{hs}}}^{\mathrm{op}}$$
given by the spectral action, where $\mathrm{cInd}_{K_p^{\mathrm{hs}}}^{G(\mathbb{Q}_p)}(\overline{\mathbb{F}}_\ell)$ is regarded as a right $H_{K_p^{\mathrm{hs}}}$-module. It is shown that after through the usual action by the unramified Hecke algebra composed with the involution $KhK \to Kh^{-1}K$ gives rise to a map which is compatible with usual L-parameters for unramified irreducible representations. In particular, the pullback of the maximal ideal $\mathfrak{m} \subset H_{K_p^{\mathrm{hs}}}$ is given by the maximal ideal $\mathfrak{m}_{\phi_{\mathfrak{m}}} \subset \mathcal{O}_{X_{\hat{G}}}(\mathfrak{X}_{\hat{G}})$. Now, by arguing as in Proposition A.3, we have an identification:
$$R\mathrm{Hom}(\mathrm{cInd}_{K_p^{\mathrm{hs}}}^{G(\mathbb{Q}_p)}(\overline{\mathbb{F}}_\ell), A_{\phi_{\mathfrak{m}}}) \simeq R\mathrm{Hom}(\mathrm{cInd}_{K_p^{\mathrm{hs}}}^{G(\mathbb{Q}_p)}(\overline{\mathbb{F}}_\ell), A)_{\mathfrak{m}_{\phi_{\mathfrak{m}}}}.$$
Using Frobenius reciprocity, this gives an identification:
$$R\Gamma(K_p^{\mathrm{hs}}, A_{\phi_{\mathfrak{m}}}) \simeq R\Gamma(K_p^{\mathrm{hs}}, A)_{\mathfrak{m}_{\phi_{\mathfrak{m}}}},$$
but the RHS identifies with $R\Gamma(K_p^{\mathrm{hs}}, A)_{\mathfrak{m}}$, as explained above. ∎

We note that we get the following Corollary of this.

Corollary 4.3. *Let A be a complex of smooth $G(\mathbb{Q}_p)$-representations which is admissible (i.e A^K is a perfect complex for all compact open $K \subset G(\mathbb{Q}_p)$). We then have a decomposition*
$$A \simeq \bigoplus_{\phi} A_\phi$$
running over semisimple L-parameters, where any irreducible constituent π of A_ϕ has Fargues-Scholze parameter equal to ϕ_π^{FS}, as conjugacy classes of parameters.

Proof. This follows by applying to Lemma 4.2 (1) and (4) to the full subcategory $\mathrm{D}(G(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell) \subset \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ ∎

Now our goal is to describe the subcategory $\mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_\phi$ more explicitly, using the results of [Ham22] in the case that ϕ is induced from a generic toral parameter ϕ_T. To do this, we will need to have some information about the Fargues-Scholze local Langlands correspondence. First, let us introduce some notation.

We let $B(G)_{\mathrm{un}} := \mathrm{Im}(B(T) \to B(G))$. We recall that these are precisely the elements $b \in B(G)$ such that the σ-centralizer J_b is quasi-split ([Ham22, Lemma 2.12]). In particular, the fixed choice of Borel $B \subset G$ transfers to a Borel subgroup B_b for all $b \in B(G)_{\mathrm{un}}$, and $J_b \simeq M_b$ under the inner twisting, where $M_b \subset G$ is the Levi subgroup of G determined by the centralizer of the slope homomorphism of b in G. We let δ_{P_b} denote the modulus character of the standard parabolic P_b of G with Levi factor M_b transferred to J_b under the inner twisting. We set $W_b := W_G/W_{M_b}$ to be the quotient of the relative Weyl group of G by the relative Weyl group of M_b. We identify W_b with a choice of representatives in $w \in W_G$ of minimal length. We set $\rho_{b,w} := i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{-1/2}$ to be the normalized parabolic induction of χ^w, where χ is the character of $T(\mathbb{Q}_p)$ attached to a toral parameter ϕ_T under local class field theory and δ_{P_b} is the modulus character of $M_b \simeq J_b$.

We will need to assume the following properties of the Fargues-Scholze local Langlands correspondence, as in [Ham22, Assumption 7.5].

Assumption 4.4. *For a connected reductive group $H/\mathbb{Q}_p$, we have*

- *the set $\Pi(H)$ of smooth irreducible $\overline{\mathbb{Q}}_\ell$-representations of $H(\mathbb{Q}_p)$,*
- *the set $\Phi(H)$ of conjugacy classes of continuous maps*
$$\mathrm{WD}_{\mathbb{Q}_p} \to {}^L H(\overline{\mathbb{Q}}_\ell)$$
where $\overline{\mathbb{Q}}_\ell$ has the discrete topology, $\mathrm{SL}(2, \overline{\mathbb{Q}}_\ell)$ acts via an algebraic representation, and the map respects the action of $W_{\mathbb{Q}_p}$ on ${}^L H(\overline{\mathbb{Q}}_\ell)$, the L-group of H,
- *the set $\Phi^{\mathrm{ss}}(H)$ of continuous semi-simple homomorphisms*
$$W_{\mathbb{Q}_p} \to {}^L H(\overline{\mathbb{Q}}_\ell),$$
- *and the semi-simplification map $(-)^{\mathrm{ss}} : \Phi(H) \to \Phi^{\mathrm{ss}}(H)$ defined by precomposition with*
$$W_{\mathbb{Q}_p} \to W_{\mathbb{Q}_p} \times \mathrm{SL}(2, \overline{\mathbb{Q}}_\ell)$$
$$g \mapsto (g, \begin{pmatrix} |g|^{1/2} & 0 \\ 0 & |g|^{-1/2} \end{pmatrix}).$$

Then, we assume, for all $b \in B(G)$, that there exists a map
$$\mathrm{LLC}_b : \Pi(J_b) \to \Phi(J_b)$$
$$\rho \mapsto \phi_\rho$$
satisfying the following properties:

(1) *The diagram*
$$\begin{tikzcd} \Pi(J_b) \arrow[r, "\mathrm{LLC}_b"] \arrow[dr, "\mathrm{LLC}_b^{\mathrm{FS}}"'] & \Phi(J_b) \arrow[d, "(-)^{\mathrm{ss}}"] \\ & \Phi^{\mathrm{ss}}(J_b) \end{tikzcd}$$
commutes, where $\mathrm{LLC}_b^{\mathrm{FS}}$ is the Fargues-Scholze local Langlands correspondence for J_b.

(2) *Consider ϕ_ρ as an element of $\Phi(G)$ given by composing with the twisted embedding ${}^L J_b(\overline{\mathbb{Q}}_\ell) \simeq {}^L M_b(\overline{\mathbb{Q}}_\ell) \to {}^L G(\overline{\mathbb{Q}}_\ell)$ (as defined in [FS21, Section IX.7.1]). Then ϕ_ρ factors through the natural embedding ${}^L T \to {}^L G$ if and only if $b \in B(G)_{\mathrm{un}}$.*

(3) *If ρ is a representation such that $W_{\mathbb{Q}_p} \times \mathrm{SL}(2, \overline{\mathbb{Q}}_\ell) \to {}^L J_b(\overline{\mathbb{Q}}_\ell) \to {}^L G(\overline{\mathbb{Q}}_\ell)$ factors through ${}^L T$, where the last map is the twisted embedding then, by (2), the element b is unramified, and we require that ρ is isomorphic to an irreducible constituent of $\rho_{b,w}$ for $w \in W_b$ and χ the character attached to the induced toral parameter ϕ_T.*

The importance of this assumption is that it allows us to deduce the following Proposition.

Proposition 4.5. *Assuming 4.4, we have that the following is true for a parameter ϕ induced from a generic parameter ϕ_T. Given any Schur-irreducible object $A \in \mathrm{D}(\mathrm{Bun}_G^b, \overline{\mathbb{F}}_\ell)_\phi \subset \mathrm{D}(\mathrm{Bun}_G^b, \overline{\mathbb{F}}_\ell) \simeq \mathrm{D}(J_b(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)$ then A is non-zero if and only if $b \in B(G)_{\mathrm{un}}$, and in this case it must be an irreducible sub-quotient of $\rho_{b,w}$, for some $w \in W_b$.*

Proof. This follows from combining the proof of [Ham22, Corollary 7.7] with Lemma 4.2 (1). □

Since we want some flexibility in the groups for which we have the above results, we discuss how Assumption 4.4 behaves under central isogenies.

4.1.1. *Assumption 4.4 under Central Isogenies.* We consider an injective map $\psi : G' \hookrightarrow G$ of connected reductive groups which induces an isomorphism of adjoint and derived groups, and the induced map $\psi_{\mathrm{Bun}} : B(G') \to B(G)$ on the associated Kottwitz sets. We now have the following lemma.

Lemma 4.6. *If $\psi : G' \to G$ is an injective map which induces an isomorphism on adjoint and derived groups then it follows that $\psi_{\mathrm{Bun}} : B(G') \to B(G)$ induces an injection $J_{b'} \to J_b$ which is an isomorphism of the derived group and adjoint groups for all $b = \psi_{\mathrm{Bun}}(b')$ and $b' \in B(G')$.*

Proof. Since ψ is an inclusion it easily follows that it induces an inclusion $J_{b'} \to J_b$ of σ-centralizers. To see that it induces an isomorphism on derived/adjoint groups, recall that J_b is an inner form of a Levi subgroup M_b of G given by the centralizer of the slope homomorphism of b. The preimage of M_b under ϕ defines a Levi subgroup $M_{b'}$ of G which will be the centralizer of the slope homomorphism of b', since ϕ induces an isomorphism on adjoint groups. Moreover, the inner twisting from J_b to M_b and $J_{b'}$ to $M_{b'}$ are compatible with the inclusion in the sense that the inclusion $J_{b'} \to J_b$ is given by applying the inner twist of M_b to the inclusion $M_{b'} \to M_b$. Since the formation of derived/adjoint groups respects inner twists, this reduces us to showing that the map $M_{b'} \to M_b$ on Levi subgroups induces an isomorphism on the derived/adjoint groups, and this is clear. □

We now consider a map $\mathrm{LLC}_{\mathrm{Bun}_G} : \bigsqcup_{b \in B(G)} \Pi(J_b) \to \Phi(G)$ determined by components $\mathrm{LLC}_b : \Pi(J_b) \to \Phi(J_b)$ and satisfying Assumption 4.4. We now wish to define $\mathrm{LLC}_{\mathrm{Bun}_{G'}} : \bigsqcup_{b' \in B(G')} \Pi(J_b) \to \Phi(G')$ in terms of $\mathrm{LLC}_{\mathrm{Bun}_G}$, and show that it also satisfies Assumption 4.4. To do this, we note that, for varying $b' \in B(G')$, we define $\mathrm{LLC}_{b'} : \Pi(J_{b'}) \to \Phi(J_{b'})$ to be the correspondence that makes the following diagram

$$\begin{tikzcd} \Pi(J_b) \arrow[r, "\mathrm{LLC}_b"] \arrow[d, dashed] & \Phi(J_b) \arrow[d] \\ \Pi(J_{b'}) \arrow[r, "\mathrm{LLC}_{b'}"] & \Phi(J_{b'}) \end{tikzcd}$$

commute, where $b := \psi_{\mathrm{Bun}}(b')$. Here the right vertical arrow is given by composing a parameter $\phi : \mathrm{WD}_{\mathbb{Q}_p} \to {}^L J_b(\overline{\mathbb{Q}}_\ell)$ with the induced map ${}^L J_b \to {}^L J_{b'}$ on the dual groups, and the left vertical arrow is not a map at all, it is a correspondence defined by the subset of $\Pi(J_b) \times \Pi(J_{b'})$ consisting of pairs $(\pi_b, \pi_{b'})$ such that $\pi_{b'}$ is a constituent of the restriction of π_b to $J_{b'}(\mathbb{Q}_p)$. We will now show that this gives rise to a well-defined map under our assumptions on ψ. Given a representation $\pi_{b'} \in \Pi(J_{b'})$, it follows by [GK82, Lemma 2.3] and the previous Lemma that we can find a lift $\pi_b \in \Pi(J_b)$ such that $\pi_{b'}$ is an irreducible constituent of $\pi_b|_{J_{b'}(\mathbb{Q}_p)}$. It also follows from [GK82, Lemma 2.1] and [Tad92, Proposition 2.4, Corollary 2.5] that the set $\Pi_{\pi_b}(J_{b'})$ of representations of $J_{b'}$ occurring in the restriction of π_b is finite. Now, using the previous Lemma, we have the following.

Lemma 4.7. *[GK82, Lemma 2.4] For the map $J_{b'} \to J_b$ of σ-centralizers induced by a map ψ as above, and $\pi_b^1, \pi_b^2 \in \Pi(J_b)$ the following are equivalent.*

(1) *There exists a character $\chi \in (J_b(\mathbb{Q}_p)/J_{b'}(\mathbb{Q}_p))^\vee$ such that $\pi_b^1 \simeq \pi_b^2 \otimes \chi$, where $(-)^\vee$ denotes the Pontryagin dual.*
(2) $\Pi_{\pi_b^1}(J_{b'}) \cap \Pi_{\pi_b^2}(J_{b'}) \neq \emptyset$
(3) $\Pi_{\pi_b^1}(J_{b'}) = \Pi_{\pi_b^2}(J_{b'})$.

Now we can use this to define $\mathrm{LLC}_{b'} : \Pi(J_{b'}) \to \Phi(J_{b'})$ in terms of $\mathrm{LLC}_b : \Pi(J_b) \to \Phi(J_b)$ for $\mathrm{LLC}_{\mathrm{Bun}_G}$ satisfying Assumption 4.4. Namely, for $\pi_{b'} \in \Pi(J_{b'})$, we let $\pi_b \in \Pi(J_b)$ be a representation such that $\pi_{b'}$ occurs as an irreducible constituent of $\pi_b|_{J_{b'}(\mathbb{Q}_p)}$. We set $\phi_{\pi_{b'}}$ to be the parameter ϕ_{π_b} attached to π_b under LLC_b composed with the map ${}^L J_b \to {}^L J_{b'}$ on dual groups induced by ψ. By the previous Lemma, any two choices of lifts π_b^1 and π_b^2 of $\pi_{b'}$ will differ by a character twist of $\chi \in (J_b(\mathbb{Q}_p)/J_{b'}(\mathbb{Q}_p))^\vee$. We note that the Fargues-Scholze local Langlands correspondence is compatible with character twists [FS21, Theorem I.9.6 (ii)]. Since LLC_b is compatible with the Fargues-Scholze local Langlands after semi-simplification by assumption, it follows that the same is true for LLC_b. Therefore, $\phi_{\pi_b^1}$ and $\phi_{\pi_b^2}$ differ by a character twist that becomes trivial after composing with ${}^L J_b \to {}^L J_{b'}$, and so ϕ_{π_b} does not depend on the choice of lift. We let $\mathrm{LLC}_{\mathrm{Bun}_{G'}} : \bigsqcup_{b' \in B(G')} \Pi(J_{b'}) \to \Phi(G)$ be the local Langlands correspondence defined by the $\mathrm{LLC}_{b'}$ for b' varying. We now prove that our assumption is comaptible with central isogenies.

Proposition 4.8. *Suppose we have an injective map $\psi : G' \hookrightarrow G$ of quasi-split connected reductive groups inducing an isomorphism on adjoint and derived groups. Assume we have a local Langlands correspondence $\mathrm{LLC}_{\mathrm{Bun}_G} : \bigsqcup_{b \in B(G)} \Pi(J_b) \to \Phi(G)$ such that Assumption 4.4 holds. If we let $\mathrm{LLC}_{\mathrm{Bun}_{G'}} : \bigsqcup_{b' \in B(G')} \Pi(J_{b'}) \to \Phi(G')$ be the local Langlands correspondence induced by $\mathrm{LLC}_{\mathrm{Bun}_G}$ and ψ as above then $\mathrm{LLC}_{\mathrm{Bun}_{G'}}$ satisfies Assumption 4.4 as well.*

Proof. We note, since the Fargues-Scholze local Langlands correspondence is compatible with maps $G' \to G$ that induce an isomorphism of adjoint groups [FS21, Theorem I.9.6 (v)], it follows by the above construction that if Assumption 4.4 (1) holds true for $\mathrm{LLC}_{\mathrm{Bun}_G}$ then it also holds true for $\mathrm{LLC}_{\mathrm{Bun}_{G'}}$. Suppose we have $b' \in B(G')$ mapping to $b \in B(G)_{\mathrm{un}}$. We let $B_b \subset J_b$ be the corresponding Borel then, since the map $J_{b'} \to J_b$ induces an isomorphism on adjoint groups by Lemma 4.6, it follows that $B \cap J_{b'} =: B_{b'} \subset J_{b'}$ is a Borel of $J_{b'}$. In particular, b' must be an unramified element of $B(G')_{\mathrm{un}}$. Now, the map $J_{b'} \to J_b$ induces an isomorphism

$$J_{b'}/B' \simeq J_b/B.$$

If we let T be the maximal split torus of J_b then the preimage T' under ϕ is a maximal torus of $J_{b'}$, and the previous isomorphism of flag varieties implies that, given a character $\chi : T(\mathbb{Q}_p) \to \overline{\mathbb{Q}}_\ell^*$, we have an isomorphism:

$$i_{B_b}^{J_b}(\chi)|_{J_{b'}(\mathbb{Q}_p)} \simeq i_{B_{b'}}^{J_{b'}}(\chi|_{T'(\mathbb{Q}_p)}).$$

Given $\pi_{b'}$ and a lift π_b to J_b then, by definition of ϕ_{π_b}, we have that it is equal to

$$\mathrm{WD}_{\mathbb{Q}_p} \xrightarrow{\phi_{\pi_b}} {}^L J_b(\overline{\mathbb{Q}}_\ell) \to {}^L J_{b'}(\overline{\mathbb{Q}}_\ell)$$

as a conjugacy class of parameters for $J_{b'}$. Therefore, $\phi_{\pi_{b'}}$ factors through ${}^L T'$ if and only if ϕ_{π_b} factors through the preimage of ${}^L T'$ under the map ${}^L J_b(\overline{\mathbb{Q}}_\ell) \to {}^L J_{b'}(\overline{\mathbb{Q}}_\ell)$ of L-groups, but this is precisely ${}^L T$, and so Assumption 4.4 (2) holds for $\mathrm{LLC}_{\mathrm{Bun}_{G'}}$. Moreover, by Assumption 4.4 (3) for $\mathrm{LLC}_{\mathrm{Bun}_G}$, we have that, in the above situation, π_b is an irreducible sub-quotient of $i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{-1/2}$, but this implies that $\pi_{b'}$ is an irreducible constituent of the restriction $i_{B_{b'}}^{J_{b'}}(\chi^w|_{T'(\mathbb{Q}_p)}) \otimes \delta_{P_{b'}}^{-1/2}$. From this, it follows that Assumption 4.4 (3) also holds for holds for $\mathrm{LLC}_{\mathrm{Bun}_{G'}}$. $\square$

Now that we have shown this compatibility assumption is somewhat flexible, we can state the groups we know to satisfy Assumption 4.4. This result is largely contained in [Ham21; FS21; HKW22; BHN22], but we also want to consider an additional group GU_2, where we have the following construction of $\mathrm{LLC}_{\mathrm{Bun}_{\mathrm{GU}_2}}$. Recall that GU_2/L can be written as

$$\mathrm{GU}_2 := (\mathrm{GL}_2 \times \mathrm{Res}_{L'/L}\mathbb{G}_m)/\mathbb{G}_m,$$

where $\mathbb{G}_m$ is embedded in $H := \mathrm{GL}_2 \times \mathrm{Res}_{L'/L}(\mathbb{G}_m)$ via $a \mapsto (\mathrm{diag}(a, a), a^{-1})$, and L'/L is an unramified quadratic extension. Let $\psi : B(H) \to B(\mathrm{GU}(2))$ and let $\tilde{\psi} : B(H) \to B(\mathrm{GL}_2)$ be the map of Kottwitz sets. Given $b \in B(H)$, let $b' = \psi(b)$, $\tilde{b} = \tilde{\psi}(b)$.

Lemma 4.9. *There is a bijection between $\Pi(J_{b'})$ and the set of pairs $(\tilde{\pi}, \chi)$ such that $\tilde{\pi} \in \Pi(J_{\tilde{b}})$ and χ is a character of $(L')^\times$ such that $\chi|_{L^\times} = \omega_{\tilde{\pi}}|_{L^\times}$, where $\omega_{\tilde{\pi}}$ is the central character of $\tilde{\pi}$.*

Proof. We will show that we have an isomorphism

$$J_{b'} \simeq (J_{\tilde{b}} \times \mathrm{Res}_{L'/L}\mathbb{G}_m)/\mathbb{G}_m$$

of groups over L. In particular, we see that $J_b = J_{\tilde{b}} \times \mathrm{Res}_{L'/L}\mathbb{G}_m$, and the quotient map $J_b \to J_{b'}$ induces an isomorphism on adjoint and derived subgroups of $J_{b'}$. Moreover, by Hilbert's Theorem 90, we have $H^1(L, \mathbb{G}_m) = 0$, and thus we also have a surjection on L-points, from which the lemma follows. To see this isomorphism, recall that J_b (resp. $J_{b'}, J_{\tilde{b}}$) is an inner form of M_b (resp. $M_{b'}, M_{\tilde{b}}$), the Levi subgroup of H (resp. $\mathrm{GU}_2, \mathrm{GL}_2$) given by the centralizer of the slope homomorphism of b (resp. $b', \tilde{b}$). In particular, we see that we have an isomorphism

$$M_{b'} \simeq (M_{\tilde{b}} \times \mathrm{Res}_{L'/L}\mathbb{G}_m)/\mathbb{G}_m$$

and thus we have a surjective map $M_b \to M_{b'}$, since $M_b = M_{\tilde{b}} \times \mathrm{Res}_{L'/L}\mathbb{G}_m$. Moreover, we see that under these maps, we have isomorphisms $M_b^{\mathrm{ad}} \simeq M_{b'}^{\mathrm{ad}} \simeq M_{\tilde{b}}^{\mathrm{ad}}$, and the inner twist $H^1(L, M_b^{\mathrm{ad}})$ corresponding to J_b is, under this identification, the inner twist inducing $J_{b'}$ and $J_{\tilde{b}}$. The identification of $J_{b'}$ then follows. $\square$

Moreover, we observe that we have an exact sequence of dual groups

$$1 \to \widehat{J}_{b'} \to \widehat{J}_b \to \mathbb{G}_m \to 1,$$

where the map $p : \widehat{J}_b \to \mathbb{G}_m$ is defined as follows. We can write $\widehat{J}_b = \widehat{J}_{\tilde{b}} \times \mathbb{G}_m^2$, and we have maps $\hat{i}_1 : \widehat{J}_{\tilde{b}} \to \mathbb{G}_m$, $\hat{i}_2 : \widehat{\mathrm{Res}_{L'/L}}\mathbb{G}_m = \mathbb{G}_m^2 \to \mathbb{G}_m$ induced from the inclusion maps $i_1 : \mathbb{G}_m \hookrightarrow J_{\tilde{b}}$ and $i_2 : \mathbb{G}_m \hookrightarrow \mathrm{Res}_{L'/L}\mathbb{G}_m$, and $p(g, h) = \hat{i}_1(g)\hat{i}_2(h)^{-1}$.

Now, we want to define $\mathrm{LLC}_{\mathrm{Bun}_{\mathrm{GU}_2}}$ in terms of $\mathrm{LLC}_{\mathrm{Bun}_H}$. More precisely, for any $b' \in B(\mathrm{GU}_2)$ we define $\mathrm{LLC}_{b'} : \Pi(J_{b'}) \to \Phi(J_{b'})$ in terms of $\mathrm{LLC}_b : \Pi(J_b) \to \Phi(J_b)$ for $\mathrm{LLC}_{\mathrm{Bun}_H}$. For $\pi_{b'} = (\tilde{\pi}, \chi) \in \Pi(J_{b'})$, we consider the image $\phi = \mathrm{LLC}_{\mathrm{Bun}_H}((\tilde{\pi}, \chi))$, and we want to show that $\phi : \mathrm{WD}_L \to {}^L J_b(\overline{\mathbb{Q}}_\ell)$ factors through ${}^L J_{b'}(\overline{\mathbb{Q}}_\ell)$. To see this, from the exact sequence above, it suffices to show that the composition of ϕ with the map ${}^L J_b(\overline{\mathbb{Q}}_\ell) \to {}^L\mathbb{G}_m(\overline{\mathbb{Q}}_\ell)$ is trivial. However, we observe that the condition that $\chi|_{L^\times} = \omega_{\tilde{\pi}}|_{L^\times}$ exactly implies that this image is trivial, since the composition is the L-parameter associated with the character $\omega_{\tilde{\pi}}\chi^{-1}|_{L^\times}$. Thus, we have an L-parameter $\phi' : \mathrm{WD}_L \to {}^L J_{b'}(\overline{\mathbb{Q}}_\ell)$. We thus define the map $\mathrm{LLC}_{b'}$ to take $\pi_{b'}$ to ϕ'.

We now have the following theorem about the groups we know to satisfy Assumption 4.4.

Theorem 4.10. *[Ham21; FS21; HKW22; BHN22] Assumption 4.4 is true and ℓ is very good in the following cases.*

1. *The group $\mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GSp}_4)$ with $p > 2$ and $[L : \mathbb{Q}_p] \geq 2$ or $L = \mathbb{Q}_p$ for all p. In both cases, we need to assume that $\ell \nmid 2[L : \mathbb{Q}_p]$.*
2. *The groups GU_n or U_n for n odd and defined with respect to an unramified quadratic extension $E/\mathbb{Q}_p$, and $\ell \neq 2$.*
3. *The group $\mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GU}_2)$ defined with respect to an unramified quadratic extension L'/L, and ℓ such that $\ell \nmid [L : \mathbb{Q}_p]$.*
4. *The group $\mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GL}_n)$ for all p and ℓ such that $\ell \nmid [L : \mathbb{Q}_p]$.*

Proof. We first start with the conditions on ℓ. If G is of type A_n then all ℓ are very good. However, when G is a unitary group with $n > 2$, we also need to impose the additional assumption that the action of $W_{\mathbb{Q}_p}$ on $\hat{G}$ is of order prime to ℓ, and this gives us the condition that $\ell \neq 2$. Similarly, this gives rise to the condition that $\ell \nmid [L : \mathbb{Q}_p]$ in all of the cases. If $G = \mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GSp}_4)$ then it is of type C and we also need to impose the additional condition that $\ell \neq 2$.

Now, we turn to Assumption 4.4 (1). For GL_n, this follows from [FS21, Theorem I.9.6] and [HKW22, Theorem 1.0.3], where LLC_b is given by the Harris-Taylor correspondence. For $\mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GSp}_4)$ and $L/\mathbb{Q}_p$ as described above, this follows from [Ham21, Theorem 1.1], where LLC_b is given by Harris-Taylor for the non-basic b and Gan-Takeda [GT11] and Gan-Tantono [GT14] for the basic elements[6]. For U_n or GU_n, this is [BHN22, Theorem 1.1], where LLC_b for $b \in B(G)$ was constructed by Mok [Mok15] and Kaletha-Minguez-Shin-White [Kal+14]. For GU_2 this follows from the compatibility for GL_2, and the fact that the Fargues-Scholze local Langlands correspondence is compatible with taking products as well as maps $G' \to G$ that induces an isomorphism of adjoint groups [FS21, Theorem I.9.6 (v), (vi)].

Now we explain why Assumption 4.4 (2) is satisfied. We recall that if J_b is a non quasi-split group then the fibers of the LLC_b over an L-parameter $\phi : \mathrm{WD}_{\mathbb{Q}_p} \to {}^LG(\overline{\mathbb{Q}}_\ell)$ should be empty if ϕ factors through ${}^LM(\overline{\mathbb{Q}}_\ell)$ for a Levi subgroup $M \subset G$ which does not transfer to a Levi subgroup of J_b. In particular, such parameters are called irrelevant, and we expect the fiber to be empty if and only if ϕ is irrelevant [Kal16, Conjecture A.2]. For the Harris-Taylor correspondence, it is know that the fibers over irrelevant parameters are empty by the standard properties of Jacquet-Langlands. For GU_n or U_n, odd unitary groups and their Levi subgroups are always quasi-split, so it is reduced to the previous case of GL_n using compatibility of the correspondence with parabolic induction. For GSp_4, one needs to show this for $\mathrm{LLC}_{\mathrm{GU}_2(D)}$, where $\mathrm{GU}_2(D)$ is the unique non-split inner form of GSp_4. Here this follows from the construction of Gan-Tantono (See the discussion before the main Theorem in [GT14]). For GU_2, we observe that since $\mathrm{GU}_2, H, \mathrm{GL}_2$ all have the same adjoint group, $b' \in B(\mathrm{GU}_2)_{\mathrm{un}}$ is unramified exactly when b is unramified with notation as in Lemma 4.9. Now, let $\tilde{T}$ be a maximal split torus of GL_2, and observe that $T' = (\tilde{T} \times \mathrm{Res}_{L'/L}\mathbb{G}_m)/\mathbb{G}_m$ is a maximal torus of GU_2 over L. Given $\pi_{b'} = (\tilde{\pi}, \chi)$, and $\phi_{\pi_{b'}} : \mathrm{WD}_L \to {}^LJ_{b'}(\overline{\mathbb{Q}}_\ell)$, we see from the construction that this factors through ${}^LT'(\overline{\mathbb{Q}}_\ell)$ exactly when the associated L-parameter for H, $\phi_{(\tilde{\pi},\chi)} : \mathrm{WD}_L \to {}^LJ_b(\overline{\mathbb{Q}}_\ell)$, factors through ${}^LT(\overline{\mathbb{Q}}_\ell)$, where $T = \tilde{T} \times \mathrm{Res}_{L'/L}\mathbb{G}_m$. Since Assumption 4.4 (2) is clearly compatible with taking products, H satisfies this assumption, and thus so does GU_2.

Now we explain why Assumption 4.4 (3) is satisfied. First, note that any parameter $\phi : \mathrm{WD}_{\mathbb{Q}_p} \to {}^LG(\overline{\mathbb{Q}}_\ell)$ induced from a toral parameter ϕ_T has necessarily trivial monodromy, since ${}^LT(\overline{\mathbb{Q}}_\ell)$ consists only of semi-simple elements. Moreover, since J_b is an inner form of M_b, it follows that the set of all distinct conjugacy classes of parameters $\phi' : \mathrm{WD}_{\mathbb{Q}_p} \to {}^LJ_b(\overline{\mathbb{Q}}_\ell)$ which can give rise to ϕ under the twisted embedding ${}^LJ_b(\overline{\mathbb{Q}}_\ell) \to {}^LG(\overline{\mathbb{Q}}_\ell)$ are parameterized by a set of minimal length representatives of $W_b = W_G/W_{M_b}$ via conjugating ϕ'. We expect (See [Kal16, Conjecture A.5]) that the fiber of LLC_b over such a ϕ' inducing ϕ to be the irreducible constituents of the normalized induction of the L-packet of ϕ_T^w, which is just χ^w by local class field theory for $w \in W_b$. This is indeed true in all the cases we consider (See for example [BHN22, Section 2.3.3] for this discussed in the case of unitary groups, and for GSp_4 and its unique non-split inner form $\mathrm{GU}_2(D)$ it follows directly from the construction). We note that the twists by $\delta_{P_b}^{-1/2}$ appear to account for the half Tate twists appearing in the definition of the *twisted* embedding ${}^LJ_b(\overline{\mathbb{Q}}_\ell) \to {}^LG(\overline{\mathbb{Q}}_\ell)$. For GU_2, we see that when b' is unramified, we have isomorphisms of flag varieties

$$J_{b'}/B_{b'} \simeq J_{\tilde{b}}/B_{\tilde{b}} \simeq J_b/B_b.$$

In the above situation where $\phi_{\pi_{b'}}$ factors through ${}^LT'$, we see that since H, GL_2 satisfy Assumption 4.4 (3), the corresponding representation of $H(L)$ is of the form $(\tilde{\pi}, \chi)$, where $\tilde{\pi}$ is an irreducible constituent of $i_{B_{\tilde{b}}}^{J_{\tilde{b}}}(\chi_1^w) \otimes \delta_{P_{\tilde{b}}}^{-1/2}$, for the associated character χ_1 of $\tilde{T}$. In particular, we see that $\pi_{b'}$ is a constituent of $i_{B_{b'}}^{J_{b'}}(\chi_1^w \otimes \chi) \otimes \delta_{P_{b'}}^{-1/2}$, as desired. ☐

We now turn our attention to deriving our desired consequences.

[6]In the current version of [Ham21], the assumption that $p > 2$ is only used to invoke basic uniformization of abelian type Shimura varieties, but when $L = \mathbb{Q}_p$ one can just use Rappoport-Zink uniformization, so this assumption is unnecessary.

4.2. **Perverse t-exactness.** We recall that $\mathrm{Bun}_G^b \simeq [*/\mathcal{J}_b]$, where $\mathcal{J}_b := \mathrm{Aut}(\mathcal{E}_b)$ is the group diamond parameterizing automorphisms of the bundle $\mathcal{E}_b$ attached to $b \in B(G)$ on X. The diamond $\mathcal{J}_b$ has pure cohomological ℓ-dimension over the base (in the sense of [FS21, Definition IV.1.17]) equal to $\langle 2\rho_G, \nu_b\rangle$, where ν_b is the slope homomorphism of b. Moreover, we have that Bun_G is cohomologically smooth of pure ℓ-dimension equal to 0 over the base. This motivates the following definition.

Definition 4.11. We define a perverse t-structure $({}^{\mathrm{p}}\mathrm{D}^{\leq 0}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell), {}^{\mathrm{p}}\mathrm{D}^{\geq 0}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell))$ on $\mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ such that $A \in \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ lies in ${}^{\mathrm{p}}\mathrm{D}^{\leq 0}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ (resp. ${}^{\mathrm{p}}\mathrm{D}^{\geq 0}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$) if and only if $j_b^*(A)$ (resp. $j_b^!(A)$) sits in cohomological degrees $\leq \langle 2\rho_G, \nu_b\rangle$ ($\geq \langle 2\rho_G, \nu_b\rangle$).

For ϕ induced from a generic toral parameter, we write $({}^{\mathrm{p}}\mathrm{D}^{\geq 0}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_\phi, {}^{\mathrm{p}}\mathrm{D}^{\leq 0}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_\phi)$ for the restriction of this t-structure to $\mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_\phi$. Let $\mathrm{Perv}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_\phi$ denote the heart. We are almost ready to formulate our first big result. To do this, we need the following definition.

Definition 4.12. We say that ϕ_T is weakly normalized regular if it is generic and if χ denotes the character attached to ϕ_T under local class field theory, we have, for all $w \in W_G$ non-trivial, that

$$\chi \otimes \delta_B^{1/2} \not\simeq (\chi \otimes \delta_B^{-1/2})^w \tag{15}$$

holds. Similarly, we say ϕ_T is regular if for all $w \in W_G$ non-trivial we have that $\chi \not\simeq \chi^w$.

To motivate this, we recall that, since ϕ_T is weakly normalized regular, we have by [Ham22, Theorem 10.10] an object $\mathrm{nEis}(\mathcal{S}_{\phi_T}) \in \mathrm{Perv}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$, which is a perverse filtered Hecke eigensheaf on Bun_G, assuming 4.4 holds. Moreover, it is supported on the set of unramified elements and, for $b \in B(G)_{\mathrm{un}}$, its stalks are given by

$$\mathrm{Red}_{b,\phi}^{\mathrm{tw}} := \bigoplus_{w \in W_b} \rho_{b,w}[-\langle 2\rho_G, \nu_b\rangle],$$

where we recall that $\rho_{b,w} := i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{-1/2}$. In particular, by Proposition 4.5 it defines an object in the localized category $\mathrm{Perv}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_\phi$. To show the desired perverse t-exactness property, we would like to use the Hecke eigensheaf property of $\mathrm{nEis}(\mathcal{S}_{\phi_T})$. Given a geometric dominant cocharacter μ, we consider the highest weight tilting module $\mathcal{T}_\mu$ attached to μ. We let

$$T_\mu : \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell) \to \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)^{BW_{E_\mu}}$$

be the Hecke operator attached to the representation $\mathcal{T}_\mu$, where E_μ denotes the reflex field of μ. The sheaf $T_\mu(\mathrm{nEis}(\mathcal{S}_{\phi_T}))$ carries a filtration which, if it splits, guarantees an isomorphism $\mathrm{nEis}(\mathcal{S}_{\phi_T}) \boxtimes r_\mu \circ \phi \simeq T_\mu(\mathrm{nEis}(\mathcal{S}_{\phi_T}))$, and we say that ϕ_T is μ-regular ([Ham22, Definition 10.11]) if such a splitting exists. Here $r_\mu : \hat{G} \to \mathrm{GL}(\mathcal{T}_\mu)$ is the map defined by the tilting module $\mathcal{T}_\mu$. The condition of being μ-regular is guaranteed by the following stronger condition, using [Ham22, Theorem 1.17].

Definition 4.13. We write $(-)^\Gamma : \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \to \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$ for the natural map from geometric cocharacters to their Γ-orbits. For a toral parameter $\phi_T : W_{\mathbb{Q}_p} \to {}^LT(\overline{\mathbb{F}}_\ell)$ and a geometric dominant cocharacter μ, we say ϕ_T is strongly μ-regular if the Galois cohomology complexes

$$R\Gamma(W_{\mathbb{Q}_p}, (\nu - \nu')^\Gamma \circ \phi_T)$$

are trivial for ν,ν' defining distinct Γ-orbits of weights in the highest weight tilting module $\mathcal{T}_\mu$.

Remark 4.14. In particular, strong μ-regularity implies μ-regularity, and if we know strong μ-regularity then it implies μ'-regularity for any $\mathcal{T}_{\mu'}$ which occurs as a direct summand of the tensor product $\mathcal{T}_\mu^{\otimes n}$, by [Ham21, Proposition 10.12]. Also, as we will see, strong μ-regularity is often implied by generic for some suitably chosen μ.

More importantly, we can use this to deduce the following.

Proposition 4.15. *For any ϕ induced from a generic ϕ_T, assume, for all $b \in B(G)_{\mathrm{un}}$ and $w \in W_b$, the representations $\rho_{b,w}$ are semi-simple, and that Assumption 4.4 is true. Then we have a direct sum decomposition*

$$\bigoplus_{b \in B(G)_{\mathrm{un}}} \mathrm{D}^{\mathrm{adm}}(\mathrm{Bun}_G^b, \overline{\mathbb{F}}_\ell)_\phi \simeq \mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_\phi,$$

where $\mathrm{D}^{\mathrm{adm}}(\mathrm{Bun}_G^b, \overline{\mathbb{F}}_\ell) \subset \mathrm{D}(\mathrm{Bun}_G^b, \overline{\mathbb{F}}_\ell) \simeq \mathrm{D}(J_b(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)$ denotes the subcategory of admissible complexes.

Moreover, for any $A \in \mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_G^b, \overline{\mathbb{F}}_\ell)_\phi \simeq \mathrm{D}^{\mathrm{adm}}(J_b(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)_\phi$, we have that the ! and $*$ pushforwards agree with respect to the inclusion $j_b : \mathrm{Bun}_G^b \to \mathrm{Bun}_G$.

Proof. The first part of the Proposition follows from the second part. To see this, we use the semi-orthogonal decomposition of $\mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ into $\mathrm{D}(\mathrm{Bun}_G^b, \overline{\mathbb{F}}_\ell) \simeq \mathrm{D}(J_b(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)$ via the excision spectral sequence. Using that the ! and $*$-pushforwards agree for all objects $A \in \mathrm{D}(\mathrm{Bun}_G^b, \overline{\mathbb{F}}_\ell)_\phi$, we see that the excision spectral sequence degenerates and the first part of the claim follows. To see the second part, we now use Proposition 4.5 to see that an object $A \in \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_\phi$ can only be supported on the HN-strata Bun_G^b for $b \in B(G)_{\mathrm{un}}$, and that the restriction of A to Bun_G^b has irreducible constituents valued in subquotients of the representations $\rho_{b,w}$ for $w \in W_b$ varying. For the representations $\rho_{b,w}$, we have the following.

Proposition 4.16. *[Ham22, Proposition 11.13] For all $b \in B(G)_{\mathrm{un}}$ and $w \in W_b$, the natural map*

$$j_{b!}(\rho_{b,w}) \to Rj_{b*}(\rho_{b,w})$$

is an isomorphism assuming ϕ_T is generic.

So the ! and $*$ pushforwards agree on the $\rho_{b,w}$, and, since we are assuming the representations $\rho_{b,w}$ are semisimple, the claim follows for any constituent of $\rho_{b,w}$. This is enough to conclude the claim for any $A \in \mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_G^b, \overline{\mathbb{F}}_\ell)_\phi \simeq \mathrm{D}^{\mathrm{adm}}(J_b(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)_\phi$ using the following claim.

Lemma 4.17. *Assuming 4.4, for ϕ a generic parameter and any $A \in \mathrm{D}^{\mathrm{adm}}(J_b(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)_\phi$ the cohomology of A has finite length.*

Proof. By Assumption 4.4, we know that any irreducible constituent of the cohomology of A is an irreducible constituent of $\rho_{b,w}$ for some $w \in W_b$. It follows by [Vig96, p. II.5.13] that there are only finitely many possibilities for the possible irreducible constituents. Therefore, by choosing $K \subset G(\mathbb{Q}_p)$ a sufficiently small open compact such that all these representations have an invariant vector, we deduce, since A^K is a perfect complex by assumption, that A must have finite length cohomology. □

□

We note that the semi-simplicity of $\rho_{1,1} = i_B^G(\chi)$ is implied by the conditions discussed above.

Lemma 4.18. *Let $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\overline{\mathbb{F}}_\ell)$ be a weakly normalized regular and regular. Suppose there exists a μ which is not fixed under any $w \in W_G$ and ϕ_T is μ-regular. Then, $i_B^G(\chi)$ is irreducible.*

Proof. It follows, by [Ham22, Corollary 11.23] and the assumed μ-regularity, that we have an isomorphism $i_B^G(\chi) \simeq i_B^G(\chi^w) = i_{B^w}^G(\chi)$ for all $w \in W_G$. Here B^w is the conjugate of B by w. We write r_B^G for the normalized parabolic restriction functor. We recall that we are working with ℓ-modular coefficients in possibly non-banal characteristic so $i_B^G(\chi)$ may have cuspidal constituents. In particular, we will need the following lemma.

Lemma 4.19. *Let $w_0 \in W_G$ be the element of longest length. For a character $\chi : T(\mathbb{Q}_p) \to \overline{\mathbb{F}}_\ell^*$, if we have an isomorphism $i_B^G(\chi) \simeq i_B^G(\chi^{w_0})$ of $G(\mathbb{Q}_p)$-modules then any non-zero quotient σ' of $i_B^G(\chi)$ satisfies $r_B^G(\sigma') \neq 0$*

Proof. We apply second adjointness [Dat+22, Corollary 1.3] to the map

$$i^G_{B^{w_0}}(\chi) \xrightarrow{\simeq} i^G_B(\chi) \to \sigma'$$

to conclude the existence of a non-zero map $\chi \to r^G_B(\sigma')$, which implies the claim. □

Now suppose for the sake of contradiction that $i^G_B(\chi)$ is not irreducible. Then there exists an exact sequence

$$0 \to \sigma \to i^G_B(\chi) \to \sigma' \to 0.$$

Since parabolic restriction is exact (for example by using second adjointness), we get an exact sequence

$$0 \to r^G_B(\sigma) \to r^G_B i^G_B(\chi) \to r^G_B(\sigma') \to 0.$$

This allows us to conclude an equality of lengths of representations:

$$\ell(r^G_B(\sigma)) + \ell(r^G_B(\sigma')) = \ell(r^G_B(i^G_B(\chi))) \leq |W_G|,$$

where the inequality follows from the geometric Lemma [Dat05, Section 2.8][7]. By the previous lemma, we conclude that $\ell(r^G_B(\sigma)) < W_G$. Now, since we know that $\sigma \subset i^G_B(\chi) \simeq i^G_B(\chi^w)$ for all $w \in W_G$, Frobenius reciprocity implies that we have non-zero maps $r^G_B(\sigma) \to \chi^w$ for all $w \in W_G$. This gives a contradiction by the regularity of χ. □

We now have the following key claim.

Theorem 4.20. *Let μ be a geometric dominant cocharacter. We write*

$$T_\mu : \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell) \to \mathrm{D}\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)^{BW_{\mathbb{Q}_p}}$$

for the Hecke operator attached to the highest weight tilting module $\mathcal{T}_\mu$ of highest weight μ. Then the operator restricted to $\mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_\phi$ is perverse t-exact if ϕ_T is weakly normalized regular, Assumption 4.4 is true, the $\rho_{b,w}$ are semi-simple for all $b \in B(G)_{\mathrm{un}}$ and $w \in W_b$, and ϕ_T is μ-regular.

Proof. Using Lemma 4.17, the commutation of Hecke operators with colimits, Proposition 4.5, Proposition 4.15, and semi-simplicity of the representations $\rho_{b,w}$, we can reduce to showing, for all $b \in B(G)_{\mathrm{un}}$, that if we consider the complex

$$\mathrm{Red}^{\mathrm{tw}}_{b,\phi} := \bigoplus_{w \in W_b} i^{J_b}_{B_b}(\chi^w) \otimes \delta^{-1/2}_{P_b}[-\langle 2\rho_G, \nu_b \rangle] \in \mathrm{Perv}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_\phi$$

then we have a containment

$$T_\mu(j_{b!}(\mathrm{Red}^{\mathrm{tw}}_{b,\phi})) \in \mathrm{Perv}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_\phi$$

for the fixed μ. However, $\mathrm{Red}^{\mathrm{tw}}_{b,\phi}$ are the stalks of the perverse filtered Hecke eigensheaf $\mathrm{nEis}(\mathcal{S}_{\phi_T})$ and, since ϕ_T is μ-regular by assumption, we have an isomorphism:

$$T_\mu(\mathrm{nEis}(\mathcal{S}_{\phi_T})) \simeq \mathrm{nEis}(\mathcal{S}_{\phi_T}) \boxtimes r_\mu \circ \phi \in \mathrm{Perv}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)^{BW_{E_\mu}}_\phi.$$

This gives the desired claim. □

We are almost ready to deduce the result we need for torsion vanishing. To do this, we will first need to discuss when the additional assumptions of weak normalized regularity and μ-regularity are superfluous, possibly under certain assumptions on ℓ.

[7] Note that this bound however fails without taking normalized restriction because of the aforementioned cuspidal constituents of $i^G_B(\chi)$ in non-banal characteristic (cf. [Dat05, Page 48]).

4.3. **Verification of additional assumptions.** We first need the following lemma which will allow us to base change to splitting fields.

Lemma 4.21. *Let G be a quasi-split connected reductive group with splitting field E. If ϕ_T is generic then $R\Gamma(W_E, \tilde{\alpha} \circ \phi_T|_{W_E})$ is trivial for all absolute coroots $\tilde{\alpha} \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$.*

Proof. We recall that, given a Γ-orbit of positive absolute coroots $\alpha \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+\backslash\Gamma$, if E_α denotes the reflex field of α then the representation of ${}^LT$ defined by α is given by choosing a representative $\tilde{\alpha} \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ of α, and inducing the representation of $\hat{T} \rtimes W_{E_\alpha}/W_E$ defined by it to $W_{\mathbb{Q}_p}/W_E$. This reduces the claim to Schapiro's Lemma. ☐

Other than the groups listed in Theorem 4.10, there are two more groups of interest to us. We will define them now.

Let $L/\mathbb{Q}_p$ be a finite extension. We have the similitude maps from GL_n (resp. GSp_4)

$$\nu : \mathrm{Res}_{L/\mathbb{Q}_p}\mathrm{GL}_n \to \mathrm{Res}_{L/\mathbb{Q}_p}\mathbb{G}_m$$

(resp.

$$\nu : \mathrm{Res}_{L/\mathbb{Q}_p}\mathrm{GSp}_4 \to \mathrm{Res}_{L/\mathbb{Q}_p}\mathbb{G}_m).$$

We thus define

$$G(\mathrm{SL}_{n,L}) := \mathrm{Res}_{L/\mathbb{Q}_p}\mathrm{GL}_n \times_\nu \mathbb{G}_m,$$
$$G(\mathrm{Sp}_{4,L}) := \mathrm{Res}_{L/\mathbb{Q}_p}\mathrm{GL}_n \times_\nu \mathbb{G}_m.$$

Lemma 4.22. *Let $L/\mathbb{Q}_p$ be a finite extension and G be one of the following groups:*

1. $\mathrm{Res}_{L/\mathbb{Q}_p}\mathrm{U}_n$,
2. $\mathrm{Res}_{L/\mathbb{Q}_p}\mathrm{GU}_n$,
3. $\mathrm{Res}_{L/\mathbb{Q}_p}\mathrm{GL}_n$,
4. $G(\mathrm{SL}_{n,L})$.

If ϕ_T is a generic toral parameter for G then ϕ_T is weakly normalized regular and regular. Moreover, for $(1)-(3)$, ϕ_T will be μ-regular for all μ, while, for (4), ϕ_T will be μ-regular for μ which are of the form $\prod_{\tau : L \hookrightarrow \overline{\mathbb{Q}}_p} \mu'$ for μ' a cocharacter of GL_n.

Proof. We establish weak normalized regularity, and suppress giving the proof that ϕ_T is regular as it is strictly easier. For cases (1), (2), and (3), we may assume for simplicity that $L = \mathbb{Q}_p$ with the proof in general essentially being the same. If $G = \mathrm{GL}_n$ then this is [Ham22, Lemma 3.10].

We now consider the case of $G = \mathrm{U}_n$ defined with respect to a quadratic extension $E/\mathbb{Q}_p$. Suppose there exists a non-trivial $w \in W_G$ such that we have an isomorphism:

$$\chi \otimes \delta_B^{1/2} \simeq (\chi \otimes \delta_B^{-1/2})^w$$

of characters on $T(\mathbb{Q}_p)$. We recall that $G_E \simeq \mathrm{GL}_{n,E}$ where $E/\mathbb{Q}_p$ denotes the quadratic extension defining the unitary group. By the definition of the modulus character in terms of the transformation character of Haar measures, we observe that the precomposition of δ_B with the Norm map $T(E) \to T(\mathbb{Q}_p)$ gives the modulus character on the Borel of $\mathrm{GL}_{n,E}$. Therefore, by precomposing the previous isomorphism with this norm map, we obtain an analogous relationship of characters on the torus $T(E)$, which is the maximal torus of $\mathrm{GL}_{n,E}$. Then Lemma 4.21 reduces us to the GL_n case.

The case of GU_n similarly reduces to the U_n case by setting the coordinate on $T(\mathbb{Q}_p)$ corresponding to the similitude factor to be equal to 1.

For case (4), let $d = [L : \mathbb{Q}_p]$. Observe that we have an isomorphism $G(\mathrm{SL}_{n,L})_L \simeq H_L$, where

$$H = \left\{ (g_i) \in \prod_{L \hookrightarrow \overline{\mathbb{Q}}_p} \mathrm{GL}_n : \det(g_i) = \det(g_j)\ \forall i,j \right\}.$$

Applying Lemma 4.21 again and arguing as for unitary groups, it suffices to work with H_L. We assume $L = \mathbb{Q}_p$ for notation simplicity. The maximal torus T' in H can be identified with

$$\mathbb{G}_m^d \times \mathbb{G}_m,$$

via the map $(t_1, \ldots, t_d, t) \mapsto (\mathrm{diag}(t_i, tt_i^{-1}))$. Since $W_H = \prod W_{\mathrm{GL}_2}$, consider any element $w' \in W_G$, which we assume for notational simplicity is of the form

$$(w, \ldots, w, \mathrm{id}, \ldots, \mathrm{id}),$$

where w is the non-trivial element of the Weyl group of GL_2, and we have w in the first k entries, for some integer $0 \leq k \leq d$. The general case follows similarly. Observe that the isomorphism

$$\chi \otimes \delta_B^{1/2} \simeq (\chi \otimes \delta_B^{-1/2})^{w'}$$

becomes

$$\prod_{i=1}^{k} \chi_{i1}(t_i^2 t^{-1}) \chi_{i2}(tt_i^{-2}) \simeq \prod_{i=k+1}^{d} |t_i^{-2} t|.$$

If we substitute $t = x, t_i = x$ for $i = 1, \ldots, k$, while for $k+1 \leq i \leq d$ we set $t_i = 1$ if $i - k$ is odd and $t_i = x$ if $i - k$ is even, we see that we get $\prod_{i=1}^{k} \chi_{i1}(x) \chi_{i2}^{-1}(x)$ is isomorphic to either the trivial representation $\mathbf{1}$ or $|\cdot|$, which is a contradiction to genericity.

We now show μ-regularity. Again, for cases (1), (2), and (3), observe that if G is of the form $\mathrm{Res}_{L/\mathbb{Q}_p} G'$ and T' denotes the maximal torus of G' then we have an isomorphism

$$\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \simeq \prod_{\phi \in \mathrm{Hom}_{\mathbb{Q}_p}(L, \overline{L})} \mathbb{X}_*(T'_{\overline{L}})$$

where $\overline{L}$ is an algebraic closure of L. Using this, we can without loss of generality assume that $L = \mathbb{Q}_p$. In the case that $G = \mathrm{GL}_n, \mathrm{U}_n$, or GU_n, this follows as in the proof of [Ham22, Corollary 10.16]. We recall briefly how this goes.

One can consider the geometric dominant cocharacter $\mu = (1, 0, \ldots, 0, 0)$ of GL_n. This defines the standard representation V_{std} of $\hat{G} \simeq \mathrm{GL}_n$. This cocharacter is in particular minuscule so the weights form a closed Weyl group orbit with representative $(1, 0, \ldots, 0, 0)$. From here, it easily follows that the difference of the weights appearing in V_{std} define coroots of G. In particular, it follows that, if ϕ_T is generic then it is strongly μ-regular for $\mu = (1, 0, \ldots, 0)$ in the sense of Definition 4.13, and this implies the filtration on $T_\mu(\mathrm{nEis}(\mathcal{S}_{\phi_T}))$ splits by [Ham22, Theorem 10.10] for this μ. Now, the tilting modules $\mathcal{T}_{\omega_i} = \Lambda^i(V_{\mathrm{std}})$ attached to the other fundamental coweights $\omega_i = (1^i, 0^{n-i})$ of G can be realized as direct summands of $V_{\mathrm{std}}^{\otimes i}$, and it follows that ϕ_T is μ-regular for $\mu = \omega_i$ by [Ham22, Proposition 10.12]. Since any dominant cocharacter can be written as a linear combination of fundamental weights, the claim for any μ now follows from [Ham22, Corollary 10.13]. The case of GU_n and U_n follows in a very similar way, using Lemma 4.21.

For case (4), observe that as before, we can base change to L, and since H is a subgroup of $\prod \mathrm{GL}_n$, all cocharacters μ of H define products of cocharacters for GL_n. Now, consider a cocharacter of the form $\mu = \prod_\tau \mu_\tau$, where $\mu_{\tau'} = \mu_\tau$ for all τ, τ' and μ_τ is a cocharacter of GL_n. Note that every dominant minuscule cocharacter of H will be of this form. This is because a cocharacter $\mu = \prod_\tau \mu_\tau$ of $\prod \mathrm{GL}_n$ factors through H exactly when the composition with the determinant is equal for all τ, and we see that for μ to be minuscule, μ_τ must be one of the fundamental coweights ω_i, which, after composing with the determinant, give different characters for $i \neq j$. Now, we observe that the same argument as above holds to show that the difference of Weyl conjugates define a coroot of H for the cocharacter $\mu_1 = \prod(1, 0, \ldots, 0)$, while for all other cocharacters of the form $\mu = \prod \mu'$, where μ' is a fundamental weight of GL_n, they appear as weights in some tensor power of the highest weight representation corresponding to μ_1. The claim for any $\mu = \prod \mu'$, where μ' is a dominant cocharacter of GL_n, follows by the same argument as above, using [Ham22, Corollary 10.13]. $\square$

Lemma 4.23. *Let $L/\mathbb{Q}_p$ be a finite extension, and G be one of the following groups:*

(1) $\mathrm{Res}_{L/\mathbb{Q}_p}\mathrm{GSp}_4$
(2) $G(\mathrm{Sp}_{4,L})$.

Suppose moreover that $\ell \neq 2$ and ℓ is banal with respect to L (i.e. $(q^4-1, l) = 1$, where q is the size of the residue field of L). If ϕ_T is a generic toral parameter for G then ϕ_T is weakly normalized regular and regular. Moreover, for (1), ϕ_T will be μ-regular for all μ, and, for (2), ϕ_T will be μ-regular for μ which are of the form $\prod_{\tau: L \hookrightarrow \overline{\mathbb{Q}}_p} \mu'$ for μ' a cocharacter of GSp_4.

Proof. We will first establish weak normalized regularity, and again suppress giving the proof that ϕ_T is regular as it is strictly easier. Again, for (1), we assume that $L = \mathbb{Q}_p$ for this part with the proof in general being more or less the same. We will show this by contradiction. Suppose on the contrary that there exists some $w \in W_G$ such that we have an isomorphism

$$\chi \otimes \delta_B^{1/2} \simeq (\chi \otimes \delta_B^{-1/2})^w. \tag{16}$$

For case (1), consider the following parametrization of the maximal torus T

$$a : (\mathbb{Q}_p^*)^2 \times \mathbb{Q}_p^* \to T(\mathbb{Q}_p) \tag{17}$$

$$(t_1, t_2, t) \mapsto \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & tt_2^{-1} & 0 \\ 0 & 0 & 0 & tt_1^{-1} \end{pmatrix}$$

as in [Tad94, Page 135]. This allows us to write the character $\chi : T(\mathbb{Q}_p) \to \overline{\mathbb{F}}_\ell^*$ as $\chi_1(t_1)\chi_2(t_2)\nu(t)$, for characters $\mathbb{Q}_p^* \to \overline{\mathbb{F}}_\ell^*$. Similarly, we can express the modulus character as

$$\delta_B(t_1, t_2, t) = |t_1|^4|t_2|^2|t|^{-3}$$

where $|\cdot|$ is the norm character. We now check that (16) cannot hold for all seven non-trivial elements of the Weyl group.

Consider the Weyl group element corresponding to the translation:

$$w_1 : a(t_1, t_2, t) \mapsto a(t_2, t_1, t)$$

If we consider equation (16) with respect to this element and evaluate on $(x, 1, x) = (t_1, t_2, t)$ then we obtain the equation

$$\chi_1(x)|x|^2|x|^{-3/2} \simeq \chi_2(x)|x|^{-1}|x|^{3/2}$$

which gives an isomorphism $\chi_1\chi_2^{-1}(x) \simeq \mathbf{1}$ contradicting genericity.

Similarly, if we consider the simple Weyl group element

$$w_2 : a(t_1, t_2, t) \mapsto a(t_1, t_2^{-1}t, t)$$

then evaluating equation (16) for this relationship reduces to

$$\chi_1(t_1)\chi_2(t_2)\nu(t)|t_1|^2|t_2||t|^{-3/2} \simeq \chi_1(t_1)\chi_2(t_2^{-1}t)\nu(t)|t_1|^{-2}|t_2||t|^{-1}|t|^{3/2}$$

cancelling terms we obtain that

$$\chi_2(t)^{-1}\chi_2(t_2)^2 \simeq |t_1|^{-4}|t|^2$$

so if we evaluate at $(t_1, t_2, t) = (x^3, x^2, x^4)$ then we obtain

$$\mathbf{1} \simeq |x|^{-4}$$

which contradicts the assumption that $(p^4 - 1, \ell) = 1$.

Consider now the Weyl group element

$$w_3 : a(t_1, t_2, t) \mapsto a(t_2^{-1}t, t_1, t)$$

if we evaluate equation (16) then we obtain

$$\chi_1(t_1)\chi_2(t_2)\nu(t)|t_1|^2|t_2||t|^{-3/2} \simeq \chi_1(t_2)^{-1}\chi_1(t)\chi_2(t_1)\nu(t)|t_2|^2|t|^{-2}|t_1|^{-1}|t|^{3/2}$$

rearranging and cancelling terms we obtain

$$\chi_1\chi_2^{-1}(t_1)\chi_2\chi_1(t_2)\chi_1(t)^{-1} \simeq |t_1|^{-3}|t_2||t|$$

so if we evaluate at $(t_1, t_2, t) = (1, 1, x)$ we obtain that

$$\chi_1^{-1}(x) \simeq |x|$$

which contradicts genericity (See [Tad94, Page 167] for the enumeration of 1-parameter subgroups attached to the coroots in the parametrization (17). Note that we could also have substituted $(t_1, t_2, t) = (x, x, x)$ to obtain

$$\chi_1(x) \simeq |x|^{-1}.$$

Consider the reflection

$$w_4 : a(t_1, t_2, t) \mapsto a(t_1^{-1}t, t_2, t)$$

then equation (16) becomes

$$\chi_1(t_1)\chi_2(t_2)\nu(t)|t_1|^2|t_2||t|^{-3/2} \simeq \chi_1(t_1^{-1}t)\chi_2(t_2)\nu(t)|t_1|^2|t|^{-2}|t_2|^{-1}|t|^{3/2}$$

which gives

$$\chi_1(t_1^2)\chi_1(t)^{-1} \simeq |t_2|^{-2}|t|$$

so if we evaluate at $(t_1, t_2, t) = (1, 1, x)$, this becomes

$$\chi_1(x)^{-1} \simeq |x|$$

which contradicts genericity. Note that we could also have substituted $(t_1, t_2, t) = (1, x, x)$ to obtain

$$\chi_1(x)^{-1} \simeq |x|^{-1}.$$

Now consider the Weyl group element

$$w_5 : a(t_1, t_2, t) \mapsto a(t_2, t_1^{-1}t, t)$$

then equation (16)

$$\chi_1(t_1)\chi_2(t_2)\nu(t)|t_1|^2|t_2||t|^{-3/2} \simeq \chi_1(t_2)\chi_2(t_1^{-1}t)\nu(t)|t_2|^{-2}|t_1||t|^{-1}|t|^{3/2}$$

which simplifies to

$$\chi_2\chi_1^{-1}(t_2)\chi_1\chi_2(t_1)\chi_2(t)^{-1} \simeq |t_1|^{-1}|t_2|^{-3}|t|^2$$

so if we evaluate at $(t_1, t_2, t) = (x, 1, x)$ then this gives

$$\chi_1(x) \simeq |x|$$

which contradicts genericity. Note that we could also have substituted $(t_1, t_2, t) = (1, x, x)$ to obtain

$$\chi_1^{-1}(x) \simeq |x|^{-1}.$$

Now consider the Weyl group element

$$w_6 : a(t_1, t_2, t) \mapsto a(t_1^{-1}t, t_2^{-1}t, t)$$

then equation (16) becomes

$$\chi_1(t_1)\chi_2(t_2)\nu(t)|t_1|^2|t_2||t|^{-3/2} \simeq \chi_1(t_1^{-1}t)\chi_2(t_2^{-1}t)\nu(t)|t_1|^2|t|^{-2}|t_2||t|^{-1}|t|^{3/2}$$

which simplifies to

$$\chi_1^2(t_1)\chi_2^2(t_2)\chi_1\chi_2(t)^{-1} \simeq \mathbf{1}$$

so if we evaluate at $(t_1, t_2, t) = (1, 1, x)$ then this becomes

$$\chi_1\chi_2(x) \simeq \mathbf{1}$$

which contradicts genericity.

Now finally we consider

$$w_7 : a(t_1,t_2,t) \mapsto a(t_2^{-1}t, t_1^{-1}t, t)$$

then equation (16) becomes

$$\chi_1(t_1)\chi_2(t_2)\nu(t)|t_1|^2|t_2||t|^{-3/2} \simeq \chi_1(t_2^{-1}t)\chi_2(t_1^{-1}t)\nu(t)|t_2|^2|t|^{-2}|t_1||t|^{-1}|t|^{3/2}$$

which simplifies to

$$\chi_1\chi_2(t_1)\chi_1\chi_2(t_2)\chi_1\chi_2(t^{-1}) \simeq |t_1|^{-1}|t_2|$$

evaluated at $(t_1,t_2,t)=(1,1,x)$ simplifies to

$$\chi_1\chi_2(x) \simeq \mathbf{1}$$

which contradicts genericity.

This concludes our discussion of weakly normalized regularity for $\mathrm{Res}_{L/\mathbb{Q}_p}\mathrm{GSp}_4$.

We now turn to the case of $G(\mathrm{Sp}_{4,L})$. As in the proof of the previous lemma, observe that if we let

$$H = \{(g_i) \in \prod_{L\hookrightarrow\overline{\mathbb{Q}}_p} \mathrm{GSp}_4 \text{ such that } \nu(g_i)=\nu(g_j), \forall i,j\}, \tag{18}$$

then we have $H_L \simeq G(\mathrm{Sp}_{4,L})_L$. Thus, we may reduce to the case of H. Since $H \subset \prod \mathrm{GSp}_4$, we may also use the parametrization in [Tad94, p. 135] to see that the maximal torus T' is given by a parametrization

$$(\mathbb{Q}_p^*)^{2d} \times \mathbb{Q}_p^* \to T'(\mathbb{Q}_p),$$

$$((t_{\tau 1},t_{\tau 2})_{\tau:L\hookrightarrow\overline{\mathbb{Q}}_p}, t) \mapsto \begin{pmatrix} t_{\tau 1} & 0 & 0 & 0 \\ 0 & t_{\tau 2} & 0 & 0 \\ 0 & 0 & tt_{\tau 2}^{-1} & 0 \\ 0 & 0 & 0 & tt_{\tau 1}^{-1} \end{pmatrix}_{\tau:L\hookrightarrow\overline{\mathbb{Q}}_p}$$

where we note that the common similitude factor is the last coordinate t.

Since δ_B is just the restriction of the character for the Borel of $\prod \mathrm{GSp}_4$ from the torus $\prod T$ to T', we see that the modulus character is

$$\delta_B((t_{\tau 1},t_{\tau 2})_\tau, t) = |t|^{-3d}\prod_\tau |t_{\tau 1}|^4|t_{\tau 2}|^2.$$

Since $W_G = \prod_\tau W_{\mathrm{GSp}_4}$, consider any element $w=(w_\tau)\in W_G$, where $w_\tau \in W_{\mathrm{GSp}_4}$. Observe that the expression obtained from the isomorphism (16) for $w=(w_\tau)$ is simply the product of the isomorphisms for GSp_4 each w_τ. Thus, if we wanted to argue by contradiction, using the notation of the proof above, when $w_\tau = w_i$ for $i=1,\dots,7$, we should substitute for $t_{\tau 1}, t_{\tau 2}, t$ the values we considered above, subject to the additional constraint that we must have t, the similitude factor, being equal for all τ.

We thus have two possibilities: either some w_τ is the Weyl group element w_2 (i.e. corresponding to the reflection

$$w_2 : a(t_1,t_2,t)\mapsto a(t_1,t_2^{-1}t,t)$$

or none of the w_τ are this element.

In the first situation, suppose that for some τ, w_τ is the Weyl reflection w_2. If we consider the equation (16), evaluated on the element $t_{\tau 1}=x, t_{\tau 2}=x^2$,and $t_{\tau' 1}=t_{\tau' 2}=x^2, t=x^4$ for all $\tau'\neq\tau$, then equation (16) simplifies to

$$\mathbf{1} \simeq |x|^{-4}, \tag{19}$$

since one can check that substituting $t_1=t_2=x^2, t=x^4$ into the isomorphism (16) for GSp_4 for all the Weyl elements not equal to w_2 above simply gives the isomorphism $\mathbf{1}\simeq\mathbf{1}$ after simplification, and thus does not matter when taking products). This contradicts the banality assumption that $(p^4-1,\ell)=1$.

Now, we suppose we are in the second situation, i.e. no $w_\tau = w_2$. For any $w = (w_\tau)$, let $J = \{\tau : L \hookrightarrow \overline{\mathbb{Q}}_p : w_\tau = \mathrm{id}, w_3, w_4, w_5\}$, and $J' = \{\tau : L \hookrightarrow \overline{\mathbb{Q}}_p : w_\tau \neq \mathrm{id}, w_3, w_4, w_5\}$. For some choice of $((t_{\tau 1}, t_{\tau 2})_\tau, t)$ we see the equation (16) becomes

$$\prod_{\tau \in J'} \chi_{\tau 1}\chi_{\tau 2} \prod_{\tau \in J} \chi_{\tau 1}^{-1} \simeq \mathbf{1} \tag{20}$$

or

$$\prod_{\tau \in J'} \chi_{\tau 1}\chi_{\tau 2} \prod_{\tau \in J} \chi_{\tau 1}^{-1} \simeq |\cdot| \tag{21}$$

which contradicts genericity. Indeed, if we chose some $((t_{\tau 1}, t_{\tau 2})_\tau, t)$ for each w_τ such that we derived a contradiction in the case of GSp_4 as above, then we see that the right-hand side of the isomorphism (16) simplifies to $|\cdot|^n$, for some n. If $n \neq 0, 1$, then we see that by changing the values of $(t_{\tau 1}, t_{\tau 2})$ for some $\tau \in J$, to the other choice of substitution, the right-hand side evaluates to $|\cdot|^{n-2}$, and continuing this process as necessary we get either equation (20) or (21).

We now show μ-regularity. As in the proof of the previous lemma, for case (1) it suffices to check the claim when $G = \mathrm{GSp}_4$, the Langlands dual group is given by GSpin_5, which is isomorphic to GSp_4. The spin representation

$$\mathrm{spin} : \mathrm{GSpin}_5 \to \mathrm{GL}_4(V_{\mathrm{spin}})$$

defines a minuscule highest weight representation, which, under the isomorphism $\mathrm{GSpin}_5 \simeq \mathrm{GSp}_4$, identifies with the defining representation of GSp_4. From here it is easy to see that the differences of the weights are roots of GSp_4 (= coroots of GSpin_5). For example, by using the parametrization of the maximal torus, as in (17), and the description of the roots in this parametrization provided on [Tad94, Page 167]. Therefore, genericity guarantees strong μ-regularity for this representation which implies μ-regularity as before. The other fundamental tilting module of GSpin_5 is given by the defining representation $\mathrm{GSpin}_5 \to \mathrm{SO}_5 \to \mathrm{GL}_5$ assuming $\ell \neq 2$ (See [Ham22, Appendix B]). Moreover, this occurs as a 5-dimensional summand of $V_{\mathrm{spin}} \otimes V_{\mathrm{spin}}$, and it follows by [Ham22, Proposition 10.12] that we know μ-regularity for this representation as well. Therefore, since we know μ-regularity for the fundamental coweights, we are now done by [Ham22, Corollary 10.13] as before.

Now, for case (2), note that, as in the previous lemma, all cocharacters of H are of the form $\mu = \prod_\tau \mu_\tau$ for some cocharacters μ_τ of GSp_4. The argument given above for GSp_4 shows that if we let μ' be one of the fundamental coweights of GSp_4, then if we take $\mu = \prod \mu'$, (i.e. $\mu_\tau = \mu'$ for all τ) then we are μ-regular for such μ. Applying [Ham22, Corollary 10.13] again shows that if $\mu = \prod_\tau \mu'$, where μ' is a dominant cocharacter of GSp_4, then ϕ_T will be μ-regular for such μ. Note that, as in the case of $G(\mathrm{SL}_n)$, every dominant minuscule cocharacter of H is of this form. $\square$

Now let us package our final result in a nice form. We first consider the following table, summarizing the groups and primes for which our results apply. We have left the entry blank if no constraint is imposed, and just mentioned the groups that appear as local constituents of global groups that admit a Shimura datum and for which G is unramified.

(22)

G	Constraint on G	ℓ	p
$\mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GL}_n)$	$L/\mathbb{Q}_p$ unramified	$(\ell, [L:\mathbb{Q}_p]) = 1$	
$\mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GSp}_4)$	$L = \mathbb{Q}_p$	$(\ell, 2(p^4-1)) = 1$	
	$L/\mathbb{Q}_p$ unramified	$(\ell, 2[L:\mathbb{Q}_p](p^{4[L:\mathbb{Q}_p]}-1)) = 1$	$p \neq 2$
$\mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GU}_2)$	$L/\mathbb{Q}_p$ unramified	$(\ell, [L:\mathbb{Q}_p]) = 1$	
$G = \mathrm{U}_n(L/\mathbb{Q}_p)$	n odd L unramified	$\ell \neq 2$	
$G = \mathrm{GU}_n(L/\mathbb{Q}_p)$	n odd L unramified	$\ell \neq 2$	
$G(\mathrm{SL}_{2,L})$	$L/\mathbb{Q}_p$ unramified	$(\ell, [L:\mathbb{Q}_p]) = 1$	
$G(\mathrm{Sp}_{4,L})$	$L/\mathbb{Q}_p$ unramified, $L \neq \mathbb{Q}_p$	$(\ell, 2[L:\mathbb{Q}_p](p^{4[L:\mathbb{Q}_p]}-1)) = 1$	$p \neq 2$

We now apply Theorem 4.20.

Corollary 4.24. *Assume G is a product of the groups appearing in Table (22) with p and ℓ satisfying the corresponding conditions. Then, we have that the natural map*

$$j_1^* T_\mu(-) : \mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}} \to \mathrm{D}^{\mathrm{adm}}(G(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}}$$

is exact with respect to the perverse t-structure on the source and the natural t-structure (= perverse t-structure) on the target for all minuscule μ.

Proof. First note that, using the decomposition $\mathrm{Bun}_{G_1 \times G_2} := \mathrm{Bun}_{G_1} \times \mathrm{Bun}_{G_2}$, we can assume that G is isomorphic to one of the groups appearing in Table (22).

Observe that all the groups in Table (22) satisfy Assumption 4.4, where the first five rows follows from Theorem 4.10, and the last two from Proposition 4.8. We now apply Theorem 4.20. To do this, we also need to check that if ϕ_T is a generic toral parameter then is also weakly normalized regular, and μ-regular for all minuscule μ. This follows from Lemma 4.22 and Lemma 4.23.

Lastly, we need to check that the representations $\rho_{b,w} := i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{-1/2}$ are semi-simple for all $b \in B(G)_{\mathrm{un}}$ and $w \in W_b$. We claim that they are in fact irreducible. Recall that $J_b \simeq M_b \subset G$, where M_b is a Levi of G. Moreover, we note that any such Levi M_b is a product of groups also appearing in (22). Therefore, the desired irreducibility follows from the μ-regularity, weak normalized regularity, and regularity of ϕ_T combined with Lemma 4.18. Note that in the case of $G(\mathrm{SL}_{2,L})$ and $G(\mathrm{Sp}_{4,L})$, we can always find a cocharacter μ of the form $\prod_\tau \mu'$ which is not fixed by the Weyl group, since we can simply look at any cocharacter of μ' of GL_2 (resp. GSp_4) which is not fixed by the Weyl group, and take μ to be the product of these μ'. □

We also have the following.

Corollary 4.25. *Assume G is a product of the groups appearing in Table (22) with p and ℓ satisfying the corresponding conditions. Then, for $\mathfrak{m} \subset H_{K_p}^{\mathrm{hs}}$ a generic maximal ideal, we have that*

$$\mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}} \simeq \bigoplus_{b \in B(G)_{\mathrm{un}}} \mathrm{D}^{\mathrm{adm}}(J_b(\mathbb{Q}_p, \overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}}.$$

Moreover, the ! and $*$ pushforwards agree for any $A \in \mathrm{D}^{\mathrm{adm}}(J_b(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}} \simeq \mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_G^b, \overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}}$

Proof. This follows from Proposition 4.15, where the semisimplicity of the $\rho_{b,w}$ follows as in the proof of the Previous Corollary. □

5. The Proof of Theorems 1.15 and 1.17

5.1. **Proof of Theorems 1.15 and 1.17.** Throughout this section, we assume $(\mathbf{G}, X)$ is a PEL datum of type A or C such that $\mathbf{G}_{\mathbb{Q}_p}$ is a product of simple groups as in Table (1) with p and ℓ satisfying the corresponding conditions, and that Assumption 1.11 holds.

Proof. (Theorem 1.15) By Corollary 3.17, the complex $R\Gamma_c(\mathcal{S}(\mathbf{G},X)_{K^p,C},\overline{\mathbb{F}}_\ell)$ has a $G(\mathbb{Q}_p)\times W_E$-equivariant filtration with graded pieces isomorphic to $j_1^*T_\mu j_{b!}(V_b)[-d](-\frac{d}{2})$. The cohomology of Igusa varieties V_b and the global Shimura variety is admissible [Zha23, Proposition 8.21], so we can apply the results of the previous section to them. We let $\mathfrak{m}$ be a generic maximal ideal of the spherical Hecke algebra, and consider the localization

$$(j_1^*T_\mu j_{b!}(V_b))_{\phi_{\mathfrak{m}}}[-d](-\frac{d}{2}).$$

This defines a filtration on $R\Gamma_c(\mathcal{S}(\mathbf{G},X)_{K^p,C},\overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}}$. The filtration on $R\Gamma_c(\mathcal{S}(\mathbf{G},X)_{K^p,C},\overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}}$ considered above comes from applying $(-)_{\phi_{\mathfrak{m}}}$ to $R\Gamma([\mathcal{F}\ell_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}],i_{b!}i_b^*(R\pi_{\mathrm{HT}!}(\overline{\mathbb{F}}_\ell)))$ viewed as a $G(\mathbb{Q}_p)$-representation. Using Corollary 4.25, we see that these graded pieces are also isomorphic to

$$(j_1^*T_\mu j_{b*}(V_b))_{\phi_{\mathfrak{m}}}[-d](-\frac{d}{2})$$

via the natural transformation $j_{b!}\to j_{b*}$ and are trivial for $b\in B(G,\mu)_{\mathrm{un}}$. However, using Lemma 3.14, this implies that the natural map

$$R\Gamma([\mathcal{F}\ell_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}],i_{b!}i_b^*(R\pi_{\mathrm{HT}!}(\overline{\mathbb{F}}_\ell)))_{\phi_{\mathfrak{m}}}\to R\Gamma([\mathcal{F}\ell_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}],i_{b*}i_b^*(R\pi_{\mathrm{HT}!}(\overline{\mathbb{F}}_\ell)))_{\phi_{\mathfrak{m}}}$$

is an isomorphism, using Remark 3.15. Therefore, we see that the edge maps in the excision spectral sequence actually degenerate after applying $(-)_{\phi_{\mathfrak{m}}}$, giving us a direct sum decomposition

$$R\Gamma_c(\mathcal{S}(\mathbf{G},X)_{K^p,C},\overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}}\simeq\bigoplus_{b\in B(G,\mu)_{\mathrm{un}}}j_1^*T_\mu j_{b!}(V_b)[-d](-\frac{d}{2})_{\phi_{\mathfrak{m}}}.$$

By applying $R\Gamma(K_p^{\mathrm{hs}},-)$ and invoking Lemma 4.2 (3), we obtain that

$$R\Gamma(K_p^{\mathrm{hs}},R\Gamma_c(\mathcal{S}(\mathbf{G},X)_{K^p},\overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}})\simeq R\Gamma_c(\mathcal{S}(\mathbf{G},X)_{K^pK_p^{\mathrm{hs}}},\overline{\mathbb{F}}_\ell)_{\mathfrak{m}}$$

has a filtration with graded pieces isomorphic to

$$R\Gamma(K_p^{\mathrm{hs}},j_1^*T_\mu j_{b!}(V_b))_{\mathfrak{m}}[-d](-\frac{d}{2}).$$

Just as in the proof of Corollary 3.17, we can rewrite this as

$$(R\Gamma_c(\mathrm{Sht}(G,b,\mu)_{\infty,C}/\underline{K_p^{\mathrm{hs}}},\overline{\mathbb{F}}_\ell(d_b))_{\mathfrak{m}}\otimes^{\mathbb{L}}_{\mathcal{H}(J_b)}V_b)[2d_b],$$

as desired. ☐

Proof. (Theorem 1.17) We recall, by Proposition 3.7, that V_b is a complex of smooth $J_b(\mathbb{Q}_p)$-representations concentrated in degree $\leq d_b$. It follows that we have an inclusion $\bigoplus_{b\in B(G,\mu)}j_{b!}(V_b)_{\phi_{\mathfrak{m}}}\in{}^{\mathrm{p}}\mathrm{D}^{\leq 0,\mathrm{ULA}}(\mathrm{Bun}_G,\overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}}$, using Proposition A.5. Corollary 4.24 implies that

$$j_1^*T_\mu j_{b!}(V_b)[-d]_{\phi_{\mathfrak{m}}}\in\mathrm{D}^{\leq d}(G(\mathbb{Q}_p),\overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}}$$

after forgetting the Weil group action. Therefore, we conclude that

$$R\Gamma_c(\mathcal{S}(\mathbf{G},X)_{K^p,C},\overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}}$$

is concentrated in degrees $0\leq i\leq d$. By applying Poincaré duality at finite level and Corollary A.7, this allows us to conclude that the non-compactly supported cohomology

$$R\Gamma(\mathcal{S}(\mathbf{G},X)_{K^p,C},\overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}^\vee}$$

localized at $\phi_{\mathfrak{m}}^\vee$ is concentrated in degrees $d\leq i\leq 2d$, where we define this to be the colimit over the non-compactly supported cohomology of finite levels (cf. Remark 3.1). Moreover, we note that generic is preserved under the Chevalley involution, since it just exchanges the role of positive and negative roots. It therefore follows that

$$R\Gamma(K_p^{\mathrm{hs}},R\Gamma(\mathcal{S}(\mathbf{G},X)_{K^p,C},\overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}^\vee})$$

is also concentrated in degrees $\geq d$, but this isomorphic to

$$R\Gamma(\mathcal{S}(\mathbf{G},X)_{K^pK_p^{\mathrm{hs}},C},\overline{\mathbb{F}}_\ell)_{\mathfrak{m}^\vee}$$

by Lemma 4.2 (3). This establishes Theorem 1.17, by applying Poincaré duality to the Shimura variety at finite level again. □

5.2. **Proof of Corollary 1.19.** We would like to obtain the main theorem for Shimura varities of non-PEL type, especially Hilbert-Siegel modular varieties (attached to $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$ or $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GSp}_4$). We will show this in a more general setup, as follows.

Let $(\mathbf{G},X)$, $(\mathbf{G}_2,X_2)$ be a pair of abelian type Shimura data such that $\mathbf{G},\mathbf{G}_2$ are centrally isogenous, and we have an isomorphism of derived subgroups

$$\mathbf{G}^{\mathrm{der}}\xrightarrow{\sim}\mathbf{G}_2^{\mathrm{der}},$$

as well as adjoint quotients. Consider the associated Shimura varieties $\mathrm{Sh}(\mathbf{G},X)_K$ and $\mathrm{Sh}(\mathbf{G}_2,X_2)_{K_2}$, where we choose the level K,K_2 such that the level at p, satisfies that we have an equality $K_p\cap G^{\mathrm{der}}(\mathbb{Q}_p)=K_{2,p}\cap G_2^{\mathrm{der}}(\mathbb{Q}_p)$.

We now assume K_p and $K_{2,p}$ are both hyperspecial. Observe that this implies that $K_p'=K_p\cap G^{\mathrm{der}}(\mathbb{Q}_p)$ is also hyperspecial. By the Satake isomorphism, we have an isomorphism of $\overline{\mathbb{F}}_\ell$-algebras

$$H_{K_p}\simeq\overline{\mathbb{F}}_\ell[X_*(T)]^{W_G},$$

and, since G,G^{der} have isomorphic adjoint groups, the inclusion of cocharacters $X_*(T')\subset X_*(T)$ induces an inclusion of Hecke algebras $H'_{K_p'}\subset H_{K_p}$, where T' denotes the torus $T\cap G^{\mathrm{der}}$, and $H'_{K_p'}$ denotes the spherical Hecke algebra for G^{der}. Moreover, given a maximal ideal $\mathfrak{m}\subset H_{K_p}$, then $\mathfrak{m}'=\mathfrak{m}\cap H'_{K_p'}$ is a maximal ideal of $H'_{K_p'}$.

Fix a connected component $X^+\subset X$. This also fixes a $X_2^+\subset X_2$, and an isomorphism $X^+\simeq X_2^+$, since $\mathbf{G},\mathbf{G}_2$ have isomorphic adjoint quotients. For any compact open subgroup $K\subset\mathbf{G}(\mathbb{A}_f)$, we let $\mathrm{Sh}^+(\mathbf{G},X)_K$ be the geometrically connected component which is the image of $X^+\times 1$. Moreover, we will let

$$\mathrm{Sh}^+(\mathbf{G},X)_{K_p}=\lim_{\substack{\leftarrow\\K^p}}\mathrm{Sh}^+(\mathbf{G},X)_{K_pK^p}.$$

Note that since all the transition morphisms are finite étale and hence affine, $\mathrm{Sh}^+(\mathbf{G},X)_{K_p}$ is also a qcqs scheme by [Sta23, Lemma 01YX].

Since $\mathbf{G},\mathbf{G}_2$ have isomorphic derived subgroups, this implies that we have an isomorphism of geometric connected components

$$\mathrm{Sh}^+(\mathbf{G},X)_{K_p}\simeq\mathrm{Sh}^+(\mathbf{G}_2,X')_{K_{2,p}}.$$

Moreover, we see that the action of $H'_{K_p'}$ on $\mathrm{Sh}(\mathbf{G},X)_{K_p}$ preserves the geometric connected component, since we see that $\mathrm{Sh}^+(\mathbf{G},X)_{K_p}$ is simply the connected Shimura variety associated to $(\mathbf{G}^{\mathrm{der}},X^+)$. Indeed, we see that the set of $\mathbb{C}$-points of $\mathrm{Sh}^+(\mathbf{G},X)_{K_p}$ is given by

$$\mathbf{G}^{\mathrm{der}}(\mathbb{Q})_+^{(p),\mathrm{cl}}\backslash X^+\times(\mathbf{G}^{\mathrm{der}}(\mathbb{A}_f^p)),$$

where $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})_+^{(p),\mathrm{cl}}$ denotes the closure in $\mathbf{G}^{\mathrm{der}}(\mathbb{A}_f^p)$ of $\mathbf{G}^{\mathrm{der}}(\mathbb{R})_+\cap\mathbf{G}^{\mathrm{der}}(\mathbb{Q})\cap K_p$, and $\mathbf{G}^{\mathrm{der}}(\mathbb{R})_+$ denotes the preimage of the neutral connected component $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+$ in $\mathbf{G}^{\mathrm{der}}(\mathbb{R})$ under the quotient map.

Following the description of the connected components of Shimura varities from [Del79, §2], and using the notation of [Kis10, §3.3] we see that there exist groups $\mathscr{A}(\mathbf{G})^\circ,\mathscr{A}(\mathbf{G}_2),\mathscr{A}(\mathbf{G})$ such that we have $\mathbf{G}(\mathbb{A}_f)$ (resp. $\mathbf{G}_2(\mathbb{A}_f)$) equivariant isomorphisms

$$\mathrm{Sh}(\mathbf{G},X)_{K_p}\simeq[\mathscr{A}(\mathbf{G})\times\mathrm{Sh}^+(\mathbf{G},X)_{K_p}/\mathscr{A}(\mathbf{G})^\circ]$$

and

$$\mathrm{Sh}(\mathbf{G}_2, X_2)_{K_{2,p}} \simeq [\mathscr{A}(\mathbf{G}_2) \times \mathrm{Sh}^+(\mathbf{G}, X)_{K_p}/\mathscr{A}(\mathbf{G})^\circ].$$

Observe that, since $\mathscr{A}(\mathbf{G})^\circ = \mathscr{A}(\mathbf{G}^{\mathrm{der}})^\circ$ is a subgroup of $\mathscr{A}(\mathbf{G}_2)$, the Shimura variety $\mathrm{Sh}(\mathbf{G}_2, X_2)_{K_{2,p}}$ is simply an (infinite) union of copies of $\mathrm{Sh}^+(\mathbf{G}, X)_{K_p}$. Moreover, we see that the action of $H'_{K'_p}$ on the right-hand side of the above isomorphism is given by the action on $\mathrm{Sh}^+(\mathbf{G}, X)_{K_p}$. In particular, we observe that

$$H^i(\mathrm{Sh}^+(\mathbf{G}, X)_{K_p}, \overline{\mathbb{F}}_\ell)_{\mathfrak{m}'}$$

vanishes if and only if

$$H^i(\mathrm{Sh}(\mathbf{G}_2, X)_{K_{2,p}}, \overline{\mathbb{F}}_\ell)_{\mathfrak{m}'}$$

does as well. We thus have the following proposition.

Proposition 5.1. *Suppose that $(\mathbf{G}, X)$ is of PEL type A or C satisfying the conditions in Theorem 1.17. Then, Conjecture 1.2 also holds for $(\mathbf{G}_2, X_2)$.*

Proof. Since Conjecture 1.2 is true for $(\mathbf{G}, X)$, we will first show that a maximal ideal $\mathfrak{m}$ of H_{K_p} is generic if and only if the maximal ideal $\mathfrak{m}'$ of $H'_{K'_p}$ is generic. To see this, we will reformulate this in terms of L-parameters. This is equivalent to showing that an L-parameter

$$\phi : W_{\mathbb{Q}_p} \to {}^L T(\overline{\mathbb{F}}_\ell)$$

is generic if and only if the composition ϕ' with the map $g : {}^L T(\overline{\mathbb{F}}_\ell) \to {}^L T'(\overline{\mathbb{F}}_\ell)$ induced by the inclusion of tori $T' \hookrightarrow T$ is generic. (Here, $T' = G^{\mathrm{der}} \cap T$). This follows from the observation that any coroot α factors through G^{der}, and hence the composition $\alpha \circ \phi$ is equal to $\alpha \circ \phi'$.

Now, we consider the limit

$$\mathrm{Sh}(\mathbf{G}, X) := \varprojlim_{K^p} \mathrm{Sh}(\mathbf{G}, X)_{K^p}.$$

Note that since all schemes appearing in the limit are qcqs, by [Sta23, Theorem 09YQ] we have an isomorphism of cohomology groups

$$H^i(\mathrm{Sh}(\mathbf{G}, X), \overline{\mathbb{F}}_\ell) \simeq \varinjlim_{K^p} H^i(\mathrm{Sh}(\mathbf{G}, X)_{K^p}, \overline{\mathbb{F}}_\ell).$$

We now have the following lemma.

Lemma 5.2. *Let $G' \to G$ be a map inducing an isomorphism on adjoint groups with $g : {}^L G \to {}^L G'$, the induced map on dual groups. For $\phi : W_{\mathbb{Q}_p} \to {}^L G(\overline{\mathbb{F}}_\ell)$ a L-parameter and A an admissible complex of $G(\mathbb{Q}_p)$-modules, there is a natural isomorphism of $G'(\mathbb{Q}_p)$-modules*

$$(A|_{G'(\mathbb{Q}_p)})_{\phi'} \simeq \bigoplus_{\substack{\phi \\ \phi' = g \circ \phi}} A_\phi|_{G'(\mathbb{Q}_p)},$$

with notation as in Corollary 4.3.

Proof. By applying Corollary 4.3, we obtain a decomposition

$$A \simeq \bigoplus_\phi A_\phi$$

of A as a $G(\mathbb{Q}_p)$-module. We restrict to $G'(\mathbb{Q}_p)$ and apply the localization map $(-)_{\phi'}$. This gives an isomorphism

$$(A|_{G'(\mathbb{Q}_p)})_{\phi'} \simeq \bigoplus_\phi (A_\phi|_{G'(\mathbb{Q}_p)})_{\phi'},$$

where we have used that localization commutes with direct sums since it is a left adjoint by definition. Now, using the compatibility of the Fargues-Scholze correspondence with central isogenies [FS21, Theorem IX.6.1], either $\phi' = g \circ \phi$ and $A_\phi|_{G'(\mathbb{Q}_p)} \in \mathrm{D}(G'(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)_{\phi'}$ and, by the idempotence of the localization map, we have that $(A_\phi|_{G'(\mathbb{Q}_p)})_{\phi'} = A_\phi|_{G'(\mathbb{Q}_p)}$ or $(A_\phi|_{G'(\mathbb{Q}_p)})_{\phi'}$ is 0. The claim follows. □

By the previous lemma applied to $G^{\mathrm{der}} = G' \subset G$, we have a natural decomposition

$$H^i(\mathrm{Sh}(\mathbf{G}, X)_{K^p}, \overline{\mathbb{F}}_\ell)_{\phi'} \simeq \bigoplus_{\substack{\phi \\ \phi' = g \circ \phi}} H^i(\mathrm{Sh}(\mathbf{G}, X)_{K^p}, \overline{\mathbb{F}}_\ell)_{\phi}.$$

Taking limits over K^p, we obtain

$$H^i(\mathrm{Sh}(\mathbf{G}, X), \overline{\mathbb{F}}_\ell)_{\phi'} \simeq \bigoplus_{\substack{\phi \\ \phi' = g \circ \phi}} H^i(\mathrm{Sh}(\mathbf{G}, X), \overline{\mathbb{F}}_\ell)_{\phi}.$$

Hence, we see that $H^i(\mathrm{Sh}(\mathbf{G}, X), \overline{\mathbb{F}}_\ell)_{\phi'}$ vanishes for $i < d$, since all the ϕ appearing on the right-hand side are generic, and hence we can apply Theorem 1.17 and take limits to see that all the direct summands vanish. Applying $R\Gamma(K_p, -)$ and Lemma 4.2 (3), we see that

$$H^i(\mathrm{Sh}(\mathbf{G}, X)_{K_p}, \overline{\mathbb{F}}_\ell)_{\mathfrak{m}'}$$

vanishes for $i < d$. Thus, we see that $H^i(\mathrm{Sh}^+(\mathbf{G}, X)_{K_p}, \overline{\mathbb{F}}_\ell)_{\mathfrak{m}'}$ vanishes for $i < d$, and therefore the same is true for $H^i(\mathrm{Sh}(\mathbf{G}_2, X_2)_{K_{2,p}}, \overline{\mathbb{F}}_\ell)_{\mathfrak{m}'}$ from the discussion above. By the Hochschild-Serre spectral sequence, we see that for all sufficiently small K_2^p, $H^i_c(\mathrm{Sh}(\mathbf{G}_2, X_2)_{K_{2,p}K_2^p}, \overline{\mathbb{F}}_\ell)_{\mathfrak{m}'}$ vanishes for $i < d$.

Now, consider a generic maximal ideal $\mathfrak{m}_2$ for the spherical Hecke algebra $H_{K_{2,p}}$ of G_2. This corresponds to a generic maximal ideal $\mathfrak{m}'$ of $H'_{K'_p}$. It remains to observe that for any finitely-generated $H_{K_{2,p}}$-module A, if the localization $A_{\mathfrak{m}'} = 0$, then there is some element in $r \in H'_{K'_p} \backslash \mathfrak{m}'$ such that $rA = 0$. Thus we must have $A_{\mathfrak{m}_2} = 0$ as well since $H'_{K'_p} \backslash \mathfrak{m}' \subset H_{K_{2,p}} \backslash \mathfrak{m}_2$. □

As a corollary, we can strengthen previous results of Caraiani-Tamiozzo [CT21, Theorem B], who previously showed torsion vanishing for Hilbert modular varieties under the additional assumption that p was split in the totally real field F (though we also note that they showed torsion vanishing under a hypothesis on $\mathfrak{m}$ which is weaker than the genericity considered here, see Remark 1.7).

Corollary 5.3. *Conjecture 1.2 is true for Hilbert-Siegel Shimura varieties (attached to* $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$, $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GSp}_4$*) and quaternionic Shimura varieties.*

Proof. Observe that for Hilbert-Siegel Shimura varieties (attached to $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$, $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GSp}_4$), there is a cover by a PEL-type Shimura variety with local group G of the form $G(\mathrm{SL}_2)$ and $G(\mathrm{Sp}_4)$ respectively. For the case of quaternionic Shimura varieties, we can relate their geometric connected components to unitary PEL-type Shimura varieties with local group GU_2, as described in [TX16, Corollary 3.11]. Therefore, the result follows from Theorem 1.17. □

6. Conjectures and Concluding Remarks

6.1. **Relationship to Xiao-Zhu.** Assume that the basic element $b \in B(G, \mu)_{\mathrm{un}}$ is unramified (See [XZ17, Remark 4.2.11] for a classification). Let us look at the middle degree cohomology $H^d(R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p, C}, \overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}})$. By Theorem 1.15, it has a summand isomorphic to

$$H^d(R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} R\Gamma_{c-\partial}(\mathrm{Ig}^b, \overline{\mathbb{F}}_\ell)).$$

To describe this, let $\mathbf{G}'$ be the unique $\mathbb{Q}$-inner form of $\mathbf{G}$ such that $\mathbf{G}(\mathbb{A}^{p\infty}) \simeq \mathbf{G}'(\mathbb{A}^{p\infty})$, $\mathbf{G}'(\mathbb{R})$ is compact modulo center, and $\mathbf{G}_{\mathbb{Q}_p} \simeq J_b$ (See [Han20, Proposition 3.1] for the existence).

We write $C(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p, \overline{\mathbb{F}}_\ell)$ for the set of all continuous functions on the profinite set $\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p$. It is easy to show that one has an isomorphism

$$C(K^p\backslash\mathbf{G}'(\mathbb{A}_f)/\mathbf{G}'(\mathbb{Q}), \overline{\mathbb{F}}_\ell) \simeq R\Gamma_{c-\partial}(\mathrm{Ig}^b, \overline{\mathbb{F}}_\ell)$$

for example by combining [Han20, Theorem 3.4] and Corollary 3.6. We let $V_\mu \in \mathrm{Rep}_{\overline{\mathbb{F}}_\ell}(\hat{G})$ be the usual highest weight module of highest weight μ, which in particular agrees with the highest weight tilting module, since μ is minuscule. We let b_T denote the unique (since b is basic) reduction of $b \in B(G)$ to $B(T)$, and regard it as an element in $B(T) \simeq \mathbb{X}^*(\hat{T}^\Gamma)$ in what follows. It should be the case that, under possible additional constraints on $\mathfrak{m}$ depending on μ (See for example [Ham22, Conjecture 1.25] and [XZ17, Definition 1.4.2]), we have an isomorphism
(23)

$$C(K^p\backslash\mathbf{G}'(\mathbb{A}_f)/\mathbf{G}'(\mathbb{Q}), \overline{\mathbb{F}}_\ell)\otimes^{\mathbb{L}} R\Gamma_c(\mathrm{Sht}(G,b,\mu)_{\infty,C}/\underline{K_p^{\mathrm{hs}}}, \overline{\mathbb{F}}_\ell)_{\mathfrak{m}} \simeq C(K^p\backslash\mathbf{G}'(\mathbb{A}_f)/\mathbf{G}'(\mathbb{Q}), \overline{\mathbb{F}}_\ell)_{\mathfrak{m}}\otimes V_\mu|_{\hat{G}^\Gamma}(b_T)[-d](-\frac{d}{2})$$

of $G(\mathbb{Q}_p)$-representations[8], where we note that $J_b \simeq G$ if $b \in B(G,\mu)_{\mathrm{un}}$ since b is basic, and J_b must be quasi-split since b is unramified. In particular, by arguing as in Koshikawa [Kos21, Page 6], we know that $R\Gamma_c(\mathrm{Sht}(G,b,\mu)_{\infty,C}/\underline{K_p^{\mathrm{hs}}}, \overline{\mathbb{F}}_\ell)_{\mathfrak{m}}$ will have irreducible constituents given by the representations of $J_b(\mathbb{Q}_p)$ with Fargues-Scholze parameter equal $\phi_{\mathfrak{m}}$ as conjugacy classes of parameters. Moreover, using that Assumption 4.4 holds for the groups appearing in Table (1), we know by Propsoition 4.5 that they have to be constituents of $i_B^G(\chi)$, which will also be irreducible under the generic assumption and the constraints appearing in Table (1) (See the proof of Corollary 4.24). Then [Ham22, Conjecture 1.25] would imply that $R\Gamma_c(G,b,\mu)[i_B^G(\chi)] \simeq i_B^G(\chi) \otimes V_\mu|_{\hat{G}^\Gamma}(b_T)[-d](\frac{-d}{2})$ as $G(\mathbb{Q}_p)$-modules. Assume ℓ is banal (i.e coprime to the pro-order of K_p^{hs}) then passing to K_p^{hs}-invariants, recalling that it is exact under the banal hypothesis, gives us the isomorphism (23).[9]

Remark 6.1. If $B(G,\mu)_{\mathrm{un}}$ consists of only the basic element and the μ-ordinary element and ϕ_T is strongly μ-regular (Definition 4.13) then [Ham22, Conjecture 1.25] is true. In particular, it follows from [Ham22, Theorem 1.27] that the isomorphism (23) can be made unconditional.

We note that this description of the middle degree cohomology on the generic fiber of the Shimura variety at hyperspecial level parallels Theorem [XZ17, Theorem 1.14 (1)], describing the middle degree cohomology on the special fiber of the natural integral model.

6.2. **A General Torsion Vanishing Conjecture.** Consider now a general Shimura datum $(\mathbf{G}, X)$. Let $\Lambda \in \{\overline{\mathbb{Q}}_\ell, \overline{\mathbb{F}}_\ell\}$. If $\Lambda = \overline{\mathbb{F}}_\ell$ assume that ℓ is very good with respect to $G := \mathbf{G}_{\mathbb{Q}_p}$, as in [FS21, Page 33]. We can then look at the $G(\mathbb{Q}_p) \times W_{E_{\mathfrak{p}}}$-representation.

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p,C}, \Lambda)$$

defined by the cohomology at infinite level. By applying Corollary 4.3, we obtain a $G(\mathbb{Q}_p) \times W_{E_{\mathfrak{p}}}$-equivariant decomposition of this

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p,C}, \Lambda) = \bigoplus_{\phi} R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p,C}, \Lambda)_\phi$$

running over semi-simple L-parameters $\phi : W_{\mathbb{Q}_p} \to {}^LG(\Lambda)$. For such a ϕ, we let (ϕ_M, M) denote a cuspidal support. I.e M is a Levi of G and $\phi_M : W_{\mathbb{Q}_p} \to {}^LM(\Lambda)$ is a supercuspidal L-parameter such that ϕ is induced by composing with the natural embedding ${}^LM(\Lambda) \to {}^LG(\Lambda)$. We want to describe the degrees of cohomology that $R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p,C}, \Lambda)_\phi$ sits in for suitably nice ϕ. The

[8]One should also be able describe the Weil group action, as in [Ham22, Conjecture 1.25].

[9]For this comparison, it would have been more natural to consider an analogue of Theorem 1.15 with $\overline{\mathbb{Q}}_\ell$-coefficients. This is indeed doable assuming that $\phi_{\mathfrak{m}}$ admits a $\overline{\mathbb{Z}}_\ell$-lattice as in [Ham22, Theorem 1.17]. This integrality condition is however an artifiact of the theory of solid $\overline{\mathbb{Q}}_\ell$-sheaves not being properly understood (e.g excision fails) and should be removable with more technology.

case where ϕ factors through $M = T$ is covered by Conjecture 1.2. To go beyond this, we give the following definition.

Definition 6.2. For a semi-simple L-parameter ϕ with a cuspidal support (M, ϕ_M), we let P be a parabolic with Levi factor M and unipotent radical N. We consider the representation r given by looking at the action of ${}^LM$ on the Lie algebra of ${}^LN$ via the adjoint action. We say ϕ is of Langlands-Shahidi type if the Galois cohomology groups

$$R\Gamma(W_{\mathbb{Q}_p}, r \circ \phi_M)$$

and

$$R\Gamma(W_{\mathbb{Q}_p}, r \circ \phi_M^\vee)$$

are trivial. Similarly, we say ϕ is of weakly Langlands-Shahidi type if

$$H^2(R\Gamma(W_{\mathbb{Q}_p}, r \circ \phi_M))$$

and

$$H^2(R\Gamma(W_{\mathbb{Q}_p}, r \circ \phi_M^\vee)$$

are trivial.

Remark 6.3. We note that since we enforced this condition on both $r \circ \phi_M$ and $r \circ \phi_M^\vee$ that this is independent of the choice of parabolic P and the choice of cuspidal support. Moreover, it is easy to check that, if $M = T$, this precisely recovers Definition 1.1.

The terminology of "Langlands-Shahidi type" comes from the fact that the representation $r \circ \phi_M$ is precisely the representation which appears in the description of the constant term of the usual Eisenstein series via the Langlands-Shahidi method. The motivation for this definition comes from considering the behavior of geometric Eisenstein series over the Fargues-Fontaine curve for general parabolics, by making analogies with the classical theory over function fields, as developed in [BG02; Lau90]. In particular, this should be the correct definition that guarantees that the eigensheaves $\mathcal{S}_\phi$ on Bun_G with eigenvalue ϕ are as simple as possible, and the analysis carried out in [Ham22] generalizes to the non-principal case. This is discussed in more detail in [Ham23, Chapter 3]. In addition, we expect that the consequences derived from the analysis in [Ham22] in the principal case should also generalize. More precisely, we conjecture the following generalization of Proposition 4.15 and Corollary 4.24

Conjecture 6.4. *Let $B(G)_M := \mathrm{Im}(B(M)_{\mathrm{basic}} \to B(G))$ be the set of M-reducible elements, and let ϕ be a semi-simple L-parameter of Langlands-Shahidi type with cuspidal support (M, ϕ_M). The category $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\phi$ of ϕ-local lisse-étale Λ-sheaves (as defined in Appendix A) breaks up as direct sum*

$$\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\phi \simeq \bigoplus_{b \in B(G)_M} \mathrm{D}(\mathrm{Bun}_G^b, \Lambda)_\phi$$

via excision, and the ! and $*$ pushhforwards agree for any smooth irreducible representation ρ of $J_b(\mathbb{Q}_p)$ lying in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G^b, \Lambda)_\phi$ for $b \in B(G)_M$.

Given a tilting module $V \in \mathrm{Tilt}_\Lambda({}^LG^I)$, if ϕ is of weakly Langlands-Shahidi type then the map induced by associated the Hecke operator

$$T_V : \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\phi \to \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\phi^{BW_{\mathbb{Q}_p}^I}$$

is perverse t-exact, where the fact the Hecke operator preserves this subcategory is proven as in Lemma 4.2 (2).

Remark 6.5. During the preparation of this manuscript, Hansen formulated similar conjectures with rational coefficients [Han23]. He refers to Langlands-Shahidi parameters as generous parameters [Han23, Definition 2.5] and to weakly Langlands-Shahidi parameters as generic semi-simple parameters [Han23, Section 2.3]. One can show that these two definitions are equivalent. Indeed, note that the Galois cohomology $H^1(R\Gamma(W_{\mathbb{Q}_p}, r \circ \phi_M))$ controls the lifts of a semi-simple parameter $\phi_M : W_{\mathbb{Q}_p} \to {}^LM(\Lambda)$ to a ${}^LP(\Lambda)$-valued parameter and that such lifts correspond to finding parameters whose semi-simplification is equal to ϕ. Moreover, insisting that $H^1(R\Gamma(W_{\mathbb{Q}_p}, r \circ \phi_M))$ is trivial is equivalent to insisting that $R\Gamma(W_{\mathbb{Q}_p}, r \circ \phi_M)$ is trivial using local Tate-duality and that the Euler-Poincaré characteristic of this complex is 0. This shows the equivalence of the generous condition with the Langlands-Shahidi type condition, using that the stack of Langlands parameters with rational coefficients is reduced. Lastly, the set of such lifts coming from classes in $H^0(R\Gamma(W_{\mathbb{Q}_p}, r \circ \phi_M))$ will give rise to non Frobenius semi-simple L-parameters allowing one to see that weakly Langlands-Shahidi is equivalent to generic semi-simple.

In particular, by combining this with a generalization of Theorem 1.13 to arbitrary Shimura varieties and the analysis carried out in §5, we could deduce the following as a consequence.

Conjecture 6.6. *Let ϕ be a semi-simple L-parameter of weakly Langlands-Shahidi type with cuspidal support (M, ϕ_M). Then the complex $R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p,C}, \Lambda)_\phi$ (resp. $R\Gamma(\mathcal{S}(\mathbf{G}, X)_{K^p,C}, \Lambda)_\phi$) is concentrated in degrees $0 \leq i \leq d$ (resp. $d \leq i \leq 2d$).*

Remark 6.7. For $(\mathbf{G}, X)$ of PEL type A or C, assuming 1.11 and that ϕ of Langlands-Shahidi type, we should also obtain a $W_{E_{\mathfrak{p}}} \times G(\mathbb{Q}_p)$-equivariant direct sum decomposition

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p,C}, \Lambda)_\phi \simeq \bigoplus_{b \in B(G,\mu)_M} (R\Gamma_c(G, b, \mu)_\phi \otimes^{\mathbb{L}} V_b)[2d_b],$$

where $R\Gamma_c(G, b, \mu) := \mathrm{colim}_{K_p \to \{1\}} R\Gamma_c(\mathrm{Sht}(G, b, \mu)_{\infty,C}/\underline{K_p}, \Lambda(d_b))$ and $R\Gamma_c(G, b, \mu)_\phi$ is the projection applied to the complex viewed as a $G(\mathbb{Q}_p)$-representation. This should also generalize once one has appropriate general definitions of Ig^b and $\mathrm{Ig}^{b,*}$ so that one can actually define $V_b := R\Gamma_{c-\partial}(\mathrm{Ig}^b, \Lambda)$. Under possible additional constraints on ϕ, one should also be able to describe the contribution of $R\Gamma_c(G, b, \mu)_\phi$ in terms of the decomposition $V_\mu|_{Z(\hat{M}^\Gamma)} = \mathcal{T}_\mu|_{Z(\hat{M}^\Gamma)}$ for $b \in B(G)_M$ (along the lines of [Ham22, Conjecture 1.25]), as is explained in the toral case in §6.1. It would be interesting to formulate an optimal conjecture.

Remark 6.8. We believe that this conjecture should be true under just the weakly Langlands-Shahidi condition. However, we strongly suspect that the splitting of the semi-orthogonal decomposition and in turn the splitting of Mantovan's filtration discussed in the previous Remark should not hold unless the set $B(G, \mu)_M$ is a singleton. In particular, in [Han23, Section 2.2] Hansen conjectures the existence of perverse sheaves lying $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\phi$, for which the semi-orthogonal decomposition does not split. Nonetheless, one still expects perverse t-exactness of Hecke operators to hold in these cases [Han23, Conjecture 2.32].

Appendix A. Spectral Decomposition of Sheaves on Bun_G, by David Hansen

Let $G/\mathbb{Q}_p$ be a connected reductive group, $\Lambda/\mathbf{Z}_\ell$ an algebraically closed field. If $\mathrm{char}(\Lambda) \neq 0$ we assume ℓ is very good for G.

Set $\mathrm{D}(\mathrm{Bun}_G) = \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ to be the derived category of lisse-étale Λ-sheaves, regarded as a stable ∞-category whenever convenient. Let $\mathfrak{X}_{\hat{G}} = Z^1(W_E, \hat{G})_\Lambda/\hat{G}$ be the stack of L-parameters over Λ, and let $X_{\hat{G}}$ be its coarse moduli space, $q : \mathfrak{X}_{\hat{G}} \to X_{\hat{G}}$ the natural map. We will regard $\mathfrak{X}_{\hat{G}}$ as a disjoint union of finite type algebraic stacks over Λ, and $X_{\hat{G}}$ as a disjoint union of finite type affine Λ-schemes. As in [FS21], we have the spectral action of $\mathrm{Perf}(\mathfrak{X}_{\hat{G}})$ on $\mathrm{D}(\mathrm{Bun}_G)$, and there is a natural map $\Psi_G : \mathcal{O}(\mathfrak{X}_{\hat{G}}) = \mathcal{O}(X_{\hat{G}}) \to \mathfrak{Z}(\mathrm{D}(\mathrm{Bun}_G)) := \pi_0(\mathrm{id}_{\mathrm{D}(\mathrm{Bun}_G)})$, where we recall that

$Z^1(W_E, \hat{G})_\Lambda$ is a disjoint union of affine schemes by [FS21, Theorem VIII.1.3]. These two structures are compatible (as proven by Zou [Zou22, Theorem 5.2.1]).

By [FS21, Prop. VIII.3.8], the set of closed points $X_{\hat{G}}(\Lambda)$ is naturally in bijection with the set of isomorphism classes of semisimple L-parameters $\phi : W_E \to {}^LG(\Lambda)$. Let $\mathfrak{m}_\phi \subset \mathcal{O}(X_{\hat{G}})$ be the maximal ideal associated with a given ϕ.

Definition A.1. Given any ϕ as above, $\mathrm{D}(\mathrm{Bun}_G)_\phi \subset \mathrm{D}(\mathrm{Bun}_G)$ is the full subcategory of sheaves $A \in \mathrm{D}(\mathrm{Bun}_G)$ such that for every $f \in \mathcal{O}(X_{\hat{G}}) \smallsetminus \mathfrak{m}_\phi$, $A \xrightarrow{\cdot f} A$ is an isomorphism. Here $\cdot f$ is the endomorphism of A induced by Ψ_G.

We will call objects of $\mathrm{D}(\mathrm{Bun}_G)_\phi$ ϕ-*local* sheaves.

By construction, $\mathrm{D}(\mathrm{Bun}_G)_\phi$ is a full subcategory of $\mathrm{D}(\mathrm{Bun}_G)$ stable under arbitrary limits and colimits, and the tautological inclusion functor $\iota_\phi : \mathrm{D}(\mathrm{Bun}_G)_\phi \hookrightarrow \mathrm{D}(\mathrm{Bun}_G)$ commutes with limits and colimits. By the ∞-categorical adjoint functor theorem [Lur09, Cor. 5.5.2.9.(2)], it therefore admits a left adjoint $\mathcal{L}_\phi : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_G)_\phi$.[10] The unit of the adjunction gives a map $A \to \iota_\phi \mathcal{L}_\phi A =: A_\phi$ functorially in A. Since ι_ϕ is fully faithful, $\mathcal{L}_\phi \iota_\phi = \mathrm{id}$, so $(A_\phi)_\phi = A_\phi$, i.e. the endofunctor $A \rightsquigarrow A_\phi$ is idempotent. We remark that $\mathrm{D}(\mathrm{Bun}_G)_\phi$ is a Bousfield localization of $\mathrm{D}(\mathrm{Bun}_G)$, and the map $A \to A_\phi$ is the initial map from A to a ϕ-local sheaf.

Proposition A.2. *The full subcategory* $\mathrm{D}(\mathrm{Bun}_G)_\phi$ *is preserved by the spectral action, and* $A \rightsquigarrow A_\phi$ *commutes with the spectral action. Moreover,* $\mathrm{supp}(A_\phi) \subseteq \mathrm{supp}(A)$.

Proof. The first claim is clear, since the spectral action commutes with the action of $\mathcal{O}(X_{\hat{G}})$. For the remaining claims (and some later arguments), it is useful to give an explicit formula for A_ϕ. Let $\mathcal{I}_\phi$ be the diagram category whose objects are elements of $\mathcal{O}(X_{\hat{G}}) \smallsetminus \mathfrak{m}_\phi$ and where a morphism $f \to g$ is an element $h \in \mathcal{O}(X_{\hat{G}}) \smallsetminus \mathfrak{m}_\phi$ such that $g = fh$. This is clearly cofiltered. Let $F \in \mathrm{Fun}(\mathcal{I}_\phi, \mathrm{D}(\mathrm{Bun}_G))$ be the functor sending f to A and sending a morphism $h \in \mathrm{Mor}(f, g)$ to $\cdot h \in \mathrm{End}(A)$. Then $A_\phi = \mathrm{colim}_{i \in \mathcal{I}_\phi} F(i)$. The remaining claims are now immediate. □

To make sense of the next proposition, note that for any $A, B \in \mathrm{D}(\mathrm{Bun}_G)$, $\mathrm{Hom}(B, A)$ is naturally a $\mathfrak{Z}(\mathrm{D}(\mathrm{Bun}_G))$-module, whence a $\mathcal{O}(X_{\hat{G}})$-module.

Proposition A.3. *If* $C \in \mathrm{D}(\mathrm{Bun}_G)$ *is compact, then* $\mathrm{Hom}(C, A_\phi) \cong \mathrm{Hom}(C, A)_{\mathfrak{m}_\phi}$ *functorially in* A *and* C*, where the RHS is the usual localization as an* $\mathcal{O}(X_{\hat{G}})$*-module.*

Proof. Notation as in the previous proof, we have

$$\begin{aligned}\mathrm{Hom}(C, A_\phi) &\cong \mathrm{Hom}(C, \mathrm{colim}_{i \in \mathcal{I}_\phi} F(i)) \\ &\cong \mathrm{colim}_{i \in \mathcal{I}_\phi} \mathrm{Hom}(C, F(i)) \\ &\cong \mathrm{Hom}(C, A)_{\mathfrak{m}_\phi}\end{aligned}$$

where the second isomorphism follows from the compactness of C and the third isomorphism is immediate from the definition of $(-)_{\mathfrak{m}_\phi}$. □

Proposition A.4. *If* A *is ULA, then also* A_ϕ *is ULA.*

Proof. Recall from [FS21, Prop. VII.7.9] that $B \in \mathrm{D}(\mathrm{Bun}_G)$ is ULA iff $R\mathrm{Hom}(C, B) \in \mathrm{Perf}(\Lambda)$ is a perfect complex for all compact objects $C \in \mathrm{D}(\mathrm{Bun}_G)$. Now, if C is compact, $R\mathrm{Hom}(C, -)$ commutes with filtered colimits, so

$$\begin{aligned}R\mathrm{Hom}(C, A_\phi) &\simeq R\mathrm{Hom}(C, \mathrm{colim}_{i \in \mathcal{I}_\phi} F(i)) \\ &\simeq \mathrm{colim}_{i \in \mathcal{I}_\phi} R\mathrm{Hom}(C, F(i))\end{aligned}$$

[10] To see that ι_ϕ is accessible, use [Lur09, Prop. 5.4.7.7] together with the fact that ι_ϕ admits a right adjoint, which follows from [Lur09, Cor. 5.5.2.9.(1)].

with notation as in the proof of Proposition A.2. Since $F(i) \simeq A$ for all i, $\mathrm{colim}_{i \in \mathcal{I}_\phi} R\mathrm{Hom}(C, F(i))$ is a filtered colimit of perfect complexes P_i which vanish outside a finite interval independent of n, and with $\dim_\Lambda(H^j(P_i))$ bounded independently of i. It then easily follows that $\mathrm{colim}_{i \in \mathcal{I}_\phi} R\mathrm{Hom}(C, F(i))$ is perfect, whence the claim. $\square$

Proposition A.5. *If A is ULA, the natural maps $A \to \prod_\phi A_\phi \leftarrow \oplus_\phi A_\phi$ are isomorphisms, where the direct sum and direct product are taken over all semi-simple L-parameters. In particular, A_ϕ is functorially a direct summand of A for ULA sheaves A, and the functor $(-)_\phi$ on ULA sheaves is perverse t-exact.*

Remark A.6. The isomorphism $\oplus_\phi A_\phi \xrightarrow{\sim} \prod_\phi A_\phi$ may be surprising at first glance. To put this in context, we remind the reader that if $(\pi_i)_{i \in I}$ is a collection of admissible smooth $\Lambda[G(\mathbb{Q}_p)]$-modules whose product $\prod_i \pi_i$ is admissible, then $\oplus_i \pi_i \xrightarrow{\sim} \prod_i \pi_i$ automatically, because admissibility of $\prod_i \pi_i$ implies that for any given compact open subgroup $K \subset G(\mathbb{Q}_p)$ we have $\pi_i^K = 0$ for all but finitely many i. A similar argument occurs in the following proof, which actually shows that if $(A_i)_{i \in I}$ is any collection of ULA sheaves on Bun_G whose product $\prod_i A_i$ is ULA, then $\oplus_i A_i \xrightarrow{\sim} \prod_i A_i$ automatically.

Proof. We first show that $A \to \prod_\phi A_\phi$ is an isomorphism. Let C be any compact object. It suffices to prove that the natural map

$$\mathrm{Hom}(C, A) \to \prod_\phi \mathrm{Hom}(C, A_\phi) \cong \mathrm{Hom}(C, \prod_\phi A_\phi)$$

is an isomorphism, since $\mathrm{D}(\mathrm{Bun}_G)$ is compactly generated [FS21, Theorem I.5.1 (iii)]. As in the previous proof, $R\mathrm{Hom}(C, A)$ is a perfect complex, so $\mathrm{Hom}(C, A)$ is a finite Λ-vector space. In particular, it is a finite length $\mathcal{O}(X_{\hat{G}})$-module supported at a finite set of closed points $S \subset X_{\hat{G}}(\Lambda)$, so if $\phi \notin S$ then $\mathrm{Hom}(C, A_\phi) = \mathrm{Hom}(C, A)_{\mathfrak{m}_\phi} = 0$ using Proposition A.3. We then conclude that

$$\begin{aligned} \mathrm{Hom}(C, A) &= \oplus_{\phi \in S} \mathrm{Hom}(C, A)_{\mathfrak{m}_\phi} \\ &= \oplus_{\phi \in S} \mathrm{Hom}(C, A_\phi) \\ &= \prod_\phi \mathrm{Hom}(C, A_\phi) \end{aligned}$$

where the first equality follows from general nonsense about finite length modules over commutative rings, the second equality follows from Proposition A.3, and the third equality follows from the vanishing of $\mathrm{Hom}(C, A_\phi)$ for all but finitely many ϕ. This also shows that $\mathrm{Hom}(C, \oplus_\phi A_\phi) \cong \oplus_\phi \mathrm{Hom}(C, A_\phi) \to \prod_\phi \mathrm{Hom}(C, A_\phi)$ is an isomorphism (here again the first isomorphism follows from compactness of C), which implies that $\oplus_\phi A_\phi \xrightarrow{\sim} \prod_\phi A_\phi$ is an isomorphism. $\square$

Next, recall the Verdier duality functor $\mathbb{D}_{\mathrm{Bun}_G}$ on $\mathrm{D}(\mathrm{Bun}_G)$, which induces an involutive anti-equivalence on the subcategory of ULA sheaves. Recall also that, for any A, the diagram

$$\begin{tikzcd} \mathcal{O}(X_{\hat{G}}) \arrow[r, "\Psi_G"] \arrow[d, "f \mapsto f^\vee"] & \mathrm{End}(A) \arrow[d] \\ \mathcal{O}(X_{\hat{G}}) \arrow[r, "\Psi_G"] & \mathrm{End}(\mathbb{D}_{\mathrm{Bun}_G}(A)) \end{tikzcd}$$

commutes, where $f \mapsto f^\vee$ is the involution of $\mathcal{O}(X_{\hat{G}})$ induced by composition with the Chevalley involution at the level of L-parameters. Since $f \in \mathfrak{m}_\phi$ iff $f^\vee \in \mathfrak{m}_{\phi^\vee}$, we deduce that if A is ϕ-local then $\mathbb{D}_{\mathrm{Bun}_G}(A)$ is $\phi^\vee$-local. Using biduality, we also get that if A is ULA then A is ϕ-local if and only if $\mathbb{D}_{\mathrm{Bun}_G}(A)$ is $\phi^\vee$-local.

Corollary A.7. *If A is ULA, then $\mathbb{D}_{\mathrm{Bun}_G}(A_\phi) \cong (\mathbb{D}_{\mathrm{Bun}_G}(A))_{\phi^\vee}$.*

Proof. By Proposition A.5 and the remarks preceding its proof, the decomposition $A = \oplus_\psi A_\psi$ dualizes to a decomposition

$$\mathbb{D}_{\mathrm{Bun}_G}(A) = \prod_\psi \mathbb{D}_{\mathrm{Bun}_G}(A_\psi) \cong \oplus_\psi \mathbb{D}_{\mathrm{Bun}_G}(A_\psi)$$

where the second isomorphism follows from the discussion in Remark A.6. On the other hand, applying Proposition A.5 directly to $\mathbb{D}_{\mathrm{Bun}_G}(A)$ gives a decomposition

$$\mathbb{D}_{\mathrm{Bun}_G}(A) \cong \oplus_{\psi'}(\mathbb{D}_{\mathrm{Bun}_G}(A))_{\psi'},$$

so comparing these we get a natural isomorphism

$$\oplus_\psi \mathbb{D}_{\mathrm{Bun}_G}(A_\psi) \cong \oplus_{\psi'}(\mathbb{D}_{\mathrm{Bun}_G}(A))_{\psi'}.$$

Applying $(-)_{\phi^\vee}$ to both sides, we get $\mathbb{D}_{\mathrm{Bun}_G}(A_\phi)$ on the left side (using that $\mathbb{D}_{\mathrm{Bun}_G}(A_\phi)$ is $\phi^\vee$-local), and $(\mathbb{D}_{\mathrm{Bun}_G}(A))_{\phi^\vee}$ on the right side. This gives the claim. □

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