Geometrization of Local Langlands and the Cohomology of Shimura Varieties

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Abstract

Fargues and Scholze showed that a candidate for the local Langlands correspondence for p-adic reductive groups $G/\mathbb{Q}_p$ could be realized in terms of a geometric Langlands correspondence occurring over Bun_G, the moduli stack of G-bundles on the Fargues-Fontaine curve. This builds a bridge between Shimura varieties and the trace formula and shtukas and geometric Langlands, since the relevant shtuka spaces can be shown to uniformize Shimura varieties. The goal of this thesis is to build on this connection with the aim of describing the cohomology of Shimura varieties. The basic strategy is as follows. First, show that the Fargues-Scholze local Langlands correspondence agrees with more classical instances of the correspondence, by describing the Galois action on global Shimura varieties and using uniformization to relate this to shtukas. Second, combine such compatibility results with techniques in geometric Langlands to explicitly describe eigensheaves on Bun_G. Third, use this description to describe the cohomology of shtukas, and then use uniformization in the other direction to describe the cohomology of the global Shimura variety.

In chapter 1, we showcase a strategy for showing that the Fargues-Scholze correspondence agrees with more classical instances of local Langlands in the particular case of GSp_4 and its inner form. We use this compatibility to describe eigensheaves with eigenvalue ϕ, a supercuspidal L-parameter, and in turn prove new cases of the Kottwitz conjecture. In chapter 2, we build on this paradigm, and show how, assuming such compatibility results, we can parabolically induce the eigensheaves on Bun_T for a maximal torus $T \subset G$ to eigensheaves on Bun_G with eigenvalue factoring through a maximal torus. We do this under a generic assumption on the parameter, as in the work of Caraiani-Scholze. In chapter 3, we discuss joint work in progress where one combines the results of chapters 1 and 2 to extend the torsion vanishing results of Cariani-Scholze to several new cases. Motivated by this, we formulate several new conjectures on the cohomology of global Shimura varieties, and explain how these conjectures would follow from generalizing the analysis in chapter 2 to describe the eigensheaves with eigenvalue ϕ induced from a supercuspidal L-parameter factoring through the dual group of a general Levi subgroup of G.

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Contents

Introduction

0.1 Chapter 1

Fix distinct primes $\ell \neq p$. Let $\mathbb{A}$ (resp. $\mathbb{A}_f$) denote the adeles (resp. finite adeles) of the rational numbers. Let $G/\mathbb{Q}_p$ be a split (for simplicity) connected reductive group over the p-adic numbers, and let $W_{\mathbb{Q}_p}$ denote the Weil group. We set $\mathbb{C}_p$ to be the completed algebraic closure of $\mathbb{Q}_p$ and $\overline{\mathbb{Q}}_\ell$ to be the algebraic closure of the ℓ-adic numbers. The local Langlands correspondence is a conjectural map

$$\mathrm{LLC}_G : \Pi(G) \to \Phi(G)$$

$$\pi \mapsto \phi_\pi$$

from the set $\Pi(G)$ of isomorphism classes of smooth irreducible representations of the p-adic group $G(\mathbb{Q}_p)$ to the set $\Phi(G)$ of conjugacy classes of parameters $\phi : W_{\mathbb{Q}_p} \times \mathrm{SL}_2(\overline{\mathbb{Q}}_\ell) \to \hat{G}(\overline{\mathbb{Q}}_\ell)$, where $\hat{G}$ is the reductive group with root datum dual to G. There are many different ways of constructing and characterizing such correspondences. However, in all its different guises, the correspondence should always be able to characterize the local constituents at p of an automorphic representation of $\mathbf{G}(\mathbb{A})$, where $\mathbf{G}$ is a global group whose base-change to $\mathbb{Q}_p$ is G. For a Shimura datum $(\mathbf{G}, X)$ attached to $\mathbf{G}$, this global automorphic spectrum is intimately related to a tower $\{\mathscr{S}(\mathbf{G}, X)_K\}$ of p-adic Shimura varieties over $\mathbb{C}_p$, where K is some varying compact open subgroup of $G(\mathbb{A}_f)$. If we write $K := K_pK^p$, where $K_p \subset G(\mathbb{Q}_p)$ is the level at p and $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$ is the level away from p, then we can look at the complex

$$R\Gamma_c(\mathscr{S}(\mathbf{G}, X)_{K^p}, \overline{\mathbb{Q}}_\ell) := \mathrm{colim}_{K_p \to \{1\}} R\Gamma_c(\mathscr{S}(\mathbf{G}, X)_{K^pK_p}, \overline{\mathbb{Q}}_\ell)$$

defined by the ℓ-adic cohomology of this tower. This has an action of $G(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$, and, given $\pi \in \Pi(G)$, we can look at the isotypic part

$R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p},\overline{\mathbb{Q}}_\ell)[\pi]$ with respect to π. The complex $R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p},\overline{\mathbb{Q}}_\ell)[\pi]$ has a leftover $W_{\mathbb{Q}_p}$-action, and we expect the action to be given (up to multiplicity) by $r_\mu \circ \phi_\pi^{\mathrm{ss}}$ for any suitable candidate of LLC_G. Here r_μ is the representation of $\hat{G}$ of highest weight μ and ϕ_π^{ss} is the semi-simplification of ϕ_π; the composite of ϕ_π with the map

$$W_{\mathbb{Q}_p} \to W_{\mathbb{Q}_p} \times \mathrm{SL}_2(\overline{\mathbb{Q}}_\ell)$$
$$g \mapsto (g, \begin{pmatrix} |g|^{1/2} & 0 \\ 0 & |g|^{-1/2} \end{pmatrix})$$

where $|\cdot| : W_{\mathbb{Q}_p} \to \overline{\mathbb{Q}}_\ell^*$ denotes the norm character.

The first goal of this thesis is to compare the more classical story explained above with a recent general construction of Fargues-Scholze for the local Langlands correspondence. To explain this, we consider the Fargues-Fontaine curve X and Bun_G, the moduli stack of G-bundles on it. We let $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G,\overline{\mathbb{Q}}_\ell)$ be the derived category of lisse-étale $\overline{\mathbb{Q}}_\ell$-sheaves on it. The space Bun_G is stratified by elements of the Kottwitz set $B(G)$. An element $b \in B(G)$ gives rise to an inner form of a Levi subgroup of G, denoted J_b, as well as a locally closed Harder-Narasimhan strata $j_b : \mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$, which is (up to an ℓ-adically contractible unipotent part) the classifying stack $[*/\underline{J_b(\mathbb{Q}_p)}]$ of the p-adic points of the reductive group J_b. It follows that $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G^b,\overline{\mathbb{Q}}_\ell) \simeq \mathrm{D}(J_b(\mathbb{Q}_p),\overline{\mathbb{Q}}_\ell)$ identifies with the unbounded derived category of smooth $\overline{\mathbb{Q}}_\ell$-representations of $J_b(\mathbb{Q}_p)$, and, by applying excision on Bun_G with respect to this stratification, that we can think of the category of lisse-étale $\overline{\mathbb{Q}}_\ell$-sheaves on Bun_G as a complex of smooth irreducible representations $\rho_b \in \Pi(J_b)$ for all $b \in B(G)$ together with some gluing datum. In particular, a representation $\pi \in \Pi(G)$ gives rise to a sheaf $\mathscr{F}_\pi \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G,\overline{\mathbb{Q}}_\ell)$ given by !-extending along the strata $j_1 : \mathrm{Bun}_G^1 \hookrightarrow \mathrm{Bun}_G$ corresponding to the trivial G-bundle.

The point of considering category $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G,\overline{\mathbb{Q}}_\ell)$ is that it carries a lot of symmetries. In particular, given any cocharacter μ of G, we get a map

$$T_\mu : \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G,\overline{\mathbb{Q}}_\ell) \to \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G,\overline{\mathbb{Q}}_\ell)^{BW_{\mathbb{Q}_p}}$$

called a Hecke operator, where $\mathrm{D}(\mathrm{Bun}_G,\overline{\mathbb{Q}}_\ell)^{BW_{\mathbb{Q}_p}}$ can be thought of as a collection of representations ρ_b of the J_b tensored by a representation of $W_{\mathbb{Q}_p}$. By considering the $W_{\mathbb{Q}_p}$-action for varying μ on $T_\mu(\mathscr{F}_\pi)$, this determines enough information to uniquely specify an element ϕ_π^{FS} of $\Phi^{\mathrm{ss}}(G)$, the conjugacy classes of semi-simple maps $W_{\mathbb{Q}_p} \to \hat{G}(\overline{\mathbb{Q}}_\ell)$. The resulting map

$$\mathrm{LLC}_G^{\mathrm{FS}} : \Pi(G) \to \Phi^{\mathrm{ss}}(G)$$

$$\pi \mapsto \phi_\pi^{\mathrm{FS}}$$

is the Fargues-Scholze local Langlands correspondence of G.

It is natural to wonder to what extent the two kinds of correspondences discussed above are compatible. Ideally, we want to show the diagram

$$\begin{tikzcd} \Pi(G) \arrow[r, "\mathrm{LLC}_G"] \arrow[dr, "\mathrm{LLC}_G^{\mathrm{FS}}"'] & \Phi(G) \arrow[d, "(-)^{\mathrm{ss}}"] \\ & \Phi^{\mathrm{ss}}(G) \end{tikzcd}$$

commutes for all G, where $(-)^{\mathrm{ss}}$ is the semi-simplification map described above. This requires establishing a link between p-adic global Shimura varieties and the Hecke operators T_μ described above. If we fix $b \in B(G)$ then T_μ is computed in terms of spaces of modifications $\mathscr{E}_b \dashrightarrow \mathscr{E}_0$ of meromorphy $\leq \mu$ for b varying in the subset $B(G,\mu) \subset B(G)$. Here $\mathscr{E}_b$ is the G-bundle on X corresponding to b and $\mathscr{E}_0$ is the trivial G-bundle. If we let $\mathrm{Sht}(G,b,\mu)_\infty$ be the space of such modifications then its cohomology valued in the sheaf attached to μ by geometric Satake determines a complex $R\Gamma_c(G,b,\mu)$ of $W_{\mathbb{Q}_p} \times G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$-modules. We consider the isotypic part $R\Gamma_c(G,b,\mu)[\pi]$ with respect to π. The leftover $W_{\mathbb{Q}_p}$-action on $R\Gamma_c(G,b,\mu)[\pi]$ for varying $b \in B(G,\mu)$ computes the $W_{\mathbb{Q}_p}$ action on $T_\mu(\mathscr{F}_\pi)$.

The complex $R\Gamma_c(G,b,\mu)[\pi]$ looks visually similar to the complex $R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p},\overline{\mathbb{Q}}_\ell)[\pi]$ described above. The Shimura varieties $\mathscr{S}(\mathbf{G},X)_K$ are certain moduli spaces of abelian varieties over $\mathbb{C}_p$. To such an abelian variety, one can attach a p-adic Hodge filtration, a linear algebraic datum capturing information about the variety. It was observed by Scholze-Weinstein [SW13; SW20a] that this linear algebraic datum can actually be described by modifications of the form described above. This allows one to show that, if μ is the minuscule cocharacter of G associated to the X defining the Shimura datum then the complex $R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p},\overline{\mathbb{Q}}_\ell)[\pi]$ is in fact built from the complexes $R\Gamma_c(G,b,\mu)[\pi]$ and the cohomology of certain perfect schemes Ig^b called Igusa varieties, for $b \in B(G,\mu)$ varying. This builds the required bridge between the two correspondences, and suggests that, by describing the Galois action on the complex $R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p},\overline{\mathbb{Q}}_\ell)[\pi]$, one can show an equality $r_\mu \circ \phi_\pi^{\mathrm{FS}} = r_\mu \circ \phi_\pi^{\mathrm{ss}}$, and, in certain cases, this will imply an equality $\phi_\pi^{\mathrm{FS}} = \phi_\pi^{\mathrm{ss}}$ as desired.

For GL_n and its inner forms, the local Langlands correspondence was first constructed in full generality by Harris-Taylor [HT01], and here the desired compatibility result was shown by Fargues-Scholze [FS21] and Hansen-Kaletha-Weinstein [HKW22], where it essentially reduces to the work of Harris-Taylor describing the cohomology groups $R\Gamma_c(G,b,\mu)[\pi]$ in terms of LLC_G. The case of a general group G introduces several new difficulties, related to the phenomenon of endoscopy, which can be thought of as a measure of the failure of LLC_G to be a bijection. In the first chapter, we overcome these difficulties in the particular case of the group GSp_4 and its inner form. Namely, by carrying out the strategy sketched above, we show that the Fargues-Scholze correspondence is the semi-simplification of the local Langlands correspondence constructed by Chan-Gan-Takeda-Tantono [GT11; GT14; CG15] (Theorem 1.1.2).

Such compatibility results allow one to combine the geometric constructions of the Fargues-Scholze construction with the more detailed knowledge known about classical instances of the local Langlands correspondence to describe the cohomology of global Shimura varieties. The point is again that the complexes $R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p},\overline{\mathbb{Q}}_\ell)[\pi]$ are built from the complexes $R\Gamma_c(G,b,\mu)[\pi]$ for minuscule μ, and the cohomology of the perfect schemes Ig^b. The second main goal of this thesis is to use compatibility results to compute $R\Gamma_c(G,b,\mu)[\pi]$ as explicitly as possible in terms of LLC_G and deduce new consequences for $R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p},\overline{\mathbb{Q}}_\ell)[\pi]$. To illustrate this, one of the main open conjectures on the complexes $R\Gamma_c(G,b,\mu)[\pi]$ is the Kottwitz Conjecture. This claims that if ϕ_π is supercuspidal in the sense that the semi-simplification ϕ_π^{ss} does not factor through $\hat{M}$, for M a proper Levi subgroup of G, then complex $R\Gamma_c(G,b,\mu)[\pi]$ should be concentrated in middle degree and be expressible in terms of $r_\mu \circ \phi_\pi^{\mathrm{ss}}$ pairing with representations of $J_b(\mathbb{Q}_p)$ according to character identities that LLC_G should always satisfy. The complexes $R\Gamma_c(G,b,\mu)[\pi]$ can be studied from the point of view of the Hecke operator $T_\mu(\mathscr{F}_\pi)$, which lives in the world of the geometric Langlands correspondence. Using this dictionary, Fargues [Far16] explained that the Kottwitz Conjecture would follow from constructing a perverse sheaf $\mathscr{S}_\phi$ on Bun_G expressible in terms of $\pi \in \Pi(G)$ such that $\phi_\pi = \phi$ under LLC_G for fixed supercuspidal ϕ. Namely, he considers the sheaf

$$\mathscr{S}_\phi := \bigoplus_{b \in B(G)_{basic}} \bigoplus_{\pi \in \Pi_\phi(J_b)} j_{b!}(\pi) \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell),$$

where $B(G)_{basic}$ are the elements such that J_b is an inner form of G (not just a Levi), and $\Pi_\phi(J_b)$ are the fibers of LLC_G over ϕ. The key insight was that the Kottwitz conjecture would follow from showing $\mathscr{S}_\phi$ is an eigensheaf with eigenvalue ϕ in the sense that, for all dominant cocharacters μ of G, one has an isomorphism: $T_\mu(\mathscr{S}_\phi) \simeq \mathscr{S}_\phi \boxtimes r_\mu \circ \phi^{\mathrm{ss}}$.

Moving to this new perspective of eigensheaves allows us to use new tools from geometric Langlands to study the complex $R\Gamma_c(G,b,\mu)[\pi]$. For example, the geometric Langlands program gives a recipe for the eigensheaf $\mathscr{S}_\phi$ in terms of what is called the spectral action. In the case that ϕ is a supercuspidal parameter, the spectral action can be made fairly explicit. If one combines this explicit description with the compatibility results, such as the one we prove for GSp_4, then one can fully compute $\mathscr{S}_\phi$, and the Kottwitz conjecture follows. Using this perspective, we conclude the first chapter by showing that a version of the Kottwitz Conjecture holds for GSp_4 (Theorem 1.8.2), which establishes this Conjecture in a new case.

0.2 Chapter 2

In the previous section, we discussed how, by using that the Fargues-Scholze local Langlands correspondence agrees with the semi-simplification of more classical instances of the correspondence, we could, for a supercuspidal parameter ϕ, explicitly compute eigensheaves $\mathscr{S}_\phi$ with eigenvalue ϕ^{ss}. In the second chapter of this thesis, we expand on this paradigm. Namely, we consider a maximal torus and Borel $T \subset B \subset G$, respectively, and let $\phi \in \Phi(G)$ be an L-parameter whose semisimplification ϕ^{ss} is induced from a toral parameter $\phi_T : W_{\mathbb{Q}_p} \to \hat{T}(\overline{\mathbb{Q}}_\ell)$. Following Fargues' vision, we might hope that, to such a parameter, we can attach a perverse Hecke eigensheaf $\mathscr{S}_\phi$ with eigenvalue ϕ^{ss}. Moreover, we expect that the stalks of this eigensheaf at the HN-strata $j_b : \mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$ are given by smooth representations of $J_b(\mathbb{Q}_p)$ whose L-parameter under LLC_G is ϕ, and that the eigensheaf property describes how representations π with parameter ϕ under LLC_G contribute to the cohomology of the complexes $R\Gamma_c(G,b,\mu)$.

Since we are assuming that ϕ^{ss} factors through a maximal torus T, a naive guess is that the sought after perverse eigensheaf $\mathscr{S}_\phi$ can only be supported on the $b \in B(G)$ such that that J_b must be quasi-split with Borel B_b and that $\mathscr{S}_\phi|_{\mathrm{Bun}_G^b}$ is valued in sub-quotients of the normalized parbaolic induction $i_{B_b}^{J_b}(\chi)$, where χ is the character attached to the toral parameter ϕ_T via local class field theory. The elements $b \in B(G)$ for which J_b is quasi-split are the set of unramified elements $B(G)_{\mathrm{un}} := \mathrm{Im}(B(T) \to B(G))$, as studied in Xiao-Zhu [XZ17]. Thinking through this more carefully, one is lead to consider the perverse sheaf

$$\mathscr{S}_\phi := \bigoplus_{b \in B(G)_{\mathrm{un}}} \bigoplus_{w \in W_b} j_{b!}(\rho_{b,w})[-\langle 2\rho_G, \nu_b \rangle] \tag{1}$$

on Bun_G, where $M_b \simeq J_b$ is the centralizer of the slope homorphism of b, $W_b := W_G/W_{M_b}$ is a quotient of Weyl groups identified with a set of representatives in W_G of minimal length, $\rho_{b,w} := i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{-1/2}$, and δ_{P_b} is the modulus character of the parabolic P_b with Levi factor M_b transferred to J_b. Here we note that the shifts by $-\langle 2\rho_G, \nu_b \rangle$ are equal to the dimension of the ℓ-adically contractible unipotent part appearing in the strata Bun_G^b and in particular make the sheaf $\mathscr{S}_\phi$ perverse. We formulate the following naive conjecture.

Conjecture 0.2.1. *(Naive) For $\phi \in \Phi(G)$ an L-parameter such that ϕ^{ss} is induced from a toral parameter $\phi_T : W_{\mathbb{Q}_p} \to \hat{T}(\overline{\mathbb{Q}}_\ell)$, the sheaf $\mathscr{S}_\phi$ defined by (1) is an eigensheaf with eigenvalue ϕ^{ss}.*

Unfortunately, this is too naive; in particular, there can exist representations of non quasi-split groups, whose L-parameter under LLC_G factors through a maximal torus T after semi-simplification (e.g the trivial representation of $D^*_{\frac{1}{2}}$, units in the quaternion division algebra). These representations should appear in the stalks of the eigensheaf $\mathscr{S}_\phi$; so, for our conjecture to have a chance of being true, we impose the following condition on ϕ_T.

Definition 0.2.2. We say ϕ_T is *generic* if, for all coroots α, the character $\alpha \circ \phi_T$ of $W_{\mathbb{Q}_p}$ is not isomorphic to the trivial representation or the norm character $|\cdot|$; equivalently, this holds if the Galois cohomology $R\Gamma(W_{\mathbb{Q}_p}, \alpha \circ \phi_T)$ is trivial for all coroots α.

This implies that $\phi|_{W_{\mathbb{Q}_p}} = \phi^{\mathrm{ss}}$ (i.e the parameter ϕ has no non-trivial monodromy), and, assuming the Fargues-Scholze local Langlands correspondence satisfies certain expected properties, one can show that the stalks of an eigensheaf with eigenvalue ϕ^{ss} can only be of the form described above. The main theorem of chapter 2 is as follows.

Theorem 0.2.3. *(Theorem 2.10.10) If ϕ^{ss} is induced from a generic toral parameter ϕ_T then, assuming certain properties of the Fargues-Scholze local Langlands correspondence (Assumption 2.7.5), and possible additional constraints on ϕ_T, Conjecture 0.2.1 is true.*

To see why this could be true, we consider the diagram of spaces

$$\begin{tikzcd} \mathrm{Bun}_B \arrow[r, "j"] \arrow[d, "\mathfrak{q}"] \arrow[dr, "\mathfrak{p}"] & \overline{\mathrm{Bun}}_B \arrow[d, "\overline{\mathfrak{p}}"] \\ \mathrm{Bun}_T & \mathrm{Bun}_G \end{tikzcd},$$

where Bun_B is the moduli space parameterizing B-structures on G-bundles and $\overline{\mathrm{Bun}}_B$ is a compactification of Bun_B. Using this diagram, one can define a sheaf $\mathrm{nEis}_B(\mathscr{S}_{\phi_T}) \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ by pulling back along $\mathfrak{q}$ and taking ! push-forwards along $\mathfrak{p}$. Here $\mathscr{S}_{\phi_T}$ is the eigensheaf attached to ϕ_T by geometric local class field theory [Zou22].

Following work of Braverman-Gaitsgory-Laumon [BG02; Lau90] in classical geometric Langlands, the true candidate should be given by a sheaf $\overline{\mathrm{Eis}}_B(\mathscr{S}_{\phi_T})$, defined by ! push-forwarding along $\overline{\mathfrak{p}}$ instead of $\mathfrak{p}$. Unfortunately, to obtain the correct definition one needs to tensor by a kernel sheaf $\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$, the intersection cohomology of the Drinfeld compactification, and, in the geometric context we are working in, defining this sheaf and showing it has good properties is a very difficult problem.

Nevertheless, in classical geometric Langlands, there exists a map

$$\mathrm{nEis}_B(\mathscr{S}_{\phi_T}) \to \overline{\mathrm{Eis}}_B(\mathscr{S}_{\phi_T})$$

which should be an isomorphism for ϕ_T generic, essentially because the Galois cohomogy groups $R\Gamma(W_{\mathbb{Q}_p}, \alpha \circ \phi_T)$ for coroots α appear in the cone of this map and are killed by the generic assumption. This suggests that the sheaf $\mathrm{nEis}_B(\mathscr{S}_{\phi_T}) \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$, which is computable and understandable in the Fargues-Scholze setting, should be the sought after eigensheaf, at least when ϕ_T is generic. It also suggests that, assuming ϕ_T is generic, the sheaf $\mathrm{nEis}_B(\mathscr{S}_{\phi_T})$ should satisfy the same good properties that $\overline{\mathrm{Eis}}_B(\mathscr{S}_{\phi_T})$ does classically. Namely, it should satisfy a functional equation with respect to the action of the Weyl group ([BG02, Theorem 2.24]), and behave well under Verdier duality on Bun_G. If these two properties were to hold for $\mathrm{nEis}_B(\mathscr{S}_{\phi_T})$ it would immediately imply Conjecture 0.2.1. The point is that Bun_B and Bun_T admit a decomposition into connected components

$$\mathrm{Bun}_B := \bigsqcup_{\nu \in B(T)} \mathrm{Bun}_B^\nu$$

$$\mathrm{Bun}_T := \bigsqcup_{\nu \in B(T)} \mathrm{Bun}_T^\nu,$$

which gives a decomposition

$$\mathrm{nEis}_B(\mathscr{S}_{\phi_T}) := \bigoplus_{\nu \in B(T)} \mathrm{nEis}_B^\nu(\mathscr{S}_{\phi_T})$$

of the geometric Eisenstein series. Given $b \in B(G)_{\mathrm{un}}$, there exists a unique element with G-dominant slopes $b_T \in i^{-1}(b)$ for $i : B(T) \to B(G)$ the natural map.

The connected component $\mathrm{Bun}_B^{b_T}$ essentially parametrizes split B-bundles, and therefore it is relatively easy to show that

$$\mathrm{nEis}_B^{b_T}(\mathscr{S}_{\phi_T}) \simeq j_{b!}(\rho_{b,1})[-\langle 2\rho_G, \nu_b\rangle].$$

Now the Weyl group W_G acts on $B(T) \simeq \mathbb{X}_*(T)$, and this acts transitively on the fiber $i^{-1}(b)$ with stabilizer W_{M_b}. Therefore, the fiber is parameterized by the Weyl group $W_b = W_G/W_{M_b}$ appearing above, and thus, the functional equation holding for $\mathrm{nEis}_B(\mathscr{S}_{\phi_T})$ under the generic assumption on ϕ_T implies that

$$\bigoplus_{w \in W_b} \mathrm{nEis}_B^{w(b_T)}(\mathscr{S}_{\phi_T}) \simeq \bigoplus_{w \in W_b} j_{b!}(\rho_{b,w})[-\langle 2\rho_G, \nu_b\rangle]$$

which varying over all elements $b \in B(G)_{\mathrm{un}}$ gives us conjecture 0.2.1.

The eigensheaf property holding for $\mathscr{S}_\phi$, as in equation (1), implies the following formula for the cohomology of local Shimura varieties/shtuka spaces

$$\bigoplus_{b \in B(G,\mu)_{\mathrm{un}}} \bigoplus_{w \in W_b} R\Gamma_c(G,b,\mu)[\rho_{b,w} \otimes \delta_{P_b}][\langle 2\rho_G, \nu_b\rangle] \simeq i_B^G(\chi) \otimes r_\mu \circ \phi^{\mathrm{ss}}.$$

The first sanity check that this is reasonable is that it is compatible with more classical work of Shin [Shi12] (See Appendix 2.11.11) after passing to the Grothendieck group of $G(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$-representations, usually obtained by stabilizing the trace formula on Igusa varieties. To give a deeper sense for what this is saying, we note that there is a bijection

$$B(G,\mu)_{\mathrm{un}} \leftrightarrow \{\text{Weyl group orbits of weights in } V_\mu\}. \tag{2}$$

In particular, this suggests a natural conjecture matching the summands $\bigoplus_{b \in B(G,\mu)_{\mathrm{un}}} \bigoplus_{w \in W_b} R\Gamma_c(G,b,\mu)[\rho_{b,w} \otimes \delta_{P_b}]$ appearing on the LHS with the summands appearing on the RHS: $i_B^G(\chi) \otimes r_\mu \circ \phi^{\mathrm{ss}} \simeq i_B^G(\chi) \otimes \bigoplus_{\nu \in \mathbb{X}_*(T)} \nu \circ \phi_T \otimes V_\mu(\nu)$, where $V_\mu(\nu)$ denotes the multiplicity of the representation of $\hat{T}$ defined by ν in $V_\mu|_{\hat{T}}$ (Conjecture 2.11.18). It is fairly easy to verify this conjecture for the Weyl group orbit of the highest weight, which corresponds to the maximal element $b_\mu \in B(G,\mu)_{\mathrm{un}}$ under the natural partial ordering on $B(G)$. In this case, it is easy to compute the contribution of the summands indexed by b_μ, using work of Boyer [Boy99a] (or rather it's generalization considered in [GI16]). In particular, this tells us that the summands on the LHS coming from b_μ must be of the form $i_B^G(\chi^w)$ for $w \in W_G$, and it thereby follows that one must have an intertwiner

$i_B^G(\chi^w) \simeq i_B^G(\chi)$ for this formula to hold. Such an isomorphism does not always exist, but will exist under the generic hypothesis on ϕ_T (Proposition A.1.3). In other words, our geometric functional equation for $\mathrm{nEis}_B(\mathscr{S}_{\phi_T})$ implies the usual "functional equation" for the principal series representations $i_B^G(\chi)$.

While this is very conceptually satisfying, since every weight of V_μ for μ minuscule is a Weyl group orbit of the highest weight, the previous calculation essentially tells us that we haven't really said anything new about the local Shtuka spaces which uniformize global Shimura varieties. However, this changes if we drop the assumption that G is split and just assume that G is quasi-split. In this case, we have an isomorphism $B(T) \simeq \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma \simeq \mathbb{X}^*(\hat{T}^\Gamma)$ where $\Gamma := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, and the same description of $\mathrm{nEis}_B(\mathscr{S}_{\phi_T})$ holds, but we need to modify the bijection (2) to become

$$B(G,\mu)_{\mathrm{un}} \leftrightarrow \{\text{Weyl group orbits of weights in } V_\mu|_{\hat{G}^\Gamma}\}.$$

In particular, even if μ is minuscule, the restriction $V_\mu|_{\hat{G}^\Gamma}$ might not be a minuscule representation of $\hat{G}^\Gamma$. For example, in the case that G is a unitary group or a restriction of scalars of a split group. In this case, we can have that the basic element $b \in B(G,\mu)_{\mathrm{un}}$ is unramified. If b is basic then $J_b \simeq G$ under the inner twisting and the above formula suggests that the complex $R\Gamma_c(G,b,\mu)[i_B^G(\chi)]$ is concentrated in degree 0 (= middle degree under the non-perverse normalization), with Weil group aciton specified by the central weight space of $V_\mu|_{\hat{G}^\Gamma}$. This situation where the basic element is unramified is the one considered by Xiao-Zhu [XZ17]. In particular, they show that the central weight space of $V_\mu|_{\hat{G}^\Gamma}$, describes the irreducible constituents of affine Deligne-Lusztig varieties, and this is precisely the special fiber of the natural integral model of the shtuka space $\mathrm{Sht}(G,b,\mu)_\infty$ at hyperspecial level, for b the basic element.

0.3 Chapter 3

Given a L-parameter $\phi \in \Phi(G)$ such that the semi-simplification ϕ^{ss} is induced from a toral parameter $\phi_T : W_{\mathbb{Q}_p} \to \hat{T}(\overline{\mathbb{Q}}_\ell)$ we saw how, assuming the Fargues-Scholze correspondence behaves reasonably (i.e by showing a local-global compatibility result, as in Section 1), we could construct an eigensheaf using geometric Eisenstein series which captures how representations ρ_b of J_b for varying $b \in B(G)$ with L-parameter ϕ contribute to the cohomology of local Shtuka spaces, assuming that ϕ_T is generic. In the previous chapters we worked with rational coefficients for the sake of simplicity, but the results we discussed in the previous section work just as well in the coefficient systems $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell\}$, assuming that the prime ℓ is very good with respect to G in the sense of [FS21, Page 33]. In particular, given a toral parameter ϕ_T we say that it is generic if, for all coroots $\alpha \in \mathbb{X}_*(T)$, the character $\alpha \circ \phi_T$ of $W_{\mathbb{Q}_p}$ is not the trivial or norm character as before, and under this condition we prove analogous results to those discussed in the previous section. Similarly, one can formulate the condition for a general quasi-split group by saying that the complex $R\Gamma(W_{\mathbb{Q}_p}, \alpha \circ \phi_T)$ is trivial for all Γ-orbits of coroots α in $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ for a maximal (not necessarily split) torus $T \subset G$. Here α denotes the corresponding representation of ${}^L T$.

Assume that G is an unramified reductive group, and let $K_p^{\mathrm{hs}} \subset G(\mathbb{Q}_p)$ denote a hyperspecial subgroup. We let

$$H_{K_p^{\mathrm{hs}}} := \overline{\mathbb{F}}_\ell[K_p^{\mathrm{hs}} \backslash G(\mathbb{Q}_p)/K_p^{\mathrm{hs}}]$$

be the spherical Hecke algebra with $\overline{\mathbb{F}}_\ell$-coefficients. If we fix a maximal ideal $\mathfrak{m} \subset H_{K_p^{\mathrm{hs}}}$ then this defines for us an unramified semi-simple L-parameter $\phi_{\mathfrak{m}} : W_{\mathbb{Q}_p} \to \hat{G}(\overline{\mathbb{F}}_\ell)$ which factors through a parameter $\phi_{\mathfrak{m}}^T : W_{\mathbb{Q}_p} \to \hat{T}(\overline{\mathbb{F}}_\ell)$. It is easy to check that $\mathfrak{m}$ is a decomposed generic maximal ideal in the sense of Caraiani-Shcolze ([CS17, Definition 1.9]) if and only if $\phi_{\mathfrak{m}}^T$ is generic in our sense. Conisder now $(\mathbf{G}, X)$ a Shimura datum with $\mathbf{G}/\mathbb{Q}$ a global group such that $\mathbf{G}_{\mathbb{Q}_p} =: G$ is unramified and $\mathfrak{m} \subset H_{K_p^{\mathrm{hs}}}$ a generic maximal ideal (i.e $\phi_{\mathfrak{m}}^T$ is generic in the above sense). We consider the cohomology

$$R\Gamma_c(\mathscr{S}(\mathbf{G}, X)_{K^p K_p^{\mathrm{hs}}}, \overline{\mathbb{F}}_\ell)$$

for some sufficiently small level $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$ away from p, and look at the localization

$$R\Gamma_c(\mathscr{S}(\mathbf{G}, X)_{K^p K_p^{\mathrm{hs}}}, \overline{\mathbb{F}}_\ell)_{\mathfrak{m}}.$$

We have the following conjecture motivated by the torsion vanishing results of [CS17; CS19; Kos21b; San23].

Conjecture 0.3.1. *(Conjecture 3.1.2) Let $(\mathbf{G}, X)$ be a Shimura datum such that $G = \mathbf{G}_{\mathbb{Q}_p}$ is unramified and $K = K_p K^p \subset \mathbf{G}(\mathbb{A}_f)$ is a sufficiently small level with $K_p = K_p^{\mathrm{hs}}$ hyperspecial. If $\mathfrak{m} \subset H_{K_p}^{\mathrm{hs}}$ is a generic maximal ideal in the above sense then the cohomology of $R\Gamma(\mathscr{S}(\mathbf{G}, X)_K, \overline{\mathbb{F}}_\ell)_{\mathfrak{m}}$ (resp. $R\Gamma_c(\mathscr{S}(\mathbf{G}, X)_K, \overline{\mathbb{F}}_\ell)_{\mathfrak{m}}$) is concentrated in degrees $d \leq i \leq 2d$ (resp. $0 \leq i \leq d$), where $d := \dim(\mathscr{S}(\mathbf{G}, X)_K)$.*

In the final chapter of this thesis we discuss some results of joint work in progress with Si-Ying Lee [HL23], where we prove the above conjecture for some PEL type Shimrua varieties of type A or C such that G is centrally isogenous to groups given by products of $\mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GL}_n)$ for $L/\mathbb{Q}_p$ unramified, $\mathrm{GU}_n(E/\mathbb{Q}_p)$ for $E/\mathbb{Q}_p$ unramified and n odd, and $\mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GSp}_4)$ for $L/\mathbb{Q}_p$ unramified (Theorem 3.1.12). The key point being that these are the cases where we can compare the Fargues-Scholze correspondence to more classical instances of Langlands, and thereby invoke the results of chapter 2.

The heart of our method for proving these results relies on a technique of Koshikawa [Kos21b]. In fact, his argument for proving torsion vanishing was part of the inspiration for the ideas discussed in chapter 2. The point is, as before, that the cohomology of $R\Gamma_c(\mathscr{S}(\mathbf{G},X)_K,\overline{\mathbb{F}}_\ell)_\mathfrak{m}$ should be built from the complexes $R\Gamma_c(G,b,\mu)$ and the cohomology of the Igusa varieties Ig^b for $b \in B(G,\mu)$ varying. What Koshikawa observed was, assuming the Fargues-Schole correspondence behaves reasonably with $\overline{\mathbb{Q}}_\ell$-coefficients, that only the elements lying in $B(G,\mu)_{\mathrm{un}} := B(G,\mu) \cap B(G)_{\mathrm{un}}$ contribute to this generic localization. In the previous cases where torsion vanishing results have been proven, it is always the case that the local group G is split, and so, as observed above, the set $B(G,\mu)_{\mathrm{un}}$ is a singleton consisting of the μ-ordinary element b_μ. The shtuka space $R\Gamma_c(G,b_\mu,\mu)$ is rather pathological in this case, and the problem entirely reduces to controlling the cohomology of the μ-ordinary Igusa variety Ig^{b_μ}, which can be controlled through Artin vanishing in the case that the Shimura variety is compact or through some kind of semi-perversity result in the case that the Shimura variety is not compact. However, in the cases we consider, where the group is non-split (e.g non-trivial restrictions of scalars and odd unitary groups), the set $B(G,\mu)_{\mathrm{un}}$ can have multiple elements, and for these $b \in B(G,\mu)_{\mathrm{un}}$ the complex $R\Gamma_c(G,b,\mu)$ is no longer pathological, and one needs to control the degrees of cohomology that representations with L-parameter $\phi_\mathfrak{m}$ contribute to this complex. However, as seen at the end of the previous section, such control is supplied by the theory of geometric Eisenstein series, and this allows us to prove new cases of Conjecture 3.1.2.

The strategy we provide for proving these torsion vanishing results is quite flexible. In particular, upon showing that the Fargues-Scholze correspondence for G behaves like usual instances of the correspondence, one should be able to deduce some form of Conjecture 0.3.1 from the theory of Geometric Eisenstein series developed in chapter 2. It also suggests a wider class of generalizations of Conjecture 0.3.1, which we discuss at the end of chapter 3. To formulate this properly, for a general Shimura datum $(\mathbf{G},X)$, we consider the $G(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$-

representation

$$R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p},\overline{\mathbb{F}}_\ell).$$

In the appendix, we construct, using the spectral action, a $G(\mathbb{Q}_p)\times W_{\mathbb{Q}_p}$-equivariant decomposition of this complex

$$R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p},\overline{\mathbb{F}}_\ell)\simeq\bigoplus_{\phi\in\Phi^{\mathrm{ss}}(G)}R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p},\overline{\mathbb{F}}_\ell)_\phi \tag{3}$$

ranging over semi-simple L-parameters, and satisfying the property that, any representation occurring in the RHS, has Fargues-Scholze parameter equal to ϕ as conjugacy classes of parameters (Corollary B.1.8). By passing to K_p^{hs}-invariants, the previous conjecture can be interpreted as saying that the summand corresponding to the parameters $\phi_{\mathfrak{m}}:W_{\mathbb{Q}_p}\to\hat{G}(\overline{\mathbb{F}}_\ell)$ coming from generic maximal ideals $\mathfrak{m}$ are concentrated in degrees $0\leq i\leq d$. It is now natural to ask for a general condition on a semi-simple L-parameter ϕ that would guarantee this to be true. If ϕ were supercuspidal then this essentially reduces to the Kottwitz conjecture proven in the first chapter, since the non-basic strata will not contribute to the cohomology of the global Shimura variety localized at a supercuspidal parameter ϕ (using compatibility). Moreover, the Igusa variety in this case will essentially just be a profinite set (cf. Definition 1.4.1). Assume now that ϕ is induced from a supercuspidal L-parameter $\phi_M:W_{\mathbb{Q}_p}\to\hat{M}(\overline{\mathbb{F}}_\ell)$ factoring through the dual group of a proper Levi subgroup M of G. We call the pair (M,ϕ_M) a cuspidal support for the parameter ϕ. It is natural to ask for a generalization of the generic condition in the case that $M=T$. As discussed in the previous section, one of the motivations for this condition in the toral case was that this should be the correct condition guaranteeing that the geometric Eisenstein series $\mathrm{nEis}_B(\mathscr{S}_{\phi_T})$ agrees with the true candidate $\overline{\mathrm{nEis}}_B(\mathscr{S}_{\phi_T})$ for the Hecke Eigensheaf with eigenvalue ϕ. In particular, in classical geometric Langlands, this should happen precisely when ϕ_T is generic. Similarly, given an eigensheaf $\mathscr{S}_{\phi_M}$ on Bun_M with supercuspidal eigenvalue ϕ_M, we can form the analogue of the geometric Eisenstein functor $\mathrm{nEis}_P(\mathscr{S}_{\phi_M})$ considered in chapter 2 for a parabolic P with Levi factor M, by using the analogous diagram

$$\begin{tikzcd} \mathrm{Bun}_P \arrow[r, "j_P"] \arrow[d, "\mathfrak{q}_P"] \arrow[dr, "\mathfrak{p}_P"] & \widetilde{\mathrm{Bun}}_P \arrow[d, "\widetilde{\mathfrak{p}}_P"] \\ \mathrm{Bun}_M & \mathrm{Bun}_G \end{tikzcd},$$

and pulling back along $\mathfrak{q}_P$ and taking !-pushforwards along $\mathfrak{p}_P$. Here $\widetilde{\mathfrak{p}}_P:\widetilde{\mathrm{Bun}}_P\to\mathrm{Bun}_G$ is a compactification of $\mathfrak{p}_P$ generalizing $\overline{\mathrm{Bun}}_B$, and using this one should be able to define an analogous sheaf $\widetilde{\mathrm{nEis}}_P(\mathscr{S}_{\phi_M})$ as in classical geometric Langlands. This should be the true candidate for the eigensheaf, and there should be a natural map

$$\mathrm{nEis}_P(\mathscr{S}_{\phi_M}) \to \widetilde{\mathrm{nEis}}_P(\mathscr{S}_{\phi_M}).$$

We can now ask what is the correct condition on ϕ_M guaranteeing that this map is an isomorphism, by consdering the geometry of the compactification $\widetilde{\mathrm{Bun}}_P$. In chapter 3, we explain this, and are lead to the following generalization of the generic condition.

Definition 0.3.2. (Definition 3.2.5) Let ϕ be a semi-simple L-parameter with cuspidal support (M, ϕ_M). We consider the representation V_{ad}^N of $\hat{M}$ given by looking at the adjoint action of $\hat{M}$ on $\hat{N}$ the dual of the unipotent radical of P, and write $r_{\mathrm{ad}}^N : \hat{M} \to \mathrm{GL}(V_{\mathrm{ad}}^N)$ for the corresponding map. We say that ϕ is of Langlands-Shahidi type if the Galois cohomology complexes

$$R\Gamma(W_{\mathbb{Q}_p}, r_{\mathrm{ad}}^N \circ \phi_M)$$

and

$$R\Gamma(W_{\mathbb{Q}_p}, r_{\mathrm{ad}}^N \circ \phi_M^{\vee})$$

are trivial, where $(-)^{\vee}$ denotes the dual.

With this condition pinned down, we formulate conjectural generalizations of the results on principal geometric Eisenstein series obtained in chapter 2 in the non-principal case. Since these results are the key input used in the paper [HL23] on torsion vanishing, these conjectures lead us to the following very general Conjecture on the structure of the torsion of global Shimura varieties.

Conjecture 0.3.3. *If ϕ is a semi-simple L-parameter of Langlands-Shahidi type then the summand*

$$R\Gamma_c(\mathscr{S}(\mathbf{G}, X)_{K^p}, \overline{\mathbb{F}}_\ell)_\phi,$$

appearing in the decomposition (3), is concentrated in degrees $0 \leq i \leq d$.

Chapter 1

Local-Global Compatibility of the Fargues-Scholze Local Langlands Correspondence

1.1 Introduction

1.1.1 Background and Main Theorems

Fix distinct primes $\ell \neq p$, let $\mathbb{Q}_p$ denote the p-adic numbers, and let $G/\mathbb{Q}_p$ be a connected reductive group. Set $\mathbb{C}_p := \hat{\overline{\mathbb{Q}}}_p$ to be the completion of the algebraic closure of $\mathbb{Q}_p$. We fix an isomorphism $i : \overline{\mathbb{Q}}_\ell \xrightarrow{\simeq} \mathbb{C}$. Let $W_{\mathbb{Q}_p}$ be the Weil group of $\mathbb{Q}_p$ and set $\hat{G}$ to be the reductive group over $\overline{\mathbb{Q}}_\ell$ with root datum dual to G. Let Q be the finite quotient through which $W_{\mathbb{Q}_p}$ acts on $\hat{G}$. We define the L-group ${}^LG := Q \ltimes \hat{G}$. We let $\Pi(G)$ denote the set of isomorphism classes of smooth irreducible representations of the p-adic group $G(\mathbb{Q}_p)$, and let $\Phi(G)$ denote the set of L-parameters, i.e the set of conjugacy classes of homomorphisms

$$\phi : W_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG(\mathbb{C})$$

where $\mathrm{SL}_2(\mathbb{C})$ acts via an algebraic representation and $W_{\mathbb{Q}_p}$ acts via a continuous semisimple homomorphism in a way that commutes with the natural projection ${}^LG(\mathbb{C}) \to Q$, where ${}^LG(\mathbb{C})$ is endowed with the discrete topology. The local Langlands correspondence is a conjectural map

$$\mathrm{LLC}_G : \Pi(G) \to \Phi(G)$$

$$\pi \mapsto \phi_\pi$$

that builds a bridge between L-parameters and the smooth irreducible representations of $G(\mathbb{Q}_p)$. Conjecturally (under some additional constraints on $\Phi(G)$ if G is not split), these maps should be surjective with finite fibers called L-packets and satisfy various properties such as compatibility with products, maps of L-groups, character twists, as well as L, ε, and γ-factors. Moreover, one expects that the correspondence is uniquely characterized by some such finite list of properties.

In general, the existence and uniqueness of such a correspondence is completely unknown. However, very recently, Fargues and Scholze [FS21], using the action of the excursion algebra on the moduli space of G-bundles on the Fargues-Fontaine curve, were able to construct a completely general candidate, analogous to the work of V. Lafforgue in the function field setting [Laf18]. Namely, they construct a map

$$\mathrm{LLC}_G^{\mathrm{FS}} : \Pi(G) \to \Phi^{\mathrm{ss}}(G)$$

$$\pi \mapsto \phi_\pi^{\mathrm{FS}}$$

where $\Phi^{\mathrm{ss}}(G)$ denotes the set of conjugacy classes of continuous semisimple maps

$$\phi : W_{\mathbb{Q}_p} \to {}^L G(\overline{\mathbb{Q}}_\ell)$$

that commute with the projection ${}^L G(\overline{\mathbb{Q}}_\ell) \to Q$ as above. Fargues and Scholze showed that their map has several good properties such as compatibility with parabolic induction; however, one would also like to check that this correspondence agrees with known instances of the local Langlands correspondence. Precisely, given a candidate for the local Langlands correspondence

$$\mathrm{LLC}_G : \Pi(G) \to \Phi(G)$$

$$\pi \mapsto \phi_\pi$$

we expect a commutative diagram of the form

$$\begin{tikzcd}
\Pi(G) \arrow[r, "\mathrm{LLC}_G"] \arrow[dr, "\mathrm{LLC}_G^{\mathrm{FS}}"'] & \Phi(G) \arrow[d, "(-)^{\mathrm{ss}}"] \\
& \Phi^{\mathrm{ss}}(G)
\end{tikzcd}$$

where the semisimplification map $(-)^{\mathrm{ss}}$ precomposes an L-parameter $\phi \in \Phi(G)$ with the map

$$g \in W_{\mathbb{Q}_p} \mapsto (g, \begin{pmatrix} |g|^{\frac{1}{2}} & 0 \\ 0 & |g|^{\frac{-1}{2}} \end{pmatrix}) \in W_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C})$$

and then applies the fixed isomorphism $i^{-1} : \mathbb{C} \xrightarrow{\simeq} \overline{\mathbb{Q}}_\ell$, where $|\cdot| : W_{\mathbb{Q}_p} \to W_{\mathbb{Q}_p}^{ab} \simeq \mathbb{Q}_p^* \to \mathbb{C}^*$ is the norm character. We make the following definition.

Definition 1.1.1. For $\pi \in \Pi(G)$, we say that a local Langlands correspondence LLC_G is compatible with the Fargues-Scholze local Langlands correspondence if we have an equality: $\phi_\pi^{\mathrm{FS}} = \phi_\pi^{\mathrm{ss}}$, as conjugacy classes of semi-simple L-parameters.

For GL_n, the local Langlands correspondence was constructed by Harris-Taylor/Henniart [He14; HT01] and is uniquely characterized by the preservation of L, ε, and γ-factors. In this case, compatibility with the Fargues-Scholze local Langlands correspondence follows from the description of the cohomology of the Lubin-Tate and Drinfeld towers proven in [HT01] and was verified by Fargues and Scholze [FS21, Theorem I.9.6]. The main goal of this note is to extend compatibility of the correspondence to GSp_4 and its inner form. To this end, we now fix a finite extension $L/\mathbb{Q}_p$ and set G to be $\mathrm{Res}_{L/\mathbb{Q}_p}\mathrm{GSp}_4$ and J to be its unique non-split inner form $\mathrm{Res}_{L/\mathbb{Q}_p}\mathrm{GU}_2(D)$, where D/L is the quaternion division algebra. In this case, the local Langlands correspondence has been constructed by Gan-Takeda and Gan-Tantono, respectively [GT11; GT14]. It is constructed from the local Langlands correspondence for GL_n and theta lifting and admits a similar unique characterization in terms of the preservation of L, ε, and γ factors. We note that we can and do identify $\Phi(G)$ and $\Phi(J)$ with a subset of homomorphisms:

$$\phi : W_L \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G}(\mathbb{C}) = \mathrm{GSpin}_5(\mathbb{C}) \simeq \mathrm{GSp}_4(\mathbb{C})$$

This allows us to introduce a bit of terminology. Namely, we say that a parameter ϕ in $\Phi(G)$ or $\Phi(J)$ is supercuspidal if the $\mathrm{SL}_2(\mathbb{C})$-factor acts trivially and ϕ does not factor through any proper Levi subgroup of GSp_4. This terminology is justified by the fact that this is precisely the case when the L-packets over ϕ contain only supercuspidal representations. In what follows, we will often abuse notation and drop the superscript $(-)^{\mathrm{ss}}$ when speaking about such parameters, as in this case it merely corresponds to forgetting the trivially acting $\mathrm{SL}_2(\mathbb{C})$-factor and applying the isomorphism i^{-1}. We now come to our main theorem.

Theorem 1.1.2. *The following is true.*

1. *For any $\pi \in \Pi(G)$ (resp. $\rho \in \Pi(J)$) such that the Gan-Takeda (resp. Gan-Tantono) parameter is not supercuspidal, we have that the Gan-Takeda (resp. Gan-Tantono) correspondence is compatible with the Fargues-Scholze correspondence.*

2. *If $L/\mathbb{Q}_p$ is unramified and $p > 2$, we have, for all $\pi \in \Pi(G)$ (resp. $\rho \in \Pi(J)$) such that the Gan-Takeda (resp. Gan-Tantono) parameter is supercuspidal, that the Gan-Takeda (resp. Gan-Tantono) correspondence is compatible with the Fargues-Scholze correspondence.*

Remark 1.1.3. As will be explained more below, the restrictions in the case where the parameter is supercuspidal are necessary to apply basic uniformization of the generic fiber of abelian type Shimura varieties due to Shen [She17]. If one were not to impose this assumption, the relevant Shimura varieties would have bad reduction at p, which, to the best of our knowledge, prevents the methods of Shen from working. In particular, if one could establish the expected description of basic locus in the sense of the isomorphism (2) of Definition 4.1, for Shimura varieties associated to the group $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}$, where $\mathbf{G}$ is an inner form of GSp_4 over a totally real field F with an inert prime p such that $F_p \simeq L$ for L any extension then our result would hold in complete generality.

As mentioned above, the proof of compatibility for GL_n uses the results of Harris-Taylor [HT01] on the cohomology of the Lubin-Tate/Drinfeld Towers. In particular, if one looks at the rigid generic fiber of the Lubin-Tate tower

$$\lim_{m\to\infty} \mathrm{LT}_{n,m,\breve{\mathbb{Q}}_p}$$

a tower of $n-1$-dimensional rigid spaces over $\breve{\mathbb{Q}}_p$, for fixed $n \geq 1$ and varying $m \geq 1$, where $\breve{\mathbb{Q}}_p$ denotes the completion of the maximal unramified extension of $\mathbb{Q}_p$. The cohomology of this tower

$$R\Gamma_c(\mathrm{LT}_{n,\infty}, \overline{\mathbb{Q}}_\ell) := \mathrm{colim}_{m\to\infty} R\Gamma_c(\mathrm{LT}_{n,m,\mathbb{C}_p}, \overline{\mathbb{Q}}_\ell)$$

based changed to $\mathbb{C}_p$ carries commuting actions of $\mathrm{GL}_n(\mathbb{Q}_p)$ and $D^*_{\frac{1}{n}}$, the units in the division algebra over $\mathbb{Q}_p$ of invariant $\frac{1}{n}$, as well as an action of the Weil group $W_{\mathbb{Q}_p}$. In particular, given $\pi \in \Pi(\mathrm{GL}_n)$ (resp. $\rho \in \Pi(D^*_{\frac{1}{n}})$), we can consider the complexes

$$R\Gamma_c(\mathrm{LT}_{n,\infty}, \overline{\mathbb{Q}}_\ell)[\pi] := R\Gamma_c(\mathrm{LT}_{n,\infty}, \overline{\mathbb{Q}}_\ell) \otimes^{\mathbb{L}}_{\mathscr{H}(\mathrm{GL}_n)} \pi$$

and

$$R\Gamma_c(\mathrm{LT}_{n,\infty},\overline{\mathbb{Q}}_\ell)[\rho] := R\Gamma_c(\mathrm{LT}_{n,\infty},\overline{\mathbb{Q}}_\ell)\otimes^{\mathbb{L}}_{\mathscr{H}(D^*_{\frac{1}{n}})}\rho$$

where $\mathscr{H}(\mathrm{GL}_n) := C_c^\infty(\mathrm{GL}_n(\mathbb{Q}_p),\overline{\mathbb{Q}}_\ell)$ (resp. $\mathscr{H}(D^*_{\frac{1}{n}})$) is the usual smooth Hecke algebra of G (resp. $D^*_{\frac{1}{n}}$). Then the key result of Harris-Taylor and later refined by Boyer and Dat [Boy99a; Dat05] is as follows.

Theorem 1.1.4. *[HT01; Boy99a; Dat05] Fix $\pi \in \Pi(\mathrm{GL}_n)$, a supercuspidal representation of $\mathrm{GL}_n(\mathbb{Q}_p)$, let*

$$\mathrm{JL} : \Pi(D^*_{\frac{1}{n}}) \to \Pi(\mathrm{GL}_n(\mathbb{Q}_p))$$

be the map defined by the Jacquet-Langlands correspondence and $\rho := \mathrm{JL}^{-1}(\pi) \in \Pi(D^*_{\frac{1}{n}})$ a Jacquet-Langlands lift of π. Then the complexes $R\Gamma_c(G,b,\mu)[\pi]$ and $R\Gamma_c(G,b,\mu)[\rho]$ are concentrated in middle degree $n-1$. The middle degree cohomology of $R\Gamma_c(G,b,\mu)[\pi]$ is isomorphic to

$$\rho \boxtimes \phi_\pi^\vee \otimes |\cdot|^{(1-n)/2}$$

as a $D^*_{\frac{1}{n}} \times W_{\mathbb{Q}_p}$ representation. Similarly, the middle degree cohomology of $R\Gamma_c(G,b,\mu)[\rho]$ is isomorphic to

$$\pi \boxtimes \phi_\pi \otimes |\cdot|^{(1-n)/2}$$

where $\phi_\pi \in \Phi^{\mathrm{ss}}(G)$ is the (semisimplified) L-parameter associated to π by Harris-Taylor.

To see why this result is relevant for compatibility, we invoke the observation, due to Scholze-Weinstein [SW13; SW20a], that, using Grothendieck-Messing theory, the Lubin-Tate tower at infinite level

$$\mathrm{LT}_{n,\infty} := \lim_{m\to\infty}\mathrm{LT}_{n,m,\breve{\mathbb{Q}}_p}$$

is representable by a space admitting a moduli interpretation as a space of shtukas over the Fargues-Fontaine curve; namely, the space denoted $\mathrm{Sht}(\mathrm{GL}_n,b,\mu)_\infty$ in the notation of [SW20a], where $b \in B(\mathrm{GL}_n)$ is an element in the Kottwitz set of GL_n corresponding to a rank n isocrystal of slope $\frac{1}{n}$ and $\mu = (1,0,\ldots,0,0)$ is a dominant cocharacter of GL_n. If X denotes the Fargues-Fontaine curve, then this this parametrizes modifications

$$\mathcal{O}_X(-\frac{1}{n}) \to \mathcal{O}_X^n$$

of type $(1,0,\ldots,0,0)$ (i.e this map is an embedding with cokernel a length 1 torsion sheaf on X), where $\mathcal{O}_X(-\frac{1}{n})$ is the unique rank n vector bundle on X of slope $-\frac{1}{n}$. This interpretation allows one to relate the complex $R\Gamma_c(\mathrm{LT}_{n,\infty},\overline{\mathbb{Q}}_\ell)[\pi]$ to the action of a Hecke operator $T_{\mu^{-1}}$ on Bun_G, acting on a sheaf $\mathscr{F}_\pi$ constructed from the supercuspidal representation π, where $\mu^{-1} = (0,0,\ldots,0,-1)$ is a dominant inverse of μ. The Fargues-Scholze parameter of π is built from the action of the excursion algebra on the sheaf $\mathscr{F}_\pi$, which in turn is built from Hecke operators equipped with a factorization structure coming from geometric Satake. It thus is reasonable to expect that the cohomology group $R\Gamma_c(\mathrm{LT}_{n,\infty},\overline{\mathbb{Q}}_\ell)[\pi]$ should have $W_{\mathbb{Q}_p}$-action given by the Fargues-Scholze parameter $\phi_\pi^{\mathrm{FS}} : W_{\mathbb{Q}_p} \to {}^L\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell) \simeq \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ composed with the highest weight representation of ${}^L\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ corresponding to the dominant cocharacter μ^{-1}. However, this is just the dual of the standard representation of $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$. Thus, using Theorem 1.2, we can see that

$$\phi_\pi^\vee = (\phi_\pi^{\mathrm{FS}})^\vee$$

where the twist by the norm-character $|\cdot|^{(1-n)/2}$ is cancelled out by a perverse normalization (also related to the middle degree being the relevant one) in the definition of the Hecke operator $T_{\mu^{-1}}$. This implies compatibility for supercuspidal π, which, by using compatibility of the Fargues-Scholze correspondence with parabolic induction, is enough to conclude the general case. In a similar fashion, using the description of the ρ-isotypic part one can prove compatibility for the inner form $D_{\frac{1}{n}}^*$.

One may expect, given the above sketch of compatibility for GL_n, that, to prove Theorem 1.1, one must similarly provide a description of the cohomology of the $\pi \in \Pi(G)$ (resp. $\rho \in \Pi(J)$)-isotypic part of a local Shimura variety/shtuka space at infinite level associated to G. In the case where the associated L-parameters are supercuspidal, this is the content of the Kottwitz conjecture [RV14, Conjecture 7.3]. To this end, we consider the cohomology of a Shtuka space associated to the group $G = \mathrm{Res}_{L/\mathbb{Q}_p}\mathrm{GSp}_4$. Namely, the space denoted $\mathrm{Sht}(G,b,\mu)_\infty$, where μ is the Siegel cocharacter and $b \in B(G)$ is a basic element in the Kottwitz set of G, corresponding to a rank 4 isocrystal with a polarization and automorphism group equal to $J = \mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GU}_2(D))$. This space carries a commuting $G(\mathbb{Q}_p)$ and $J(\mathbb{Q}_p)$ action. The quotients

$$\mathrm{Sht}(G,b,\mu)_K := \mathrm{Sht}(G,b,\mu)_\infty/\underline{K}$$

for varying compact open $K \subset G(\mathbb{Q}_p)$ are, as before, representable by rigid analytic varieties over $\mathrm{Spa}(\breve{L})$ of dimension 3, where $\breve{L} := L\breve{\mathbb{Q}}_p$. They define a tower of local Shimura varieties in the sense of Rapoport-Viehmann [RV14], which uniformize the basic locus of certain global Shimura varieties analogous to the Lubin-Tate case described above. Letting $\mathrm{Sht}(G,b,\mu)_{K,\mathbb{C}_p}$ be the base-change of these spaces to $\mathbb{C}_p$, we can then consider the analog of the complexes described above

$$R\Gamma_c(G,b,\mu) := \mathrm{colim}_{K\to 1} R\Gamma_c(\mathrm{Sht}(G,b,\mu)_{K,\mathbb{C}_p}, \overline{\mathbb{Q}}_\ell)$$

This complex is concentrated in degrees $0 \leq i \leq 6 = 2\dim(\mathrm{Sht}(G,b,\mu)_K)$ and admits an action of $G(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \times W_L$. This allows one to consider the ρ and π-isotypic parts, i.e we set

$$R\Gamma_c(G,b,\mu)[\rho] := R\Gamma_c(G,b,\mu) \otimes^{\mathbb{L}}_{\mathscr{H}(J)} \rho$$

and

$$R\Gamma_c(G,b,\mu)[\pi] := R\Gamma_c(G,b,\mu) \otimes^{\mathbb{L}}_{\mathscr{H}(G)} \pi$$

where $\mathscr{H}(G)$ (resp. $\mathscr{H}(J)$) are the usual smooth Hecke algebra of G (resp. J). To deduce compatibility, one needs to realize the (semi-simplified) L-parameter ϕ_π (resp. ϕ_ρ) of Gan-Takeda (resp. Gan-Tantono) in these two cohomology groups. We will sketch how to do this in the next section using uniformization and global methods. Interestingly, after knowing compatibility, one can use ideas from the geometry of the Fargues-Scholze construction to provide a more precise of the complexes $R\Gamma_c(G,b,\mu)[\rho]$ and $R\Gamma_c(G,b,\mu)[\pi]$. Namely, recent work of Hansen [Han20] allows us to deduce that if ϕ_ρ (resp. ϕ_π) is supercuspidal the complexes $R\Gamma_c(G,b,\mu)[\rho]$ (resp. $R\Gamma_c(G,b,\mu)[\pi]$) are concentrated in middle degree 3. It then follows from work of Hansen-Kaletha-Weinstein [HKW22] on a weakening of the Kottwitz conjecture and work of Fargues-Scholze [FS21, Section X.2] describing the Hecke action on objects with supercuspidal Fargues-Scholze parameter that one can actually deduce a strong form of the Kottwitz conjecture for these representations. To state this result, we first note that, if we are given a supercuspidal parameter $\phi : W_L \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_\ell)$, even though the parameter ϕ is irreducible, its composition with the standard embedding $\mathrm{std} : \mathrm{GSp}_4(\overline{\mathbb{Q}}_\ell) \hookrightarrow \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ may not be. In particular, the size of the L-packets $\Pi_\phi(G) := \mathrm{LLC}_G^{-1}(\phi)$ and $\Pi_\phi(J) := \mathrm{LLC}_J^{-1}(\phi)$ over ϕ are governed by this.

1. (stable) $\mathrm{std} \circ \phi$ is irreducible. In this case, the L-packets each contain one supercuspidal member.

2. (endoscopic) $\mathrm{std} \circ \phi \simeq \phi_1 \oplus \phi_2$, where $\phi_i : W_L \to \mathrm{GL}_2(\bar{\mathbb{Q}}_\ell)$ are distinct irreducible 2-dimensional representations with $\det(\phi_1) = \det(\phi_2)$. In this case, the L-packets over ϕ each contain two supercuspidal members.

This allows us to state our main consequence of Theorem 1.1, which (almost) verifies the strong form of the Kottwitz conjecture for GSp_4/L and $\mathrm{GU}_2(D)/L$.

Theorem 1.1.5. *Let $L/\mathbb{Q}_p$ be an unramified extension with $p > 2$. Let π (resp. ρ) be members of the L-packet over a supercuspidal parameter $\phi : W_L \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_\ell)$ as above. Then the complexes*

$$R\Gamma_c(G,b,\mu)[\pi]$$

and

$$R\Gamma_c(G,b,\mu)[\rho]$$

are concentrated in middle degree 3*.*

1. *If ϕ is stable supercuspidal, with singleton L-packets $\{\pi\} = \Pi_\phi(G)$ and $\{\rho\} = \Pi_\phi(J)$, then the cohomology of $R\Gamma_c(G,b,\mu)[\pi]$ in middle degree is isomorphic to*
$$\rho \boxtimes (\mathrm{std} \circ \phi)^\vee \otimes |\cdot|^{-3/2}$$
as a $J(\mathbb{Q}_p) \times W_L$-module, and the cohomology of $R\Gamma_c(G,b,\mu)[\rho]$ in middle degree is isomorphic to
$$\pi \boxtimes \mathrm{std} \circ \phi \otimes |\cdot|^{-3/2}$$
as a $G(\mathbb{Q}_p) \times W_L$-module.

2. *If ϕ is an endoscopic parameter, with L-packets $\Pi_\phi(G) = \{\pi^+, \pi^-\}$ and $\Pi_\phi(J) = \{\rho_1, \rho_2\}$*[1]*, the cohomology of $R\Gamma_c(G,b,\mu)[\pi]$ in middle degree is isomorphic to*
$$\rho_1 \boxtimes \phi_1^\vee \otimes |\cdot|^{-3/2} \oplus \rho_2 \boxtimes \phi_2^\vee \otimes |\cdot|^{-3/2}$$
or
$$\rho_1 \boxtimes \phi_2^\vee \otimes |\cdot|^{-3/2} \oplus \rho_2 \boxtimes \phi_1^\vee \otimes |\cdot|^{-3/2}$$

[1] For an explanation of the notation, see the discussion at the end of section 2.2.

as a $J(\mathbb{Q}_p) \times W_L$-module. Similarly, the cohomology of $R\Gamma_c(G,b,\mu)[\rho]$ in middle degree is isomorphic to

$$\pi^+ \boxtimes \phi_1 \otimes |\cdot|^{-3/2} \oplus \pi^- \boxtimes \phi_2 \otimes |\cdot|^{-3/2}$$

or

$$\pi^+ \boxtimes \phi_2 \otimes |\cdot|^{-3/2} \oplus \pi^- \boxtimes \phi_1 \otimes |\cdot|^{-3/2}$$

as a $G(\mathbb{Q}_p) \times W_L$-module. Here we write $\mathrm{std} \circ \phi \simeq \phi_1 \oplus \phi_2$, with ϕ_i distinct irreducible 2-dimensional representations of W_L and $\det(\phi_1) = \det(\phi_2)$.

Moreover, both possibilities for the cohomology of $R\Gamma_c(G,b,\mu)[\rho]$ (resp. $R\Gamma_c(G,b,\mu)[\pi]$) in the endoscopic case occur for some choice of representation $\rho \in \Pi_\phi(J)$ (resp. $\pi \in \Pi_\phi(G)$). In particular, knowing the precise form of either $R\Gamma_c(G,b,\mu)[\rho]$ or $R\Gamma_c(G,b,\mu)[\pi]$ for some $\rho \in \Pi_\phi(J)$ or $\pi \in \Pi_\phi(G)$ determines the precise form of the cohomology in all other cases.

Remark 1.1.6.

1. Results of this form when $L = \mathbb{Q}_p$ have also been shown by Ito-Meida [IM21].

2. If one knew Arthur's multiplicity formula for inner forms of GSp_4 over totally real fields, one should be able to determine the cohomology in the endoscopic case more precisely, using basic uniformization and the more precise description of the cohomology of the global Shimura variety this multiplicity formula would provide (See for example [Ngu19, Section 3.2] for this kind of analysis in the case of unitary groups.). However, to our knowledge the multiplicity formula is unknown in this case. In the case that $L = \mathbb{Q}_p$, one can apply what is known about the multiplicity formula for $\mathrm{GSp}_4/\mathbb{Q}$ [Art04; GT19]. This is carried out by Ito-Mieda [IM21]. The correct answer, for the ρ-isotypic part, should be that, if $\rho = \rho_1$, we are in the first case, and if $\rho = \rho_2$, we are in the second case. Similarly, for the π-isotypic part, if $\pi = \pi^+$ is the unique generic member of the L-packet for a fixed choice of Whittaker datum, we should be in the first case and, if $\pi = \pi^-$, we should be in the second case. It might also be possible to show this using a weaker argument. Our analysis reduces us to checking that $R\Gamma_c(G,b,\mu)[\rho_1]$ admits a sub-quotient isomorphic to $\pi_+ \boxtimes \phi_1 \otimes |\cdot|^{-3/2}$, which may be possible to show through basic uniformization and a small global argument.

3. We hope that the perspective we take on the Kottwitz conjecture in this paper will help provide further advancements in our knowledge of the cohomology of local Shimura varieties. In particular, by invoking the use of these very general geometric tools from the Fargues-Scholze construction, we require global input only to show compatibility, which, as we will see in the next section, requires substantially less than the input needed to determine the precise form of the cohomology such as a multiplicity formula for the automorphic spectrum.

We will now conclude the introduction by providing a sketch of the proof of Theorem 1.1.

1.1.2 Proof Sketch of the Main Theorems

As before, we set $G = \mathrm{Res}_{L/\mathbb{Q}_p}\mathrm{GSp}_4$ and $J = \mathrm{Res}_{L/\mathbb{Q}_p}\mathrm{GU}_2(D)$. Similar to the case of GL_n, the idea behind proving compatibility for G and J is to use the compatibility of the Fargues-Scholze local Langlands correspondence with parabolic induction to reduce to the case where π is a supercuspidal representation. However, this is a little bit more subtle than the case of GL_n. Unlike GL_n, the local Langlands correspondence for these groups is not a bijection. As seen before, the L-packets can be either of size 1 or 2. Given an L-parameter $\phi : W_L \times \mathrm{SL}_2(\overline{\mathbb{Q}}_\ell) \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_\ell)$, there are three distinct possibilities.

1. The L-packets $\Pi(G)_\phi$ and $\Pi(J)_\phi$ do not contain any supercuspidal representations.

2. The L-packets $\Pi(G)_\phi$ and $\Pi(J)_\phi$ contain a mix of supercuspidal and non-supercuspidal representations.

3. The L-packets $\Pi(G)_\phi$ and $\Pi(J)_\phi$ contain only supercuspidals.

Case (1) is straight forward. Since compatibility is known for GL_n and its inner forms and any proper Levi subgroup of G (resp. J) is a product of such groups, it follows from compatibility of the Fargues-Scholze correspondence with parabolic induction and products that the correspondences are compatible for any representation lying in such an L-packet.

Case (2) is a bit more subtle, here ϕ^{ss} factors through a Levi subgroup of $\mathrm{GSp}_4(\overline{\mathbb{Q}}_\ell)$, but ϕ itself does not. In particular, the restriction to the SL_2 factor of ϕ is non-trivial. In this case, we can write $\Pi_\phi(G) = \{\pi_{disc}, \pi_{sc}\}$ (resp.

$\Pi_\phi(J) := \{\rho_{disc}, \rho_{sc}\}$ or $\Pi_\phi(J) = \{\rho^1_{disc}, \rho^2_{disc}\}$, depending on whether the parameter is of Saito-Kurokawa or Howe-Piatetski–Schapiro type), where π_{disc} (resp. ρ^1_{disc}, ρ^2_{disc}, and ρ_{disc}) are non-supercuspidal (essentially) discrete series representation of G (resp. J), and π_{sc} (resp. ρ_{sc}) is a supercuspidal representation of $G(\mathbb{Q}_p)$ (resp. $J(\mathbb{Q}_p)$). The key observation is that π_{disc} (resp. ρ_{disc}) is an irreducible sub-quotient of a parabolic induction, so, in this case, we can apply the same argument as in case (1) to deduce compatibility of the two correspondences. It remains to see that the same is true for π_{sc}. To do this, we use a description of the ρ-isotypic part of the Shtuka space $\mathrm{Sht}(G,b,\mu)_\infty$ introduced in section 1.1. Namely, we consider the complex

$$R\Gamma_c(G,b,\mu)[\rho_{disc}] \simeq R\Gamma_c(G,b,\mu) \otimes^{\mathbb{L}}_{\mathscr{H}(G)} \rho_{disc}$$

of $G(\mathbb{Q}_p) \times W_L$-modules. Recent work of Hansen-Kaletha-Weinstein [HKW22] then tells us the form of this cohomology group (or rather a small variant thereof) as a $J(\mathbb{Q}_p)$-representation. In particular, if we let $R\Gamma_c(G,b,\mu)[\rho_{disc}]_{sc}$ denote the summand of $R\Gamma_c(G,b,\mu)[\rho_{disc}]$ where $J(\mathbb{Q}_p)$ acts via a supercuspidal representation, then in the Grothendieck group of admissible $J(\mathbb{Q}_p)$-representations of finite length $R\Gamma_c(G,b,\mu)[\rho_{disc}]_{sc}$ is equal to $-2\pi_{sc}$.

Similar to the case of $G = \mathrm{GL}_n$, this complex describes the action of the Hecke operator T_μ acting on a sheaf $\mathscr{F}_{\rho_{disc}}$ constructed from ρ_{disc} on Bun_G. Moreover, the complex $R\Gamma_c(G,b,\mu)[\rho_{disc}]_{sc}$ can be interpreted as a complex of sheaves on the open Harder-Narasimhan(=HN)-strata $\mathrm{Bun}^{\mathbf{1}}_G \subset \mathrm{Bun}_G$ corresponding to the trivial G-bundle on the Fargues-Fontaine curve X. It follows from the above description in the Grothendieck group that the excursion algebra will act on this complex via eigenvalues valued in the parameter $\phi^{\mathrm{FS}}_{\pi_{sc}}$. However, since the excursion algebra is built from Hecke operators, it will also commute with the action of Hecke operators on $\mathscr{F}_{\rho_{disc}}$. This allows us to conclude that it also must act via eigenvalues valued in $\phi^{\mathrm{FS}}_{\rho_{disc}}$ giving a chain of equalities

$$\phi^{\mathrm{FS}}_{\pi_{sc}} = \phi^{\mathrm{FS}}_{\rho_{disc}} = \phi^{\mathrm{ss}}_{\rho_{disc}} = \phi^{\mathrm{ss}}_{\pi_{sc}}$$

where the first equality follows from the previous argument and the second equality follows from the above analysis of induced representations. Similarly, one deduces compatibility for ρ_{sc} by applying a similar argument to the π-isotypic part.

Case (3) is by far the most involved and takes up the majority of the paper.

This is the case in which the L-parameter ϕ is supercuspidal. First, one can make a reduction to showing compatibility for just $\rho \in \Pi(J)$ with supercuspidal Gan-Tantono parameter ϕ, by using the commutation of Hecke operators and the excursion algebra similar to what was done in case (2). Now, the key point again is that the complex $R\Gamma_c(G,b,\mu)[\rho]$ describes the action of the Hecke operator T_μ on a sheaf $\mathscr{F}_\rho$ on Bun_G. To make further progress towards compatibility, we use that the Hecke operators can be in turn described using the spectral action of the derived category of perfect complexes on the stack of L-parameters, as constructed in [FS21, Chapter X]. In particular, a Hecke operator defines a vector bundle on the stack of L-parameters, whose action on the sheaf $\mathscr{F}_\rho$ via the spectral action is precisely T_μ. Using this, we argue using the support of the spectral action of certain averaging operators, considered by [AL21a] in the case of GL_n, to show that, if $\mathrm{std} \circ \phi \otimes |\cdot|^{-3/2}$ occurs as a W_L-stable sub-quotient of the complex

$$\bigoplus_{\rho' \in \Pi_\phi(J)} R\Gamma_c(G,b,\mu)[\rho']$$

we have an equality:

$$\mathrm{std} \circ \phi_\rho^{\mathrm{FS}} = \mathrm{std} \circ \phi$$

for all $\rho \in \Pi_\phi(J)$. Now a GSp_4-valued parameter is in turn determined by its composition with std and its similitude character, which is precisely the central character of ρ. Therefore, since the Fargues-Scholze correspondence is compatible with central characters, this is enough to conclude that $\phi = \phi_\rho^{\mathrm{FS}}$. This reduces the question of showing compatibility for $\rho \in \Pi(\mathrm{GU}_2(D))$ with supercuspidal Gan-Tantono parameter ϕ to the following.

Proposition 1.1.7. *Let ϕ be a supercuspidal parameter with associated L-packet $\Pi_\phi(J)$. Then the direct summand of*

$$\bigoplus_{\rho' \in \Pi_\phi(J)} R\Gamma_c(G,b,\mu)[\rho']$$

where $G(\mathbb{Q}_p)$ acts via a supercuspidal representation

$$\bigoplus_{\rho' \in \Pi_\phi(J)} R\Gamma_c(G,b,\mu)[\rho']_{sc}$$

is concentrated in middle degree 3 *and admits a non-zero W_L-stable sub-quotient with W_L-action given by* $\mathrm{std} \circ \phi \otimes |\cdot|^{-3/2}$.

Just as one does in proving Theorem 1.2, the key idea is to directly relate the complex

$$\bigoplus_{\rho'\in\Pi_\phi(J)} R\Gamma_c(G,b,\mu)[\rho']_{sc}$$

to the cohomology of a global Shimura variety using basic uniformization of the generic fiber as proven by Shen [She17] and an analogue of Boyer's trick [Boy99b]. This allows us to in turn prove Proposition 1.4 using global results on Galois representations in the cohomology of these Shimura varieties due to Kret-Shin [KS16] and Sorensen [Sor10]. More specifically, in this case the relevant Shimura datum is given by $(\mathbf{G},X)$, where $\mathbf{G}$ is a $\mathbb{Q}$-inner form of $\mathbf{G}^* := \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GSp}_4)$ for $F/\mathbb{Q}$ a totally real extension with p inert and $F_p \simeq L$. The relevant uniformization result is then applicable if $L/\mathbb{Q}_p$ is an unramified extension and $p > 2$. To state the key consequence of this uniformization result, we introduce some notation. We let $\mathbb{A}$ and $\mathbb{A}_f$ denote the adeles and finite adeles of $\mathbb{Q}$, respectively. If $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$ denotes the level away from p and $K_p \subset G(\mathbb{Q}_p)$ denotes the level at p, we let $\mathscr{S}(\mathbf{G},X)_{K_pK^p}$ be the rigid analytic Shimura variety over $\mathbb{C}_p$ of level K_pK^p. We set ξ be a regular weight of an algebraic representation $\mathscr{V}_\xi$ of $\mathbf{G}$ over $\mathbb{Q}$ and let $\mathscr{L}_\xi$ denote the associated $\overline{\mathbb{Q}}_\ell$ local system on $\mathscr{S}(\mathbf{G},X)_{K_pK^p}$. We then define

$$R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p},\mathscr{L}_\xi) := \mathrm{colim}_{K_p\to\{1\}} R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^pK_p},\mathscr{L}_\xi)$$

The basic uniformization result of Shen then furnishes a $\mathbb{Q}$-inner form $\mathbf{G}'$ of $\mathbf{G}$ satisfying that $\mathbf{G}'_{\mathbb{Q}_p} \simeq J$ together with a $G(\mathbb{Q}_p)\times W_L$-invariant map

$$\Theta : R\Gamma_c(G,b,\mu) \otimes^{\mathbb{L}}_{\mathscr{H}(J_b)} \mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p,\mathscr{L}_\xi) \to R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p},\mathscr{L}_\xi)$$

functorial in the level K^p. Here $\mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p,\mathscr{L}_\xi)$ denotes the space of algebraic automorphic forms of level K^p valued in the algebraic representation $\mathscr{V}_\xi$ in the sense of [Gro99]. We want to use this uniformization map to apply global results on the cohomology of $R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p},\mathscr{L}_\xi)$ to study the action of W_L on $R\Gamma_c(G,b,\mu)$. To do this, we show an analogue of Boyer's trick, which says that the non-basic Newton strata of the adic flag variety $\mathscr{F}\ell_{G,\mu^{-1}} := (G/P_{\mu^{-1}})^{ad}$ are parabolically induced as spaces with $G(\mathbb{Q}_p)$-action. Using the Hodge-Tate period map from the Shimura variety $\mathscr{S}(\mathbf{G},X)_{K^p}$ to $\mathscr{F}\ell_{G,\mu^{-1}}$, this implies that, if we pass to the part of the cohomology on both sides where $G(\mathbb{Q}_p)$ acts via a supercuspidal representation, we get a $W_L\times G(\mathbb{Q}_p)$-equivariant isomorphism:

$$\Theta_{sc} : R\Gamma_c(G,b,\mu)_{sc} \otimes^{\mathbb{L}}_{\mathscr{H}(J_b)} \mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p,\mathscr{L}_\xi) \xrightarrow{\simeq} R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p},\mathscr{L}_\xi)_{sc}$$

After showing this, we fix a ρ having supercuspidal Gan-Tantono parameter ϕ, and, via an argument using the simple trace formula, choose a globalization of ρ to a cuspidal automorphic representation Π' of $\mathbf{G}'$, which occurs as a $J(\mathbb{Q}_p)$-stable direct summand of $\mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p,\mathscr{L}_\xi)$ and is an unramified twist of Steinberg at some non-empty set of places S_{st}, for some sufficiently large regular weight ξ and sufficiently small level K^p. We set S to be a finite set of places outside of which Π' is unramified. The Hecke eigenvalues of Π' then define a maximal ideal $\mathfrak{m} \subset \mathbb{T}^S$ in the abstract commutative Hecke algebra of $\mathbf{G}'$ away from the finite places S. Regarding both sides of Θ as $\mathbb{T}^S$-modules, we can localize at $\mathfrak{m}$ to get a map

$$\Theta_{\mathfrak{m}} : (R\Gamma_c(G,b,\mu)\otimes^{\mathbb{L}}_{\mathscr{H}(J_b)}\mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p,\mathscr{L}_\xi))_{\mathfrak{m}} \to R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p},\mathscr{L}_\xi)_{\mathfrak{m}}$$

We write $K^p = K^{S_{st}\cup\{p\}}K_{\{p\}\cup S_{st}}$ for $K^{S_{st}\cup\{p\}} \subset \mathbf{G}(\mathbb{A}_f^{S_{st}\cup\{p\}})$. Taking colimits on both sides as $K_{\{p\}\cup S_{st}} \to \{1\}$, we see that $\Theta_{\mathfrak{m}}$ induces a map:

$$(R\Gamma_c(G,b,\mu)\otimes^{\mathbb{L}}_{\mathscr{H}(J_b)}\mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^{S_{st}\cup\{p\}},\mathscr{L}_\xi))_{\mathfrak{m}} \to R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^{S_{st}\cup\{p\}}},\mathscr{L}_\xi)_{\mathfrak{m}}$$

Since we know that Θ_{sc} is an isomorphism, we have an isomorphism

$$(R\Gamma_c(G,b,\mu)\otimes^{\mathbb{L}}_{\mathscr{H}(J_b)}\mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^{S_{st}\cup\{p\}},\mathscr{L}_\xi))^{st}_{\mathfrak{m}} \xrightarrow{\simeq} R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^{S_{st}\cup\{p\}}},\mathscr{L}_\xi)^{st}_{\mathfrak{m}}$$

Noting that $\mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p,\mathscr{L}_\xi)$ is semi-simple, we can project to the summand where $\mathbf{G}(F_v) \simeq \mathbf{G}'(F_v)$ acts via an unramified twist of the Steinberg representation for all $v \in S_{st}$. This implies that we have an isomorphism

$$\Theta^{st}_{\mathfrak{m},sc} : (R\Gamma_c(G,b,\mu)_{sc}\otimes^{\mathbb{L}}_{\mathscr{H}(J_b)}\mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^{S_{st}\cup\{p\}},\mathscr{L}_\xi))^{st}_{\mathfrak{m}} \xrightarrow{\simeq} R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^{S_{st}\cup\{p\}}},\mathscr{L}_\xi)^{st}_{\mathfrak{m},sc}$$

The key point is now, by analyzing the simple twisted trace formula of Kottwitz-Shelstad [KS99] and stable trace formulas of Arthur [Art02], we can prove a strong multiplicity one type result (Proposition 5.4), for cuspidal automorphic representations that are unramified twists of Steinberg at some sufficiently large non-empty set of places and regular of weight ξ at infinity. This implies that the representations of $\mathbf{G}'$) occurring on LHS of $\Theta^{st}_{\mathfrak{m}}$ must have local constituent at p with Langlands parameter ϕ, since we localized at the Hecke eigensystem defined by Π' at the unramified places. and, since the local constituents of the automorphic representations of $\mathbf{G}'$ at p occurring in the LHS are all in the L-packet $\Pi_\phi(J)$ by the strong multipicity one result, we can reduce Proposition 1.4 to showing that $R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p},\mathscr{L}_\xi)^{st}_{\mathfrak{m},sc}$ is concentrated in degree 3 and has W_L-action given (up to multiplicity) by $\mathrm{std} \circ \phi \otimes |\cdot|^{-3/2}$. This follows from the analysis carried out in Kret-Shin [KS16]. In particular, it follows from their results that this complex will be concentrated in degree 3 and that the traces of Frobenius in $\Gamma_F := \mathrm{Gal}(\overline{F}/F)$ on the étale cohomology of the associated global Shimura variety over $\overline{F}$ are given by $\mathrm{std} \circ \phi_{\tau_v}$, where τ_v are the local constituents of some weak transfer τ of Π' to an automorphic representation of $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GSp}_4 =: \mathbf{G}^*$ and ϕ_{τ_v} is the associated Gan-Takeda parameter. This allows one, up to multiplicities, to describe the Galois action on the global Shimura variety in terms of the composition $\mathrm{std} \circ \rho_\tau$, where ρ_τ is a global $\mathrm{GSp}_4(\overline{\mathbb{Q}}_\ell)$-valued representation of the absolute Galois group of F constructed by Sorensen [Sor10] from τ characterized by the property that $i\mathrm{WD}(\mathrm{std} \circ \rho_\tau)|_{W_{F_v}}^{F-s.s.} \simeq \phi_{\tau_v} \otimes |\cdot|^{-3/2}$ for all but finitely many places v of F. This would give one precisely the desired description of the W_L-action on $R\Gamma_c(G,b,\mu)[\rho]_{sc}$ if one knew that $\phi_{\tau_p} = \phi_\rho$. Since Π' is globalization of ρ, one needs to choose τ to be a strong transfer of Π' at the prime p. This latter goal is accomplished using analysis of the simple trace formula as done in Kret-Shin [KS16, Section 6] combined with the character identities proven by Chan-Gan [CG15]. These results on strong transfers also aid us in deducing the strong multiplicity one type result mentioned above.

In section 2, we give an overview of the Gan-Takeda and Gan-Tantono local Langlands correspondence, putting it in the framework of the refined local Langlands correspondence of Kaletha in preparation for applications to the Kottwitz conjecture. In section 3, we describe the Fargues-Scholze local Langlands correspondence and related ideas, giving the proof of compatibility in cases (1) and (2) and reducing case (3) to Proposition 1.4, via some properties of the spectral action discussed in section 3.2. In section 4, we discuss basic uniformization of the relevant Shimura varieties and prove the aforementioned analogue of Boyer's trick, showing that the uniformization map Θ_{sc} is an isomorphism. In section 5, we analyze the simple trace formula with fixed central character in a fashion similar to Kret-Shin [KS16] to deduce the existence of the required strong transfers, as well as combine this with analysis of the simple twisted trace formula to deduce the required strong multiplicity one result. In section 6, we apply the results of section 5 combined with results of Kret-Shin [KS16] and Sorensen [Sor10] to compute the relevant Galois action on the global Shimura variety. Finally, in section 7, we put the results of the previous sections together to prove Proposition 1.4. We then conclude with the application to the proofs of Theorem 1.1 and 1.3, as well as formally deduce compatibility for the local Langlands correspondence for Sp_4 and its non quasi-split inner form $\mathrm{SU}_2(D)$, as constructed by Gan-Takeda [GT10] and Choiy [Cho17], respectively. We finish the section with a brief discussion of an application to the cohomology of the related (non-minuscule) local Shtuka spaces.

Conventions and Notations

For a diamond or v-stack, we freely use the formalism in [Sch18; FS21] of ℓ-adic cohomology of diamonds and v-stacks. We will fix isomorphisms $i : \overline{\mathbb{Q}}_\ell \xrightarrow{\simeq} \mathbb{C}$ and $j : \overline{\mathbb{Q}}_p \xrightarrow{\simeq} \mathbb{C}$ and use the (geometric) normalization of local class field theory that sends the Frobenius to the inverse of the uniformizer. For a supercuspidal L-parameter, we will often abuse notation and use ϕ to denote both the L-parameter and the semisimplified parameter ϕ^{ss}, as in this case this merely corresponds to forgetting the trivially acting $\mathrm{SL}_2(\mathbb{C})$-factor and applying the isomorphism i. For a reductive group $H/\mathbb{Q}_p$, we will write $R\mathscr{H}om_{H(\mathbb{Q}_p)}(-,\overline{\mathbb{Q}}_\ell) : \mathrm{D}(H(\mathbb{Q}_p),\overline{\mathbb{Q}}_\ell)^{\mathrm{op}} \to \mathrm{D}(H(\mathbb{Q}_p),\overline{\mathbb{Q}}_\ell)$ for the derived smooth duality functor, where $\mathrm{D}(H(\mathbb{Q}_p),\overline{\mathbb{Q}}_\ell)$ is the unbounded derived category of smooth representations. Namely, it is derived functor induced by the left exact smooth duality functor $V \mapsto (V^*)^{\mathrm{sm}}$, where $(V^*)^{\mathrm{sm}}$ is the set of all functions $f : V \to \overline{\mathbb{Q}}_\ell$ such that there exists $K \subset H(\mathbb{Q}_p)$ a compact open such that for all $v \in V$ and $k \in K$ we have that $f(kv) = f(v)$. Normally, in the literature the space $\mathrm{Sht}(G,b,\mu)_\infty$ parametrizes modifications $\mathscr{E}_0 \dashrightarrow \mathscr{E}_b$ with meromorphy μ. For us, it will denote the space parametrizing modifications of type μ^{-1}. This convention limits the appearances of duals (cf. Remark 3.8).

1.2 Local Langlands for GSp_4 and $\mathrm{GU}_2(D)$

1.2.1 Local Langlands for GSp_4

In this section, we will describe the local Langlands correspndence of Gan-Takeda for the group $G := \mathrm{GSp}_4/L$, where $L/\mathbb{Q}_p$ is a finite extension. We fix a choice of Whittaker datum $\mathfrak{m} := (B,\psi)$ throughout section 2, where B is the Borel and ψ is a generic character of L.

As before, we consider the set $\Phi(G)$ of admissible homomorphisms

$$\phi : W_L \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G}(\mathbb{C}) = \mathrm{GSpin}_5(\mathbb{C}) \simeq \mathrm{GSp}_4(\mathbb{C})$$

taken up to $\hat{G}$-conjugacy, where W_L acts via a continuous semisimple homomorphism with respect to the discrete topology and $\mathrm{SL}_2(\mathbb{C})$ acts via an algebraic representation. Similarly, let $\Pi(G)$ denote the isomorphism classes of smooth irreducible representations of the group $G(L)$. We can now state the main theorem of Gan-Takeda.

Theorem 1.2.1. *[GT11] There is a surjective finite to one map*

$$\mathrm{LLC}_G : \Pi(\mathrm{GSp}_4) \to \Phi(\mathrm{GSp}_4)$$

$$\pi \mapsto \phi_\pi$$

with the following properties:

1. *π is an (essentially) discrete series representation of $\mathrm{GSp}_4(L)$ if and only if its L-parameter does not factor through any proper Levi subgroup of $\mathrm{GSp}_4(\mathbb{C})$.*

2. *Given an L-parameter ϕ, we set $S_\phi := Z_{\hat{G}}(\mathrm{Im}(\phi))$ to be the centralizer of ϕ. The fiber $\Pi_\phi(G)$ can be naturally parametrized by the set of irreducible characters of the component group*

$$A_\phi := \pi_0(S_\phi) \simeq \pi_0(S_\phi / Z(\mathrm{GSp}_4))$$

which is either trivial or equal to $\mathbb{Z}/2\mathbb{Z}$. When $A_\phi = \mathbb{Z}/2\mathbb{Z}$, exactly one of the two representations in $\Pi_\phi(G)$ is generic for the fixed choice of Whittaker datum, and is indexed by the trivial character of A_ϕ.

3. *The similitude character $\mathrm{sim}(\phi_\pi)$ of $\mathrm{GSp}_4(L)$ is equal to the central character ω_π via the isomorphism given by local class field theory.*

4. *Given a character χ of L^* and letting $\lambda : \mathrm{GSp}_4 \to L^*$ be the similitude character of $\mathrm{GSp}_4(L)$, we have, via local class field theory, that the L-parameter of $\pi \otimes (\chi \circ \lambda)$ is equal to $\phi_\pi \otimes \chi$.*

5. *If $\pi \in \Pi(\mathrm{GSp}_4)$ is a representation, for any smooth irreducible representation σ of $\mathrm{GL}_r(L)$, we have that*

$$\gamma(s, \pi \times \sigma, \psi) = \gamma(s, \phi_\pi \otimes \phi_\sigma, \psi)$$

$$L(s, \pi \times \sigma, \psi) = L(s, \phi_\pi \otimes \phi_\sigma, \psi)$$

$$\varepsilon(s,\pi\times\sigma,\psi)=\varepsilon(s,\phi_\pi\otimes\phi_\sigma,\psi)$$

where the RHS are the Artin local factors associated to the representations of $W_L\times \mathrm{SL}_2(\mathbb{C})$ and the LHS are the local factors of Shahidi [Sha90] with respect to the morphisms of L-groups defined in [GT11, Section 4] in the case that π is a generic supercuspidal or non-supercuspidal and are the local factors defined by Townsend [Tow13] if π is a non-generic supercuspidal representation.

The map LLC_G *is uniquely determined by the properties (1),(3), and (5), where one can take* $r\leq 2$ *in* (5).

Remark 1.2.2. When the paper of Gan-Takeda was released there was no good theory of L,ε, and γ factors for nongeneric supercuspidal representations satisfying the usual properties. (See the 10-Commandents in [LR05]) Instead, to uniquely characterize the correspondence for these representations, they use an equality between the Plancharel measure on the family of inductions from $\mathrm{GSpin}_5(L)\times\mathrm{GL}_r(L)\simeq\mathrm{GSp}_4(L)\times\mathrm{GL}_r(L)$ to $\mathrm{GSpin}_{2r+5}(L)$ for $r\leq 2$. However, this theory of L,ε, and γ factors was later constructed by Nelson Townsend in his PhD thesis [Tow13].

We now make the following definition.

Definition 1.2.3. Write $\mathrm{std}:\mathrm{GSp}_4\hookrightarrow\mathrm{GL}_4$ for the standard embedding. We say a discrete L-parameter is stable if the L-packet $\Pi_\phi(G)$ has size 1 and is endoscopic if it has size 2. Equivalently, by Theorem 2.1 (2), this is equivalent to saying that the character group $A_\phi^\vee$ of the component group A_ϕ has cardinality 1 or 2, respectively. By [GT11, Lemma 6.2], this can be characterized as follows.

- (stable) $\mathrm{std}\circ\phi$ is an irreducible representation of $W_L\times\mathrm{SL}_2(\mathbb{C})$. In this case, $S_\phi=Z(\hat{G})=\mathbb{G}_m$, so A_ϕ is trivial.
- (endoscopic) $\mathrm{std}\circ\phi\simeq\phi_1\oplus\phi_2$, where the $\phi_i:W_L\times\mathrm{SL}_2(\mathbb{C})\to\mathrm{GL}_2(\mathbb{C})$ for $i=1,2$ are distinct irreducible 2-dimensional representations of $W_L\times\mathrm{SL}_2(\mathbb{C})$ with $\det(\phi_1)=\det(\phi_2)$. In this case, $A_\phi\simeq\mathbb{Z}/2\mathbb{Z}$. We recall that $\mathrm{GSp}_4(\mathbb{C})$ has a unique endoscopic group $\mathrm{GSO}_{2,2}$, and the endoscopic parameters lie in the image of the map $\Phi(\mathrm{GSO}_{2,2})\to\Phi(\mathrm{GSp}_4)$. More specifically, the dual group of $\mathrm{GSO}_{2,2}$ is GSpin_4 and one has an identification $\mathrm{GSpin}_4\simeq(\mathrm{GL}_2(\mathbb{C})\times\mathrm{GL}_2(\mathbb{C}))^0:=\{(g_1,g_2)\in\mathrm{GL}_2(\mathbb{C})\times\mathrm{GL}_2(\mathbb{C})|\det(g_1)=\det(g_2)\}$ and the map $\Phi(\mathrm{GSO}_{2,2})\to\Phi(\mathrm{GSp}_4)\xrightarrow{\mathrm{std}\circ}$

$\Phi(\mathrm{GL}_4)$ comes from the inclusion $(\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}))^0 \subset \mathrm{GL}_4(\mathbb{C})$. Using this, we can compute that one has an identification

$$S_\phi \simeq \{(a,b) \in \mathbb{C}^* \times \mathbb{C}^* | a^2 = b^2\} \subset (\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}))^0$$

where the center $Z(\mathrm{GSp}_4)(\mathbb{C}) \simeq \mathbb{C}^*$ embeds diagonally.

For an L-parameter ϕ we see, by Theorem 2.1 (2), that the size of the L-packet $\Pi_\phi(G)$ is at most 2, this allows us to subdivide into three cases:

1. The L-packet $\Pi_\phi(G)$ does not contain any supercuspidal representations.

2. The L-packet $\Pi_\phi(G)$ contains one supercuspidal and one non-supercuspidal.

3. The L-packet contains only supercuspidals.

In Case (1) the parameter will not be discrete. Case (2) is where the parameter ϕ does not factor through a Levi-subgroup so it is discrete, but its semisimplification ϕ^{ss} as defined in section 1 does. Case (2) does not occur when the parameter is a stable discrete parameter, by definition. The relevant case is when the parameter is discrete endoscopic. To understand this, we let $\nu(n)$ denote the unique n-dimension irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ then there are two cases:

1. (Saito-Kurokawa Type) We have $\mathrm{std} \circ \phi = \phi_0 \oplus \chi \boxtimes \nu(2)$, where ϕ_0 is a 2-dimensional irreducible representation of W_L and χ is a character, with $\chi^2 = \det(\phi_0)$. Therefore, the semisimplification ϕ^{ss} satisfies: $\mathrm{std} \circ \phi^{\mathrm{ss}} = \phi_0 \oplus \chi \otimes |\cdot|^{\frac{1}{2}} \oplus \chi \otimes |\cdot|^{-\frac{1}{2}}$.

2. (Howe-Piatetski–Shapiro Type) We have $\mathrm{std} \circ \phi = \chi_1 \boxtimes \nu(2) \oplus \chi_2 \boxtimes \nu(2)$, where χ_1 and χ_2 are distinct characters of W_L satisfying $\chi_1^2 = \chi_2^2$. Therefore, the semisimplification ϕ^{ss} satisfies: $\mathrm{std} \circ \phi^{\mathrm{ss}} = \chi_1 \otimes |\cdot|^{\frac{1}{2}} \oplus \chi_1 \otimes |\cdot|^{-\frac{1}{2}} \oplus \chi_2 \otimes |\cdot|^{\frac{1}{2}} \oplus \chi_2 \otimes |\cdot|^{-\frac{1}{2}}$.

Remark 1.2.4.
1. The terminology here is explained by Arthur's classification [Art04] of the global automorphic representations of GSp_4 appearing in the papers [Kur78] and [HP79], respectively.

2. We will mention in the next section how to distinguish these two cases via the number of supercuspidals in the L-packet $\Pi_\phi(\mathrm{GU}_2(D))$ defined by the Gan-Tantono local Langlands correspondence.

Case (3) is the situation where the parameter ϕ is supercuspidal as defined in the introduction. In particular, in the supercuspidal case the restriction of the parameter ϕ to the $\mathrm{SL}_2(\mathbb{C})$ factor is trivial, so the irreducible representations occurring in the decomposition of $\mathrm{std} \circ \phi$ are just representations of W_L.

For the purposes of applying the weak form of the Kottwitz Conjecture proven in Hansen-Kaletha-Weinstein [HKW22], we formulate this correspondence in terms of the refined local Langlands of Kaletha [Kal16] with respect to the fixed choice of Whittaker datum $\mathfrak{m}$. Now, given a parameter ϕ that is either mixed supercuspidal or supercuspidal, we have by Theorem 2.1 (2) a correspondence between the L-packet $\Pi_{\phi}(G)$ and the set of irreducible characters $A_{\phi}^{\vee}$. This in turn gives rise to an irreducible character of the group S_{ϕ} via the composition:

$$S_{\phi} \to \pi_0(S_{\phi}) = A_{\phi}$$

This allows us to make the following definition.

Definition 1.2.5. For ϕ a supercuspidal or mixed-supercuspidal parameter ϕ as above and $\pi \in \Pi_{\phi}(G)$, we denote the character of S_{ϕ} described above by τ_{π}.

1.2.2 Local Langlands for $\mathrm{GU}_2(D)$

In this section, we describe the local Langlands correspondence for the unique non-split inner form $J = \mathrm{GU}_2(D)$, the group of similitudes of the unique 2-dimensional Hermitian vector space over the quaternion division algebra D/L. As in the previous section, we let $\Pi(J)$ denote the set of irreducible admissible representations of J, and $\Phi(J)$ be the set of L-parameters of J. This is a subset of the previous set $\Phi(\mathrm{GSp}_4)$ as we will now explain. J has a unique up to conjugacy minimal parabolic whose Levi factor is

$$D^* \times \mathrm{GL}_1.$$

This defines a form of the Siegel parabolic of GSp_4 and it determines a dual parabolic subgroup $P^{\vee}(\mathbb{C})$ in the dual group $\mathrm{GSp}_4(\mathbb{C})$ of $\mathrm{GU}_2(D)$. This is the Heisenberg parabolic subgroup of $\mathrm{GSp}_4(\mathbb{C})$, its conjugacy class is said to be relevant for J while all other conjugacy classes of proper parabolics are said to be irrelevant. We say $\phi \in \Phi(\mathrm{GSp}_4)$ is relevant if it does not factor through any irrelevant parabolic subgroups of $\mathrm{GSp}_4(\mathbb{C})$. We define $\Phi(J)$ to be the subset of relevant ϕ in $\Phi(\mathrm{GSp}_4)$. We set $B_{\phi} := \pi_0(Z_{\mathrm{Sp}_4}(\mathrm{Im}(\phi)))$. One has an exact sequence:

$$\langle \pm 1 \rangle \to B_{\phi} \to A_{\phi} \to 0$$

Implying that one has an injection on the group of irreducible characters $\hat{A}_\phi \hookrightarrow \hat{B}_\phi$, which identifies $\hat{A}_\phi$ as the subgroup of (index at most 2) of characters trivial on the image of the center $Z(\mathrm{Sp}_4)(\mathbb{C})$. One can check that $\hat{B}_\phi \neq \hat{A}_\phi$ if and only if ϕ is relevant for $\mathrm{GU}_2(D)$. Now we can state the main theorem of Gan and Tantono.

Theorem 1.2.6. *[GT14] There is a natural surjective finite-to-one map*

$$\mathrm{LLC}_J : \Pi(\mathrm{GU}_2(D)) \to \Phi(\mathrm{GU}_2(D))$$

$$\rho \mapsto \phi_\rho$$

with the following properties:

1. *ρ is an (essentially) discrete series representation of $\mathrm{GU}_2(D)$ if and only if its parameter ϕ_ρ does not factor through any proper Levi subgroup of $\mathrm{GSp}_4(\mathbb{C})$.*

2. *For an L-parameter ϕ, the fiber $\Pi_\phi(J)$ can be naturally parametrized by the set $\hat{B}_\phi \setminus \hat{A}_\phi$. This set has size either 1 or 2.*

3. *The similitude character $\mathrm{sim}(\phi_\rho)$ of ϕ_ρ is equal to the central character ω_ρ of ρ, via the isomorphism given by local class field theory.*

4. *Given a character χ of L^* and letting $\lambda : \mathrm{GU}_2(D) \to L^*$ be the similitude character of $\mathrm{GU}_2(D)$, we have, via local class field theory, that the L-parameter of $\rho \otimes (\chi \circ \lambda)$ is equal to $\phi_\rho \otimes \chi$.*

5. *If $\rho \in \Pi(\mathrm{GU}_2(D))$ is a non-supercuspidal representation then, for any smooth irreducible representation σ of $\mathrm{GL}_r(L)$, we have that*

$$\gamma(s, \rho \times \sigma, \psi) = \gamma(s, \phi_\rho \otimes \phi_\sigma, \psi)$$

$$L(s, \rho \times \sigma, \psi) = L(s, \phi_\rho \otimes \phi_\sigma, \psi)$$

$$\varepsilon(s, \pi \times \sigma, \psi) = \varepsilon(s, \phi_\rho \otimes \phi_\sigma, \psi)$$

where the RHS are the Artin local factors associated to the representations of $W_L \times \mathrm{SL}_2(\mathbb{C})$ and the LHS are the local factors of Shahidi, as defined in [GT14, Section 8].

6. *Suppose that ρ is a supercuspidal representation. For any irreducible supercuspidal representation σ of $\mathrm{GL}_r(L)$ with L-parameter ϕ_σ, if $\mu(s, \rho \boxtimes \sigma, \psi)$ denotes the Plancharel measure associated to the family of induced representations $I_P(\pi \boxtimes \sigma, s)$ on $\mathrm{GSpin}_{r+4,r+1}$, where we have regarded $\rho \boxtimes \sigma$ as a representation of the Levi subgroup $\mathrm{GSpin}_{4,1} \times \mathrm{GL}_r \simeq \mathrm{GU}_2(D) \times \mathrm{GL}_r$, then $\mu(s, \rho \boxtimes \sigma)$ is equal to*

   $$\gamma(s, \phi_\rho^\vee \otimes \phi_\sigma, \psi) \cdot \gamma(-s, \phi_\rho \otimes \phi_\sigma^\vee, \overline{\psi}) \cdot \gamma(2s, \mathrm{Sym}^2 \phi_\sigma \otimes \mathrm{sim}(\phi_\rho)^{-1}, \psi)) \cdot \gamma(-2s, \mathrm{Sym}^2 \phi_\sigma^\vee \otimes \mathrm{sim}(\phi_\rho), \overline{\psi})$$

   The map LLC_J is uniquely determined by the properties (1), (3), (5), and (6), with $r \leq 4$ in (5) and (6).

We now further elaborate on the structure of the L-packets $\Pi_\phi(J) := \mathrm{LLC}_J^{-1}(\phi)$ in the case where the parameter ϕ is mixed supercuspidal. If the parameter ϕ is of this form, then, it follows from [GT14, Proposition 5.4] and the description of LLC_J provided in [GT14, Section 7], that the L-packet $\Pi_\phi(J)$ has following structure, as alluded to in Remark 2.2 (2).

1. (Saito-Kurokawa Type) The L-packet $\Pi_\phi(J) = \{\rho_{disc}, \rho_{sc}\}$ contains one supercuspidal representation ρ_{sc} and one non-supercuspidal representation ρ_{disc}.
2. (Howe-Piatetski–Shapiro Type) The L-packet $\Pi_\phi(J) = \{\rho^1_{disc}, \rho^2_{disc}\}$ contains no supercuspidal representations.

We now would also like to briefly comment on the structure of the set $\hat{B}_\phi \setminus \hat{A}_\phi$, confirming the expectation that the size of the L-packets $\Pi_\phi(G)$ and $\Pi_\phi(J)$ is always the same.

- (stable) In the case that the L-parameter ϕ is stable, we have that $B_\phi = \mathbb{Z}/2\mathbb{Z}$ and, as noted in section 2.1, $A_\phi = 1$. This means the set $\hat{B}_\phi \setminus \hat{A}_\phi$ consists of one element corresponding to the non-trivial character.
- (endoscopic) In the case that the parameter ϕ is endoscopic, we have that the decomposition $\mathrm{std} \circ \phi \simeq \phi_1 \oplus \phi_2$ induces an exact sequence

  $$A_\phi = \mathbb{Z}/2\mathbb{Z} \xrightarrow{\Delta} B_\phi = Z(\mathrm{SL}_2) \times Z(\mathrm{SL}_2) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

  and so the set $\hat{B}_\phi \setminus \hat{A}_\phi$ has size 2 and is indexed by two characters η_{+-} and η_{-+} each non-trivial on one of the two $\mathbb{C}^*$-factors under the isomorphism

  $$S_\phi \simeq \{(a,b) \in \mathbb{C}^* \times \mathbb{C}^* | a^2 = b^2\} \subset (\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}))^0$$

  from section 2.1.

We now wish to put this local Langlands correspondence for the inner form in the context of the refined local Langlands correspondence of Kaletha [Kal16]. We consider the Kottwitz set $B(G)$ [Kos21b; RR96] and let $b \in B(G)$ be the basic element whose associated σ-centralizer $J_b = J$. We take this to be the basic element whose slope homorphism is the dominant rational cocharacter of G given by $(1/2, 1/2, 1/2, 1/2)$. Let $Z(\mathrm{GSp}_4) \simeq \mathbb{G}_m$ be the center. We recall that we have an isomorphism $\pi_1(G) \simeq X_*(Z(\hat{G})) \simeq \mathbb{Z}$ and that the κ-invariant of b is sent to the element $1 \in \mathbb{Z}$ under this isomorphism. This indexes the identity representation of $\mathbb{G}_m$, denoted $id_{\mathbb{G}_m}$. Thus, given a discrete parameter (= supercuspidal or mixed supercuspidal) $\phi : W_L \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GSp}_4(\mathbb{C})$, the refined local Langlands correspondence asserts bijections

$$\Pi_\phi(G) \longleftrightarrow \{\text{irreducible algebraic representations } \tau \text{ of } S_\phi \text{ s.t } \tau|_{Z(\hat{G})} = \mathbf{1}\}$$

$$\pi \mapsto \tau_\pi$$

$$\Pi_\phi(J) \longleftrightarrow \{\text{irreducible algebraic representations } \tau \text{ of } S_\phi \text{ s.t } \tau|_{Z(\hat{G})} = id_{\mathbb{G}_m}\}$$

$$\rho \mapsto \tau_\rho$$

where $\mathbf{1}$ is the trivial representation. In section 2.1, we saw how for $\pi \in \Pi_\phi(G)$ to construct the desired τ_π. Here it is uniquely pinned down by the property that the trivial representation corresponds to the unique $\mathfrak{m}$-generic representation. In the case of the inner form, the situation is a bit more tricky. Consider $\rho \in \Pi(J)$ with associated L-parameter ϕ_ρ. If ϕ_ρ is stable then $S_\phi = \mathbb{G}_m$ and τ_ρ is simply $id_{\mathbb{C}^*}$. If ϕ is endoscopic, then, as noted in section 2.1, we have an inclusion:

$$Z(\mathrm{GSp}_4)(\mathbb{C}) = \mathbb{C}^* \xrightarrow{\Delta} S_\phi \simeq \{(a,b) \in \mathbb{C}^* \times \mathbb{C}^* | a^2 = b^2\}$$

We consider the characters $\tau_i : S_\phi \to \mathbb{C}^*$ for $i = 1, 2$ given by projecting to the first and second coordinate. These satisfy the property that $\tau_i|_{\mathbb{C}^*} = id_{\mathbb{C}^*}$ on the diagonally embedded center as desired. Similarly, under the parametrization of Gan-Tantono $\Pi_\phi(J) = \{\rho_{+-}, \rho_{-+}\}$, where ρ_{+-} and ρ_{-+} correspond to the characters η_{+-} and η_{-+} described above. Specifically, if π_1 and π_2 are the unique discrete series representations of $\mathrm{GL}_2(L)$ in the L-packet over ϕ_1 and ϕ_2 then

$$\rho_{+-} := \theta(\mathrm{JL}^{-1}(\tau_2) \boxtimes \tau_1) \text{ and } \rho_{-+} := \theta(\mathrm{JL}^{-1}(\tau_1) \boxtimes \tau_2)$$

where θ denotes the non-zero local theta lift from $D^* \times \mathrm{GL}_2(F)$ to $\mathrm{GU}_2(D)$, as in [GT14, Proposition 5.4], and $\mathrm{JL} : \Pi(D^*) \to \Pi(\mathrm{GL}_2)$ is the Jacquet-Langlands correspondence. Now we would like to match these two representations with

τ_1 and τ_2 under the refined local Langlands of Kaletha. Suppose we fix such a matching. Now, we consider a refined endoscopic datum $\mathfrak{c}$ for the quasi-split reductive group G, which we recall is a tuple $(H, s, \mathscr{H}, \eta)$ which consists of

- a quasi-split group H over F,
- an extension $\mathscr{H}$ of W_F by $\widehat{H}$ such that the map $W_F \to \mathrm{Out}(\widehat{H})$ coincides with the map $\rho_H : W_F \to \mathrm{Out}(\widehat{H})$ induced by the action of W_F on $\widehat{H} \subset {}^L H$,
- an element $s \in Z(\widehat{H})^\Gamma$,
- an L-homomorphism $\eta : \mathscr{H} \to {}^L G$,

satisfying the condition:

- we have $\eta(\widehat{H}) = Z_{\widehat{G}}(s)^\circ$.

Considering J as the σ-centralizer of the unique basic element $b_1 \in B(G)$ of the Kottwitz invariant 1, this defines for us an extended pure inner twisting of $(\xi, b) : G \to J_b$ in the sense of [Kot97a, Section 5.2], and we can attach a canonical transfer factor $\Delta[\mathfrak{m}, \mathfrak{c}, b_1]$ to this datum, as defined in [Kal16, Section 4.1]. Given a test function $f \in C_c^\infty(J(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$, we can use these transfer factors to say what it means for $f^{\mathfrak{c}} \in C_c^\infty(H(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ to be matching in the sense that their stable orbital integrals normalized with respect to these transfer factors match up.

Now, suppose we have a discrete parameter $\phi \in \Phi(J)$ and a refined endoscopic datum $\mathfrak{c}$, such that $\phi = \eta \circ \phi^{\mathfrak{c}}$ as conjugacy classes of parameters for an L-parameter $\phi^{\mathfrak{c}} : W_L \times \mathrm{SL}(2, \overline{\mathbb{Q}}_\ell) \to \mathscr{H}$. Then the matching is uniquely described using the endoscopic character identities. This assert an equality

$$\Theta^1_{\phi^{\mathfrak{c}}}(f^{\mathfrak{c}}) = \sum_{\pi \in \Pi_{\phi^{\mathfrak{c}}}(H)} \mathrm{tr}(1|\tau_\pi)\theta_\pi(f^{\mathfrak{c}}) = e(J) \sum_{\rho \in \Pi_\phi(J)} \mathrm{tr}(s|\tau_\rho)\theta_\pi(f) = \Theta^s_\phi(f)$$

where $e(J)$ is the Kottwitz sign of J, as defined in [Kot97b] and θ_π denotes the Harish-Chandra character of π. Using the linear independence of the distributions Θ_π and the fact that the packets $\Pi_\phi(J)$ are disjoint, we can see that the matching between $\rho \mapsto \tau_\rho$ is uniquely characterized by these relations. To show that there exists a matching between the representations τ_1 and τ_2 and ρ_{+-} and ρ_{-+}, we need to show that these identities are satisfied under the parametrization of Gan-Takeda-Tantono. This will follow from the endoscopic character identities verified by Chan-Gan [CG15]. Namely, in the case that the parameter ϕ is stable, one only needs to consider the trivial endoscopic datum $\mathfrak{c}_{\mathrm{triv}} = (G, 1, {}^L G, \mathrm{id})$, and these identities follow from [CG15, Proposition 11.1 (1)]. In the case that the parameter ϕ is endoscopic, the case $\mathfrak{c}_{\text{triv}} = (G, 1, {}^L G, \text{id})$ follows from [CG15, Proposition 11.1 (1)], but one also needs to consider the refined endoscopic datum given by $\mathfrak{c} = (\text{GSO}_{2,2}, (1,-1), \text{GSpin}_4, i : \text{GSpin}_4 \to \text{GSpin}_5 \simeq \text{GSp}_4)$, where i is the map from before and $(1,-1) \in S_\phi \subset (\text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C}))^0$. In this case, the identities (up to a sign) follow from combining [CG15, Proposition 11.1 (1)] as before and [CG15, Proposition 11.1 (2)], where we note that since τ_1 and τ_2 are by definition two projections their traces against $(-1,1)$ have the opposite sign, which is consistent with [CG15, Proposition 11.1 (2)]. This shows the refined local Langlands correspondence of Kaletha holds for the group G; however, in order to describe the exact matching between ρ_{+-} and ρ_{-+} with τ_1 and τ_2 one needs to exactly compare the signs of the ad-hoc transfer factors fixed in the statement of [CG15, Proposition 11.1 (2)] with the canonical ones $\Delta(\mathfrak{m}, \mathfrak{c}, b_1)$ constructed by Kaletha. The above argument only shows that there is some matching; nonetheless, for our purposes the choice ends up being irrelevant, so we denote the representations in the L-packet $\Pi_\phi(J)$ corresponding to the projections τ_1 and τ_2 by ρ_1 and ρ_2, respectively. Similarly, for the representations obtained by pre-composing a character with the composition

$$S_\phi \to A_\phi$$

we denote the elements of the L-packet $\Pi_\phi(G)$ corresponding to the trivial (non-trivial) character of A_ϕ by π^+ (resp. π^-). We note that, by Theorem 2.1 (2), π^+ can be characterized by the unique $\mathfrak{m}$-generic representation of this L-packet.

Definition 1.2.7. Given a supercuspidal or mixed supercuspidal L-parameter ϕ as above and $\rho \in \Pi_\phi(J)$, we let τ_ρ be the irreducible representation of S_ϕ associated to it via the matching described above. Given $\pi \in \Pi_\phi(G)$ and $\rho \in \Pi_\phi(J)$, we set

$$\delta_{\pi,\rho} := \tau_\pi^\vee \otimes \tau_\rho$$

where $\tau_\pi^\vee$ denotes the contragredient.

Remark 1.2.8. Changing the choice of Whittaker datum scales the representations by a 1-dimensional character of S_ϕ that is trivial when restricted to the center, so in particular this pairing is independent of the choice of Whittaker datum (See [HKW22, Lemma 2.3.3]).

1.3 The Fargues-Scholze Local Langlands Correspondence

We will now discuss the Fargues-Scholze local Langlands correspondence and deduce compatibility in the cases where the Gan-Takeda/Gan-Tantono parameter is not supercuspidal. We will then conclude by reducing the question of compatibility in the supercuspidal case to Proposition 1.4.

1.3.1 Overview of the Fargues-Scholze Local Langlands Correspondence

For now, let G be any connected reductive group over $\mathbb{Q}_p$. Since we are going to be using geometric Satake, we fix a choice of the square root of p in $\overline{\mathbb{Q}}_\ell$, so that half Tate-twists are well-defined. For us, we will always take $i^{-1}(\sqrt{p})$, where $i : \overline{\mathbb{Q}}_\ell \xrightarrow{\simeq} \mathbb{C}$ is the fixed isomorphism. Fargues-Scholze [FS21] consider the moduli space of G-bundles on the Fargues-Fontaine curve X, denoted Bun_G. This moduli space is an Artin v-stack (in the sense of [FS21, Section IV.I]) and has the structure that the underlying points of its topological space $|\mathrm{Bun}_G|$ are in natural bijection with elements of the Kottwitz set $B(G)$, where the slopes of the G-isocrystal associated to $b \in B(G)$ are the negatives of the slopes of the associated vector bundle $\mathscr{E}_b$ and the specializations between points of $|\mathrm{Bun}_G|$ is dictated by the partial ordering on $B(G)$ induced by the kappa invariant and the slope homomorphism [Vie]. In particular, the connected components of Bun_G are in bijection with $B(G)_{basic} \xrightarrow{\simeq} \pi_1(G)_\Gamma$. Specifically, for any $b \in B(G)_{basic}$, there is a unique open Harder-Narasimhan strata $\mathrm{Bun}_G^b \subset \mathrm{Bun}_G$ dense inside the associated connected component. We recall that the elements of $B(G)_{basic}$ parametrize extended pure inner forms of G, via sending an element $b \in B(G)_{basic}$ to its σ-centralizer $J_b/\mathbb{Q}_p$. For such a basic b, we have an identification $\mathrm{Bun}_G^b \simeq [*/\underline{J_b(\mathbb{Q}_p)}] =: B\underline{J_b(\mathbb{Q}_p)}$ of the HN-strata defined by b and the classifying stack of $\underline{J_b(\mathbb{Q}_p)}$. For any Artin v-stack Z, Fargues-Scholze define a triangulated category $\mathrm{D}_\blacksquare(Z, \overline{\mathbb{Q}}_\ell)$ of solid $\overline{\mathbb{Q}}_\ell$-sheaves [FS21, Section VII.1] and isolate a nice full subcategory $\mathrm{D}_{\mathrm{lis}}(Z, \overline{\mathbb{Q}}_\ell) \subset \mathrm{D}_\blacksquare(Z, \overline{\mathbb{Q}}_\ell)$ of lisse-étale $\overline{\mathbb{Q}}_\ell$-sheaves [FS21, Section VII.6.], which may be roughly thought of as the unbounded derived category of étale $\overline{\mathbb{Q}}_\ell$ sheaves on Z, where one has made an enlargement to capture information about the topology of p-adic groups. In any case, the key point for us is that we have the following basic result.

Lemma 1.3.1. *[FS21, Proposition VII.7.1] There is an equivalence of categories*

$$\mathrm{D}_{\mathrm{lis}}(B\underline{J_b(\mathbb{Q}_p)}, \overline{\mathbb{Q}}_\ell) \simeq \mathrm{D}(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$$

where the RHS denotes unbounded derived category of smooth $J_b(\mathbb{Q}_p)$-representations with coefficients in $\overline{\mathbb{Q}}_\ell$. Under this equivalence, Verdier duality corresponds to smooth duality.

Remark 1.3.2. The main reason for constructing this category $\mathrm{D}_{\mathrm{lis}}$ is that, if one were to take the usual definition for the category of étale $\overline{\mathbb{Q}}_\ell$-sheaves on $B\underline{J_b(\mathbb{Q}_p)}$, this equivalence would no longer be true. In particular, one would obtain the bounded derived category of representations of $J_b(\mathbb{Q}_p)$ admitting a $J_b(\mathbb{Q}_p)$-stable $\overline{\mathbb{Z}}_\ell$-lattice, where the representation is continuous with respect to the ℓ-adic topology on the target. This would limit the scope of the Fargues-Scholze LLC as, in general, one wants to consider smooth $\overline{\mathbb{Q}}_\ell$-representations of $J_b(\mathbb{Q}_p)$, and hence the need for the enlargement of the derived category to $\mathrm{D}_{\mathrm{lis}}$.

Lemma 3.1 tells us that, given an irreducible smooth representation π of $G(\mathbb{Q}_p)$, we can consider the associated sheaf, denoted $\mathscr{F}_\pi$, on $\mathrm{Bun}_G^{\mathbf{1}}$ the open HN-strata corresponding to the trivial element $\mathbf{1} \in B(G)$, and take the extension by zero along the open inclusion $j_!(\mathscr{F}_\pi)$[2]. This realizes the representation π in terms of a sheaf on the moduli space Bun_G in an analogous way to how the function-sheaf dictionary realizes cuspidal automorphic forms as functions associated to sheaves in the context of curves over finite fields. Following V. Lafforgue [Laf18], Fargues and Scholze construct a semisimple L-parameter associated to this sheaf by looking at the action of the excursion algebra on this category $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$. This relies on a form of the geometric Satake correspondence for the B_{dR}^+-affine Grassmannians. For any finite set I, let X^I be the product of I-copies of the diamond $X = \mathrm{Spd}(\breve{\mathbb{Q}}_p)/\mathrm{Frob}^{\mathbb{Z}}$. We then have the Hecke stack

$$\begin{tikzcd} & \mathrm{Hck} \arrow[dl, "h^{\leftarrow}"'] \arrow[dr, "h^{\rightarrow} \times supp"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times X^I \end{tikzcd}$$

defined as the functor that parametrizes, for S a perfectoid space in characteristic p together with a map $S \to X^I$ defining a tuple of Cartier divisors in the relative

[2] The shriek push-forward is not in general well-defined in the context of solid $\overline{\mathbb{Q}}_\ell$-sheaves. However, for the inclusion of HN-strata into Bun_G, its existence follows from [FS21, Proposition VII.7.3], using [FS21, Proposition VII.6.7].

Fargues-Fontaine X_S over S, corresponding to characteristic 0 untilts $S_i^\sharp$ for $i \in I$ of S, a pair of G-torsors $\mathscr{E}_1$, $\mathscr{E}_2$ together with an isomorphism

$$\beta : \mathscr{E}_1|_{X_S \setminus \bigcup_{i\in I} S_i^\sharp} \xrightarrow{\simeq} \mathscr{E}_2|_{X_S \setminus \bigcup_{i\in I} S_i^\sharp}$$

where $h^{\leftarrow}((\mathscr{E}_1, \mathscr{E}_2, i, (S_i^\sharp)_{i\in I})) = \mathscr{E}_1$ and $h^{\rightarrow} \times supp((\mathscr{E}_1, \mathscr{E}_2, \beta, (S_i^\sharp)_{i\in I})) = (\mathscr{E}_2, (S_i^\sharp)_{i\in I})$. We set ${}^LG^I$ to be I-copies of the Langlands dual group of G, i.e ${}^LG = Q \ltimes \hat{G}(\overline{\mathbb{Q}}_\ell)$, where $\hat{G}$ is the reductive group having dual root datum to G and is viewed as a reductive group over $\overline{\mathbb{Q}}_\ell$. The Weil group acts on $\hat{G}$ via the induced action on root datum through some finite quotient Q, which we now fix. Let $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^LG^I)$ denote the category of algebraic $\overline{\mathbb{Q}}_\ell$-representations of I-copies of ${}^LG$. For each element $W \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^LG^I)$, the geometric Satake correspondence of Fargues-Scholze [FS21, Chapter VI] furnishes a solid $\overline{\mathbb{Q}}_\ell$-sheaf $\mathscr{S}_W$ on Hck. This allows us to define Hecke operators.

Definition 1.3.3. For each $W \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^LG^I)$, we define the Hecke operator

$$T_W : \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell) \to \mathrm{D}_{\blacksquare}(\mathrm{Bun}_G \times X^I)$$

$$A \mapsto R(h^{\rightarrow} \times supp)_\natural (h^{\leftarrow *}(A) \otimes^{\mathbb{L}} \mathscr{S}_W)$$

where $\mathscr{S}_W$ is a solid $\overline{\mathbb{Q}}_\ell$-sheaf and the functor $R(h^{\rightarrow} \times supp)_\natural$ is the natural push-forward. I.e the left adjoint to the restriction functor in the category of solid $\overline{\mathbb{Q}}_\ell$-sheaves [FS21, Proposition VII.3.1].

Remark 1.3.4. These satisfy various compatibilities with respect to composition and restriction to the diagonal. In particular, given two representations $V, W \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^LG)$, we have that

$$(T_V \times id)(T_W)(\cdot)|_\Delta \simeq T_{V\otimes W}(\cdot)$$

where $\Delta : X \to X^2$ is the diagonal map.

We then consider $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^{BW^I_{\mathbb{Q}_p}}$, the category of objects in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ with continuous action by $W^I_{\mathbb{Q}_p}$. Examples of objects in this category are objects of $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ tensored by a continuous representation of $W^I_{\mathbb{Q}_p}$, for a more precise description see [FS21, Section IX.1]. With this in hand, we then have the following theorem of Fargues-Scholze.

Theorem 1.3.5. *[FS21, Theorem I.7.2, Proposition IX.2.1, Corollary IX.2.3] The Hecke operator T_W for $W \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^LG^I)$*

$$T_W : \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell) \to \mathrm{D}_{\blacksquare}(\mathrm{Bun}_G \times X^I)$$

induces a functor

$$\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell) \to \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^{BW^I_{\mathbb{Q}_p}}$$

and the induced endofunctors of $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ *given by forgetting the Weil group action preserve compact and ULA objects.*

Remark 1.3.6. This should be thought of as a manifestation of Drinfeld's Lemma, where (roughly) the étale fundamental group of $\mathrm{Spd}(\breve{\mathbb{Q}}_p)/\mathrm{Frob}^{\mathbb{Z}} = X$ should be the same as $W_{\mathbb{Q}_p}$.

From now on, when talking about Hecke operators we shall always refer to this induced functor, which we will also abusively denote by T_W. Theorem 3.2 has direct implications for the cohomology of local Shimura varieties. To study this, consider a minuscule cocharacter μ with field of definition E, and let $b \in B(G,\mu)$ be the unique basic element in the μ-admissible locus (See [RV14, Definition 2.3]). We say that the triple (G,b,μ) defines a local Shimura datum in the sense of Rapoport-Viehmann [RV14]. Attached to such a data, Scholze-Weinstein [SW20a] construct a tower of diamonds

$$p_K : (\mathrm{Sht}(G,b,\mu)_K)_{K \subset G(\mathbb{Q}_p)} \to \mathrm{Spd}(\breve{E})$$

for varying open compact $K \subset G(\mathbb{Q}_p)$. This is obtained by considering the space $\mathrm{Sht}(G,b,\mu)_\infty$ which parametrizes modifications $\mathscr{E}_b \to \mathscr{E}_0$ with meromorphy bounded by μ, where $\mathscr{E}_b$ (resp. $\mathscr{E}_0$) is the bundle corresponding to $b \in B(G)$ (resp. the trivial bundle) on the Fargues-Fontaine curve. It has commuting actions by $G(\mathbb{Q}_p)$ and $J_b(\mathbb{Q}_p)$ given by acting via automorphisms on $\mathscr{E}_0$ and $\mathscr{E}_b$, respectively. The tower is then given by considering the quotients of this space for varying open compact $K \subset G(\mathbb{Q}_p)$ under the action of $G(\mathbb{Q}_p)$.

Definition 1.3.7. Let $\mathrm{Sht}(G,b,\mu)_{K,\mathbb{C}_p}$ be the base-change of the above tower to $\mathbb{C}_p$. We define the complex

$$R\Gamma_c(G,b,\mu) := \mathrm{colim}_{K \to \{1\}} R\Gamma_c(\mathrm{Sht}(G,b,\mu)_{K,\mathbb{C}_p}, \overline{\mathbb{Q}}_\ell)$$

a colimit of smooth $W_E \times J_b(\mathbb{Q}_p)$-modules with a $G(\mathbb{Q}_p)$-action, where W_E is the Weil group of E. A priori it only has an action by the inertia group, but this space admits a non-effective Frobenius descent datum. We then define, for ρ a smooth admissible $J_b(\mathbb{Q}_p)$-representation, the complex

$$R\Gamma_c(G,b,\mu)[\rho] := \mathrm{colim}_{K\to\{1\}} R\Gamma_c(G,b,\mu)_K \otimes^{\mathbb{L}}_{\mathscr{H}(J_b)} \rho$$

where $\mathscr{H}(J_b)$ is the usual smooth Hecke algebra. We also define

$$R\Gamma^{\flat}_c(G,b,\mu)[\rho] := \mathrm{colim}_{K\to\{1\}} R\mathscr{H}om_{J_b(\mathbb{Q}_p)}(R\Gamma_c(G,b,\mu)_K,\rho)[-2d](-d)$$

Similarly, for π a smooth admissible $G(\mathbb{Q}_p)$-representation, we define $R\Gamma_c(G,b,\mu)[\pi]$ and $R\Gamma^{\flat}_c(G,b,\mu)[\pi]$.

Remark 1.3.8. We note that, by Hom-Tensor duality, $R\mathscr{H}om(R\Gamma_c(G,b,\mu)[\rho],\overline{\mathbb{Q}}_\ell)[-2d](-d)$ is isomorphic to $R\Gamma^{\flat}_c(G,b,\mu)[\rho^*]$, where ρ^* is the contragredient. We will end up using both of these cohomology groups throughout this manuscript. The former is more natural from the point of view of basic uniformization, while the latter is disposable to the results of Hansen-Kaletha-Weinstein [HKW22] on the Kottwitz conjecture.

To study these complexes, we specialize the above discussion of Hecke operators to the case where $W = V_{\mu^{-1}}$ is specified by the highest weight representation of highest weight μ^{-1} a dominant inverse of μ and $I = \{*\}$ is a singleton. The sheaf $\mathscr{S}_W$ will then be supported on the closed subspace $\mathrm{Hck}_{\leq\mu^{-1}} = \mathrm{Hck}_{\mu^{-1}}$ of Hck, parametrizing modifications with meromorphy bounded by or equal to μ^{-1}, where the equality follows by the minuscule assumption. The space $\mathrm{Hck}_{\mu^{-1}}$ is cohomologically smooth of dimension $d := \langle 2\rho_G,\mu\rangle$, and the sheaf $\mathscr{S}_W$, as in the geometric Satake correspondence of [MV07], behaves like the intersection cohomology of this space, so we have $\mathscr{S}_W \simeq \overline{\mathbb{Q}}_\ell[d](\frac{d}{2})$. This implies that, to study the action of the Hecke operator T_W on Bun_G, we can look at the restriction of the diagram defining the Hecke correspondence to this subspace

$$\begin{tikzcd} & \mathrm{Hck}_{\mu^{-1}} \arrow[dl, "h^{\leftarrow}_{\mu^{-1}}"'] \arrow[dr, "h^{\rightarrow}_{\mu^{-1}}\times supp"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times \mathrm{Spd}(\breve{E})/\mathrm{Frob}^{\mathbb{Z}} \end{tikzcd}$$

In particular, we have an isomorphism:

$$T_{\mu^{-1}}(A) := T_W(A) \simeq R(h^{\rightarrow}_{\mu^{-1}} \times supp)_{\natural}(h^{\leftarrow *}_{\mu^{-1}}(A))[d](\frac{d}{2})$$

Now consider a smooth admissible representation π of $G(\mathbb{Q}_p)$ and apply the Hecke operator to the sheaf:

$$j_!(\mathscr{F}_\pi)$$

Then the fiber of $\mathrm{Hck}_{\mu^{-1}}$ of $h^{\leftarrow}_{\mu^{-1}}$ over $\mathrm{Bun}_G^{\mathbf{1}}$ is identified with

$$[Gr_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}]$$

the Schubert cell/variety associated to μ^{-1} in the B_{dR}^+-affine Grassmannian, quotiented out by $G(\mathbb{Q}_p)$ acting on the trivial bundle via automorphisms. The sheaf

$$T_{\mu^{-1}} j_!(\mathscr{F}_\pi)$$

is then supported on the HN-strata given by the Kottwitz elements in $B(G,\mu)$ since, by [Rap18, Proposition A.9], any G-bundle occurring as a modification of type μ^{-1} of the trivial bundle has associated Kottwitz element lying in this set. We then consider the restriction

$$j_b^* T_{\mu^{-1}} j_!(\mathscr{F}_\pi) \in \mathrm{D}(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)^{BW_E}$$

where $j_b : \mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$ is the inclusion of the open HN-strata defined by b. The complex $j_b^* T_{\mu^{-1}} j_!(\mathscr{F}_\pi)$ will be computed in terms of the cohomology of sheaves supported on the Newton strata

$$[Gr^b_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}]$$

parametrizing modifications of type μ^{-1} of the trivial bundle such that the resulting bundle has associated Kottwitz element of type b after pulling back to each geometric point, modulo automorphisms of the trivial bundle. The space $\mathrm{Sht}(G,b,\mu)_\infty$ defined above is a pro-étale $\underline{J_b(\mathbb{Q}_p)}$-torsor with respect to the $J_b(\mathbb{Q}_p)$-action by automorphisms of $\mathscr{E}_b$

$$\mathrm{Sht}(G,b,\mu)_\infty \to Gr^b_{G,\mu^{-1}}$$

over this Newton strata. Using this description of the infinite level Shimura variety, it then follows from base change and the fact that the sheaves $\mathscr{S}_W$ are ULA over X (See [FS21, Chapter IX.3] for details) that we have an isomorphism

$$R\Gamma_c(G,b,\mu)[\pi][d](\frac{d}{2}) \simeq j_b^* T_{\mu^{-1}} j_!(\mathscr{F}_\pi) \in \mathrm{D}(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)^{BW_E}$$

of $J_b(\mathbb{Q}_p) \times W_E$-modules. For our purposes, it will be also useful to have a description of the ρ-isotypic part of this cohomology in terms of Hecke operators, for ρ a smooth irreducible representation of $J_b(\mathbb{Q}_p)$. In particular, analysis similar to the above gives us an isomorphism

$$R\Gamma_c(G,b,\mu)[\rho][d](\frac{d}{2}) \simeq j_{\mathbf{1}}^{*}T_{\mu}j_{b!}(\mathscr{F}_{\rho})$$

as $G(\mathbb{Q}_p) \times W_E$-modules. We record these two isomorphisms as a corollary of the above discussion.

Corollary 1.3.9. *Given a local Shimura datum (G,b,μ) as above and π (resp. ρ) a smooth irreducible representation of $G(\mathbb{Q}_p)$ (resp. $J_b(\mathbb{Q}_p)$). There exists an isomorphism*

$$R\Gamma_c(G,b,\mu)[\rho][d](\frac{d}{2}) \simeq j_{\mathbf{1}}^{*}T_{\mu}j_{b!}(\mathscr{F}_{\rho})$$

of complexes of $G(\mathbb{Q}_p) \times W_E$-modules and an isomorphism

$$R\Gamma_c(G,b,\mu)[\pi][d](\frac{d}{2}) \simeq j_{b}^{*}T_{\mu^{-1}}j_{\mathbf{1}!}(\mathscr{F}_{\pi})$$

of complexes of $J_b(\mathbb{Q}_p) \times W_E$-modules.

We have the following basic structural result which, in more generality, follows from the analysis in Fargues-Scholze, but, in the case of a local Shimura datum, also partially follows from standard finiteness results for rigid spaces (See [RV14, Section 6]). In particular, one can show the following.

Theorem 1.3.10. *[FS21, Corollary I.7.3, Page 317] For a local Shimura datum (G,b,μ) as above, the cohomology groups of $R\Gamma_c^{\flat}(G,b,\mu)[\rho]$ and $R\Gamma_c(G,b,\mu)[\rho]$ are valued in smooth admissible $G(\mathbb{Q}_p)$-representations of finite length with an action of W_E. Moreover, they are concentrated in degrees $0 \leq i \leq 2d$.*

Remark 1.3.11. A sheaf $\mathscr{F} \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_{\ell})$ being ULA is equivalent to its stalks at different HN-strata being valued in complexes of smooth admissible representations [FS21, Theorem V.7.1, Proposition VII.7.9], so indeed the admissibility of the above complex is a consequence of Theorem 3.2 and Corollary 3.3.

Fargues-Scholze use the endofunctors defined by the Hecke algebra on $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_{\ell})$ to define the excursion algebra.

Definition 1.3.12. For a finite set I, a representation $W \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^LG^I)$, maps $\alpha : \overline{\mathbb{Q}}_\ell \to \Delta^*W$ and $\beta : \Delta^*W \to \overline{\mathbb{Q}}_\ell$, and elements $\gamma_i \in W_{\mathbb{Q}_p}$ for $i \in I$, one defines the excursion operator on $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ to be the composition:

$$id = T_{\overline{\mathbb{Q}}_\ell} \xrightarrow{\alpha} T_{\Delta^*W} = T_W \xrightarrow{(\gamma_i)_{i\in I}} T_W = T_{\Delta^*W} \xrightarrow{\beta} T_{\overline{\mathbb{Q}}_\ell} = id$$

where $\Delta^*(W)$ is the precomposition of W with the diagonal embedding ${}^LG \to {}^LG^I$.

This defines a natural endomorphism of the identity functor on $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$. If one looks at the induced endofunctor given by the inclusion $\mathrm{D}(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell) \subset \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ induced by the open immersion $j : \mathrm{Bun}_G^{\mathbf{1}} \hookrightarrow \mathrm{Bun}_G$ then one obtains a natural endomorphism of the identity functor on $\mathrm{D}(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$. In other words, we get a family of compatible endomorphisms for all complexes of smooth representations of $G(\mathbb{Q}_p)$; namely, an element of the Bernstein center. One can verify that this excursion algebra satisfies similar properties to that considered by V. Lafforgue, so, using Lafforgue's reconstruction theorem [Laf18, Proposition 11.7], one can show the following.

Theorem 1.3.13. *To an irreducible smooth $\overline{\mathbb{Q}}_\ell$-representation π of $G(\mathbb{Q}_p)$ (or more generally $A \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ any Schur-irreducible object (i.e $\mathrm{End}(A) = \overline{\mathbb{Q}}_\ell$)), there is a unique continuous semisimple map*

$$\phi_\pi^{\mathrm{FS}} : W_{\mathbb{Q}_p} \to {}^LG(\overline{\mathbb{Q}}_\ell)$$

characterized by the property that for all I, W, α, β, and $\gamma_i \in W_{\mathbb{Q}_p}$ for $i \in I$, the corresponding endomorphism of π defined above is given by multiplication by the scalar that results from the composite

$$\overline{\mathbb{Q}}_\ell \xrightarrow{\alpha} \Delta^*W = W \xrightarrow{(\phi_\pi(\gamma_i))_{i\in I}} W = \Delta^*W \xrightarrow{\beta} \overline{\mathbb{Q}}_\ell$$

By further studying the geometry of Bun_G and the Hecke stacks, one can deduce various good properties of this correspondence.

Theorem 1.3.14. *[FS21, Theorem I.9.6] The mapping defined above*

$$\pi \mapsto \phi_\pi^{\mathrm{FS}}$$

enjoys the following properties:

1. *(Compatibility with Local Class Field Theory) If $G = T$ is a torus, then $\pi \mapsto \phi_\pi$ is the usual local Langlands correspondence*

2. *The correspondence is compatible with character twists, passage to contragredients, and central characters.*

3. *(Compatibility with products) Given two irreducible representations π_1 and π_2 of two connected reductive groups G_1 and G_2 over $\mathbb{Q}_p$, respectively. We have*

$$\pi_1 \boxtimes \pi_2 \mapsto \phi^{\mathrm{FS}}_{\pi_1} \times \phi^{\mathrm{FS}}_{\pi_2}$$

under the Fargues-Scholze local Langlands correspondence for $G_1 \times G_2$.

4. *(Compatibility with parabolic induction) Given a parabolic subgroup $P \subset G$ with Levi factor M and a representation π_M of M, then the semisimple L-parameter corresponding to any sub-quotient of $ind_P^G(\pi_M)$ the (normalized) parabolic induction is the composition*

$$W_{\mathbb{Q}_p} \xrightarrow{\phi^{\mathrm{FS}}_{\pi_M}} {}^L M(\overline{\mathbb{Q}}_\ell) \to^L G(\overline{\mathbb{Q}}_\ell)$$

where the map ${}^L M(\overline{\mathbb{Q}}_\ell) \to {}^L G(\overline{\mathbb{Q}}_\ell)$ is the natural embedding.

5. *(Compatibility with Harris-Taylor/Henniart LLC) For $G = \mathrm{GL}_n$ or an inner form of G the semisimple L-parameter associated to π is the (semi-simplified) parameter ϕ^{ss}_{π}. associated to π by Harris-Taylor/Henniart.*

6. *(Compatibility with Restriction of Scalars) The above story works the same for G' a connected reductive group over any finite extension $E'/\mathbb{Q}_p$, where one then gets a semisimple L-parameter valued on $W_{E'}$. If $G = \mathrm{Res}_{E'/\mathbb{Q}_p} G'$ is the Weil restriction of some G'/E' then L-parameters for $G/\mathbb{Q}_p$ agree with L-parameters for G'/E' in the usual sense.*

7. *(Compatibility with Isogenies) If $G' \to G$ is a map of reductive groups inducing an isomorphism of adjoint groups, π is an irreducible smooth representation of $G(E)$ and π' is an irreducible constituent of $\pi|_{G'(E)}$ then $\phi_{\pi'}$ is the image of ϕ_π under the induced map $\hat{G} \to \hat{G}'$.*

Remark 1.3.15. In (5), the compatibility of the Fargues-Scholze local Langlands correspondence with the Harris-Taylor/Henniart local Langlands correspondence for an arbitrary inner form of GL_n is not included in the paper of Fargues-Scholze [FS21]. However, it follows from the work of Hansen-Kaletha-Weinstein on the Kottwitz conjecture [HKW22, Theorem 1.0.3].

1.3.2 The Spectral Action

With these basic structural properties out of the way, we turn our attention to the "spectral action" on $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$, which will be very important to proving compatibility of the two correspondences in the case where the parameter is supercuspidal, as well as deducing applications to the Kottwitz conjecture. We recall that an L-parameter over $\overline{\mathbb{Q}}_\ell$ can be thought of as a continuous (not necessarily semisimple) homomorphism

$$\phi : W_{\mathbb{Q}_p} \to {}^L G(\overline{\mathbb{Q}}_\ell)$$

commuting with the natural projection to Q. One can use the classical construction of Grothendieck-Deligne to see that this coincides with the definition given in section 2 for GSp_4 after applying the isomorphism i, where the monodromy operation is recovered through the exponential of the action of $W_{\mathbb{Q}_p}$ on the ℓ-power roots of unity, and assuming Frobenius semi-simplicity of ϕ. Such a continuous map can be thought of as a continuous 1-cocycle $W_{\mathbb{Q}_p} \to \hat{G}(\overline{\mathbb{Q}}_\ell)$, with respect to the action of $W_{\mathbb{Q}_p}$ on $\hat{G}(\overline{\mathbb{Q}}_\ell)$. If we let $A/\mathbb{Z}_\ell$ be any $\mathbb{Z}_\ell$-algebra endowed with a topology given by writing $A = \mathrm{colim}_{A' \subset A} A'$, where A' is a finitely generated $\mathbb{Z}_\ell$-module with its ℓ-adic topology, then we can defined a moduli space, denoted $\mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})$, over $\mathbb{Z}_\ell$, whose A-points are the continuous 1-cocycles $W_{\mathbb{Q}_p} \to \hat{G}(A)$ with respect to the natural action of $W_{\mathbb{Q}_p}$ on $\hat{G}(A)$. This defines a scheme considered in [Dat+20] and [Zhu20] which, by [FS21, Theorem I.9.1], can be written as a union of open and closed affine subschemes $\mathscr{Z}^1(W_{\mathbb{Q}_p}/P, \hat{G})$ as P runs through subgroups of wild inertia of W_E, where each $\mathscr{Z}^1(W_{\mathbb{Q}_p}/P, \hat{G})$ is a flat local complete intersection over $\mathbb{Z}_\ell$ of dimension $\dim(G)$. This allows us to consider the Artin stack quotient $[\mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})/\hat{G}]$, where $\hat{G}$ acts via conjugation. We then consider the base change to $\overline{\mathbb{Q}}_\ell$, denoted $[\mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}]$ and referred to as the stack of Langlands parameters, as well as the category $\mathrm{Perf}([\mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}])$ of perfect complexes of coherent sheaves on this space. We let $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$ denote the triangulated sub-category of compact objects in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ (which are precisely the objects with quasi-compact support on Bun_G and which restrict to compact objects in $\mathrm{D}(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ for all $b \in B(G)$ by [FS21, Theorem V.4.1, Proposition VII.7.4]). We then have the key theorem of Fargues-Scholze.

Theorem 1.3.16. *[FS21, Corollary X.I.3] There exists a natural compactly supported $\overline{\mathbb{Q}}_\ell$-linear action of* $\mathrm{Perf}(\mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G})$ *on* $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$ *satisfying the property that the restriction along the map*

$$\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L G) \to \mathrm{Perf}([\mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}])^{BW_{\mathbb{Q}_p}}$$

induces the action of Hecke operators

$$\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^LG^I) \to \mathrm{End}(\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega)^{BW^I_{\mathbb{Q}_p}}$$

for a varying finite index set I.

Remark 1.3.17.

1. Here the map

$$\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^LG) \to \mathrm{Perf}([\mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}])^{BW_{\mathbb{Q}_p}}$$

associates to a representation V, with associated map $r_V : {}^LG(\overline{\mathbb{Q}}_\ell) \to GL(V)(\overline{\mathbb{Q}}_\ell)$ a vector bundle on $[\mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G})]$ of rank equal to $\dim(V)$ with $W_{\mathbb{Q}_p}$-action. This bundle, denoted C_V, has the property that its evaluation at a $\overline{\mathbb{Q}}_\ell$-point corresponding to a parameter $\phi : W_{\mathbb{Q}_p} \to {}^LG(\overline{\mathbb{Q}}_\ell)$ is precisely $r_V \circ \phi$.

2. The compactly supported condition means that, for all $\mathscr{F} \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$, the functor $\mathrm{Perf}([\mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}]) \to \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$ induced by acting on $\mathscr{F}$ factors through an action of $\mathrm{Perf}([\mathscr{Z}^1(W_{\mathbb{Q}_p}/P, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G})])$, where P is a subgroup of wild inertia. Fargues and Scholze state this action in terms of a $(\infty,1)$-category acting on an $(\infty,1)$-category, we suppress this technicality for simplicity. However, we note that this enhanced action has the concrete implication that if one has a morphism, cone, or homotopy limit/colimit in the derived category $\mathrm{Perf}([\mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}])$ that acting on an object $A \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$ will produce a corresponding morphism, cone, or homotopy limit/colimit.

3. In fact, Fargues-Scholze show that giving such a compactly supported $\overline{\mathbb{Q}}_\ell$-linear action is equivalent (when properly formulated) to giving a $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(Q^I)$-linear monoidal functor

$$\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^LG^I) \to \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega)^{BW^I_{\mathbb{Q}_p}}$$

In the case that $I = \{*\}$, the fact that the Hecke action satisfies this monoidal property is precisely Remark 3.2.

To study this spectral action, we consider, as in [FS21, Section VIII.3.], the coarse quotient in the category of schemes

$$\mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}//\hat{G}$$

of $\mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}$ by the action of $\hat{G}$ via conjugation. Given an L-parameter $\phi : W_{\mathbb{Q}_p} \to {}^LG(\overline{\mathbb{Q}}_\ell)$, it follows by [Dat+20, Proposition 4.13] or [FS21, Proposition VIII.3.2], that the $\hat{G}$-orbit of ϕ defines a closed $\overline{\mathbb{Q}}_\ell$-point of $\mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}//\hat{G}$ if and only if ϕ is a semisimple parameter. Moreover, the natural map

$$\pi : [\mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}] \to \mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}//\hat{G}$$

evaluated on a $\overline{\mathbb{Q}}_\ell$-point in the stack quotient defined by an L-parameter ϕ defines a closed $\overline{\mathbb{Q}}_\ell$-point in the coarse moduli space given by its semisimplification ϕ^{ss}. We can fit excursion operators into this picture as follows. We let $\mathscr{Z}^{spec}(G, \overline{\mathbb{Q}}_\ell) := \mathscr{O}(\mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell})^{\hat{G}}$ be the ring of functions on the stack of L-parameters/the coarse moduli space, which we refer to as the spectral Bernstein center. As noted above, the excursion operators define a family of commuting endomorphisms of the identity functor on $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$. We let $\mathscr{Z}^{geom}(G, \overline{\mathbb{Q}}_\ell)$ be the ring of such endomorphisms as in [FS21, Definition IX.0.2], which we refer to as the geometric Bernstein center. In [FS21, Corollary IX.0.3], Fargues and Scholze construct a canonical map of rings

$$\mathscr{Z}^{spec}(G, \overline{\mathbb{Q}}_\ell) \to \mathscr{Z}^{geom}(G, \overline{\mathbb{Q}}_\ell)$$

which is given by excursion operators in the following sense. By [FS21, Theorem VIII.3.6], there is an identification between $\mathscr{Z}^{spec}(G, \overline{\mathbb{Q}}_\ell)$ and the algebra of excursion operators. In particular, an excursion operator, as in Definition 3.3, associated to the datum I, W, α, β, and $\gamma_i \in W_{\mathbb{Q}_p}$ for $i \in I$ defines a function $f_{I,W,\alpha,\beta,(\gamma_i)_{i\in I}} \in \mathscr{O}_{[Z^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}]} = \mathscr{Z}^{spec}(G, \overline{\mathbb{Q}}_\ell)$ on $\mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}//\hat{G}$, whose evaluation on the closed point of the coarse moduli space associated to a semisimple parameter $\phi : W_{\mathbb{Q}_p} \to {}^LG(\overline{\mathbb{Q}}_\ell)$ is precisely the scalar that results from the endomorphism:

$$\overline{\mathbb{Q}}_\ell \xrightarrow{\alpha} \Delta^* W = W \xrightarrow{(\phi(\gamma_i))_{i\in I}} W = \Delta^* W \xrightarrow{\beta} \overline{\mathbb{Q}}_\ell$$

We note that multiplication by $f_{I,W,\alpha,\beta,(\gamma_i)_{i\in I}}$ defines an endomorphism

$$\mathscr{O}_{[Z^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}]} \to \mathscr{O}_{[Z^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}]}$$

of the structure sheaf on the Artin stack $[Z^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}]$. If we act on a Schur-irreducible object $A \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$ then we obtain an endomorphism

$$\{\mathscr{O}_{[Z^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}]} \star A = A \to \mathscr{O}_{[Z^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}]} \star A = A\} \in \mathrm{End}(A) = \overline{\mathbb{Q}}_\ell$$

which will be precisely the scalar given by evaluating ϕ_A^{FS} on the excursion datum (See also [Zou22, Theorem 5.2]). In this way, we see that the action of excursion operators can be obtained through the spectral action. We will leverage this interpretation of excursion operators to prove the following key lemma.

Lemma 1.3.18. *Let $A \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$ be any Schur-irreducible object with Fargues-Scholze parameter ϕ_A^{FS}. Set x to be the closed point defined by the parameter ϕ_A^{FS} in the coarse moduli space $\mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}//\hat{G}$ and let $\pi^{-1}(x)$ denote the closed subset defined by the preimage in $[\mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}]$. Suppose we have $C \in \mathrm{Perf}([\mathscr{Z}^1(W_E, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}])$ with support disjoint from $\pi^{-1}(x)$ then C acts by zero on A via the spectral action.*

Proof. If we look at the action on $A \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ via the map

$$\mathscr{Z}^{spec}(G, \overline{\mathbb{Q}}_\ell) \to \mathscr{Z}^{geom}(G, \overline{\mathbb{Q}}_\ell)$$

given by excursion operators this factors through the maximal ideal $\mathfrak{m}_A \subset \mathscr{Z}^{spec}(G, \overline{\mathbb{Q}}_\ell) = \mathscr{O}(Z^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell})^{\hat{G}}$ defined by the closed point ϕ_A^{FS} in the coarse moduli space. By the conditions on the support of C, this implies that we can write the identity element as $1 = 1_C + 1_A \in \mathscr{Z}^{spec}(G, \overline{\mathbb{Q}}_\ell)$, where 1_C is a function that annihilates C and 1_A is in the annihilator of $\mathscr{Z}^{spec}(G, \overline{\mathbb{Q}}_\ell)/\mathfrak{m}_A$. We consider the spectral action of C on A

$$C \star A \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$$

and look at the endomorphism induced by multiplication by 1 on C

$$C \star A \to C \star A$$

which is just the identity. However, since 1_C annihilates C, this is the same as the action of 1_A on $C \star A$, but, it follows by the above discussion that acting via multiplication by 1_A is the same as acting via the map

$$\mathscr{Z}^{spec}(G, \overline{\mathbb{Q}}_\ell) \to \mathscr{Z}^{geom}(G, \overline{\mathbb{Q}}_\ell)$$

given by excursion operators, and the action of 1_A after applying this map is zero. This would lead to a contradiction unless $C \star A$ is also zero. $\square$

To take advantage of this lemma, we now introduce the following endofunctors of $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$, which are analogues of the averaging operators considered by [AL21a] in the Fargues-Scholze geometric Langlands correspondence for GL_n and by [FGV02; Gai04] in the classical geometric Langlands correspondence over function fields.

Definition 1.3.19. Let ϕ be a representation of $W_{\mathbb{Q}_p}$ and V a representation of ${}^LG$ with T_V the associated Hecke operator. We consider the endofunctor of $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$

$$A \mapsto R\Gamma(W_{\mathbb{Q}_p}, T_V(A) \otimes \phi^\vee)$$

where $R\Gamma(W_{\mathbb{Q}_p}, -) : \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^{BW_{\mathbb{Q}_p}} \to \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ is the derived functor given by continuous group cohomology with respect to $W_{\mathbb{Q}_p}$. We denote this endofunctor by $Av_{V,\phi} : \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell) \to \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$.

We now would like to realize the functor $Av_{V,\phi}$ as the spectral action of an object in $\mathrm{Perf}([\mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G})])$ similar to [AL21a, Section 5.5]. An obvious guess would be that one should take the vector bundle C_V corresponding to the Hecke operator T_V, as in Remark 3.7 (1), and then twist this by the constant sheaf defined by $\phi^\vee$, which we denote by

$$C_V \otimes \phi^\vee \in \mathrm{Perf}([\mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}])^{BW_{\mathbb{Q}_p}}$$

More precisely, this is the vector bundle with $W_{\mathbb{Q}_p}$-action whose evaluation at a $\overline{\mathbb{Q}}_\ell$-point corresponding to a L-parameter $\tilde{\phi} : W_{\mathbb{Q}_p} \to {}^LG(\overline{\mathbb{Q}}_\ell)$ is the vector space with $W_{\mathbb{Q}_p}$-action given by tensoring the representation

$$\tilde{\phi} : W_{\mathbb{Q}_p} \to {}^LG(\overline{\mathbb{Q}}_\ell) \xrightarrow{r_V} GL(V)$$

with $\phi^\vee$. To obtain the desired perfect complex, it is natural to apply $R\Gamma(W_{\mathbb{Q}_p}, -)$ to the vector bundle to $C_V \otimes \phi^\vee$, which we denote by $\mathscr{A}v_{V,\phi}$. We note that $\mathscr{A}v_{V,\phi}$ is a perfect complex. Indeed, as p is invertible in $\overline{\mathbb{Q}}_\ell$, the wild inertia $P \subset W_{\mathbb{Q}_p}$ will always act through a finite quotient on $C_V \otimes \phi^\vee$ and has no higher cohomology which implies that the invariants

$$(C_V \otimes \phi^\vee)^P$$

are a direct summand of the vector bundle $C_V \otimes \phi^\vee$. If we choose a generator $\tau \in I/P$ in the tame quotient of the inertia subgroup $I \subset W_{\mathbb{Q}_p}$ together with a Frobenius lift $\sigma \in W_{\mathbb{Q}_p}/P$ then the complex $\mathscr{A}v_{V,\phi}$ is computed as the homotopy limit of the diagram:

$$\begin{tikzcd}
(C_V \otimes \phi^\vee)^P \arrow[r, "\tau-1"] \arrow[d, "\sigma-1"] & (C_V \otimes \phi^\vee)^P \arrow[d, "\sigma(1+\tau+\ldots+\tau^{p-1})-1"] \\
(C_V \otimes \phi^\vee)^P \arrow[r, "\tau-1"] & (C_V \otimes \phi^\vee)^P
\end{tikzcd} \tag{1.1}$$

This gives a presentation of $\mathscr{A}v_{\phi,V}$ as a perfect complex, as in [AL21a, Page 21]. To see this, note that σ and τ generate a dense discrete subgroup of $W_{\mathbb{Q}_p}/P$ subject to the relationship that $\sigma^{-1}\tau\sigma = \tau^p$, and it follows that the limit of the diagram is just $((C_V \otimes \phi^\vee)^P)^{W_{\mathbb{Q}_p}/P}$ and that the homotopy limit is in turn the derived functor $R\Gamma(W_{\mathbb{Q}_p}, C_V \otimes \phi^\vee) \simeq R\Gamma(W_{\mathbb{Q}_p}/P, (C_V \otimes \phi^\vee)^P)$, where we have used the vanishing of the higher Galois cohomology with respect to P for the last isomorphism.

Then we have the following Lemma which is a verbatim generalization of [AL21a, Lemma 5.7].

Lemma 1.3.20. *There exists a canonical identification*

$$Av_{V,\phi}(-) \simeq \mathscr{A}v_{V,\phi} \star (-)$$

of endofunctors of $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$

Proof. Equation (1) gives a diagram of perfect complexes on $[\mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}]$. Acting via the spectral action on an object $\mathscr{F} \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ then gives a diagram

$$\begin{array}{ccc} (T_V(\mathscr{F}) \otimes \phi^\vee)^P & \xrightarrow{\tau-1} & (T_V(\mathscr{F}) \otimes \phi^\vee)^P \\ \Big\downarrow{\scriptstyle \sigma-1} & & \Big\downarrow{\scriptstyle \sigma(1+\tau+\ldots+\tau^{p-1})-1} \\ (T_V(\mathscr{F}) \otimes \phi^\vee)^P & \xrightarrow{\tau-1} & (T_V \otimes \phi^\vee)^P \end{array}$$

However, if we take the homotopy limit of the diagram in (1), the claim follows from the fact that the spectral action commutes with homotopy limits, as noted in Remark 3.7 (3). $\square$

With this identification in hand, we can apply Lemma 3.8 to prove the following key consequence.

Lemma 1.3.21. *Let* $A \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$ *be an ULA Schur-irreducible object with Fargues-Scholze parameter* ϕ_A^{FS}*,* V *a representation of* ${}^LG$*, and* ϕ *an irreducible representation of* $W_{\mathbb{Q}_p}$*. If the cohomology sheaves of* $T_V(A) \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^{BW_{\mathbb{Q}_p}}$ *with respect to the standard t-structure on* $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^{BW_{\mathbb{Q}_p}}$ *have a non-zero sub-quotient as* $W_{\mathbb{Q}_p}$*-modules with* $W_{\mathbb{Q}_p}$*-action given by* ϕ *or* $\phi(1)$ *then* $r_V \circ \phi_A^{\mathrm{FS}}$ *also has such a sub-quotient.*

Remark 1.3.22. During the creation of this manuscript, a similar result was obtained by Koshikawa through a similar but simpler proof [Kos21a, Theorem 1.3].

As we will similarly conclude in Section 3.3, he shows that if the cohomology $R\Gamma_c(G,b,\mu)[\rho](\frac{d}{2})$ admits a sub-quotient with W_E-action given by an irreducible representation ϕ then the Fargues-Scholze parameter ϕ_ρ^{FS} admits a sub-quotient given by $\phi^\vee$. However, in his paper, the Shtuka space $\mathrm{Sht}(G,b,\mu)_\infty$ parametrizes modifications of the form $\mathscr{E}_0 \dashrightarrow \mathscr{E}_b$ of type μ, whereas for us it parametrizes modifications of type μ^{-1}. This explains why, under our conventions, there is no dual appearing.

Proof. We first show that the assumption on the cohomology sheaves of $T_V(A)$ implies that averaging operator

$$Av_{\phi,V}(A) = R\Gamma(W_{\mathbb{Q}_p}, T_V(A) \otimes \phi^\vee)$$

is non-trivial. Since A is compact we know by Theorem 3.2 that $T_V(A)$ is also comapct and therefore supported on a finite number of HN-strata. By applying excision with respect to the HN-strata of Bun_G [FS21, Proposition VII.7.3], we can assume, without loss of generality, that $T_V(A)$ is supported on a single stalk, where it is given by a complex of smooth irreducible representations of $J_b(\mathbb{Q}_p)$ for $b \in B(G)$. Moreover, since A is ULA, by Theorem 3.2 the sheaf $T_V(A)$ is also ULA and therefore it follows that, for all open compact $K \subset J_b(\mathbb{Q}_p)$, $T_V(A)^K$ is a perfect complex of $\overline{\mathbb{Q}}_\ell$-vector spaces. Writing $T_V(A) := \mathrm{colim}_{K \to \{1\}} T_V(A)^K$ and using that cohomology commutes with colimits, this allows us to apply results from the Galois cohomology of $W_{\mathbb{Q}_p}$ on finite-dimensional $\overline{\mathbb{Q}}_\ell$ vector spaces to $T_V(A)$. To do this, we consider the spectral sequence

$$E_2^{p,q} = H^p(W_{\mathbb{Q}_p}, H^q(T_V(A) \otimes \phi^\vee)) \Longrightarrow H^{p+q}(R\Gamma(W_{\mathbb{Q}_p}, T_V(A) \otimes \phi^\vee))$$

where cohomology is being taken with respect to the standard t-structure on $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$. Now recall that the cohomological dimension of $W_{\mathbb{Q}_p}$ acting on finite dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces is 2. Therefore, this sequence degenerates at the E_3 page. Moreover, the only non-zero degeneracy maps are given by

$$E_2^{0,q+1} = H^0(W_{\mathbb{Q}_p}, H^{q+1}(T_V(A) \otimes \phi^\vee)) \to E_2^{2,q} = H^2(W_{\mathbb{Q}_p}, H^q(T_V(A) \otimes \phi^\vee))$$

However, using local Tate-duality on the RHS, we can rewrite this differential as a map:

$$(H^{q+1}(T_V(A)) \otimes \phi^\vee)^{W_{\mathbb{Q}_p}} \to H^0(W_{\mathbb{Q}_p}, H^q(T_V(A) \otimes \phi^\vee)^\vee(1))^\vee \simeq ((H^q(T_V(A))^\vee \otimes \phi(1))^{W_{\mathbb{Q}_p}})^\vee$$

Now the term on the RHS (resp. LHS) will only be non-zero if $\phi(1)$ (resp. ϕ) occurs as a sub-quotient of $H^q(T_V(A))$ (resp. $H^{q+1}(T_V(A))$). By assumption, this will be true for some value of q, but now, since the Euler-Poincaré characteristic of $W_{\mathbb{Q}_p}$ acting on $\overline{\mathbb{Q}}_\ell$-vector spaces is 0, one of these values being non-zero implies the $H^1(W_{\mathbb{Q}_p}, -)$ of some cohomology sheaf of T_V must also be non-zero. This will then give rise to a non-zero contribution to the cohomology of $R\Gamma(W_{\mathbb{Q}_p}, T_V(A) \otimes \phi^\vee)$ so the averaging operator $Av_{V,\phi}(A)$ applied to A is non-zero. Lemma 3.9 therefore tells us that the spectral action of the perfect complex $\mathscr{A}v_{V,\phi}$ on A is non-trivial. If x denotes the closed $\overline{\mathbb{Q}}_\ell$-point in the coarse moduli space of Langlands parameters defined by ϕ_A^{FS} with preimage $\pi^{-1}(x)$ in the stack of Langlands parameters, Lemma 3.8 tells us that $\mathscr{A}v_{V,\phi}$ must have non-zero support on $\pi^{-1}(x)$. The $\overline{\mathbb{Q}}_\ell$-points of $\pi^{-1}(x)$ correspond to the set of Langlands parameters whose semisimplification is precisely ϕ_A^{FS}. The previous analysis tells us that the evaluation of the perfect complex $R\Gamma(W_{\mathbb{Q}_p}, C_V \otimes \phi^\vee)$ at some such point, corresponding to an L-parameter $\tilde{\phi} : W_{\mathbb{Q}_p} \to {}^LG(\overline{\mathbb{Q}}_\ell)$, must be non-zero. This evaluation is precisely the complex

$$R\Gamma(W_{\mathbb{Q}_p}, r_V \circ \tilde{\phi} \otimes \phi^\vee)$$

However, by again applying local Tate-duality, this can only be the case if $r_V \circ \tilde{\phi}$ has a sub-quotient isomorphic to ϕ or $\phi(1)$. Since ϕ is irreducible, this can only happen if $r_V \circ \tilde{\phi}^{\mathrm{ss}} = r_V \circ \phi_A^{\mathrm{FS}}$ has this property. $\square$

We conclude this section by reviewing how the spectral action behaves on objects with supercuspidal Fargues-Scholze parameter along the lines of [FS21, Section X.2]. These results will be used to deduce a strong form of the Kottwitz conjecture from compatibility. We recall that a supercuspidal parameter, viewed as a continuous 1-cocyle with respect to the $W_{\mathbb{Q}_p}$-action on $\hat{G}(\overline{\mathbb{Q}}_\ell)$, denoted $\phi : W_{\mathbb{Q}_p} \to \hat{G}(\overline{\mathbb{Q}}_\ell)$, satisfies the property that it doesn't factor through any $\hat{P}(\overline{\mathbb{Q}}_\ell)$ for P a parabolic subgroup of G. Equivalently, this is the same as insisting that, if $S_\phi := Z_{\hat{G}}(\mathrm{Im}(\phi))$ as before, then the quotient $S_\phi / Z(\hat{G})^\Gamma$ is finite, where $\Gamma := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. From this, it follows by deformation theory (See the proof of [Dat+20, Theorem 1.6] for more details) that the unramified twists of the parameter ϕ define a connected component

$$C_\phi \hookrightarrow [\mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell} / \hat{G}]$$

giving rise to a direct summand

$$\mathrm{Perf}(C_\phi) \hookrightarrow \mathrm{Perf}([\mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell} / \hat{G}])$$

Therefore, the spectral action gives rise to a corresponding direct summand

$$\mathrm{D}_{\mathrm{lis}}^{C_\phi}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega \subset \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$$

such that the Schur-irreducible objects in this subcategory all have Fargues-Scholze parameter given by an unramified twist of ϕ. Now, it follows by Proposition 3.14 in the next section and Theorem 3.6 (4), that, since ϕ is supercuspidal, the restriction of any object in $\mathrm{D}_{lis}^{C_\phi}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$ to any non-basic HN-strata of Bun_G must be zero. Therefore, one obtains a decomposition

$$\mathrm{D}_{\mathrm{lis}}^{C_\phi}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega \simeq \bigoplus_{b \in B(G)_{basic}} \mathrm{D}^{C_\phi}(J_b(E), \overline{\mathbb{Q}}_\ell)^\omega$$

where J_b is the σ-centralizer of b and $\mathrm{D}^{C_\phi}(J_b(E), \overline{\mathbb{Q}}_\ell)^\omega \subset \mathrm{D}(J_b(E), \overline{\mathbb{Q}}_\ell)^\omega$ is a full subcategory of the derived category of compact objects in smooth representations of $J_b(E)$. It also follows again by Theorem 3.6 (4) that the Schur-irreducible objects of any $\mathrm{D}^{C_\phi}(J_b(E), \overline{\mathbb{Q}}_\ell)^\omega$ must lie only in the supercuspidal components of the Bernstein center. Now to further analyze this we fix a character χ of $Z(G)(\mathbb{Q}_p)$ and consider the subcategory

$$\mathrm{D}_{\mathrm{lis},\chi}^{C_\phi}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega \simeq \bigoplus_{b \in B(G)_{basic}} \mathrm{D}_\chi^{C_\phi}(J_b(E), \overline{\mathbb{Q}}_\ell)^\omega$$

where $\mathrm{D}_\chi^{C_\phi}(J_b(E), \overline{\mathbb{Q}}_\ell)^\omega$ is the derived subcategory of $\mathrm{D}^{C_\phi}(J_b(E), \overline{\mathbb{Q}}_\ell)^\omega$ generated by compact objects with fixed central character χ, via the natural isomorphism $Z(J_b)(\mathbb{Q}_p) \simeq Z(G)(\mathbb{Q}_p)$ (where we recall that J_b is an extended pure inner form of G). One can see that the spectral action of $\mathrm{Perf}(C_\phi)$ preserves this subcategory. Indeed, it follows by [FS21, Theorem I.8.2] that $\mathrm{Perf}([\mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}])$ and in turn $\mathrm{Perf}(C_\phi)$ is generated under cones and retracts by the image of $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^LG)$ in $\mathrm{Perf}([\mathscr{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}])$. This reduces us to checking that Hecke operators preserve this subcategory, which, in turn reduces to the observation that, if one looks at the simultaneous action of $J_b(\mathbb{Q}_p) \times J_{b'}(\mathbb{Q}_p)$ on the space parametrizing modifications $\mathscr{E}_b \to \mathscr{E}_{b'}$, for b and b' in $B(G)_{basic}$ that, under the canonical identification $Z(J_{b'})(\mathbb{Q}_p) \simeq Z(J_b)(\mathbb{Q}_p)$, the diagonally embedded center acts trivially. This follows since an element in the center of $J_b(\mathbb{Q}_p)$ acts on the modification by the inverse of an element in the corresponding center of $J_{b'}(\mathbb{Q}_p)$, where the inverse appears from the fact that $J_b(\mathbb{Q}_p)$ is acting on the left and $J_{b'}(\mathbb{Q}_p)$ is acting on the right.

Now, via local class field theory, we take χ to be the central character determined by ϕ and local class field theory (as in [Bor79, Section 10.1]). Since all Schur-irreducible objects in $\mathrm{D}_\chi^{C_\phi}(J_b(E), \overline{\mathbb{Q}}_\ell)$ lie in the supercuspidal component of the Bernstein-center and have fixed central character χ and, by [Cas95, Theorem 5.4.1], supercuspidal representations are injective/projective in the category of smooth representations with fixed central character, we can write $\mathrm{D}_{\chi}^{C_\phi}(J_b(E),\overline{\mathbb{Q}}_\ell)^\omega = \bigoplus_\pi \mathrm{Perf}(\overline{\mathbb{Q}}_\ell)\otimes\pi$, where π runs over all supercuspidal representations of $J_b(E)$ with central character χ which, a priori, have Fargues-Scholze parameter given by an unramified twist of ϕ, but, by Theorem 3.6 (2), must indeed be equal to ϕ.

Now, the closed point of C_ϕ determined by the parameter ϕ gives rise to a closed embedding

$$[\overline{\mathbb{Q}}_\ell/S_\phi] \hookrightarrow C_\phi$$

and in turn a fully faithful embedding

$$\mathrm{Perf}([\overline{\mathbb{Q}}_\ell/S_\phi]) \hookrightarrow \mathrm{Perf}(C_\phi)$$

The above discussion and Lemma 3.8 imply that the action of $\mathrm{Perf}(C_\phi)$ factors over this subcategory in the sense that everything not in the image of this must act by zero. All in all, we conclude that we have a decomposition

$$\mathrm{D}_{\mathrm{lis},\chi}^{C_\phi}(\mathrm{Bun}_G,\overline{\mathbb{Q}}_\ell)^\omega = \bigoplus_{b\in B(G)_{basic}}\bigoplus_{\pi_b}\mathrm{Perf}(\overline{\mathbb{Q}}_\ell)\otimes\pi_b$$

where the π_b runs over all supercuspidal representations of $J_b(E)$ with Fargues-Scholze parameter $\phi_{\pi_b}^{\mathrm{FS}} = \phi$. Moreover, the RHS carries an action of $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(S_\phi)$, the category of finite-dimensional $\overline{\mathbb{Q}}_\ell$-representations of S_ϕ. Therefore, given $W \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(S_\phi)$, we get an object:

$$\mathrm{Act}_W(\pi_b) \in \bigoplus_{b\in B(G)_{basic}}\bigoplus_{\pi_b}\mathrm{Perf}(\overline{\mathbb{Q}}_\ell)\otimes\pi_b$$

Assume that $W|_{Z(\hat{G})^\Gamma}$ is isotypic, given by some character $\eta : Z(\hat{G})^\Gamma \to \overline{\mathbb{Q}}_\ell^*$. As $Z(\hat{G})^\Gamma$ is diagonalizable with characters given by $B(G)_{basic} \xrightarrow{\simeq} \pi_1(G)_\Gamma$ via the κ map, we obtain an element $b_\eta \in B(G)_{basic}$. Then $\mathrm{Act}_W(\pi_b)$ is concentrated on the basic HN-strata given by $b' = b + b_\eta$. Therefore, we get an isomorphism

$$\mathrm{Act}_W(\pi_b) \simeq \bigoplus_{\pi_{b'}} V_{\pi_{b'}}\otimes\pi_{b'}$$

where $V_{\pi_{b'}} \in \mathrm{Perf}(\overline{\mathbb{Q}}_\ell)$ and $\pi_{b'}$ runs over all supercuspidals of $J_{b'}$ with Fargues-Scholze parameter $\phi^{\mathrm{FS}}_{\pi_{b'}} = \phi$. With this in hand, we can elucidate the $W_{\mathbb{Q}_p}$-action on the Hecke operator applied to a smooth irreducible object with supercuspidal Fargues-Scholze parameter, similar to what was done in Lemma 3.10 in the general case. Namely, given $V \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^LG)$, we obtain a vector bundle on $[\overline{\mathbb{Q}}_\ell/S_\phi]$ with $W_{\mathbb{Q}_p}$-action given by ϕ. In other words, we have a functor:

$$\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^LG) \to \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(S_\phi)^{BW_{\mathbb{Q}_p}}$$

Theorem 3.7 and the above discussion imply that the action of the image of V in $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(S_\phi)^{BW_{\mathbb{Q}_p}}$ acting via the spectral action on $\mathrm{D}^{C_\phi}_{\mathrm{lis},\chi}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ is precisely the Hecke operator T_V. This tells us that, if we decompose $r_V \circ \phi$ viewed as a representation of S_ϕ as a direct sum $\bigoplus_{i\in I} W_i \otimes \sigma_i$ where $W_i \in Rep(S_\phi)$ is irreducible and σ_i is a continuous finite-dimensional representation of $W_{\mathbb{Q}_p}$, then we have an isomorphism

$$T_V(\pi) \simeq \bigoplus_{i\in I} \mathrm{Act}_{W_i}(\pi) \otimes \sigma_i$$

as $J_{b'}(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$-modules. We now summarize the above discussion as a corollary for future use.

Corollary 1.3.23. *Let ϕ be a supercuspidal parameter of G, $b \in B(G)_{basic}$ a basic elment, $V \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^LG)$ an irreducible representation of some highest weight μ with dominant inverse μ^{-1}, and π_b a representation of $J_b(E)$ with Fargues-Scholze parameter equal to ϕ. We set $b_\mu \in B(G,\mu^{-1})$ to be the unique basic element and $b' = b + b_\mu$. If we decompose $r_V \circ \phi$ viewed as representation of S_ϕ as a direct sum $\bigoplus_{i\in I} W_i \otimes \sigma_i$, where $W_i \in Rep(S_\phi)$ is irreducible and σ_i is a continuous finite-dimensional representation of $W_{\mathbb{Q}_p}$, then there exists an isomorphism of $W_{\mathbb{Q}_p} \times J_{b'}(\mathbb{Q}_p)$-modules*

$$T_\mu(\pi_b) \simeq \bigoplus_{i\in I} \mathrm{Act}_{W_i}(\pi) \otimes \sigma_i$$

where $\mathrm{Act}_{W_i}(\pi) \simeq \bigoplus_{\pi_{b'}} V_{\pi_{b'}} \otimes \pi_{b'}$ with $V_{\pi_{b'}} \in \mathrm{Perf}(\overline{\mathbb{Q}}_\ell)$ and $\pi_{b'}$ ranging over supercuspidal representation of $J_{b'}(\mathbb{Q}_p)$ with Fargues-Scholze parameter equal to ϕ.

Remark 1.3.24. As we will start to see in the next section, the work of Hansen [Han20], Hansen-Kaletha-Weinstein [HKW22], and compatibility of the Fargues-Scholze and Gan-Takeda/Gan-Tantono local Langlands correspondence will allow us to use this Corollary to prove Theorem 1.3. This is suggested already by

Corollary 3.3, which shows us that $T_\mu(\pi_b)$ can be computed explicitly using the cohomology of local Shimura varieties.

1.3.3 Compatibility with the Local Langlands for GSp_4 and $\mathrm{GU}_2(D)$

With the results of the previous section in place, we can now start making progress towards our goal of proving compatibility. So we again let $G = \mathrm{Res}_{L/\mathbb{Q}_p}\mathrm{GSp}_4$ and $J = \mathrm{Res}_{L/\mathbb{Q}_p}\mathrm{GU}_2(D)$, for $L/\mathbb{Q}_p$ a finite extension. As mentioned in section 1, the case where a representation $\pi \in \Pi(G)$ (resp. $\rho \in \Pi(J)$), is a sub-quotient of a parabolic induction easily follows from Theorem 3.6 (3), (4), (5), and compatibility of the (semi-simplified) Gan-Takeda (resp. Gan-Tantono) parameter with parabolic induction. We record this as a corollary now.

Corollary 1.3.25. *Let $\pi \in \Pi(G)$ (resp. $\rho \in \Pi(J)$) be representations occurring as a sub-quotient of a parabolic induction. For such π (resp. ρ), the Fargues-Scholze and Gan-Takeda (resp. Gan-Tantono) local Langlands correspondences are compatible.*

To tackle the remaining cases where the L-parameter ϕ is mixed supercuspidal or supercuspidal, we note that these are the cases where the L-parameter is discrete (i.e the L-parameter does not factor through a Levi subgroup). This case is disposable to the results of Hansen-Kaletha-Weinstein [HKW22]. We will now let μ be the Siegel cocharacter of G so that the σ-centralizer of the unique basic element $b \in B(G,\mu)$ is J. We now state the main result of [HKW22] specialized to the case of the Shimura datum (G,b,μ).

Theorem 1.3.26. *[HKW22, Theorem 1.0.2] Let ϕ be a mixed supercuspidal or supercuspidal parameter and $S_\phi := Z_{\hat{G}}(\mathrm{Im}(\phi))$ as before. Let $\Pi_\phi(G)$ and $\Pi_\phi(J)$ denote the L-packets over ϕ. Set $\pi \in \Pi_\phi(G)$ (resp. $\rho \in \Pi_\phi(J)$) to be smooth irreducible representations of G (resp. J). If ϕ is supercuspidal or mixed supercuspidal, we have the following equality in the Grothendieck group $K_0(G(\mathbb{Q}_p))^{ell}$ of elliptic admissible representations of $G(\mathbb{Q}_p)$ of finite length*

$$[R\Gamma_c^\flat(G,b,\mu)[\rho]] = -\sum_{\pi\in\Pi_\phi(G)} Hom_{S_\phi}(\delta_{\pi,\rho}, \mathrm{std}\circ\phi)\pi$$

and the following equality

$$[R\Gamma_c^\flat(G,b,\mu)[\pi]] = -\sum_{\rho\in\Pi_\phi(J)} Hom_{S_\phi}(\delta_{\pi,\rho}^\vee, (\mathrm{std}\circ\phi)^\vee)\rho$$

in the Grothendieck group of elliptic admissible $J(\mathbb{Q}_p)$-representations of finite length, where $\delta_{\pi,\rho}$ is the algebraic representation of S_ϕ in Definition 2.3. Moreover, if the Fargues-Scholze parameter of π (resp. ρ) is supercuspidal this is true in the Grothendieck group $K_0(G(\mathbb{Q}_p))$ (resp. $K_0(J_b(\mathbb{Q}_p))$) of all admissible representations of finite length.

Remark 1.3.27.

1. To deduce the result for the π-isotypic part, we have implicitly used the two towers isomorphism $\mathrm{Sht}(G,b,\mu)_\infty \simeq \mathrm{Sht}(J,\hat{b},\mu^{-1})_\infty$, where μ^{-1} is a dominant inverse to μ and $\hat{b} = b^{-1} \in B(G,\mu^{-1})$ is the unique basic element [SW20a, Corollary 23.3.2.], and $B(G,\mu^{-1}) \simeq B(J,\mu^{-1})$ under the inner twisting. This inverse explains the appearance of duals in the formula for the π-isotypic part.

2. We see that, via Corollary 3.3, this, in the case that ϕ is supercuspidal, should provide us insight into the multiplicity spaces $V_{\pi_{b'}}$ appearing in Corollary 3.11, assuming compatibility of the Fargues-Scholze and Gan-Tantono/Gan-Takeda local Langlands correspondences. Namely, we will see later (Theorem 3.17 and 3.18) that $R\Gamma_c(G,b,\mu)[\rho] \simeq R\Gamma_c^\flat(G,b,\mu)[\rho]$ and is concentrated in middle degree 3 if ϕ_ρ^{FS} is supercuspidal. Assuming compatibility, Corollary 3.11 will therefore tell us that $R\Gamma_c(G,b,\mu)[\rho]$ will be a direct sum over representations $\pi \in \Pi_\phi(G)$ with W_L-action given by $\mathrm{std} \circ \phi_\rho^{\mathrm{FS}} = \mathrm{std} \circ \phi_\rho$ decomposed as a representation of S_ϕ. The summands in the decomposed S_ϕ-representation correspond to the weight spaces appearing in the above description in the Grothendieck group.

We now wish to write out the precise formula for the ρ and π-isotypic parts, using the refined local Langlands discussed in section 2.

1. (ϕ stable) In this case, the L-packet $\Pi_\phi(G) = \{\pi\}$ is a singleton so the RHS of the above formula for the ρ-isotypic part has one term

$$-\pi Hom_{S_\phi}(\delta_{\pi,\rho}, \mathrm{std} \circ \phi_\rho)$$

In this case, $S_\phi = \mathbb{G}_m$ and $\delta_{\pi,\rho}$ is simply the identity representation. Thus, this Hom space gets identified with the characters of GL_4, so the formula reduces to

$$-4\pi$$

2. (ϕ endoscopic) In this case, the L-packet has size 2 and, as seen in section 2.1, $\Pi_\phi(G) = \{\pi^+, \pi^-\}$, where π^+ (resp. π^-) corresponds to the trivial

(resp. non-trivial) character of the component group. ρ can be either of the two representations corresponding to the irreducible representation of S_ϕ given by τ_i for $i = 1,2$ the projection to the two coordinates of S_ϕ. However, the RHS remains the same regardless of which one it corresponds to. So, without loss of generality, we assume that $\rho = \rho_1$. Then the RHS of the above formula for the ρ-isotypic part has two terms

$$\pi^+ Hom_{S_\phi}(\tau_1, \mathrm{std} \circ \phi_\rho)$$

and

$$\pi^- Hom_{S_\phi}(\tau_1 \otimes \tau_{\pi^-} \simeq \tau_2, \mathrm{std} \circ \phi_\rho)$$

However, writing $\mathrm{std} \circ \phi_\rho \simeq \phi_1 \oplus \phi_2$, these get identified with

$$-\pi^+ Hom_{\overline{\mathbb{Q}}_\ell^*}(\overline{\mathbb{Q}}_\ell^*, \phi_1)$$

and

$$-\pi^- Hom_{\overline{\mathbb{Q}}_\ell^*}(\overline{\mathbb{Q}}_\ell^*, \phi_2)$$

which will both be identified with characters of GL_2. Thus, the RHS is equal to

$$-2\pi^+ - 2\pi^-$$

Similarly, for the π-isotypic part, we get that the RHS of the above formula is given by

$$-4\rho$$

in the stable case and

$$-2\rho_1 - 2\rho_2$$

in the endoscopic case.

As mentioned in section 1.2, we will now use the previous result to perform a bootstrap to the supercuspidal representations occurring in the L-packets $\Pi_\phi(G)$ (resp. $\Pi_\phi(J)$), for ϕ a mixed supercuspidal parameter. For this, we will mention one last result from the Fargues-Scholze local Langlands correspondence.

Proposition 1.3.28. *[FS21, Section IX.7.1] For G any connected reductive group over $\mathbb{Q}_p$, the action of the excursion algebra on $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ commutes with Hecke operators. Moreover, it is compatible with restriction to the HN-strata Bun_G^b for $b \in B(G)$ in the following sense. Given a Schur irreducible object $A \in$*

$\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ *and* $b \in B(G)$*, if we let* B *be a smooth irreducible constituent of* $j_b^*(A)$ *then we can view it as a sheaf on the neutral strata of* $\mathrm{Bun}_{J_b}^1$ *and let* ϕ_B^{FS} *denote its Fargues-Scholze parameter with respect to the excursion algebra on* Bun_{J_b}*. Then the Fargues-Scholze parameter of* A *is the composition*

$$W_{\mathbb{Q}_p} \xrightarrow{\phi_B^{\mathrm{FS}}} {}^L J_b(\overline{\mathbb{Q}}_\ell) \to {}^L G(\overline{\mathbb{Q}}_\ell)$$

where ${}^L J_b(\overline{\mathbb{Q}}_\ell) \to {}^L G(\overline{\mathbb{Q}}_\ell)$ *is the* twisted *embedding, as defined in [FS21, Page 327].*

Remark 1.3.29. The commutation of Hecke operators and the excursion algebra follows from the interpretation of the excursion algebra in terms of endomorphisms coming from multiplication by the ring of global functions of the stack of L-parameters, as discussed in Section 3.2.

From this, we can deduce the following useful corollary.

Corollary 1.3.30. *For* G *any connected reductive group with* (G, b, μ) *a local Shimura datum and* $\pi \in \Pi(G)$ *and* $\rho \in \Pi(J_b)$ *smooth irreducible representations. All smooth irreducible representations occurring in the cohomology of* $R\Gamma_c(G, b, \mu)[\pi]$ *have Fargues-Scholze parameter equal to* ϕ_π^{FS}*. Similarly, all smooth irreducible representations occurring in the cohomology of* $R\Gamma_c(G, b, \mu)[\rho]$ *have Fargues-Scholze parameter equal to* ϕ_ρ^{FS}*. The same is also true for* $R\Gamma_c^\flat(G, b, \mu)[\rho]$ *and* $R\Gamma_c^\flat(G, b, \mu)[\pi]$.

Proof. The first part follows immediately from Proposition 3.14 and Corollary 3.3. It remains to see the same is true for the complexes $R\Gamma_c^\flat(G, b, \mu)[\rho]$ and $R\Gamma_c^\flat(G, b, \mu)[\pi]$. This can be done by writing them as $j_1^* T_\mu R j_{b*}(\rho)$ and $j_b^* T_{\mu^{-1}} R j_{1*}(\pi)$, as in [FS21, Section IX.7.1], where it again follows from Proposition 3.14 and Corollary 3.3. □

We now exploit this corollary to deduce compatibility in the mixed supercuspidal case.

Corollary 1.3.31. *Let* ϕ *be an L-parameter of Howe-Piatetski–Schapiro or Saito-Kurokawa type. Then, for any* $\pi \in \Pi_\phi(G)$ *(resp.* $\rho \in \Pi_\phi(J)$*), the Fargues-Scholze and Gan-Takeda (resp. Gan-Tantono) local Langlands correspondences are compatible.*

Proof. We give the proof for the Gan-Takeda local Langlands correspondence with the proof for the Gan-Tantono correspondence being completely analogous. If ϕ is of Saito-Kurokawa type then, as seen in section 2.2, we can write $\Pi_{\phi}(G) = \{\pi_{sc}, \pi_{disc}\}$ and $\Pi_{\phi}(J) = \{\rho_{sc}, \rho_{disc}\}$, where π_{sc} (resp. ρ_{sc}) is a supercuspidal representation of G (resp. J) and π_{disc} (resp. ρ_{disc}) is a non-supercuspidal representaiton. We note that the Gan-Takeda (resp. Gan-Tantono) correspondences are compatible with the Fargues-Scholze correspondence for π_{disc} (resp. ρ_{disc}), by Corollary 3.12.

If we let μ be the Siegel cocharacter and $b \in B(G,\mu)$ be the unique basic element. Then the σ-centralizer J_b is isomorphic to J and we can consider the complex

$$R\Gamma_c^{\flat}(G,b,\mu)[\rho_{disc}]$$

of $J(\mathbb{Q}_p) \times W_L$-representations. We then let $R\Gamma_c^{\flat}(G,b,\mu)[\rho_{disc}]_{sc}$ denote the direct summand of $R\Gamma_c^{\flat}(G,b,\mu)[\rho_{disc}]$, where $J(\mathbb{Q}_p)$ acts via a supercuspidal representation. Theorem 3.13 tells us that we can describe this complex in the Grothendieck group of admissible $G(\mathbb{Q}_p)$-representations of finite length as

$$[R\Gamma_c^{\flat}(G,b,\mu)[\rho_{disc}]_{sc}] = -2\pi_{sc}$$

which tells us that π_{sc}, occurs as a non-zero sub-quotient of the complex $R\Gamma_c^{\flat}(G,b,\mu)[\rho_{disc}]$. By Corollary 3.15, we know that we have an equality:

$$\phi^{\mathrm{FS}}_{\rho_{disc}} = \phi^{\mathrm{FS}}_{\pi_{sc}}$$

However, Corollary 3.12 tells us that $\phi^{\mathrm{FS}}_{\rho_{disc}} = \phi^{\mathrm{ss}}_{\rho_{disc}}$, which is equal to $\phi^{\mathrm{ss}}_{\pi_{sc}}$, so we get the desired equality. The analysis in the Howe-Piatetski–Schapiro case is the same, where one can look at the ρ-isotypic part for any of the two non-supercuspidals in $\Pi_{\phi}(J)$. $\square$

In the remaining part of this section, we address proving compatibility in the case where the parameter ϕ is supercuspidal. Before tackling the question of compatibility, we address some geometric properties of the sheaves $\mathscr{F}_{\rho}$, for ρ with supercuspidal Fargues-Scholze parameter. This will be leveraged in proving the strong form of the Kottwitz Conjecture for the ρ and π-isotypic parts in section 8, as mentioned in Remark 3.10 (2).

Now, considering again G a general connected reductive group, $b \in B(G)_{basic}$

a basic element, and a smooth irreducible representation ρ of the σ-centralizer $J_b(\mathbb{Q}_p)$. We will now address some further consequences of the Fargues-Scholze parameter ϕ_ρ^{FS} being supercuspidal. It turns out that the sheaves defined by representations with these parameters have interesting geometric properties, which were leveraged in [Han20] to prove various general results on the cohomology groups. In particular, Hansen shows the following:

Theorem 1.3.32. *[Han20, Theorem 1.1.] Let (G,b,μ) be a basic local Shimura datum with E the reflex field of μ as before, and let ρ be a smooth irreducible representation of $J_b(\mathbb{Q}_p)$. Suppose the following conditions hold:*

1. *The spaces $(\mathrm{Sht}(G,b,\mu)_K)_{K\subset G(\mathbb{Q}_p)}$ occur in the basic uniformization at p of a global Shimura variety in the sense of Definition 4.1.*
2. *The Fargues-Scholze parameter $\phi_\rho^{\mathrm{FS}} : W_{\mathbb{Q}_p} \to {}^L G(\overline{\mathbb{Q}}_\ell)$ is supercuspidal.*

Then the complex $R\Gamma_c(G,b,\mu)[\rho]$ defined above is concentrated in middle degree $d = \dim(\mathrm{Sht}(G,b,\mu)_\infty) = \langle 2\rho_G, \mu\rangle$.

One of the key ideas in the argument is to exploit the behavior of the sheaf $j_{b!}(\mathscr{F}_\rho)$ under Verdier duality, where $j_b : \mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$ is the inclusion of the open HN-strata corresponding to $b \in B(G)_{basic}$. In particular, by Proposition 3.14 and Theorem 3.6 (4), one can see that the natural map $j_{b!}(\mathscr{F}_\rho) \to Rj_{b*}(\mathscr{F}_\rho)$ is an isomorphism. Namely, Proposition 3.14 implies that a non-zero restriction of $Rj_{b*}(\mathscr{F}_\rho)$ to any non-basic HN-strata must be valued in representations having Fargues-Scholze parameter ϕ_ρ^{FS} under the relevant twisted embedding, which is impossible since the σ-centralizers of non-basic elements are extended pure inner forms of proper Levi subgroups of G and, by assumption, the parameter ϕ_ρ^{FS} is supercuspidal. This implies that, if we apply Verdier duality to both sides of the isomorphism

$$j_{\mathbf{1}}^* T_\mu j_{b!}(\mathscr{F}_\rho) \simeq R\Gamma_c(G,b,\mu)[\rho][d](\frac{d}{2})$$

supplied by Corollary 3.3, we see that the LHS is isomorphic to

$$j_{\mathbf{1}}^* T_\mu j_{b!}(\mathscr{F}_{\rho^*}) \simeq R\Gamma_c(G,b,\mu)[\rho^*][d](\frac{d}{2})$$

On the other hand, on the RHS we act through Verdier duality on the tower $(\mathrm{Sht}(G,b,\mu)_K)_{K\subset G(\mathbb{Q}_p)}$, which are smooth rigid spaces of dimension d. So, in particular, the dualizing object is isomorphic to $\overline{\mathbb{Q}}_\ell[2d](d)$. This allows one to deduce the following consequence for the cohomology groups $R\Gamma_c(G,b,\mu)[\rho]$.

Theorem 1.3.33. *[Han20, Theorem 1.3, Theorem 2.23] Fix a basic local Shimura datum (G,b,μ) and let ρ be representation of $J_b(\mathbb{Q}_p)$ with supercuspidal Fargues-Scholze parameter. Then there is a natural isomorphism*

$$R\mathscr{H}om(R\Gamma_c(G,b,\mu)[\rho],\overline{\mathbb{Q}}_\ell) \simeq R\Gamma_c(G,b,\mu)[\rho^*][2d](d)$$

as W_E-equivariant objects of $\mathrm{D}(J_b(\mathbb{Q}_p),\overline{\mathbb{Q}}_\ell)$, where ρ^* is the contragredient of ρ. In particular, we have a natural W_E-equivariant isomorphism of admissible $G(\mathbb{Q}_p)$-representations for all $0 \leq i \leq 2d$

$$H^i(R\Gamma_c(G,b,\mu)[\rho])^* \simeq H^{2d-i}(R\Gamma_c(G,b,\mu)[\rho^*])(d)$$

Remark 1.3.34. As noted in Remark 3.4, the LHS of the above formula is isomorphic to $R\Gamma_c^\flat(G,b,\mu)[\rho^*][2d](d)$, so it follows by cancelling the shifts and Tate twists and relaxing contragradients that one has an isomorphism

$$R\Gamma_c^\flat(G,b,\mu)[\rho] \simeq R\Gamma_c(G,b,\mu)[\rho]$$

as $J_b(\mathbb{Q}_p) \times W_E$-representations for all such ρ.

Now we turn our attention to the question of showing compatibility for supercuspidal parameters assuming Proposition 1.4. So again let $L/\mathbb{Q}_p$ be a finite extension and let $G := \mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GSp}_4)$ be the restriction of scalars of GSp_4 and $J := \mathrm{Res}_{L/\mathbb{Q}_p}\mathrm{GU}_2(D)$ the unique non-split inner form as before. As we will see in section 8, it essentially follow from Theorem 3.13 and Corollary 3.15 that showing compatibility for $\rho \in \Pi(J)$ with supercuspidal Gan-Tantono parameter implies the corresponding statement for $\pi \in \Pi(G)$ with supercuspidal Gan-Takeda parameter. So we fix such a ρ and assume that the Gan-Tantono parameter ϕ_ρ is endoscopic supercuspidal with the stable case being strictly easier. We will write $\mathrm{std} \circ \phi_\rho \simeq \phi_1 \oplus \phi_2$ for ϕ_i distinct irreducible 2-dimensional representations of W_L and let μ be the Siegel cocharacter. The Shtuka space $\mathrm{Sht}(G,b,\mu)_\infty$ in this case will have dimension $d := \langle 2\rho_G, \mu \rangle = 3$. We will assume for the rest of this section that Proposition 1.4 is true.

Proposition 1.3.35. *Let ϕ be a supercuspidal parameter with associated L-packet $\Pi_\phi(J)$. Then the direct summand of*

$$\bigoplus_{\rho' \in \Pi_\phi(J)} R\Gamma_c(G,b,\mu)[\rho']$$

where $G(\mathbb{Q}_p)$ acts via a supercuspidal representation

$$\bigoplus_{\rho' \in \Pi_\phi(J)} R\Gamma_c(G,b,\mu)[\rho']_{sc}$$

is concentrated in middle degree 3 *and admits a non-zero W_L-stable sub-quotient with W_L-action given by* $\mathrm{std} \circ \phi \otimes |\cdot|^{-3/2}$.

First, we combine this with the following lemma.

Lemma 1.3.36. *Let ϕ be a supercuspidal parameter then all representations in the L-packet $\Pi_\phi(J)$ have the same Fargues-Scholze parameter.*

Proof. We choose a $\pi \in \Pi_\phi(G)$ and then apply Corollary 3.15 to deduce that all representations occurring in the cohomology of $R\Gamma_c^\flat(G,b,\mu)[\pi]$ have Fargues-Scholze parameter equal to ϕ_π^{FS}. However, by Theorem 3.13, we have that all representations in $\rho \in \Pi_\phi(J)$ occur in the cohomology of $R\Gamma_c^\flat(G,b,\mu)[\pi]$, so their Fargues-Scholze parameters are the same as desired. $\square$

With this in hand, we are ready to prove the key consequence of Proposition 1.4 using the results on the spectral action obtained in section 3.2.

Corollary 1.3.37. *Assume that $L/\mathbb{Q}_p$ is an unramified extension and that $p > 2$ and that Proposition 1.4 is true. Then, for ρ a smooth irreducible representation of $J(\mathbb{Q}_p)$ with supercuspidal Gan-Tantono parameter ϕ, the Gan-Tantono and Fargues-Scholze correspondences coincide.*

Proof. As mentioned in the introduction, the key will be the isomorphism

$$\bigoplus_{\rho' \in \Pi_\phi(J)} j_{\mathbf{1}}^* T_\mu j_{b!}(\mathscr{F}_{\rho'}) \simeq \bigoplus_{\rho' \in \Pi_\phi(J)} R\Gamma_c(G,b,\mu)[\rho'][3](\frac{3}{2})$$

supplied by Corollary 3.3. Now Proposition 1.4 tells us that one of the summands on the RHS admits a sub-quotient with W_L-action given by ϕ_1 and one of them admits a sub-quotient with W_L-action given by ϕ_2. Applying Lemma 3.20 and 3.10 therefore tells us that $\mathrm{std} \circ \phi_\rho^{\mathrm{FS}}$ admits a sub-quotient isomorphic to ϕ_1 or $\phi_1(1)$ and a sub-quotient isomorphic to ϕ_2 or $\phi_2(1)$. This gives four possibilities for what the parameter $\mathrm{std} \circ \phi_\rho^{\mathrm{FS}}$ is. Since ϕ_ρ^{FS} is a GSp_4-valued parameter only two of these are possible; namely, $\phi_1 \oplus \phi_2$ or $(\phi_1 \oplus \phi_2)(1)$. However, the second possibility can be ruled out since the similitude character of $\mathrm{std} \circ \phi_\rho^{\mathrm{FS}}$ must coincide with the central character of ρ by Theorem 3.6 (2), which agrees with the similitude character of $\mathrm{std}\circ\phi_\rho = \phi_1 \oplus \phi_2$. Therefore, we conclude that $\mathrm{std}\circ\phi = \mathrm{std}\circ\phi_\rho^{\mathrm{FS}}$ and, by the aforementioned equality of similitude characters of these two parameters and [GT11, Lemma 6.1], this is enough to conclude that $\phi_\rho = \phi_\rho^{\mathrm{FS}}$, as conjugacy classes of GSp_4-valued parameters. □

1.4 Basic Uniformization

In this section, we will briefly review what basic uniformization of the generic fiber of a global Shimura variety means, following [Han20, Section 3.1]. Then we will apply it to our particular case and derive an analogue of Boyer's trick, providing useful consequences for the proof of Proposition 1.4.

1.4.1 A Review of Basic Uniformization

We now recall briefly what basic uniformization means. Let $\mathbf{G}/\mathbb{Q}$ be a connected reductive group over $\mathbb{Q}$ and let $(\mathbf{G},X)$ be a Shimura datum, with associate conjugacy class of Hodge cocharacters $\mu : \mathbb{G}_{m,\mathbb{C}} \to \mathbf{G}_{\mathbb{C}}$. Fix a prime p, and set $G := \mathbf{G}_{\mathbb{Q}_p}$. Using our fixed isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_p$, we can and do regard μ as a conjugacy class of cocharacters $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$. This allows us to consider the μ-admissible locus $B(G,\mu)$ in the Kottwitz set of G. Let $\mathbb{A}$ (resp. $\mathbb{A}_f$) denote the adeles (resp. finite adeles) of $\mathbb{Q}$ and $\mathbb{A}_f^p$ denote the finite adeles away from p. For any compact open subgroup $K \subset \mathbf{G}(\mathbb{A}_f)$, let $\mathscr{S}(\mathbf{G},X)_K$ be the associated rigid analytic Shimura variety over $\mathbb{C}_p$ of level K. We let $K = K^pK_p$, where $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$ and $K_p \subset G(\mathbb{Q}_p)$ are open compact subgroups. We set

$$\mathscr{S}(\mathbf{G},X)_{K^p} = \lim_{K_p\to\{1\}} \mathscr{S}(\mathbf{G},X)_{K^pK_p}$$

If $(\mathbf{G},X)$ is of pre-abelian type, this is (up to completing the structure sheaf) representable by a perfectoid space and in general it is a diamond. By the results of [Han20], there exists a canonical $G(\mathbb{Q}_p)$-equivariant Hodge-Tate period map

$$\pi_{HT} : \mathscr{S}(\mathbf{G},X)_{K^p} \to \mathscr{F}\ell_{G,\mu^{-1}}$$

where $\mathscr{F}\ell_{G,\mu^{-1}} := (G_{\mathbb{C}_p}/P_{\mu^{-1}})^{ad}$ is the adic space associated to the flag variety defined by the parabolic $P_{\mu^{-1}} \subset G_{\mathbb{C}_p}$ given by μ^{-1} via the dynamical method. By the $G(\mathbb{Q}_p)$-equivariance, π_{HT} descends to a map:

$$\pi_{HT,K_p} : \mathscr{S}(\mathbf{G},X)_{K_pK_p} \to [\mathscr{F}\ell_{G,\mu^{-1}}/\underline{K_p}]$$

We let $b \in B(G,\mu)$ be the unique basic element, and let $\mathscr{F}\ell^b_{G,\mu^{-1}}$ be the basic Newton stratum. This parametrizes, for S a perfectoid space in characterstic p, modifications $\mathscr{E}_0 \dashrightarrow \mathscr{E}$ of type μ^{-1} between the trivial G-bundle $\mathscr{E}_0$ on the relative Fargues-Fontaine curve X_S and $\mathscr{E}$ a bundle isomorphic to the G-bundle $\mathscr{E}_b$ corresponding to $b \in B(G)$ after pulling back to a geometric point of S. (G,b,μ) defines a local Shimura datum, as in section 3.1, so we may consider the infinite level Shimura variety/Shtuka space $\mathrm{Sht}(G,b,\mu)_\infty$ and its base change $\mathrm{Sht}(G,b,\mu)_{\infty,\mathbb{C}_p}$. By pulling back along π_{HT}, we get an open subspace $\mathscr{S}(\mathbf{G},X)^b_{K^p} \subset \mathscr{S}(\mathbf{G},X)_{K^p}$, which descends to an open subspace $\mathscr{S}(\mathbf{G},X)^b_K$, for $K \subset \mathbf{G}(\mathbb{A}_f)$ an open compact. We now have the key definition.

Definition 1.4.1. We say a global Shimura datum $(\mathbf{G},X)$ satisfies basic uniformization at p if there exists (in fact this is always true see [Han20, Proposition 3.1]) a unique up to isomorphism $\mathbb{Q}$-inner form $\mathbf{G}'$ of $\mathbf{G}$ satisfying

- $\mathbf{G}'_{\mathbb{A}^p_f} \simeq \mathbf{G}_{\mathbb{A}^p_f}$ as algebraic groups over $\mathbb{A}^p_f$,
- $\mathbf{G}'_{\mathbb{Q}_p} \simeq J_b$, where J_b is the inner form of G given by the σ-centralizer of the basic element $b \in B(G,\mu)$,
- $\mathbf{G}'(\mathbb{R})$ is compact modulo center,

and a $\mathbf{G}(\mathbb{A}_f)$-equivariant isomorphism of diamonds over $\mathbb{C}_p$

$$\lim_{K^p \to \{1\}} \mathscr{S}(\mathbf{G},X)^b_{K^p} \simeq (\underline{\mathbf{G}'(\mathbb{Q})} \backslash \underline{\mathbf{G}'(\mathbb{A}_f)} \times_{\mathrm{Spd}(\mathbb{C}_p)} \mathrm{Sht}(G,b,\mu)_{\infty,\mathbb{C}_p})/\underline{J_b(\mathbb{Q}_p)} \tag{1.2}$$

where $J_b(\mathbb{Q}_p)$ acts diagonally, such that, under the identification $\mathscr{F}\ell^b_{G,\mu^{-1}} \simeq \mathrm{Sht}(G,b,\mu)_{\infty,\mathbb{C}_p}/\underline{J_b(\mathbb{Q}_p)}$, the morphism

$$\pi_{HT} : \lim_{K^p \to \{1\}} \mathscr{S}(\mathbf{G},X)^b_{K^p} \to \mathscr{F}\ell^b_{G,\mu^{-1}}$$

identifies with the projection

$$(\underline{\mathbf{G}'(\mathbb{Q})} \backslash \underline{\mathbf{G}'(\mathbb{A}_f)} \times_{\mathrm{Spd}(\mathbb{C}_p)} \mathrm{Sht}(G,b,\mu)_{\infty,\mathbb{C}_p})/\underline{J_b(\mathbb{Q}_p)} \to \mathrm{Sht}(G,b,\mu)_{\infty,\mathbb{C}_p}/\underline{J_b(\mathbb{Q}_p)}$$

where $\mathbf{G}(\mathbb{A}_f) \simeq \mathbf{G}'(\mathbb{A}^p_f) \times G(\mathbb{Q}_p)$ acts on the RHS via the natural action of $\mathbf{G}'(\mathbb{A}^p_f)$ on $\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)$ and $G(\mathbb{Q}_p)$ acts on $\mathrm{Sht}(G,b,\mu)_{\infty,\mathbb{C}_p}$. Moreover, if the reflex field of the cocharacter $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$ is $E/\mathbb{Q}_p$ then this isomorphism descends to an isomorphism of diamonds over $\breve{E} := E\breve{\mathbb{Q}}_p$.

We now mention some consequences of uniformization which will be key to us in what follows. Let $\mathscr{H}(J_b) := C_c^{\infty}(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ be the usual smooth Hecke algebra. We fix an algebraic representation of $\mathbf{G}/\mathbb{Q}$, denoted $\mathscr{V}_\xi$, of some regular highest weight ξ. The isomorphism $i : \overline{\mathbb{Q}}_\ell \xrightarrow{\simeq} \mathbb{C}$ then determines a $\overline{\mathbb{Q}}_\ell$-local system $\mathscr{L}_\xi$ on the Shimura variety $\mathscr{S}(G,X)_{K^p}$. We now consider the space of algebraic automorphic forms valued in $\mathscr{V}_\xi$:

$$\mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash \mathbf{G}'(\mathbb{A}_f)/K^p, \mathscr{L}_\xi) := \operatorname{colim}_{K_p \to \{1\}} \mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash \mathbf{G}'(\mathbb{A}_f)/K^p K_p, \mathscr{L}_\xi)$$

in the sense of Gross [Gro99]. Namely, it is the space of all continuous functions $\phi : \mathbf{G}'(\mathbb{A}_f) \to \mathscr{V}_\xi(\overline{\mathbb{Q}}_\ell)$ with respect to the pro-finite topology on the target and the discrete topology on the source such that, for all $k \in K^p$, $\gamma \in G(\mathbb{Q})$, and $g \in \mathbf{G}'(\mathbb{A}_f)$, we have that $\phi(gk) = \phi(g)$ and $\phi(\gamma g) = \gamma\phi(g)$, where γ acts via $\mathscr{V}_\xi$. The isomorphism (2) then allows us to deduce an isomorphism

$$R\Gamma_c(G,b,\mu) \otimes^{\mathbb{L}}_{\mathscr{H}(J_b)} \mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash \mathbf{G}'(\mathbb{A}_f)/K^p, \mathscr{L}_\xi) \xrightarrow{\simeq} R\Gamma_c(\mathscr{S}(\mathbf{G},X)^b_{K^p}, \mathscr{L}_\xi)$$

of $G(\mathbb{Q}_p) \times W_E$-modules, which when composed with the morphism

$$R\Gamma_c(\mathscr{S}(\mathbf{G},X)^b_{K^p}, \mathscr{L}_\xi) \to R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p}, \mathscr{L}_\xi)$$

coming from excision with respect to the open strata $\mathscr{S}(\mathbf{G},X)^b_{K^p} \hookrightarrow \mathscr{S}(\mathbf{G},X)_{K^p}$, gives rise to the uniformization map mentioned in the introduction. We show this now.

Corollary 1.4.2. *Assume that $(\mathbf{G},X)$ satisfies basic uniformization at p, then there exists a $G(\mathbb{Q}_p) \times W_E$-equivariant map*

$$\Theta : R\Gamma_c(G,b,\mu) \otimes^{\mathbb{L}}_{\mathscr{H}(J_b)} \mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash \mathbf{G}'(\mathbb{A}_f)/K^p, \mathscr{L}_\xi) \to R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p}, \mathscr{L}_\xi)$$

functorial in the level K^p.

Proof. We fix a level at $K_p \subset \mathbf{G}(\mathbb{Q}_p)$ and write $\pi_{HT,K_p} : \mathscr{S}(\mathbf{G},X)_{K^pK_p} \to [\mathscr{F}\ell_{G,\mu^{-1}}/\underline{K_p}]$ for the induced Hodge-Tate period map. Then we have an isomorphism:

$$R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p}, \mathscr{L}_\xi) \simeq \operatorname{colim}_{K_p \to \{1\}} R\Gamma([\mathscr{F}\ell_{G,\mu^{-1}}/\underline{K_p}], R\pi_{HT,K_p!}(\mathscr{L}_\xi))$$

Write $j_{K_p} : [\mathscr{F}\ell^b_{G,\mu^{-1}}/\underline{K_p}] \hookrightarrow [\mathscr{F}\ell_{G,\mu^{-1}}/\underline{K_p}]$ for the open inclusion of the basic locus quotiented out by K_p. Now, we claim that, under this identification, the desired map is given by

$$R\Gamma_c([\mathscr{F}\ell^b_{G,\mu^{-1}}/\underline{K_p}], j^*_{K_p} R\pi_{HT,K_p!}(\mathscr{L}_\xi)) \to R\Gamma([\mathscr{F}\ell_{G,\mu^{-1}}/\underline{K_p}], R\pi_{HT,K_p!}(\mathscr{L}_\xi))$$

coming from applying excision with respect to j_{K_p} for varying K_p. Here we stress that, since we are finite level, this is happening in the usual category of étale $\overline{\mathbb{Q}}_\ell$-sheaves, where excision is well-defined. The claim would now follow from showing a natural identification

$$R\Gamma_c([\mathscr{F}\ell^b_{G,\mu^{-1}}/\underline{K_p}], j^*_{K_p}R\pi_{HT,K_p!}(\mathscr{L}_\xi)) \simeq R\Gamma_c(G,b,\mu)_{K_p}\otimes^{\mathbb{L}}_{\mathscr{H}(J_b)}\mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p,\mathscr{L}_\xi)$$

for all K_p. There is a natural map

$$q^b_{K_p} : [\mathscr{F}\ell^b_{G,\mu^{-1}}/\underline{K_p}] \to [\mathrm{Spd}(\mathbb{C}_p)/\underline{J_b(\mathbb{Q}_p)}]$$

which is cohomologically smooth as in [Han20, Proposition 2.16]. Since we are working with the usual category of étale $\overline{\mathbb{Q}}_\ell$-sheaves, the derived category of $\overline{\mathbb{Q}}_\ell$-sheaves on the target is not identified with the unbounded derived category of smooth representations of $J_b(\mathbb{Q}_p)$, but rather the unbounded derived category of smooth ℓ-complete representations (See Remark 3.1). However, by fixing a K^p stable lattice in the $\overline{\mathbb{Z}}_\ell$-lattice in the $\overline{\mathbb{Q}}_\ell$ realization of the algebraic representation $\mathscr{V}_\xi$, we can endow $\Pi := \mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p,\mathscr{L}_\xi)$ with the structure of such a $J_b(\mathbb{Q}_p)$-representation, and we write $\mathscr{F}_\Pi$ for this sheaf. By [Han20, Corollary 3.7], the fact that we know basic uniformization at p implies we have a natural in K^p isomorphism:

$$(q^{b*}_{K_p})(\mathscr{F}_\Pi) \simeq j^*_{K_p}R\pi_{HT,K_p!}(\mathscr{L}_\xi)$$

Applying $R\Gamma_c([\mathscr{F}\ell^b_{G,\mu^{-1}}/K_p],-)$, we obtain

$$R\Gamma_c([\mathscr{F}\ell^b_{G,\mu^{-1}}/K_p],(q^{b*}_{K_p})(\mathscr{F}_\Pi)) \simeq R\Gamma_c([\mathscr{F}\ell^b_{G,\mu^{-1}}/K_p], j^*_{K_p}R\pi_{HT,K_p!}(\mathscr{L}_\xi))$$

but now, by [Han20, Proposition 2.17], we have an isomorphism

$$R\Gamma_c(\mathrm{Sht}(G,b,\mu)_\infty/\underline{K_p},\overline{\mathbb{Q}}_\ell)\otimes^{\mathbb{L}}_{\mathscr{H}(J_b)}\mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p,\mathscr{L}_\xi) \simeq R\Gamma_c([\mathscr{F}\ell^b_{G,\mu^{-1}}/K_p],(q^{b*}_{K_p})(\mathscr{F}_\Pi))$$

and this gives the desired result. □

1.4.2 Boyer's Trick

We will now be interested in applying uniformization to the situation we are interested in, proving an analogue of Boyer's trick [Boy99a] and deducing some relevant consequences. The relevant results are due to Shen.

Theorem 1.4.3. *[She17] If $(\mathbf{G},X)$ is a Shimura datum of abelian type and $p>2$ is a prime where G is unramified then $(\mathbf{G},X)$ satisfies basic uniformization at p.*

Remark 1.4.4. To see a full proof of this exact statement for the full integral model at hyperspecial level, one can look at the proof of [Li-22, Theorem D].

Now, consider a Shimura datum $(\mathbf{G},X)$, where $\mathbf{G}$ is a $\mathbb{Q}$-inner form of $\mathbf{G}^* := \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GSp}_4$, with $F/\mathbb{Q}$ a totally real field such that p is totally inert and $F_p \simeq L$, a fixed unramified extension of $\mathbb{Q}_p$ and assume that $\mathbf{G}_{\mathbb{Q}_p} \simeq \mathrm{Res}_{L/\mathbb{Q}_p}\mathrm{GSp}_4 = G$. We fix a level $K = K_pK^p \subset \mathbf{G}(\mathbb{A}_f)$ as before, and assume from now on that $(\mathbf{G},X)$ is such that the corresponding cocharacter μ is the Siegel cocharacter. Therefore, the unique basic $b \in B(G,\mu)$ will have σ-centralizer given by $\mathrm{Res}_{L/\mathbb{Q}_p}\mathrm{GU}_2(D)$, with D/L the quaternionic division algebra. Since $L/\mathbb{Q}_p$ is unramified and $p>2$, we can apply Theorem 4.2 to deduce basic uniformization at p. Let $\mathbf{G}'$ be the $\mathbb{Q}$-inner form defined above. Now we prove the following result, which plays a similar role to Boyer's trick [Boy99a] in the study of the cohomology of the Lubin-Tate/Drinfeld towers.

Lemma 1.4.5. *For $b \in B(G,\mu)$ non-basic, the adic Newton strata $\mathscr{F}\ell^b_{G,\mu^{-1}}$ is parabolically induced as a space with $G(\mathbb{Q}_p)$-action from a proper parabolic subgroup $P = LU$ of G and a diamond $\mathscr{S}_P$ with $\underline{P(\mathbb{Q}_p)}$-action. Moreover, the action of $U(\mathbb{Q}_p)$ on the ℓ-adic cohomology of $R\Gamma_c(\mathscr{S}_P,\overline{\mathbb{Q}}_\ell)$ is trivial.*

Proof. We recall [RV14, Definition 4.28] that we say $b \in B(G,\mu)$ is Hodge-Newton reducible if there exists a proper Levi subgroup L together with a basic element $b_L \in B(L,\mu_L)$ mapping to $b \in B(G,\mu)$ under the natural map $B(L) \to B(G)$, where μ_L is a choice of representative for μ as a geometric dominant cocharacter of L. Given such a b, we let P be the standard parabolic of G with respect to a choice of Borel such that its Levi factor is L, and fix μ_L to be the conjugacy class of dominant cocharacters of L that is dominant respect to B. It now follows by [GI16, Proposition 4.13] that $P(\mathbb{Q}_p) \subset G(\mathbb{Q}_p)$ stabilizes a subspace $\mathscr{C}^\mu_b$ and that we have a $L(\mathbb{Q}_p)$-equivariant isomorphism:

$$\mathscr{C}^{\mu_L}_{b_L} \times^{P(\mathbb{Q}_p)} G(\mathbb{Q}_p) \simeq \mathscr{F}\ell^b_{G,\mu^{-1}}$$

We recall that, if $\mathscr{J}_b$ denotes the group diamond parametrizing automorphisms of the bundle corresponding to $b \in B(G)$, this has a semi-direct product decomposition

$$\mathscr{J}_b := \underline{J_b(\mathbb{Q}_p)} \ltimes \mathscr{J}^U_b$$

given by the splitting of the HN-filtration of $\mathscr{E}_b$ [FS21, Proposition III.5.1]. By [GI16, Proposition 4.24], the space $\mathscr{C}_b^{\mu_L}$ is isomorphic to

$$\mathscr{F}\ell^{b_L}_{L,\mu_L^{-1}} \times \mathscr{J}_b^U$$

and, by [GI16, Lemma 4.28], it follows that the action of $U_b(\mathbb{Q}_p)$ on $R\Gamma_c(\mathscr{C}_{b_L}^{\mu_L}, \overline{\mathbb{Q}}_\ell)$ is trivial. Therefore, the claim will follow from checking that the non-basic elements in $B(G,\mu)$ are Hodge-Newton reducible. This follows immediately from the classification of this condition in [GHN19, Theorem 2.5]. For clarity, we recall how this works in our case. There are two non-basic elements $b \in B(G,\mu)$. The μ-ordinary element, defined by the maximal element $b^{max} \in B(G,\mu)$ with respect to the partial ordering on $B(G,\mu)$, and the intermediate strata corresponding to the element lying between the basic element and b^{max} with respect to the partial ordering on $B(G)$. The element $b^{\max}$ admits a reduction to the unique element $b_T \in B(T,\mu_T)$, where T is the maximal torus inside GSp_4/L. Similarly, the element $b \in B(G,\mu)$ corresponding to the intermediate strata admits a reduction to the unique basic element $b_L \in B(L,\mu_L)$, where L is the Levi factor of the Klingen parabolic of GSp_4. □

We now consider some irreducible representation ξ as above and look at the uniformization map

$$\Theta : R\Gamma_c(G,b,\mu) \otimes^{\mathbb{L}}_{\mathscr{H}(J_b)} \mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p, \mathscr{L}_\xi) \to R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p}, \mathscr{L}_\xi)$$

furnished by Corollary 4.1 and Theorem 4.2. We let $R\Gamma_c(G,b,\mu)_{sc}$ and $R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p}, \mathscr{L}_\xi)_{sc}$ be the direct summands where $G(\mathbb{Q}_p)$ acts via a supercuspidal representation. Then we have the following key consequence of the previous lemma, which justifies why we are referring to this as Boyer's trick.

Proposition 1.4.6. *The uniformization map* Θ *induces an isomorphism*

$$\Theta_{sc} : R\Gamma_c(G,b,\mu)_{sc} \otimes^{\mathbb{L}}_{\mathscr{H}(J_b)} \mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p, \mathscr{L}_\xi) \xrightarrow{\simeq} R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p}, \mathscr{L}_\xi)_{sc}$$

on the summand where $G(\mathbb{Q}_p)$ *acts via a supercuspidal representation.*

Proof. We let $\mathscr{F}\ell^{nbas}_{G,\mu^{-1}}$ denote the closed complement of $\mathscr{F}\ell^{b}_{G,\mu^{-1}}$ for $b \in B(G,\mu)$ the unique basic element. The space $\mathscr{F}\ell^{nbas}_{G,\mu^{-1}}$ is stratified by $\mathscr{F}\ell^{b}_{G,\mu^{-1}}$ for $b \in B(G,\mu)$ non-basic. For varying $K_p \subset G(\mathbb{Q}_p)$, we set $[\mathscr{F}\ell^{nbas}_{G,\mu^{-1}}/\underline{K_p}]$ to be the v-stack quotient. We let $\mathscr{S}(\mathbf{G},X)^{nbas}_{K^pK_p}$ denote the preimage of $[\mathscr{F}\ell^{nbas}_{G,\mu^{-1}}/\underline{K_p}]$ under the map $\pi_{HT,K_p}: \mathscr{S}(\mathbf{G},X)_{K^pK_p} \to [\mathscr{F}\ell_{G,\mu^{-1}}/\underline{K_p}]$. It follows from the proof of Corollary 4.1 that the cone of Θ is identified with

$$R\Gamma_c(\mathscr{S}(\mathbf{G},X)^{nbas}_{K^p}, \mathscr{L}_\xi) := \mathrm{colim}_{K_p \to \{1\}} R\Gamma_c(\mathscr{S}(\mathbf{G},X)^{nbas}_{K^pK_p}, \mathscr{L}_\xi).$$

We want to show that the cohomology is parabolically induced as a $G(\mathbb{Q}_p)$ representation from proper Levi subgroups of G. For this, we consider the Hodge-Tate period morphism:

$$\pi_{HT,K_p}: \mathscr{S}(\mathbf{G},X)^{nbas}_{K_pK^p} \to [\mathscr{F}\ell^{nbas}_{G,\mu^{-1}}/\underline{K_p}]$$

The Cartan-Leray spectral sequence then gives us

$$E_2^{p,q} = H^p_c([\mathscr{F}\ell^{nbas}_{G,\mu^{-1}}/K_p], R^q\pi_{HT,K_p!}(\mathscr{L}_\xi)) \Longrightarrow H^{p+q}_c(\mathscr{S}(\mathbf{G},X)^{nbas}_{K_pK^p}, \mathscr{L}_\xi).$$

Using the $G(\mathbb{Q}_p)$-equivariance of the Hodge-Tate period map, the colimit over $K_p \to \{1\}$ gives rise to a spectral sequence

$$\mathrm{colim}_{K_p \to \{1\}} H^p_c([\mathscr{F}\ell^{nbas}_{G,\mu^{-1}}/K_p], R^q\pi_{HT,K_p!}(\mathscr{L}_\xi)) \Longrightarrow H^{p+q}_c(\mathscr{S}(\mathbf{G},X)^{nbas}_{K_p}, \mathscr{L}_\xi)$$

with $G(\mathbb{Q}_p)$-equivariant maps. Therefore, we are reduced to showing the following.

Lemma 1.4.7. *For all integers $p,q \geq 0$ the cohomology of $\mathrm{colim}_{K_p \to \{1\}} H^p_c([\mathscr{F}\ell^{nbas}_{G,\mu^{-1}}/\underline{K_p}], R^q\pi_{HT,K_p!}(\mathscr{L}_\xi))$ is parabolically induced as a $G(\mathbb{Q}_p)$-representation from a proper Levi subgroup of G.*

Proof. We apply excision with respect to the locally closed stratification given by $\mathscr{F}\ell^b_{G,\mu^{-1}}$ for $b \in B(G,\mu)$ which is not basic. We write $j_{b,K_p}: [\mathscr{F}\ell^b_{G,\mu^{-1}}/\underline{K_p}] \hookrightarrow [\mathscr{F}\ell_{G,\mu^{-1}}/\underline{K_p}]$ for the associated locally closed immersion at level K_p. Since the Newton strata $\mathscr{F}\ell^b_{G,\mu^{-1}}$ are stable under the $G(\mathbb{Q}_p)$ action, this reduces us to showing that the cohomology of

$$\mathrm{colim}_{K_p \to \{1\}} H^p_c([\mathscr{F}\ell^b_{G,\mu^{-1}}/\underline{K_p}], j^*_{b,K_p} R^q\pi_{HT,K_p!}(\mathscr{L}_\xi))$$

is parabolically induced as a $G(\mathbb{Q}_p)$-representation. However, this now follows from Lemma 4.3, where one uses the second part of the claim to see the unipotent radical acts trivially. □

□

This result will be the key tool in allowing us to describe the W_L-action on the local Shimura variety by global methods. We start this global analysis by constructing strong transfers between GSp_4 and its inner forms over a number field F and proving a strong multiplicity one result.

1.5 Existence of Strong Transfers and a Strong Multiplicity One Result

In this section, we will show the existence of strong transfers of certain automorphic representations of an inner form of GSp_4 over a number field F, using the analysis of the trace formula similar to that of [KS16, Section 6]. We will then combine this with analysis of the simple twisted trace formula of Kottwitz-Shelstad [KS99], to deduce a kind of strong multiplicity one result for inner forms of GSp_4.

1.5.1 The Simple Trace Formula and Existence of Strong Transfers

In order to describe the Galois action on the global Shimura variety, we will need to construct strong transfers for inner forms of GSp_4/F over a totally real field F. This will allow us to compute the traces of Frobenius on the global Shimura variety in terms of the Langlands parameters of the strong transfer. The construction of strong transfers will be accomplished by applying the elliptic part of the stable trace formula with respect to the Lefschetz functions constructed by Kret-Shin [KS16] at the Steinberg/Infinite places and pseudo-coefficients at some finite number places where the representation has supercuspidal L-parameter, applying the character identities of Chan-Gan [CG15] to conclude equality of the orbital integrals at these latter places. First, we recall the key results of Kret-Shin on the trace formula with fixed central character. For now, we will work generally. Let $\mathbf{G}$ denote a connected reductive group over a number field F with center Z, write A_Z for the maximal $\mathbb{Q}$-split torus of $\mathrm{Res}_{F/\mathbb{Q}}Z$, and set $A_{Z,\infty} = A_Z(\mathbb{R})^0$ to be the connected component of the identity. Let $\mathbb{A}_F$ be the adeles of F and write $\mathbf{G}(\mathbb{A}_F)^1$ for a choice of subgroup so that $\mathbf{G}(\mathbb{A}_F) = \mathbf{G}(\mathbb{A}_F)^1 \times A_{Z,\infty}$, as in [Art81, Page 11]. We consider a closed subgroup $\mathfrak{X} \subset Z(\mathbb{A}_F)$ which contains $A_{Z,\infty}$ such that $Z(F)\mathfrak{X}$ is closed inside $Z(\mathbb{A}_F)$ and a continuous character $\chi : (\mathfrak{X} \cap Z(F))\backslash\mathfrak{X} \to \mathbb{C}^*$.

We write Pl_F for the set of places of F, and, for $v \in \mathrm{Pl}_F$, let $\mathfrak{X}_v \subset Z(F_v)$ denote a closed subgroup. We let $\chi_v : \mathfrak{X}_v \to \mathbb{C}^*$ be a smooth character. Write $\mathscr{H}(\mathbf{G}(F_v), \chi_v^{-1})$ for the space of smooth compactly supported functions modulo center on $\mathbf{G}(F_v)$ which transform under $\mathfrak{X}_v$ via χ_v^{-1}. We also require the functions to be K_v-finite for some maximal compact subgroup K_v of $\mathbf{G}(F_v)$ if v is archimedean. We now take $\mathfrak{X}_v := \mathfrak{X}(F_v)$, where $\mathfrak{X}$ is as above. Given a semisimple element $\gamma_v \in \mathbf{G}(F_v)$ and an admissible representation π_v of $\mathbf{G}(F_v)$ with central character χ_v on $\mathfrak{X}_v$, we define the orbital integral and trace character for $f_v \in \mathscr{H}(\mathbf{G}(F_v), \chi_v^{-1})$ as follows. Let I_{γ_v} denote the connected centralizer of γ_v in $\mathbf{G}$. We have

$$O_{\gamma_v}(f_v) := \int_{I_{\gamma_v}(F_v)\backslash \mathbf{G}(F_v)} f_v(x^{-1}\gamma_v x)dx$$

and

$$\mathrm{tr}(f_v|\pi_v) := \mathrm{tr}(\int_{\mathbf{G}(F_v)/Z(F_v)} f_v(g)\pi_v(g)dg)$$

where we have fixed compatible choices of Haar measure on $\mathbf{G}$ and Z throughout. We note that this operator is well defined since the above operator is of finite rank if v is finite and is of trace class if v is infinite, by the K_v-finiteness assumption.

We define the adelic Hecke algebra $\mathscr{H}(\mathbf{G}(\mathbb{A}_F), \chi^{-1})$, as well as the global orbital integrals by taking a restricted tensor product over the local Hecke algebras defined above and products of the local integrals. Write $\Gamma_{ell}(\mathbf{G})$ to be the set of F-elliptic conjugacy classes in $\mathbf{G}(F)$. Let $\mathbb{A}_{f,F}$ denote the finite adeles of F. For our purposes, it will suffice to consider a central character datum $(\mathfrak{X}, \chi)$, where $\mathfrak{X} = Z(\mathbb{A}_F)$, and we write $\chi = \bigotimes_{v\in \mathrm{Pl}_F} \chi_v$ for smooth characters $\chi_v : Z(F_v) \to \mathbb{C}^*$ and all $v \in \mathrm{Pl}_F$. We let $L^2_{disc,\chi}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F))$ denote the space of functions on $\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F)$ transforming under $\mathfrak{X}$ by χ and square-integrable on $\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F)^1/\mathfrak{X}(\mathbb{A}_F) \cap G(\mathbb{A}_F)^1$. Write $\mathscr{A}_{cusp,\chi}(\mathbf{G})$ for the set of isomorphism classes of cuspidal automorphic representations of $\mathbf{G}(\mathbb{A}_F)$ whose central characters restricted to $\mathfrak{X}$ are χ. For $f \in \mathscr{H}(\mathbf{G}(\mathbb{A}_F), \chi^{-1})$, define the invariant distributions $T^{\mathbf{G}}_{ell,\chi}$ and $T^{\mathbf{G}}_{disc,\chi}$ by

$$T^{\mathbf{G}}_{ell,\chi}(f) := \sum_{\gamma\in\Gamma_{ell}(\mathbf{G})} i(\gamma)^{-1} vol(I_\gamma(F)\backslash I_\gamma(\mathbb{A}_F)/\mathfrak{X}(\mathbb{A}_F))O_\gamma(f)$$

$$T^{\mathbf{G}}_{disc,\chi}(f) := \mathrm{tr}(f|L^2_{disc,\chi}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F)))$$

where $i(\gamma)$ is the number of connected components in the centralizer of γ. Analogously, we define $T^{\mathbf{G}}_{cusp,\chi}(f)$ by taking the trace on the space of square-integrable cusp forms whose central character restricted to $\mathfrak{X}$ is χ. Let $\mathbf{G}^*$ denote the quasi-split inner form of $\mathbf{G}$ over F, with a fixed inner twist $\mathbf{G}^* \simeq \mathbf{G}$ over $\overline{F}$. Since Z is canonically identified with the center of $\mathbf{G}^*$, we may view the character χ as a central character datum for $\mathbf{G}^*$. We then let f^* denote a Langlands-Shelstad transfer of f to $\mathbf{G}^*$. One can construct such a transfer by lifting f along the surjection

$\mathscr{H}(\mathbf{G}(\mathbb{A}_F)) \to \mathscr{H}(\mathbf{G}(\mathbb{A}_F), \chi^{-1})$ applying the transfer with trivial central character due to Waldspurger and then taking the image along the analogous surjection for $\mathbf{G}^*(\mathbb{A}_F)$. We let $\Sigma_{ell,\chi}(\mathbf{G}^*)$ denote the set of $Z(\mathbb{A}_F)$ orbits of stable F-elliptic conjugacy classes in $\mathbf{G}^*(F)$. We define the stable elliptic distribution

$$ST^{\mathbf{G}^*}_{ell,\chi}(f^*) := \tau(\mathbf{G}^*) \sum_{\gamma \in \Sigma_{ell,\chi}(\mathbf{G}^*)} \tilde{i}(\gamma)^{-1} SO^{\mathbf{G}^*}_{\gamma,\chi}(f)$$

where $SO^{\mathbf{G}^*}_{\gamma,\chi}(f^*)$ denotes the stable orbital integral of f^* at γ, $\tau(\mathbf{G}^*)$ is the Tamagawa number of $\mathbf{G}^*$, and $\tilde{i}(\gamma)$ is the number of Galois fixed connected components of the centralizer of γ in $\mathbf{G}^*$. Let ξ be an irreducible representation of $\mathbf{G}_{F_\infty}$. Denote by $\chi_\xi : Z(F_\infty) \to \mathbb{C}^*$ the restriction of ξ to $Z(F_\infty)$. Write $f^{\mathbf{G}}_\xi \in \mathscr{H}(\mathbf{G}(F_\infty), \chi_\xi^{-1})$ for a Lefschetz function associated with ξ. In other words, a function such that $\mathrm{tr}(f^{\mathbf{G}}_\xi|\pi_\infty)$ computes the Euler-Poincaré characteristic for the relative Lie algebra cohomology of $\pi_\infty \otimes \xi$ for every irreducible admissible representation π_∞ of $\mathbf{G}(F_\infty)$ with central character χ_ξ. It follows by the Vogan-Zuckerman classification [VZ84] that, if ξ is regular, this will be non-zero if and only if π_∞ is an (essentially) discrete series representation cohomological of regular weight ξ. For a finite place v_{st} of F, we let $f_{v_{st}} := f^{\mathbf{G}}_{Lef,v_{st}} \in \mathscr{H}(\mathbf{G}(F_{v_{st}}))$ denote a Lefschetz function at v_{st}. Morally, this should be characterized by the property that $\mathrm{tr}(f_{v_{st}}|\pi_{v_{st}})$ computes the Euler-Poincaré characteristic of the continuous group cohomology of $\mathbf{G}(F_{v_{st}})$ valued in $\pi_{v_{st}}$. In the case that the center is anisotropic, it follows from the computations in [BW80, Theorem XI.3.9] that this means that the trace of $f^{\mathbf{G}}_{Lef,v_{st}}$ will only be non-zero if π_v is 1-dimensional or an unramified twist of the Steinberg representation, for all irreducible admissible unitary π_v. For finite places $v = v_{st}$, they were originally constructed by Kottwitz; however, these results do not apply for the desired application, since the center of GSp_4 is not compact. For the construction of these functions in this case and the proof of the property that their traces detect when a representation is 1-dimensional or an unramified twist of Steinberg, see [KS16, Appendix A].

We now assume for simplicity that the center Z of $\mathbf{G}$ is split, which will be the case in all our applications. We consider a character $\eta : \mathbf{G}(F) \to \mathbb{C}^*$ such that $\eta|_{Z(F_{v_{st}})} = \chi_{v_{st}}$. We can define a function $f^{\mathbf{G}}_{Lef,\eta} \in \mathscr{H}(\mathbf{G}(F), \chi_{v_{st}}^{-1})$ as $f^{\mathbf{G}}_{Lef,\eta}(g) := \eta^{-1}(g) f^{G/Z}_{Lef}(\overline{g})$, where $\overline{g} \in G(F_{v_{st}})/Z(F_{v_{st}})$ denotes the image of g under the quotient map. This (up to scaling by a non-zero constant) forms a pseudo-coefficient for the Steinberg representation twisted by η, as constructed in [Kaz86; SS97]. In particular, by [KS16, Corollary A.8 (2)], we have, for $\pi_{v_{st}}$

an irreducible (essentially) unitary representation of $\mathbf{G}(F_{v_{st}})$ with central character $\chi_{v_{st}}$, that $\mathrm{tr}(f^{\mathbf{G}}_{Lef,\eta}|\pi_{v_{st}}) \neq 0$ if and only if $\pi_{v_{st}}$ is isomorphic to η or Steinberg twisted by η.

Now we have the key lemma, which tells us that with respect to these choices of test functions we get the following "simple trace formula".

Lemma 1.5.1. *[KS16, Lemma 6.1, 6.2] Fix a central character datum $(Z(\mathbb{A}_F),\chi)$ with $\chi_\xi = \chi|_{Z(F_\infty)}$ as above for χ_ξ attached to some regular weight ξ. For an element $f \in \mathscr{H}(\mathbf{G}(\mathbb{A}_F),\chi^{-1})$ in the global Hecke algebra as above, assume that $f_\infty = f^{\mathbf{G}}_\xi \in \mathscr{H}(\mathbf{G}(F_\infty),\chi^{-1})$ is a Lefschetz function at ∞ and assume that $f_{v_{st}} = f^{\mathbf{G}}_{Lef,\eta}$ is the Lefschetz function described above at v_{st} for a character η of $\mathbf{G}(F_{v_{st}})$ such that $\eta|_{Z(F_{v_{st}})} = \chi_{v_{st}}$. Then we have an equality:*

$$ST^{\mathbf{G}^*}_{ell,\chi}(f^*) = T^{\mathbf{G}}_{ell,\chi}(f) = T^{\mathbf{G}}_{disc,\chi}(f) = T^{\mathbf{G}}_{cusp,\chi}(f)$$

Proof. Strictly speaking, the cited Lemmas only prove this in the case where the central character datum is $(Z(F_\infty),\chi_\xi)$; however, the result in this case easily follows using [KS16, Corollary A.8 (3)] (cf. the proof of [KS16, Corollary 8.5]). □

Let S_{st} and S_{sc} be disjoint finite sets of finite places of F, and let S_0 be a finite set of places contained in $S_{st} \cup S_{sc}$. Let S_∞ denote the infinite places of F. Set S to be a finite set of places containing $S_{st} \cup S_{sc} \cup S_\infty$. We assume that the inner twist $\mathbf{G}^*$ of $\mathbf{G}$ is trivialized away from S_0 and S_∞ (i.e for all $v \notin S_0 \cup S_\infty$, the inner twisting gives rise to an isomorphism $\mathbf{G}_{F_v} \simeq \mathbf{G}^*_{F_v}$). In particular, $\mathbf{G}$ is unramified outside S and we can fix a reductive model for $\mathbf{G}$ and $\mathbf{G}^*$ over $\mathscr{O}_F[1/S]$. By abuse of notation, we write $\mathbf{G}$ and $\mathbf{G}^*$ for the integral models of these groups. The inner twist gives an isomorphism $\mathbf{G}^*_{F_v} \simeq \mathbf{G}_{F_v}$ and isomorphisms $\mathbf{G}^*_{\mathscr{O}_{F_v}} \simeq \mathbf{G}_{\mathscr{O}_{F_v}}$ of the hyperspecial subgroups determined by this model for the finite places $v \notin S$. The notion of unramified local representation on either side will be defined with respect to this fixed choice of hyperspecial level.

For the rest of the section, we will assume that $\mathbf{G}^* = \mathrm{GSp}_4$. We assume throughout that π is a global cuspidal automorphic representation of the group $\mathbf{G}(\mathbb{A}_F)$ satisfying the following properties:

1. π is cohomological of some regular weight ξ at infinity.

2. π_v is unramified at all finite places $v \notin S$.

3. π_v has supercuspidal L-parameter for $v \in S_{sc}$.

4. π_v is an unramified twist of Steinberg at all finite places $v \in S_{st}$.

Note, as in Definition 2.1, we can further partition S_{sc} into S_{ssc}, where $\mathrm{std} \circ \phi_{\pi_v}$ is irreducible (ssc = stable supercuspidal) and S_{esc}, where $\mathrm{std} \circ \phi_{\pi_v}$ is reducible (esc = endoscopic supercuspidal).

The main result of this section shows the existence of strong transfers from $\mathbf{G}$ to $\mathbf{G}^*$ at the places in $S_{sc} \cup S_{st} \cup S_\infty$ for a certain class of automorphic representations of $\mathbf{G}$. It is essentially a more refined version of [KS16, Proposition 6.3] in the particular case that $\mathbf{G}^* = \mathrm{GSp}_4$.

Theorem 1.5.2. *Suppose that S_{st} is non-empty. Given a π as above, there exists a cuspidal automorphic representation τ of $\mathbf{G}^*(\mathbb{A}_F)$ satisfying the following:*

- $\tau^S \simeq \pi^S$.
- *At all $v \in S_{sc} \cup S_{st} \cup S_\infty$, τ_v has the same Langlands parameter as π_v.*

Moreover, we can choose τ to be globally generic. For the first part, the same is true with the roles of τ and π reversed.

Proof. First off note that, since π is an unramified twist of Steinberg at some finite place, τ, if it exists, is automatically (essentially) tempered at all places (cf. Remark 5.1). It follows, by [GT19, Remark 7.4.7], that the global L-packet of τ therefore contains a globally generic representation. So, if we can show the existence of some τ with the desired properties, that means we can find τ globally generic with the same properties. For the former, we now apply the trace formula.

Let $\mathfrak{X} = Z(\mathbb{A}_F)$ and χ be the central character of π. We set $f = \bigotimes_{v \in F} f_v$ to be a test function on $\mathbf{G}(\mathbb{A}_F)$ satisfying the following:

1. $f_\infty = f_\xi^{\mathbf{G}}$ is a Lefschetz/Euler-Poincaré function of weight ξ of $\mathbf{G}(F_\infty)$.
2. At $v \in S_{st}$, $f_v = f_{Lef,\eta_v}^{\mathbf{G}}$ is a Lefschetz function for $\mathbf{G}(F_v)$, where η_v is the unique character such that π_v is the Steinberg twisted by η_v.
3. At $v \in S_{ssc}$, $f_v = f_{\pi_v}$ is the pseudo-coefficient of π_v, as constructed in [Kaz86; SS97].
4. At $v \in S_{esc}$, $f_v = f_{\pi_v^+} + f_{\pi_v^-}$, where $\{\pi_v^+, \pi_v^-\}$ is the L-packet over ϕ_{π_v} and $f_{\pi_v^\pm}$ is the pseudo-coefficient of $\pi_v^\pm$.

5. At the finite places $v \notin S$, f_v is an arbitrary element of the unramified Hecke algebra.

6. For $v \in S \setminus S_{st} \cup S_\infty \cup S_{sc}$, choose f_v to be an arbitrary function such that $\mathrm{tr}(f_v|\pi_v) > 0$.

Given such a f, we choose a test function $f^* = \bigotimes_{v \in F} f_v^*$ on $\mathbf{G}^*(\mathbb{A}_F)$ satisfying the following:

1. $f_\infty^* = f_\xi^{\mathbf{G}^*}$ is a Lefschetz/Euler-Poincaré for the representation ξ of $\mathbf{G}^*(F_\infty)$.

2. At $v \in S_{st}$, $f_v^* = f_{Lef,\eta_v}^{\mathbf{G}^*}$ is the Lefschetz function for $\mathbf{G}^*(F_v)$.

3. At $v \in S_{ssc}$, $f_v^* = f_{\tau_v}$ is a pseudo-coefficient of τ_v, where τ_v is the unique supercuspidal representation of $\mathbf{G}^*(F_v)$ with Langlands parameter ϕ_{π_v}.

4. At $v \in S_{esc}$, $f_v^* = f_{\tau_v^+} + f_{\tau_v^-}$, where $\{\tau_v^+, \tau_v^-\}$ is the L-packet over ϕ_{π_v} of $\mathbf{G}^*(F_v)$ and $f_{\tau_v^\pm}$ is the pseudo-coefficient of $\tau_v^\pm$.

5. At the finite places $v \notin S$, $f_v^* = f_v$ is the same element of the unramified Hecke algebra.

6. For $v \in S \setminus S_{st} \cup S_\infty \cup S_{sc}$, choose $f_v^* = f_v$.

Now we wish to check that f and f^* are matching up to a non-zero constant c, in the sense of [KS16, Section 5.5]. We can check this place by place. For the finite places $v \notin S_{st} \cup S_{sc} \cup S_\infty$, this is tautological. For all $v \in S_\infty \cup S_{st}$, this follows from [KS16, Lemma A.4, A.11]. For $v \in S_{sc}$, this follows from the character identities of Chan-Gan [CG15, Proposition 11.1]. Namely, recall for a regular semisimple elliptic element γ the orbital integrals of the pseudo-coefficients of a discrete series representation π is given by the Harish-Chandra character of π evaluated at γ, and it vanishes if γ is non-elliptic [Kaz86, Theorem K] [KST20, Proposition 3.2]. It follows that the orbital integrals for a sum of pseudo-coefficients over an L-packet of GSp_4 or its inner form is already a stable distribution, by [CG15, Main Theorem] combined with [CG15, Proposition 11.1 (1)] for the inner form. Therefore, the desired equality of stable orbital integrals reduces to showing an equality of the sum of Harish-Chandra characters over the L-packets of supercuspidal parameters, and this is precisely [CG15, Proposition 11.1 (1)].

Since the test functions are matching and $S_{st} \neq \emptyset$, we can apply Lemma 5.1 to conclude:

$$T_{cusp,\chi}^{\mathbf{G}^*}(f^*) = ST_{ell,\chi}^{\mathbf{G}^*}(f^*) = cT_{cusp,\chi}^{\mathbf{G}}(f)$$

Then, by linear independence of characters, we have a relationship

$$\sum_{\substack{\Pi' \in \mathscr{A}_{cusp,\chi}(\mathbf{G}^*) \\ \Pi'^S \simeq \pi^S}} m(\Pi')\mathrm{tr}(f_S^*|\Pi'_S) = c \cdot \sum_{\substack{\Pi \in \mathscr{A}_{cusp,\chi}(\mathbf{G}) \\ \Pi^S \simeq \pi^S}} m(\Pi)\mathrm{tr}(f_S|\Pi_S)$$

where $m(\Pi)$ (resp. $m(\Pi')$) denotes the multiplicity of Π (resp. Π') in the cuspidal spectrum of $\mathbf{G}$ (resp. $\mathbf{G}^*$). Now at the infinite places, as soon as $\mathrm{tr}(f_v|\Pi_v) \neq 0$ at $v \mid \infty$ the regularity condition on ξ implies that Π_v is an (essentially) discrete series representation cohomological of regular weight ξ and that $\mathrm{tr}(f_v|\Pi_v) = (-1)^{q(\mathbf{G}_v)}$ by the Vogan-Zuckerman classification of unitary cohomological representations, where $q(\mathbf{G})$ is the F-rank of the derived group of $\mathbf{G}$. At $v_{st} \in S_{st}$ it follows by [KS16, Corollary A.8] that $\Pi_{v_{st}}$ is either the $\eta_{v_{st}}$-twist of the Steinberg or trivial representation. If $\Pi_{v_{st}}$ were one-dimensional then the global representation would also be one-dimensional by a strong-approximation argument [KST20, Lemma 6.2], implying that Π_∞ cannot be tempered, which would contradict the fact Π_∞ is an (essentially) discrete series representation. Therefore, $\Pi_{v_{st}}$ is always the $\eta_{v_{st}}$ twist of the Steinberg representation. At the remaining $v \in S_{ssc}$ (resp. $v \in S_{esc}$), it follows from the definition of pseudo-coefficients that, if $\mathrm{tr}(f_v|\Pi_v) > 0$, we have $\Pi_v \simeq \pi_v$ (resp. $\Pi_v \in \{\pi_v^+, \pi_v^-\}$). Similar considerations apply for $\Pi' \in \mathscr{A}_{cusp,\chi}(\mathbf{G}^*)$ occurring non-trivially in the LHS. In summary, by the above analysis, we can deduce that the RHS of the previous equation is non-zero for the term corresponding to π and that all the non-trivial terms on the RHS have the same sign. Therefore, the LHS is also non-zero, and we see, by choosing any non-zero term, that we obtain the desired τ. The converse direction works similarly, where the role of $\mathbf{G}$ and $\mathbf{G}^*$, are swapped. $\square$

Remark 1.5.3. We note that we crucially used at the places $v \in S_{sc}$ that ϕ_{π_v} was supercuspidal. Otherwise, $\mathrm{tr}(f_v|\Pi_v) \neq 0$ wouldn't necessarily imply that Π_v lies in the L-packet over ϕ_{π_v} without assuming that Π_v is tempered. However, by [KS16, Lemma 2.7] any representation of GSp_4 that is Steinberg at some non-empty finite set of places is tempered at all places. Therefore, we can relax this assumption at least for the forward direction of Theorem 5.2 to just assuming that τ_v is a discrete series representation at all $v \in S_{sc}$.

1.5.2 The Stable and σ-twisted Simple Trace Formula

For the proof of strong multiplicity one, we will need some more refined analysis of trace formulae. Namely, we will be interested in the discrete part of the stable trace formula in the particular case of $\mathbf{G}^* = \mathrm{GSp}_4/F$, as discussed in [CG15, Section 7.1]. To this end, fix a central character datum $(\mathfrak{X},\chi) = (Z(\mathbb{A}_F),\chi)$ as before. We recall that the unique elliptic proper endoscopic group of GSp_4/F is $C = \mathrm{GSO}_{2,2} \simeq (\mathrm{GL}_2 \times \mathrm{GL}_2)/\{(t,t^{-1})|t \in \mathrm{GL}_1\}$. Then, for a test function f^* on $\mathbf{G}^*(\mathbb{A}_F)$ as above, the discrete part of the stable trace formula is an equality:

$$I^{\mathbf{G}^*}_{disc,\chi}(f^*) = ST^{\mathbf{G}^*}_{disc,\chi}(f^*) + \frac{1}{4}ST^{C}_{disc,\chi}(f^C)$$

for f^C a matching test function on C. Here

$$I^{\mathbf{G}^*}_{disc,\chi}(f^*) = \sum_M |W(G,M)|^{-1} \cdot \sum_{s \in W(M,G)_{reg}} |\det(s-1)_{\mathfrak{a}_M/\mathfrak{a}_G}|^{-1} \cdot \mathrm{tr}(M_P(s,0) \cdot I^P_{disc,\chi}(0,f^*))$$

is a sum indexed over classes of standard Levi subgroups of $\mathbf{G}^*$. The precise definition of the terms will not be important for our purposes, but the interested reader can look at [Art02, Section 3]. We simply note that the term corresponding to $M = \mathbf{G}^*$ is precisely equal to $T^{\mathbf{G}^*}_{disc,\chi}(f^*)$, as defined in section 5.1. $ST^{\mathbf{G}^*}_{disc,\chi}$ is a stable distribution on $\mathbf{G}^*$, similar to $ST^{\mathbf{G}^*}_{ell,\chi}$, and $ST^{C}_{disc,\chi}$ is the analogous stable distribution on $C(\mathbb{A}_F)$. However, since C has no proper elliptic endoscopic group, we have

$$ST^{C}_{disc,\chi}(f^C) = I^{C}_{disc,\chi}(f^C) = T^{C}_{disc,\chi}(f^C) + (\text{other terms})$$

with the other terms indexed by proper standard Levi subgroups of C as above.

We will be interested in combining this with the elliptic part of the twisted trace formula as described by Kottwitz-Shelstad [KS99] for the particular group $\tilde{\mathbf{G}} := \mathrm{GL}_4 \times \mathrm{GL}_1/F$ with respect to involution

$$\sigma : (g,e) \mapsto (J^t g^{-1} J^{-1}, e\det(g))$$

where

$$J := \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}$$

We can enumerate the elliptic σ-twisted endoscopic groups as follows.

1. $\mathbf{G}^* = \mathrm{GSp}_4$

2. $C_E = \mathrm{Res}_{E/F}\mathrm{GL}_2' := \{(g_1,g_2) \in \mathrm{Res}_{E/F}\mathrm{GL}_2 | \det(g_1) = \det(g_2)\}$

3. $C_+^E = (\mathrm{GL}_2 \times \mathrm{Res}_{E/F}\mathrm{GL}_1)/\mathrm{GL}_1$

where E is an étale quadratic F-algebra and E is not split in case (3). The simple stable twisted trace formula says that if $\tilde{f}$ is a test function on $\tilde{\mathbf{G}}$ whose twisted orbital integral is supported on the regular elliptic set at at least 3 finite places, then we have an identity

$$I_{disc,\chi}^{\tilde{\mathbf{G}},\sigma}(\tilde{f}) = \frac{1}{2} ST_{disc,\chi}^{\mathbf{G}^*}(f^*) + \frac{1}{4}\sum_E ST_{disc,\chi}^{C_E}(f^{C_E}) + \frac{1}{8}\sum_{E \neq F^{\oplus 2}} ST_{disc,\chi}^{C_+^E}(f^{C_+^E})$$

where

- $\sum_E$ is a sum over étale quadratic F-algebras E,
- $\tilde{f}$, f^*, f^{C_E}, and $f^{C_+^E}$ are matching test functions,
- $ST_{disc,\chi}^{\mathbf{G}^*}$, $ST_{disc,\chi}^{C_E}$, and $ST_{disc,\chi}^{C_+^E}$ are the stable distributions appearing in the discrete part of the stable trace formula, as described above,
- $I_{disc,\chi}^{\tilde{\mathbf{G}},\sigma}$ is the invariant distribution which is the twisted analogue of $I_{disc,\chi}^{\mathbf{G}^*}$. It is given by [LW13, Theorem 14.3.1 and Proposition 14.3.2] and has the form

  $$I_{disc,\chi}^{\tilde{\mathbf{G}},\sigma}(\tilde{f}) = \sum_M |W(G,M)|^{-1} \cdot \sum_{s \in W(M,G)_{reg}} |\det(s-1)_{\mathfrak{a}_M/\mathfrak{a}_G}^{s\cdot\sigma}|^{-1} \cdot \mathrm{tr}(M_P(s,0) \cdot I_{disc,\chi}^P(0,\tilde{f}) I_{P,disc}(\sigma))$$

  where the sum runs over standard Levi subgroups M of G.

Now we want to apply these trace formulae with respect to appropriately chosen test functions. We will assume that S_{st} is a finite set of places such that $|S_{st}| \geq 3$. Then, for all $v \in S_{st}$, we let f_v^* be a pseudo-coefficient for the unramified twist of Steinberg by η_v, where η_v is a character of $\mathbf{G}(F_v)$ such that the restriction to $Z(F_v)$ is χ_v. In particular, we will take f_v^* to be the Lefschetz function $f_{Lef,\eta_v}^{\mathbf{G}^*} \in \mathscr{H}(\mathbf{G}^*(F_v), \chi_v^{-1})$ considered in the previous section which is a pseudo-coefficient for Steinberg up to scaling. It follows, by [CG15, Corollary 10.8], that we can choose the local constituent of the matching function $\tilde{f}$ at v to be the σ-twisted pseudo-coefficient of the Steinberg twisted by η_v, as defined in [MW18]. These functions orbital integrals are supported on the regular elliptic set (See the construction in [MW18, Section 7.2]) and therefore we can apply the simple twisted trace formula. Moreover, the twisted orbital integral of $\tilde{f}_v$ is a stable function, and hence the κ-orbital integral of $\tilde{f}_v$ is zero for all $\kappa \neq 1$. Therefore, it follows that the transfers of f_v to all elliptic twisted endoscopic groups $(\tilde{\mathbf{G}}_{F_v}, \sigma)$ vanish, except possibly for $\mathbf{G}^*_{F_v}$. Thus, the simple twisted trace formula simplifies giving an equality:

$$I^{\tilde{\mathbf{G}},\sigma}_{disc,\chi}(\tilde{f}) = \frac{1}{2} ST^{\mathbf{G}^*}_{disc,\chi}(f^*)$$

Now we apply the discrete part of the stable trace formula for $\mathbf{G}^*$ to the RHS this gives us an equality:

$$I^{\mathbf{G}^*}_{disc,\chi}(f^*) - \frac{1}{4} ST^{C}_{disc,\chi}(f^C) = ST^{\mathbf{G}^*}_{disc,\chi}(f^*)$$

However, for any $v \in S_{st}$, the orbital integral of f^*_v is again stable, so we see that its transfer to the endoscopic group C is zero. Hence, the second term vanishes. All in all, we obtain the following lemma.

Lemma 1.5.4. *For S_{st} a finite set of finite places with $|S_{st}| \geq 3$, f^* and $\tilde{f}$ matching test functions on $\mathbf{G}^*$ and $\tilde{\mathbf{G}}$, respectively, such that f^*_v is a pseudo-coefficient for the (essentially) discrete series Steinberg representation twisted by an unramified character η_v such that $\eta_v|_{Z(F_v)} = \chi_v$ and $\tilde{f}_v$ is the σ-twisted pseudo-coefficient for the Steinberg representation of $\tilde{\mathbf{G}}_{F_v}$ twisted by η_v, we have an equality:*

$$\frac{1}{2} I^{\mathbf{G}^*}_{disc,\chi}(f^*) = I^{\tilde{\mathbf{G}},\sigma}_{disc,\chi}(\tilde{f})$$

relating spectral information on $\mathbf{G}^*$ to $\tilde{\mathbf{G}}$.

1.5.3 Strong Multiplicity One

We now would like to combine the analysis of sections 5.1 and 5.2 to deduce a strong multiplicity one result for $\mathbf{G}^* = \mathrm{GSp}_4/F$ and certain inner forms. Our analysis is very similar to [CG15, Sections 10.5 and 10.6] and benefited from reading the proofs of [RW18, Proposition 10.1 and Theorem 11.4] in a paper of Rosner and Weissauer, where they prove a similar multiplicity one result using Weselmann's topological twisted trace formula [Wes12] instead of the simple twisted trace formula of Kottwitz-Shelstad. Let S_{st} and S_{sc} be disjoint finite sets of finite places. Let S_∞ denote the set of infinite places. Set $S_0 \subset S_{st} \cup S_{sc}$ and $S_{sc} \cup S_{st} \cup S_\infty \subset S$ to be finite sets of places as before. We let $\mathbf{G}$ be an inner form over F, as in Theorem 5.2, trivialized outside of $S_0 \cup S_\infty$. We have the following.

Proposition 1.5.5. *Assume that $|S_{st}| \geq 3$. Let π be a cuspidal automorphic representation of $\mathbf{G}^* = \mathrm{GSp}_4/F$ or the above inner form $\mathbf{G}$ satisfying the following:*

1. π is cohomological of regular weight ξ at infinity,

2. π is unramified outside of S,

3. π is an unramified twist of Steinberg at all places in S_{st},

4. π has supercuspidal L-parameter at all places in S_{sc}.

If π' is a cuspidal automorphic representation of $\mathbf{G}^*$ satisfying (1), (3), and $\pi'^S \simeq \pi^S$ then its Langlands parameter at all places in S agrees with π. If π' is a cuspidal automorphic representation of $\mathbf{G}$ satisfying conditions (1), (3), and $\pi'^S \simeq \pi^S$ then its Langlands parameter at all places in $S_{st} \cup S_{sc} \cup S_\infty$ agrees with π.

Proof. Set χ to be the central character of π. We apply the above trace formulae with central character datum $(Z(\mathbb{A}_F), \chi)$. If π is a representation of $\mathbf{G}^* = \mathrm{GSp}_4/F$, we take τ to be a globally generic member of the global L-packet of π, as in the proof of Theorem 5.2. If π is a representation of the inner form $\mathbf{G}$ then, using Theorem 5.2, we take τ to be a globally generic strong transfer τ of π to a cuspidal automorphic representation of $\mathbf{G}^*$, with Langlands parameter equal to ϕ_{π_v} at all places in $v \in S_{sc} \cup S_{st} \cup S_\infty$. Now we apply [GT11, Section 13] to τ to deduce the existence of a strong transfer to a globally generic automorphic representation of $\mathrm{GL}_4(\mathbb{A}_F)$, denoted $\tilde{\tau}$. It satisfies the following:

1. $\tilde{\tau}$ is a global theta lift of τ.

2. $\tilde{\tau}^\vee \otimes \chi \simeq \tilde{\tau}$.

3. For all places v, we have that $\phi_{\tilde{\tau}_v} = \mathrm{std} \circ \phi_{\tau_v}$ as conjugacy classes of parameters.

4. Its form falls into the two cases:

   (a) $\tilde{\tau}$ is cuspidal

   (b) $\tilde{\tau} = \sigma \boxplus \sigma'$ for $\sigma \neq \sigma'$ a cuspidal automorphic representation of GL_2.

   In the latter case, τ is the theta lift of a cusp form $\sigma \otimes \sigma'$ on $C = \mathrm{GSO}_{2,2}$.

To distinguish these two cases, we say that $\tilde{\tau}$ is a stable or endoscopic lift. We choose matching test functions $\tilde{f}$ and f^* on $\tilde{\mathbf{G}}$ and $\mathbf{G}^*$, respectively, such that, for $v \in S_{st}$, they are pseudo-coefficients for Steinberg twisted by η_v, as in Lemma 5.3, where η_v the unramified character that $\pi_{v_{st}}$ is a twist of Steinberg of, where we can, up to scaling, take this to be the Lefschetz function $f^{\mathbf{G}^*}_{Lef,\eta_{v_{st}}}$ considered in section 5.1. We let f^*_∞ be a Lefschetz function for the discrete series L-packet given by ξ as before. Lemma 5.3 then gives us an equality:

$$\frac{1}{2}I^{\mathbf{G}^*}_{disc,\chi}(f^*) = I^{\tilde{\mathbf{G}},\sigma}_{disc,\chi}(\tilde{f})$$

We first treat the case where $\tilde{\tau}$ is a stable lift, and consider the part of the RHS corresponding to the cuspidal representation $\tilde{\tau}$ constructed above. By using linear independence of the unramified characters and the strong multiplicity one property for $\tilde{\mathbf{G}}$, the above identity implies an equality

$$c_1 \cdot \sum_{\Pi' \simeq \pi^S} m(\Pi')\mathrm{tr}(f^*_S|\Pi'_S) = \mathrm{tr}_\sigma(\tilde{\tau}_S|\tilde{f}_S) \tag{1.3}$$

where c_1 is a non-zero constant. Here $\mathrm{tr}_\sigma(\tau_S|\tilde{f}_S)$ is the σ-twisted trace, as defined in [CG15, Section 5.16]. The LHS runs over automorphic representations satisfying the following:

1. Π' has non-zero contribution to the discrete part of the trace formula $I^{\mathbf{G}^*}_{disc,\chi}$.
2. The coefficient $m(\Pi')$ is the coefficient associated with the trace of Π' in $I^{\mathbf{G}^*}_{disc,\chi}$.

We can further simplify the LHS of (3) by noting that non-discrete spectrum representations which intervene in $I^{\mathbf{G}^*}_{disc,\chi}$ are parabolically induced from the discrete spectrum of proper Levi subgroups of $\mathbf{G}^*$. By [CG15, Section 5.8], we know that parabolically induced representations of $\mathbf{G}^*$ lift to parabolically induced representations of $\tilde{\mathbf{G}}$. Therefore, since $\tilde{\tau}$ is a stable lift and therefore cuspidal, all terms occurring in in the LHS must all come from the discrete spectrum $T^{\mathbf{G}^*}_{disc,\chi}(f^*)$, by strong multiplicity one for $\tilde{\mathbf{G}}$. Moreover, the coefficients $m(\Pi')$ must then be the multiplicities of Π' in the discrete spectrum. However, as in Lemma 5.1, we have an equality:

$$T^{\mathbf{G}^*}_{disc,\chi}(f^*) = T^{\mathbf{G}^*}_{cusp,\chi}(f^*)$$

In other words, we may assume that the sum on the LHS of (3) ranges over $\Pi' \in \mathscr{A}_{cusp,\chi}(\mathbf{G}^*)$, and that $m(\Pi')$ denotes the multiplicity in the cuspidal automorphic spectrum. In other words, we can rewrite the LHS as

$$\sum_{\substack{\Pi' \in \mathscr{A}_{cusp,\chi}(\mathbf{G}^*) \\ \Pi'^S \simeq \pi^S}} m(\Pi')\mathrm{tr}(f^*_S|\Pi'_S)$$

Now, for the RHS, we apply the local character identities of Chan-Gan [CG15, Proposition 9.1], this tells us that we have an equality:

$$\mathrm{tr}_\sigma(\tilde{\tau}_S|\tilde{f}_S) = c_2 \prod_{v\in S} \sum_{\pi'_v \in \Pi_{\phi_{\tau_v}}(\mathbf{G}^*_{F_v})} \mathrm{tr}(f^*_v|\pi'_v)$$

for some non-zero constant c_2, where we have used property (3) of the representation $\tilde{\tau}$. In summary, we have concluded

$$\sum_{\substack{\Pi'\in\mathscr{A}_{cusp,\chi}(\mathbf{G}^*)\\ \Pi'^S\simeq\pi^S}} m(\Pi')\mathrm{tr}(f^*_S|\Pi'_S) = c\cdot\prod_{v\in S}\sum_{\pi'_v\in\Pi_{\phi_{\tau_v}}(\mathbf{G}^*_{F_v})} \mathrm{tr}(f^*_v|\pi'_v)$$

for some non-zero constant c. If π was a representation of $\mathbf{G}^*$, we know by our choice of τ that $\phi_{\tau_v} = \phi_{\pi_v}$ for all $v \in S$, so, by linear independence of characters at the places $v \in S\setminus S_\infty \cup S_{st}$, this tells us that the local constituents of some Π' occurring in the LHS with non-zero trace at $S_\infty \cup S_{st}$ are described by members of the L-packet over ϕ_{π_v} occurring with some multiplicity. Since the representation π' is by assumption cohomological of regular weight ξ and an unramified twist of Steinberg at all places in S_{st}, by arguing as in proof of Theorem 5.2, we have that $\mathrm{tr}(f_{S_{st}\cup S_\infty}|\pi'_{S_{st}\cup S_\infty}) \neq 0$, and this gives us the desired claim for $\mathbf{G}^* = \mathrm{GSp}_4$. Now, if π' is a representation of the inner form, we apply the character identities of Chan-Gan [CG15, Proposition 11.1]. This tells us that the RHS of the previous equation is equal to

$$c_3 \prod_{v\in S}\sum_{\rho_v\in\Pi_{\phi_{\tau_v}}(\mathbf{G}_{F_v})} \mathrm{tr}(f_v|\rho_v)$$

for some non-zero constant c_3. Now, to rewrite the LHS, we apply the trace formula as in the proof of Theorem 5.2. By linear independence of characters, we obtain a relationship

$$\sum_{\substack{\Pi'\in\mathscr{A}_{cusp,\chi}(\mathbf{G}^*)\\ \Pi'^S\simeq\pi^S}} m(\Pi')\mathrm{tr}(f^*_S|\Pi'_S) = c_4\cdot \sum_{\substack{\Pi\in\mathscr{A}_{cusp,\chi}(\mathbf{G})\\ \Pi^S\simeq\pi^S}} m(\Pi)\mathrm{tr}(f_S|\Pi_S)$$

for some non-zero constant c_4. All in all, we obtain that

$$\cdot\sum_{\substack{\Pi\in\mathscr{A}_{cusp,\chi}(\mathbf{G})\\ \Pi^S\simeq\pi^S}} m(\Pi)\mathrm{tr}(f_S|\Pi_S) = c'\prod_{v\in S}\sum_{\rho_v\in\Pi_{\phi_{\tau_v}}(\mathbf{G}_{F_v})} \mathrm{tr}(f_v|\rho_v)$$

for some non-zero constant c'. We know by our choice of τ that $\phi_{\tau_v} = \phi_{\pi_v}$ for all $v \in S_{sc} \cup S_{st} \cup S_\infty$. From here, the claim follows.

In the case that $\tilde{\tau}$ is an endoscopic lift. We apply the stable trace form for $\mathbf{G}^*$ and a test function f^* of $\mathbf{G}^*(\mathbb{A}_F)$. We recall that this is an identity:

$$ST^{\mathbf{G}^*}_{disc,\chi}(f^*) = I^{\mathbf{G}^*}_{disc,\chi}(f^*) - \frac{1}{4} ST^C_{disc,\chi}(f^C)$$

for a f^C a matching test function on $C(\mathbb{A}_F)$. Moreover, since C has no proper elliptic endoscopic groups, we have

$$ST^C_{disc,\chi}(f^C) = I^C_{disc,\chi}(f^C).$$

We look at the $\sigma_1 \otimes \sigma_2$-isotypic part, where $\sigma_1 \otimes \sigma_2$ is the representation of C whose theta lift is τ. and use linear independence of characters at the unramified places to obtain a semi-local identity

$$ST^{\mathbf{G}^*}_{\sigma_1\otimes\sigma_2}(f^*_S) = \sum_{\Pi'^S \simeq \pi^S} m(\Pi')\mathrm{tr}(f^*_S|\Pi'_S) - \frac{1}{4} \sum_{(\sigma'_1\otimes\sigma'_2)^S \simeq (\sigma_1\otimes\sigma_2)^S} m(\sigma\otimes\sigma')\mathrm{tr}(f^C_S|(\sigma'_1\otimes\sigma'_2)_S)$$

where the LHS is a stable distribution on $\mathbf{G}^*(\mathbb{A}_S)$, as in [CG15, Equation 8.5]. The first term in the RHS is a sum over discrete automorphic representations Π' of $\mathbf{G}^*(\mathbb{A}_F)$ with the coefficient $m(\Pi')$ being the multiplicity in the discrete spectrum, and the second term is a sum over all automorphic representations $\sigma'_1 \otimes \sigma'_2$ of $C(\mathbb{A}_F)$. It follows by [CG15, Corollary 8.6] that the LHS is equal to

$$\frac{1}{2} \prod_{v\in S} \sum_{\pi'_v \in \Pi_{\phi_{\tau_v}}(\mathbf{G}^*_{F_v})} \mathrm{tr}(f^*_v|\pi'_v)$$

On the other hand, if, for $v \in S_{st}$, we take $f^*_v = f^{\mathbf{G}^*}_{Lef,\eta_v}$ to be the Lefschetz function as above, we see that

$$-\frac{1}{4} \sum_{(\sigma_1\otimes\sigma'_2)^S \simeq (\sigma_1\otimes\sigma_2)^S} m(\sigma\otimes\sigma')\mathrm{tr}(f^C_S|(\sigma'_1\otimes\sigma'_2)_S)$$

vanishes, since the orbital integral of f^*_v is stable and $S_{st} \neq \emptyset$. In summary, we have concluded an identity

$$c'' \prod_{v\in S} \sum_{\pi'_v \in \Pi_{\phi_{\tau_v}}(\mathbf{G}^*_{F_v})} \mathrm{tr}(f^*_S|\pi'_v) = \sum_{\Pi'^S \simeq \pi^S} m(\Pi')\mathrm{tr}(f^*_S|\Pi'_S)$$

for some non-zero constant c''. Now taking f^*_∞ to be our Lefschetz function at ∞, we can argue just as in the stable case. $\square$

Remark 1.5.6. Strong multiplicity one for globally generic automorphic representations of GSp_4 has been proven by Jiang-Soudry [JS07]. So, in the particular case that π and π' are representations of GSp_4, we could have just assumed that S_{st} is non-empty and then applied their results to a globally generic member in the global L-packets of π and π' to deduce the desired claim.

1.6 Galois Representations in the Cohomology of Shimura varieties

We now would like to combine the results of the previous section with results of Sorensen [Sor10] on the Galois representations associated to automorphic representations of $\mathbf{G}^* = \mathrm{GSp}_4/F$ to say something about the Galois action of the global Shimura varieties occurring in basic uniformization. Let $F/\mathbb{Q}$ be a totally real field and $\mathbb{A}_{f,F}$ the finite adeles of F. Throughout, we will assume that τ is a cuspidal automorphic representation of $\mathbf{G}^*$ satisfying the same properties as in the previous section.

1. τ_∞ is cohomological of some regular weight ξ of $\mathbf{G}^*(F_\infty)$.
2. τ_v is unramified at all finite places outside of S.
3. τ_v is an unramified twist of Steinberg at some finite set of finite places S_{st}.

We have the following key result of Sorensen.

Theorem 1.6.1. *[Sor10, Theorem A] Fix a globally generic τ as above such that S_{st} is non-empty. Then there exists, a unique (after fixing the isomorphism $i : \overline{\mathbb{Q}}_\ell \xrightarrow{\simeq} \mathbb{C}$) irreducible continuous representation $\rho_\tau : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_\ell)$ characterized by the property that, for each finite place $v \nmid \ell$ of F, we have*

$$i\mathrm{WD}(\rho_\tau|_{W_{F_v}})^{F-s.s} \simeq \phi_{\tau_v} \otimes |\cdot|^{-3/2}$$

where $(-)^{F-s.s}$ denotes the Frobenius semisimplification and ϕ_{τ_v} is the Gan-Takeda parameter of τ_v.

Now let us fix τ with associated ρ_τ as above and assume that τ is a strong transfer of some cuspidal automorphic representation π of $\mathbf{G}$, as in Theorem 5.2. We assume that S_{st} contains q an odd inert prime in the number field F and choose the inner form $\mathbf{G}$ to be of the following form, as in Kret-Shin [KS16, Section 8],

- $\mathbf{G}(\mathbb{R}) \simeq \mathrm{GSp}_4(\mathbb{R}) \times \mathrm{GU}_2(\mathbb{H})^{[F:\mathbb{Q}]-1}$,
- $\mathbf{G}_{F_v} \simeq \mathrm{GSp}_4/F_v$ at all finite places v if $[F:\mathbb{Q}]$ is odd,
- $\mathbf{G}_{F_v} \simeq \mathrm{GSp}_4/F_v$ at all but the finite place q if $[F:\mathbb{Q}]$ is even,

where $\mathbb{H}$ is the Hamilton quaternions. Let $A(\pi)$ be the set of isomorphism classes of cuspidal automorphic representations Π of $\mathbf{G}$ such that, for all $v \in S_{st}$, Π_v is an unramified twist of Steinberg, Π_∞ is ξ cohomological, and, for all $v \notin S_\infty \cup S_{st}$, $\Pi_v \simeq \pi_v$. Our main task now is to show that ρ_τ is realized in the π^∞ isotypic component of the Shimura variety associated to a Shimura datum $(\mathbf{G}, X)$, where X is as in [KS16, Pages 41-42]. Let $\mathrm{Sh}(\mathbf{G},X)_{K,\overline{F}}$ be the associated Shimura variety over $\overline{F}$ which we recall is 3-dimensional. We set $\mathscr{L}_\xi$ to be the $\overline{\mathbb{Q}}_\ell$ local system associated to a irreducible representation of $\mathbf{G}$ over F of highest weight ξ on it as before, and let $H^i_c(\mathrm{Sh}(\mathbf{G},X)_K, \mathscr{L}_\xi)_{ss}$ denote the semisimplification as a Hecke module of the compactly supported etale cohomology valued in $\mathscr{L}_\xi$. Choose $K \subset \mathbf{G}(\mathbb{A}_{f,F})$ a sufficiently small compact open subgroup such that π^∞ has a non-zero K-invariant vector. Let S_{bad} denote the set of prime numbers p for which either $p = 2$, the group $\mathbf{G}$ is ramified, or $K_p = \prod_{v|p} K_v$ is not hyperspecial. Then we define the virtual Galois representation

$$\rho^\pi_{shim} := (-1)^3 \sum_{\Pi \in A(\pi)} \sum_{i=0}^{6} (-1)^i [Hom_{\mathbf{G}(\mathbb{A}_{f,F})}(\Pi^\infty, H^i_c(\mathrm{Sh}(\mathbf{G},X)_{K,\overline{F}}, \mathscr{L}_\xi)_{ss})] \in K_0(\overline{\mathbb{Q}}_\ell(\Gamma_F)) \tag{1.4}$$

where $K_0(\overline{\mathbb{Q}}_\ell(\Gamma_F))$ denotes the Grothendieck group of continuous $\Gamma_F := \mathrm{Gal}(\overline{F}/F)$-representations with coefficients in $\overline{\mathbb{Q}}_\ell$. We now define the rational number

$$a(\pi) := (-1)^3 N_\infty^{-1} \sum_{\Pi \in A(\pi)} m(\Pi) \cdot ep(\Pi_\infty \otimes \xi)$$

where

1. $m(\Pi)$ is the multiplicity of Π in the automorphic spectrum of $\mathbf{G}$,
2. $N_\infty = |\Pi_\xi^{G(F_\infty)}| \cdot |\pi_0(\mathbf{G}(F_\infty)/Z(F_\infty)| = 4$, where $\Pi_\xi^{G(F_\infty)}$ denotes the discrete series L-packet of representations of $G(F_\infty)$ cohomological of weight ξ,
3. $ep(\Pi_\infty \otimes \xi) := \sum_{i=0}^\infty (-1)^i \dim(H^i(Lie(\mathbf{G}(F_\infty)), K_\infty; \Pi_\infty \otimes \xi)$.

Then we have the following proposition of Kret-Shin.

Proposition 1.6.2. *[KS16, Proposition 8.2] With notation as above, for almost all finite F-places v not dividing a prime number in S_{bad} and all sufficiently large integers j, we have*

$$\mathrm{tr}(\rho^{\pi}_{shim}(\mathrm{Frob}^{j}_{v})) = a(\pi) q_v^{j\frac{3}{2}} \mathrm{tr}(\mathrm{std} \circ \phi_{\pi_v})(\mathrm{Frob}^{j}_{v})$$

Moreover, the virtual representation ρ^{π}_{shim} is a true representation. In particular, the only non-zero term appearing in the above alternating sum occurs in middle degree ($= 3$).

Remark 1.6.3. The claim about it occurring in middle degree is part of the proof of the Proposition not the statement. (See the discussion after equation (8.13) in [KS16])

We use this to deduce the following corollary.

Corollary 1.6.4. *The π^{∞}-isotypic component of $R\Gamma_c(\mathrm{Sh}(\mathbf{G},X)_{K^p,\overline{F}}, \mathscr{L}_{\xi})$ is concentrated in degree 3 and has Γ_F-action given (up to multiplicity) by the semi-simplification of* $\mathrm{std} \circ \rho_{\tau}$.

Proof. The first part follows immediately from the previous Proposition, and the second part follows from the identification of the traces. In particular, by the Brauer-Nesbitt Theorem, Cheboratev density theorem, and the condition characterizing ρ_{τ}, we can identify (up to multiplicity) the Galois representation ρ^{π}_{shim} with the semi-simplifaction of the Galois representation $\mathrm{std} \circ \rho_{\tau}$. $\square$

1.7 Proof of the Key Proposition

We will now combine the results of the previous three sections to deduce some key consequences that will be used to derive Proposition 1.4. For this, using Krasner's lemma, we now fix a totally real number field F with two odd totally inert primes p and q such that $F_p \simeq L$ the fixed unramified extension of $\mathbb{Q}_p$. We fix the $\mathbb{Q}$-inner form $\mathbf{G}$ of $\mathbf{G}^* = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GSp}_4$ defined in section 6, and let $\mathbf{G}'$ be the inner form of $\mathbf{G}$ seen in Definition 4.1. We take $(\mathbf{G},X)$ to be the Shimura datum considered in [KS16, Pages 41-42] as in section 6. We note that this forces the associated geometric dominant cocharacter μ of $G := \mathbf{G}_{\mathbb{Q}_p}$ to be the Siegel cocharacter; in particular, we can apply the results of section 4.2. Set ξ to be a regular weight of an algebraic representation of $\mathbf{G}$ over $\mathbb{Q}$. Let $K^p \subset \mathbf{G}(\mathbb{A}^{p\infty})$ be an open compact subgroup. We set $S_{sc} = \{p\}$ and S_{st} to be a disjoint finite set of finite places of $\mathbb{Q}$ containing q. We consider the uniformization map

$$\Theta : R\Gamma_c(G,b,\mu) \otimes^{\mathbb{L}}_{\mathscr{H}(J_b)} \mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash \mathbf{G}'(\mathbb{A}_f)/K^p, \mathscr{L}_\xi) \to R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p}, \mathscr{L}_\xi) \tag{1.5}$$

supplied by Theorem 4.2 and Corollary 4.1. Now fix a smooth irreducible supercuspidal representation ρ of $J = \mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GU}_2(D))(\mathbb{Q}_p) = \mathrm{GU}_2(D)(L)$. We have the following lemma.

Lemma 1.7.1. *Suppose ρ is a supercuspidal representation of $J(\mathbb{Q}_p)$, then, for sufficiently regular ξ and sufficiently small K^p, we can find a lift Π' to a cuspidal automorphic representation of $\mathbf{G}'$, such that Π'^{∞} occurs as a $J(\mathbb{Q}_p)$-stable direct summand of $\mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash \mathbf{G}'(\mathbb{A}_f))/K^p, \mathscr{L}_\xi)$. Moreover, for all places in S_{st}, we can assume that the local constituents at $v \in S_{st}$ are unramified twists of the Steinberg representation.*

Proof. This follows from an argument using the simple trace formula. See for example [Han20, Proposition 2.9] or [Shi12]. We note in particular that cuspidality is vacuous, since $\mathbf{G}'(\mathbb{R})$ is compact modulo center by construction. □

So let Π' be a globalization of a fixed supercuspidal ρ to a cuspidal automorphic representation of $\mathbf{G}'$ for some sufficiently regular ξ and sufficiently small K^p. We can and do regard $\Pi'^{p\infty}$ as a representation of $\mathbf{G}(\mathbb{A}_f^p) \simeq \mathbf{G}'(\mathbb{A}_f^p)$. We set $K^p = K_S^p K^S$, where $K^S \subset \mathbf{G}(\mathbb{A}_f^S)$ is an open compact in the finite adeles away from S, for $S \subset \mathrm{Pl}_F$ some finite set of places of $\mathbb{Q}$ containing $S_{st} \cup \{p\} \cup \{\infty\}$, as in section 5. We assume that S is sufficiently large such that outside of S the automorphic representation Π' is unramified, so, in particular, the subgroup $K^S \subset \mathbf{G}(\mathbb{A}_f^S)$ is a product of hyperspecial subgroups away from S. We consider the abstract commutative Hecke algebra

$$\mathbb{T}^S := Z[\mathbf{G}(\mathbb{A}_f^S)//K^S)]$$

of bi-invariant compactly supported smooth functions on $\mathbf{G}(\mathbb{A}_f^S)$. We regard both sides of (5) as $\mathbb{T}^S$-modules and consider the maximal ideal $\mathfrak{m}$ defined by the Hecke eigenvalues of Π'^S. We then localize both sides of (5) at $\mathfrak{m}$ to obtain a map:

$$\Theta_{\mathfrak{m}} : (R\Gamma_c(G,b,\mu) \otimes^{\mathbb{L}}_{\mathscr{H}(J_b)} \mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash \mathbf{G}'(\mathbb{A}_f)/K^p, \mathscr{L}_\xi))_{\mathfrak{m}} \to R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p}, \mathscr{L}_\xi)_{\mathfrak{m}}$$

We would like to apply Propositions 4.4 and 5.4 to the representations occurring on both sides of this map. However, to apply these results we need to make some more modifications. In particular, the automorphic representations of $\mathbf{G}'$ occurring in the LHS (resp. RHS) of $\Theta_{\mathfrak{m}}$ are not necessarily unramified twists of Steinberg at all places in S_{st}.

To remedy this, we set $K^p = K_{\{p\}\cup S_{st}} K^{\{p\}\cup S_{st}}$, where $K^{\{p\}\cup S_{st}} \subset \mathbf{G}(\mathbb{A}_f^{\{p\}\cup S_{st}}) \simeq \mathbf{G}'(\mathbb{A}_f^{\{p\}\cup S_{st}})$ is an open compact subgroup. Then we consider the colimits

$$R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^{\{p\}\cup S_{st}}}, \mathscr{L}_\xi) := \mathrm{colim}_{K_{\{p\}\cup S_{st}} \to \{1\}} R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K_{\{p\}\cup S_{st}} K^{\{p\}\cup S_{st}}}, \mathscr{L}_\xi)$$

and

$$\mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash \mathbf{G}'(\mathbb{A}_f)/K^{\{p\}\cup S_{st}}, \mathscr{L}_\xi) := \mathrm{colim}_{K_{\{p\}\cup S_{st}} \to \{1\}} \mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash \mathbf{G}'(\mathbb{A}_f)/K_{\{p\}\cup S_{st}} K^{\{p\}\cup S_{st}}, \mathscr{L}_\xi)$$

Since $S_{st} \subset S$, the map $\theta_{\mathfrak{m}}$ gives rise to a map

$$(R\Gamma_c(G,b,\mu) \otimes^{\mathbb{L}}_{\mathscr{H}(J_b)} \mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash \mathbf{G}'(\mathbb{A}_f)/K^{S_{st}\cup\{p\}}, \mathscr{L}_\xi))_{\mathfrak{m}} \to R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^{S_{st}\cup\{p\}}}, \mathscr{L}_\xi)_{\mathfrak{m}}$$

By Proposition 4.4, we obtain an isomorphism

$$(R\Gamma_c(G,b,\mu)_{sc} \otimes^{\mathbb{L}}_{\mathscr{H}(J_b)} \mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash \mathbf{G}'(\mathbb{A}_f)/K^{S_{st}\cup\{p\}}, \mathscr{L}_\xi))_{\mathfrak{m}} \xrightarrow{\simeq} R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^{S_{st}\cup\{p\}}}, \mathscr{L}_\xi)_{\mathfrak{m},sc}$$

Now, for all $v \in S_{st}$, we can project to summand of the LHS where $\mathbf{G}(F_v) \simeq \mathbf{G}'(F_v)$ acts via an unramified twist of Steinberg, noting that the LHS and hence the RHS is semisimple.

This gives an isomorphism:

$$\Theta^{st}_{\mathfrak{m},sc} : (R\Gamma_c(G,b,\mu)_{sc} \otimes^{\mathbb{L}}_{\mathscr{H}(J_b)} \mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash \mathbf{G}'(\mathbb{A}_f)/K^{S_{st}\cup\{p\}}, \mathscr{L}_\xi))^{st}_{\mathfrak{m}} \xrightarrow{\simeq} R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^{S_{st}\cup\{p\}}}, \mathscr{L}_\xi)^{st}_{\mathfrak{m},sc}$$

We now apply Proposition 5.4 to obtain the following.

Proposition 1.7.2. *Let $\rho \in \Pi(J)$ be a representation with supercuspidal Gan-Tantono parameter ϕ. Assume that $|S_{st}| \geq 3$. Then, for Π' a choice of globalization of ρ as in Lemma 7.1, unramified outside S with associated maximal ideal $\mathfrak{m} \subset \mathbb{T}^S$ in the Hecke algebra defined by the Hecke eigenvalues of Π'^S, the representations of $\mathbf{G}'(\mathbb{A}_f)$ occurring in the LHS of the map*

$$\Theta^{st}_{\mathfrak{m},sc} : (R\Gamma_c(G,b,\mu)_{sc} \otimes^{\mathbb{L}}_{\mathscr{H}(J_b)} \mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash \mathbf{G}'(\mathbb{A}_f)/K^{S_{st}\cup\{p\}}, \mathscr{L}_\xi))^{st}_{\mathfrak{m}} \xrightarrow{\simeq} R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^{S_{st}\cup\{p\}}}, \mathscr{L}_\xi)^{st}_{\mathfrak{m},sc}$$

are isomorphic to $\overline{\Pi}'^{\infty}$ for $\overline{\Pi}'$ a cuspidal automorphic representation of $\mathbf{G}'(\mathbb{A})$ satisfying the following:

- $\overline{\Pi}'^S \simeq \Pi'^S$,
- $\overline{\Pi}'$ *is cohomological of regular weight* ξ *at* ∞,
- $\overline{\Pi}'$ *is unramified twist of Steinberg at all* $v \in S_{st}$,
- $\overline{\Pi}'$ *has local constituent at* p *with associated L-parameter* ϕ.

Proof. We localized at $\mathfrak{m}$ corresponding to Π'^S and are considering algebraic automorphic representations of $\mathbf{G}'$ valued in the algebraic representation defined by ξ. Therefore, it is clear that any representation occurring in the LHS is of the form $\overline{\Pi}'^{\infty}$, where $\overline{\Pi}'$ is a cuspidal automorphic form of $\mathbf{G}'$ satisfying (1) and (2). Here cuspidality is automatic since $\mathbf{G}'(\mathbb{R})$ is compact modulo center. Moreover, by construction, it follows that $\mathbf{G}'(\mathbb{Q}_v)$ acts on the LHS via representations which are an unramified twist of Steinberg for $v \in S_{st}$. This allows us to apply proposition 5.4, since $|S_{st}| \geq 3$ by assumption. Proposition 5.4 applied to the inner form $\mathbf{G}'$ of $\mathbf{G}^*$ and the cuspidal automorphic representation Π' of $\mathbf{G}'$ tells us that $\overline{\Pi}'$ must have Langlands parameter at $\{p\} = S_{sc}$ given by ϕ, which was the desired claim. □

We now combine this with Corollary 6.3 to deduce the following.

Corollary 1.7.3. *With notation as above, the map*

$$\Theta^{st}_{\mathfrak{m},sc} : (R\Gamma_c(G,b,\mu)_{sc} \otimes^{\mathbb{L}}_{\mathscr{H}(J_b)} \mathscr{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^{S_{st}\cup\{p\}}, \mathscr{L}_\xi))^{st}_{\mathfrak{m}} \xrightarrow{\simeq} R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^{S_{st}\cup\{p\}}}, \mathscr{L}_\xi)^{st}_{\mathfrak{m},sc}$$

is an isomorphism of complexes of $G(\mathbb{Q}_p) \times W_L$*-modules concentrated in degree* 3 *with* W_L*-action given, up to multiplicity and semi-simplification as a* W_L*-module, by* $\mathrm{std} \circ \phi \otimes |\cdot|^{-3/2}$.

Proof. Proposition 7.2 tells us that the LHS of $\Theta^{st}_{\mathfrak{m},sc}$ breaks up as a direct sum of $G(\mathbb{Q}_p) \times W_L$-modules of the form

$$R\Gamma_c(G,b,\mu)_{sc} \otimes^{\mathbb{L}}_{\mathscr{H}(J_b)} \overline{\Pi}'^{\{p,\infty\}\cup S_{st}}$$

for $\overline{\Pi}'$ a cuspidal automorphic representation of $\mathbf{G}'$ that has L-parameter ϕ at p, and is also cohomological of regular weight ξ at infinity and an unramified twist of Steinberg at all places in S_{st}. It suffices to prove the claim for each one of these summands. This summand will map to the $\overline{\Pi}'^{\{p,\infty\}\cup S_{st}}$-isotypic part of the RHS by construction. Let τ denote a strong transfer of Π' to a cuspidal automorphic representation of $\mathbf{G}^*$ given by Theorem 5.2, with associated Galois representation ρ_τ given by Theorem 6.1. Applying Theorem 5.2 again, we consider a strong transfer of τ to $\mathbf{G}$ given by Π. We note, by Corollary 6.3, that the $\overline{\Pi}'^{\{p,\infty\}\cup S_{st}} \simeq \Pi^{\{p,\infty\}\cup S_{st}}$-isotypic part will be concentrated in degree 3 and have W_L-action given (up to multiplicity and semi-simplification) by $\mathrm{std} \circ \phi \otimes |\cdot|^{-3/2}$, by the property characterizing ρ_τ and the fact that τ was a strong transfer. $\square$

With this in hand, we are finally ready conclude our key Proposition.

Proposition 1.7.4. *Let ϕ be a supercuspidal parameter with associated L-packet $\Pi_\phi(J)$. Then the direct summand of*

$$\bigoplus_{\rho' \in \Pi_\phi(J)} R\Gamma_c(G,b,\mu)[\rho']$$

where $G(\mathbb{Q}_p)$ acts via a supercuspidal representation

$$\bigoplus_{\rho' \in \Pi_\phi(J)} R\Gamma_c(G,b,\mu)[\rho']_{sc}$$

is concentrated in middle degree 3 *and admits a non-zero W_L-stable sub-quotient with W_L-action given by* $\mathrm{std} \circ \phi \otimes |\cdot|^{-3/2}$.

Proof. This is an immediate consequence of Proposition 7.2 and Corollary 7.3. $\square$

In particular, using Corollary 3.21, we can deduce the following.

Corollary 1.7.5. *If $p > 2$ and $L/\mathbb{Q}_p$ is an unramified extension, then, for all $\rho \in \Pi(J)$ with supercuspidal Gan-Tantono parameter ϕ_ρ, the Fargues-Scholze and Gan-Tantono correspondences are compatible.*

1.8 Applications

We will now apply Corollary 7.5 to deduce some applications to the strong form of the Kottwitz conjecture and conclude the proof of Theorem 1.1. We begin with the latter.

Theorem 1.8.1. *The following is true.*

1. *For any $\pi \in \Pi(G)$ (resp. $\rho \in \Pi(J)$) such that the Gan-Takeda (resp. Gan-Tantono) parameter is not supercuspidal, we have that the Gan-Takeda (resp. Gan-Tantono) correspondence is compatible with the Fargues-Scholze correspondence.*

2. *If $L/\mathbb{Q}_p$ is unramified and $p > 2$, we have, for all $\pi \in \Pi(G)$ (resp. $\rho \in \Pi(J)$) such that the Gan-Takeda (resp. Gan-Tantono) parameter is supercuspidal, that the Gan-Takeda (resp. Gan-Tantono) correspondence is compatible with the Fargues-Scholze correspondence.*

Proof. Part (1) follows by Corollary 3.12 and Corollary 3.16. Part (2) for the Gan-Tantono local Langlands is precisely Corollary 7.5. It remains to show that for $L/\mathbb{Q}_p$ unramified and $p > 2$, π a smooth irreducible representation of GSp_4/L with supercuspidal Gan-Takeda ϕ_π parameter that the two correspondences are compatible. To show this, we consider the complex

$$R\Gamma_c^\flat(G,b,\mu)[\pi]$$

of $J(\mathbb{Q}_p) \times W_L$-representations. We know, by Theorem 3.13, that this admits subquotients as a $J(\mathbb{Q}_p)$-module given by ρ, for all ρ whose Gan-Tantono parameter ϕ_ρ is equal to the Gan-Takeda parameter ϕ_π of π. However, by Corollary 3.15, we know that these representations must have Fargues-Scholze parameter equal to ϕ_π^{FS}. Therefore, we get a chain of equalities

$$\phi_\pi^{\mathrm{FS}} = \phi_\rho^{\mathrm{FS}} = \phi_\rho = \phi_\pi$$

where we have used compatibility of the Gan-Tantono and the Fargues-Scholze correspondence for the middle equality. □

Now, with this out of the way, we turn our attention to proving some strong forms of the Kottwitz conjecture for these representations, verifying Theorem 1.3.

Theorem 1.8.2. *Let $L/\mathbb{Q}_p$ be an unramified extension with $p > 2$. Let π (resp. ρ) be members of the L-packet over a supercuspidal parameter $\phi : W_L \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_\ell)$. Then the complexes*

$$R\Gamma_c(G,b,\mu)[\pi]$$

and

$$R\Gamma_c(G,b,\mu)[\rho]$$

are concentrated in middle degree 3.

1. *If ϕ is stable supercuspidal, with singleton L-packets $\{\pi\} = \Pi_{\phi}(G)$ and $\{\rho\} = \Pi_{\phi}(J)$, then the cohomology of $R\Gamma_c(G,b,\mu)[\pi]$ in middle degree is isomorphic to*
$$\rho \boxtimes (\mathrm{std} \circ \phi)^{\vee} \otimes |\cdot|^{-3/2}$$
as a $J(\mathbb{Q}_p) \times W_L$-module, and the cohomology of $R\Gamma_c(G,b,\mu)[\rho]$ in middle degree is isomorphic to
$$\pi \boxtimes \mathrm{std} \circ \phi \otimes |\cdot|^{-3/2}$$
as a $G(\mathbb{Q}_p) \times W_L$-module.

2. *If ϕ is an endoscopic parameter, with L-packets $\Pi_{\phi}(G) = \{\pi^+, \pi^-\}$ and $\Pi_{\phi}(J) = \{\rho_1, \rho_2\}$, the cohomology of $R\Gamma_c(G,b,\mu)[\pi]$ in middle degree is isomorphic to*
$$\rho_1 \boxtimes \phi_1^{\vee} \otimes |\cdot|^{-3/2} \oplus \rho_2 \boxtimes \phi_2^{\vee} \otimes |\cdot|^{-3/2}$$
or
$$\rho_1 \boxtimes \phi_2^{\vee} \otimes |\cdot|^{-3/2} \oplus \rho_2 \boxtimes \phi_1^{\vee} \otimes |\cdot|^{-3/2}$$
as a $J(\mathbb{Q}_p) \times W_L$-module. Similarly, the cohomology of $R\Gamma_c(G,b,\mu)[\rho]$ in middle degree is isomorphic to
$$\pi^+ \boxtimes \phi_1 \otimes |\cdot|^{-3/2} \oplus \pi^- \boxtimes \phi_2 \otimes |\cdot|^{-3/2}$$
or
$$\pi^+ \boxtimes \phi_2 \otimes |\cdot|^{-3/2} \oplus \pi^- \boxtimes \phi_1 \otimes |\cdot|^{-3/2}$$
as a $G(\mathbb{Q}_p) \times W_L$-module. Here we write $\mathrm{std} \circ \phi_{\rho} \simeq \phi_1 \oplus \phi_2$*, with ϕ_i distinct irreducible* 2*-dimensional representations of W_L and* $\det(\phi_1) = \det(\phi_2)$.

   Moreover, both possibilities for the cohomology of $R\Gamma_c(G,b,\mu)[\rho]$ (resp. $R\Gamma_c(G,b,\mu)[\pi]$) in the endoscopic case occur for some choice of representation $\rho \in \Pi_{\phi}(J)$ (resp. $\pi \in \Pi_{\phi}(G)$). In particular, knowing the precise form of either $R\Gamma_c(G,b,\mu)[\rho]$ or $R\Gamma_c(G,b,\mu)[\pi]$ for some $\rho \in \Pi_{\phi}(J)$ or $\pi \in \Pi_{\phi}(G)$ determines the precise form of the cohomology in all other cases.

Proof. We show the proof in the endoscopic case, with the stable case being strictly easier. We first note that, since $\phi = \phi_{\rho}^{\mathrm{FS}}$ by Theorem 8.1, it follows by assumption that the Fargues-Scholze parameter $\phi_{\rho}^{\mathrm{FS}}$ of ρ is supercuspidal. Therefore, by Remark 3.12, we have an isomorphism

$$R\Gamma_c(G,b,\mu)[\rho] \simeq R\Gamma_c^{\flat}(G,b,\mu)[\rho]$$

of $G(\mathbb{Q}_p) \times W_L$-modules. Moreover, by Theorem 3.17, we see that both are concentrated in middle degree 3. Applying Theorem 3.13, we get the following chain of equalities in the Grothendieck group of admissible $G(\mathbb{Q}_p)$-representations of finite length

$$-[H^3(R\Gamma_c(G,b,\mu)[\rho])] = [R\Gamma_c(G,b,\mu)[\rho]] = [R\Gamma_c^{\flat}(G,b,\mu)[\rho]] = -\sum_{\pi \in \Pi_{\phi}(G)} Hom_{S_{\phi}}(\delta_{\pi,\rho}, \mathrm{std} \circ \phi_{\rho})\pi$$

Now we saw in the discussion proceeding Theorem 3.13 that the RHS takes the form:

$$-2\pi^+ - 2\pi^-$$

We set p_1 and p_2 to be the two representations of $S_{\phi} \simeq \{(a,b) \in \overline{\mathbb{Q}}_{\ell}^* \times \overline{\mathbb{Q}}_{\ell}^* | a^2 = b^2\} \subset (\mathrm{GL}_2(\overline{\mathbb{Q}}_{\ell}) \times \mathrm{GL}_2(\overline{\mathbb{Q}}_{\ell}))^0$ given by projecting to the first and second $\overline{\mathbb{Q}}_{\ell}^*$-factor, respectively. By Corollary 3.3, Corollary 3.11, and Theorem 8.1, we have an isomorphism of $G(\mathbb{Q}_p) \times W_L$-modules

$$R\Gamma_c(G,b,\mu)[\rho] \simeq \mathrm{Act}_{p_1}(\rho)[-3] \boxtimes \phi_1 \otimes |\cdot|^{-3/2} \oplus \mathrm{Act}_{p_2}(\rho)[-3] \boxtimes \phi_2 \otimes |\cdot|^{-3/2}$$

where $\mathrm{Act}_{p_1}(\rho)$ and $\mathrm{Act}_{p_2}(\rho)$ are a priori direct sum of shifts of supercuspidal representations of $G(\mathbb{Q}_p)$ with Fargues-Scholze (= Gan-Takeda) parameter equal to ϕ. However, since we know that the LHS is a complex concentrated in middle degree 3, this implies, by the above description in the Grothendieck group, that one of the $\mathrm{Act}_{p_1}(\rho)$ and $\mathrm{Act}_{p_2}(\rho)$ is isomorphic to π^+ and the other is isomorphic to π^-. Without loss of generality, assume that

$$\mathrm{Act}_{p_1}(\rho_1) \simeq \pi^+$$

and

$$\mathrm{Act}_{p_2}(\rho_1) \simeq \pi^-$$

We let p_+ and p_- be the representation of S_{ϕ} determined by the trivial and non-trivial characters of the component group, respectively. Now, given two representations of S_{ϕ}, denoted W and W', it follows from Remark 3.7 (3) that we have an isomorphism:

$$\mathrm{Act}_W \circ \mathrm{Act}_{W'}(\cdot) \simeq \mathrm{Act}_{W \otimes W'}(\cdot)$$

In turn, we get

$$\mathrm{Act}_{p_1^\vee}(\pi^+) \simeq \mathrm{Act}_{p_1^\vee} \circ \mathrm{Act}_{p_1}(\rho_1) \simeq \mathrm{Act}_{p_+}(\rho_1) \simeq \rho_1$$

where the last isomorphism follows since p_+ is the trivial representation. Similarly, depending on the values of $\mathrm{Act}_{p_1}(\rho_2)$ and $\mathrm{Act}_{p_2}(\rho_2)$ we can deduce that $\mathrm{Act}_{p_2^\vee}(\pi^+)$ is isomorphic to ρ_1 or ρ_2. Now by Corollary 3.3, Corollary 3.11, and Theorem 8.1, we have an isomorphism

$$R\Gamma_c(G,b,\mu)[\pi^+] \simeq \mathrm{Act}_{p_1^\vee}(\pi^+)[-3] \boxtimes \phi_1^\vee \otimes |\cdot|^{-3/2} \oplus \mathrm{Act}_{p_2^\vee}(\pi^+)[-3] \boxtimes \phi_2^\vee \otimes |\cdot|^{-3/2}$$

Since $\mathrm{Act}_{p_1^\vee}(\pi^+) \simeq \rho_1$ it therefore follows, by Theorem 3.13 and Remark 3.11, that $\mathrm{Act}_{p_2^\vee}(\pi^+)$ must be isomorphic to ρ_2. Moreover, we know that $\mathrm{Act}_{p_-} \circ \mathrm{Act}_{p_1}(\rho_1) \simeq \mathrm{Act}_{p_2}(\rho_1)$ and $\mathrm{Act}_{p_-} \circ \mathrm{Act}_{p_2}(\rho_1) \simeq \mathrm{Act}_{p_1}(\rho_1)$. Therefore, we obtain that $\mathrm{Act}_{p_-}(\pi^+) \simeq \pi^-$ and $\mathrm{Act}_{p_-}(\pi^-) \simeq \pi^+$. This allows us to determine that

$$\mathrm{Act}_{p_1^\vee}(\pi^-) \simeq \mathrm{Act}_{p_2^\vee} \circ \mathrm{Act}_{p_-}(\pi^-) \simeq \mathrm{Act}_{p_2^\vee}(\pi^+) \simeq \rho_2$$

and

$$\mathrm{Act}_{p_2^\vee}(\pi^-) \simeq \mathrm{Act}_{p_1^\vee} \circ \mathrm{Act}_{p_-}(\pi^-) \simeq \mathrm{Act}_{p_1^\vee}(\pi^+) \simeq \rho_1$$

which will determine the cohomology of $R\Gamma_c(G,b,\mu)[\pi^-]$. It only remains to show that the value of $R\Gamma_c(G,b,\mu)[\rho_2]$ is determined. However, this follows since

$$\mathrm{Act}_{p_2}(\rho_2) \simeq \mathrm{Act}_{p_2} \circ \mathrm{Act}_{p_2^\vee}(\pi^+) \simeq \pi^+$$

and

$$\mathrm{Act}_{p_1}(\rho_2) \simeq \mathrm{Act}_{p_1} \circ \mathrm{Act}_{p_1^\vee}(\pi^-) \simeq \pi^-$$

□

To conclude this section, we use Theorem 8.1 to deduce compatibility with the local Langlands correspondence for Sp_4 and its unique non quasi-split inner form $SU_2(D)$. These correspondences are described in the papers [GT10] and [Cho17] by Gan-Takeda and Choiy, respectively. For Sp_4, this is described as the unique correspondence which sits in the commutative diagram:

$$\begin{array}{ccc} \Pi(\mathrm{GSp}_4) & \xrightarrow{\mathrm{LLC}_{\mathrm{GSp}_4}} & \Phi(\mathrm{GSp}_4) \\ \downarrow & & \downarrow \alpha \\ \Pi(\mathrm{Sp}_4) & \xrightarrow{\mathrm{LLC}_{\mathrm{Sp}_4}} & \Phi(\mathrm{Sp}_4) \end{array}$$

Here, the left vertical arrow is not a map at all, it is a correspondence defined by the subset of $\Pi(\mathrm{GSp}_4) \times \Pi(\mathrm{Sp}_4)$ consisting of pairs (π, ω) such that ω is a constituent of the restriction of π to Sp_4, and the right vertical arrow is the map on L-parameters induced by the natural map $\mathrm{GSpin}_5(\mathbb{C}) \to \mathrm{SO}_5(\mathbb{C})$. One has a similar characterization of the local Langlands correspondence for $\mathrm{SU}_2(D)$. With this description of the correspondence, compatibility for Sp_4 and $\mathrm{SU}_2(D)$ follows from Theorem 8.1 and Theorem 3.6 (7).

Corollary 1.8.3. *For π (resp. ρ) a smoooth irreducible representation of Sp_4/L (resp. $\mathrm{SU}_2(D)$), with associated Gan-Takeda (resp. Choiy) parameter $\phi_\pi : W_L \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_5(\mathbb{C})$ (resp. ϕ_ρ) we have that:*

1. *The Fargues-Scholze and Gan-Takeda (resp. Choiy) local Langlands correspondences are compatible for any representation π (resp. ρ) such that ϕ_π (resp. ϕ_ρ) is not supercuspidal.*

2. *If $L/\mathbb{Q}_p$ is unramified and $p > 2$ then the Fargues-Scholze and Gan-Takeda (resp. Choiy) local Langlands correspondences are compatible for any representation π (resp. ρ) such that ϕ_π (resp. ϕ_ρ) is supercuspidal.*

Remark 1.8.4. We note that Corollary 3.11, Theorem 3.18, Remark 3.11, and [HKW22, Theorem 1.0.2] apply to a triple (G, b, μ), where μ is any cocharacter and $b \in B(G, \mu)$ is the unique basic element. Therefore, by applying the same kind of analysis as in the proof of Theorem 8.2, we can prove the analogue of Theorem 8.2 in the Grothendieck group of finite length admissible representations with a smooth action of W_E (cf. [HKW22, Conjecture 1.0.1]) for the cohomology of the local Shtuka spaces defined by the triple (G, b, μ). By Corollary 8.3, this works even in the case when $G = \mathrm{Sp}_4$ and there are no Shimura varieties that these spaces uniformize. Moreover, in the case that $G = \mathrm{GSp}_4$, one can also deduce from Theorem 8.2 that it is concentrated in middle degree $\langle 2\rho_G, \mu \rangle$, using the monoidal property of the Act-functors and Corollary 3.11, for μ any cocharacter. This in particular will imply some form of Fargues' Conjecture for these groups (See e.g. [BHN22, Pages 37-40], for this worked out in the more complicated case of $G = \mathrm{U}_n$).

Chapter 2

Geometric Eisenstein Series over the Fargues-Fontaine Curve

2.1 Introduction

2.1.1 Geometric Eisenstein Series over Function Fields

In the Langlands program, Eisenstein series are a way of describing the non-cuspidal automorphic spectrum of a group in terms of the cuspidal automorphic spectrum of its proper Levi subgroups. Over function fields these objects have several geometric incarnations, as first studied extensively by Laumon [Lau90] in the case of GL_n and later refined by Braverman-Gaitsgory [BG02] for general reductive groups. In particular, if one is interested in the function field of a curve Y over a finite field k then one replaces the functions defining Eisenstein series by certain automorphic sheaves. Namely, let G/k be a split connected reductive group; then one wishes to construct "Eisenstein sheaves" on Bun_G the moduli stack of G-bundles on Y. To do this, for $P \subset G$ a proper parabolic subgroup with Levi factor M one considers the following diagram of moduli stacks of bundles

$$\begin{tikzcd} \mathrm{Bun}_P \arrow[d, "\mathfrak{q}_P"] \arrow[dr, "\mathfrak{p}_P"] & \\ \mathrm{Bun}_M & \mathrm{Bun}_G \end{tikzcd}$$

where Bun_P is the moduli stack of P-bundles $\mathscr{G}_P$ on Y and the maps $\mathfrak{p}_P$ (resp. $\mathfrak{q}_P$) send $\mathscr{G}_P$ to the G-bundle (resp. M-bundle) $\mathscr{G}_P \times^P G$ (resp. $\mathscr{G}_P \times^P M$). Using this, one can define an Eisenstein functor which (up to shifts and twists) is given by

$$\mathrm{Eis}_P(-) := \mathfrak{p}_{P!}\mathfrak{q}_P^*(-),$$

and takes ℓ-adic sheaves on Bun_M to ℓ-adic sheaves on Bun_G. Under the function-sheaf dictionary, the values of this functor give rise to the classical Eisenstein series.

In this geometric context, one can ask for even more. Namely, in the geometric Langlands correspondence one is interested in constructing Hecke eigensheaves on Bun_G, and it is natural to ask whether one could upgrade Eis_P to a functor that is well-behaved with respect to the eigensheaf property. In particular, if $\hat{M}$ (resp. $\hat{G}$) denotes the Langlands dual group of M (resp. G) one can consider a $\hat{M}$-local system $E_{\hat{M}}$ and a Hecke eigensheaf $\mathscr{S}_{E_{\hat{M}}}$ with eigenvalue $E_{\hat{M}}$. One then considers the induced $\hat{G}$ local system $E_{\hat{G}}$ given by the natural embedding $\hat{M} \hookrightarrow \hat{G}$, and one would like to construct a functor that produces a eigensheaf with eigenvalue $E_{\hat{G}}$ from the Hecke eigensheaf $\mathscr{S}_{E_{\hat{M}}}$. One might hope that $\mathrm{Eis}_P(\mathscr{S}_{E_{\hat{M}}})$ works; however, this is too naive. Namely, one expects such sheaves to be well-behaved under Verdier duality, and one can easily check that $\mathrm{Eis}_P(-)$ will not commute with Verdier duality, since the map $\mathfrak{p}_P$ is not proper. To remedy this, one considers relative Drinfeld compactifications of the map $\mathfrak{p}_P$, denoted $\widetilde{\mathrm{Bun}}_P$ and $\overline{\mathrm{Bun}}_P$, respectively. These compactifications are equipped with open immersions $\widetilde{j} : \mathrm{Bun}_P \hookrightarrow \widetilde{\mathrm{Bun}}_P$ and $j : \mathrm{Bun}_P \hookrightarrow \overline{\mathrm{Bun}}_P$, which realize Bun_P as an open and dense subspace, and are defined by considering parabolic structures with torsion at finitely many Cartier divisors. Moreover, they both have maps

$$\overline{\mathfrak{p}}_P : \overline{\mathrm{Bun}}_P \to \mathrm{Bun}_G$$

and

$$\widetilde{\mathfrak{p}}_P : \widetilde{\mathrm{Bun}}_P \to \mathrm{Bun}_G$$

which are proper after restricting to a connected component and extend $\mathfrak{p}_P$, as well as maps $\widetilde{\mathfrak{q}}_P : \widetilde{\mathrm{Bun}}_P \to \mathrm{Bun}_M$ and $\overline{\mathfrak{q}}_P : \overline{\mathrm{Bun}}_P \to \mathrm{Bun}_{M^{ab}}$ extending the natural maps $\mathfrak{q}_P : \mathrm{Bun}_P \to \mathrm{Bun}_M$ and $\mathfrak{q}_P^\dagger : \mathrm{Bun}_P \xrightarrow{\mathfrak{q}_P} \mathrm{Bun}_M \to \mathrm{Bun}_{M^{ab}}$, respectively.

To obtain a sheaf that interacts well with Verdier duality, one needs to take account for the singularities of the compactification. Namely, if one considers the intersection cohomology sheaf $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$ of $\widetilde{\mathrm{Bun}}_P$ then the desired functor is given by

$$\widetilde{\mathrm{Eis}}_P(-) := \widetilde{\mathfrak{p}}_{P*}(\widetilde{\mathfrak{q}}_P^*(-) \otimes \mathrm{IC}_{\widetilde{\mathrm{Bun}}_P})$$

One of the main results of Braverman-Gaitsgory [BG02] is that this satisfies the desired Hecke eigensheaf property when applied to $\mathscr{S}_{E_{\hat{M}}}$. One may also wonder what this corresponds to at the level of functions. We recall that the classical Eisenstein series is known to satisfy a functional equation after multiplying by an appropriate ratio of L-values. The compactified Eisenstein series corresponds to this completed Eisenstein series under the function-sheaf dictionary. In fact, in certain cases one can see the usual functional equation at the sheaf-theoretic level. If $P = B$ is the Borel and we consider the appropriately normalized Hecke eigensheaf $\mathscr{S}_{E_{\hat{T}}}$ associated to $E_{\hat{T}}$ via geometric class field theory then $\widetilde{\mathrm{Eis}}(\mathscr{S}_{\phi_T})$ satisfies a functional equation under a regularity hypothesis on the local system $E_{\hat{T}}$. Now let $w \in W_G$ be an element of the Weyl group of G with $\widetilde{w} \in N(T)$ a choice of representative. Then $\widetilde{w}$ acts on Bun_T, and, if we write $\mathscr{S}_{E_{\hat{T}}}^{w}$ for the pullback of $\mathscr{S}_{E_{\hat{T}}}$ along this automorphism, we have the following result.

Theorem 2.1.1. *[BG02, Theorem 2.2.4] For $E_{\hat{T}}$ a regular $\hat{T}$-local system on Y and a choice of representative $\widetilde{w} \in N(\hat{T})$ of $w \in W_G$, we have an isomorphism*

$$\widetilde{\mathrm{Eis}}_B(\mathscr{S}_{E_{\hat{T}}}) \simeq \widetilde{\mathrm{Eis}}_B(\mathscr{S}_{E_{\hat{T}}}^{w})$$

of ℓ-adic sheaves on Bun_G.

As alluded to above, one can see that, after passing to functions, this gives precisely the well-known functional equation for the Eisenstein series multiplied by the appropriate ratio of L-values ([BG02, Section 2.2]). Moreover, by [BG08, Theorem 1.5], under the regularity assumption the sheaf $\widetilde{\mathrm{Eis}}_B(\mathscr{S}_{E_{\hat{T}}})$ is perverse.

The main goal of this note is to explore what this theory of geometric Eisenstein series has to tell us in the context of the recent geometric Langlands correspondence constructed by Fargues and Scholze.

2.1.2 Hecke Eigensheaves over the Fargues-Fontaine Curve

Consider two distinct primes $\ell \neq p$. Let $G/\mathbb{Q}_p$ be a quasi-split connected reductive group with simply connected derived group over the p-adic numbers, and set $\breve{\mathbb{Q}}_p$ to be the completion of the maximal unramified extension of $\mathbb{Q}_p$ with Frobenius σ. Let $W_{\mathbb{Q}_p} \subset \Gamma := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ denote the Weil group. In [FS21], Fargues and Scholze developed the geometric framework required to make sense of objects like Bun_G, the moduli stack of G-bundles on the Fargues-Fontaine curve X, and show that the local Langlands correspondence for G can be viewed as a geometric Langlands correspondence over X. Namely, they prove a version of Geometric Satake in this setup, allowing them to construct Hecke operators and in turn excursion operators. Their Hecke operators take sheaves on Bun_G to sheaves on Bun_G with a continuous $W_{\mathbb{Q}_p}$-action, via some version of Drinfeld's Lemma. As a consequence, they were able to construct semi-simple Langlands parameters $\phi_\pi^{\mathrm{FS}} : W_{\mathbb{Q}_p} \to {}^L G(\overline{\mathbb{Q}}_\ell)$ attached to any smooth irreducible representation π of $G(\mathbb{Q}_p)$.

Predating the construction of the Fargues-Scholze local Langlands correspondence was Fargues' conjecture as formulated in [Far16, Conjecture 4.4]. This asserted the existence of Hecke Eigensheaves attached to supercuspidal L-parameters $\phi : W_{\mathbb{Q}_p} \to {}^L G(\overline{\mathbb{Q}}_\ell)$, where ${}^L G = \hat{G} \ltimes W_{\mathbb{Q}_p}$ is the L-group of G. More precisely, given such a ϕ, Fargues conjectured the existence of a $\overline{\mathbb{Q}}_\ell$-sheaf $\mathscr{S}_\phi$ on Bun_G such that, if one acts via a Hecke operator T_V corresponding to a representation $V \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L G)$ then there is an isomorphism

$$T_V(\mathscr{S}_\phi) \simeq \mathscr{S}_\phi \boxtimes r_V \circ \phi$$

of sheaves on Bun_G with continuous $W_{\mathbb{Q}_p}$-action satisfying natural compatiblities. The moduli stack Bun_G is stratified by elements of the Kottwitz set $B(G) := G(\breve{\mathbb{Q}}_p)/(b \sim gb\sigma(g)^{-1})$, giving rise to Harder-Narasimhan (abbv. HN)-strata Bun_G^b for all $b \in B(G)$. It was conjectured that the sheaf $\mathscr{S}_\phi$ must be supported on the basic locus $\bigsqcup_{b \in B(G)_{basic}} \mathrm{Bun}_G^b$ or, in bundle-theoretic terms, the locus defined by semistable bundles. Each of the basic strata Bun_G^b are isomorphic to the classifying stack $[*/\underline{J_b(\mathbb{Q}_p)}]$, where J_b is the σ-centralizer attached to $b \in B(G)$. The σ-centralizers of the basic elements parameterize extended pure inner forms of G in the sense of Kottwitz [Kot97b], and the restrictions of the sheaf to these classifying stacks can be interpreted as smooth representations of $J_b(\mathbb{Q}_p)$. Using this, Fargues also gave a conjectural description for what the restrictions of the sheaf should be given by. In particular, for $b \in B(G)_{basic}$ the restriction to Bun_G^b should be a direct sum (up to multiplicities)[1] $\bigoplus_{\pi \in \Pi_\phi(J_b)} \pi$, where $\Pi_\phi(J_b)$ is the L-packet over ϕ as conjectured by Kaletha's refined local Langlands correspondence for G [Kal16]. Assuming the refined local Langlands, the verification of the Hecke eigensheaf property ultimately reduces to a strong form of the Kottwitz conjecture for the cohomology of a space of shtukas parameterizing modifications

$$\mathscr{F}_b \to \mathscr{F}_{b'}$$

[1]There are no higher multiplicities if the centralizer of ϕ is abelian.

on X bounded by a geometric dominant cocharacter μ, where $b, b' \in B(G)_{basic}$ are appropriately chosen basic elements with respect to μ, and $\mathscr{F}_b$ and $\mathscr{F}_{b'}$ denote the bundles on X corresponding to $b, b' \in B(G)$.

Since the work of Fargues-Scholze, the construction of this eigensheaf has been carried out in several cases. For tori, it follows from the work of Fargues [Far16; Far20], and Zou [Zou22]. For $G = \mathrm{GL}_n$, this is a result of Anschütz and Le-Bras [AL21a]. For general reductive groups, a somewhat general strategy for constructing this eigensheaf in the particular case of the group GSp_4 is layed out in [Ham21b], by showing compatibility of the Fargues-Scholze correspondence with the refined local Langlands correspondence of Kaletha [Kal16], and then using this to deduce the non-minuscule cases of the Kottwitz conjecture required for the verification of the Hecke eigensheaf property via the spectral action [FS21, Section X.2]. This strategy is carried out for odd unitary groups in the paper [BHN22]. We recall that the fibers of the local Langlands correspondence over supercuspidal parameters should solely consist of supercuspidal representations. Therefore, the above eigensheaves can be thought of as analogous to supercuspidal representations in the classification of smooth irreducible representations. To have a more definitive connection between the theory of smooth representations and Fargues' eigensheaves, it becomes desirable to construct eigensheaves that serve as the analogues of parabolic inductions of supercuspidals, which will analogously be "parabolically induced" from the eigensheaves attached to supercuspidal parameters. This will be precisely what carrying over the theory of the previous section to the Fargues-Fontaine setting gives us.

2.1.3 Geometric Eisenstein Series over the Fargues-Fontaine Curve

Let $A \subset T \subset B \subset G$ be a choice of maximal split torus, maximal torus, and Borel, respectively. We will assume that G is quasi-split with simply connected derived group. In this paper, we will be concerned with studying the geometric Eisenstein functor over the Fargues-Fontaine curve in the principal case (i.e where the parabolic $P \subset G$ is the Borel). Restricting to the principal case has several advantages. For one, one can unconditionally speak about the Hecke eigensheaf attached to a parameter $\phi_T : W_{\mathbb{Q}_p} \to {}^L T$ valued in the maximal torus. Additionally, in this case there exists only one Drinfeld compactification $\overline{\mathrm{Bun}}_B$ of the moduli space of B-structures Bun_B, which has a fairly manageable geometry. As mentioned in §1.1, there are in general two compactifications $\overline{\mathrm{Bun}}_P$ and $\widetilde{\mathrm{Bun}}_P$. The compactification $\overline{\mathrm{Bun}}_P$ has a relatively simple geometry and can be understood more or less the same as $\overline{\mathrm{Bun}}_B$. The problem is that $\overline{\mathrm{Bun}}_P$ admits only a map to $\mathrm{Bun}_{M^{ab}}$ and not to Bun_M. This means we can prove analogous results for the Hecke eigensheaves attached to characters induced from $\phi : W_{\mathbb{Q}_p} \to {}^L M^{ab}$ which correspond to generalized principal series representations induced from M. We have chosen not to do this in this note for simplicity. However, if one wants to consider inductions of Hecke eigensheaves attached to a general supercuspidal parameter $\phi_M : W_{\mathbb{Q}_p} \to {}^L M$ then one is forced to understand the much more complicated geometry of the space $\widetilde{\mathrm{Bun}}_P$. Certainly, some analogues of the results proven in this paper should be possible, but there are many technical hurdles that need to be overcome.

Geometric Eisenstein Series

Throughout, we will let $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\Lambda)$ be a continuous parameter valued in the L-group of T, where $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$ has the discrete topology. Our aim is to construct an eigensheaf (Definition 2.3.1) with eigenvalue ϕ, the composition of ϕ_T with the natural embedding ${}^L T(\Lambda) \to {}^L G(\Lambda)$. This will be an object in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ the category of lisse-étale solid Λ-sheaves, as defined in [FS21, Chapter VII]. We do not work directly with this category of solid sheaves in our argument, as the usual six functors are not as well behaved in this case. Instead, we will first restrict to the case where $\Lambda = \overline{\mathbb{F}}_\ell$ and one has an isomorphism $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell) \simeq \mathrm{D}_{\text{ét}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ with the usual category of étale $\overline{\mathbb{F}}_\ell$ sheaves as defined in [Sch18]. We then construct the lisse-étale sheaves with $\overline{\mathbb{Z}}_\ell$ and $\overline{\mathbb{Q}}_\ell$ coefficients from this sheaf. For the first part of this section, we will always assume that $\Lambda = \overline{\mathbb{F}}_\ell$ unless otherwise stated, and to simplify the notation, we adopt the convention that, when working with $\Lambda = \overline{\mathbb{F}}_\ell$, we will denote the derived category of étale sheaves on a v-stack or diamond Z by simply writing $\mathrm{D}(Z)$. We will assume that the prime ℓ is very good with respect to the group G ([FS21, Page 33]) throughout, to avoid complications in this ℓ-modular setting.

We let $\mathscr{S}_{\phi_T}$ be the eigensheaf on the moduli stack Bun_T parameterizing T-bundles on X attached to ϕ_T by Fargues [Far20; Far20] and Zou [Zou22]. Our aim is to construct an eigensheaf with respect to the parameter ϕ by applying a geometric Eisenstein functor to $\mathscr{S}_{\phi_T}$. To do this, one needs to show that the relevant geometric objects used in defining geometric Eisenstein series are well-behaved in this framework. Namely, one can show that the moduli stack of B-bundles on X, denoted Bun_B, gives rise to an Artin v-stack and the maps in the natural diagram

$$\begin{tikzcd} \mathrm{Bun}_B \arrow[d, "\mathfrak{q}"] \arrow[dr, "\mathfrak{p}"] & \\ \mathrm{Bun}_T & \mathrm{Bun}_G \end{tikzcd} \tag{2.1}$$

have good geometric properties; $\mathfrak{q}$ is a cohomologically smooth (non-representable) map of Artin v-stacks, and $\mathfrak{p}$ is compactifiable and representable in locally spatial diamonds (See §2.5.1). It follows that one has a well-defined functor given by $\mathfrak{p}_!\mathfrak{q}^* : \mathrm{D}(\mathrm{Bun}_T) \to \mathrm{D}(\mathrm{Bun}_G)$ using the functors constructed in [Sch18].

In order to make the functor $\mathfrak{p}_!\mathfrak{q}^*$ behave well with respect to Verdier duality, one needs to take into account the appropriate shifts and twists coming from the dualizing object on Bun_B. Namely, using the cohomological smoothness of $\mathfrak{q}$ and Bun_T, it is easy to see that the moduli stack Bun_B is cohomologically smooth of some ℓ-dimension given by a locally constant function $\dim(\mathrm{Bun}_B) : |\mathrm{Bun}_B| \to \mathbb{Z}$, where $|\mathrm{Bun}_B|$ is the underlying topological space of Bun_B. It follows that, v-locally on Bun_B, the dualizing object is given by $\Lambda[2\dim(\mathrm{Bun}_B)]$. This leads us to our first attempt to define the Eisenstein functor as $\mathrm{Eis}(-) := \mathfrak{p}_!(\mathfrak{q}^*(-)[\dim(\mathrm{Bun}_B)])$. While this definition is closer to what we want, it has one key flaw; even though the dualizing object on Bun_B is v-locally isomorphic to $\Lambda[2\dim(\mathrm{Bun}_B)]$ it is not equal to this sheaf on the nose. In particular, usually one would need to include some Tate twists. However, all these spaces are defined over the base $* = \mathrm{Spd}\,\overline{\mathbb{F}}_p$, so one cannot naively make sense of this. Typically, this would be encoded via some kind of Frobenius descent datum, but here the answer is even more interesting. Understanding this requires one to explicitly compute the dualizing object on Bun_B. This can be accomplished by noting that each of the connected components of Bun_B are related to Banach-Colmez spaces, the diamonds parameterizing global sections $H^0(X,\mathscr{E})$ and $H^1(X,\mathscr{E})$ for $\mathscr{E}$ a bundle on the Fargues-Fontaine curve X, as studied in [FS21, Chapter 2] and [Le 18]. These objects have a relatively explicit description in terms of pro-étale quotients of perfectoid open unit discs and also have absolute versions defined over $\mathrm{Spd}\,\mathbb{F}_{p^s}$, for some $s \geq 1$. Using this explicit description, one can compute that the dualizing object on these absolute spaces is given by the constant sheaf with the appropriate shift and Tate twist by the dimension. However, these Tate twists do not disappear over the algebraic closure $\overline{\mathbb{F}}_p$; namely, the action of geometric Frobenius is manifestly related to the action of $p^{\mathbb{Z}} \in H^0(X,\mathscr{O}_X)^* = \mathbb{Q}_p^*$ by scaling of global sections, essentially by definition of the Fargues-Fontaine curve. This allows one to see that the dualizing object on the Banach-Colmez space over $\overline{\mathbb{F}}_p$ is isomorphic to a shift of the constant sheaf together with a descent datum with respect to the scaling action by $\mathbb{Q}_p^*$ given by the norm character $|\cdot| : \mathbb{Q}_p^* \to \Lambda^*$, which serves the role of a Frobenius descent datum. Unravelling this all, this will ultimately tell us that the dualizing object on Bun_B is related to the modulus character δ_B. One way of nicely encoding these modulus character twists is to consider the sheaf $\Delta_B^{1/2}$ on Bun_T, which will be the Hecke eigensheaf attached to the L-parameter

$$\hat{\rho} \circ |\cdot| : W_{\mathbb{Q}_p} \to {}^L T(\Lambda)$$

where $\hat{\rho}$ denotes the half sum of all positive roots of G and we abusively write $|\cdot| : W_{\mathbb{Q}_p} \to \Lambda^*$ for the norm character of $W_{\mathbb{Q}_p}$. As a sheaf on Bun_T, the stalks on the connected components $\mathrm{Bun}_T^{\nu} \simeq [*/\underline{T(\mathbb{Q}_p)}]$ will just be given by the character $\delta_B^{1/2} : T(\mathbb{Q}_p) \to \Lambda^*$. Similarly, we write Δ_B for the sheaf given by the L-parameter $2\hat{\rho} \circ |\cdot|$. We can now state our first Theorem.

Theorem 2.1.2. *(Theorem 2.6.1) The dualizing object on* Bun_B *is isomorphic to*

$$\mathfrak{q}^*(\Delta_B)[2\dim(\mathrm{Bun}_B)]$$

In particular, the sheaf

$$\mathrm{IC}_{\mathrm{Bun}_B} := \mathfrak{q}^*(\Delta_B^{1/2})[\dim(\mathrm{Bun}_B)]$$

is Verdier self-dual on Bun_B*.*

With this in hand, we can refine the previous definition of the Eisenstein functor. We define the normalized Eisenstein functor to be

$$\mathrm{nEis}(-) := \mathfrak{p}_!(\mathfrak{q}^*(-) \otimes \mathrm{IC}_{\mathrm{Bun}_B})$$

This is already very suggestive. Indeed, the (unnormalized) Eisenstein functor will have stalks related to the unnormalized parabolic induction of the characters χ, and the normalized Eisenstein series will have stalks related to the normalized parabolic induction. Moreover, just as smooth duality interacts nicely with normalized parabolic induction so too does Verdier duality with the normalized Eisenstein functor. In order to study how the normalized Eisenstein functor interacts with Verdier duality, it becomes very natural to want a nice compactification of the morphism $\mathfrak{p}$, as Eis involves the functor $\mathfrak{p}_!$. As seen in §1.1, this can be accomplished by considering an analogue of the Drinfeld compactification $\overline{\mathrm{Bun}}_B$. We show that such a compactification exists and has the right properties.

Theorem 2.1.3. *(Proposition 2.5.8, Proposition 2.5.22) There exists an Artin v-stack $\overline{\mathrm{Bun}}_B$ together with an inclusion*

$$j : \mathrm{Bun}_B \hookrightarrow \overline{\mathrm{Bun}}_B$$

which realizes Bun_B as an open and dense substack. Moreover, $\overline{\mathrm{Bun}}_B$ has natural maps $\overline{\mathfrak{q}} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_T$ (resp. $\overline{\mathfrak{p}} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_G$) extending the map $\mathfrak{q}$ (resp. $\mathfrak{p}$) along j. The map $\overline{\mathfrak{p}}$ is representable in nice diamonds and proper after restricting to connected components.

Now, we would like to claim that the sheaf $\mathrm{nEis}(\mathscr{S}_{\phi_T})$ is a Hecke eigensheaf with respect to the parameter ϕ given by composing ϕ_T with the natural embedding ${}^LT \to {}^LG$; however, in analogy with §1.1, the right object to consider in this case is not $\mathrm{nEis}(\mathscr{S}_{\phi_T})$, but rather a compactified version $\overline{\mathrm{Eis}}(\mathscr{S}_{\phi_T})$. Unfortunately, there is currently no well-behaved formalism for intersection cohomology in the context of diamonds and v-stacks. This prevents us from even defining the kernel sheaf $\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ typically used in the definition of $\overline{\mathrm{Eis}}(-)$ in any naive way. There is however a way out if we impose some conditions on our parameter ϕ_T. We write $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$ for the set of Γ-orbits of geometric cocharacters. Given an element $\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$, we can attach to it a representation of ${}^LT$ by inducing the representation of $\hat{T}$ defined by a representative of the orbit ν in $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$. We consider the composition $\nu \circ \phi_T$, and we will say that ϕ_T is generic if the Galois cohomology complexes

$$R\Gamma(W_{\mathbb{Q}_p}, \alpha \circ \phi_T)$$

are trivial for all $\alpha \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$ defined by the Γ-orbits of coroots in $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$. This condition may appear mysterious; but there are several ways to see why it is the morally correct condition. Perhaps the most compelling comes from local representation theory. As mentioned in §1.1, the compactified Eisenstein series in the function field setting corresponds to the completed Eisenstein series under the function sheaf dictionary, while the non-compactified Eisenstein series corresponds to just the usual Eisenstein series. In particular, the sheaf $\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ encodes the zeros and poles of the meromorphic continuation of the Eisenstein series. If χ denotes the character of $T(\mathbb{Q}_p)$ attached to ϕ_T via local class field theory then we recall that, if $w \in W_G$ is an element of the relative Weyl group, the local analogue of the meromorphic continuation of Eisenstein series is the theory of intertwining operators. In particular, suppose that $\Lambda = \overline{\mathbb{Q}}_\ell$, then we have maps

$$i_{\chi,w} : i_B^G(\chi) \to i_B^G(\chi^w)$$

of smooth $G(\mathbb{Q}_p)$-representations, which can be viewed as meromorphic functions on the set of complex unramified characters. Now, using local Tate-duality, it is easy to see that the vanishing of the above complexes will imply that χ precomposed with coroots is not trivial or isomorphic to a power of the norm character. These are precisely the type of conditions one expects to guarantee that the intertwining operators are holomorphic and give rise to an isomorphism. In fact, we show that, for χ attached to a generic parameter ϕ_T, we always have an isomorphism $i_B^G(\chi) \simeq i_B^G(\chi^w)$ for any $w \in W_G$ (Proposition A.1.3).

This suggests that, at least heuristically, we should always have an isomorphism $\mathrm{nEis}(\mathscr{S}_{\phi_T}) \simeq \overline{\mathrm{Eis}}(\mathscr{S}_{\phi_T})$, for ϕ_T satisfying some version of genericity and any reasonable definition of $\overline{\mathrm{Eis}}(\mathscr{S}_{\phi_T})$. Indeed, this makes sense when we look at the geometry of $\overline{\mathrm{Bun}}_B$; in particular, we will show that the closed complement of Bun_B in $\overline{\mathrm{Bun}}_B$ admits a locally closed stratification given by $\mathrm{Div}^{(\overline{\nu})} \times \mathrm{Bun}_B$, where $\mathrm{Div}^{(\overline{\nu})}$ is a certain partially symmetrized version of the mirror curve Div^1 parameterizing effective Cartier divisors in X attached to $\overline{\nu}$ inside the coinvariant lattice $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$. We recall that there is a natural map $(-)_\Gamma : \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma \to \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$, from Γ-orbits to coinvariants. This map defines an injection on the Γ-orbits of the simple positive coroots. In particular, for each vertex $i \in \mathscr{J}$ of the relative Dynkin diagram of G, we get an element α_i in the coinvariant lattice, which corresponds to a Γ-orbit of positive simple coroots. The natural strata of $\overline{\mathrm{Bun}}_B$ are specified by elements $\overline{\nu} \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$ lying in the positive span of these α_i, and each strata corresponds to the locus of B-bundles with torsion specified by $\overline{\nu}$. Now, the restriction of $\overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T})$ to this strata is given by pulling back a Hecke operator on Bun_T applied to $\mathscr{S}_{\phi_T}$ along $\mathfrak{q}$. One can deduce that the factor appearing on the divisor curve $\mathrm{Div}^{(\overline{\nu})}$ will be related to $\alpha_i \circ \phi_T$, via the Hecke eigensheaf property $T_{\alpha_i}(\mathscr{S}_{\phi_T}) \simeq \alpha_i \circ \phi_T \boxtimes \mathscr{S}_{\phi_T}$ for $\mathscr{S}_{\phi_T}$, where we have identified α_i with its associated Γ-orbit via $(-)_\Gamma$. This implies that the complex $R\Gamma(W_{\mathbb{Q}_p}, \alpha_i \circ \phi_T)$ appears in $\overline{\mathfrak{p}}_!$ applied to this restriction as a tensor factor, and will in turn vanish for ϕ_T generic, suggesting an isomorphism of the form $\mathrm{nEis}(\mathscr{S}_{\phi_T}) \simeq \overline{\mathrm{Eis}}(\mathscr{S}_{\phi_T})$ in this case.

We can turn these heuristics into actual math. In particular, since we expect an isomorphism of the form $\mathrm{nEis}(\mathscr{S}_{\phi_T}) \simeq \overline{\mathrm{Eis}}(\mathscr{S}_{\phi_T})$ under some verison of genericity, we should expect for such parameters that $\mathrm{nEis}(\mathscr{S}_{\phi_T})$ behaves well under Verdier duality, is a perverse Hecke eigensheaf with eigenvalue ϕ, and satisfies the analogue of the functional equation seen in Theorem 2.1.1 in this case. The precise conditions on ϕ_T that we will need to prove our results will depend on the result and the particular group G. For this reason, we break up our condition into several parts.

Condition/Definition 2.1.4. *(Condition/Definition 2.3.7) Given a parameter $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\Lambda)$, we impose the following conditions on ϕ_T in what follows.*

1. *For all $\alpha \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$ defined by the Γ-orbits of simple coroots in $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$, the Galois cohomology complex $R\Gamma(W_{\mathbb{Q}_p}, \alpha \circ \phi_T)$ is trivial.*
2. *For all $\alpha \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$ defined by the Γ-orbits of coroots in $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$, the Galois cohomology complex $R\Gamma(W_{\mathbb{Q}_p}, \alpha \circ \phi_T)$ is trivial.*
3. *If χ is the character attached to ϕ_T under local class field theory. We have, for all $w \neq 1$ in the relative Weyl group W_G of G, that*

$$\chi \otimes \delta_B \not\simeq (\chi \otimes \delta_B^{-1/2})^w$$

If ϕ_T satisfies (1) we say that it is weakly generic, and if it satisfies (2) then we say it is generic. If it satisfies (2)-(3) we say that it is weakly normalized regular.

Remark 2.1.5. The relationship between these various conditions appears to be complicated in general, and is related to the behavior of the principal series representations $i_B^G(\chi)$ of G. Roughly speaking, Condition (2) guarantees the irreducibility of non-unitary principal series representations and that the intertwining operators for $i_B^G(\chi)$ are isomorphisms, while Condition (3) guarantees the irreducibility of certain unitary principal series representations. As we will see in Appendix A.1, genericity is enough to guarantee that one has an isomorphism $i_B^G(\chi) \simeq i_B^G(\chi^w)$ for all $w \in W_G$. However, one could still have an isomorphism of this form if $i_B^G(\chi)$ is the reducible induction of a unitary character χ. This can happen if the character χ is not regular (i.e it is fixed by some $w \in W_G$). Condition (3) is like such a regularity condition. To illustrate this, note that, for $G = \mathrm{GL}_n$, if we write $\phi_T = \bigoplus_{i=1}^n \phi_i$ as a sum of characters then genericity is equivalent to supposing that

$$R\Gamma(W_{\mathbb{Q}_p}, \phi_i^\vee \otimes \phi_j)$$

is trivial for all $i \neq j$. We note that, by local Tate-duality and using that the Euler-Poincaré characteristic of this complex is 0, genericity is equivalent to assuming that ϕ_i is not isomorphic to ϕ_j or $\phi_j(1)$. If we write $\chi = \chi_1 \otimes \ldots \otimes \chi_n$ as a product of characters of $\mathbb{Q}_p^*$ then this implies that $\chi_i^{-1}\chi_j \not\simeq |\cdot|^{\pm 1}$ for all $i > j$, which is precisely the condition guaranteeing irreducibility. On the other hand, if $G = \mathrm{SL}_2$, and we write χ for the character of $\mathbb{Q}_p^*$ attached to ϕ_T via local class field theory, we need that $\chi \not\simeq |\cdot|^{\pm 1}$ and $\chi^2 \not\simeq \mathbf{1}$ to guarantee irreducibility of $i_B^G(\chi)$.

The condition $\chi \not\simeq |\cdot|^{\pm 1}$ is guaranteed by Condition (2), and Condition (3) is equivalent to $\chi^2 \not\simeq \mathbf{1}$.

Remark 2.1.6. The choice of calling a parameter satisfying Condition (2) generic is motivated by the analogous notion of decomposed generic considered by Caraiani and Scholze [CS17, Definition 1.9]. In particular, we note that if ϕ_T is unramified and $G = \mathrm{GL}_n$ then, by the previous remark, we see that Condition (2) is precisely equivalent to ϕ_T being decomposed generic.

Weak genericity will be required to show our Eisenstein functor commutes with Verdier duality, while weak normalized regularity will be needed to compute the stalks of our Eisenstein series and show that it satisfies the functional equation. Unfortunately, in general to get the Hecke eigensheaf property for $\mathrm{nEis}(\mathscr{S}_{\phi_T})$ we still need more. Write $(-)^{\Gamma} : \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \to \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$ for the natural map from geometric cocharacters to their Galois orbits. Given a geometric dominant cocharacter $\mu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ with Galois orbit $\mu^{\Gamma} \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+/\Gamma$, we have an associated representation $V_{\mu^{\Gamma}} \in \mathrm{Rep}_{\Lambda}({}^LG)$. The weights of $V_{\mu^{\Gamma}}|_{{}^LT}$ can be interpreted in terms of the representations corresponding to the Galois orbits of weights in the usual highest weight representation V_{μ} of $\hat{G}$. The following condition will guarantee the Hecke eigensheaf property for $\mathrm{nEis}(\mathscr{S}_{\phi_T})$, and the Hecke operator defined by $V_{\mu^{\Gamma}}$.

Definition 2.1.7. (Definition 2.3.14) For a toral parameter $\phi_T : W_{\mathbb{Q}_p} \to {}^LT(\Lambda)$ and a geometric dominant cocharacter μ, we say ϕ_T is strongly μ-regular if the Galois cohomology complexes

$$R\Gamma(W_{\mathbb{Q}_p}, (\nu - \nu')^{\Gamma} \circ \phi_T)$$

are trivial for ν,ν' defining distinct Γ-orbits of weights the highest weight representation of $\hat{G}$ of highest weight μ.

Remark 2.1.8. We note that genericity usually implies strong μ-regularity for some suitably chosen cocharacters $\mu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$. In particular, note that if $\mu = (1,0,\ldots,0,0)$ and $G = \mathrm{GL}_n$ then, since μ is minuscule, the possible weights of V_{μ} in $\hat{T}$ are all given by Weyl group orbits of μ, and the possible differences between the weights will consequently be precisely the coroots of GL_n.

Let's now see how these conditions manifest in our results on Eisenstein series. We begin with studying how Verdier duality interacts with Eisenstein series. To do this, we need to make the following assumption.

Assumption 2.1.9. *(Assumption 2.8.1) If $j : \mathrm{Bun}_B \hookrightarrow \overline{\mathrm{Bun}}_B$ is the open inclusion into the Drinfeld compactification the sheaf $j_!(\mathrm{IC}_{\mathrm{Bun}_B})$ is ULA with respect to $\overline{\mathfrak{q}}$.*

Remark 2.1.10. This is a precise analogue of [BG02, Theorem 5.1.5]. The proof in this case seems a bit subtle, since being ULA over a point in this case is not a trivial condition, but should appear in upcoming work [HHS] on geometric Eisenstein series and the Harris-Viehmann conjecture.

The relevance for studying how Verdier duality interacts with $\mathrm{nEis}(-)$ is as follows. It follows that $j_!(\mathrm{IC}_{\mathrm{Bun}_B})$ is reflexive with respect to Verdier duality on Bun_B. In particular, using the Verdier self-duality of $\mathrm{IC}_{\mathrm{Bun}_B}$, this allows us to see that the Verdier dual of $j_*(\mathrm{IC}_{\mathrm{Bun}_B})$ is isomorphic to $j_!(\mathrm{IC}_{\mathrm{Bun}_B})$. This reduces the problem of how Eisenstein series interact with Verdier duality to the problem of describing the cone of the map

$$j_!(\mathfrak{q}^*(\mathscr{S}_{\phi_T})) \to \overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T})$$

after applying $\overline{\mathfrak{p}}_!$. As already mentioned above, if we look at the restriction of $\overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T})$ to the strata $\mathrm{Div}^{(\overline{\nu})} \times \mathrm{Bun}_B$ described above, this vanishes for weakly generic ϕ_T after applying $\overline{\mathfrak{p}}_!$, as the Galois cohomology complexes $R\Gamma(W_{\mathbb{Q}_p}, \alpha \circ \phi_T)$ will appear for $\alpha \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$ corresponding to a Γ-orbit of simple coroots. In particular, if $\mathbb{D}_Z$ denotes Verdier duality on a v-stack or diamond Z, we can show the following.

Theorem 2.1.11. *(Theorem 2.8.3) For ϕ_T a weakly generic toral parameter, there is an isomorphism of objects in* $\mathrm{D}(\mathrm{Bun}_G)$

$$\mathbb{D}_{\mathrm{Bun}_G}(\mathrm{nEis}(\mathscr{S}_{\phi_T})) \simeq \mathrm{nEis}(\mathbb{D}_{\mathrm{Bun}_T}(\mathscr{S}_{\phi_T}))$$

where we note that $\mathbb{D}_{\mathrm{Bun}_T}(\mathscr{S}_{\phi_T}) \simeq \mathscr{S}_{\phi_T^\vee}$, if $\phi_T^\vee$ is the parameter dual to ϕ_T (i.e the Chevalley involution applied to ϕ_T).

We will assume the validity of the ULA Theorem and thereby the validity of this theorem for the rest of the section. We now turn our attention to the Hecke eigensheaf property. In particular, consider a finite index set I and a representation $V \in \mathrm{Rep}_\Lambda({}^LG^I)$. Given $(\nu_i)_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma)^I$, we write $V((\nu_i)_{i\in I})$ for the multiplicity of the weight space of the corresponding representation of ${}^LT^I$ in $V|_{{}^LT^I}$, and write $T_{(\nu_i)_{i\in I}}$ for the associated Hecke operator. By applying excision to the aforementioned locally closed stratification of $\overline{\mathrm{Bun}}_B$ and combining it with the geometric Satake correspondence of Fargues-Scholze [FS21, Chapter VI], we can show the following result.

Theorem 2.1.12. *(Theorem 2.7.1) For $\mathscr{F} \in \mathrm{D}(\mathrm{Bun}_T)$, I a finite index set, and $V \in \mathrm{Rep}_\Lambda({}^LG^I)$ with associated Hecke operator T_V, the sheaf $T_V(\mathrm{nEis}(\mathscr{F}))$ on*

Bun_G *with continuous* $W^I_{\mathbb{Q}_p}$*-action has a* $W^I_{\mathbb{Q}_p}$*-equivariant filtration indexed by* $(\nu_i)_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma)^I$*. The filtration's graded pieces are isomorphic to*

$$\bigoplus_{(\nu_i)_{i\in I}\in(\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma)^I} \mathrm{nEis}(T_{(\nu_i)_{i\in I}}(\mathscr{F}))\otimes V((\nu_i)_{i\in I})$$

Moreover, the filtration is natural in I and V, as well as compatible with compositions and exterior tensor products.

The argument for proving this "filtered eigensheaf property" is very similar to that given by [BG02] in their proof of the Hecke eigensheaf property for the compactified geometric Eisenstein functor in the function field setting. However, there Braverman and Gaitsgory rely on the decomposition theorem applied to the perverse sheaf $\mathrm{IC}_{\overline{\mathrm{Bun}_B}}$, which would not make sense in this context. Nevertheless, we still have access to the excision spectral sequence one usually uses in proving the decomposition theorem. In particular, in the proof of the decomposition theorem one uses the excision spectral sequence and then invokes the theory of weights to show that it degenerates. Something similar happens here. Namely, if we apply this to the sheaf $\mathscr{F} = \mathscr{S}_{\phi_T}$ then we see that $\mathrm{nEis}(\mathscr{S}_{\phi_T})$ is a filtered eigensheaf in the sense that, up to passing to the graded pieces of this filtration, it is an eigensheaf with eigenvalue ϕ, and later we can see, by looking at the Weil group action, this filtration must always split under some version of genericity on ϕ_T.

Even without knowing that the filtration on $\mathrm{nEis}(\mathscr{S}_{\phi_T})$ splits, the filtered eigensheaf property can already be used to tell us a lot about the structure of $\mathrm{nEis}(\mathscr{S}_{\phi_T})$. In particular, given $b \in B(G)$, the restriction $\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^b}$ to the HN-strata Bun_G^b defines a complex of smooth $J_b(\mathbb{Q}_p)$-representations, and we are interested in describing this restriction. Here J_b is the σ-centralizer of b and it is an extended pure inner form of a Levi subgroup M_b of G. The fact that $\mathrm{nEis}(\mathscr{S}_{\phi_T})$ is a filtered Hecke eigensheaf with eigenvalue ϕ implies that, if ρ is an irreducible constituent of this restriction then the Fargues-Scholze parameter ϕ_ρ^{FS} must be equal to ϕ under the twisted embedding ${}^LJ_b \to {}^LG$. If we believe that the Fargues-Scholze local Langlands correspondence is the true local Langlands correspondence then this seems to suggest that ρ should be given by a normalized parabolic induction of the character χ attached to ϕ_T via local class field theory. In particular, we note, by deformation theory, that a toral parameter being generic implies that every lift of ϕ_T to $\overline{\mathbb{Z}}_\ell$ factors through ${}^LT$ and that the induced $\overline{\mathbb{Q}}_\ell$-parameter cannot come from the semi-simplification of a parameter with non-trivial monodromy (Lemma 2.3.18). This imposes a very rigid constraint on J_b. In particular, the Borel $B \cap M_b$ of M_b should transfer to a Borel B_b of J_b. The elements where this occurs are the elements in the image of the map $B(T) \to B(G)$, called the unramified elements $B(G)_{\mathrm{un}}$, as studied in the work of Xiao-Zhu [XZ17].

Corollary 2.1.13. *(Corollary 2.7.7) For ϕ_T a generic parameter and $b \in B(G)$, assuming compatibility of the Fargues-Scholze and the conjectural local Langlands correspondence (Assumption 2.7.5), the restriction* $\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^b}$ *vanishes unless $b \in B(G)_{\mathrm{un}}$.*

We now assume compatibility (Assumption 2.7.5) in addition to validity of the ULA theorem (Assumption 2.8.1) for the rest of the section. The previous corollary tells us that if we are interested in understanding the complex of $J_b(\mathbb{Q}_p)$-representations defined by the stalks $\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^b}$ then we can restrict to the case where $b \in B(G)_{\mathrm{un}}$, and here we expect the stalks to be given by the inductions $i_{B_b}^{J_b}(\chi) \otimes \delta_{P_b}^{-1/2}$, where δ_{P_b} is the modulus character of the parabolc P_b with Levi factor M_b transferred to J_b.[2]. This is indeed the case. To understand this, we use that each element $b \in B(G)_{\mathrm{un}}$ has a unique element b_T such that the slopes are G-dominant. Similarly, we write b_T^- for the unique element whose isocrystal slopes are anti-dominant and will refer to it as the HN-dominant element, where we recall that the isocrystal slopes are the negative of the Harder-Narasimhan slopes of the associated bundles. The set of elements in $B(T)$ mapping to b can be described as $w(b_T^-)$, where we identify $w \in W_b := W_G/W_{M_b}$ with a set of representatives of minimal length in W_G. The connected components of Bun_B and Bun_T are indexed by elements in $B(T) \simeq \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$, giving a direct sum decomposition:

$$\mathrm{nEis}(\mathscr{S}_{\phi_T}) = \bigoplus_{\overline{\nu} \in B(T)} \mathrm{nEis}^{\overline{\nu}}(\mathscr{S}_{\phi_T})$$

After restricting to Bun_G^b, the point is that only the summands indexed by $\overline{\nu} = w(b_T^-)$ for $w \in W_b$ survive. It is fairly easy to see this when $\overline{\nu} = b_T^-$. In particular, the connected component $\mathrm{Bun}_B^{b_T^-}$ will parametrize split B-structures since the HN-

[2]These twists by the modulus character come from the fact that the excursion algebra on Bun_G acts on a smooth irreducible representation $\rho \in \mathrm{D}(J_b(\mathbb{Q}_p), \Lambda) \simeq \mathrm{D}(\mathrm{Bun}_G^b) \subset \mathrm{D}(\mathrm{Bun}_G)$ via the Fargues-Scholze parameter $\phi_\rho^{FS} : W_{\mathbb{Q}_p} \to {}^L J_b(\Lambda)$ of a smooth irreducible representation ρ of J_b, composed with the *twisted* embedding ${}^L J_b(\Lambda) \to {}^L G(\Lambda)$, as in [FS21, Section IX.7.1].

slopes are dominant, and the diagram (2.1) (essentially) becomes

$$\begin{tikzcd} \left[*/\underline{B_b(\mathbb{Q}_p)}\right] \arrow[d, "\mathfrak{q}"] \arrow[dr, "\mathfrak{p}"] & \\ \left[*/\underline{T(\mathbb{Q}_p)}\right] & \left[*/\underline{J_b(\mathbb{Q}_p)}\right] \end{tikzcd}$$

Using this, it is easy to see that $\mathfrak{p}_!\mathfrak{q}^*(\chi)$ will be given by compactly supported functions of $J_b/B_b(\mathbb{Q}_p)$ which transform under $B_b(\mathbb{Q}_p)$ via χ. In other words, the unnormalized induction $\mathrm{Ind}_{B_b}^{J_b}(\chi)$. When one accounts for the twists coming from the dualizing object as well as the sign switch between isocrystals and G-bundles, one finds that the exact formula becomes

$$\mathrm{nEis}^{b_T^-}(\mathscr{S}_{\phi_T}) \simeq j_{b!}(i_{B_b}^{J_b}(\chi^{w_0}) \otimes \delta_{P_b}^{-1/2})[-\langle 2\hat{\rho}, \nu_b\rangle]$$

where $j_b : \mathrm{Bun}_G^b \to \mathrm{Bun}_G$ is the inclusion of the HN-strata corresponding to b and $w_0 \in W_b$ is a minimal length representative of the element of longest length. Now, what about the connected components $\overline{\nu} = w(b_T^-)$ with w non-trivial? Here the HN-slopes of $\overline{\nu}$ are at least partially anti-dominant, and therefore $\mathrm{Bun}_B^{w(b_T^-)}$ will parameterize some non-split extensions. Nonetheless, one finds that $\mathrm{nEis}^{w(b_T^-)}(\mathscr{S}_{\phi_T})$ behaves similarly to the contribution of the connected component given by the HN-dominant reduction. Note that, a priori, the complex $\mathrm{nEis}^{w(b_T^-)}(\mathscr{S}_{\phi_T})$ could be supported on all $b' \in B(G)$ with $b \succeq b'$ in the natural partial ordering on $B(G)$. If one imposes the previous compatibility assumption and assumes ϕ_T is generic then one can use the previous corollary to assume that $b' \in B(G)_{\mathrm{un}}$. In this case, the complex $\mathrm{nEis}^{w(b_T^-)}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^{b'}}$ can be computed in terms of the cohomology of the space of simulatenous reductions of a G-bundle $\mathscr{F}_G$ to two B-bundles with underlying T-bundles given by $w(b_T^-)$ and a Weyl group translate of b'_T, where b'_T is the HN-dominant reduction of b'. This space admits a locally closed stratification by the generic relative position of these two reductions coming from the Bruhat decomposition of $B\backslash G/B$. If $b' \neq b$ then each of the non-empty strata admit a map to a positive symmetric power of the mirror curve Div^1, and are locally modelled by a semi-infinite flag space called a Zastava space, as studied in the function field setting by Feign, Finkelberg, Kusnetzov, and Mirković [Fei+99]. By combining a study of these Zastava spaces with Condition (3) on ϕ_T and induction on b, we can show that the restriction of $\mathrm{nEis}^{w(b_T^-)}(\mathscr{S}_{\phi_T})$

to each of the locally closed strata indexed by $b' \neq b$ vanishes, from which we can conclude that the restriction $\mathrm{nEis}^{w(b_T^-)}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^{b'}}$ vanishes unless $b' = b$, where again only the contribution of the split B-structure matters. All in all, we conclude an isomorphism:

$$\mathrm{nEis}^{w(b_T^-)}(\mathscr{S}_{\phi_T}) \simeq j_{b!}(i_{B_b}^{J_b}(\chi^{ww_0}) \otimes \delta_{P_b}^{-1/2})[-\langle 2\hat{\rho}, \nu_b \rangle]$$

This parallel behavior between the HN-dominant connected component and the connected components in its Weyl group orbit is no accident. In analogy with §1.1, we expect, for a choice of representative $\tilde{w} \in N(T)$ of $w \in W_G$ in the relative Weyl group, to have an isomorphism

$$\mathrm{nEis}(\mathscr{S}_{\phi_T}) \simeq \mathrm{nEis}(\mathscr{S}_{\phi_T}^{\tilde{w}})$$

where $\mathscr{S}_{\phi_T}^{\tilde{w}}$ is the pullback of $\mathscr{S}_{\phi_T}$ along the map $\mathrm{Bun}_T \to \mathrm{Bun}_T$ induced by $\tilde{w}$. This involution sends the connected component $\mathrm{Bun}_T^{b_T}$ to $\mathrm{Bun}_T^{w(b_T)}$ and sends the character χ to χ^w. In particular, we see that, by our previous description of stalks, this gives the precise analogue of Theorem 2.1.1. We summarize the above discussion as follows.

Theorem 2.1.14. *(Corollary 2.9.2) Consider ϕ_T a weakly normalized regular parameter with associated character $\chi : T(\mathbb{Q}_p) \to \Lambda^*$. Given $b \in B(G)_{\mathrm{un}}$, we consider J_b, M_b, B_b, and W_b as defined above. For $b \in B(G)$, the stalk $\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^b} \in \mathrm{D}(\mathrm{Bun}_G^b) \simeq \mathrm{D}(J_b(\mathbb{Q}_p), \Lambda)$ is given by*

1. *an isomorphism $\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^b} \simeq \bigoplus_{w \in W_b} i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{-1/2}[-\langle 2\hat{\rho}, \nu_b \rangle]$ if $b \in B(G)_{\mathrm{un}}$,*
2. *an isomorphism $\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^b} \simeq 0$ if $b \notin B(G)_{\mathrm{un}}$.*

In particular, $\mathrm{nEis}(\mathscr{S}_{\phi_T})$ is a perverse sheaf on Bun_G with respect to the standard t-structure defined by the HN-strata using Theorem 1.11.

We note that the previous Corollaries imply that the stalks of the sheaf $\mathrm{nEis}(\mathscr{S}_{\phi_T})$ are valued in smooth admissible representation when ϕ_T is weakly normalized regular. This implies that the sheaf $\mathrm{nEis}(\mathscr{S}_{\phi_T})$ is ULA with respect to the structure map $\mathrm{Bun}_G \to *$, using the characterization given in [FS21, Theorem V.7.1]. This ULA property allows us to extend the construction of $\mathrm{nEis}(\mathscr{S}_{\phi_T})$ to $\overline{\mathbb{Z}}_\ell$ and $\overline{\mathbb{Q}}_\ell$-coefficients, where passing to the world of lisse-étale solid sheaves is a non-trivial matter because of the difference between ℓ-adic and discrete topologies. We will need to work with ϕ_T that is integral in the sense that, if $\Lambda = \overline{\mathbb{Q}}_\ell$, it is of the form $\overline{\phi}_T \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{Q}}_\ell$ for a $\overline{\mathbb{Z}}_\ell$-valued parameter $\overline{\phi}_T$. Given an integral parameter ϕ_T such that the mod ℓ-reduction is weakly normalized regular, we get a sheaf $\mathrm{nEis}(\mathscr{S}_{\phi_T}) \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ for all $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$. The description of the stalks, the filtered eigensheaf property with eigenvalue ϕ, and the commutation with Verdier duality all extend in a natural way to these coefficient systems, and we would now like to say that the filtered eigensheaf property implies it is a genuine eigensheaf under the conditions on ϕ_T. For a representation $V \in \mathrm{Rep}_\Lambda({}^L G)$, the filtered eigensheaf property tells us that $T_V(\mathrm{nEis}(\mathscr{S}_{\phi_T}))$ has a filtration whose graded pieces have Weil group action given by $\nu^\Gamma \circ \phi_T$, for $\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ a non-zero weight of V in $\hat{T}$. In order to see this splits, it suffices to show for ν, ν' defining distinct Galois orbits of weights of V in $\hat{T}$ that the extension group $H^1(W_{\mathbb{Q}_p}, (\nu - \nu')^\Gamma \circ \phi_T)$ vanishes for the Γ-orbit $(\nu - \nu')^\Gamma$ defined by $\nu - \nu'$. However, this is equivalent to saying that the entire complex

$$R\Gamma(W_{\mathbb{Q}_p}, (\nu - \nu')^\Gamma \circ \phi_T)$$

is trivial, and this was precisely the kind of vanishing result that strong μ-regularity of ϕ_T guaranteed. Moreover, by the vanishing of the H^0 the splitting will be unique. In particular, if $V = V_{\mu^\Gamma}$ for $\mu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ such that ϕ_T is strongly μ-regular then this allows us to see that we get a unique splitting for $V = V_{\mu^\Gamma}$. Now, as noted in Remark 2.1.8, we see that strong μ-regularity is usually guaranteed under generic for a sufficiently nicely chosen cocharacter. We would like to use this to conclude that the filtration on $T_{V_{\mu^\Gamma}}$ splits in more generality. Suppose we are given cocharacters $\mu_1, \dots, \mu_k$ and we know that the filtration splits for the V_{μ_i} then, using the compatibilities of the filtration, we can show the splitting for any representation V realized as a direct summand of $\bigotimes_{k=1}^m V_{\mu_k}^{\otimes n_i}$, but in this case we cannot guarantee that this splitting is unique.

We need to be a bit careful when running the argument sketched above. In particular, if $\Lambda = \overline{\mathbb{Q}}_\ell$, then the category $\mathrm{Rep}_\Lambda({}^L G)$ is semi-simple with irreducible objects parametrized by Γ-orbits of dominant cocharacters $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+/\Gamma$ and the above argument goes through. If $\Lambda \in \{\overline{\mathbb{Z}}_\ell, \overline{\mathbb{F}}_\ell\}$ this is no longer true. However, in these cases, we can replace $\mathrm{Rep}_\Lambda({}^L G)$ by a sub-category of tilting modules $\mathrm{Tilt}_\Lambda({}^L G)$ ([Mat00],[Jan03, Appendix E]), which will be semi-simple with indecomposable objects parameterized by $\mu^\Gamma \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+/\Gamma$, denoted $\mathscr{T}_{\mu^\Gamma} \in \mathrm{Tilt}_\Lambda({}^L G)$. Extending the theory of tilting modules to the full L-group

${}^LG$ is a bit subtle. However, this is precisely what our assumption that ℓ is very good with respect to G will allow us to do. This category is preserved under taking tensor products, and therefore we can define the notion of a "tilting eigensheaf" (Definition 2.10.5) by replacing $\mathrm{Rep}_\Lambda({}^LG)$ with $\mathrm{Tilt}_\Lambda({}^LG)$ in the usual definition. For $V \in \mathrm{Rep}_\Lambda({}^LG^I)$, we write $r_V : {}^LG^I \to \mathrm{GL}(V)$ for the associated map. This allows us to define the following.

Definition 2.1.15. For a finite index set I, we say a tuple of cocharacters $(\mu_i)_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+)^I$ is $(\mu_i)_{i\in I}$-regular if the filtration on $T_V(\mathrm{nEis}(\mathscr{S}_{\phi_T})$ splits for the tilting module $V = \boxtimes_{i\in I}\mathscr{T}_{\mu_i^\Gamma}$.

Remark 2.1.16. The argument sketched above allows us to show if ϕ_T is $(\mu_{1i})_{i\in I}$ and $(\mu_{2i})_{i\in I}$ regular then this implies that ϕ_T is regular for the cocharacter attached to any highest weight tilting module occurring in the tensor product $\boxtimes_{i\in I}\mathscr{T}_{\mu_{1i}^\Gamma} \otimes \mathscr{T}_{\mu_{2i}^\Gamma}$ (Proposition 2.10.12). In particular, this implies that ϕ_T is $(\mu_{1i}+\mu_{2i})_{i\in I}$-regular by considerations of highest weight. This property often allows us to see that we get μ-regularity just under the generic hypothesis on ϕ_T. For example, for $G = \mathrm{GL}_n$, and $\Lambda = \overline{\mathbb{Q}}_\ell$ then, as observed in Remark 2.1.8, genericity will imply strong μ-regularity for $\mu = (1,0,\ldots,0)$, and this will imply the filtration splits uniquely for this cocharacter. This allows us to see that the filtration splits uniquely for the standard representation, and this in turn allows us to see that we get a splitting for the representations corresponding to the other fundamental coweights $\omega_i = (1_i, 0_{n-i})$ using the decomposition $V_{\mathrm{std}}^{\otimes i} = \mathrm{Sym}^i(V) \bigoplus \cdots \bigoplus \Lambda^i(V)$ of the tensor powers of the standard representation V_{std} of GL_n. From here, we can show μ-regularity for all μ using the fact that we can realize $\mathscr{T}_\mu$ as a direct summand of a tensor product of the representations corresponding to fundamental weights. This argument also works with torsion coefficients since $\Lambda^i(V) = V_{\omega_i}$ will always be tilting by virtue of ω_i being minuscule for all $i = 1,\ldots,n$ when $G = \mathrm{GL}_n$ (Corollary 2.10.16).

Our main theorem is then as follows.

Theorem 2.1.17. *(Theorem 2.10.10) For $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$, we consider $\phi_T : W_{\mathbb{Q}_p} \to {}^LT(\Lambda)$ an integral parameter such that its mod ℓ-reduction is weakly normalized regular. There then exists a perverse sheaf* $\mathrm{nEis}(\mathscr{S}_{\phi_T}) \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ *which is a filtered eigensheaf with eigenvalue ϕ. If $V \in \mathrm{Tilt}_\Lambda({}^LG)$ is a direct sum of tilting modules $\boxtimes_{i\in I}\mathscr{T}_{\mu_i^\Gamma}$ for geometric dominant cocharacters μ_i, and ϕ_T is μ_i-regular (resp. strongly μ_i-regular), the filtration on $T_V(\mathrm{nEis}(\mathscr{S}_{\phi_T})$ splits (resp.*

splits uniquely), and we have a natural isomorphism

$$T_V(\mathrm{nEis}(\mathscr{S}_{\phi_T})) \simeq \mathrm{nEis}(\mathscr{S}_{\phi_T}) \boxtimes r_V \circ \phi$$

of sheaves in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G)^{BW^I_{\mathbb{Q}_p}}$. *In particular, if* ϕ_T *is* μ*-regular (resp. strongly* μ*-regular) for all geometric dominant cocharacters* μ *then* $\mathrm{nEis}(\mathscr{S}_{\phi_T})$ *is a weak tilting eigensheaf (resp. tilting eigensheaf). For* $b \in B(G)$*, the stalk* $\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}^b_G} \in \mathrm{D}(\mathrm{Bun}^b_G) \simeq \mathrm{D}(J_b(\mathbb{Q}_p), \Lambda)$ *is given by*

1. *an isomorphism* $\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}^b_G} \simeq \bigoplus_{w \in W_b} i^{J_b}_{B_b}(\chi^w) \otimes \delta_{P_b}^{-1/2}[-\langle 2\hat{\rho}, \nu_b \rangle]$ *if* $b \in B(G)_{\mathrm{un}}$,
2. *an isomorphism* $\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}^b_G} \simeq 0$ *if* $b \notin B(G)_{\mathrm{un}}$.

Moreover, if $\mathbb{D}_{\mathrm{Bun}_G}$ *denotes Verdier duality on* Bun_G*, we have an isomorphism*

$$\mathbb{D}_{\mathrm{Bun}_G}(\mathrm{nEis}(\mathscr{S}_{\phi_T})) \simeq \mathrm{nEis}(\mathscr{S}_{\phi_T^\vee})$$

of sheaves in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$.

Remark 2.1.18. The notion of a weak tilting eigensheaf means that we always have isomorphisms

$$\eta_{V,I} : T_V(\mathrm{nEis}(\mathscr{S}_{\phi_T})) \simeq \mathrm{nEis}(\mathscr{S}_{\phi_T}) \boxtimes r_V \circ \phi$$

for $V \in \mathrm{Tilt}({}^LG^I)$ and a finite index set I, but do not necessarily know that the desired compatibilities with respect to I and V. Even though we know these compatibilities for the filtration, it is not necessarily clear that the splitting we produce through our argument respects these compatibilities without assuming strong μ-regularity. Only knowing the compatibilities of the splittings under such restrictive conditions is a bit unfortunate; fortunately, for most of the applications to local Shtuka spaces with one leg it suffices to only know a splitting exists.

This eigensheaf has several suprising applications to the cohomology of local Shimura varieties and shtuka spaces. To formalize this, we define, for $b \in B(G)$, a complex of $J_b(\mathbb{Q}_p)$-representations denoted $\mathrm{Red}_{b,\phi}$. If $b \notin B(G)_{\mathrm{un}}$ we set this to be equal to 0 and if $b \in B(G)_{\mathrm{un}}$ to be equal to $\bigoplus_{w \in W_b} i^{J_b}_{B_b}(\chi^w) \otimes \delta_{P_b}^{-1/2}[-\langle 2\hat{\rho}, \nu_b \rangle]$. Now let's consider μ a geometric dominant cocharacter of G with reflex field E and set $B(G,\mu) \subset B(G)$ to be the subset of μ-admissible elements (Definition 2.2.5). We let $\mathscr{T}_\mu$ be the associated highest weight tilting module of $\hat{G}$.

This defines a representation of $W_E \ltimes \hat{G}$ with associated Hecke operator T_μ. We write $r_\mu : W_E \ltimes \hat{G} \to \mathrm{GL}(\mathscr{T}_\mu)$ for the associated map. We consider the cohomology of the local shtuka spaces $\mathrm{Sht}(G,b,\mu)_\infty$, as defined in [SW20b]. In particular, the representation $\mathscr{T}_\mu$ attached to a dominant inverse defines a sheaf $\mathscr{S}_\mu$ on $\mathrm{Sht}(G,b,\mu)_\infty$ via geometric Satake, and we can consider the complex $R\Gamma_c(G,b,\mu)$ of $J_b(\mathbb{Q}_p) \times G(\mathbb{Q}_p) \times W_E$-modules attached to cohomology valued in this sheaf.

Remark 2.1.19. We note that, since we have used the tilting module $\mathscr{T}_\mu$ in the definition of $R\Gamma_c(G,b,\mu)$ instead of the usual highest weight representation V_μ this is slightly different than the usual definition appearing in the literature. The two definitions will coincide when the representation V_μ defines a tilting module, which is equivalent to V_μ being irreducible with coefficients in Λ. We say such a μ is tilting if this holds. This will always hold if $\Lambda = \overline{\mathbb{Q}}_\ell$ or if μ is minuscule, and we study this notion more carefully in Appendix A.2.

We can use $\mathrm{nEis}(\mathscr{S}_{\phi_T})$ to describe the cohomology of $R\Gamma_c(G,b,\mu)$. Assume that ϕ_T is μ-regular, the Hecke eigensheaf property then tells us that we have an isomorphism

$$T_\mu(\mathrm{nEis}(\mathscr{S}_{\phi_T})) \simeq \mathrm{nEis}(\mathscr{S}_{\phi_T}) \boxtimes r_\mu \circ \phi|_{W_E}$$

of sheaves with continuous W_E-action. If we restrict to the open HN-strata $j_{\mathbf{1}} : \mathrm{Bun}_G^{\mathbf{1}} \to \mathrm{Bun}_G$ defined by the trivial element $\mathbf{1} \in B(G)$ then this gives an isomorphism

$$j_{\mathbf{1}}^* T_\mu(\mathrm{nEis}(\mathscr{S}_{\phi_T})) \simeq i_B^G(\chi) \boxtimes r_\mu \circ \phi|_{W_E}$$

of complexes of $G(\mathbb{Q}_p) \times W_E$-modules. Now the point is that only the elements $b \in B(G,\mu)$ occur as a modifications $\mathscr{F}_b \to \mathscr{F}_G^0$ of type μ, where $\mathscr{F}_G^0$ is the trivial G-bundle. Therefore, only these stalks contribute to the LHS. By applying excision to the locally closed stratification given by Bun_G^b for $b \in B(G,\mu)$, we find that the LHS has a filtration with graded pieces isomorphic to $j_{\mathbf{1}}^* T_\mu(j_{b!}(\mathrm{Red}_{b,\phi}))$, but these are related to the isotypic parts $R\Gamma_c^\flat(G,b,\mu)[\mathrm{Red}_{b,\phi}]$. From the above analysis, we deduce the following.

Theorem 2.1.20. *(Theorem 2.11.7) For $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\Lambda)$ an integral toral parameter such that its mod ℓ-reduction is weakly normalized regular and any geometric dominant cocharacter μ such that ϕ_T is μ-regular, we have an equality*

$$\sum_{b \in B(G,\mu)} [R\Gamma_c^\flat(G,b,\mu)[\mathrm{Red}_{b,\phi}]] = [r_\mu \circ \phi|_{W_E} \boxtimes i_B^G(\chi)]$$

in the Grothendieck group $K_0(G(\mathbb{Q}_p) \times W_E, \Lambda)$ of smooth admissible $G(\mathbb{Q}_p)$-representations with a continuous action of W_E.

If we now consider the case where $\Lambda = \overline{\mathbb{Q}}_\ell$ then it follows that the averaging formula is valid for all weakly normalized regular and μ-regular parameters $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\overline{\mathbb{Q}}_\ell)$, which admit a $\overline{\mathbb{Z}}_\ell$-lattice. Moreover, with $\overline{\mathbb{Q}}_\ell$-coefficients, we can interpret both sides as trace forms on $K_0(T(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$, and use that the set of characters obtained from such parameters is Zariski dense in the variety of unramified characters to conclude the following more general claim.

Theorem 2.1.21. *(Theorem 2.11.10) For $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\overline{\mathbb{Q}}_\ell)$ any toral parameter and μ any geometric dominant cocharacter of G, we have an equality*

$$\sum_{b \in B(G,\mu)} [R\Gamma_c^\flat(G,b,\mu)[\mathrm{Red}_{b,\phi}]] = [r_\mu \circ \phi|_{W_E} \boxtimes i_B^G(\chi)]$$

in $K_0(G(\mathbb{Q}_p) \times W_E, \overline{\mathbb{Q}}_\ell)$.

If μ is minuscule and $G = \mathrm{GL}_n$ then this recovers special cases of an averaging formula of Shin [Shi12], which was formalized for more general reductive groups by Alexander Bertoloni-Meli [Ber21]. In particular, for all χ the induction $i_B^G(\chi)$ defines a class $[i_B^G(\chi)] \in K_0^{st}(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ in the subgroup of the Grothendieck group with stable character sum. To such a class, the averaging formula gives a description of the RHS in terms of an average over $B(G,\mu)$ of the isotypic parts of $R\Gamma_c(G,b,\mu)$ with respect to $\mathrm{Red}_b^{\mathfrak{c}}(i_B^G(\chi))$, where $\mathfrak{c}$ is a refined endoscopic datum (Definition A.3.1). In Appendix A.3, we verify that this indeed agrees with the conjectured averaging formula when $\mathfrak{c}$ is the trivial endoscopic datum. This is rather remarkable. Such formulae are typically proven in the minuscule case by stabilizing the trace formula on the Igusa varieties indexed by $b \in B(G,\mu)$, and our analysis gives a more conceptual explanation for them. By combining our work here with the compatibility results proven in [Ham21b] and [BHN22], this gives a proof of this averaging formula in cases where the non-basic Igusa varieties haven't even been properly defined yet![3] We recall that, in the proof of the averaging formula, we used excision to produce a filtration whose graded pieces were isomorphic to

$$j_{\mathbf{1}}^* T_\mu(j_{b!}(\mathrm{Red}_{b,\phi})) \simeq j_{\mathbf{1}}^* T_\mu(j_{b!}(\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^b}))$$

By using the isomorphism $\mathbb{D}_{\mathrm{Bun}_G}(\mathrm{nEis}(\mathscr{S}_{\phi_T})) \simeq \mathrm{nEis}(\mathscr{S}_{\phi_T^\vee})$, we can show (See Corollary 2.11.15) that we have an isomorphism: $j_{\mathbf{1}}^* T_\mu(j_{b!}(\mathrm{Red}_{b,\phi})) \simeq$

[3]However, in the case where G is split and μ is minuscule this formula can actually be checked by hand (See Proposition 2.11.20), but in the case of unitary groups or restrictions of scalars and μ minuscule this already gives new information.

$j_{\mathbf{1}}^{*}T_{\mu}(j_{b*}(\mathrm{Red}_{b,\phi}))$ of objects in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)$. This implies that the excision spectral sequence degenerates, allowing us to conclude the following refined averaging formula.

Theorem 2.1.22. *(Theorem 2.11.16) For ϕ_T an integral parameter with weakly normalized regular mod ℓ-reduction, and μ any geometric dominant cocharacter such that ϕ_T is μ-regular, we have an isomorphism*

$$\bigoplus_{b\in B(G,\mu)_{\mathrm{un}}}\bigoplus_{w\in W_b} R\Gamma_c^{\flat}(G,b,\mu)[\rho_{b,w}][-\langle 2\hat{\rho},\nu_b\rangle]\simeq i_B^G(\chi)\boxtimes r_{\mu}\circ\phi|_{W_E}$$

of complexes of $G(\mathbb{Q}_p)\times W_E$-modules, where $\rho_{b,w}:=i_{B_b}^{J_b}(\chi^w)\otimes\delta_{P_b}^{-1/2}$.

We now assume that ϕ_T is an integral parameter with weakly normalized regular mod ℓ reduction in all that follows. The previous theorem leads to a very explicit descriptions of the complexes $R\Gamma_c(G,b,\mu)[\rho_{b,w}\otimes\delta_{P_b}]$ and the degrees of cohomology they sit in.

Corollary 2.1.23. *(Corollary 2.11.17) For μ a geometric dominant cocharacter with reflex field E such that ϕ_T is μ-regular, fixed $b\in B(G,\mu)_{\mathrm{un}}$, and varying $w\in W_b$, the complex $R\Gamma_c^{\flat}(G,b,\mu)[\rho_{b,w}]$ is isomorphic to $\phi_{b,w}^{\mu}\boxtimes\sigma[\langle 2\hat{\rho},\nu_b\rangle]$, for $\phi_{b,w}^{\mu}$ a representation of W_E and σ a sub representation of $i_B^G(\chi)$. Moreover, we have an isomorphism*

$$\bigoplus_{b\in B(G,\mu)_{\mathrm{un}}}\bigoplus_{w\in W_b}\phi_{b,w}^{\mu}\simeq r_{\mu}\circ\phi|_{W_E}$$

of W_E-representations.

Remark 2.1.24. When $G=\mathrm{GL}_n$, we can deduce these consequences for all μ under the assumption that ϕ_T is generic (cf. Remark 2.1.5). We anticipate that, by combining this statement with the approach to torsion vanishing taken by Koshikawa [Kos21b] via using compatibility of the Fargues-Scholze and the usual local Langlands correspondence, it should lead to generalizations of Caraiani and Scholze's results.

It is now natural to wonder what the representations $\phi_{b,w}^{\mu}$ exactly are. It was already observed by Xiao-Zhu [XZ17] that the elements of the set $B(G,\mu)_{\mathrm{un}}$ correspond to Weyl group orbits of weights of the highest weight module $\mathscr{T}_{\mu}$ or rather its restriction $\mathscr{T}_{\mu}|_{\hat{G}^{\Gamma}}$ (Corollary 2.2.9). If we let $b_T\in B(T)$ be the dominant reduction of an element $b \in B(G,\mu)_{\mathrm{un}}$ then the orbit of the character b_T under the Weyl group W_G can be described as $w(b_T)$ for $w \in W_b$ varying. For varying $b \in B(G,\mu)_{\mathrm{un}}$, this describes the set of non-zero weights which can occur in the representation $\mathscr{T}_\mu|_{\hat{G}^\Gamma}$. In particular, given such a $\overline{\nu} \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$, we can look at the direct sum of weight spaces

$$\bigoplus_{\substack{\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \\ \nu_\Gamma = \overline{\nu}}} \mathscr{T}_\mu(\nu)$$

and this coincides with the weight space of $\mathscr{T}_\mu|_{\hat{G}^\Gamma}(\overline{\nu})$ via the isomorphism $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma \simeq \mathbb{X}^*(\hat{T}^\Gamma)$. The refined averaging formula suggests the following relationship.

Conjecture 2.1.25. *(Conjecture 2.11.18) For all geometric dominant cocharacters μ such that ϕ_T is μ-regular, an unramified element $b \in B(G,\mu)_{\mathrm{un}}$, and a Weyl group element $w \in W_b$, we have an isomorphism*

$$\bigoplus_{\substack{\widetilde{w(b_T)} \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \\ \widetilde{w(b_T)}_\Gamma = w(b_T)}} \widetilde{w(b_T)} \circ \phi_T|_{W_{E'}} \otimes \mathscr{T}_\mu(\widetilde{w(b_T)}) \simeq \phi^\mu_{b,w}|_{W_{E'}}$$

of $W_{E'}$-representations, where $E'|E$ denotes the splitting field of G.

We verify this conjecture in some particular cases, by noting that the contribution from the μ-ordinary locus can be explicitly computed using a shtuka analogue of Boyer's trick [Boy99a], as studied by Gaisin-Imai [GI16]. To do this, we note that we have a distinguished element in $B(G,\mu)_{\mathrm{un}}$ called the μ-ordinary element, which we denote by b_μ. It is the maximal element in the partial ordering on $B(G,\mu)$, and we let b_{μ_T} be its dominant reduction. The conjecture suggests that this should correspond to the Weyl group orbit of the highest weight of $\mathscr{T}_\mu$. In this case, the space $\mathrm{Sht}(G,b_\mu,\mu)_{\infty,\mathbb{C}_p}$ with its $G(\mathbb{Q}_p)\times J_b(\mathbb{Q}_p)$-action is determined by the space $\mathrm{Sht}(T,b_{\mu_T},\mu)_{\infty,\mathbb{C}_p}$ with its $T(\mathbb{Q}_p)\times T(\mathbb{Q}_p)$-action. In particular, using this we can deduce the following isomorphism (See Proposition 2.11.20):

$$R\Gamma_c^\flat(G,b_\mu,\mu)[\rho_{b_\mu,w}] \simeq i_B^G(\chi^{ww_0}) \boxtimes w(\mu)\circ\phi_T|_{W_E}[\langle 2\hat{\rho},\nu_b\rangle],$$

where w and w_0 are minimal length representatives of W_b in W_G. This calculation has a very interesting consequence. In particular, when combined with the refined averaging formula, we see that we must have an isomorphism $i_B^G(\chi^w) \simeq i_B^G(\chi)$. So, by choosing μ to be sufficiently regular so that $W_{b_\mu} = W_G$, we can deduce the following.

Theorem 2.1.26. *(Corollary 2.11.22) For ϕ_T an integral parameter with weakly normalized regular mod ℓ-reduction such that there exists a μ which is not fixed under W_G and ϕ_T is μ-regular, we have an isomorphism*

$$i_{\chi,w} : i_B^G(\chi) \simeq i_B^G(\chi^w)$$

of smooth $G(\mathbb{Q}_p)$-representations for all $w \in W_G$.

This showcases the strong connection between the theory of geometric Eisenstein series and the theory of intertwining operators and the Langlands quotient that has been our philosophical guide throughout. A relation that holds even with mod ℓ-coefficients! With mod ℓ coefficients, there is no good theory of intertwining operators or the Langlands quotient (See however [Dat05], for the current state of the art), and we suspect that further developing the theory of geometric Eisenstein series should provide some insights into these notions in the ℓ-modular setting.

We saw above that our previous conjecture on $\phi_{b,w}^{\mu}$ can be completely verified using Boyer's trick in the case that the only weights of $\mathscr{T}_\mu|_{\hat{G}^\Gamma}$ are orbits of the highest weight. This will be the case when the image $\mu_\Gamma \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma^+ \simeq \mathbb{X}^*(\hat{T}^\Gamma)^+$ of μ is minuscule with respect to the pairing with $\mathbb{X}_*(\hat{T}^\Gamma)$. If we combine this with the refined averaging formula then we can also deduce the claim when $B(G,\mu)$ has two elements. I.e the case where $\mathscr{T}_\mu|_{\hat{G}^\Gamma}$ has two weight spaces; one corresponding to the μ-ordinary element and the other corresponding to the basic element. This will prove the previous conjecture in all cases where $\mu_\Gamma \in \mathbb{X}^*(\hat{T}^\Gamma)^+$ is minuscule or quasi-minuscule with respect to the pairing with the cocharacters $\mathbb{X}_*(\hat{T}^\Gamma)$.

Theorem 2.1.27. *(Corollary 2.11.27) For μ a geometric dominant cocharacter and ϕ_T strongly μ-regular such that $\mu_\Gamma \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma^+$ is minuscule or quasi-minuscule with respect to the pairing with $\mathbb{X}_*(\hat{T}^\Gamma)$, the previous conjecture is true.*

Remark 2.1.28. Even for μ minuscule it can be the case that the image $\mu_\Gamma \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma^+$ is no longer minuscule with respect to the above pairing, as this corresponds to restricting the highest weight representation of $\hat{G}$ defined by μ to $\hat{G}^\Gamma$. Therefore, even for μ minuscule, we can still have that the basic element

$b \in B(G,\mu)$ is unramified (See [XZ17, Remark 4.2.11]) In these cases, a very analogous result was proven by [XZ17], where they describe the irreducible components of affine Deligne-Lusztig varieties in terms of the weight space defined by the basic element. These affine Deligne-Lusztig varieties describe the special fibers of the local shtuka spaces $\mathrm{Sht}(G,b,\mu)_{\infty}/\underline{K}$ in the case that G is unramified, and K is a hyperspecial level. Moreover, we suspect that, by using nearby cycles, one could deduce some special cases of their result from ours.

Throughout our results, we have introduced various technical conditions on ϕ_T. We suspect that some of these conditions are artifacts of the proofs we have used to overcome the technical geometry of Bun_B and its compacitifications in this diamond world. While the conditions are manageable for specific applications to specific groups it leaves one wanting for a more conceptually clear picture. In particular, we conjecture that the following is true, which (modulo checking the compatibilities of the isomorphisms in the eigensheaf property) our methods show for GL_n and integral parameters (Corollary 2.10.16).

Conjecture 2.1.29. *For $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$ and $\phi_T : W_{\mathbb{Q}_p} \to {}^LT(\Lambda)$ a generic toral L-parameter, there exists a sheaf $\mathrm{nEis}(\mathscr{S}_{\phi_T}) \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)$ which is a perverse Hecke eigensheaf with eigenvalue ϕ such that one has an isomorphism $\mathbb{D}_{\mathrm{Bun}_G}(\mathrm{nEis}(\mathscr{S}_{\phi_T})) \simeq \mathrm{nEis}(\mathscr{S}_{\phi_T^\vee})$, and its stalk at all $b \in B(G)$ is isomorphic to $\mathrm{Red}_{b,\phi} \otimes \delta_{P_b}^{-1}$.*

This conjecture would follow from knowing the existence of $\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ and in turn the compactified Eisenstein functor $\overline{\mathrm{Eis}}$ with all the various desiderata proven by Braverman-Gaitsgory [BG02] in the function field setting. In particular, we expect that $\overline{\mathrm{Eis}}(-)$ should commute with Verdier duality, and satisfy the functional equation if $\alpha \circ \phi_T$ is non-trivial for all Γ-orbits of roots. Moreover, by the analogue of the results of [BG08], there should be a natural map

$$\mathrm{nEis}(\mathscr{S}_{\phi_T}) \to \overline{\mathrm{Eis}}(\mathscr{S}_{\phi_T})$$

whose cone should be given by Eisenstein functors tensored with complexes admitting a filtration isomorphic to $R\Gamma(W_{\mathbb{Q}_p}, \alpha \circ \phi_T)$ for α a Γ-orbit of coroots of G. In particular, we should have an isomorphism $\overline{\mathrm{Eis}}(\mathscr{S}_{\phi_T}) \simeq \mathrm{nEis}(\mathscr{S}_{\phi_T})$ precisely when ϕ_T is generic. It follows by our above analysis that this would imply an isomorphism $i_B^G(\chi) \simeq i_B^G(\chi^w)$ for all generic χ, which is precisely what we show in the appendix (Proposition A.1.3).

In §2, we start by defining the set of unramified elements in $B(G)$ and discussing their relationship with highest weight theory, as in [XZ17]. In §3, we review the construction of eigensheaves on Bun_T attached to parameters ϕ_T, introducing the conditions on our parameter ϕ_T and working through some useful lemmas and examples related to them. In §4, we review the geometric Satake correspondence of Fargues-Scholze, recalling the key results and relating the highest weight theory of ${}^LG$ to the cohomology of semi-infinite Schubert cells. In §5, we introduce Drinfeld's compactifications over the Fargues-Fontaine curve and establish Theorem 2.1.3. We also introduce a locally closed stratification of $\overline{\mathrm{Bun}}_B$ and show it is well-behaved. In §6, we move into the sheaf theory introducing the normalized Eisenstein functor and establishing Theorem 1.2. In §7, we will study how the Eisenstein functor interacts with Hecke operators, establishing Theorem 1.12. This will ultimately be done via a key diagram relating the action of Hecke operators of Bun_G base-changed along the map $\mathfrak{p} : \mathrm{Bun}_B \to \mathrm{Bun}_G$ to semi-infinite Schubert cells, where it reduces to the results in §4. In §8, we study Verdier duality and show Theorem 1.11. In §9, we will carry out the computation of the stalks of the Eisenstein series $\mathrm{nEis}(\mathscr{S}_{\phi_T})$ establishing Theorem 1.14. In §10, we describe the theory of tilting modules for the L-group ${}^LG$, constructing $\mathrm{nEis}(\mathscr{S}_{\phi_T})$ with $\overline{\mathbb{Z}}_\ell$ and $\overline{\mathbb{Q}}_\ell$-coefficients and showing Theorem 1.15. Finally, in §11, we deduce the applications to the cohomology of local shtuka spaces showing Theorems 1.18, 1.20, 1.24, and 1.25.

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2.2 Notation

- Let $\ell \neq p$ be distinct primes.
- Let G be a quasi-split connected reductive group with simply connected derived group.
- We let $\overline{\mathbb{Q}}_\ell$ denote the algebraic closure of the ℓ-adic numbers, with residue field $\overline{\mathbb{F}}_\ell$ and ring of integers $\overline{\mathbb{Z}}_\ell$, endowed with the discrete topology. Throughout, we will assume that, for our fixed G, ℓ is very good in the sense of [FS21, Page 33] and banal with respect to T.
- Let Γ be the absolute Galois group of $\mathbb{Q}_p$, and let $W_{\mathbb{Q}_p} \subset \Gamma$ be the Weil group of $\mathbb{Q}_p$.
- We set $\mathscr{L}_{\mathbb{Q}_p} := W_{\mathbb{Q}_p} \times \mathrm{SL}_2(\overline{\mathbb{Q}}_\ell)$ to be the Weil-Deligne group.
- Fix choices $A \subset T \subset B \subset G$ of maximal split torus, maximal non-split torus, and Borel. We use U to denote the unipotent radical of B.
- We let W_G be the relative Weyl group of G and w_0 be the element of longest length.
- We write $\mathrm{Ind}_B^G(-)$ for the unnormalized parabolic induction functor from B to G. We let δ_B be the modulus character defined by B so that $i_B^G(-) := \mathrm{Ind}_B^G(- \otimes \delta_B^{1/2})$ is the normalized induction. In other words, δ_B is defined by the transformations of the space of right Haar measures.
- Let $\breve{\mathbb{Q}}_p$ be the completion of the maximal unramified extension of $\mathbb{Q}_p$ with Frobenius σ. For $E/\mathbb{Q}_p$ a finite extension, we set $\breve{E}$ to be the compositum $E\breve{\mathbb{Q}}_p$.
- Set $\mathbb{C}_p$ to be the completion of the algebraic closure of $\mathbb{Q}_p$.
- Let $B(G) = G(\breve{\mathbb{Q}}_p)/(g \sim hg\sigma(h)^{-1})$ denote the Kottwitz set of G.
- For $b \in B(G)$, we write J_b for the σ-centralizer of b.

- We will always work over the fixed base $* := \mathrm{Spd}\,\overline{\mathbb{F}}_p$, unless otherwise stated.

- Let Perf denote the category of (affinoid) perfectoid spaces in characteristic p over $*$ endowed with the v-topology. For $S \in \mathrm{Perf}$, let Perf_S denote the category of affinoid perfectoid spaces over it.

- For $S \in \mathrm{Perf}$, let X_S denote the relative (schematic) Fargues-Fontaine curve over S.

- For $\mathrm{Spa}(F, \mathscr{O}_F) \in \mathrm{Perf}$ a geometric point, we will often drop the subscript on X_F and just write X for the associated Fargues-Fontaine curve.

- For $b \in B(G)$, we write $\mathscr{F}_b$ for the associated G-bundle on X.

- For $S \in \mathrm{Perf}$, we let $\mathscr{F}_G^0$ denote the trivial G-bundle on X_S.

- We consider coefficient systems $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$, with a fixed choice of square root of $p \in \Lambda$. We define all half Tate twists with respect to this choice.

- For an Artin v-stack X, we write $\mathrm{D}_{\blacksquare}(X, \Lambda)$ for the condensed ∞-category of solid Λ-valued sheaves on X, and write $\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \subset \mathrm{D}_{\blacksquare}(X, \Lambda)$ for the full sub-category of Λ-valued lisse-étale sheaves, as defined in [FS21, Chapter VII].

- For a v-stack or diamond X, when working with torsion coefficients, we will indicate this by just writing $\mathrm{D}(X) := \mathrm{D}_{\text{ét}}(X, \Lambda)$ for the category of étale Λ-sheaves on X, as defined [Sch18]. If X is an Artin v-stack ([FS21, Definition IV.V.1]) admitting a separated cohomologically smooth surjection $U \to X$ from a locally spatial diamond U such that the etale site has a basis with bounded ℓ-cohomological dimension (which will always be the case for our applications) then we will regard it as a condensed ∞-category via the identification $\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \simeq \mathrm{D}(X)$ when viewed as objects in $\mathrm{D}_{\blacksquare}(X, \Lambda)$ [FS21, Proposition VII.6.6].

- We let $\hat{G}$ denote the Langlands dual group of G with fixed splitting $(\hat{T}, \hat{B}, \{X_\alpha\})$.

- If E denotes the splitting field of G then the action of $W_{\mathbb{Q}_p}$ factors through $Q := W_{\mathbb{Q}_p}/W_E$. We let ${}^L G := \hat{G} \rtimes Q$ denote the L-group.

- For I a finite index set, we let $\mathrm{Rep}_\Lambda({}^LG^I)$ denote the category of finite-dimensional algebraic representations of ${}^LG^I$.

- To any condensed ∞-category $\mathscr{C}$, we write $\mathscr{C}^{BW^I_{\mathbb{Q}_p}}$ for the category of objects with continuous $W^I_{\mathbb{Q}_p}$-action, as defined in [FS21, Section IX.1].

- We will let $\mathrm{Div}^1 := \mathrm{Spd}\,\breve{\mathbb{Q}}_p/\mathrm{Frob}^{\mathbb{Z}}$ denote the mirror curve, and, for a finite extension $E/\mathbb{Q}_p$, we write Div^1_E for the base-change to E.

- For I a finite index set, we let Div^I denote $|I|$-copies of the mirror curve. For $n \in \mathbb{Z}$, we let $\mathrm{Div}^{(n)} = (\mathrm{Div}^1)^n/S_n$, denote the nth symmetric power of the mirror curve, where S_n is the symmetric group on n letters.

- For a reductive group $H/\mathbb{Q}_p$, we write $\mathrm{D}(H(\mathbb{Q}_p),\Lambda)$ for the unbounded derived category of smooth Λ-representations.

- We say a map of v-stacks $f : X \to Y$ is representable in nice diamonds if it is representable in locally spatial diamonds, is compactifiable, and (locally) $\mathrm{tr.deg}(f) < \infty$.

- All 6-functors will be implicitly derived unless otherwise stated.

- For a locally pro-p group H, we write $\underline{H}$ for the functor sending $S \in \mathrm{Perf}$ to $\mathrm{Cont}(|S|,H)$, the set of continuous maps from the underlying topological space of S to H.

  Remark 2.2.1. At various points, we will need to consider the functors $f_! : \mathrm{D}(X) \to \mathrm{D}(Y)$ and $f^! : \mathrm{D}(Y) \to \mathrm{D}(X)$ for certain "stacky" morphisms of Artin v-stacks $f : X \to Y$. The correct definitions of these functors in this case are given in the work of [GHW22]. In particular, they extend the 6-functors studied in [Sch18; FS21] to fine maps [GHW22, Definition 1.3] of decent v-stacks [GHW22, Definition 1.2]. In general being a decent v-stack is stronger than being Artin. However, it is easy to check that all the stacks (resp. morphisms) we consider these functors for will be decent (resp. fine). To see this, one can use [GHW22, Proposition 4.11] which states that if $f : X \to Y$ is a map of v-stacks which is representable in nice diamonds and Y is decent then X is also decent and f is fine. In the cases we consider, one can apply this if one takes $Y = \mathrm{Bun}_G$. To see that Bun_G is decent, one can use the charts studied in [FS21, Section V.3], and take advantage of the fact that the maps defining the charts are formally smooth [FS21, Definition IV.3.1] by [FS21, Proposition IV.4.24]. This in particular allows one to see that these charts map strictly surjectively [GHW22, Defition 4.1] to Bun_G. It remains to explain why the maps appearing in our context are fine, to do this one can combine the previous analysis with [GHW22, Proposition 4.10 (iii)], which says that fine morphisms satisfy the 2 out of 3 property.

- When speaking about such fine maps of decent v-stacks we will often just cite theorems that only apply to the setting where f is representable in nice diamonds, and leave it to the reader to check that one can deduce the analogous results from the cited result and the formal properties of the 6-functors defined in [GHW22].
- Given a decent v-stack $X \to *$ such that X is fine over $*$, we let $K_X := f^!(\Lambda) \in \mathrm{D}(X)$ denote the dualizing object of X. Similarly, for $\mathscr{F} \in \mathrm{D}(X)$, we will write $R\Gamma_c(X, \mathscr{F}) := f_!(\mathscr{F}) \in D(\Lambda)$. We write $\mathbb{D}_X(-) := R\mathscr{H}om(-, K_X)$ for the Verdier duality functor. For a fine map $f : X \to S$ of decent v-stacks, we write $\mathbb{D}_{X/S} := R\mathscr{H}om(-, f^!(\Lambda))$ for relative Verdier duality.
- We will use the geometric normalization of local class field theory. For $n \in \mathbb{Z}$, we write (n) for the nth power of the ℓ-adic cyclotomic character of $W_{\mathbb{Q}_p}$. We note that, under this normalization, (1) is sent to the norm character $|\cdot| : \mathbb{Q}_p^* \to \Lambda^*$, which acts trivially on $\mathbb{Z}_p^*$ and sends p to $p^{-1} \in \Lambda^*$.

Before introducing the rest of the notation, we discuss the relationship between unramified elements in $B(G)$ and the representation theory of the dual group.

2.2.1 Unramified Elements in $B(G)$ and Highest Weight Theory

In this section, we will study the set of unramified elements in the Kottwitz set of G. As we will show, these elements are connected to the highest weight theory of the Langlands dual group $\hat{G}$, as discussed in [XZ17, Section 4.2.1]. First, we recall that the Kottwitz set $B(G)$ of a connected reductive group $G/\mathbb{Q}_p$ is equipped with two maps:

- The slope homomorphism

$$\nu : B(G) \to \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_{\mathbb{Q}}^{+,\Gamma}$$

$$b \mapsto \nu_b$$

where $\Gamma := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+_{\mathbb{Q}}$ is the set of rational dominant cocharacters of G.

- The Kottwitz invariant

$$\kappa_G : B(G) \to \pi_1(G)_\Gamma$$

where $\pi_1(G)$ denotes the algebraic fundamental group of Borovoi.

Now, given a geometric cocharacter μ of G with reflex field E, we can define the element:

$$\tilde{\mu} := \frac{1}{[E:\mathbb{Q}_p]} \sum_{\gamma \in \mathrm{Gal}(E/\mathbb{Q}_p)} \gamma(\mu) \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^{+,\Gamma}_{\mathbb{Q}}$$

We let $\mu^\flat$ be the image of μ in $\pi_1(G)_\Gamma \simeq X^*(Z(\hat{G})^\Gamma)$. Via the isomorphim $B(G)_{\mathrm{basic}} \simeq \pi_1(G)_\Gamma$, we regard it as a basic element of $B(G)$, which are the minimal elements in the natural partial ordering on $B(G)$. Now we recall that, for a torus T, we have an isomorphism $B(T) \simeq \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$. We can use this isomorphism to give a nice description of a certain piece of $B(G)$.

Definition 2.2.2. [XZ17, Section 4.2.1] We let $B(G)_{\mathrm{un}} \subset B(G)$ denote the image of the natural map $B(T) \to B(G)$. We refer to this as the set of unramified elements.

We now have the following Lemma. We write $(-)_\Gamma$ for the natural quotient map $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \to \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$.

Lemma 2.2.3. *[XZ17, Lemma 4.2.2] Let $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma \simeq B(T) \to B(G)$ be the natural map. Then this induces an isomorphism:*

$$\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma / W_G \simeq B(G)_{\mathrm{un}}$$

Proof. Strictly speaking, the proof given by Xiao-Zhu is only in the case that G is unramified. We remedy this now. Note that it is clear that this map is surjective, so it suffices to check injectivity. Let $\mu_1, \mu_2 \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ be two elements with b_1, b_2 their images in $B(G)$ under the natural composite

$$\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \to \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma \simeq B(T) \to B(G),$$

and suppose that $b_1 = b_2$. Since $\kappa_G(b_1) = \kappa_G(b_2)$, it follows that we have $\mu_1 - \mu_2 = (\gamma - 1)\nu + \alpha$ for some coroot $\alpha \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ and $\gamma \in \Gamma$. We may, without loss of generality, replace μ_1 by $\mu_1 + (\gamma - 1)\nu$, and therefore assume that $\mu_1 - \mu_2 = \alpha$. Since the slope homomorphisms of ν_{b_1} and ν_{b_2} are equal by assumption, we can assume, after conjugating by an element of W_G, that $\tilde{\mu}_1 = \tilde{\mu}_2$. Therefore, it follows that, if E_α denotes the reflex field of α, we have an equality

$$\sum_{g \in \mathrm{Gal}(E_\alpha/\mathbb{Q}_p)} g(\alpha) = 0$$

which in turn implies that

$$\sum_{g \in \mathrm{Gal}(E_\alpha/\mathbb{Q}_p)} (1-g)(\alpha) = |\mathrm{Gal}(E_\alpha/\mathbb{Q}_p)|\alpha$$

This would imply that α_Γ vanishes in $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$ assuming that α_Γ isn't torsion. However, Γ permutes the simple coroots, which form a basis of all coroots. Therefore, it follows that α_Γ is not torsion. $\square$

Now we would like to describe $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma/W_G$ slightly differently. To do this, we consider the natural pairing

$$\langle -,- \rangle : \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma \times \mathbb{X}^*(T_{\overline{\mathbb{Q}}_p})^\Gamma \to \mathbb{Z}$$

induced by the usual pairing between cocharacters and characters. We let $\hat{\Delta} \subset \mathbb{X}^*(T_{\overline{\mathbb{Q}}_p})$ (resp. $\Delta \subset \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$) be the set of (absolute) simple roots (resp. coroots) of G. Then we define $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma^+$ to be the set of elements in $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$ whose inner product with $\mathrm{Im}(\hat{\Delta} \to \mathbb{X}^*(T_{\overline{\mathbb{Q}}_p})^\Gamma)$ under the natural averaging map is positive. The natural map

$$\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma^+ \to \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma/W_G$$

is an isomorphism. We also note that we have a natural partial ordering on $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$. In particular, given $\overline{\nu}, \overline{\nu}' \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$ we say that $\overline{\nu} \geq \overline{\nu}'$ if $\overline{\nu} - \overline{\nu}'$ is a positive integral combination of α_Γ for $\alpha \in \Delta$. We note that we have a natural injective order preserving map:

$$\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma^+ \to \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_{\mathbb{Q}}^{\Gamma,+} \times \pi_1(G)_\Gamma$$

With this, we can reformulate the previous lemma as follows.

Lemma 2.2.4. *[XZ17, Lemma 4.2.3] The following diagram is commutative and respects the partial ordering*

$$\begin{tikzcd} B(G)_{\mathrm{un}} \arrow[r, "\simeq"] \arrow[d] & \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_{\Gamma}^{+} \arrow[d] \\ B(G) \arrow[r, "\nu\times\kappa"] & \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_{\mathbb{Q}}^{\Gamma,+} \times \pi_1(G)_{\Gamma} \end{tikzcd}$$

Now recall that, for μ a geometric dominant cocharacter of G, we have the following.

Definition 2.2.5. We define $B(G,\mu) \subset B(G)$ to be subset of $b \in B(G)$ for which $\nu_b \leq \tilde{\mu}$ with respect to the Bruhat ordering and $\kappa(b) = \mu^{\flat}$.

The previous lemma allows us to interpret the unramified elements in this set as follows.

Corollary 2.2.6. *[XZ17, Corollary 4.2.4] Under the identifications $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_{\Gamma}^{+} \simeq \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_{\Gamma}/W_G \simeq B(G)_{\mathrm{un}}$, we have an equality:*

$$B(G,\mu)_{\mathrm{un}} := B(G)_{\mathrm{un}} \cap B(G,\mu) = \{\lambda_{\Gamma} \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_{\Gamma}^{+} \mid \lambda_{\Gamma} \leq \mu_{\Gamma}\}$$

We now would like to connect this set with the highest weight theory for $\hat{G}$. If the group is not split then the unramified elements are naturally connected with the highest weight theory of the subgroup $\hat{G}^{\Gamma}$. Even though $\hat{G}^{\Gamma}$ is possibly disconnected its representation theory behaves like a connected reductive group. To see this, first we note that the subgroup $\hat{T}^{\Gamma}$ defined by the maximal torus has character group isomorphic to $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_{\Gamma}$, and the partial order described above allows one to talk about the highest weight of a representation. In particular, if we let $\hat{T}^{\Gamma,\circ}$ (resp. $\hat{G}^{\Gamma,\circ}$) denote the neutral component of $\hat{T}^{\Gamma}$ (resp. $\hat{G}^{\Gamma}$). Then one can use that the natural map $\hat{T}^{\Gamma}/\hat{T}^{\Gamma,\circ} \to \hat{G}^{\Gamma}/\hat{G}^{\Gamma,\circ}$ is an isomorphism ([Zhu15, Lemma 4.6]) to see that usual highest weight theory extends to $\hat{G}^{\Gamma}$. In particular, we have the following.

Lemma 2.2.7. *[Zhu15, Lemma 4.10] For $\overline{\mu} \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_{\Gamma}^{+}$, there is a unique up to isomorphism irreducible representation of $V_{\overline{\mu}} \in \mathrm{Rep}_{\overline{\mathbb{Q}}_{\ell}}(\hat{G}^{\Gamma})$ of highest weight $\overline{\mu}$, which give rise to all the irreducible representations in $\mathrm{Rep}_{\overline{\mathbb{Q}}_{\ell}}(\hat{G}^{\Gamma})$ for varying $\overline{\mu}$. Moreover, the multiplicity of the $\overline{\mu}$ weight space in $V_{\overline{\mu}}$ is 1, and the non-zero weights $\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_{\Gamma}$ of $V_{\overline{\mu}}$ lie in the convex hull of the W_G-orbit of $\overline{\mu}$.*

To a geometric dominant cocharacter μ, we can attach an irreducible representation $V_\mu \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\hat{G})$. This defines a natural representation of $\hat{G} \rtimes W_{E_\mu}$ as in [Kot97a, Lemma 2.1.2], where E_μ is the reflex field of μ. An element $\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ defines a representation of $\hat{T}$, and we write $V_\mu(\nu)$ for the corresponding weight space of V_μ. If we consider the restriction $V_\mu|_{\hat{G}^\Gamma}$ then the weight space $V_\mu(\nu)$ gives rise to a ν_Γ weight space, where we write $(-)_\Gamma : \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \to \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$ for the map given by taking coinvariants. Using this, it is easy to see we have the following relationship.

Lemma 2.2.8. *For $\mu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ and $\overline{\nu} \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$, we have the following equality:*

$$\dim(V_\mu(\overline{\nu})) = \sum_{\substack{\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \\ \nu_\Gamma = \overline{\nu}}} \dim(V_\mu(\nu))$$

We will combine this lemma with the following, which follows from the above discussion.

Corollary 2.2.9. *([XZ17, Lemma 4.26]) For μ a geometric dominant cocharacter, under the identification $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+_\Gamma \simeq B(G)_{\mathrm{un}}$ the elements $\overline{\nu} \in B(G,\mu)_{\mathrm{un}}$ correspond to W_G-orbits of the possible non-zero weights in $V_\mu|_{\hat{G}^\Gamma}$.*

Let's study this now more carefully. For $b \in B(G)$, we want to use the above discussion to understand the fiber of the map:

$$i : B(T) \to B(G)_{\mathrm{un}} \subset B(G)$$

Recall that, given $b \in B(G)$, since G is quasi-split the σ-centralizer J_b is an extended pure inner form (in the sense of [Kot97b, Section 5.2]) of a Levi subgroup M_b of G [Kot97b, Section 6.2], which is the centralizer of the slope homomorphism ν_b of b. We make the following definition.

Definition 2.2.10. For $b \in B(G)$, we let W_{M_b} denote the relative Weyl group of M_b and set $W_b := W_G/W_{M_b}$. We will fix a set of representatives $w \in W_G$ of minimal length, as in [BZ77, Section 2.11], and abuse notation by writing w for both the representative and the corresponding element.

When combining the above discussion with Lemma 2.2.3, we can deduce the following Corollary.

Corollary 2.2.11. *For fixed $b \in B(G)_{\mathrm{un}}$, the fiber $i^{-1}(b)$ has a unique element, denoted b_T, whose κ-invariant lies in $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+_\Gamma$. Moreover, we have an equality $i^{-1}(b) = \{w(b_T) \mid w \in W_b\}$.*

Now, given a parabolic P with Levi factor M, the element $b \in B(G)$ admits a reduction to a Levi subgroup M if and only if the parabolic $P \cap M_b$ of M_b transfers to a parabolic of J_b under the inner twisting (apply [CFS21, Pages 13, 28] to the basic reduction of b to M_b). We record this specialized to the case of the Borel for future use.

Lemma 2.2.12. *An element $b \in B(G)$ lies in $B(G)_{\mathrm{un}}$ if and only if $B \cap M_b$ defines, via the inner twisting, a Borel subgroup of J_b.*

Remark 2.2.13. We note that, when $b \in B(G)_{\mathrm{un}}$, we have an isomorphism $M_b \simeq J_b$ because J_b will be a quasi-split inner form of M_b.

We will end with an important observation about the map $(-)_\Gamma : \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma \to \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$ from orbits to coinvariants. Let $\tilde{\mathscr{J}}$ (resp. $\mathscr{J}$) denote the vertices of the absolute (resp. relative) Dynkin diagram of G. For $i \in \tilde{\mathscr{J}}$, we write $\tilde{\alpha}_i \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ for the corresponding simple absolute coroot. We recall that Γ permutes the $\tilde{\alpha}_i$, and the orbits under Γ are in bijection with elements of $\mathscr{J}$; namely, the average over the orbit is the (reduced) positive coroot corresponding to $i \in \mathscr{J}$. Therefore, for each $i \in \mathscr{J}$, we obtain an element $\alpha_i \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$ given by the common image of the elements in the orbit corresponding to $i \in \mathscr{J}$ under the map $(-)_\Gamma$. This allows us to make the following definition.

Definition 2.2.14. We denote the group of coinvariants by $\Lambda_{G,B} := \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$, and set $\Lambda^{pos}_{G,B}$ to be the semi-group spanned by the elements $\alpha_i \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$ corresponding to the Γ-orbit of coroots indexed by $i \in \mathscr{J}$.

Now we introduce the rest of the notation motivated by the discussion above.

- We let $\Lambda^+_G := \mathbb{X}_*(A)^+$ (resp. $\hat{\Lambda}^+_G := \mathbb{X}^*(A)^+$) be the semi-group of dominant cocharacters (resp. characters), viewed as elements of the positive Weyl chamber in $\Lambda_G := \mathbb{X}_*(A) = \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^\Gamma$ (resp. $\hat{\Lambda}_G = \mathbb{X}^*(A) = \mathbb{X}^*(T_{\overline{\mathbb{Q}}_p})^\Gamma$) defined by the choice of Borel.
- To a dominant character $\hat{\lambda} \in \hat{\Lambda}^+_G$, we get an associated highest weight Weyl G-module, denoted $\mathscr{V}^{\hat{\lambda}}$. It has a fixed highest weight vector $v^{\hat{\lambda}} \in \mathscr{V}^{\hat{\lambda}}$, and,

given a pair of such weights $\hat{\lambda}_1, \hat{\lambda}_2$, there is a canonical map

$$\mathscr{V}^{\hat{\lambda}_1+\hat{\lambda}_2} \to \mathscr{V}^{\hat{\lambda}_1} \otimes \mathscr{V}^{\hat{\lambda}}$$

which takes $v^{\hat{\lambda}_1+\hat{\lambda}_2}$ to $v^{\hat{\lambda}_1} \otimes v^{\hat{\lambda}_2}$.

- Let $\mathscr{J}$ be the set of vertices of the relative Dynkin diagram of $G/\mathbb{Q}_p$. For each $i \in \mathscr{J}$, we denote the corresponding element in the coinvariant lattice by $\alpha_i \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$, as in Definition 2.2.14. We warn the reader that this is different then the Γ-orbits defined by the (reduced) simple roots corresponding to $i \in \mathscr{J}$, these we will denote by $\alpha_{i,A} \in X_*(A) \subset X_*(T_{\overline{\mathbb{Q}}_p})^\Gamma$, and will be sum over the elements in the Galois orbit associated to α_i. On the other hand, in the root lattice $\hat{\Lambda}_G$ we will only be interested in the (reduced) simple positive roots corresponding to $i \in \mathscr{J}$, and so we just write this as $\hat{\alpha}_i \in \hat{\Lambda}_G$.

- We consider the natural pairing

$$\langle -,- \rangle : \hat{\Lambda}_G \times \Lambda_{G,B} \to \mathbb{Z}$$

given by the identifications $\hat{\Lambda}_G = \mathbb{X}^*(A) \simeq \mathbb{X}^*(T_{\overline{\mathbb{Q}}_p})^\Gamma$ and $\Lambda_{G,B} = \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$.

- Using the assumption on the derived group of G being simply connected, we define a set of fundamental weights $\hat{\varpi}_i \in \hat{\Lambda}_G^+$, non-uniquely characterized by the property that $\langle \hat{\varpi}_i, \alpha_j \rangle = \delta_{ij}$. Namely, this can be defined by taking the sum over the Galois orbits of fundamental weights in $\mathbb{X}^*(T_{\overline{\mathbb{Q}}_p})$.

- We will regularly use the natural quotient map

$$(-)_\Gamma : \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \to \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$$

as well as the map

$$(-)^\Gamma : \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \to \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$$

from cocharacters to their Galois orbits. We note that $(-)_\Gamma$ factorizes over $(-)^\Gamma$ as a map of sets.

- For a geometric dominant cocharacter $\mu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ with reflex field E, we write V_μ for the natural representation of $W_E \ltimes \hat{G}$ of highest weight μ, as in [Kot97a, Lemma 2.1.2]. We also write $V_{\mu^\Gamma} \in \mathrm{Rep}_\Lambda({}^L G)$ for the induction of this representation to ${}^L G$, which only depends on the associated Γ-orbit μ^Γ of μ.

- For $b \in B(G)$, we let J_b be the extended pure inner form of M_b considered above. If $b \in B(G)_{\mathrm{un}}$, we write B_b for the Borel defined by $B \cap M_b$ under the inner twisting, as in Lemma 2.2.12.
- Given $b \in B(G)_{\mathrm{un}}$, we note that, by Lemma 2.2.3, there exists a unique element $b_T \in B(T)$ with dominant slope homomorphism with respect to the choice of Borel. We refer to this as the dominant reduction. Similarly, we write b_T^- for the unique element with anti-dominant slopes and will refer to this as the HN-dominant reduction of b (Recall that there is a minus sign when comparing isocrystal slopes and HN-slopes).
- We set $W_b := W_G/W_{M_b}$, where W_{M_b} (resp. W_G) is the relative Weyl group of M_b (resp. G). We identify $w \in W_b$ with a representative of minimal length in W_G throughout.
- We will write $\hat{\rho}$ for the half sum of all positive roots.

2.3 Geometric Local Class Field Theory

2.3.1 Hecke Eigensheaves for Tori

In this section, we want to talk about geometric local class field theory. Namely, given a torus T with L-parameter

$$\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\Lambda)$$

where ${}^L T$ denotes the Langlands dual group of T, we want to construct a Hecke eigensheaf, denoted $\mathscr{S}_{\phi_T}$, on the moduli stack Bun_T. We recall that, for a representation $V \in \mathrm{Rep}_\Lambda({}^L G^I)$ of I-copies of the L-group of G for some finite index set I, a Hecke operator is a map defined by the correspondence

$$\begin{tikzcd} & \mathrm{Hck}_G^I \arrow[dl, "h^{\leftarrow} \times \pi"'] \arrow[dr, "h^{\rightarrow}"] & \\ \mathrm{Bun}_G \times \mathrm{Div}^I & & \mathrm{Bun}_G \end{tikzcd}$$

where Hck_G^I is the functor that parametrizes, for $S \to \mathrm{Div}^I$ defining a tuple of Cartier divisors in the relative Fargues-Fontaine X_S over S, corresponding to characteristic 0 untilts $S_i^\sharp$ for $i \in I$ of S, a pair of G-torsors $\mathscr{E}_1$, $\mathscr{E}_2$ together with an isomorphism

$$\beta : \mathscr{E}_1|_{X_S \setminus \bigcup_{i\in I} S_i^\sharp} \xrightarrow{\simeq} \mathscr{E}_2|_{X_S \setminus \bigcup_{i\in I} S_i^\sharp}$$

where $h^{\rightarrow}((\mathscr{E}_1, \mathscr{E}_2, (S_i^\sharp)_{i\in I})) = \mathscr{E}_1$ and $(h^{\leftarrow} \times \pi)((\mathscr{E}_1, \mathscr{E}_2, \beta, (S_i^\sharp)_{i\in I})) = (\mathscr{E}_2, (S_i^\sharp)_{i\in I})$. For an algebraic representation $V \in \mathrm{Rep}_\Lambda({}^LG^I)$, the geometric Satake correspondence of Fargues-Scholze [FS21, Chapters VI, IX.2] furnishes a sheaf $\mathscr{S}_V \in \mathrm{D}_\blacksquare(\mathrm{Hck}, \Lambda)$. Using this, we can define the Hecke operator as

$$T_V : \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \to \mathrm{D}_\blacksquare(\mathrm{Bun}_G \times \mathrm{Div}^I, \Lambda)$$
$$\mathscr{F} \mapsto (h^{\leftarrow} \times \pi)_\natural((h^{\rightarrow})^*(\mathscr{F}) \otimes \mathscr{S}_V)$$

where $(h^{\leftarrow} \times \pi)_\natural$ is the natural push-forward. I.e the left adjoint to the restriction functor in the category of solid Λ-sheaves [FS21, Proposition VII.3.1]. This induces a functor

$$T_V : \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \to \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{BW^I_{\mathbb{Q}_p}}$$

via a version of Drinfeld's Lemma [FS21, Theorem I.7.2, Proposition IX.2.1, Corollary IX.2.3]. When $\Lambda = \overline{\mathbb{F}}_\ell$, this is essentially the statement that Λ-valued local systems on Div^I are equivalent to continuous representations of $W^I_{\mathbb{Q}_p}$ on finite projective Λ-modules [FS21, Proposition VI.9.2]. In this case, we will freely pass between this perspective of local systems and $W^I_{\mathbb{Q}_p}$-representations.

With this in hand, we can define what it means for a sheaf on Bun_G to be a Hecke eigensheaf.

Definition 2.3.1. Given a continuous L-parameter $\phi : W_{\mathbb{Q}_p} \to {}^LG(\Lambda)$, we say a sheaf $\mathscr{S}_\phi \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ is a Hecke eigensheaf with eigenvalue ϕ if, for all $V \in \mathrm{Rep}_\Lambda({}^LG^I)$ with associated map $r_V : {}^LG^I \to \mathrm{GL}(V)$, we are given isomorphisms

$$\eta_{V,I} : T_V(\mathscr{S}_\phi) \simeq \mathscr{S}_\phi \boxtimes r_V \circ \phi$$

of sheaves in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{BW^I_{\mathbb{Q}_p}}$, that are natural in I and V, and compatible with compositions and exterior tensor products in V. We will similarly say that $\mathscr{S}_\phi$ is a weak eigensheaf with eigenvalue ϕ if we only know the existence of these isomorphisms.

Remark 2.3.2. We recall that Hecke operators are monoidal and functorial in (V, I). In particular, given two representations $V, W \in \mathrm{Rep}_\Lambda({}^LG)$, we have a natural isomorphism

$$(T_V \times \mathrm{id})(T_W)(\cdot)|_\Delta \simeq T_{V\otimes W}(\cdot)$$

where $\Delta : \mathrm{Div}^1 \to (\mathrm{Div}^1)^2$ is the diagonal map. The compatibilities for the isomorphisms $\eta_{V,I}$ are defined with respect to such isomorphisms.

Now let's elucidate what this means for tori. Recall that an irreducible representation of ${}^LT^I$ is parametrized by a tuple of Galois orbits $(\nu_i)_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma)^I$. Similarly, one has a decomposition

$$\mathrm{Hck}^I_T = \bigsqcup_{(\nu_i)_{i\in I}\in(\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma)^I} \mathrm{Hck}^I_{T,(\nu_i)_{i\in I}}$$

of Hck^I_T into open and closed substacks, where $\mathrm{Hck}^I_{T,(\nu_i)_{i\in I}}$ parametrizes a modification $\mathscr{E}_1 \dashrightarrow \mathscr{E}_2$ of meromorphy given by the Galois orbit ν_i over the I Cartier divisors in Div^I. If one lets $\mathscr{S}_{(\nu_i)_{i\in I}}$ be the sheaf defined by the representation of ${}^LT^I$ corresponding to $(\nu_i)_{i\in I}$, then this sheaf is simply the constant sheaf supported on the component $\mathrm{Hck}^I_{T,(\nu_i)_{i\in I}}$. Therefore, for studying the Hecke operator $T_{(\nu_i)_{i\in I}}$, we can restrict the Hecke correspondence to the diagram:

$$\begin{tikzcd} & \mathrm{Hck}^I_{T,(\nu_i)_{i\in I}} \arrow[dl, "h^{\leftarrow}_{(\nu_i)_{i\in I}}\times\pi"'] \arrow[dr, "h^{\rightarrow}_{(\nu_i)_{i\in I}}"] & \\ \mathrm{Bun}_T \times \mathrm{Div}^I & & \mathrm{Bun}_T \end{tikzcd}$$

Let E_{ν_i} denote the reflex field of the Γ-orbit $\nu_i \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$. We can consider the following base-change of Div^I:

$$\mathrm{Div}^I_{E_{(\nu_i)_{i\in I}}} := \prod_{i\in I} \mathrm{Div}^1_{E_{\nu_i}}$$

We note that, since a modification of T-bundles is uniquely determined by the locus of meromorphy, we have an isomorphism $\mathrm{Hck}^I_{T,(\nu_i)_{i\in I}} \simeq \mathrm{Bun}_T \times \mathrm{Div}^I_{E_{(\nu_i)_{i\in I}}}$. Under this identification, we have a map

$$h^{\rightarrow}_{(\nu_i)_{i\in I}} : \mathrm{Bun}_T \times \mathrm{Div}^I_{E_{(\nu_i)_{i\in I}}} \to \mathrm{Bun}_T$$

where, given $(\mathscr{F}_T, (D_i)_{i\in I})$, we denote the resulting T-bundle under applying this map as $\mathscr{F}_T(\sum_{i\in I} -\nu_i D_i)$. We note that the isomorphism class of this bundle is only determined by the image of $\nu_{i\Gamma}$ in the coinvariant lattice, and this will be important in the next section. The map $h^{\leftarrow}_{(\nu_i)_{i\in I}} \times \pi$ is defined by the natural finite étale morphism $q_{(\nu_i)_{i\in I}} : \mathrm{Div}^I_{E_{(\nu_i)_{i\in I}}} \to \mathrm{Div}^I$, and pushing forward corresponds to inducing the $\prod_{i\in I} W_{E_{\nu_i}}$ action to $\prod_{i\in I} W_{\mathbb{Q}_p}$.

Now, via local class field theory, there is a character $\chi : T(\mathbb{Q}_p) \to \Lambda^*$ attached to ϕ_T. Moreover, each connected component Bun_T^{ν} for varying $\nu \in B(T) \simeq \Lambda_{G,B}$ is isomorphic to the classifying stack $[*/\underline{T(\mathbb{Q}_p)}]$. As a consequence, we may interpret χ as a sheaf on the connected components $j_{\overline{\nu}} : \mathrm{Bun}_T^{\overline{\nu}} \to \mathrm{Bun}_T$ for $\overline{\nu} \in B(T)$. One might hope that considering

$$\mathscr{S}_{\phi_T} := \bigoplus_{\overline{\nu}\in B(T)} j_{\overline{\nu}!}(\chi)$$

the sheaf on Bun_T whose restriction to each connected component is equal to χ gives rise to the desired Hecke eigensheaf. This is indeed the case. In particular, via the realization of local class field theory in the torsion of Lubin-Tate formal groups, we have the following proposition.

Proposition 2.3.3. *[Far16, Section 9.2],[Zou22] The sheaf $\mathscr{S}_{\phi_T}$ is an eigensheaf with eigenvalue ϕ_T. In particular, for all $(\nu_i)_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma)^I$, we have an isomorphism*

$$T_{(\nu_i)_{i\in I}}(\mathscr{S}_{\phi_T}) \simeq \boxtimes_{i\in I} \nu_i \circ \phi_T \otimes \mathscr{S}_{\phi_T}$$

of objects in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_T, \Lambda)^{BW^I_{\mathbb{Q}_p}}$. More precisely, if $\tilde{\nu}_i$ is a representative of the Γ-orbit of ν_i for all $i \in I$, we have an isomorphism

$$(h^{\rightarrow}_{(\nu_i)_{i\in I}})^*(\mathscr{S}_{\phi_T}) \simeq \boxtimes_{i\in I} \tilde{\nu}_i \circ \phi_T|_{W_{E_{\nu_i}}} \otimes \mathscr{S}_{\phi_T}$$

which after applying $q_{(\nu_i)_{i\in I}*}$ gives rise to the previous identification of induced representations.

A special role will be played by the eigensheaf attached to the parameter

$$\hat{\rho} \circ |\cdot| : W_{\mathbb{Q}_p} \to {}^LT(\Lambda)$$

where we recall that $\hat{\rho}$ denotes the half sum of all positive roots with respect to the choice of Borel and $|\cdot| : W_{\mathbb{Q}_p} \to \Lambda^*$ is the norm character. We note that the value of this sheaf on each connected component is given by the representation $\delta_B^{1/2}$, where δ_B denotes the modulus character defined by B. This leads to the following definition.

Definition 2.3.4. We let $\Delta_B^{1/2}$ be the eigensheaf on Bun_T attached to the parameter

$$\hat{\rho} \circ |\cdot| : W_{\mathbb{Q}_p} \to {}^L T(\Lambda)$$

via Proposition 2.3.3. Similarly, we write Δ_B for the eigensheaf attached to $2\hat{\rho} \circ |\cdot|$, where the stalks of this sheaf are given by δ_B.

The key point is that (up to shifts) the pullback of this eigensheaf to the moduli stack Bun_B gives rise to a sheaf which we will denote by $\mathrm{IC}_{\mathrm{Bun}_B}$. We will see later that this sheaf is Verdier self-dual on Bun_B and therefore tensoring by it will give rise to the morally correct definition of the Eisenstein functor. We note that, given a parameter ϕ_T with associated eigensheaf $\mathscr{S}_{\phi_T}$, the tensor product $\mathscr{S}_{\phi_T} \otimes \Delta_B^{1/2}$ will be the eigensheaf attached to the tensor product $\phi_T \otimes \hat{\rho} \circ |\cdot|$ of L-parameters (the L-parameter whose value is equal to the product of the parameters). It therefore follows from Proposition 2.3.3 that the following is true.

Corollary 2.3.5. *For all $(\nu_i)_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma)^I$, we have an isomorphism*

$$T_{(\nu_i)_{i \in I}}(\mathscr{S}_{\phi_T} \otimes \Delta_B^{1/2}) \simeq \boxtimes_{i \in I}(\nu_i \circ \phi_T)(\langle \hat{\rho}, \nu_{i\Gamma} \rangle) \otimes (\mathscr{S}_{\phi_T} \otimes \Delta_B^{1/2})$$

of objects in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_T, \Lambda)^{BW_{\mathbb{Q}_p}^I}$.

Remark 2.3.6. Note that, for any representative $\tilde{\nu} \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ of the orbit $\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$, we have an equality: $\langle \hat{\rho}, \tilde{\nu} \rangle = \langle \hat{\rho}, \nu_\Gamma \rangle$.

Now we discuss the various conditions that we will impose on our parameter ϕ_T, as well as discuss their relationship with the irreducibility of principal series through various examples.

2.3.2 Genericity, Weak Normalized Regularity, and the Irreducibility of Principal Series

Consider the functor

$$R\Gamma(W_{\mathbb{Q}_p}, -) : \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_T, \Lambda)^{BW_{\mathbb{Q}_p}} \to \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_T, \Lambda)$$

given by taking continuous cohomology with respect to $W_{\mathbb{Q}_p}$. As we will see later, computing the Eisenstein functor applied to the eigensheaf $\mathscr{S}_{\phi_T}$ will reduce to computing the values of $R\Gamma(W_{\mathbb{Q}_p}, (h_\nu^{\leftarrow})^*(\mathscr{S}_{\phi_T}))$ and $R\Gamma(W_{\mathbb{Q}_p}, (h_\nu^{\leftarrow})^*(\mathscr{S}_{\phi_T} \otimes \Delta_B^{1/2}))$ for $\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$. In this note, we will want to restrict to the simplest case where these contributions all vanish. The exact conditions we will need are as follows.

Condition/Definition 2.3.7. *Given a parameter $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\Lambda)$, we impose the following conditions on ϕ_T in what follows.*

1. *For all Γ-orbits $\alpha \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$ of simple coroots in $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$, the Galois cohomology complex $R\Gamma(W_{\mathbb{Q}_p}, \alpha \circ \phi_T)$ is trivial.*
2. *For all Γ-orbits $\alpha \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$ of coroots in $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$, the Galois cohomology complex $R\Gamma(W_{\mathbb{Q}_p}, \alpha \circ \phi_T)$ is trivial.*
3. *If χ is the character attached to ϕ_T under local class field theory. We have, for all $w \neq 1$ in the relative Weyl group W_G of G, that*

$$\chi \otimes \delta_B \not\simeq (\chi \otimes \delta_B^{-1/2})^w$$

If ϕ_T satisfies (1) we say that it is weakly generic, and if it satisfies (2) then we say it is generic. If it satisfies (2)-(3) we say that it is weakly normalized regular.

These conditions are related to the irreducibility of the induction $i_B^G(\chi)$. To explain this, let's translate all these conditions to the character χ. Let $E/\mathbb{Q}_p$ denote the splitting field of G. Then the action of $W_{\mathbb{Q}_p}$ on $\hat{G}$ factors through $W_{\mathbb{Q}_p}/W_E$. Given $\alpha \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$ a Γ-orbit of coroots with reflex field E_α, local class field theory [Lan97] gives us a map

$$E_\alpha^* \to T(E_\alpha)$$

attached to $\tilde{\alpha} \circ \phi_T|_{W_{E_\alpha}}$, for a representative $\tilde{\alpha}$ in the Γ-orbit of α. If we post-compose with the norm map $\mathrm{Nm}_{E_\alpha/\mathbb{Q}_p}$ then we get a map

$$E_\alpha^* \to T(\mathbb{Q}_p)$$

which only depends on the Galois orbit α. We further precompose with the norm map $\mathrm{Nm}_{E/E_\alpha} : E^* \to E_\alpha^*$, giving a character:

$$E^* \to T(\mathbb{Q}_p)$$

We write $\chi_\alpha : E^* \to \overline{\mathbb{Q}}_\ell^*$ for the precomposition of χ with this map. Now, consider the complex $R\Gamma(W_{\mathbb{Q}_p}, \alpha \circ \phi_T)$, where ϕ_T is the parameter attached to χ. It follows by Schapiro's lemma that we have an isomorphism:

$$R\Gamma(W_{E_\alpha}, \tilde{\alpha} \circ \phi_T|_{W_{E_\alpha}}) \simeq R\Gamma(W_{\mathbb{Q}_p}, \alpha \circ \phi_T)$$

Applying local Tate duality and using that the Euler-Poincaré characteristic of $R\Gamma(W_{E_\alpha}, \tilde{\alpha} \circ \phi_T|_{W_{E_\alpha}})$ is 0, we see that this is equivalent to insisting that $\tilde{\alpha} \circ \phi_T|_{W_{E_\alpha}}$ is not the trivial representation or the cyclotomic character (1). From here, by using compatibility of local class field theory with restriction and the fact that $\tilde{\alpha} \circ \phi_T$ is an extension from W_E, we can see that the above conditions on ϕ_T imply the following conditions on χ.

Condition/Definition 2.3.8. *Given a smooth character $\chi : T(\mathbb{Q}_p) \to \Lambda^*$ consider the following conditions on χ.*

1. *For all Γ-orbits $\alpha \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$ of positive coroots in $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$, the character χ_α is not isomorphic to the trivial representation $\mathbf{1}$ or $|\cdot|_E^{\pm 1}$, where $|\cdot|_E$ is the norm character on E the splitting field of G.*

2. *For all $w \in W_G$ non-trivial, we have that*
$$\chi \otimes \delta_B^{1/2} \not\simeq (\chi \otimes \delta_B^{-1/2})^w$$

We say that χ is generic if (1) holds, and that it is weakly normalized regular if (1)-(2) hold.

We now illustrate how this condition is related to irreducibility of $i_B^G(\chi)$ in some examples. We will assume that $\Lambda = \overline{\mathbb{Q}}_\ell$ in all of the examples for simplicity.

Example 2.3.9. ($G = \mathrm{GL}_2$) We can write $\chi := \chi_1 \otimes \chi_2$ for $\chi_i : \mathbb{Q}_p^* \to \overline{\mathbb{Q}}_\ell^*$ smooth characters and $i = 1, 2$. We see that χ being generic implies that $\chi_1\chi_2^{-1} \not\simeq \mathbf{1}$ and $\chi_1\chi_2^{-1} \not\simeq |\cdot|^{\pm 1}$, and this latter condition guarantees that the normalized parabolic induction $i_B^{\mathrm{GL}_2}(\chi_1 \otimes \chi_2)$ is irreducible. Let's also look at Condition (2) in this case. Suppose it fails, then we have an isomorphism:
$$\chi_1(t_1)\chi_2(t_2)|t_1t_2^{-1}|^{1/2} \simeq \chi_1(t_2)\chi_2(t_1)|t_1t_2^{-1}|^{1/2}$$
Evaluating at $(t_1, t_2) = (t, 1)$ this would imply that $\chi_1\chi_2^{-1} \simeq \mathbf{1}$, which would contradict χ being generic.

In particular, we see that χ being generic is enough to guarantee irreducibility and weak normalized regularity for GL_2. In fact, this is a more general phenomenon.

Lemma 2.3.10. *If $G = \mathrm{GL}_n$ then χ being generic implies that it is weakly normalized regular.*

Proof. (Sketch) We write the character as a product $\chi := \chi_1 \otimes \chi_2 \otimes \cdots \otimes \cdots \otimes \chi_n$ of characters $\chi_i : \mathbb{Q}_p^* \to \Lambda^*$, and write $(t_1, \ldots, t_n)$ for the natural coordinates on $T(\mathbb{Q}_p) \simeq (\mathbb{Q}_p^*)^n$. We have an equality:

$$\chi \otimes \delta_B^{1/2} = \prod_{i=1}^{n} \chi_i(t_i) \otimes |t_i|^{\frac{n-1}{2}-(i-1)}$$

We visualize the coordinates as a set of vertices of a graph:

$$(t_1) \leftrightarrow (t_2) \leftrightarrow \cdots \leftrightarrow (t_{n-1}) \leftrightarrow (t_n)$$

Then this has an axis of symmetry between $t_{n/2}$ and $t_{n/2+1}$ if n is even, and an axis of symmetry going through $t_{(n+1)/2}$ if n is odd. The element w corresponds to a permutation σ of the vertices of the graph. Now suppose that σ crosses the line of symmetry. If n is odd we assume that it sends $t_{(n-1)/2}$ to $t_{(n+1)/2}$ then by evaluating the equality

$$\chi \otimes \delta_B^{1/2} \simeq \chi \otimes (\delta_B^{-1/2})^w$$

at $(1, \ldots, 1, t_{(n-1)/2} = t, 1, \ldots, 1)$ it reduces to

$$\chi_{t_{(n-1)/2}}(t)|t| \simeq \chi_{t_{(n+1)/2}}(t)$$

which implies

$$\chi_{t_{(n-1)/2}}(t)^{-1} \chi_{t_{(n+1)/2}}(t) \simeq |t|$$

contradicting χ being generic. Similarly, if n is even and the permutation sends $t_{n/2}$ to $t_{n/2+1}$ then evaluating at $(1, \ldots, 1, t_{n/2} = t, 1, \ldots, 1)$ the equality becomes

$$\chi_{t_{n/2}} \chi_{t_{n/2+1}}^{-1}(t) \simeq \mathbf{1}$$

which again contradicts χ being generic. Here the point is that the power of the norm character appearing when evaluating at $(1, \ldots, 1, t = t_i, 1, \ldots, 1)$ is given by the distance of t_i from the reflection of t_{i+1} across the line of symmetry. In general, we can consider the cycle which is closest to the line of symmetry, suppose it ends in a permutation $t_i \to t_{i+1}$. If we just evaluated at $(1, \ldots, 1, t = t_i, 1, \ldots, 1)$ then we would get that $\chi_i(t)\chi_{i+1}(t)^{-1}$ is equal to a very large power of the norm character. However, since we choose the permutation to be as close as possible to the axis of symmetry, the permutation will fix all the vertices from the axis of symmetry to t_{i+1} reflected across it. Therefore, we are free to also evaluate at a power of t on these invariant coordinates, since the characters χ_j corresponding to an invariant coordinate t_j on either side of the equation will cancel. This allows us to reduce the power of the norm character appearing to 1 or 0, and then we see that this contradicts χ being generic again. $\square$

Remark 2.3.11. Note that if $\Lambda = \overline{\mathbb{Q}}_\ell$, we could have replaced the second part of the above argument by simply evaluating the permutation σ on one of the coordinates t_i that σ leaves fixed. This would give a relationship of the form $|\cdot|^n \simeq \mathbf{1}$, which is impossible with rational coefficients. With modular coefficients, this is impossible under a banal hypothesis on the prime ℓ.

In general, genericity does not always imply weak normalized regularity. In particular, Condition 2.3.7 (3) seems to be related to the irreducibility of some unitary principal series representations.

Example 2.3.12. ($G = \mathrm{SL}_2$) In this case, $\chi : \mathbb{Q}_p^* \to \overline{\mathbb{Q}}_\ell^*$ is a character of $\mathbb{Q}_p^*$. The induction $i_B^G(\chi)$ will be irreducible if and only if $\chi \not\simeq |\cdot|^{\pm 1}$ and $\chi^2 \not\simeq \mathbf{1}$. The condition that $\chi \not\simeq |\cdot|^{\pm 1}$ is guaranteed by χ being generic but the condition $\chi^2 \not\simeq \mathbf{1}$ is not. However, we note that χ being weakly normalized regular enforces the additional condition that $\chi^2 \not\simeq \mathbf{1}$ guaranteeing irreducibility in this case.

We notice in the previous example that the Condition that

$$\chi \otimes \delta_B^{1/2} \not\simeq (\chi \otimes \delta_B^{-1/2})^w$$

for $w \in W_G$ non-trivial guaranteed that the character χ was regular in the sense that $\chi \not\simeq \chi^w$ for all $w \in W_G$. This is the type of condition that guarantees irreducibility of the unitary principal series representations. This is always true for $w_0 \in W_G$ the element of longest length.

Lemma 2.3.13. *If χ is weakly normalized regular then $\chi \not\simeq \chi^{w_0}$ for the element of longest length $w_0 \in W_G$.*

Proof. We note that the relationship

$$\chi \otimes \delta_B^{1/2} \simeq (\chi \otimes \delta_B^{-1/2})^w$$

becomes precisely the relationship

$$\chi \simeq \chi^w$$

for $w = w_0$, using the product expansion of the modulus character and the fact that w_0 sends each (reduced) root $\hat{\alpha}$ to $-\hat{\alpha}$. $\square$

We now come to the last condition on our parameter ϕ_T. In particular, the exact condition we will need is as follows.

Definition 2.3.14. For a toral parameter $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\Lambda)$ and a geometric dominant cocharacter μ, we say ϕ_T is strongly μ-regular if the Galois cohomology complexes

$$R\Gamma(W_{\mathbb{Q}_p}, (\nu - \nu')^{\Gamma} \circ \phi_T)$$

are trivial for ν, ν' distinct weights of the highest weight representation of $\hat{G}$ of highest weight μ_k for all $k = 1, \ldots, n$.

To give some flavor for this condition, we prove the following Proposition.

Lemma 2.3.15. *Suppose that $G = \mathrm{GL}_n$ and let $\mu = (1, \ldots, 0)$ denote the standard character then strong μ-regularity is implied by generic.*

Proof. We note that since μ is minuscule the standard representation V_μ has weights given by Weyl group orbits of the highest weight. From here, it easily follows that the difference of distinct weights of V_μ in $\hat{T}$ are given by coroots of G and the claim follows. $\square$

As mentioned in the introduction, weak normalized regularity and μ-regularity for a geometric dominant cocharacter which isn't fixed under any element of W_G, will imply the existence of isomorphisms $i_{\chi,w} : i_B^G(\chi) \xrightarrow{\simeq} i_B^G(\chi^w)$ for all $w \in W_G$ once the theory of geometric Eisenstein series has been developed. Similarly, we will show the following.

Proposition 2.3.16. *(Proposition A.1.3) If $\chi : T(\mathbb{Q}_p) \to \overline{\mathbb{Q}}_\ell^*$ is a generic character then, for all $w \in W_G$, we have an isomorphism $i_B^G(\chi) \simeq i_B^G(\chi^w)$.*

In fact, for regular characters χ, the existence of such isomorphisms is equivalent to irreducibility. More generally, we show that such isomorphisms exist assuming χ is generic (Proposition A.1.3), but this does not guarantee irreducibility of certain unitary principal series as seen when $G = \mathrm{SL}_2$.

Using the Langlands classification, the proof of this proposition will essentially reduce to a calculation of reducibility points in rank 1, where it reduces to Example 2.3.12 and the following example, which illustrates the behavior of our conditions in the non-split case.

Example 2.3.17. ($G = \mathrm{U}_3/E$) Let $E/\mathbb{Q}_p$ be a quadratic extension. We write $\overline{(-)}$ for the non-trivial automorphism of E over $\mathbb{Q}_p$. If e_1, e_2, e_3 is the standard basis for the cocharacter lattice $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ then $\overline{(-)}$ acts by

$$e_1 \longleftrightarrow -e_3$$

$$e_2 \longleftrightarrow -e_2$$

It follows that the simple coroot $\alpha_1 := e_1 - e_2$ is sent to the simple coroot $\alpha_2 := e_2 - e_3$ under $\overline{(-)}$. Thus, the Γ-orbits of positive coroots in $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ are given by $\{\alpha_1, \alpha_2\}$ with reflex field E and $\alpha_1 + \alpha_2$ with reflex field $\mathbb{Q}_p$. Now recall that the maximal torus $T(\mathbb{Q}_p) \subset \mathrm{U}_3(\mathbb{Q}_p)$ is isomorphic to $E^* \times E^1$, via the embedding

$$E^* \times E^1 \to \mathrm{U}_3(\mathbb{Q}_p)$$

$$t \mapsto \begin{pmatrix} t & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & \bar{t}^{-1} \end{pmatrix}$$

where E^1 denotes the set of norm 1 elements. Then if we write the character $\chi(t,s) : E^* \times E^1 \to \Lambda^*$ as $\chi(t,s) = \chi_1(t)\chi_2(ts\bar{t}^{-1})$ the reducibility of $i_B^G(\chi)$ depends solely on χ_1, as in [Rog90, Page 173], where here it reduces to the analogous question for SU_3, and there the reducibility points were studied in [Key84, Section 7]. The induction $i_B^G(\chi)$ is reducible if and only if one of the following hold:

1. $\chi_1 = \eta |\cdot|_E^{\pm 1/2}$, where $\eta|_{\mathbb{Q}_p^*} = \eta_{E/\mathbb{Q}_p}$,
2. $\chi_1 = |\cdot|_E^{\pm 1}$,
3. $\chi_1|_{\mathbb{Q}_p^*}$ is trivial, but χ is not.

Here $|\cdot|_E$ is the norm character of E which is in particular the splitting field of G, and $\eta_{E/\mathbb{Q}_p} : \mathbb{Q}_p^* \to \Lambda^*$ is the unique quadratic character with kernel given by $\mathrm{Nm}_{E/\mathbb{Q}_p}(E^*)$. We note that that the cocharacters of $T(\mathbb{Q}_p)$ given by the Γ-orbits of positive roots are

$$\{\alpha_1, \alpha_2\} : E^* \to T(\mathbb{Q}_p) = E^* \times E^1$$

$$t \mapsto (t, t^{-1}\bar{t})$$

and

$$\{\alpha_1 + \alpha_2\} : E^* \to E^* \times E^1$$

$$t \mapsto (\mathrm{Nm}_{E/\mathbb{Q}_p}(t), 1)$$

By precomposing χ with the first character, we see that χ being generic implies $\chi_1 \not\simeq \mathbf{1}$ and $\chi_1 \not\simeq |\cdot|_E^{\pm 1}$, which implies reducibility point (2) cannot occur. By precomposing χ with the second character, we see that χ being generic implies that $\chi(\mathrm{Nm}_{E/\mathbb{Q}_p}(t)) \simeq \chi_1(\mathrm{Nm}_{E/\mathbb{Q}_p}(t)) \not\simeq \mathbf{1}$ and $\chi_1(\mathrm{Nm}_{E/\mathbb{Q}_p}(t)) \not\simeq |t|_E^{\pm 1}$. Note that if $\chi \simeq \eta|\cdot|_E^{\pm 1/2}$ then we have, for all $t \in E^*$, an isomorphism:

$$\chi(\mathrm{Nm}_{E/\mathbb{Q}_p}(t)) = \chi_1(t\bar{t}) \simeq \eta(\mathrm{Nm}_{E/\mathbb{Q}_p}(t))|\mathrm{Nm}_{E/\mathbb{Q}_p}(t)|_E^{\pm 1/2} \simeq |t|_E^{\pm 1}$$

Summarizing, we see again that χ being generic guarantees irreducibility of the two non-unitary inductions. Now, if $\chi|_{\mathbb{Q}_p^*}$ is trivial then we have that

$$\chi(\mathrm{Nm}_{E/\mathbb{Q}_p}(t)) = \chi_1(t\bar{t}) \simeq \mathbf{1}$$

Thus, we see χ being generic guarantees the irreducibility of all principal series. Moreover, χ being weakly normalized regular enforces the additional constraint that

$$\chi(t\bar{t}) \not\simeq \mathbf{1}$$

which we just saw follows from χ being generic, so weak normalized regularity follows from generic in this case.

The connection between genericity and irreducibility of non-unitary principal series fits in nicely with the general Langlands philosophy. In particular, since we are inducing from a Borel, we expect that a parameter should have monodromy if it arises as a constituent of the reducible induction of a non-unitary character. We saw in the above examples that this shouldn't occur when the parameter ϕ_T attached to χ is generic. We can analyze when a parameter comes from the semi-simplification of a parameter with non-trivial monodromy and relate this to genericity.

Lemma 2.3.18. *We let $\phi : \mathscr{L}_{\mathbb{Q}_p} \to {}^LG(\overline{\mathbb{Q}}_\ell)$ be an L-parameter. Suppose that $\phi : \mathscr{L}_{\mathbb{Q}_p} \to {}^LG(\overline{\mathbb{Q}}_\ell)$ has non-trivial monodromy, and that the semi-simplification ϕ^{ss} (See Assumption 2.7.5) factors through ${}^LT \to {}^LG$ via the natural embedding. If we write $\phi_T : W_{\mathbb{Q}_p} \to {}^LT(\overline{\mathbb{Q}}_\ell)$ for the parameter induced by ϕ^{ss} then ϕ_T is not generic.*

Proof. If ϕ has non-trivial monodromy and the semi-simplification factors through ${}^LT$, there exists a lift

$$\begin{tikzcd} & {}^LB(\overline{\mathbb{Q}}_\ell) \arrow[d] \\ W_{\mathbb{Q}_p} \arrow[ur, dashed, "\tilde{\phi}"] \arrow[r, "\phi_T"] & {}^LT(\overline{\mathbb{Q}}_\ell) \end{tikzcd}$$

of ϕ_T, which is not given by the inclusion ${}^LT(\overline{\mathbb{Q}}_\ell) \to {}^LB(\overline{\mathbb{Q}}_\ell)$. Such an extension implies that there exists a non-trivial class in $R\Gamma(W_{\mathbb{Q}_p}, \alpha \circ \phi_T)$, which would in turn imply ϕ_T is not generic. $\square$

We finish this section by deducing a geometric consequence of weak genericity that will be useful for studying how Eisenstein series interact with Verdier duality.

2.3.3 A Geometric Consequence of Weak Genericity

We work with torsion coefficeints $\Lambda = \overline{\mathbb{F}}_\ell$. Consider the following easy lemma.

Lemma 2.3.19. *Assume ϕ_T is weakly generic then it follows that, for all the Γ-orbits of simple positive coroots α, the complex $R\Gamma(W_{\mathbb{Q}_p}, \alpha \circ \phi_T(\langle \hat{\rho}, \alpha_\Gamma \rangle))$ is trivial.*

Proof. We note that, since α is a Γ-orbit of simple positive coroots, we have that $\langle \hat{\rho}, \alpha_\Gamma \rangle = 1$; therefore, we want to check that

$$R\Gamma(W_{\mathbb{Q}_p}, (\alpha \circ \phi_T)(1))$$

is trivial. By using local Tate-duality and that the Euler-Poincaré characteristic of this complex is 0, this is equivalent to checking that $(\alpha \circ \phi_T)(1)$ is not the trivial representation or the cyclotomic twist by (-1), which is equivalent to insisting that $(\alpha \circ \phi_T)^{-1}$ is not the trivial representation or the cyclotomic twist by (1). This is the same condition guaranteeing that the complex

$$R\Gamma(W_{\mathbb{Q}_p}, (\alpha \circ \phi_T)^{-1})$$

is trivial, which follows from weak genericity applied to the Γ-orbit of negative simple coroots $-\alpha$. $\square$

This has an important geometric consequence, related to the vanishing of certain Galois cohomology groups appearing in the moduli space of B-bundles and its compactifications. To explain this, let $\overline{\nu}$ be an element of $\Lambda_{G,B}^{pos} \setminus \{0\}$. We can write this as a linear combination $\sum_{i \in \mathscr{J}} n_i \alpha_i$ for positive integers n_i, where the α_i correspond to the Galois orbits of simple absolute roots as in Definition 2.2.14. Given such an α_i, we can consider the reflex field E_i of the associated Galois orbit, and define the following partially symmetrized curve:

$$\mathrm{Div}^{(\overline{\nu})} := \prod_{i \in \mathscr{J}} \mathrm{Div}_{E_i}^{(n_i)}$$

Points of this curve correspond to tuples of Cartier divisors D_i over E_i of degree n_i for all $i \in \mathscr{J}$. We can consider the map

$$h_{(\overline{\nu})}^{\rightarrow} : \mathrm{Bun}_T \times \mathrm{Div}^{(\overline{\nu})} \to \mathrm{Bun}_T$$

given by sending $(\mathscr{F}_T, (D_i)_{i\in \mathscr{J}})$ to $\mathscr{F}_T(\sum -\alpha_i \cdot D_i)$, where we are identifying α_i with its corresponding Γ-orbit.

This partially symmetrized mirror curve $\mathrm{Div}^{(\overline{\nu})}$ behaves a bit strangely if G is not split. To illustrate this, consider the following example.

Example 2.3.20. Let $G = \mathrm{U}_3$ be a unitary group in 3 variables attached to a quadratic extension $E/\mathbb{Q}_p$ and write α_1 and α_2 for the two absolute positive simple roots. We recall, as in Example 2.3.17, that the Galois group exchanges α_1 and α_2. Therefore, they both map to a unique element $\alpha \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$ which spans the lattice $\Lambda_{G,B}^{pos}$ by Definition. Consider the element $\overline{\nu} := 2\alpha \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$. We note that we have an equality $\mathrm{Div}^{(\overline{\nu})} = \mathrm{Div}_E^{(2)}$ in this case. The pre-image of 2α under the natural map $(-)_\Gamma : \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma \to \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$ consists of two elements: the Γ-orbit $\{2\alpha_1, 2\alpha_2\}$ with reflex field E and the Γ-orbit of $\{\alpha_1 + \alpha_2\}$ with reflex field $\mathbb{Q}_p$. We saw in the previous section that the space of modifications defined by the Γ-orbit $\{2\alpha_1, 2\alpha_2\}$ is given by $\mathrm{Bun}_T \times \mathrm{Div}_E^1$, correspondingly we have a natural map

$$\triangle_{\{2\alpha_1,2\alpha_2\}} : \mathrm{Div}_E^1 \xrightarrow{\triangle} \mathrm{Div}_E^2 \to \mathrm{Div}_E^{(2)}$$

given by the diagonal embedding composed with the quotient map. It is easy to check we have an equality $h_{\{2\alpha_1,2\alpha_2\}}^{\rightarrow}(-) = h_{\overline{\nu}}^{\rightarrow} \circ (\mathrm{id} \times \triangle_{\{2\alpha_1,2\alpha_2\}})$. Perhaps more interestingly, attached to the Γ-orbit $\{\alpha_1 + \alpha_2\}$, we have a twisted diagonal map

$$\triangle_{\{\alpha_1+\alpha_2\}} : \mathrm{Div}_{\mathbb{Q}_p}^1 \to \mathrm{Div}_E^{(2)}$$

given by sending a Cartier divisor D to its pre-image under the natural finite-étale covering $X_{S,E} \to X_S$ of Fargues-Fontaine curves induced by the extension $E/\mathbb{Q}_p$. By [Li-22, Proposition 1.5], this map defines a closed embedding whose image lies in the complement of the image of $\triangle_{\{2\alpha_1,2\alpha_2\}}$ in $\mathrm{Div}_E^{(2)}$, and we similarly see that we have a relationship $h_{(\overline{\nu})}^{\rightarrow} \circ (\mathrm{id} \times \triangle_{\{\alpha_1+\alpha_2\}})(-) = h_{\{\alpha_1+\alpha_2\}}^{\rightarrow}(-)$.

The previous example illustrates that we can can understand $\mathrm{Bun}_T \times \mathrm{Div}^{(\overline{\nu})}$ as realizing the set of all modifications specified by Galois orbits $\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$ such that $\nu_\Gamma = \overline{\nu}$. This is indeed a general phenomenon. In particular, using that

Γ permutes the simple absolute coroots of G, given any such ν with reflex field E_ν, we can define a twisted diagonal embedding

$$\Delta_\nu : \mathrm{Div}^1_{E_\nu} \to \mathrm{Div}^{(\overline{\nu})}$$

such that we have a relationship $h^{\rightarrow}_{\nu}(-) := h^{\rightarrow}_{\overline{\nu}} \circ (\mathrm{id} \times \triangle_\nu)(-)$. If $\overline{\nu} = \alpha_i$ for $i \in \mathscr{J}$ the map Δ_ν is an isomorphism for the unique Γ-orbit of simple coroots corresponding to α_i. Therefore, the pullback of $\mathscr{S}_{\phi_T}$ along $h^{\leftarrow}_{(\overline{\nu})}$ is isomorphic to $\mathscr{S}_{\phi_T} \boxtimes \tilde{\alpha}_i \circ \phi_T$ for a choice of representative $\tilde{\alpha}_i$ of the Γ-orbit corresponding to α_i. In general, recall that given a local system $\mathbb{L}$ and n a positive integer, we can consider the symmetric powers

$$\mathbb{L}^{(n)} := \pi_*(\boxtimes_{i=1}^n \mathbb{L})^{S_n}$$

where π denotes the push-forward along the S_n-torsor:

$$\pi : (\mathrm{Div}^1)^n \to \mathrm{Div}^{(n)}$$

Using this, we can define a local system on $\mathrm{Div}^{(\overline{\nu})}$ given by

$$E^{(\overline{\nu})}_{\phi_T} := \boxtimes_{i \in \mathscr{J}} E^{(n_i)}_{\phi_i}$$

where ϕ_i is the local system on $\mathrm{Div}^1_{E_i}$ corresponding to the character $\tilde{\alpha}_i \circ \phi_T|_{W_{E_i}}$ of W_{E_i} for E_i the reflex field of the Galois orbit corresponding to α_i. With this in hand, we can describe the pullback of $\mathscr{S}_{\phi_T}$ along $h^{\leftarrow}_{(\overline{\nu})}$ as the sheaf $E^{(\overline{\nu})}_{\phi_T} \boxtimes \mathscr{S}_{\phi_T}$ by using that Hecke operators are monoidal, and the natural compatibilities of the eigensheaf. We now state a key vanishing result that will be important for studying how geometric Eisenstein series interact with Verdier duality.

Lemma 2.3.21. *If ϕ_T is weakly generic then, for all $\overline{\nu} \in \Lambda^{pos}_{G,B} \setminus \{0\}$, the complexes*

$$R\Gamma_c(\mathrm{Div}^{(\overline{\nu})}, E^{(\overline{\nu})}_{\phi_T}(\langle \hat{\rho}, \overline{\nu} \rangle))$$

and

$$R\Gamma_c(\mathrm{Div}^{(\overline{\nu})}, E^{(\overline{\nu})}_{\phi_T})$$

are trivial.

Proof. By Künneth formula, this easily reduces to showing that $R\Gamma_c(\mathrm{Div}^{(n_i)}, E^{(n_i)}_{\phi_i}(n_i))$ is trivial for all $i \in \mathscr{J}$ and $n_i \in \mathbb{N}_{>0}$. However,

$E_{\phi_i}^{(n_i)}(n_i)$ is given by taking the n_ith symmetric power of $E_{\phi_i}(1)$. Therefore, by Künneth formula again, this reduces us to showing that

$$R\Gamma(W_{\mathbb{Q}_p}, \alpha_i \circ \phi_T(1)) \simeq R\Gamma_c(W_{E_i}, \tilde{\alpha}_i \circ \phi_T|_{W_{E_i}}(1)) \simeq R\Gamma_c(\mathrm{Div}^1_{E_i}, E_{\phi_i}(1))$$

vanishes, where the first isomorphism follows from Schapiro's lemma and the second isomorphism follows from the correspondence between Λ-valued local systems on $\mathrm{Div}^1_{E_i}$ and representations of W_{E_i} [FS21, Proposition VI.9.2]. The vanishing of the LHS follows from weak genericity and Lemma 2.3.19. The second vanishing statement follows from the same argument and weak genericity. □

We will now review the next ingredient in our calculations of geometric Eisenstein series, the Geometric Satake correspondence.

2.4 Geometric Satake and the Affine Grassmannian

2.4.1 The Geometric Satake Correspondence

We will now recall some facts about the geometric Satake correspondence for the B^+_{dR} Grassmannian, as proven in [FS21, Chapter VI]. For any finite set I, we consider the local Hecke stack

$$\pi_G : \mathscr{H}\mathrm{ck}^I_G \to \mathrm{Div}^I$$

For a point $S \to \mathrm{Div}^I$, we can consider the completion of the structure sheaf $\mathscr{O}_{X_S}$ at the union of the I Cartier divisors in X_S defined by S. This defines a ring which we denote by B^+, and inverting D, we get a ring which we denote by B. The mapping sending $S \in \mathrm{Div}^I$ to $G(B^+)$ and $G(B)$ defines étale sheaves on Perf, which we denote by $L^+_{\mathrm{Div}^I}G$ and $L_{\mathrm{Div}^I}G$, respectively. We note that, for $I = \{*\}$ and $S = \mathrm{Spa}(F, \mathscr{O}_F) \to \mathrm{Div}^I$ a geometric point with associated untilt $(C, \mathscr{O}_C)$, we have $B^+ = B^+_{dR}(C, \mathscr{O}_C)$ and $B = B_{dR}(C, \mathscr{O}_C)$, the usual de-Rham period rings attached to the untilt. For simplicity, we will often just drop the subscript Div^I and just write L^+G and LG for these étale sheaves. By [SW20b, Proposition 19.1.2], the Hecke stack can be described as the quotient:

$$[L^+G \backslash LG / L^+G] \to \mathrm{Div}^I$$

In other words, for $S \in$ Perf mapping to Div^I, $\mathscr{H}\mathrm{ck}^I_G$ will parameterize a pair of G-bundles over the formal completion of the tuple of divisors $D_{S,i}$ defined by the map $S \to \mathrm{Div}^I$ together with a trivialization away from the $D_{S,i}$. It follows that this has a map to the global Hecke stack considered in §2.3, by restricting to formal completions. Later on in the paper, we will use the analogous notations introduced here for the local Hecke stack to denote their pullback to the global Hecke stack Hck_G along this map.

To study this, we can uniformize this by the quotient

$$\mathrm{Gr}_G^I := LG/L^+G \to \mathrm{Div}^I$$

which is the Fargues-Fontaine analogue of the Beilinson-Drinfeld Grassmannian. Using Beauville-Laszlo [SW20b, Lemma 5.29], this can be interpreted as parameterizing modifications $\mathscr{F}_G \dashrightarrow \mathscr{F}_G^0$ over X_S away from the tuple of Cartier divisors defined by the projection to Div^I. It follows by the results of [SW20b, Lecture XX] that the Beilinson-Drinfeld Grassmannian is a well-behaved geometric object; it can be written as a closed union of subsheaves that are proper and representable in spatial diamonds over Div^I, given by bounding the meromorphy of the modifications. We can consider the category

$$\mathrm{D}(\mathscr{H}\mathrm{ck}_G^I)^{bd}$$

of objects with support in one of the aforementioned quasi-compact subsets. This carries a monoidal structure coming from the diagram:

$$\mathscr{H}\mathrm{ck}_G^I \times_{\mathrm{Div}^I} \mathscr{H}\mathrm{ck}_G^I \xleftarrow{(p_1,p_2)} L^+G\backslash LG \times^{L^+G} LG/L^+G \xrightarrow{m} \mathscr{H}\mathrm{ck}_G^I$$

Here the middle space can be interpreted as parameterizing a pair of G-bundles $\mathscr{E}_1, \mathscr{E}_2$ together with a pair of modifications $\beta_1 : \mathscr{E}_1 \dashrightarrow \mathscr{E}_0$ and $\beta_2 : \mathscr{E}_0 \dashrightarrow \mathscr{E}_2$ to/from the trivial bundle with the same locus of meromorphy. The maps p_i are the natural projections remembering only the data $(\mathscr{E}_i, \beta_i)$ for $i = 1,2$, and m is defined by sending $(\mathscr{E}_i, \beta_i)_{i=1,2}$ to $\beta_2 \circ \beta_1 : \mathscr{E}_1 \dashrightarrow \mathscr{E}_2$. Given $A, B \in \mathrm{D}(\mathscr{H}\mathrm{ck}_G^I)^{bd}$, we define

$$A \star B := Rm_*(p_1^*(A) \otimes p_2^*(B)) \in \mathrm{D}(\mathscr{H}\mathrm{ck}_G^I)^{bd}$$

the convolution of A and B [FS21, Section VI.8]. Since the map m is a fibration in Gr_G^I it is proper over any quasi-compact subset, so by proper base-change it gives a well-defined associative monoidal structure. One can further refine our category of sheaves as follows. In particular, we note that the locus of $\mathscr{H}\mathrm{ck}_G^I$ where the meromorphy is equal to the Galois orbit of $\mu_i \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ at the ith Cartier divisor for $i \in I$ is uniformized by a cohomologically smooth diamond of relative dimension $\sum_{i\in I}\langle 2\hat{\rho},\mu_i\rangle$ over Div^I by [FS21, Proposition VI.2.4]. One can check that this gives rise to a well-defined perverse t-structure on $\mathrm{D}(\mathscr{H}\mathrm{ck}_G^I)^{bd}$ over Div^I given by insisting that !-restriction (resp. $*$-restriction) to these strata sit in cohomological degrees $\geq$ (resp. $\leq$) $-\sum_{i\in I}\langle 2\hat{\rho},\mu_i\rangle$, and that convolution preserves perversity [FS21, Proposition VI.8.1]. With this in hand, we arrive at the key definition.

Definition 2.4.1. [FS21, Definition I.6.2] We define the Satake category

$$\mathrm{Sat}_G^I \subset \mathrm{D}(\mathscr{H}\mathrm{ck}_G^I)^{bd}$$

as the category of all $A \in \mathrm{D}(\mathscr{H}\mathrm{ck}_G^I)^{bd}$ which are perverse, flat (i.e for all Λ-modules M $A\otimes M$ is also perverse), and ULA over Div^I, as defined in [FS21, Chapter V.7].

The ULA and flatness property in the above definition has the key consequence that the pullback of $A \in \mathrm{Sat}_G^I$ to Gr_G^I composed with the push-forward to Div^I has cohomology sheaves valued in Λ-valued local systems on Div^I. In particular, using an analogue of Drinfeld's lemma [FS21, Proposition VI.9.2], this gives rise to a fiber functor

$$F_G^I : \mathrm{Sat}_G^I \to \mathrm{Rep}_{W_{\mathbb{Q}_p}^I}(\Lambda)$$
$$A \mapsto \bigoplus_i \mathscr{H}^i(R\pi_{G*}(A))$$

where $\mathrm{Rep}_{W_{\mathbb{Q}_p}^I}(\Lambda)$ denotes the category of continuous representations of $W_{\mathbb{Q}_p}^I$ on finite projective Λ-modules, and $R\pi_{G*}$ is the functor given by pulling back to Gr_G^I and taking the push-forward to Div^I, as in [FS21, Definition/Proposition VI.7.10]. Now, by using the factorization structure on these Grassmannians, one can also construct an analogue of the fusion product [FS21, Section VI.9]. Let us recall its construction. We consider a partition $I = I_1 \sqcup \ldots \sqcup I_k$ of this index set. We consider the open subspace

$$\mathrm{Div}^{I;I_1,\ldots,I_k} \subset \mathrm{Div}^I$$

parameterizing a tuple of Cartier divisors $(D_i)_{i\in I}$ such that D_i and $D_{i'}$ are disjoint whenever $i,i' \in I = I_1 \sqcup \ldots \sqcup I_k$ lie in different I_js, and let $\mathscr{H}\mathrm{ck}_G^{I;I_1,\ldots,I_k}$ be the base-change of $\mathscr{H}\mathrm{ck}_G^I$ to this open subspace. We can define the subcategory $\mathrm{Sat}_G^{I;I_1,\ldots,I_k} \subset \mathrm{D}(\mathscr{H}\mathrm{ck}_G^{I;I_1,\ldots,I_k})^{\mathrm{bd}}$ in an analogous manner to Definition 2.4.1. We have a natural restriction map

$$\mathrm{Sat}_G^I \to \mathrm{Sat}_G^{I;I_1,\ldots,I_k}$$

which, by [FS21, Proposition VI.9.3], defines a fully faithful embedding. There is also an identification

$$\mathscr{H}\mathrm{ck}_G^I \times_{\mathrm{Div}^I} \mathrm{Div}^{I;I_1,\dots,I_k} \simeq \prod_{j=1}^{k} \mathscr{H}\mathrm{ck}_G^{I_j} \times_{\prod_j \mathrm{Div}^{I_j}} \mathrm{Div}^{I;I_1,\dots,I_k}$$

giving a natural map

$$\mathrm{Sat}_G^{I_1} \times \ldots \times \mathrm{Sat}_G^{I_k} \to \mathrm{Sat}_G^{I;I_1,\dots,I_k}$$

via taking exterior products. Then, by [FS21, Definition/Proposition VI.9.4], this lies in the full subcategory Sat_G^I, and the resulting map

$$\mathrm{Sat}_G^{I_1} \times \ldots \times \mathrm{Sat}_G^{I_k} \to \mathrm{Sat}_G^I$$

is called the fusion product. Now consider the composite

$$\mathrm{Sat}_G^I \times \mathrm{Sat}_G^I \to \mathrm{Sat}_G^{I \sqcup I} \to \mathrm{Sat}_G^I$$

where the first map is the fusion product, and the last map is given by restricting along the diagonal embedding $\mathscr{H}\mathrm{ck}^I \to \mathscr{H}\mathrm{ck}^{I \sqcup I}$. Then this is naturally isomorphic to the convolution product. By this comparison between fusion and convolution, one can deduce that the convolution product is in fact symmetric monoidal, and that the functor F_G^I takes this monoidal structure to the usual tensor product on $\mathrm{Rep}_{W_{\mathbb{Q}_p}^I}(\Lambda)$. Now, using Tannakian duality, one deduces the following.

Theorem 2.4.2. *[FS21, Theorem I.6.3] For a finite index set I, the category* Sat_G^I *is naturally in I identified with* $\mathrm{Rep}_\Lambda({}^L G^I)$ *the category of representations of* ${}^L G^I$ *on finite projective* Λ*-modules.*

Remark 2.4.3. One needs to be a bit careful here. In particular, ${}^L G$ as defined here differs from the usual definition of the L-group; namely, the usual action of $W_{\mathbb{Q}_p}$ on $\hat{G}$ is twisted by a cyclotomic character (See [FS21, Section VI.11] for more details).

One of the key points that also plays an important role in the proof of this theorem is that this construction respects the natural map

$$\mathrm{res}_T^{I,G} : \mathrm{Rep}_\Lambda({}^L G^I) \to \mathrm{Rep}_\Lambda({}^L T^I)$$

given by restricting a representation of ${}^L G^I$ to ${}^L T^I$ along the natural embedding:

$$ {}^L T^I \to {}^L G^I $$

To explain this, we need to explain how this functor is realized in the geometry of Beilinson-Drinfeld Grassmannians. First note that, as in §2.3, we have an identification

$$ \mathrm{Gr}^I_T = \bigsqcup_{(\nu_i)_{i\in I}\in(\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma)^I} \mathrm{Gr}^I_{T,(\nu_i)_{i\in I}} \simeq \bigsqcup_{(\nu_i)_{i\in I}\in(\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma)^I} \mathrm{Div}^I_{E_{(\nu_i)_{i\in I}}} $$

where $\mathrm{Gr}_{T,(\nu_i)_{i\in I}}$ parametrizes modifications of T-bundles with meromorphy equal to the Galois orbit defined by ν_i at a Cartier divisor D_i for all $i \in I$, and $\mathrm{Div}^I := \prod_{i\in I} \mathrm{Div}^1_{E_{\nu_i}}$, where E_{ν_i} denotes the reflex field of ν_i. In particular, we see that Theorem 2.4.2 is trivial in this case, as F^I_T will induce an equivalence between Sat^I_T and $(\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma)^I$-graded objects in $\mathrm{Rep}_\Lambda(W^I_{\mathbb{Q}_p})$ under this isomorphism. We consider the diagram

$$ \begin{tikzcd} & \mathrm{Gr}^I_B \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathrm{Gr}^I_T & & \mathrm{Gr}^I_G \end{tikzcd} $$

and define the functor:

$$ p_! q^* : \mathrm{D}(\mathrm{Gr}^I_G)^{bd} \to \mathrm{D}(\mathrm{Gr}^I_T)^{bd} $$

As we will discuss further in the next section, the fibers of the morphism p over the connected components $\mathrm{Gr}^I_{T,(\nu_i)_{i\in I}}$ give rise to a locally closed stratification of Gr^I_G which embed via the morphism q (cf. [FS21, Example VI.3.4]). These are the so called semi-infinite Schubert cells. If one considers $\mathbb{G}_m$ acting on Gr^I_G via a suitably chosen cocharacter μ composed with the L^+G action on Gr^I_G then one can identify these semi-infinite cells with a union $\mathbb{G}_m$-orbits (the attractor of the $\mathbb{G}_m$-action), and the fixed points will be precisely the connected components:

$$ \mathrm{Gr}^I_T = \bigsqcup_{(\nu_i)_{i\in I}\in(\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma)^I} \mathrm{Gr}^I_{T,(\nu_i)_{i\in I}} $$

This allows one to apply a diamond analogue [FS21, Theorem IV.6.5] of Braden's hyperbolic localization theorem [Bra03]. In particular, since sheaves in Sat_G^I pull-back to L^+G-equivariant sheaves on Gr_G^I, they will be $\mathbb{G}_m$-equivariant. From this, one can deduce that $p_!q^*$ is a hyperbolic localization with respect to this $\mathbb{G}_m$-action and will therefore preserve perverse, flat, and ULA objects over Div^I. We therefore get an induced functor

$$\mathrm{CT} : \mathrm{Sat}_G^I \to \mathrm{Sat}_T^I$$

called the constant term functor, as in [FS21, Proposition VI.7.13]. We now consider the following function

$$\deg : |\mathrm{Gr}_T^I| \to \mathbb{Z}$$

which has value $\langle 2\hat{\rho}, |(\nu_i)_{i\in I}|\rangle$ on the connected component indexed by $(\nu_i)_{i\in I}$, where $|(\nu_i)_{i\in I}| := \sum_{i\in I} \overline{\nu}_{i\Gamma} \in \Lambda_{G,B}$. Now, by applying excision with respect to the stratatification by semi-infinite cells one can show that, for all $A \in \mathrm{Sat}_G$, one has an isomorphism $\bigoplus_i \mathscr{H}^i(\pi_{G*}(A)) \simeq \mathscr{H}^0(\pi_{T*}(\mathrm{CT}(A)[\deg]))$ (See the proof of [FS21, Proposition VI.7.10]). This in particular gives us the following Proposition.

Proposition 2.4.4. *For all finite index sets I, there exists a commutative diagram*

$$\begin{tikzcd}
\mathrm{Sat}_G^I \arrow[r, "\mathrm{CT}[\deg]"] \arrow[d, "F_G^I"] & \mathrm{Sat}_T^I \arrow[d, "F_T^I"] \\
\mathrm{Rep}_\Lambda({}^LG^I) \arrow[r, "\mathrm{res}_T^{I,G}"] & \mathrm{Rep}_\Lambda({}^LT^I)
\end{tikzcd}$$

where here F_G^I (resp. F_T^I) is the equivalence given by Theorem 2.4.2 for G (resp. T).

Moreover, it follows by [FS21, Proposition VI.9.6], that the fusion product respects the constant term functors.

Proposition 2.4.5. *For all finite index sets I, with a partition $I_1 \sqcup \ldots \sqcup I_k$, we have a commutative diagram*

$$\begin{tikzcd}
\mathrm{Sat}_G^{I_1} \times \ldots \times \mathrm{Sat}_G^{I_k} \arrow[r] \arrow[d, "\mathrm{CT}^{I_1}[\deg]\times\ldots\times\mathrm{CT}^{I_k}[\deg]"] & \mathrm{Sat}_G^I \arrow[d, "\mathrm{CT}^I[\deg]"] \\
\mathrm{Sat}_T^{I_1} \times \ldots \times \mathrm{Sat}_T^{I_k} \arrow[r] & \mathrm{Sat}_T^I
\end{tikzcd}$$

which commutes functorially in I, where the top (resp. bottom) vertical arrow is given by the fusion product for G (resp. T).

We now turn our attention to the semi-infinite cells.

2.4.2 The Cohomology of Semi-Infinite Cells

Theorem 2.4.2, Proposition 2.4.4, and Proposition 2.4.5 have implications for the cohomology of spaces related to moduli spaces of B-structures. We will record this now. We let $E/\mathbb{Q}_p$ be the splitting field of G. Then we have an identification $\mathrm{Gr}^I_{G,E} \simeq \mathrm{Gr}^I_{G_E}$ over the base change Div^I_E of Div^I to E. Sheaves on this space will be equivalent algebraic representations of I copies of the dual group $\hat{G}$. First, we recall that we have the following natural stratification of $\mathrm{Gr}^I_{G,E}$.

Definition 2.4.6. For $(\lambda_i)_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+)^I$, we let $\mathrm{Gr}^I_{G,(\lambda_i)_{i\in I},E}$ (resp. $\mathrm{Gr}^I_{G,\leq(\lambda_i)_{i\in I},E}$) be the locally closed (resp. closed) subfunctor of Gr_G parameterizing modifications

$$\mathscr{F}^0_G \to \mathscr{F}_G$$

between the trivial G-bundle $\mathscr{F}^0_G$ and a G-bundle $\mathscr{F}_G$ of meromorphy equal to (resp. less than) then $\sum_{D_i=D_j} \lambda_i$ at a Cartier divisor D_j, for some fixed $j \in I$.

As mentioned in the previous section, $\mathrm{Gr}^I_{G,(\lambda_i)_{i\in I},E}$ is representable in nice diamonds and is cohomologically smooth of dimension equal to $\sum_{i\in I}\langle 2\hat{\rho}, \lambda_i\rangle$ over Div^I_E, by [FS21, Proposition VI.2.4]. Similarly, by [SW20b, Proposition 19.2.3], $\mathrm{Gr}_{G,\leq(\lambda_i)_{i\in I},E}$ is representable in nice diamonds and proper over Div^I_E. Fix $\boxtimes_{i\in I} V_i = V \in \mathrm{Rep}_{\Lambda}(\hat{G}^I)$ with fixed central character, and suppose the highest weight of V_i is given by λ_i for $\lambda_i \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$. Attached to this, we get a Λ-valued perverse sheaf $\mathscr{S}_V$ supported on $\mathrm{Gr}^I_{G,\leq(\lambda_i)_{i\in I},E}$ by Theorem 2.4.2 and [FS21, Proposition VI.7.5]. We now fix a geometric point $x \to \mathrm{Div}^I_E$. In what follows, for a space ? over Div^I_E we write ${}_x?$ for the base-change to x. Since a local system on Div^I_E will be determined by the W^I_E-representation given by its pullback to this geometric point, looking at this pullback will be sufficient for most calculations. We now consider another stratification of $\mathrm{Gr}^I_{G,E}$.

Definition 2.4.7. Consider the natural map:

$$p : \mathrm{Gr}^I_{B,E} \to \mathrm{Gr}^I_{T,E}$$

For $(\nu_i)_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I$, we set $\mathrm{S}^I_{G,(\nu_i)_{i\in I},E}$ to be the fiber of $\mathrm{Gr}^I_{B,E}$ over the connected component $\mathrm{Gr}^I_{T,(\nu_i)_{i\in I},E}$ in $\mathrm{Gr}^I_{T,E}$ parameterizing modifications of type $(\nu_i)_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I$. We note, by [FS21, Proposition VI.3.1], that the natural map

$$q : \mathrm{Gr}^I_{B,E} \to \mathrm{Gr}^I_{G,E}$$

is a bijection on geometric points and it defines a locally closed embedding on the $\mathrm{S}^I_{G,(\nu_i)_{i\in I},E}$. Therefore, the spaces $\mathrm{S}^I_{G,(\nu_i)_{i\in I},E}$ for varying $(\nu_i)_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I$, form a locally-closed stratification of $\mathrm{Gr}^I_{G,E}$.

Remark 2.4.8. In particular, given a modification $\beta : \mathscr{F}^0_G \dashrightarrow \mathscr{F}_G$ of G-bundles, by the Tannakian formalism this is equivalent to specifying a set of meromorphic maps

$$\mathscr{V}^{\hat{\lambda}}_{\mathscr{F}^0_G} \dashrightarrow \mathscr{V}^{\hat{\lambda}}_{\mathscr{F}_G}$$

for all dominant characters $\hat{\lambda} \in \hat{\Lambda}^+_G$ satisfying the Plücker relations (cf. Definition 2.5.6). The trivial G-bundle $\mathscr{F}^0_G$ admits a natural split B-structure whose associated T-bundle is the trivial T-bundle $\mathscr{F}^0_T$, and this defines a set of maps

$$\mathscr{L}^{\hat{\lambda}}_{\mathscr{F}^0_T} \hookrightarrow \mathscr{V}^{\hat{\lambda}}_{\mathscr{F}^0_G}$$

where $\mathscr{L}^{\hat{\lambda}} := (\mathscr{V}^{\hat{\lambda}})^U$ and U is the unipotent radical of B. For a set of divisors $(D_i)_{i\in I} \in \mathrm{Div}^I_E$, we can also consider the meromorphic map

$$\mathscr{L}^{\hat{\lambda}}_{\mathscr{F}^0_T}(\sum_{i\in I} -\langle \hat{\lambda}, \nu_{i\Gamma}\rangle \cdot D_i) \dashrightarrow \mathscr{L}^{\hat{\lambda}}_{\mathscr{F}^0_T} \to \mathscr{V}^{\hat{\lambda}}_{\mathscr{F}^0_G} \dashrightarrow \mathscr{V}^{\hat{\lambda}}_{\mathscr{F}_G}$$

defined by modifying the T-bundles by ν_i at D_i for all $i \in I$. We claim that β defines a point in $\mathrm{S}^I_{G,(\nu_i)_{i\in I},E}$ if and only if this map does not have a zero or pole for all $\hat{\lambda}$. This is easy to see. In particular, if this map does not have a zero or a pole then the maps

$$\mathscr{L}^{\hat{\lambda}}_{\mathscr{F}^0_T}(-\sum_{i\in I}\langle \hat{\lambda}, \nu_{i\Gamma}\rangle \cdot D_i) \to \mathscr{V}^{\hat{\lambda}}_{\mathscr{F}_G}$$

define a B-structure $\mathscr{F}_B$ on $\mathscr{F}_G$ whose underlying T-bundle is given by $\mathscr{F}^0_T(-\sum_{i\in I}\langle \hat{\lambda}, \nu_{i\Gamma}\rangle \cdot D_i)$. Moreover, the construction induces a map of B-bundles

$$\mathscr{F}^0_B \dashrightarrow \mathscr{F}_B$$

which when applying $\times^B G$ gives the modification β and when applying $\times^B T$ gives rise to a modification defining a point in the union of connected components $\mathrm{Gr}^I_{T,(\nu_i)_{i\in I},E}$ over the point attached to $(D_i)_{i\in I}$ in Div^I_E. In other words, $\mathscr{F}^0_B \dashrightarrow \mathscr{F}_B$ defines an element of the locally closed stratum $\mathrm{S}^I_{G,(\nu_i)_{i\in I},E}$.

For $V \in \mathrm{Rep}_\Lambda(\hat{G}^I)$, we again consider the sheaf $\mathscr{S}_V$ on $\mathrm{Gr}^I_{G,\leq(\lambda_i)_{i\in I},E}$, and pullback to a fixed geometric point $x \to \mathrm{Div}^I_E$ defined by $\mathrm{Spa}(C)$ for C an algebraically closed perfectoid field. If we write $p_{(\nu_i)_{i\in I}} : {}_x\mathrm{S}^I_{G,(\nu_i)_{i\in I},E} \to {}_x\mathrm{Gr}^I_{T,(\nu_i)_{i\in I},E} \simeq \mathrm{Spa}(C)$ for the induced map on connected components indexed by $(\nu_i)_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I$, we note that we have an isomorphism

$$p_! q^*(\mathscr{S}_V) \simeq \bigoplus_{(\nu_i)_{i\in I}\in(\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I} p_{(\nu_i)_{i\in I}!}(\mathscr{S}_V|_{{}_x\mathrm{S}^I_{G,(\nu_i)_{i\in I},E}}) = \bigoplus_{(\nu_i)_{i\in I}\in(\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I} \mathbb{H}^*_c({}_x\mathrm{S}^I_{G,(\nu_i)_{i\in I}}, \mathscr{S}_V|_{{}_x\mathrm{S}_{G,(\nu_i)_{i\in I}}})$$

(cf. [FS21, Example VI.3.6]). However, this is simply the constant term functor in the previous section. In particular, by Proposition 2.4.4, we deduce the following.

Corollary 2.4.9. *For $V = \boxtimes_{i\in I} V_i \in \mathrm{Rep}(\hat{G}^I)$, a geometric point $x \to \mathrm{Div}^I_E$, and all tuples $(\nu_i)_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I$, we have an isomorphism*

$$\mathbb{H}_c^{-\langle 2\hat{\rho}, |(\nu_i)_{i\in I}|\rangle}({}_x\mathrm{S}^I_{G,(\nu_i)_{i\in I},E}, \mathscr{S}_V|_{{}_x\mathrm{S}^I_{G,(\nu_i)_{i\in I},E}}) \simeq \boxtimes_{i\in I} V_i(\nu_i)(-\langle\hat{\rho},\nu_i\rangle))$$

of W^I_E-modules.

Remark 2.4.10. The Tate twists appearing here are due to the difference between the standard definition of ${}^LG$ and the one used in the geometric Satake equivalence, as in the remark proceeding Theorem 2.4.2.

This will be the key proposition required for the proof of the filtered Hecke eigensheaf property. More specifically, to show the compatibilities of the filtered eigensheaf property, we need to show that this isomorphism is functorial in I. In particular, consider a map of finite index sets $\pi : I \to J$. For $j \in J$, we set $I_j := \pi^{-1}(j)$ and consider the natural map $\Delta_{IJ} : \mathrm{Div}^J_E \to \mathrm{Div}^I_E$, which diagonally embeds the jth copy of Div^1_E in Div^J_E into $\mathrm{Div}^{I_j}_E$. Then, by the relationship between fusion product and tensor product under Theorem 2.4.2, we have a identification

$$\Delta^*_{IJ}(\mathscr{S}_V) \simeq \mathscr{S}_{\Delta^*_{IJ}(V)}$$

of sheaves on $\mathrm{Gr}^J_{G,E}$, where $\Delta^*_{IJ}(V)$ is given by restriction along the corresponding map $\hat{G}^J \to \hat{G}^I$. Now, by Proposition 2.4.5, we have the following.

Corollary 2.4.11. *For all finite index sets I, J with a map $f : I \to J$, a tuple $(\nu_j)_{j\in J} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^J$, a representation $V \in \mathrm{Rep}(\hat{G}^I)$, and a geometric point $x \to \mathrm{Div}^J_E$, the identification $\Delta^*_{IJ}(\mathscr{S}_V) \simeq \mathscr{S}_{\Delta^*_{IJ}(V)}$ induces an isomorphism*

$$\mathbb{H}^*_c({}_x\mathrm{S}^J_{G,(\nu_j)_{j\in J},E}, \mathscr{S}_{\Delta^*_{IJ}(V)}|_{{}_x\mathrm{S}^J_{G,(\nu_j)_{j\in J},E}}) \simeq \bigoplus_{\substack{(\nu_i)_{i\in I}\in\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \\ \sum_{i\in I_j}\nu_i=\nu_j}} \mathbb{H}^*_c({}_x\mathrm{S}_{G,(\nu_i)_{i\in I},E}, \mathscr{S}_V|_{{}_x\mathrm{S}^I_{G,(\nu_i)_{i\in I},E}})$$

of W^J_E-modules, where the action on the RHS is via the natural map $\Delta_{IJ} : W^J_E \to W^I_E$. This is compatible with the identification

$$\Delta^*_{IJ}(V((\nu_j)_{j\in J})) \simeq \bigoplus_{\substack{(\nu_i)_{i\in I}\in(\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I \\ \sum_{i\in I_j}\nu_i=\nu_j}} V((\nu_i)_{i\in I})$$

under the isomorphisms of Corollary 2.4.9.

We note that the previous result has some very useful geometric consequences. Let's explain this in the case that $I = \{*\}$ is a singleton for a fixed geometric point $x \to \mathrm{Div}^1_E$. Fix $\lambda \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$, and consider the highest weight representation $V_\lambda \in \mathrm{Rep}_\Lambda(\hat{G})$ defined by λ. Since the sheaf $\mathscr{S}_\lambda$ is supported on ${}_x\mathrm{Gr}_{G,\leq\lambda}$, we can deduce the following.

Corollary 2.4.12. *For $\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ and $\lambda \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ with associated highest weight representation $V_\lambda \in \mathrm{Rep}_\Lambda(\hat{G})$, if the weight space $V_\lambda(\nu)$ is non-trivial then the intersection ${}_x\mathrm{S}_{G,\nu,E} \cap {}_x\mathrm{Gr}_{G,\leq\lambda,E}$ is non-empty.*

Remark 2.4.13. One can also see this by using the Iwasawa decomposition of G and working explicitly with the loop group of G (See the analysis proceeding [She21, Proposition 6.4] and [She21, Remark 6.5]). For example, one can show that the intersection ${}_x\mathrm{Gr}_{G,\leq\lambda,E} \cap {}_x\mathrm{S}_{G,w_0(\lambda),E}$ is simply the point given by ξ^λ, where $\xi \in B^+_{dR}(C, \mathscr{O}_C)$ is the uniformizing parameter defined by the geometric point x. This corresponds to the lowest weight space $V_\lambda(w_0(\lambda))$.

We will now finish our analysis by recording some facts about the closure relationships for these strata.

Proposition 2.4.14. *[She21, Proposition 6.4] For $\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ and $\lambda \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$, the closure of the intersection*

$${}_x\mathrm{S}_{G,\nu,E} \cap {}_x\mathrm{Gr}_{G,\leq\lambda,E}$$

in $\mathrm{Gr}_{G,\leq\lambda,E}$ *is equal to the disjoint union*

$$\bigsqcup_{\nu'\leq\nu} {}_x\mathrm{S}_{G,\nu',E} \cap {}_x\mathrm{Gr}_{G,\leq\lambda,E}$$

where this defines a closed subspace in ${}_x\mathrm{Gr}_{G,\leq\lambda,E}$ *by [FS21, Proposition VI.3.1]. In particular, using the previous Corollary, we deduce that* ${}_x\mathrm{S}_{G,w_0(\lambda),E} \cap {}_x\mathrm{Gr}_{G,\leq\lambda,E}$ *is a closed subspace and* ${}_x\mathrm{S}_{G,\lambda,E} \cap {}_x\mathrm{Gr}_{G,\leq\lambda,E}$ *is an open subspace.*

2.5 Moduli stacks of B-structures

In this section, we will study the moduli stack of B-structures Bun_B and its basic geometric properties. This will allow us to define the geometric Eisenstein functor. For understanding many of the finer properties of this functor, it is important to consider a compactification of the natural morphism $\mathfrak{p} : \mathrm{Bun}_B \to \mathrm{Bun}_G$ taking B-bundles to their induced G-bundles. This compactification will be an analogue of Drinfeld's compactification in the function-field setting, denoted $\overline{\mathrm{Bun}}_B$. We will show that this gives rise to an Artin v-stack, which admits Bun_B as an open and dense substack, and that the natural map $\overline{\mathfrak{p}} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_G$ extending $\mathfrak{p}$ is indeed proper after restricting to connected components. We will also define a locally closed stratification of $\overline{\mathrm{Bun}}_B$. These strata will play an important role in the proof of the Hecke eigensheaf property and understanding how the Eisenstein functor interacts with Verdier duality.

2.5.1 The Geometry of Bun_B

We will start by collecting some basic facts about the moduli stack Bun_B parameterizing, for $S \in \mathrm{Perf}$, the groupoid of B-bundles on X_S. Given a B-bundle $\mathscr{G}_B$, we can send it to the induced T-bundle and G-bundle via the natural maps

$$\begin{array}{ccc} B & \longrightarrow & G \\ \downarrow & & \\ T & & \end{array}$$

which induces a diagram of v-stacks:

$$\begin{tikzcd} \mathrm{Bun}_B \arrow[r, "\mathfrak{p}"] \arrow[d, "\mathfrak{q}"] & \mathrm{Bun}_G \\ \mathrm{Bun}_T & \end{tikzcd}$$

Let's first start by breaking this up into connected components. As seen in §3, the connected components of Bun_T are indexed by elements $\overline{\nu} \in B(T) \simeq \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma = \Lambda_{G,B}$. This allows us to define the following.

Definition 2.5.1. For $\overline{\nu} \in B(T)$, we write $\mathrm{Bun}_B^{\overline{\nu}}$ for the pre-image of the connected component $\mathrm{Bun}_T^{\overline{\nu}}$ defined by $\overline{\nu}$ under the map $\mathfrak{q}^{\overline{\nu}}$. We write $\mathfrak{p}^{\overline{\nu}} : \mathrm{Bun}_B^{\overline{\nu}} \to \mathrm{Bun}_G$ and $\mathfrak{q}^{\overline{\nu}} : \mathrm{Bun}_B^{\overline{\nu}} \to \mathrm{Bun}_T^{\overline{\nu}}$ for the restriction of $\mathfrak{p}$ and $\mathfrak{q}$ to $\mathrm{Bun}_B^{\overline{\nu}}$, respectively.

We claim that this induces a decomposition of the moduli stack Bun_B into connected components. To each element $\overline{\nu}$, we define the integer $d_{\overline{\nu}} := \langle 2\hat{\rho}, \overline{\nu} \rangle$, where $2\hat{\rho}$ is the sum of all positive roots with respect to the choice of Borel. We note that, if $\overline{\nu}$ is anti-dominant with respect to the choice of Borel, $d_{\overline{\nu}}$ is negative. This will be the case where the HN-slopes are dominant so the B-bundles will split, and the negative dimension comes from quotienting out by the torsor of splittings. On the other hand, if $\overline{\nu}$ is dominant then the connected component will parametrize non-split B-structures and we see that the dimension will be positive. We have the following claim.

Proposition 2.5.2. *The map $\mathfrak{q}$ is a cohomologically smooth (non-representable) morphism of Artin v-stacks in the sense of [FS21, Definition IV.1.11]. In particular, for $\overline{\nu} \in \Lambda_{G,B}$, the map $\mathfrak{q}^{\overline{\nu}}$ is pure of ℓ-dimension equal to $d_{\overline{\nu}}$, in the sense of [FS21, Definition IV.1.17].*

Proof. This follows from [Ham21a, Proposition 4.7], where we note that Bun_T is an Artin v-stack that is cohomologically smooth of ℓ-dimension 0 (See also [AL21a, Lemma 4.1 (ii)]). $\square$

In particular, this implies using [Sch18, Proposition 23.11], that $\mathfrak{q}$ is a universally open morphism of Artin v-stacks. Moreover, one can check that the fibers of this morphism are connected (See the proof of [AL21a, Lemma 4.1 (ii)] and [Ham21a, Proposition 3.16]). As a consequence, we can deduce that, since $\bigsqcup_{\overline{\nu} \in \Lambda_{G,B}} \mathrm{Bun}_T^{\overline{\nu}}$ is a decomposition of Bun_T into connected components, the following is true.

Corollary 2.5.3. *The connected components of* Bun_B *are given by* $\mathrm{Bun}_B^{\overline{\nu}}$*, for varying* $\overline{\nu} \in B(T)$.

We now comment briefly on the geometry of the map $\mathfrak{p}$. In particular, we have the following.

Lemma 2.5.4. *[AL21a, Lemma 4.1 (ii)] The map* $\mathfrak{p}$ *is representable in nice diamonds.*

These properties allow us to define the Eisenstein functor using the derived functors defined in [Sch18].

Definition 2.5.5. We define a locally constant function

$$\dim(\mathrm{Bun}_B) : |\mathrm{Bun}_B| \to \mathbb{Z}$$

$$x \in |\mathrm{Bun}_B^{\overline{\nu}}| \mapsto d_{\overline{\nu}}$$

and with it the unnormalized Eisenstein functor

$$\mathrm{Eis} : \mathrm{D}(\mathrm{Bun}_T) \to \mathrm{D}(\mathrm{Bun}_G)$$

$$\mathscr{F} \mapsto \mathfrak{p}_!(\mathfrak{q}^*(\mathscr{F})[\dim(\mathrm{Bun}_B)])$$

In particular, since this definition involves the functor $\mathfrak{p}_!$ it is natural to consider a compactification of the morphism $\mathfrak{p}$ to understand the finer properties of $\mathfrak{p}_!$. This leads us to our study of the Drinfeld compactification.

2.5.2 The Drinfeld Compactification

The Definition and Basic Properties

We recall that classically (curve over a finite or complex field) there is a rather straight-forward way of compactifying the map:

$$\mathfrak{p} : \mathrm{Bun}_B \to \mathrm{Bun}_G$$

This is called a Drinfeld Compactification of $\mathfrak{p}$, denoted $\overline{\mathrm{Bun}}_B$. Its main property is that there exists an open immersion $\mathrm{Bun}_B \to \overline{\mathrm{Bun}}_B$, with topologically dense image, and it has a map $\overline{\mathfrak{p}} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_G$ extending $\mathfrak{p}$, which is proper after restricting to a connected component. First, as a warm up, let us explain the construction when $G = \mathrm{GL}_2$. For this, we recall that Bun_B can be viewed as parameterizing tuples $(\mathscr{M}, \mathscr{L}, \kappa : \mathscr{L} \hookrightarrow \mathscr{M})$, where $\mathscr{E}$ is a rank 2 vector bundle, $\mathscr{L}$ is a rank 1 vector bundle, and κ is an injective bundle map. To compactify this space, we will allow κ to be a map of $\mathscr{O}_{X_S}$-modules whose pullback to each geometric point is an injective map of coherent sheaves on X. In other words, we allow $\mathscr{M}/\mathscr{L}$ to have torsion. For a general G, the idea is to apply the Tannakian formalism. In particular, given a G-bundle $\mathscr{F}_G$ we get, for all $\hat{\lambda} \in \hat{\Lambda}^+_G$, an induced highest weight bundle, denoted $\mathscr{V}^{\hat{\lambda}}_{\mathscr{F}_G}$. A point of Bun_B mapping to $\mathscr{F}_G$ via $\mathfrak{p}$ then defines a set of line subbundles $\kappa^{\hat{\lambda}} : \mathscr{L}^{\hat{\lambda}} \hookrightarrow \mathscr{V}^{\hat{\lambda}}_{\mathscr{F}_G}$ which satisfy some Plücker relations. Using this interpretation, $\overline{\mathrm{Bun}}_B$ can then be defined as classifying G-bundles $\mathscr{F}_G$ together with a system of maps $\overline{\kappa}^{\hat{\lambda}} : \mathscr{L}^{\hat{\lambda}} \to \mathscr{V}^{\hat{\lambda}}_{\mathscr{F}_G}$ for all $\hat{\lambda} \in \Lambda^+_G$, which are injective after pulling back to a geometric point and satisfy the same Plücker relations.

We now explain how to construct the aforementioned compactification $\overline{\mathrm{Bun}}_B$ of Bun_B over Bun_G in the Fargues-Fontaine setting. In order to describe the Drinfeld compactification, we note that, for $S \in \mathrm{Perf}$, Bun_B can be viewed as a stack parameterizing triples:

1. A G-bundle $\mathscr{F}_G$ on X_S.
2. A T-bundle $\mathscr{F}_T$ on X_S.
3. A G-equivariant map $\kappa : \mathscr{F}_G \to G/U \times^T \mathscr{F}_T$.

By the Tannakian formalism, (3) can be described as a collection of injective bundle maps on X_S, $\kappa^{\mathscr{V}} : (\mathscr{V}^U)_{\mathscr{F}_T} \to \mathscr{V}_{\mathscr{F}_G}$ for every G-module $\mathscr{V}$ satisfying the following Plücker relations:

1. For the trivial representation $\mathscr{V}$, $\kappa^{\mathscr{V}}$ must be the identity map $\mathscr{O}_{X_S} \to \mathscr{O}_{X_S}$.
2. For a G-module map $\mathscr{V}^1 \to \mathscr{V}^2$, the induced square

$$\begin{array}{ccc}
((\mathscr{V}^1)^U)_{\mathscr{F}_T} & \xrightarrow{\kappa^{\mathscr{V}^1}} & \mathscr{V}^1_{\mathscr{F}_G} \\
\downarrow & & \downarrow \\
((\mathscr{V}^2)^U)_{\mathscr{F}_T} & \xrightarrow{\kappa^{\mathscr{V}^2}} & \mathscr{V}^2_{\mathscr{F}_G}
\end{array}$$

commutes.

3. For two G-modules $\mathscr{V}^1$ and $\mathscr{V}^2$, we have that the diagram

$$\begin{tikzcd}
((\mathscr{V}^1)^U \otimes (\mathscr{V}^2)^U)_{\mathscr{F}_T} \arrow[r, "\kappa^{\mathscr{V}^1} \otimes \kappa^{\mathscr{V}^2}"] \arrow[d] & \mathscr{V}^1_{\mathscr{F}_G} \otimes \mathscr{V}^2_{\mathscr{F}_G} \arrow[d, "id"] \\
((\mathscr{V}^1 \otimes \mathscr{V}^2)^U)_{\mathscr{F}_T} \arrow[r, "\kappa^{\mathscr{V}^1 \otimes \mathscr{V}^2}"] & \mathscr{V}^1_{\mathscr{F}_G} \otimes \mathscr{V}^2_{\mathscr{F}_G}
\end{tikzcd}$$

commutes.

As mentioned above, the idea will now be to introduce torsion in the above definition. In particular, we have the following definition for $\overline{\mathrm{Bun}}_B$.

Definition 2.5.6. We define $\overline{\mathrm{Bun}}_B$ to be the v-stack parameterizing, for $S \in \mathrm{Perf}$, triples $(\mathscr{F}_G, \mathscr{F}_T, \overline{\kappa}^{\mathscr{V}})$, where $\overline{\kappa}^{\mathscr{V}}$ is a map of $\mathscr{O}_{X_S}$-modules defined for every G-module $\mathscr{V}$

$$(\mathscr{V}^U)_{\mathscr{F}_T} \to \mathscr{V}_{\mathscr{F}_G}$$

satisfying the following conditions:

- For every geometric point $s \to S$, the pullback of $\overline{\kappa}^{\mathscr{V}}$ to the Fargues-Fontaine curve over s is an injection of coherent sheaves.
- The Plücker relations hold in the following sense:
  1. For the trivial representation $\mathscr{V}$, $\overline{\kappa}^{\mathscr{V}}$ is the identity map $\mathscr{O} \to \mathscr{O}$.
  2. For a G-module map $\mathscr{V}^1 \to \mathscr{V}^2$, the induced square

  $$\begin{tikzcd}
  ((\mathscr{V}^1)^U)_{\mathscr{F}_T} \arrow[r, "\overline{\kappa}^{\mathscr{V}^1}"] \arrow[d] & \mathscr{V}^1_{\mathscr{F}_G} \arrow[d] \\
  ((\mathscr{V}^2)^U)_{\mathscr{F}_T} \arrow[r, "\overline{\kappa}^{\mathscr{V}^2}"] & \mathscr{V}^2_{\mathscr{F}_G}
  \end{tikzcd}$$

  commutes.
  3. For two G-modules $\mathscr{V}^1$ and $\mathscr{V}^2$, we have that the diagram

  $$\begin{tikzcd}
  ((\mathscr{V}^1)^U \otimes (\mathscr{V}^2)^U)_{\mathscr{F}_T} \arrow[r, "\overline{\kappa}^{\mathscr{V}^1} \otimes \overline{\kappa}^{\mathscr{V}^2}"] \arrow[d] & \mathscr{V}^1_{\mathscr{F}_G} \otimes \mathscr{V}^2_{\mathscr{F}_G} \arrow[d, "id"] \\
  ((\mathscr{V}^1 \otimes \mathscr{V}^2)^U)_{\mathscr{F}_T} \arrow[r, "\overline{\kappa}^{\mathscr{V}^1 \otimes \mathscr{V}^2}"] & \mathscr{V}^1_{\mathscr{F}_G} \otimes \mathscr{V}^2_{\mathscr{F}_G}
  \end{tikzcd}$$

  commutes.

Remark 2.5.7. To simplify the notation, we will write $\mathscr{L}^{\hat{\lambda}} := (\mathscr{V}^{\hat{\lambda}})^U$ and $\overline{\kappa}^{\hat{\lambda}}$ for the embedding attached to the highest weight module of G of highest weight $\hat{\lambda} \in \Lambda_G^+$. It follows by construction that it suffices to consider only the embeddings induced by the highest weight Weyl G-modules attached to the fundamental weights $\hat{\varpi}_i \in \hat{\Lambda}_G^+$ for $i \in \mathscr{J}$, using our assumption that the derived group of G is simply connected.

This gives rise to a well-defined v-stack and, using this description, we get well-defined morphisms $\overline{\mathfrak{p}} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_G$ and $\overline{\mathfrak{q}} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_T$ via projecting the data $(\mathscr{F}_G, \mathscr{F}_T, \overline{\kappa})$ to the first and second factor, respectively. We also get a natural map $j : \mathrm{Bun}_B \to \overline{\mathrm{Bun}}_B$. Now, to conclude this section, we prove some basic things about its geometry.

Proposition 2.5.8. *The v-stack $\overline{\mathrm{Bun}}_B$ is an Artin v-stack, and the map $j : \mathrm{Bun}_B \to \overline{\mathrm{Bun}}_B$ is an open immersion.*

Proof. It suffices to show the claim after base-change to an algebraically closed perfectoid field $\mathrm{Spa}(F, \mathscr{O}_F)$. We write X for the associated Fargues-Fontaine curve. Recall that, given a scheme Y, one defines the affine closure to be $\overline{Y} = \mathrm{Spec}\,\Gamma(Y, \mathscr{O}_Y)$. We let $\overline{G/U}$ be the affine closure of G/U. Viewing this as a constant scheme over X, we consider the stack:

$$Z := [(\overline{G/U})/(T \times G)] \to X$$

Now, it follows by [BG02, Theorem 1.12], that, for $S \in \mathrm{Perf}$, a section

$$\begin{tikzcd} & Z \arrow[d] \\ X_S \arrow[r] \arrow[ur, dotted, "s"] & X \end{tikzcd}$$

is equivalent to the datum of a T-bundle (resp. G-bundle) $\mathscr{F}_T$ (resp. $\mathscr{F}_G$) on X_S together with a family of maps $\overline{\kappa}^{\mathscr{V}}$ of $\mathscr{O}_{X_S}$-modules, satisfying the Plücker conditions of Definition 2.5.6. Therefore, if we consider $\mathscr{M}_Z$, the moduli stack parameterizing such sections, then $\overline{\mathrm{Bun}}_B$ is the sub-functor corresponding to the locus where these maps are injective after pulling back to a geometric point. By [Ham21a, Remark 3.3], this is an open subfunctor. By [Ham21a, Theorem 1.7], $\mathscr{M}_Z$ is an Artin v-stack; therefore, the same is true for $\overline{\mathrm{Bun}}_B$. It remains to see that Bun_B is an open sub-functor. Now, by the work of [Gro83], it follows that $\overline{G/U}$ is strongly quasi-affine in the sense that $G/U \hookrightarrow \overline{G/U}$ is an open immersion. This induces an open immersion of stacks

$$[(G/U)/(T \times G)] = [X/B] \hookrightarrow Z$$

which, after passing to moduli stacks of sections, gives a natural map $\mathrm{Bun}_B \to \mathscr{M}_Z$ factoring through the open immersion $\overline{\mathrm{Bun}}_B \hookrightarrow \mathscr{M}_Z$. The map $\mathrm{Bun}_B \to \mathscr{M}_Z$ is an open immersion, since $[X/B] \hookrightarrow Z$ is an open immersion, by arguing as in [FS21, Proposition IV.4.22]. $\square$

Properness of Compactifications

We now seek to show that the morphism $\overline{\mathfrak{p}} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_G$ is indeed a compactification of the map $\mathfrak{p}$. In particular, we write $\overline{\mathrm{Bun}}_B^{\overline{\nu}}$ for the pre-image of the connected component $\mathrm{Bun}_T^{\overline{\nu}} \subset \mathrm{Bun}_T$. Later we will see that $\overline{\mathrm{Bun}}_B^{\overline{\nu}}$ defines the connected components of $\overline{\mathrm{Bun}}_B$. This will follow from Corollary 2.5.3 and showing that $\mathrm{Bun}_B^{\overline{\nu}}$ is dense inside $\overline{\mathrm{Bun}}_B^{\overline{\nu}}$ (Proposition 2.5.22). We write $\overline{\mathfrak{q}}^{\overline{\nu}} : \overline{\mathrm{Bun}}_B^{\overline{\nu}} \to \mathrm{Bun}_T^{\overline{\nu}}$ and $\overline{\mathfrak{p}}^{\overline{\nu}} : \overline{\mathrm{Bun}}_B^{\overline{\nu}} \to \mathrm{Bun}_G$ for the maps induced by $\overline{\mathfrak{q}}$ and $\overline{\mathfrak{p}}$, respectively. This section will be dedicated to proving the following.

Proposition 2.5.9. *For all $\overline{\nu} \in \Lambda_{G,B}$, the map $\overline{\mathfrak{p}}^{\overline{\nu}}$ is representable in nice diamonds and proper.*

Proof. Consider $S \in \mathrm{Perf}$ and an S-point of Bun_G corresponding to a G-bundle $\mathscr{F}_G$. We let Z denote the fiber of $\overline{\mathfrak{p}}$ over the S-point defined by $\mathscr{F}_G$. We need to show $Z \to S$ is representable in nice diamonds and proper. To this end, let $\hat{\varpi}_i$ for $i \in \mathscr{J}$ be the set of fundamental weights. The G-bundle $\mathscr{F}_G$ then defines a finite set of vector bundles $(\mathscr{V}^{\hat{\varpi}_i})_{\mathscr{F}_G} =: \mathscr{V}_i$ for $i \in \mathscr{J}$. For $\overline{\nu} \in \Lambda_{G,B}$, we set $d_i := \langle \hat{\varpi}_i, \overline{\nu} \rangle$. We consider the space

$$P_i := (\mathscr{H}^0(\mathscr{V}_i(-d_i)) \setminus \{0\})/\underline{\mathbb{Q}}_p^* \to S$$

where $\mathscr{H}^0(\mathscr{V}_i(-d_i))$ is the Banach-Colmez space parameterizing global sections of the vector bundle $\mathscr{V}_i$ twisted by $-d_i$ and $\{0\}$ denotes the 0-section. It follows, by [FS21, Proposition II.2.16], that $P_i \to S$ is representable in nice diamonds and proper. Therefore, the same is true for the product:

$$\prod_{i \in \mathscr{J}} P_i \to S$$

This will parametrize line bundles $\mathscr{L}_i$ of degree d_i together with a map

$$\mathscr{L}_i \to \mathscr{V}_i$$

of $\mathscr{O}_{X_S}$-modules whose pullback to each geometric point is an injection of coherent sheaves. Therefore, we get a natural map

$$X \to \prod_{i=1}^{r} P_i$$

remembering only the embeddings defined by the fundamental weights $\hat{\varpi}_i$. Now, by the Remark proceeding 2.5.6, it follows that this is an injective map into the subspace where the Plücker relations are satisfied. The desired claim would therefore follow from showing that this locus is closed. To see this, note that the commutativity conditions describing the Plücker relations can be expressed in terms of the vanishing of some linear combination of morphisms of $\mathscr{O}_{X_S}$-modules. Using this, one reduces to checking that, given a vector bundle $\mathscr{T}$ on X_S, the point in $\mathscr{H}^0(\mathscr{T})$ defined by the zero section of $\mathscr{T}$ is closed. By choosing an injection $\mathscr{T} \hookrightarrow \mathscr{O}(m)^N$ for sufficiently large m and N, one obtains an injective map $\mathscr{H}^0(\mathscr{T}) \to \mathscr{H}^0(\mathscr{O}(m)^N)$ of diamonds compatible with the zero section. Thus, one reduces to checking the claim for $\mathscr{T} = \mathscr{O}(m)^N$. In this case, it is nothing more than [Far20, Lemma 2.10]. $\square$

Now, we seek to describe the finer structure of these compactifications. In particular, a key role will be played by their stratifications. To do this, we will need to take a brief detour to discuss some properties of some $\mathscr{O}_{X_S}$-modules on the Fargues-Fontaine that occur as cokernels of fiberwise injective maps, as considered in the definition of the Drinfeld compactification.

$\mathscr{O}_S$-flat coherent sheaves on the Fargues-Fontaine Curve

Throughout this section, we fix $S \in \mathrm{Perf}$ and note that, in Definition 2.5.6, we imposed the condition that the intervening maps of vector bundles $\mathscr{F} \to \mathscr{F}'$ on X_S satisfy the property that their pullback to each geometric point of S is an injection of coherent sheaves on X. The stratification on the Drinfeld compactification will be given by fixing the length of the torsion of the cokernel of such morphisms. Normally, one could appeal to classical results on flat coherent sheaves in families to get a handle on the structure of such strata; however, as the Fargues-Fontaine curve X_S is non-Noetherian unless S is a field, one cannot naively define a category of coherent sheaves on it. Nonetheless, we can still define the following.

Definition 2.5.10. [AL21b, Definition 2.8] For $S \in \mathrm{Perf}$, a flat coherent $\mathscr{O}_{X_S}$-module on X_S is an $\mathscr{O}_{X_S}$-module $\mathscr{F}$ which can, locally for the analytic topology on X_S, be written as the cokernel of a fiberwise injective map of bundles on X_S. Equivalently, the map remains an injection after pulling back to any $T \in \mathrm{Perf}_S$.

Remark 2.5.11. By [AL21b, Proposition 2.6], it follows that we can always find a global presentation of an S-flat coherent sheaf $\mathscr{F}$ as a two term complex of vector bundles on X_S.

Remark 2.5.12. We note that in the classical context of a relative projective curve $X \to S$ over a reduced Noetherian scheme S the analogue of this condition for a coherent sheaf $\mathscr{F}$ is equivalent to insisting that $\mathscr{F}$ is $\mathscr{O}_S$-flat (See [AL21b, Remark 2.9]). However, as discussed above, the notion of coherent sheaves make no sense in this context, and the notion of flatness also does not make sense as $\mathscr{O}_{X_S}$ is not a module over $\mathscr{O}_S$.

We have the following easy lemma, which gives a homological criterion characterizing flat coherent $\mathscr{O}_{X_S}$-modules.

Lemma 2.5.13. *Let $\mathscr{F}$ be an $\mathscr{O}_{X_S}$-module. For integers $a \leq b$ write $\mathrm{Perf}_{[a,b]}(X_S)$ for the derived category of perfect complexes of Tor-amplitude $[a,b]$, as in [AL21b, Section 2.1]. The following conditions are equivalent.*

1. *$\mathscr{F}$ is a flat coherent $\mathscr{O}_{X_S}$-module.*

2. *$\mathscr{F}$ is represented by an object in $\mathrm{Perf}_{[-1,0]}(X_S)$ and $\mathrm{Tor}_{1,\mathscr{O}_{X_S}}(\mathscr{O}_{X_T},\mathscr{F})$ is trivial for all $T \in \mathrm{Perf}_S$.*

Proof. For the forward direction, by definition $\mathscr{F}$ is represented by an object in $\mathrm{Perf}_{[-1,0]}(X_S)$. For the other condition, we choose a presentation

$$0 \to \mathscr{F}_{-1} \to \mathscr{F}_0 \to \mathscr{F} \to 0$$

of $\mathscr{F}$ as the cokernel of a fiberwise injective map of vector bundles. Now, since the first map is injective after tensoring by $-\otimes_{\mathscr{O}_{X_S}} \mathscr{O}_{X_T}$ for any $T \in \mathrm{Perf}_S$, it follows easily by the associated long exact sequence that $\mathrm{Tor}_{1,\mathscr{O}_{X_S}}(\mathscr{O}_{X_T},\mathscr{F}) = 0$. For the converse direction, it follows from the proof of [AL21b, Proposition 2.6] that if $\mathscr{F}$ is represented by an object in $\mathrm{Perf}_{[-1,0]}(X_S)$ then it can be globally represented by a two term complex of vector bundles on X_S. Choosing such a presentation $\mathscr{F}_{-1} \to \mathscr{F}_0$, it follows that the defining map must be fiberwise injective by the vanishing of $\mathrm{Tor}_{1,\mathscr{O}_{X_S}}(\mathscr{O}_{X_T},\mathscr{F})$ for all $T \in \mathrm{Perf}_S$. $\square$

This simple lemma allows us to see the following.

Lemma 2.5.14. *Flat coherent sheaves on X_S are stable under taking kernels of surjections and cokernels of fiberwise injective $\mathscr{O}_{X_S}$-module maps.*

Proof. We explain the case of taking cokernels of injections with the case of surjections being similar. We consider two flat coherent sheaves $\mathscr{F}$ and $\mathscr{F}'$ on X_S and a short exact sequence

$$0 \to \mathscr{F} \to \mathscr{F}' \to \mathscr{F}'/\mathscr{F} \to 0$$

of $\mathscr{O}_{X_S}$-modules where the first map is fiberwise injective. It easily follows from the corresponding long exact sequences of Tors, the previous lemma applied to $\mathscr{F}$ and $\mathscr{F}'$, and the fiberwise injectivity of the first map that $\mathrm{Tor}_{i,\mathscr{O}_{X_S}}(\mathscr{O}_{X_T}, \mathscr{F}'/\mathscr{F})$ is trivial for all $T \in \mathrm{Perf}_S$ and $i \geq 1$. Therefore, by the previous lemma, it suffices to show that $\mathscr{F}'/\mathscr{F} \in \mathrm{Perf}_{[-1,0]}(X_S)$. Since $\mathscr{F}, \mathscr{F}' \in \mathrm{Perf}_{[-1,0]}(X_S)$ it follows that $\mathscr{F}'/\mathscr{F} \in \mathrm{Perf}_{[-2,0]}(X_S)$, and, by the vanishing of $\mathrm{Tor}_{i,\mathscr{O}_{X_S}}(\mathscr{O}_{X_T}, \mathscr{F}'/\mathscr{F})$ for $i \geq 2$, it must lie in $\mathrm{Perf}_{[-1,0]}(X_S)$. $\square$

Now, consider the underlying topological space of S, denoted $|S|$. We can consider a geometric point $s \in |S|$ and the pullback of such a flat coherent $\mathscr{O}_{X_S}$-module $\mathscr{F}$ to X_s. The scheme X_s will just be the usual Fargues-Fontaine curve over a geometric point so it is in particular a Dedekind scheme, and $\mathscr{F}|_{X_s}$ will just be a coherent sheaf. Therefore, we have a decomposition

$$\mathscr{F}|_{X_s} \simeq \mathscr{F}^{\mathrm{tors}}|_{X_s} \oplus \mathscr{F}^{\mathrm{vb}}|_{X_s}$$

where $\mathscr{F}^{\mathrm{tors}}|_{X_s}$ (resp. $\mathscr{F}^{\mathrm{vb}}|_{X_s}$) is a torsion sheaf (resp. vector bundle) on X_s. Given such a torsion sheaf, we write $\lambda(\mathscr{F}|_{X_s}) := \ell(\mathscr{F}^{\mathrm{tors}}|_{X_s})$ for the length of this torsion sheaf. Our main aim is to prove the following proposition. We would like to thank David Hansen for supplying the idea behind its proof.

Proposition 2.5.15. *If $\mathscr{F}$ is a flat coherent sheaf on X_S then the function*

$$|S| \to \mathbb{N}_{\geq 0}$$
$$s \mapsto \lambda(\mathscr{F}|_{X_s})$$

is upper semi-continuous. Moreover, if this function is locally constant on S, then there exists a unique short exact sequence of flat coherent sheaves on X_S

$$0 \to \mathscr{F}^{\mathrm{tors}} \to \mathscr{F} \to \mathscr{F}^{\mathrm{vb}} \to 0$$

where $\mathscr{F}^{\mathrm{vb}}$ is a vector bundle on X_S, and $\mathscr{F}^{\mathrm{tors}}$ is a torsion sheaf in the sense that its pullback to each geometric point is a torsion sheaf.

Proof. To do this, we will need to prove the following lemma.

Lemma 2.5.16. *Let $\mathscr{F}$ be a flat coherent sheaf on X_S. We consider the v-sheaf on* Perf_S*, denoted $\mathscr{H}^0(\mathscr{F})$, which sends $T \in \mathrm{Perf}_S$ to the set of global sections of $\mathscr{F}$. The following is true.*

1. *$\mathscr{H}^0(\mathscr{F}) \to S$ is separated and representable in nice diamonds.*
2. *The v-sheaf $(\mathscr{H}^0(\mathscr{F}) \setminus \{0\})/\underline{\mathbb{Q}_p^*} \to S$ given by deleting the 0-section and quotienting out by the scaling action by $\underline{\mathbb{Q}_p}^*$ is proper and representable in nice diamonds over S.*

Proof. We can check all claims analytically locally on S. First note, by [AL21b, Lemma 2.12], that analytically locally on S we can find an exact sequence

$$0 \to \mathscr{E} \to \mathscr{O}_{X_S}(-n)^m \to \mathscr{F} \to 0$$

where $\mathscr{E}$ is a vector bundle, for all n sufficiently large and fixed m. Passing to cohomology, this gives us an exact sequence of v-sheaves:

$$0 \to \mathscr{H}^0(\mathscr{F}) \hookrightarrow \mathscr{H}^1(\mathscr{E}) \to \mathscr{H}^1(\mathscr{O}(-n)^m)$$

Now note that the slopes of $\mathscr{E}$ are necessarily negative after pulling back to a geometric point for n sufficiently large. Therefore, we can assume that $\mathscr{H}^1(\mathscr{E})$ is a nice diamond by [FS21, Proposition II.3.5]. It follows by [FS21, Proposition II.2.5 (i)] that $\mathscr{H}^1(\mathscr{O}_{X_S}(-n)^m)$ is separated which implies that the injective map $\mathscr{H}^0(\mathscr{F}) \to \mathscr{H}^1(\mathscr{E})$ is a closed embedding. Therefore, $\mathscr{H}^0(\mathscr{F})$ is also nice. For the second part, we note that the closed embedding

$$\mathscr{H}^0(\mathscr{F}) \hookrightarrow \mathscr{H}^1(\mathscr{E})$$

is compatible with the 0-section and the scaling action. This reduces us to checking that $(\mathscr{H}^1(\mathscr{E}) \setminus \{0\})/\underline{\mathbb{Q}}_p^* \to S$ is proper, which follows from [FS21, Proposition II.3.5]. □

In particular, we will need the following Corollary.

Corollary 2.5.17. *Let $\mathscr{F}$ and $\mathscr{F}'$ be two flat coherent sheaves on X_S. We consider the v-sheaf on* Perf_S*, denoted $\mathscr{H}om(\mathscr{F}, \mathscr{F}')$, which sends $T \in \mathrm{Perf}_S$ to the set of $\mathscr{O}_{X_S}$-module homomorphisms $\mathscr{F} \to \mathscr{F}'$. The following is true.*

1. *$\mathscr{H}om(\mathscr{F}, \mathscr{F}') \to S$ is separated and representable in nice diamonds.*

2. *The diamond* $(\mathscr{H}om(\mathscr{F},\mathscr{F}')\setminus\{0\})/\underline{\mathbb{Q}_p^*}\to S$ *given by deleting the* 0*-map and quotienting out by the scaling action by* $\underline{\mathbb{Q}}_p^*$ *is proper over S.*

Proof. By applying [AL21b, Lemma 2.12], we have analytically locally on S a short exact sequence

$$0\to\mathscr{E}\to\mathscr{O}_{X_S}(-n)^m\to\mathscr{F}\to 0$$

for all n sufficiently large. Applying $\mathrm{Hom}(-,\mathscr{F}')$, we get an exact sequence of v-sheaves:

$$0\to\mathscr{H}om(\mathscr{F},\mathscr{F}')\to\mathscr{H}om(\mathscr{O}_{X_S}(-n)^m,\mathscr{F}')\to\mathscr{H}om(\mathscr{E},\mathscr{F})$$

In other words, $\mathscr{H}om(\mathscr{F},\mathscr{F}')$ is the fiber of the last map over the 0-section, and in turn the first map is a closed immersion by Lemma 2.5.16 (1) applied to $\mathscr{H}^0(\mathscr{E}^\vee\otimes\mathscr{F}')$. This allows us to replace $\mathscr{F}$ with a vector bundle, and the claim then follows by Lemma 2.5.16 applied to $\mathscr{F}^\vee\otimes\mathscr{F}'$. □

Now that we have gotten this out of the way we can finally turn to the proof of our claim. We consider the v-sheaf

$$\mathscr{S}_{\mathscr{F}}\to S$$

of fiberwise non-zero global sections s of $\mathscr{F}$ which are annihilated by $\mathscr{I}_D\subset\mathscr{O}_{X_S}$ for D a degree 1 relative Cartier divisor $D\subset X_S$. Alternatively, we can view this as parameterizing pairs of (D,f) of a degree 1 Cartier divisor in X_S and a point of $\mathscr{H}om(\mathscr{O}_{X_S}/\mathscr{I}_D,\mathscr{F})\setminus\{0\}$. Using this description, we can factorize the map $\mathscr{S}_{\mathscr{F}}\to S$ as $\mathscr{S}_{\mathscr{F}}\to\mathrm{Div}^1_S\to S$. We let $\mathscr{S}_{\mathscr{F}}/\underline{\mathbb{Q}}_p^*$ be the quotient of this space by the scaling action on the section f. Now the projection $\mathrm{Div}^1_S\to S$ is proper and representable in nice diamonds by combining [FS21, Corollary II.2.4] and [FS21, Proposition II.2.16]. Moreover, by the previous Corollary, we know that $\mathscr{S}_{\mathscr{F}}/\underline{\mathbb{Q}}_p^*\to\mathrm{Div}^1_S$ is proper and representable in nice diamonds because it is a fibration in the spaces $\mathscr{H}om(\mathscr{O}_{X_T}/\mathscr{I}_D,\mathscr{F}_T)\setminus\{0\}/\underline{\mathbb{Q}_p}^*$ for $T\in\mathrm{Perf}_S$ and D a degree 1 relative Cartier divisor in X_T. It follows that the image of $\mathscr{S}_{\mathscr{F}}\to S$ is closed. We note that this coincides with the locus where $\lambda(\mathscr{F})>0$, so we have deduced a special case of the upper semi-continuity result[4]. Now we argue by induction. In particular, by replacing S by its image, it follows by properness that $\mathscr{S}_{\mathscr{F}}\to S$ is a v-cover. Therefore, after v-localization, we can assume that $\mathscr{S}_{\mathscr{F}}\to S$ admits

[4]We could have also deduced this special case by arguing as in the proof of Proposition 2.5.8.

a section, which implies that there exists a degree 1 Cartier divisor $D \subset X_S$ and a fiberwise non-zero section $s : \mathscr{O}_{X_S}/\mathscr{I}_D \to \mathscr{F}$. Using Lemma 2.5.14, we can replace $\mathscr{F}$ with $\mathscr{F}' = \mathscr{F}/s(\mathscr{O}_{X_S})$, noting that the locus where $\lambda(\mathscr{F}') > 0$ coincides with the locus where $\lambda(\mathscr{F}) > 1$. Therefore, the claimed upper semi-continuity result follows. For the second claim, uniqueness is clear, and, by [AL21b, Theorem 2.11], the category of flat coherent sheaves on X_S satisfies v-descent, so it suffices to show the claim up to a v-localization. However, now via induction we can just argue as above, using the sections s produced above and Lemma 2.5.14 to give rise to the map $\mathscr{F}^{\mathrm{tors}} \to \mathscr{F}$. □

Now we can reap the fruit of our labor in this section, and use it to show that the compactification $\overline{\mathrm{Bun}}_B$ has a well-behaved stratification.

Stratifications

We now turn our attention to stratifying $\overline{\mathrm{Bun}}_B$. In particular, for an element $\overline{\nu} \in \Lambda_{G,B}^{pos} \setminus \{0\}$, we write $\overline{\nu} = \sum_{i \in \mathscr{J}} n_i \alpha_i$ as a positive linear combination of the elements corresponding to Γ-orbits of simple positive coroots α_i. We let $\mathrm{Div}^{(\overline{\nu})}$ be the partially symmetrized power of the mirror curve attached to it, as in §2.3.3. We have a map of Artin v-stacks

$$j_{\overline{\nu}} : \mathrm{Div}^{(\overline{\nu})} \times \mathrm{Bun}_B \to \overline{\mathrm{Bun}}_B$$

sending a tuple $(\{(D_i)_{i \in \mathscr{J}}\}, \mathscr{F}_G, \mathscr{F}_T, \kappa^{\hat{\lambda}})$, to the tuple

$$(\mathscr{F}_G, \mathscr{F}_T(-\sum_{i \in J} \alpha_i \cdot D_i), \overline{\kappa}^{\hat{\lambda}})$$

where $\overline{\kappa}^{\hat{\lambda}}$ is the natural composition

$$(\mathscr{L}^{\hat{\lambda}})_{\mathscr{F}_T}(-\sum_{i \in J} \langle \alpha_i, \hat{\lambda} \rangle \cdot D_i) \to \mathscr{L}^{\hat{\lambda}}_{\mathscr{F}_T} \xrightarrow{\kappa^{\hat{\lambda}}} (\mathscr{V}^{\hat{\lambda}})_{\mathscr{F}_G}$$

defined by the unique effective modification of T-bundles of the specified meremorphy and support. We now make the following definition.

Definition 2.5.18. For $\overline{\nu} \in \Lambda_{G,B}^{\mathrm{pos}} \setminus \{0\}$, we define the v-stack ${}_{\overline{\nu}}\overline{\mathrm{Bun}}_B$ (resp. ${}_{\geq \overline{\nu}}\overline{\mathrm{Bun}}_B$) to be the locus where, for all $\hat{\lambda} \in \hat{\Lambda}_G^+$, the cokernels $\mathscr{V}^{\hat{\lambda}}/\mathrm{Im}(\overline{\kappa}^{\hat{\lambda}})$ have torsion of length equal to (resp. greater than) $\langle\hat{\lambda},\overline{\nu}\rangle$ after pulling back to any geometric point. By Proposition 2.5.15 (i), ${}_{\overline{\nu}}\overline{\mathrm{Bun}}_B$ is a locally closed substack of $\overline{\mathrm{Bun}}_B$, and the closure of ${}_{\overline{\nu}}\overline{\mathrm{Bun}}_B$ in $\overline{\mathrm{Bun}}_B$ is contained in ${}_{\geq\overline{\nu}}\overline{\mathrm{Bun}}_B$ (In fact, is equal to it, as will follow from Proposition 2.5.22).

To work with these strata, we will need the following.

Proposition 2.5.19. *For $\overline{\nu} \in \Lambda_{G,B}^{pos} \setminus \{0\}$, the map $j_{\overline{\nu}}$ induces an isomorphism $\mathrm{Bun}_B \times \mathrm{Div}^{(\overline{\nu})} \simeq {}_{\overline{\nu}}\overline{\mathrm{Bun}}_B$ In particular, $j_{\overline{\nu}}$ is a locally closed embedding.*

Proof. It is clear that $j_{\overline{\nu}}$ induces a map into the locally closed stratum ${}_{\overline{\nu}}\overline{\mathrm{Bun}}_B$. We need to exhibit an inverse of this map. We first begin with the following lemma.

Lemma 2.5.20. *Let* Coh *be the v-stack parameterizing, for $S \in \mathrm{Perf}$, flat coherent sheaves on X_S, as in [AL21b, Theorem 2.11]. For $k,n \in \mathbb{N}_{\geq 0}$, we set Coh_n^k to be the locally closed (by Proposition 2.5.15 (i)) substack parameterizing flat coherent sheaves whose torsion length (resp. vector bundle rank) is equal to k (resp. n) after pulling back to a geometric point. There is a well-defined map*

$$\mathrm{Coh}_n^k \to \mathrm{Div}^{(k)}$$

of v-stacks, sending a S-flat coherent sheaf $\mathscr{F}$ with attached short exact sequence $0 \to \mathscr{F}^{\mathrm{tor}} \to \mathscr{F} \to \mathscr{F}^{\mathrm{vb}} \to 0$, as in Proposition 2.5.15 (ii), to the support of $\mathscr{F}^{\mathrm{tor}}$.

Proof. We note that, given $\mathscr{F}$ a S-flat coherent sheaf in Coh_n^k, we have, by Proposition 2.5.15 (ii), a unique short exact sequence

$$0 \to \mathscr{F}^{\mathrm{tor}} \to \mathscr{F} \to \mathscr{F}^{\mathrm{vb}} \to 0$$

where we note that $\mathscr{F}^{\mathrm{tor}}$ will be a S-flat coherent sheaf of generic rank 0, using Lemma 2.5.14. Now we can choose a presentation of this

$$0 \to \mathscr{F}_{-1} \to \mathscr{F}_0 \to \mathscr{F}^{\mathrm{tor}} \to 0$$

for two vector bundles $\mathscr{F}_{-1}$ and $\mathscr{F}_0$. Since $\mathscr{F}^{\mathrm{tor}}$ is of constant rank 0 by construction, the first map must be a fiberwise injective map of vector bundles locally of the same rank. Therefore, analytically locally on S, we can then take the top exterior power of the map $\mathscr{F}_{-1} \to \mathscr{F}_0$, and this will give rise to a fiberwise injective map of line bundles, which in turn gives rise to a relative Cartier divisor D in X_S (cf. [KM76] to see that this is independent of the choice of presentation). If $\mathscr{F}$ defines a point in Coh_n^k then D must be of degree k, and we get the desired map. □

We need to exhibit an inverse to the natural map $\mathrm{Bun}_B \times \mathrm{Div}^{(\overline{\nu})} \to {}_{\overline{\nu}}\mathrm{Bun}_B$. To do this, for $S \in \mathrm{Perf}$, consider a short exact sequence of

$$0 \to \mathscr{V}_1 \to \mathscr{V}_{n+1} \to \mathscr{V}_n \to 0$$

of $\mathscr{O}_{X_S}$-modules, where $\mathscr{V}_1$ (resp. $\mathscr{V}_{n+1}$) is a line bundle (resp. rank $n+1$ vector bundle), and the first map is a fiberwise-injective map, so that $\mathscr{V}_n$ is S-flat. Assume that $\mathscr{V}_n$ defines a point in Coh_n^k for some k, and let D be the degree k Cartier divisor in X_S defined by the previous Lemma. It follows by an application of Proposition 2.5.15 (ii) that we have a short exact sequence

$$0 \to \mathscr{V}_n^{\mathrm{tors}} \to \mathscr{V}_n \to \mathscr{V}_n^{\mathrm{vb}} \to 0$$

where $\mathscr{V}_n^{\mathrm{tors}}$ will define a point in Coh_0^k and $\mathscr{V}_n^{\mathrm{vb}}$ is a rank n vector bundle. Let $\widetilde{\mathscr{V}_1}$ denote the preimage of $\mathscr{V}_n^{\mathrm{tors}}$ in $\mathscr{V}_{n+1}$. It is then easy to see that $\mathscr{V}_1 \to \mathscr{V}_{n+1}$ gives rise to an isomorphism $\mathscr{V}_1(D) \simeq \widetilde{\mathscr{V}_1}$. Now, given a S-point of ${}_{\overline{\nu}}\mathrm{Bun}_B$, we can construct the desired inverse by applying the above argument to the short exact sequences coming from the embeddings $\overline{\kappa}^{\hat{\omega}_i}$, for the fundamental weights $\hat{\omega}_i \in \hat{\Lambda}_G^+$. □

With this locally closed stratification in hand, we can now study how Hecke correspondences base-changed to $\overline{\mathrm{Bun}}_B$ interact with it. This will play a key role in the proof of the Hecke eigensheaf property, and reduce showing the density of $\mathrm{Bun}_B \subset \overline{\mathrm{Bun}}_B$ to studying the usual closure relationships for semi-infinite Schubert cells.

The Key diagram and Density of the Compactification

We would now like to describe how Hecke correspondences on Bun_G interact with pullback along the map $\overline{\mathfrak{p}} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_G$. This will be used to show the filtered Hecke eigensheaf property for the geometric Eisenstein series, analogous to the analysis carried out in [BG02, Section 3]. In this section, we will just study the geometry of the relevant diagram and use it to deduce that the open inclusion $\mathrm{Bun}_B \subset \overline{\mathrm{Bun}}_B$ defines a dense subset in the underlying topological space of $\overline{\mathrm{Bun}}_B$. We fix a finite index set I, and consider the Hecke stacks $\mathrm{Hck}_{G,E}^I$ base-changed to the field E over which G splits. We consider the usual diagram

$$\mathrm{Bun}_G \times \mathrm{Div}_E^I \xleftarrow{h_G^{\leftarrow} \times \pi} \mathrm{Hck}_{G,E}^I \xrightarrow{h_G^{\rightarrow}} \mathrm{Bun}_G$$

as in §2.3 and §2.4. We now fix a tuple of geometric dominant characters $(\lambda_i)_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+)^I$, and restrict to the locus $\mathrm{Hck}_{G,\leq(\lambda_i)_{i\in I},E}$ where the meromorphy of this modification is bounded by $(\lambda_i)_{i\in I}$. We define the v-stack $\overline{Z}^I_{(\lambda_i)_{i\in I}}$ by the Cartesian diagram:

$$\begin{tikzcd} \overline{Z}^I_{(\lambda_i)_{i\in I}} \arrow[r, "{'h^{\rightarrow}_G}"] \arrow[d, "{'\overline{\mathfrak{p}}}"] & \overline{\mathrm{Bun}_B} \arrow[d, "\overline{\mathfrak{p}}"] \\ \mathrm{Hck}^I_{G,\leq(\lambda_i)_{i\in I},E} \arrow[r, "h^{\rightarrow}_G"] & \mathrm{Bun}_G \end{tikzcd}$$

By definition, $\overline{Z}^I_{(\lambda_i)_{i\in I}}$ parametrizes pairs of G-bundles $(\mathscr{F}_G, \mathscr{F}'_G)$ together with a modification $\mathscr{F}_G \dashrightarrow \mathscr{F}'_G$ with meromorphy bounded by λ_i at Cartier divisors D_i for $i \in I$ and an enhanced B-structure on $\mathscr{F}'_G$ specified by maps $\overline{\kappa}'^{\hat{\lambda}}$ for $\hat{\lambda} \in \hat{\Lambda}^+_G$. The fact that the modification $\mathscr{F}_G \dashrightarrow \mathscr{F}'_G$ has meromorphy bounded by $(\lambda_i)_{i\in I}$ implies that, for all $\hat{\lambda} \in \hat{\Lambda}^+_G$, we have an inclusion:

$$\mathscr{V}^{\hat{\lambda}}_{\mathscr{F}'_G} \subset \mathscr{V}^{\hat{\lambda}}_{\mathscr{F}_G}\Big(\sum_{i\in I}\langle\hat{\lambda}, -w_0(\lambda_{i\Gamma})\rangle \cdot D_i\Big)$$

Therefore, the embeddings

$$\overline{\kappa}'^{\hat{\lambda}} : \mathscr{L}^{\hat{\lambda}}_{\mathscr{F}'_T} \hookrightarrow \mathscr{V}^{\hat{\lambda}}_{\mathscr{F}'_G}$$

give rise to a map:

$$\overline{\kappa}^{\hat{\lambda}} : \mathscr{L}^{\hat{\lambda}}_{\mathscr{F}'_T}\Big(\sum_{i\in I}\langle\hat{\lambda}, w_0(\lambda_{i\Gamma})\rangle \cdot D_i\Big) \hookrightarrow \mathscr{V}^{\hat{\lambda}}_{\mathscr{F}_G}$$

This defines for us a morphism

$$\phi_{(\lambda_i)_{i\in I}} : \overline{Z}^I_{(\lambda_i)_{i\in I}} \to \overline{\mathrm{Bun}_B} \times \mathrm{Div}^I_E$$

which records the point in $\overline{\mathrm{Bun}_B} \times \mathrm{Div}^I_E$ defined by the pair $(\overline{\kappa}^{\hat{\lambda}}, (D_i)_{i\in I})$. This sits in a commutative diagram

$$\begin{tikzcd} \overline{\mathrm{Bun}_B} \times \mathrm{Div}^I_E \arrow[d, "\overline{\mathfrak{p}}\times \mathrm{id}"] & \overline{Z}^I_{(\lambda_i)_{i\in I}} \arrow[l, "\phi_{(\lambda_i)_{i\in I}}"'] \arrow[r, "{'h^{\rightarrow}_G}"] \arrow[d, "{'\mathfrak{p}}"] & \overline{\mathrm{Bun}_B} \arrow[d, "\overline{\mathfrak{p}}"] \\ \mathrm{Bun}_G \times \mathrm{Div}^I_E & \mathrm{Hck}^I_{G,\leq(\lambda_i)_{i\in I},E} \arrow[l, "h^{\leftarrow}_G\times\pi"'] \arrow[r, "h^{\rightarrow}_G"] & \mathrm{Bun}_G \end{tikzcd} \tag{2.2}$$

where we note that left square is not Cartesian. We will not use the space $\overline{Z}^I_{(\lambda_i)_{i\in I}}$ at all in our arguments, but it should be important for future applications. For our purposes, we consider $Z^I_{(\lambda_i)_{i\in I}}$, the space obtained by replacing $\overline{\mathrm{Bun}}_B$ with Bun_B on the right hand side of the diagram. This sits in an analogous diagram

$$\begin{tikzcd} \overline{\mathrm{Bun}}_B \times \mathrm{Div}^I_E \arrow[d, "\overline{\mathfrak{p}}\times\mathrm{id}"] & Z^I_{(\lambda_i)_{i\in I}} \arrow[l, "\phi_{(\lambda_i)_{i\in I}}"'] \arrow[r, "{'h^{\rightarrow}_G}"] \arrow[d, "{'\mathfrak{p}}"] & \mathrm{Bun}_B \arrow[d, "\mathfrak{p}"] \\ \mathrm{Bun}_G \times \mathrm{Div}^I_E & \mathrm{Hck}^I_{G,\leq(\lambda_i)_{i\in I},E} \arrow[l, "h^{\leftarrow}_G\times\pi"'] \arrow[r, "h^{\rightarrow}_G"] & \mathrm{Bun}_G \end{tikzcd} \tag{2.3}$$

As we will see, the proof of the filtered Hecke eigensheaf Property will ultimately reduce to contemplating the fibers of the morphism $\phi_{(\lambda_i)_{i\in I}}$. For our purposes, it will suffice to consider the pullback of this diagram to a geometric point $\mathrm{Spa}(F,\mathscr{O}_F) \to \mathrm{Div}^I_E$. We denote the resulting space by ${}_x\overline{Z}^I_{(\lambda_i)_{i\in I}}$. It sits in a diagram of the form

$$\begin{tikzcd} {}_x\overline{\mathrm{Bun}}_B \arrow[d, "\overline{\mathfrak{p}}"] & {}_x\overline{Z}^I_{(\lambda_i)_{i\in I}} \arrow[l, "\phi_{(\lambda_i)_{i\in I}}"'] \arrow[r, "{'h^{\rightarrow}_G}"] \arrow[d, "{'\overline{\mathfrak{p}}}"] & \overline{\mathrm{Bun}}_B \arrow[d, "\overline{\mathfrak{p}}"] \\ {}_x\mathrm{Bun}_G & {}_x\mathrm{Hck}^I_{G,\leq(\lambda_i)_{i\in I},E} \arrow[l, "h^{\leftarrow}_G\times\pi"'] \arrow[r, "h^{\rightarrow}_G"] & \mathrm{Bun}_G \end{tikzcd}$$

where ${}_x\mathrm{Hck}^I_{G,\leq(\lambda_i)_{i\in I},E}$ is the Hecke stack parameterizing modifications at the tuple of Cartier divisors $(D_i)_{i\in I}$ corresponding to x and ${}_x\overline{\mathrm{Bun}}_B$ (resp. ${}_x\mathrm{Bun}_G$) denotes the base change of $\overline{\mathrm{Bun}}_B$ (resp. Bun_G) to $\mathrm{Spa}(F,\mathscr{O}_F)$. Consider a tuple $(\nu_i)_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I$ and write $\overline{\nu} := \sum_{i\in I}\nu_{i\Gamma}$. Let E_{ν_i} denote the reflex field of ν_i. We view the geometric point $x \to \mathrm{Div}^I$ as a geometric point of $\mathrm{Div}^{(\overline{\nu})}$ via composing with the map

$$\Delta_{(\nu_i)_{i\in I}} : \mathrm{Div}^I_E \to \prod_{i\in I}\mathrm{Div}^1_{E_{\nu_i}} \xrightarrow{\prod_{i\in I}\triangle_{\nu_i}} \prod_{i\in I}\mathrm{Div}^{(\nu_{i\Gamma})} \to \mathrm{Div}^{(\overline{\nu})}$$

where the last map is given by taking the union of Cartier divisors and Δ_{ν_i} is the twisted diagonal embedding described in §2.3.3. We set ${}_{x,(\nu_i)_{i\in I}}\overline{\mathrm{Bun}}_B$ to be the pullback of the locally closed stratum ${}_{\overline{\nu}}\overline{\mathrm{Bun}}_B \simeq \mathrm{Div}^{(\overline{\nu})} \times \mathrm{Bun}_B$ to this geometric point. The substack ${}_{x,(\nu_i)_{i\in I}}\overline{\mathrm{Bun}}_B$ corresponds to the locus where the embeddings

$$\mathscr{L}^{\hat{\lambda}}_{\mathscr{F}_T} \to \mathscr{V}^{\hat{\lambda}}_{\mathscr{F}_G}$$

have a zero of order given by ν_i at D_i for all dominant characters $\hat{\lambda}$ of G and all $i \in I$, and a zero nowhere else. We now consider the open substack ${}_{x,0}\overline{\mathrm{Bun}}_B = {}_x\mathrm{Bun}_B$. Since the maps κ have no zero at the Cartier divisors corresponding to x, it follows that they define an L^+B torsor over ${}_{x,0}\overline{\mathrm{Bun}}_B$, which we denote by ${}_x\mathscr{B}$. We then consider the map

$$i_{(\nu_i)_{i\in I}} : \overline{\mathrm{Bun}}_B \times \mathrm{Div}^I_E \to \overline{\mathrm{Bun}}_B \times \mathrm{Div}^I_E$$

given by sending $(\mathscr{F}_G, \mathscr{F}_T, \overline{\kappa}^{\hat{\lambda}}, (D_i)_{i\in I})$ to the object $(\mathscr{F}_G, \mathscr{F}_T(-\sum_{i\in I} \overline{\nu}_i \cdot D_i), \mathscr{L}^{\hat{\lambda}}_{\mathscr{F}_T}(-\sum_{i\in I}\langle \overline{\nu}_i, \hat{\lambda}\rangle \cdot D_i) \hookrightarrow \mathscr{L}^{\hat{\lambda}}_{\mathscr{F}_T} \hookrightarrow \mathscr{V}^{\hat{\lambda}}_{\mathscr{F}_G}, (D_i)_{i\in I})$, and note that the map $i_{(\nu_i)_{i\in I}}$ defines an isomorphism between the pullbacks ${}_{x,0}\overline{\mathrm{Bun}}_B$ and ${}_{x,(\nu_i)_{i\in I}}\overline{\mathrm{Bun}}_B$. Therefore, by transport of structure, we get a L^+B-torsor, denoted ${}_x\mathscr{B}^{(\nu_i)_{i\in I}}$, over ${}_{x,(\nu_i)_{i\in I}}\overline{\mathrm{Bun}}_B$. For $(\nu_i)_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I$, we let ${}_xZ^{I,*,(\nu_i)_{i\in I}}_{(\lambda_i)_{i\in I}}$ (resp. ${}_xZ^{I,(\nu_i)_{i\in I},*}_{(\lambda_i)_{i\in I}}$) be the fibers of ${}'h^{\rightarrow}_G$ (resp. $\phi_{(\lambda_i)_{i\in I}}$) over ${}_{x,(\nu_i)_{i\in I}}\overline{\mathrm{Bun}}_B$. We now have the following Lemma describing these subspaces, which is an analogue of [BG02, Lemma 3.3.6].

Lemma 2.5.21. *For tuples $(\nu_i)_{i\in I}, (\nu'_i)_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I$, geometric dominant cocharacters $(\lambda_i)_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+)^I$, and a geometric point $x \to \mathrm{Div}^I_E$, the following is true.*

1. *There is an isomorphism*

$$ {}_xZ^{I,*,(\nu'_i)_{i\in I}}_{(\lambda_i)_{i\in I}} \simeq {}_x\mathrm{Gr}^I_{G,\leq(-w_0(\lambda_i))_{i\in I}} \times^{L^+B} {}_x\mathscr{B}^{(\nu'_i)_{i\in I}}$$

   where the L^+B action on ${}_x\mathrm{Gr}^I_{G,\leq(-w_0(\lambda_i))_{i\in I}}$ is given by the inclusion $L^+B \hookrightarrow L^+G$.

2. *Under the identification in (1), the substack ${}_xZ^{I,(\nu_i)_{i\in I},(\nu'_i)_{i\in I}}_{(\lambda_i)_{i\in I}} \hookrightarrow {}_xZ^{I,*,(\nu'_i)_{i\in I}}_{(\lambda_i)_{i\in I}}$ identifies with the substack*

$$ {}_x\mathrm{Gr}^I_{G,\leq(-w_0(\lambda_i))_{i\in I},E} \cap {}_x\mathrm{S}^I_{G,(-w_0(\lambda_i)-\nu_i+\nu'_i)_{i\in I},E} \times^{L^+B} {}_x\mathscr{B}^{(\nu'_i)_{i\in I}} \subset {}_x\mathrm{Gr}^I_{G,\leq(-w_0(\lambda_i))_{i\in I},E} \times^{L^+B} {}_x\mathscr{B}^{(\nu'_i)_{i\in I}}$$

3. *When viewed as a stack projecting to ${}_{x,(\nu_i)_{i\in I}}\overline{\mathrm{Bun}}_B$, the stack ${}_xZ^{I,(\nu_i)_{i\in I},(\nu'_i)_{i\in I}}_{(\lambda_i)_{i\in I}}$ identifies with*

$$ {}_x\mathrm{Gr}^I_{G,\leq(\lambda_i)_{i\in I},E} \cap {}_x\mathrm{S}^I_{G,(\nu_i-\nu'_i+w_0(\lambda_i))_{i\in I},E} \times^{L^+B} {}_x\mathscr{B}^{(\nu_i)_{i\in I}}$$

Proof. Follows from the definitions and the description of the semi-infinite cells mentioned in the remark proceeding Definition 2.4.7. □

As mentioned earlier, this description of the fibers will serve a key role in the proof of the Hecke eigensheaf property. For now, we content ourselves by using it to prove density.

Proposition 2.5.22. Bun_B *defines a substack of* $\overline{\mathrm{Bun}}_B$ *which is topologically dense.*

Proof. Consider a geometric point of $s \to \overline{\mathrm{Bun}}_B$ defined by a triple $(\mathscr{F}_G, \mathscr{F}_T, \overline{\kappa})$, and an open substack of $U \subset \overline{\mathrm{Bun}}_B$ containing s. It suffices to show that U contains a point in Bun_B. We assume that $\overline{\kappa}$ has a singularity ν_i defined at distinct Cartier divisors D_i for $i \in I$ corresponding to geometric points $x_i \to \mathrm{Div}^1_E$. We write $x \to \mathrm{Div}^I_E$ for the associated geometric point given by the product of the x_i, and apply the previous Lemma in the situation that $\nu'_i = 0$ for all $i \in I$. By Lemma 2.5.21 (1), we have that ${}_x\mathrm{Gr}^I_{G,\leq(-w_0(\lambda_i))_{i\in I},E} \times^{L^+B} {}_x\mathscr{B} \simeq \prod_{i\in I} {}_{x_i}\mathrm{Gr}_{G,\leq -w_0(\lambda_i),E} \times^{L^+B} {}_{x_i}\mathscr{B}$ maps to $\overline{\mathrm{Bun}}_B$ via the morphism $\phi_{(\lambda_i)_{i\in I}}$. Now, for the given $(\nu_i)_{i\in I}$, we can choose λ_i for all $i \in I$ such that the weight space $V_{\lambda_i}(w_0(\lambda_i) + \nu_i)$ is non-zero. Then, by Lemma 2.5.21 (2) and Corollary 2.4.12, the fiber of $\phi_{(\lambda_i)_{i\in I}}$ over s is non-empty. Therefore, pulling back U, we get a non-empty open subset of $\prod_{i\in I} {}_{x_i}\mathrm{Gr}_{G,\leq -w_0(\lambda_i),E} \times^{L^+B} {}_{x_i}\mathscr{B}$. By the closure relations of Proposition 2.4.14, we get that this open subset must have non-empty intersection with the open subspace $\prod_{i\in I} {}_{x_i}\mathrm{Gr}_{G,\leq -w_0(\lambda_i),E} \cap {}_{x_i}\mathrm{S}_{G,-w_0(\lambda_i),E} \times^{L^+B} {}_{x_i}\mathscr{B}$, but, by another application of Lemma 2.5.21 (2), this tells us that U must have non-trivial intersection with Bun_B. □

2.6 The Normalized Eisenstein Functor and Verdier Duality on Bun_B

Now that we have finished our geometric preparations, we can start to understand the sheaf theory on the moduli stack of B-structures. Our first order of business is to refine our definition of the Eisenstein functor given in the previous section to better respect Verdier duality on the moduli stack Bun_B.

2.6.1 The Normalized Eisenstein Functor

Before proceeding with our analysis of Eisenstein series, we refine the definition of the Eisenstein functor. There is one key problem with our definition, the sheaf

$\Lambda[\dim(\mathrm{Bun}_B)]$ is not Verdier self-dual. In particular, the dualizing object K_{Bun_B} on Bun_B is not isomorphic to $\Lambda[2\dim(\mathrm{Bun}_B)]$; this is only v-locally true on Bun_B. To elucidate the problem, note that, given $\overline{\nu} \in \Lambda_{G,B}$, we can consider the natural map $\mathfrak{q}^{\overline{\nu}} : \mathrm{Bun}_B^{\overline{\nu}} \to \mathrm{Bun}_T^{\overline{\nu}} = [*/\underline{T(\mathbb{Q}_p)}]$. Given a character $\kappa_{\overline{\nu}}$ of $T(\mathbb{Q}_p)$, we can pull this character back along $\mathfrak{q}^{\overline{\nu}}$ to get a sheaf on $\mathrm{Bun}_B^{\overline{\nu}}$. These characters give us sheaves on $\mathrm{Bun}_B^{\overline{\nu}}$, which v-locally will be constant, but are not constant on the nose. Our main Theorem is as follows.

Theorem 2.6.1. *The dualizing object on* Bun_B *is isomorphic to* $\mathfrak{q}^*(\Delta_B)[2\dim(\mathrm{Bun}_B)]$, *with* $\Delta_B \in \mathrm{D}(\mathrm{Bun}_T)$ *as in Definition 2.3.4.*

Before tackling the proof, we record the key consequence of this theorem for us.

Corollary 2.6.2. *The sheaf* $\mathfrak{q}^*(\Delta_B^{1/2})[\dim(\mathrm{Bun}_B)]$ *on* Bun_B *is Verdier self-dual.*

This motivates the definition of the normalized Eisenstein functor.

Definition 2.6.3. We let $\mathrm{IC}_{\mathrm{Bun}_B} := \mathfrak{q}^*(\Delta_B^{1/2})[\dim(\mathrm{Bun}_B)]$. We define the normalized Eisenstein functor:

$$\mathrm{nEis} : \mathrm{D}(\mathrm{Bun}_T) \to \mathrm{D}(\mathrm{Bun}_G)$$
$$\mathscr{F} \mapsto \mathfrak{p}_!(\mathfrak{q}^*(\mathscr{F}) \otimes \mathrm{IC}_{\mathrm{Bun}_B})$$

In particular, we note that we have a natural isomorphism:

$$\mathrm{nEis}(-) \simeq \mathrm{Eis}(- \otimes \Delta_B^{1/2}).$$

We now tackle the proof of Theorem 2.6.1.

2.6.2 The Proof of Theorem 2.6.1

As a warm up, we explain the proof in the case that $G = \mathrm{GL}_2$, proving some key lemmas along the way. We recall that in this case $B(T) \simeq \Lambda_{G,B} \simeq \mathbb{Z}^2$ via the Kottwitz invariant, and so we can index the connected components of Bun_B by a pair of integers. We write $d_{\overline{\nu}}$ for the ℓ-cohomological dimension of the connected component $\mathrm{Bun}_B^{\overline{\nu}}$.

Example 2.6.4. We first consider the case that $\overline{\nu} = (d,d)$ for $d \in \mathbb{Z}$, so $\overline{\nu}$. Note that the connected component $\mathrm{Bun}_B^{\overline{\nu}}$ parametrizes split reductions, and is isomorphic to $[*/\underline{B(\mathbb{Q}_p)}]$. Now we have the following lemma, which is [HKW22, Example 4.2.4].

Lemma 2.6.5. *f Let H be a locally pro-p group then $K_{[*/H]} \simeq Haar(H,\Lambda)^*$, where $Haar(H,\Lambda)$ is the space of Λ-valued right Haar measures on H.*

Therefore, the dualizing object on $\mathrm{Bun}_B^{\overline{\nu}}$ is isomorphic to $Haar(B(\mathbb{Q}_p),\Lambda)^*$. We note that $T(\mathbb{Q}_p)$ acts on this space by δ_B^{-1} by definition of the modulus character. However, the natural action of $T(\mathbb{Q}_p)$ on the dualizing complex will be via the right conjugation action $t \mapsto t^{-1}ut$ on $U(\mathbb{Q}_p)$. Thus, Theorem 2.6.1 is true in this case.

Given a diamond or v-stack $X \to \mathrm{Spd}\,\mathbb{F}_p$, we define the local systems $\Lambda(d)$ by pulling back the local system on $\mathrm{Spd}\,\mathbb{F}_p$ given by the representation of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ for which the geometric Frobenius acts via scaling by $p^{-d} \in \Lambda^*$. We consider the following key example.

Example 2.6.6. We let $Z_{\mathbb{F}_p} := \mathscr{H}^0(\mathscr{O}(1))_{\mathbb{F}_p} \to \mathrm{Spd}\,\mathbb{F}_p$ be the absolute positive Banach-Colmez space parmeterizing sections of the line bundle $\mathscr{O}(1)$ on X_S for $S \in \mathrm{Perf}_{\mathbb{F}_p}$, as in [FS21, Section II.2.2]. The space has a right action of $\mathbb{Q}_p^*$ by the inverse of the scaling map on global sections, and this is the most natural one to consider for the dualizing complex. The diamond $Z_{\mathbb{F}_p}$ is isomorphic to $\mathrm{Spd}\,\mathbb{F}_p[|x^{1/p^\infty}|]$ [FS21, Proposition II.2.5 (iv)]. We let $\mathrm{Frob}_{Z_{\mathbb{F}_p}}$ denote the geometric Frobenius sending $x \mapsto x^p$. Then, by [Han21, Proposition 4.8], we have a natural isomorphism

$$K_{Z_{\mathbb{F}_p}} \simeq \Lambda[2](1)$$

of étale sheaves on $Z_{\mathbb{F}_p}$. We now show that the right action of $\mathbb{Z}_p^*$ on the dualizing complex is trivial. To do this, consider the open subspace $U_{\mathbb{F}_p} = \mathscr{H}^0(\mathscr{O}(1))_{\mathbb{F}_p} \setminus \{0\} \subset Z_{\mathbb{F}_p}$ given by the complement of the 0-section, so that the action of $\mathbb{Q}_p^*$ is in particular free. It suffices to show the analogous claim for $K_{U_{\mathbb{F}_p}}$. We let $K_{U_{\mathbb{F}_p},\overline{\mathbb{Z}}_\ell}$ denote the dualizing complex on $U_{\mathbb{F}_p}$ with respect to étale (not solid sheaves) with $\overline{\mathbb{Z}}_\ell$-coefficients, as defined in [Sch18, Section 26]. Since U is cohomologically smooth, we have that

$$K_{U_{\mathbb{F}_p}} \simeq K_{U_{\mathbb{F}_p},\overline{\mathbb{Z}}_\ell} \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell,$$

implying that it suffices to prove the analogous claim for $K_{U_{\mathbb{F}_p},\overline{\mathbb{Z}}_\ell}$. Moreover, using that the map

$$\mathrm{Aut}(\overline{\mathbb{Z}}_\ell^*) \to \mathrm{Aut}(\overline{\mathbb{Q}}_\ell^*)$$

is injective, it suffices to establish the analogous claim for $K_{U_{\mathbb{F}_p},\overline{\mathbb{Q}}_\ell}$, the dualizing complex with $\overline{\mathbb{Q}}_\ell$-coefficients. Now, to do this, consider the v-stack quotient by the right $\mathbb{Z}_p^*$ action:

$$q : U_{\mathbb{F}_p} \to [U_{\mathbb{F}_p}/\underline{\mathbb{Z}_p^*}].$$

By [Han21, Proposition 4.6], fixing a Haar measure on $\mathbb{Z}_p^*$ determines a unique isomorphism:

$$(q_*(K_{U_{\mathbb{F}_p},\overline{\mathbb{Q}}_\ell}))^{\mathbb{Z}_p^*} \simeq K_{[U_{\mathbb{F}_p}/\underline{\mathbb{Z}_p^*}],\overline{\mathbb{Q}}_\ell},$$

which in particular implies the $\mathbb{Z}_p^*$ action is trivial on $q^*(K_{[U_{\mathbb{F}_p}/\underline{\mathbb{Z}_p^*}],\overline{\mathbb{Q}}_\ell}) \simeq K_{U_{\mathbb{F}_p},\overline{\mathbb{Q}}_\ell}$[5], where this isomorphism follows from [Sch18, Proposition 24.2].

Now, it remains to determine the value of $\kappa(p)$. To elucidate this, we note that, under the identification $Z_{\mathbb{F}_p} \simeq \mathrm{Spd}\,\mathbb{F}_p[|x^{1/p^\infty}|]$, the element $p^{-1} \in \mathbb{Q}_p^*$ acts via the geometric Frobenius $\mathrm{Frob}_{Z_{\mathbb{F}_p}}$ on $Z_{\mathbb{F}_p}$ (since we are looking at the right action). It follows by the previous isomorphism that the value of the character $\kappa(p)$ is determined by the action of $\mathrm{Frob}_{Z_{\mathbb{F}_p}}$ on $\Lambda(1)$, which is just the multiplication by p^{-1} map on Λ. In summary, we have concluded an identification $K_{Z_{\mathbb{F}_p}} \simeq |\cdot|^{-1}[2]$ of sheaves with the right $\mathbb{Q}_p^*$-action scaling action, where $|\cdot|$ is the rank 1 local system on $Z_{\mathbb{F}_p}$ with right $\mathbb{Q}_p^*$-action given by the norm character. From here, we conclude the analogous isomorphism

$$K_{Z_{\overline{\mathbb{F}}_p}} \simeq |\cdot|^{-1}[2]$$

of sheaves with right $\mathbb{Q}_p^*$-action over the algebraic closure.

Let's now push this a bit further and consider the case of a general positive absolute Banach-Colmez space $Z_{\mathbb{F}_p} := \mathscr{H}^0(\mathscr{O}(d))_{\mathbb{F}_p} \to \mathrm{Spd}\,\mathbb{F}_p$ for $d \geq 1$. Now we don't have such a simple presentation as in the case that $d = 1$; however, we claim that we still have a similar relationship between the geometric Frobenius and the scaling action by $p^{\mathbb{Z}} \in \mathbb{Q}_p^*$ on $Z_{\mathbb{F}_p}$. To understand this, we invoke the following explicit description. Recall that, for $S \in \mathrm{Perf}$, we can view the adic Fargues-Fontaine curve $\mathscr{X}_S$ as given by gluing the open Fargues-Fontaine curve $\mathscr{Y}_{S,[1,p]}$ along the map $\phi : \mathscr{Y}_{S,[1,1]} \simeq \mathscr{Y}_{S,[p,p]}$ induced by the geometric Frobenius, with notation as in [FS21, Section II.1]. We write $B_{S,[1,p]}$ (resp. $B_{S,[1,1]}$) for the ring of functions of $\mathscr{Y}_{S,[1,p]}$ (resp. $\mathscr{Y}_{S,[1,1]}$). We make use of the following lemma, which follows from the analysis in [FS21, Section II.2].

[5] We note that these claims only apply to the maximal pro-p subgroup $\mathbb{Z}_p \subset \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p \simeq \mathbb{Z}_p^*$; however, since we are applying this in the setting where $p-1$ is invertible in $\overline{\mathbb{Q}}_\ell$ it is easy to see the proof extends to further quotienting out by this finite group.

Lemma 2.6.7. *For $d \in \mathbb{Z}$, let $\mathscr{O}(d)_{X_S}$ be the natural line bundle of degree d on X_S. We have an isomorphism:*

$$R\Gamma(X_S, \mathscr{O}(d)) \simeq \{B_{S,[1,p]} \xrightarrow{\phi - p^d} B_{S,[1,1]}\}$$

Therefore, if we write $\mathbf{B}_{[1,p]}$ for the sheaf of rings on $\mathrm{Perf}_{\mathbb{F}_p}$ defined by sending $S \in \mathrm{Perf}_{\mathbb{F}_p}$ to the global sections of $Y_{S,[1,p]}$ then we have an isomorphism

$$\mathscr{H}^0(\mathscr{O}(d)) \simeq \mathbf{B}_{[1,p]}^{\phi = p^d}$$

of diamonds over $\mathrm{Spd}(\mathbb{F}_p)$. As in the above example, this identification tells us that action of the geometric Frobenius $\mathrm{Frob}_{Z_{\mathbb{F}_p}}$ on $Z_{\mathbb{F}_p}$ is the same as the right action by $p^d \in \mathbb{Q}_p^*$ on $Z_{\mathbb{F}_p}$ under the right action. Again, by [Han21, Proposition 4.8], we have an isomorphism

$$K_{Z_{\mathbb{F}_p}} \simeq \Lambda[2d](d)$$

and, it follows that $p^d \in \mathbb{Q}_p^*$ acts on the sheaf $\Lambda(d)$ by $p^{-d} \in \Lambda^*$, which, arguing as above, allows us to conclude an isomorphism

$$K_{Z_{\mathbb{F}_p}} \simeq |\cdot|^{-1}[2d]$$

of sheaves with right $\mathbb{Q}_p^*$-action.

We record the content of the above example as a Lemma for future use.

Lemma 2.6.8. *Let $d \in \mathbb{N}_{\geq 1}$ be a positive integer and consider the absolute positive Banach-Colmez space $\mathscr{H}^0(\mathscr{O}(d))_{\mathbb{F}_p} \to \mathrm{Spd}\,\mathbb{F}_p$. Then we have an isomorphism $K_{\mathscr{H}^0(\mathscr{O}(d))_{\mathbb{F}_p}} \simeq |\cdot|^{-1}[2d]$ (resp. $K_{\mathscr{H}^0(\mathscr{O}(d))_{\overline{\mathbb{F}}_p}} \simeq |\cdot|^{-1}[2d]$) as sheaves under the right $\mathbb{Q}_p^*$-action, where $|\cdot|$ denotes the rank 1 sheaf on $\mathscr{H}^0(\mathscr{O}(d))_{\mathbb{F}_p}$ (resp. $\mathscr{H}^0(\mathscr{O}(d))_{\overline{\mathbb{F}}_p}$) that transforms under the right scaling action by $\mathbb{Q}_p^*$ via the norm character. Similarly, if $\lambda = \frac{r}{s} > 0$ with $(r,s) = 1$ then we have an isomorphism $K_{\mathscr{H}^0(\mathscr{O}(\lambda))_{\mathbb{F}_p}} \simeq |\cdot|^{-s}[2r]$, where $\mathscr{O}(\lambda)$ is the unique stable bundle of slope λ on X.*

Proof. The case of line bundles follows from the above discussion. To deal with the general case, we can replace $\mathbb{Q}_p$ with the unramified extension $\mathbb{Q}_{p^s}$ to reduce to the case of line bundles. □

Remark 2.6.9. In what follows, it will be important to formalize this in terms of v-stacks. If we consider the v-stack quotient

$$[\mathscr{H}^0(\mathscr{O}(d))_{\overline{\mathbb{F}}_p}/\underline{\mathbb{Q}}_p^*] \to \operatorname{Spd}\overline{\mathbb{F}}_p$$

then this admits a natural map $q : [\mathscr{H}^0(\mathscr{O}(d))/\underline{\mathbb{Q}}_p^*] \to [\operatorname{Spd}\overline{\mathbb{F}}_p/\underline{\mathbb{Q}}_p^*]$ to a classifying stack. Then the above isomorphism descends to an identification $K_{[\mathscr{H}^0(\mathscr{O}(d))_{\overline{\mathbb{F}}_p}/\underline{\mathbb{Q}}_p^*]} \simeq q^*(|\cdot|^{-1})[2d]$, where we recall that there is an identification $\mathrm{D}([\operatorname{Spd}\overline{\mathbb{F}}_p/\underline{\mathbb{Q}}_p^*]) \simeq \mathrm{D}(\mathbb{Q}_p^*, \Lambda)$ of the derived category of sheaves on the classifying stack $[\operatorname{Spd}\overline{\mathbb{F}}_p/\underline{\mathbb{Q}}_p^*]$ with the derived category of smooth $\mathbb{Q}_p^*$-representations on Λ-modules.

Let's now push this a bit further and prove an analogous claim for negative Banach-Colmez spaces.

Lemma 2.6.10. *Let $d \in \mathbb{N}_{\geq 1}$ and consider the negative absolute Banach-Colmez space $\mathscr{H}^1(\mathscr{O}(-d)) \to \operatorname{Spd}\mathbb{F}_p$. Then we have isomorphisms $K_{\mathscr{H}^1(\mathscr{O}(d))_{\mathbb{F}_p}} \simeq |\cdot|[2d]$ (resp. $K_{\mathscr{H}^1(\mathscr{O}(d))_{\overline{\mathbb{F}}_p}} \simeq |\cdot|[2d]$) as sheaves with right $\mathbb{Q}_p^*$-action. Similarly, if $\lambda = \frac{r}{s} < 0$ with $(r,s) = 1$ then we have an isomorphism $K_{\mathscr{H}^1(\mathscr{O}(\lambda))_{\overline{\mathbb{F}}_p}} \simeq |\cdot|^s[2r]$ of sheaves with right $\mathbb{Q}_p^*$ action.*

Proof. Using the explicit description above, we know that, if we consider the sheaf of rings $\mathbf{B}_{[1,p]}$ and $\mathbf{B}_{[1,1]}$ on $\mathrm{Perf}_{\mathbb{F}_p}$ given by taking global sections of $Y_{S,[1,p]}$ (resp. $Y_{S,[1,1]}$), there is an isomorphism

$$\mathscr{H}^1(\mathscr{O}(-d))_{\mathbb{F}_p} \simeq \operatorname{Coker}(\mathbf{B}_{[1,p]} \xrightarrow{\phi - p^{-d}} \mathbf{B}_{[1,1]})$$

of v-sheaves on $\mathrm{Perf}_{\mathbb{F}_p}$. As a consequence of this, we can deduce that the geometric Frobenius $\mathrm{Frob}_{Z_{\mathbb{F}_p}}$ on $Z_{\mathbb{F}_p} = \mathscr{H}^0(\mathscr{O}(-d))_{\mathbb{F}_p}$ agrees with the right scaling action by $p^d \in \mathbb{Q}_p^*$. Moreover, as in the positive case, we have an isomorphism

$$K_{\mathscr{H}^1(\mathscr{O}(d))_{\mathbb{F}_p}} \simeq \Lambda[2d](d)$$

of étale sheaves. One can deduce this from the positive case [Han21, Proposition 4.8(ii)] and the proof of [FS21, Proposition II.2.5 (i)]. Now, arguing as in the case of positive Banach-Colmez spaces, we deduce that

$$K_{\mathscr{H}^1(\mathscr{O}(d))_{\mathbb{F}_p}} \simeq |\cdot|[2d]$$

as sheaves with $\mathbb{Q}_p^*$-action. The claim over $\overline{\mathbb{F}}_p$ follows. This finishes the case of line bundles. We can reduce to this case by replacing $\mathbb{Q}_p$ by the unramified extension $\mathbb{Q}_{p^s}$. $\square$

With these two lemmas in hand, let's continue our calculation of the dualizing object on Bun_B in the case that $G = \mathrm{GL}_2$ and B is the upper triangular Borel. We will return to working over the base $* = \mathrm{Spd}\,\overline{\mathbb{F}}_p$ in all that follows.

Example 2.6.11. Consider the case where $\overline{\nu} = (-d, -e)$ is anti-dominant, so that $d > e$. In this case, $d_{\overline{\nu}} = e - d$. We note that $d_{\overline{\nu}}$ is negative, so to keep track of the sign change, we consider the absolute value $|d_{\overline{\nu}}|$. The connected component $\mathrm{Bun}_B^{\overline{\nu}}$ just parametrizes the split extensions of $\mathscr{O}(d)$ by $\mathscr{O}(e)$. Therefore, its topological space is just a point. More precisely, if b corresponds to the bundle $\mathscr{O}(d) \oplus \mathscr{O}(e)$ then it is isomorphic $[*/\mathscr{J}_b]$, where $\mathscr{J}_b$ parametrizes automorphisms of $\mathscr{O}(d) \oplus \mathscr{O}(e)$. The group diamond $\mathscr{J}_b$ is isomorphic to

$$\begin{pmatrix} \mathrm{Aut}(\mathscr{O}(d)) & \mathscr{H}om(\mathscr{O}(e), \mathscr{O}(d)) \\ 0 & \mathrm{Aut}(\mathscr{O}(e)) \end{pmatrix} \simeq \begin{pmatrix} \underline{\mathbb{Q}}_p^* & \mathscr{H}^0(\mathscr{O}(|d_{\overline{\nu}}|)) \\ 0 & \underline{\mathbb{Q}}_p^* \end{pmatrix}$$

In particular, note that in this case $J_b(\mathbb{Q}_p) = T(\mathbb{Q}_p) \simeq \mathbb{Q}_p^* \times \mathbb{Q}_p^*$. We need to compute the dualizing object of $[*/\mathscr{J}_b]$. To do this, we consider the natural map

$$p : [*/\mathscr{J}_b] \to [*/\underline{T(\mathbb{Q}_p)}]$$

induced via the map $\mathscr{J}_b \to \underline{T(\mathbb{Q}_p)}$ given by quotienting out by the unipotent part. This map has a section given by the inclusion $\underline{T(\mathbb{Q}_p)} \subset \mathscr{J}_b$, and we denote this by $s : [*/\underline{T(\mathbb{Q}_p)}] \to [*/\mathscr{J}_b]$. First off note, by Lemma 2.6.5, that the dualizing object on $[*/\underline{T(\mathbb{Q}_p)}]$ can be identified with the set of Haar measures on $T(\mathbb{Q}_p)$. Now, since $\underline{T(\mathbb{Q}_p)}$ is unimodular, this implies that, as a $T(\mathbb{Q}_p)$-representation, the sheaf is trivial. Therefore, we are reduced to computing $p^!(\Lambda)$. To do this, note that the section s is a fibration in the positive Banach-Colmez space $\mathscr{H}^0(\mathscr{O}(|d_{\overline{\nu}}|))$. It therefore follows by the proof of [FS21, Proposition V.2.1] that adjunction induces an isomorphism:

$$s^! s_!(\Lambda) \simeq \Lambda$$

However, since s is a section of p, that gives us a natural isomorphism:

$$s_!(\Lambda) \simeq p^!(\Lambda)$$

Therefore, we are reduced to computing $s_!(\Lambda)$. We now have the following.

Lemma 2.6.12. *There is a natural isomorphism*

$$s^!(\Lambda) \simeq \delta_B^{-1}[2|d_{\overline{\nu}}|]$$

of sheaves on $[*/\underline{J_b(\mathbb{Q}_p)}] \simeq [*/\underline{T(\mathbb{Q}_p)}]$*, where* $s : [*/\underline{T(\mathbb{Q}_p)}] \to [*/\mathscr{J}_b]$ *is the natural map.*

Proof. We consider the Cartesian diagram:

$$\begin{tikzcd} \mathscr{H}^0(|d_{\overline{\nu}}|) \arrow[r, "\tilde{s}"] \arrow[d, "\tilde{q}"] & * \arrow[d, "q"] \\ \left[*/\underline{T(\mathbb{Q}_p)}\right] \arrow[r, "s"] & [*/\mathscr{J}_b] \end{tikzcd}$$

By base-change, we have a natural isomorphism $\tilde{q}^* s^!(\Lambda) \simeq \tilde{s}^!(\Lambda)$ and, by Lemma 2.6.8, the RHS is isomorphic to $|\cdot|^{-1}[2|d_{\overline{\nu}}|]$ as a sheaf with the right $\mathbb{Q}_p^*$-action. However, we want to understand how this transforms as a $T(\mathbb{Q}_p)$ representation. To do this, we note that the $T(\mathbb{Q}_p)$ action on $\mathscr{H}^0(\mathscr{O}(|d_{\overline{\nu}}|)) \simeq \mathscr{H}^0(\mathscr{O}(e)^\vee \otimes \mathscr{O}(d))$ comes from the semi-direct product structure on $\mathscr{J}_b$

$$\mathscr{J}_b = \begin{pmatrix} \mathrm{Aut}(\mathscr{O}(d)) & \mathscr{H}om(\mathscr{O}(e), \mathscr{O}(d)) \\ 0 & \mathrm{Aut}(\mathscr{O}(e)) \end{pmatrix} = \begin{pmatrix} \mathbb{Q}_p^* & \mathscr{H}^0(\mathscr{O}(e)^\vee \otimes \mathscr{O}(d)) \\ 0 & \mathbb{Q}_p^* \end{pmatrix} \simeq T(\mathbb{Q}_p) \ltimes \mathscr{H}^0(\mathscr{O}(d-e$$

It follows that if we consider the map

$$T(\mathbb{Q}_p) \to \mathbb{Q}_p^*$$

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mapsto ad^{-1}$$

then $T(\mathbb{Q}_p)$ acts via composing this map with the right scaling action on $\mathscr{H}^0(\mathscr{O}(|d_{\overline{\nu}}|))$. This implies that $\tilde{q}^* s^!(\Lambda) \simeq \tilde{s}^!(\Lambda) \simeq \delta_B^{-1}[2|d_{\overline{\nu}}|]$ as a sheaf with right $T(\mathbb{Q}_p)$-action. This was the desired claim. $\square$

Using the previous adjunction, the lemma tells us that $p^!(\Lambda) \simeq s_!(\Lambda) \simeq \delta_B[-2|d_{\overline{\nu}}|]$. This was the desired claim.

Now let's consider the anti-dominant case.

Example 2.6.13. Let $\overline{\nu} = (-d,-e)$ be dominant so that $d < e$. Then $d_{\overline{\nu}} = e-d = |d_{\overline{\nu}}|$ is positive. The connected component $\mathrm{Bun}_B^{\overline{\nu}}$ is isomorphic to $[\mathscr{H}^1(\mathscr{O}(-d_{\overline{\nu}}))/\underline{T(\mathbb{Q}_p)}]$, where $T(\mathbb{Q}_p)$ acts via interpreting $\mathscr{H}^1(\mathscr{O}(-d_{\overline{\nu}}))$ as a space parameterizing extensions

$$0 \to \mathscr{O}(d) \to \mathscr{E} \to \mathscr{O}(e) \to 0$$

on X_S, and $T(\mathbb{Q}_p)$ acts on the right via the identification $T(\mathbb{Q}_p) \simeq (\mathrm{Aut}(\mathscr{O}(d)), \mathrm{Aut}(\mathscr{O}(e)))$. Now we can factor the structure map as

$$[\mathscr{H}^1(\mathscr{O}(-d_{\overline{\nu}}))/\underline{T(\mathbb{Q}_p)}] \xrightarrow{f} [*/\underline{T(\mathbb{Q}_p)}] \to *$$

and, using that $T(\mathbb{Q}_p)$ is unimodular, we reduce to computing $f^!(\Lambda)$ as before. We consider the Cartesian diagram

$$\begin{tikzcd}
\mathscr{H}^1(\mathscr{O}(-d_{\overline{\nu}})) \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{q}"] & * \arrow[d, "q"] \\
\left[\mathscr{H}^1(\mathscr{O}(-d_{\overline{\nu}}))/\underline{T(\mathbb{Q}_p)}\right] \arrow[r, "f"] & \left[*/\underline{T(\mathbb{Q}_p)}\right]
\end{tikzcd}$$

and again, by base-change, this gives us an isomorphism $\tilde{q}^* f^!(\Lambda) \simeq \tilde{f}^!(\Lambda)$. Now, applying Lemma 2.6.10, we deduce that $\tilde{q}^* f^!(\Lambda)$ is isomorphic to $|\cdot|[2d_{\overline{\nu}}]$ as a sheaf with right $\mathbb{Q}_p^*$ action. We note that the $T(\mathbb{Q}_p)$-action comes from the identification $\mathscr{H}^1(\mathscr{O}(d) \otimes \mathscr{O}(e)^\vee) \simeq \mathscr{H}^1(\mathscr{O}(-d_{\overline{\nu}}))$. Therefore, $T(\mathbb{Q}_p)$ acts via the map

$$T(\mathbb{Q}_p) \to \mathbb{Q}_p^*$$
$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \to ad^{-1}$$

composed with the right action of $\mathbb{Q}_p^*$ on $\mathscr{H}^1(\mathscr{O}(-d_{\overline{\nu}}))$ by scaling of global sections. As a consequence, we deduce that, as a $T(\mathbb{Q}_p)$-representation, $\Lambda[2d_{\overline{\nu}}](d_{\overline{\nu}})$ is isomorphic to $\delta_B[2d_{\overline{\nu}}]$. This gives the desired claim.

We will need some formalism to move beyond the case of GL_2. Let G be a connected quasi-split reductive group over $\mathbb{Q}_p$ as before.

We let $b \in B(G)_{\mathrm{un}}$ be an unramified element, and set b_T to be the dominant reduction as before. We let $\mathscr{J}_b$ be the group diamond parameterizing automorphisms of the bundle $\mathscr{F}_b$ associated to b. Now, we can write $\mathscr{J}_b^{\geq\lambda}$ for the sub-diamond of automorphisms γ of $\mathscr{F}_b$ such that

$$(\gamma - 1)(\rho_*(\mathscr{F}_b))^{\geq\lambda'} \subset (\rho_*\mathscr{F}_b)^{\geq\lambda'+\lambda}$$

for all representations ρ of G. We set $\mathscr{J}_b^{>\lambda} = \cup_{\lambda'>\lambda}\mathscr{J}_b^{\geq\lambda'}$. We note that, since $J_b(\mathbb{Q}_p)$ is the automorphism group of the isocrystal defined by b, and, for $S \in \mathrm{Perf}$, there is an identification: $H^0(X_S, \mathscr{O}_{X_S}) = \underline{\mathbb{Q}_p}(S)$, we have a natural injection

$$\underline{J_b(\mathbb{Q}_p)} \hookrightarrow \mathscr{J}_b$$

which has a natural section given by letting $\mathscr{J}_b$ act on the graded-pieces of the HN-filtration. This gives us a semi-direct product decomposition

$$\mathscr{J}_b \simeq \underline{J_b(\mathbb{Q}_p)} \ltimes \mathscr{J}_b^{>0}$$

of group diamonds (See [FS21, Proposition III.5.1]).

We can relate this to B-bundles as follows. In a similar vein, we consider a HN-dominant reduction $b_T^- \in B(T)$ of $b \in B(G)$ with G anti-dominant isocrystal slopes. We consider the torsor $Q := \mathscr{F}_{b_T^-} \times^T B$ over X_S for $S \in \mathrm{Perf}$. We suppose now that T is the centralizer of the slope homomorphism of b, so that b_T is the canonical basic reduction of $b \in B(G)$. Then, as in [FS21, Proposition III.5.1.1], we have an isomorphism

$$\mathscr{J}_b(S) \simeq Q(X_S)$$
$$\mathscr{J}_b^{>0}(S) \simeq (\mathscr{E}_{b_T^-} \times^T U)(X_S) = R_u Q(X_S)$$

where U is the unipotent radical B. We note that Fargues-Scholze instead consider the B^--torsor $\mathscr{E}_{b_T} \times^T B^-$. Both of these provide descriptions of $\mathscr{J}_b$, but the actions of $J_b(\mathbb{Q}_p) \simeq T(\mathbb{Q}_p)$ on the unipotent parts will be intertwined by conjugation by the element of longest length. To distinguish them, we will write $\mathscr{G}_{b_T^-}$ for the group diamond such that $Q(X_S) = \mathscr{G}_{b_T^-}(S)$. We let $\mathscr{G}_{b_T^-}^{=0}$ be the slope 0 part and $\mathscr{G}_{b_T}^{>0}$ be the positive slope under this identification.

Remark 2.6.14. This identification is a manifestation of the following easy fact, that we will use implicitly throughout. Given a bundle $\mathscr{E}$ on X_S, the set of $\mathscr{E}$-torsors on X_S is parametrized by $H^1(X_S, \mathscr{E})$, and the automorphisms of such $\mathscr{E}$-torsors are parametrized by $H^0(X_S, \mathscr{E})$.

In particular, since Q defines a B-structure on $\mathscr{F}_b$, every such global section induces an automorphism of $\mathscr{F}_b$. Moreover, since T is equal to M_b this will imply that every element of $\mathscr{J}_b(S)$ can be obtained in this way. Let's now consider the case where b_T is dominant, but we only have a proper inclusion $T \subset M_b$. In other words, there exist some positive roots $\hat{\alpha}$ such that $\langle \hat{\alpha}, \nu_b \rangle = 0$ (cf. Example 2.6.4). In this case, all we can conclude is that $Q(X_S)$ is isomorphic to a subset of $\mathscr{J}_b(S)$. In particular, if we write $Q^{>0}(X_S)$ for the subset mapping to $\mathscr{J}_b^{>0}$ then we still have an isomorphism

$$\mathscr{J}_b^{>0}(S) \simeq Q^{>0}(X_S)$$

but, on the slope 0 part, $Q^{=0}(X_S)$ is only isomorphic to a proper subset of $\underline{J_b(\mathbb{Q}_p)}(S)$. The points of $Q^{=0}(S)$ will be identified with $\underline{B_b(\mathbb{Q}_p)}(S)$, where $B_b \subset J_b$ is the Borel subgroup defined in Lemma 2.2.12, and $R_u Q^{=0}$ is identified with the unipotent radical of this Borel. More precisely, if J_b is an inner twisting of $M_b \subset G$ then M_b is the Levi subgroup in G corresponding to the positive simple roots $\hat{\alpha}_i$ such that $\langle \hat{\alpha}_i, \nu_{b_T} \rangle = 0$, by definition of M_b as the centralizer of the slope homomorphism of b. Now let's refine this further, recall that U has a filtration by commutator subgroups

$$(1) \subset \cdots \subset U_{i+1} \subset U_i \subset \cdots \subset U_1 \subset U_0 = U$$

where $U_i/U_{i-1} \simeq \mathbb{G}_a^{d_i}$. If we let $\mathbb{D}$ be the protorus with character group $\mathbb{Q}$ then the slope homomorphism ν_b defines an action of $\mathbb{D}$ on $\mathscr{E}_{b_T^-} \times^T U_i/U_{i+1}$, which factors through the action of the maximal split torus A on $\mathscr{E}_{b_T^-} \times^T U_i/U_{i+1}$. This gives rise to a filtration on $R_u Q$ whose graded pieces will be a direct sum of the semistable vector bundles

$$Q_{\hat{\alpha}}$$

of degree equal to $\langle \hat{\alpha}, \nu_b \rangle$ and rank equal to $\dim(\mathfrak{g}_{\hat{\alpha}})$, where $\mathfrak{g}_{\hat{\alpha}}$ is the $\hat{\alpha}$ root space[6]. This allows us to write $\mathscr{G}_{b_T^-}^{>0}$ as an iterated fibration of positive Banach-Colmez spaces

$$\mathscr{H}^0(Q_{\hat{\alpha}})$$

for $\hat{\alpha}$ such that $\deg(Q_{\hat{\alpha}}) = \langle \hat{\alpha}, \nu_b \rangle > 0$. The subgroup $T(\mathbb{Q}_p) \subset B_b(\mathbb{Q}_p)$ will act on $\mathscr{G}_{b_T^-}^{>0}$ on the right. To describe this action, let P_b denote the standard parabolic with Levi factor M_b. By construction, the action will factor as

$$T(\mathbb{Q}_p) \to \mathrm{Aut}(R_u(P_b))_b(\mathbb{Q}_p).$$

[6]Recall that these are not one dimensional if the group is not split.

Here $\mathrm{Aut}(R_u(P_b))_b$ is the σ-centralizer of the image of the canonical reduction $b_{M_b} \in B(M_b)_{\mathrm{basic}}$ under the map $B(M_b) \to B(\mathrm{Aut}(R_u(P_b)))$ given by left conjugation $t \mapsto tut^{-1}$, and $\mathrm{Aut}(R_u(P_b))_b(\mathbb{Q}_p)$ acts on the right of $\mathscr{G}_b^{>0}$ by automorphisms and taking inverses. In particular, if we restrict to the subgroup $A(\mathbb{Q}_p)$ given by the maximal split torus then this will preserve the decomposition of $\mathrm{Aut}(R_u(P_b))$ into root eigenspaces. Therefore, for a reduced root $\hat{\alpha} > 0$, the induced action of $A(\mathbb{Q}_p)$ on $\mathscr{H}^0(Q_{\hat{\alpha}})$ will factor as

$$A(\mathbb{Q}_p) \xrightarrow{\hat{\alpha}} \mathbb{Q}_p^*$$

composed with the right scaling action by global sections on $\mathscr{H}^0(Q_{\hat{\alpha}})$.

These observations combined with the above Lemmas relating the dualizing objects on positive Banach-Colmez to norm characters give us everything we need to pin down the dualizing object on $\mathrm{Bun}_B^{\overline{\nu}}$ in the case that $\overline{\nu}$ is antidominant.

Proposition 2.6.15. *Let $b_T^- \in B(T)$ be an element with anti-dominant slopes with respect to B. The dualizing object $K_{\mathrm{Bun}_B^{b_T^-}}$ on $\mathrm{Bun}_B^{b_T^-}$ is isomorphic to $(\mathfrak{q}^{b_T^-})^*(\delta_B)[2d_{b_T^-}]$.*

Proof. Since b_T^- has G anti-dominant isocrystal slopes its HN-slopes are G-dominant, and therefore $\mathrm{Bun}_B^{b_T^-}$ parametrizes split reductions. In particular, its underlying topological space is just a point. More specifically, it is isomorphic to $[*/\mathscr{G}_{b_T^-}]$, where $\mathscr{G}_{b_T^-}(S) := Q(X_S)$, and Q is the torsor defined above. The semi-direct product structure on $\mathscr{J}_b$ and the above discussion imply that we have a semi-direct product structure

$$\mathscr{G}_{b_T^-} \simeq \underline{B_b(\mathbb{Q}_p)} \ltimes \mathscr{G}_{b_T^-}^{>0}$$

on $\mathscr{G}_{b_T^-}$. Therefore, we have a natural map

$$[*/\mathscr{G}_{b_T^-}] \to [*/\underline{B_b(\mathbb{Q}_p)}] \to [*/\underline{T(\mathbb{Q}_p)}],$$

and the induced right action of $T(\mathbb{Q}_p)$ on $K_{\mathrm{Bun}_B^{b_T^-}}$ will be given by the action described above.

Now, by Lemma 2.6.5, the dualizing object on $[*/\underline{B_b(\mathbb{Q}_p)}]$ is identified with the modulus character $\delta^{=0} := \delta_{B_b} : T(\mathbb{Q}_p) \to \Lambda^*$ under this right action. Here

δ_{B_b} will be given by the unique (See [Cas95, Lemma 1.6.1]) rational character of $T(\mathbb{Q}_p)$ such that, when restricted to $A(\mathbb{Q}_p)$, it becomes

$$t \mapsto \prod_{\substack{\hat{\alpha}>0 \\ \langle \hat{\alpha}, \nu_b \rangle = 0}} \det(\mathrm{Ad}(t|\mathfrak{g}_{\hat{\alpha}}))$$

composed with the norm character. We now consider the dualizing object with respect to the map:

$$p : [*/\mathcal{G}_{b_T^-}] \to [*/\underline{B_b(\mathbb{Q}_p)}]$$

This map has a natural section

$$s : [*/\underline{B_b(\mathbb{Q}_p)}] \to [*/\mathcal{G}_{b_T^-}]$$

the fibers of which are given by $\mathcal{J}_b^{>0}$. Arguing as in Example 2.6.11, this reduces us to showing that

$$s^!(\Lambda) \simeq (\delta^{\neq 0})^{-1}[-2d_{b_T^-}]$$

where $\delta^{\neq 0} : B_b(\mathbb{Q}_p) \to \Lambda^*$ is determined by the unique rational character of $T(\mathbb{Q}_p)$ such that when restricted to $A(\mathbb{Q}_p)$ it is given by

$$t \mapsto \prod_{\substack{\hat{\alpha}>0 \\ \langle \hat{\alpha}, \nu_b \rangle \neq 0}} |\det(\mathrm{Ad}(t|\mathfrak{g}_{\hat{\alpha}}))|^{-1},$$

so that we have $\delta_B = \delta^{=0}\delta^{\neq 0}$. To see this, we consider the Cartesian diagram

$$\begin{tikzcd}
\mathcal{J}_b^{>0} \arrow[r, "\tilde{s}"] \arrow[d, "\tilde{q}"] & * \arrow[d, "q"] \\
\left[*/\mathcal{G}_{b_T^-}\right] \arrow[r, "s"] & \left[*/\underline{B_b(\mathbb{Q}_p)}\right]
\end{tikzcd}$$

which, by base-change, gives us an isomorphism $\tilde{q}^* s^!(\Lambda) \simeq \tilde{s}^!(\Lambda)$. As above, we can write $\mathcal{G}_{b_T^-}^{>0}$ as an iterated fibration of the positive Banach-Colmez spaces $\mathcal{H}^0(Q_{\hat{\alpha}})$ for $\hat{\alpha} > 0$ such that $\langle \hat{\alpha}, \nu_b \rangle > 0$. By Lemma 2.6.8, the dualizing object on $\mathcal{H}^0(Q_{\hat{\alpha}})$ is isomorphic to $|\cdot|^{-\dim(\mathfrak{g}_{\hat{\alpha}})}[2\langle \hat{\alpha}, \nu_b \rangle]$ as a $\mathbb{Q}_p^*$ representation under the right scaling action, and $A(\mathbb{Q}_p)$ acts by $\hat{\alpha}$ composed with the right scaling action on $\mathcal{H}^0(Q_{\hat{\alpha}})$. Hence, we obtain that the dualizing object on $\mathcal{H}^0(Q_{\hat{\alpha}})$ as a right $A(\mathbb{Q}_p)$-representation is given by $|\det(\mathrm{Ad}(t|\mathfrak{g}_{\hat{\alpha}}))|^{-1}$. Therefore, by using the formula

$$(g\circ f)^!(\Lambda) = g^!(\Lambda)\otimes g^*(f^!(\Lambda))$$

for cohomologically smooth morphisms f and g, we deduce that the dualizing object on $\mathscr{J}_b^{>0}$ is isomorphic to

$$\Lambda[\sum_{\substack{\hat{\alpha}>0\\ \langle\hat{\alpha},\nu_b\rangle\neq 0}} \langle\hat{\alpha},\nu_b\rangle](\prod_{\substack{\hat{\alpha}>0\\ \langle\hat{\alpha},\nu_b\rangle\neq 0}} |\det(\mathrm{Ad}(t|\mathfrak{g}_{\hat{\alpha}})|)|^{-1} = \Lambda[2\langle 2\hat{\rho},\nu_b\rangle](\delta^{\neq 0})^{-1}|_{A(\mathbb{Q}_p)} = \Lambda[-2d_{b_T^-}](\delta^{\neq 0})^{-1}|_{A(\mathbb{Q}_p)}$$

as a sheaf with $A(\mathbb{Q}_p)$-action. Now we need to show that, as a sheaf with $B_b(\mathbb{Q}_p)$-action, that this is isomorphic to $\delta^{\neq 0}$. This follows since we know that the action of $B_b(\mathbb{Q}_p)$ on $\mathscr{G}_{b_T^-}^{>0}$ factors as

$$B_b(\mathbb{Q}_p)\to T(\mathbb{Q}_p)\to \mathrm{Aut}(R_u(P_b))_b(\mathbb{Q}_p))$$

composed with the right action by $\mathrm{Aut}(R_u(P_b))_b(\mathbb{Q}_p)$ on $\mathscr{G}_{b_T^-}^{>0}$ by automorphisms and taking inverses. In particular, by [Cas95, Lemma 1.61] and its proof, the action of $B_b(\mathbb{Q}_p)$ on the sheaf is uniquely determined by its restriction to $A(\mathbb{Q}_p)$, since the left conjugation map $T\to \mathrm{Aut}(R_u(P_b))$ factors through the split quotient. □

We now push this further. Let $\overline{\nu}\in B(T)$ be an element mapping to $b\in B(G)_{\mathrm{un}}$. We write $\overline{\nu} = w(b_T)\in B(T)$ mapping to $b\in B(G)_{\mathrm{un}}$ for $w\in W_b$ a representative of minimal length and b_T with dominant isocrystal slopes. We consider the torsor

$$Q_{\overline{\nu}} = \mathscr{F}_{\overline{\nu}}\times^T B$$

and

$$R_uQ_{\overline{\nu}} = \mathscr{F}_{\overline{\nu}}\times^T U$$

Now, since $Q_{\overline{\nu}}$ defines a reduction of $\mathscr{F}_b$, we have as before that $Q_{\overline{\nu}}(X_S)$ is isomorphic to a subgroup of $\mathscr{J}_b(S)$ giving rise to a slope filtration on $Q_{\overline{\nu}}$. This gives the following definition.

Definition 2.6.16. For $\overline{\nu} = w(b_T)\in B(T)$ mapping to $b\in B(G)_{\mathrm{un}}$ as above, we let $\mathscr{G}_{\overline{\nu}}$ be the group diamond on Perf whose S-valued points are equal to $Q_{\overline{\nu}}(X_S)$. For any $\lambda\in\mathbb{Q}$, we define subdiamonds $\mathscr{G}_{\overline{\nu}}^{\geq\lambda}$ and $\mathscr{G}_{\overline{\nu}}^{>\lambda}$, as above. We get a semi-direct product structure

$$\mathscr{G}_{\overline{\nu}}^{>0}\rtimes\mathscr{G}_{\overline{\nu}}^{=0}$$

where $\mathscr{G}_{\overline{\nu}}^{>0}$ injects into $\mathscr{J}_b^{>0}$ and $\mathscr{G}_{\overline{\nu}}^{=0}$ injects into $\underline{J_b(\mathbb{Q}_p)}$.

In particular, $\mathscr{G}_{\overline{\nu}}^{=0}$ will be identified with the Borel subgroup $\underline{B_b(\mathbb{Q}_p)}$. Now, as above, $R_u Q_{\overline{\nu}}$ has a filtration given by the filtration on U

$$(1) \subset \cdots \subset U_{i+1} \subset U_i \subset \cdots \subset U_1 \subset U_0 = U$$

by commutator subgroups, where $U_{i-1}/U_i \simeq \mathbb{G}_a^{d_i}$. The action of $\mathbb{D}$ via ν_b on these graded pieces allows us to write $\mathscr{G}_{\overline{\nu}}^{>0}$ as an iterated fibration of the semistable bundles $Q_{\hat{\alpha}}$ for $\hat{\alpha}$ such that $w^{-1}(\hat{\alpha}) < 0$ and $\langle \hat{\alpha}, \nu_b \rangle \neq 0$. Here the fact that $w^{-1}(\hat{\alpha}) < 0$ appears is due to the minus sign when passing between isocrystal slopes and G-bundles slopes. In particular, if $\overline{\nu}$ is anti-dominant (so that $\mathrm{Bun}_B^{\overline{\nu}}$ only parametrizes split B-structures) then all the roots $\hat{\alpha}$ such that $\langle \hat{\alpha}, \nu_b \rangle \neq 0$ appear, as described above. The group $A(\mathbb{Q}_p)$ will act on the right via $\hat{\alpha}$ composed with the natural right action by scaling of global sections. Now, with this in hand, let's prove Theorem 2.6.1.

Proof. (Theorem 2.6.1) We begin with the following Lemma.

Lemma 2.6.17. *If $\overline{\nu} = w(b_T) \in B(T)$ is an element mapping to $b \in B(G)_{\mathrm{un}}$ then the ℓ-dimension $d_{\overline{\nu}}$ of the connected component $\mathrm{Bun}_B^{\overline{\nu}}$ is equal to*

$$\langle 2\hat{\rho}, \nu_b \rangle - 2\langle 2\hat{\rho}^w, \nu_b \rangle$$

where $2\hat{\rho}^w$ is the sum of the positive roots $\hat{\alpha}$ such that $w^{-1}(\hat{\alpha}) < 0$.

Proof. By definition $d_{\overline{\nu}} = \langle 2\hat{\rho}, w(b_T) \rangle$, and this is easily identified with the above quantity recalling that $\nu_{b_T} = \nu_b$ by definition. $\square$

Write $\overline{\nu} = w(b_T)$ as in the lemma. The key point is that we can factor the structure morphism $\mathrm{Bun}_B^{\overline{\nu}} \to *$ as

$$\mathrm{Bun}_B^{\overline{\nu}} \xrightarrow{f} [*/\mathscr{G}_{\overline{\nu}}] \to [*/\underline{B_b(\mathbb{Q}_p)}] \to *.$$

We decompose the modulus character δ_B as follows to reflect this factorization, we have after restricting to $A(\mathbb{Q}_p)$ that

$$t \mapsto \prod_{\hat{\alpha}>0} |\det(\mathrm{Ad}(t|\mathfrak{g}_{\hat{\alpha}}))| = \prod_{\substack{\hat{\alpha}>0 \\ \langle \hat{\alpha}, \nu_b \rangle = 0}} |\det(\mathrm{Ad}(t|\mathfrak{g}_{\hat{\alpha}}))| \prod_{\substack{\hat{\alpha}>0 \\ w^{-1}(\hat{\alpha})>0 \\ \langle \hat{\alpha}, \nu_b \rangle \neq 0}} |\det(\mathrm{Ad}(t|\mathfrak{g}_{\hat{\alpha}}))| \prod_{\substack{\hat{\alpha}>0 \\ w^{-1}(\hat{\alpha})<0 \\ \langle \hat{\alpha}, \nu_b \rangle \neq 0}} |\det(\mathrm{Ad}(t|\mathfrak{g}_{\hat{\alpha}}))|$$

where the products over all positive roots $\hat{\alpha} > 0$. We call these rational characters $\delta^{=0}$, $\delta^{>0}$, and $\delta^{<0}$, respectively.

By arguing as in the proof of Proposition 2.6.15, we can see that the dualizing complex with respect to the map $[*/\mathscr{G}_{\overline{\nu}}] \to [*/\underline{B_b(\mathbb{Q}_p)}]$ is given by $\delta^{<0}[-2\langle 2\hat{\rho}_G^w, \nu_b\rangle)]$ and similarly the dualizing complex on $[*/\underline{B_b(\mathbb{Q}_p)}]$ is given by $\delta^{=0}$ under the relevant right action. Thus, it suffices to show that

$$f^!(\Lambda) \simeq \delta^{>0}[2(\langle 2\hat{\rho}, \nu_b\rangle - \langle 2\hat{\rho}^w, \nu_b\rangle)],$$

as sheaves with right $B_b(\mathbb{Q}_p)$-action. To do this, we should proceed analogously to Example 2.6.13. Namely, define the space $X^{\overline{\nu}}$ by the Cartesian diagram

$$\begin{tikzcd} X^{\overline{\nu}} \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{q}"] & * \arrow[d, "q"] \\ \mathrm{Bun}_B^{\overline{\nu}} \arrow[r, "f"] & {[*/\mathscr{G}_{\overline{\nu}}]} \end{tikzcd}$$

and we need to elucidate the space $X^{\overline{\nu}}$. We claim it is an iterated fibration of negative Banach-Colmez spaces. To see this, we consider $Q_{\overline{\nu}}$ as above, and look at the negative slope part $Q_{\overline{\nu}}^{\leq 0}$. We can identify $X^{\overline{\nu}}(S)$ with the set of $Q_{\overline{\nu}}^{\leq 0}$-torsors over X_S (cf. the proof of [FS21, Proposition V.3.5]). By considering the filtration of U by commutator subgroups, we can write $X^{\overline{\nu}}$ as an iterated fibration of the negative Banach-Colmez spaces

$$\mathscr{H}^1(Q_{-\hat{\alpha}})$$

for positive roots $\hat{\alpha} > 0$ such that $w^{-1}(\hat{\alpha}) > 0$ and $\langle\hat{\alpha}, \nu_b\rangle \neq 0$. Then, using Lemma 2.6.10, we deduce that the dualizing object on $\mathscr{H}^1(Q_{-\hat{\alpha}})$ is equal to $|\cdot|^{\dim(\mathfrak{g}_{\hat{\alpha}})}[2\langle\hat{\alpha}, \nu_b\rangle]$ as a sheaf with right $\mathbb{Q}_p^*$-action. Now $A(\mathbb{Q}_p)$ acts on $\mathscr{H}^1(Q_{-\hat{\alpha}})$ on the right via the right scaling action on $\mathscr{H}^1(Q_{-\hat{\alpha}})$ pre-composed with the character

$$A(\mathbb{Q}_p) \xrightarrow{\hat{\alpha}} \mathbb{Q}_p^*$$

for varying $\hat{\alpha} > 0$.

This tells us that we have an isomorphism

$$\tilde{f}^!(\Lambda) \simeq \delta^{>0}|_{A(\mathbb{Q}_p)}[2(\langle 2\hat{\rho}, \nu_b\rangle - \langle 2\hat{\rho}_G^w, \nu_b\rangle)]$$

as sheaves with $A(\mathbb{Q}_p)$ action. However, as in Proposition 2.6.15, this is enough to establish that we have the analogous isomorphism as sheaves with $B_b(\mathbb{Q}_p)$-action, since it follows from the construction that the action of $B_b(\mathbb{Q}_p)$ on $X^{\overline{\nu}}$ factors through the adjoint action as before. The claim follows. $\square$

2.7 The Filtered Eigensheaf Property

2.7.1 Proof of the Filtered Eigensheaf Property

We would now like the describe how Hecke correspondences on Bun_G interact with the Eisenstein functor. This will be used to show the Hecke eigensheaf property. Our analysis is heavily inspired by [BG02, Section 3], where an analogous claim is proven in the classical case. However, unlike the arguments there, we cannot appeal to the decomposition theorem, as the usual formalism of weights doesn't exist in this context. Nonetheless, we still have the excision spectral sequence, which will give us a filtration on the Eisenstein series. In §2.10, we will show that, under the condition of μ-regularity (Definition 2.3.14), this filtration splits for the Hecke operator defined by $V_{\mu^\Gamma} \in \mathrm{Rep}_\Lambda({}^LG)$ when applied to $\mathscr{F} = \mathscr{S}_{\phi_T}$.

Our goal is the following Theorem.

Theorem 2.7.1. *For $\mathscr{F} \in \mathrm{D}(\mathrm{Bun}_T)$, I a finite index set, and $V \in \mathrm{Rep}_\Lambda({}^LG^I)$, the sheaf $T_V(\mathrm{Eis}(\mathscr{F}))$ has a $W^I_{\mathbb{Q}_p}$-equivariant filtration indexed by $(\nu_i)_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma)^I$. The filtration's graded pieces are isomorphic to*

$$\mathrm{Eis}(T_{(\nu_i)_{i\in I}}(\mathscr{F})) \otimes V((\nu_i)_{i\in I})(-\langle\hat{\rho}, \sum_{i\in I} \nu_{i\Gamma}\rangle)$$

as sheaves in $\mathrm{D}(\mathrm{Bun}_G)^{BW^I_{\mathbb{Q}_p}}$. Moreover, the filtration is natural in I and V, as well as compatible with compositions and exterior tensor products in V.

Let $E/\mathbb{Q}_p$ be an extension over which G splits. The value of $T_V(\mathrm{nEis}(\mathscr{S}_{\phi_T}))$ is determined by the value of the Hecke correspondence base-changed to E, using [FS21, Corollary V.2.3] (See [FS21, Page 314]). Here the correspondence is determined by a representation $V = \boxtimes_{i\in I} V_i \in \mathrm{Rep}_\Lambda(\hat{G}^I)$ of I-copies of the dual group. We let $\lambda_i \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ be the highest weight of V_i. By taking direct sums, we can assume WLOG that V has a fixed central character. We let $\mathscr{S}_V$ be the Λ-valued sheaf on $\mathrm{Hck}^I_{G,\leq(\lambda_i)_{i\in I},E}$ defined via Theorem 2.4.2. Our aim is to construct a W^I_E-equivariant filtration on the sheaf:

$$T_V(\mathrm{Eis}(\mathscr{F})) = (h^{\rightarrow}_G \times \pi)_!(h^{\rightarrow *}_G(\mathfrak{p}_!\mathfrak{q}^*(\mathscr{F})[\dim(\mathrm{Bun}_B)]) \otimes \mathscr{S}_V) \in \mathrm{D}(\mathrm{Bun}_G)^{BW^I_E}$$

This will be accomplished by contemplating the diagram

$$\begin{tikzcd} \overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I \arrow[d, "\overline{\mathfrak{p}}\times\mathrm{id}"] & Z^I_{(\lambda_i)_{i\in I}} \arrow[l, "\phi_{(\lambda_i)_{i\in I}}"'] \arrow[r, "{'h_G^{\rightarrow}}"] \arrow[d, "{'\mathfrak{p}}"] & \mathrm{Bun}_B \arrow[d, "\mathfrak{p}"] \\ \mathrm{Bun}_G \times \mathrm{Div}_E^I & \mathrm{Hck}^I_{G,\leq(\lambda_i)_{i\in I},E} \arrow[l, "h_G^{\leftarrow}\times\pi"'] \arrow[r, "h_G^{\rightarrow}"] & \mathrm{Bun}_G \end{tikzcd}$$

as defined in §2.5.2 (2.3). Using base-change on the right Cartesian square, we get an isomorphism

$$T_V(\mathrm{Eis}(\mathscr{F})) \simeq (h_G^{\rightarrow}\times\pi)_!({}'\mathfrak{p}_!{}'h_G^{\rightarrow *}\mathfrak{q}^*(\mathscr{F})[\dim(\mathrm{Bun}_B)]\otimes\mathscr{S}_V)$$

but, applying the projection formula with respect to $\mathfrak{p}'$, this becomes

$$T_V(\mathrm{Eis}(\mathscr{F})) \simeq (h_G^{\rightarrow}\times\pi)_!{}'\mathfrak{p}_!({}'h_G^{\rightarrow *}(\mathfrak{q}^*(\mathscr{F})[\dim(\mathrm{Bun}_B)])\otimes{}'\mathfrak{p}^*(\mathscr{S}_V))$$

We define

$$K_V := {}'h_G^{\rightarrow *}(\Lambda[\dim(\mathrm{Bun}_B)])\otimes{}'\mathfrak{p}^*(\mathscr{S}_V)$$

allowing us to rewrite our formula nicely as

$$T_V(\mathrm{Eis}(\mathscr{F})) \simeq (h_G^{\rightarrow}\times\pi)_!{}'\mathfrak{p}_!({}'h_G^{\rightarrow *}\mathfrak{q}^*(\mathscr{F})\otimes K_V) = (\overline{\mathfrak{p}}\times\mathrm{id})_!\phi_{(\lambda_i)_{i\in I}!}({}'h_G^{\rightarrow *}\mathfrak{q}^*(\mathscr{F})\otimes K_V) \tag{2.4}$$

Now we would like to reduce the claim to applying excision to $\phi_{(\lambda_i)_{i\in I}!}(K_V)$ with respect to a locally closed stratification of $\overline{\mathrm{Bun}}_B\times\mathrm{Div}_E^I$. The claim should then follow from Corollary 2.4.9 and Lemma 2.5.21. In order to do this, let's further rewrite the formula. We recall that, for $(\nu_i)_{i\in I}\in(\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I$, we have a map

$$i_{(\nu_i)_{i\in I}} : \overline{\mathrm{Bun}}_B\times\mathrm{Div}_E^I \to \overline{\mathrm{Bun}}_B\times\mathrm{Div}_E^I$$

sending the tuple $(\mathscr{F}_G,\mathscr{F}_T,\overline{\kappa}^{\hat{\lambda}},(D_i)_{i\in I})$ to the tuple $(\mathscr{F}_G,\mathscr{F}_T(-\sum_{i\in I}\nu_i\cdot D_i),\mathscr{L}^{\hat{\lambda}}_{\mathscr{F}_T}(-\sum_{i\in I}\langle\nu_i,\hat{\lambda}\rangle\cdot D_i)\hookrightarrow\mathscr{L}^{\hat{\lambda}}_{\mathscr{F}_T}\hookrightarrow\mathscr{V}^{\hat{\lambda}}_{\mathscr{F}_G},(D_i)_{i\in I})$. We also have the map

$$h^{\rightarrow}_{(\nu_i)_{i\in I}} : \mathrm{Bun}_T\times\mathrm{Div}_E^I \simeq \mathrm{Hck}^I_{T,(\nu_i)_{i\in I}} \to \mathrm{Bun}_T$$

as in §2.3 which is given by modifying a T-bundle by $(\nu_i)_{i\in I}$ at a tuple of divisors $(D_i)_{i\in I}$ defining a point in Div_E^I. Then, for $\mathscr{F}\in\mathrm{D}(\mathrm{Bun}_T)$, we recall that we have an identification $(h^{\rightarrow}_{(\nu_i)_{i\in I}})^*(\mathscr{F}) = T_{(\nu_i)_{i\in I}}(\mathscr{F})$ of sheaves in $\mathrm{D}(\mathrm{Bun}_T)^{BW_E^I}$. Now, we can verify the following easy Lemma, which follows from the definition of $\phi_{(\lambda_i)_{i\in I}}$.

Lemma 2.7.2. *The following is true.*

1. *The maps* $h^{\rightarrow}_{(w_0(\lambda_i))_{i\in I}} \circ (\overline{\mathfrak{q}} \times \mathrm{id}) \circ \phi_{(\lambda_i)_{i\in I}}$ *and* $\mathfrak{q} \circ {}' h^{\rightarrow}_G$ *from* $Z^I_{(\lambda_i)_{i\in I}}$ *to* Bun_T *coincide.*

2. *For every* $(\nu_i)_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I$*, the maps* $(\overline{\mathfrak{q}} \times \mathrm{id}) \circ i_{(\nu_i)_{i\in I}}$ *and* $(h^{\rightarrow}_{(\nu_i)_{i\in I}} \times \mathrm{id}) \circ (\overline{\mathfrak{q}} \times \mathrm{id})$ *from* $\overline{\mathrm{Bun}}_B \times \mathrm{Div}^I_E$ *to* $\mathrm{Bun}_T \times \mathrm{Div}^I_E$ *coincide.*

Now, with this in hand, let's revisit equation (4):

$$T_V(\mathrm{Eis}(\mathscr{F})) \simeq (\overline{\mathfrak{p}} \times \mathrm{id})_! \phi_{(\lambda_i)_{i\in I}!}({}' h^{\rightarrow *}_G \mathfrak{q}^*(\mathscr{F}) \otimes K_V)$$

Using Lemma 2.7.2 (1), we have that

$${}' h^{\rightarrow *}_G \mathfrak{q}^*(\mathscr{F}) \simeq \phi^*_{(\lambda_i)_{i\in I}} (\overline{\mathfrak{q}} \times \mathrm{id})^* (h^{\rightarrow}_{(w_0(\lambda_i))_{i\in I}})^*(\mathscr{F})$$

substituting this in and applying projection formula with respect to $\phi_{(\lambda_i)_{i\in I}}$, we can rewrite the RHS as

$$(\overline{\mathfrak{p}} \times \mathrm{id})_!((\overline{\mathfrak{q}} \times \mathrm{id})^* (h^{\rightarrow}_{(w_0(\lambda_i))_{i\in I}})^*(\mathscr{F}) \otimes \phi_{(\lambda_i)_{i\in I}!}(K_V))$$

Now we claim that we have the following description of $\phi_{(\lambda_i)_{i\in I}!}(K_V)$.

Theorem 2.7.3. *The sheaf* $\phi_{(\lambda_i)_{i\in I}!}(K_V) \in \mathrm{D}(\overline{\mathrm{Bun}}_B)^{BW^I_E}$ *has a* W^I_E*-equivariant filtration indexed by* $(\nu_i)_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I$*. The graded pieces of this filtration are given by*

$$\boxtimes_{i\in I}(i_{\nu_i !}(j \times \mathrm{id})_!(\Lambda[\dim(\mathrm{Bun}_B)])) \otimes V_i(w_0(\lambda_i) + \nu_i)(-\langle \hat{\rho}, w_0(\lambda_i) + \nu_i \rangle)$$

Assuming this for now, we get that $T_V(\mathrm{Eis}(\mathscr{F}))$ has a W^I_E-equivariant filtration with graded pieces given by

$$\boxtimes_{i\in I}(\overline{\mathfrak{p}} \times \mathrm{id})_!((\overline{\mathfrak{q}} \times \mathrm{id})^* h^{\rightarrow *}_{w_0(\lambda_i)}(\mathscr{F}) \otimes (i_{\nu_i !}(j \times \mathrm{id})_!(\Lambda[\dim(\mathrm{Bun}_B)])) \otimes V_i(w_0(\lambda_i) + \nu_i)(-\langle \hat{\rho}, w_0(\lambda_i) + \nu_i \rangle)))$$

Applying projection formula with respect to i_{ν_i}, we obtain

$$\boxtimes_{i\in I}(\overline{\mathfrak{p}} \times \mathrm{id})_! i_{\nu_i !}(i^*_{\nu_i}(\overline{\mathfrak{q}} \times \mathrm{id})^* h^{\rightarrow *}_{w_0(\lambda_i)}(\mathscr{F}) \otimes (j \times \mathrm{id})_!(\Lambda[\dim(\mathrm{Bun}_B)])) \otimes V_i(w_0(\lambda_i) + \nu_i)(-\langle \hat{\rho}, w_0(\lambda_i) + \nu_i \rangle)$$

but now, by Lemma 2.7.2 (2), we have

$$i^*_{\nu_i}(\overline{\mathfrak{q}} \times \mathrm{id})^* h^{\rightarrow *}_{w_0(\lambda_i)}(\mathscr{F}) \simeq (\overline{\mathfrak{q}} \times \mathrm{id})^* h^{\rightarrow *}_{w_0(\lambda_i)+\nu_i}(\mathscr{F}) \simeq (\overline{\mathfrak{q}} \times \mathrm{id})^*(T_{w_0(\lambda_i)+\nu_i}(\mathscr{F}))$$

so substituting this into the previous formula we get

$$\boxtimes_{i\in I}(\overline{\mathfrak{p}}\times \mathrm{id})_! i_{\nu_i !}((\overline{\mathfrak{q}}\times \mathrm{id})^*(T_{w_0(\lambda_i)+\nu_i}(\mathscr{F})\otimes (j\times \mathrm{id})_!(\Lambda[\dim(\mathrm{Bun}_B)])))\otimes V_i(w_0(\lambda_i)+\nu_i)(-\langle\hat{\rho}, w_0(\lambda_i)+\nu_i\rangle)$$

Now i_{ν_i} does nothing to the G-bundle $\mathscr{F}_G$ and the copy of Div^I_E. Therefore, this becomes

$$\boxtimes_{i\in I}(\overline{\mathfrak{p}}\times \mathrm{id})_!((\overline{\mathfrak{q}}\times \mathrm{id})^*(T_{w_0(\lambda_i)+\nu_i}(\mathscr{F}))\otimes (j\times \mathrm{id})_!(\Lambda[\dim(\mathrm{Bun}_B)]))\otimes V_i(w_0(\lambda_i)+\nu_i)(-\langle\hat{\rho}, w_0(\lambda_i)+\nu_i\rangle)$$

which is just

$$\boxtimes_{i\in I}(\mathrm{Eis}\boxtimes \mathrm{id})(T_{w_0(\lambda_i)+\nu_i}(\mathscr{F}))\otimes V_i(w_0(\lambda_i)+\nu_i)(-\langle\hat{\rho}, w_0(\lambda_i)+\nu_i\rangle)$$

by an application of projection formula to $j\times\mathrm{id}$. Since $w_0(\lambda_i)$ is the lowest weight of V_i this implies the desired result. Thus, to construct the filtration all we have to do is prove Theorem 2.7.3.

Proof. (Theorem 2.7.3) For $(\nu_i)_{i\in I}\in(\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I$, let $\overline{\nu} := \sum_{i\in I}\nu_{i\Gamma}$. Consider the locally closed stratum ${}_{\overline{\nu}}\overline{\mathrm{Bun}}_B\times\mathrm{Div}^I_E \simeq \mathrm{Bun}_B\times\mathrm{Div}^{(\overline{\nu})}\times\mathrm{Div}^I_E\subset\overline{\mathrm{Bun}}_B\times\mathrm{Div}^I_E$. We have the natural projection

$$p_1 : {}_{\overline{\nu}}\overline{\mathrm{Bun}}_B\times\mathrm{Div}^I_E \simeq \mathrm{Bun}_B\times\mathrm{Div}^{(\overline{\nu})}\times\mathrm{Div}^I_E\to\mathrm{Div}^{(\overline{\nu})}$$

as well as the map

$$p_2 : {}_{\overline{\nu}}\overline{\mathrm{Bun}}_B\times\mathrm{Div}^I_E\to\mathrm{Div}^I_E\xrightarrow{\Delta_{(\nu_i)_{i\in I}}}\mathrm{Div}^{(\overline{\nu})}$$

where the first map is the natural projection and $\Delta_{(\nu_i)_{i\in I}}$ is as defined in §2.5.2. Since $\mathrm{Div}^{(\overline{\nu})}$ is proper using [FS21, Proposition II.1.21] and in particular separated, it follows that if we let ${}_{(\nu_i)_{i\in I}}(\overline{\mathrm{Bun}}_B\times\mathrm{Div}^I_E)\to{}_{\overline{\nu}}\overline{\mathrm{Bun}}_B\times\mathrm{Div}^I_E$ be the pullback of the diagonal morphism $\Delta_{\mathrm{Div}^{(\overline{\nu})}}$ along (p_1,p_2) that this is a closed immersion. Therefore, we see that the composite ${}_{(\nu_i)_{i\in I}}(\overline{\mathrm{Bun}}_B\times\mathrm{Div}^I_E)\to{}_{\overline{\nu}}\overline{\mathrm{Bun}}_B\to\overline{\mathrm{Bun}}_B\times\mathrm{Div}^I_E$ is a locally closed immersion parameterizing $(\overline{\kappa}^{\hat{\lambda}} : (\mathscr{L}^{\hat{\lambda}})_{\mathscr{F}_T}\to(\mathscr{V}^{\hat{\lambda}})_{\mathscr{F}_G}, \hat{\lambda}\in\hat{\Lambda}^+_G, (D_i)_{i\in I})$ such that $\mathrm{Coker}(\overline{\kappa}^{\hat{\lambda}})$ has torsion of length $\langle\hat{\lambda},\nu_{i\Gamma}\rangle$ supported at D_i, in the sense of Lemma 2.5.20, and a 0 nowhere else for all $\hat{\lambda}\in\hat{\Lambda}^+_G$ and $i\in I$. If we let $j_{(\nu_i)_{i\in I}} := i_{(\nu_i)_{i\in I}}\circ(j\times\mathrm{id})$ then we see that this maps isomorphically onto ${}_{(\nu_i)_{i\in I}}(\overline{\mathrm{Bun}}_B\times\mathrm{Div}^I_E)$, and for varying $(\nu_i)_{i\in I}\in(\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I$ these form a locally closed stratification of $\overline{\mathrm{Bun}}_B\times\mathrm{Div}^I_E$. Now, by applying the excision with respect to this locally closed stratification, we obtain a filtration on $\phi_{(\lambda_i)_{i\in I}!}(K_Z)$ whose graded pieces are isomorphic to:

$$j_{(\nu_i)_{i\in I}!}j^*_{(\nu_i)_{i\in I}}\phi_{(\lambda_i)_{i\in I}!}(K_V)$$

Moreover, since the maps $j_{(\nu_i)_{i\in I}}$ are defined over the projection to Div^I_E, it follows that this filtration is W^I_E-equivariant. It remains to determine the W^I_E-action on the graded pieces. To do this, we can consider the pullback to a geometric point $x = \mathrm{Spa}(C, \mathscr{O}_C) \to \mathrm{Div}^I_E$, where we can regard x as a point of $\mathrm{Div}^{(\overline{\nu})}$, as in §2.5.2. We recall that by definition

$$K_V := {}^{'}\mathfrak{p}^*(\mathscr{S}_V) \otimes ({}'h^{\rightarrow}_{G})^*(\Lambda[\dim(\mathrm{Bun}_B)])$$

so, by Lemma 2.5.21 (3), this identifies with

$$R\Gamma_c({}_x\mathrm{Gr}_{G,\leq(\lambda_i)_{i\in I},E} \cap {}_x\mathrm{S}_{G,(\nu_i+w_0(\lambda_i))_{i\in I},E}, \mathscr{S}_V|_{{}_x\mathrm{Gr}_{G,\leq(\lambda_i)_{i\in I},E}\cap {}_x\mathrm{S}_{G,(\nu_i+w_0(\lambda_i))_{i\in I},E}})[-\langle 2\hat{\rho}, \sum_{i\in I}(\nu_i + w_0(\lambda_i))\rangle + \dim(\mathrm{B}$$

We need to justify that the contribution of the L^+B torsor ${}_x\mathscr{B}$ in Lemma 2.5.21 (3) is isomorphic to $\Lambda[-\langle 2\hat{\rho}, \sum_{i\in I}(\nu_i + w_0(\lambda_i))\rangle]$. To see this, consider the case where $I = \{*\}$ is a singleton, and we have elements $\lambda \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ and $\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$. We consider the L^+B-action on ${}_x\mathrm{Gr}_{G,\leq\lambda,E} \cap {}_x\mathrm{S}_{G,\nu+w_0(\lambda),E}$. We note that L^+T will act trivially on this space, as can be seen from the definition of the semi-infinite cells. Consider the remaining unipotent part L^+U. Using the filtration on U by commutator subgroups, we can write L^+U as an iterated fibration of $L^+\mathbb{G}_{a,\hat{\alpha}}$ indexed by the positive roots $\hat{\alpha}$ of G. One can check that the $L^+\mathbb{G}_{a,\hat{\alpha}}$ action on ${}_x\mathrm{Gr}_{G,\leq\lambda,E} \cap {}_x\mathrm{S}_{G,\nu+w_0(\lambda),E}$ factors through the truncated loop group $L^+_{n_{\hat{\alpha}}}\mathbb{G}_{a,\hat{\alpha}}$, where $n_{\hat{\alpha}} = \langle \hat{\alpha}, \nu + w_0(\lambda)\rangle$ (See for example the proof of [FS21, Proposition VI.2.4]). If we let $\mathscr{O}_{X,x}$ denote the completed local ring at the fixed untilt x, with uniformizing parameter t_x then, by writing $\mathscr{O}_{X,x}/t_x^{n_{\hat{\alpha}}}$ as an iterated extension of $\mathscr{O}_{X,x}/t_x \simeq C$, we can describe $L^+_{n_{\hat{\alpha}}}\mathbb{G}_{a,\hat{\alpha}}$ as a fibration of $(\mathbb{A}^1_C)^\diamond$ iterated $n_{\hat{\alpha}}$ times. This tells us that the compactly supported cohomology of $L^+_{n_{\hat{\alpha}}}\mathbb{G}_{a,\hat{\alpha}}$ over $\mathrm{Spa}(C, \mathscr{O}_C)$ is isomorphic to $\Lambda[-2n_{\hat{\alpha}}]$. As a consequence, we deduce that the contribution of ${}_x\mathscr{B}$ to the above formula is $\Lambda[-\sum_{\hat{\alpha}>0} n_{\hat{\alpha}}] = \Lambda[-\langle 2\hat{\rho}, \nu + w_0(\lambda)\rangle]$, by Künneth. Now, by Corollary 2.4.9, $j^*_{(\nu_i)_{i\in I}}\phi_{(\lambda_i)_{i\in I}!}(K_V)$ identifies with

$$\boxtimes_{i\in I} V_i(w_0(\lambda_i)+\nu_i)(-\langle\hat{\rho}, (w_0(\lambda_i)+\nu_i)\rangle)[-\langle 2\hat{\rho}, \sum_{i\in I}(w_0(\lambda_i)+\nu_i)\rangle + \langle 2\hat{\rho}, \sum_{i\in I}(w_0(\lambda_i)+\nu_i)\rangle + \dim(\mathrm{Bun}_B)]$$

or rather

$$\boxtimes_{i\in I} V_i(w_0(\lambda_i)+\nu_i)(-\langle\hat{\rho}, w_0(\lambda_i)+\nu_i\rangle)[\dim(\mathrm{Bun}_B)]$$

and so we get the desired result. $\square$

It remains to see that this filtration satisfies the desired compatibilities. Consider a map of finite index sets $\pi : I \to J$. For $j \in J$, we set $I_j := \pi^{-1}(j)$ and consider the natural map $\Delta_{IJ} : \mathrm{Div}_E^J \to \mathrm{Div}_E^I$, which diagonally embeds the jth copy of Div_E^1 in Div_E^J into $\mathrm{Div}_E^{I_j}$. Attached to this, we have a Cartesian diagram

$$\begin{tikzcd} \overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^J \arrow[d, "\mathrm{id}\times\Delta_{IJ}"] & Z^J_{(\lambda_j)_{j\in J}} \arrow[l, "\phi_{(\lambda_j)_{j\in J}}"'] \arrow[d, "\tilde{\Delta}_{IJ}"] \\ \overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I & Z^I_{(\lambda_i)_{i\in I}} \arrow[l, "\phi_{(\lambda_i)_{i\in I}}"'] \end{tikzcd}$$

where $\lambda_j := \sum_{i\in I_j}\lambda_i$ for all $j \in J$. Base change gives us a natural isomorphism:

$$(\mathrm{id}\times\Delta_{IJ})^*\phi_{(\lambda_i)_{i\in I}!}(K_{\boxtimes_{i\in I}V_i}) \simeq \phi_{(\lambda_j)_{j\in J}!}\tilde{\Delta}_{IJ}^*(K_{\boxtimes_{i\in I}V_i}) \tag{2.5}$$

However, by the relationship between fusion product and tensor product under Theorem 2.4.2, we deduce that $\tilde{\Delta}_{IJ}^*(K_{\boxtimes_{i\in I}V_i}) \simeq K_{\boxtimes_{j\in J}V_j}$, where $V_j := \otimes_{i\in I_j}V_i$. We now compare the two filtrations on the LHS and the RHS of this isomorphism. To do this, we define $(\nu_j)_{j\in J}$ by $\nu_j := \sum_{i\in I_j}\nu_i$. We note that we have a natural Cartesian diagram

$$\begin{tikzcd} \overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^J \arrow[r, "i_{(\nu_j)_{j\in J}}"] \arrow[d, "\mathrm{id}\times\Delta_{IJ}"] & \overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^J \arrow[d, "\mathrm{id}\times\Delta_{IJ}"] \\ \overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I \arrow[r, "i_{(\nu_i)_{i\in I}}"] & \overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I \end{tikzcd}$$

On the LHS of (5), we have a filtration with graded pieces isomorphic to

$$(\mathrm{id}\times\Delta_{IJ})^* i_{(\nu_i)_{i\in I}!}(j\times\mathrm{id})_!(\Lambda[\dim(\mathrm{Bun}_B)])\otimes\boxtimes_{i\in I}V_i(w_0(\lambda_i)+\nu_i)(-\langle\hat{\rho},w_0(\lambda_i)+\nu_i\rangle)$$

which is naturally isomorphic to

$$i_{(\nu_j)_{j\in J}!}(\mathrm{id}\times\Delta_{IJ})^*(j\times\mathrm{id})_!(\Lambda[\dim(\mathrm{Bun}_B)])\otimes\boxtimes_{i\in I}V_i(w_0(\lambda_i)+\nu_i)(-\langle\hat{\rho},w_0(\lambda_i)+\nu_i\rangle)$$

by base-change applied to the previous Cartesian square. We can further rewrite this as

$$i_{(\nu_j)_{j\in J}!}\circ(j\times\mathrm{id})_!(\Lambda[\dim(\mathrm{Bun}_B)])\otimes\boxtimes_{j\in J}\otimes_{i\in I_j}V_i(w_0(\lambda_i)+\nu_i)(-\langle\hat{\rho},w_0(\lambda_i)+\nu_i\rangle))$$

On the other hand, for such a $(\nu_j)_{j\in J}$, the RHS of (5) has a filtration with graded pieces isomorphic to

$$i_{(\nu_j)_{j\in J}!}(j\times \mathrm{id})_!(\Lambda[\dim(\mathrm{Bun}_B)])\otimes \boxtimes_{j\in J} V_j(w_0(\lambda_j)+\nu_j)(-\langle\hat{\rho}, w_0(\lambda_j)+\nu_j\rangle))$$

but now note that

$$V_j(w_0(\lambda_j)+\nu_j)(-\langle\hat{\rho}, w_0(\lambda_j)+\nu_j\rangle)) = \bigoplus_{\substack{(\nu_i)_{i\in I}\in\Lambda^I_{G,B}\\ \sum_{i\in I_j}\nu_i=\nu_j}} \bigotimes_{i\in I_j} V_i(w_0(\lambda_i)+\nu_i)(-\langle\hat{\rho}, w_0(\lambda_i)+\nu_i\rangle))$$

for all $j\in J$. Therefore, the graded piece indexed by $(\nu_j)_{j\in J}$ on the RHS have a split filtration with graded pieces isomorphic to the graded pieces coming from the filtration on the LHS. The compatibility of these two filtrations now follows from Corollary 2.4.11, and the fact that the filtration came from restricting the sheaf $\mathscr{S}_V$ to semi-infinite cells. Now, we can reap the fruit of this section using the filtered eigensheaf property to get some control on the stalks of $\mathrm{nEis}(\mathscr{S}_{\phi_T})$.

2.7.2 Consequences of the Filtered Eigensheaf Property

First, we note, by applying Theorem 2.7.1 when $\mathscr{F}=\mathscr{S}_{\phi_T}\otimes\Delta_B^{1/2}$ together with Corollary 2.3.5, we obtain the following.

Corollary 2.7.4. *For all finite index sets I and $V=\boxtimes_{i\in I}V_i\in\mathrm{Rep}_\Lambda({}^LG^I)$, the sheaf $T_V(\mathrm{nEis}(\mathscr{S}_{\phi_T}))$ admits a $W^I_{\mathbb{Q}_p}$-equivariant filtration indexed by $(\nu_i)_{i\in I}\in(\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma)^I$. The filtration's graded pieces are isomorphic to $\mathrm{nEis}(\mathscr{S}_{\phi_T})\otimes\boxtimes_{i\in I}(\nu_i\circ\phi_T)\otimes V_i(\nu_i)$. The filtration is natural in I and V, as well as compatible with compositions and exterior tensor products in V.*

In particular, we note that the direct sum of the graded pieces of the filtration on $T_V(\mathrm{nEis}(\mathscr{S}_{\phi_T}))$ is isomorphic to

$$\mathrm{nEis}(\mathscr{S}_{\phi_T})\otimes\bigoplus_{(\nu_i)_{i\in I}\in(\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma)^I}\boxtimes_{i\in I}\nu_i\circ\phi_T\otimes V_i(\nu_i)\simeq\mathrm{nEis}(\mathscr{S}_{\phi_T})\boxtimes r_V\circ\phi$$

as sheaves in $\mathrm{D}(\mathrm{Bun}_G)^{BW^I_{\mathbb{Q}_p}}$, where ϕ is the parameter ϕ_T composed with the natural embedding ${}^LT\to{}^LG$. In other words, $T_V(\mathrm{nEis}(\mathscr{S}_{\phi_T}))$ is a filtered eigensheaf with eigenvalue ϕ. We now would like to use this to deduce some consequences about the stalks of the Eisenstein series $\mathrm{nEis}(\mathscr{S}_{\phi_T})$. In particular, let's consider some Schur irreducible constituent A of $\mathrm{nEis}(\mathscr{S}_{\phi_T})$. We will now need our assumption that the prime ℓ is very good. Under this assumption, the excursion algebra will define endomorphims of A which determine and are determined by the parameter ϕ_A^{FS}, as in [FS21, Propositions I.9.1, I.9.3]. Since the excursion algebra is determined by natural transformations of Hecke operators, the filtered Hecke eigensheaf property tells us that these scalars must be specified by ϕ, so that we have an equality: $\phi = \phi_A^{\mathrm{FS}}$, as conjugacy classes of semi-simple parameters. In particular, if we consider $b \in B(G)$ and look at the restriction $\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^b}$ to the locally closed HN-strata $\mathrm{Bun}_G^b \subset \mathrm{Bun}_G$ indexed by b. Then, by [FS21, Proposition V.2.2], we have a natural isomorphism $\mathrm{D}(\mathrm{Bun}_G^b) \simeq \mathrm{D}(J_b(\mathbb{Q}_p), \Lambda)$. By the previous discussion and compatibility of the Fargues-Scholze correspondence with restriction to J_b [FS21, Section IX.7.1], we deduce that any irreducible constituent ρ of the restriction $\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^b}$ has Fargues-Scholze parameter $\phi_\rho^{\mathrm{FS}} : W_{\mathbb{Q}_p} \to {}^L J_b(\Lambda)$ equal to ϕ under the appropriately Tate twisted embedding:

$$ {}^L J_b(\Lambda) \to {}^L G(\Lambda) $$

Now, since our parameter ϕ is induced from the maximal torus, we would like to say that this is impossible unless J_b itself admits a maximal torus, which is in turn equivalent to assuming that b is unramified. Here we need to be a bit careful. In particular, if we consider $G = \mathrm{GL}_2$ and b the element of slope $\frac{1}{2}$ then $J_b = D^*_{\frac{1}{2}}$ the units in the quaternion division algebra. The trivial representation $\mathbf{1}$ of $D^*_{\frac{1}{2}}$ has Fargues-Scholze parameter given by

$$ W_{\mathbb{Q}_p} \to \mathrm{GL}_2(\Lambda) $$

$$ g \mapsto \begin{pmatrix} |g|^{1/2} & 0 \\ 0 & |g|^{-1/2} \end{pmatrix} $$

This parameter is induced from a maximal torus of GL_2; however, it is not generic. In particular, the composite of this parameter with the unique simple root defined by the upper triangular Borel gives a Galois representation isomorphic to the norm character $|\cdot|$. Therefore, one might hope that assuming compatibility of some suitably nice form of the local Langlands correspondence for G with the Fargues-Scholze correspondence together with genericity of ϕ_T is enough to give us the desired description of the stalks. This is indeed the case. The assumption we need is as follows.

Assumption 2.7.5. *For a connected reductive group $H/\mathbb{Q}_p$, we have:*

- $\Pi(H)$ *the set of smooth irreducible $\overline{\mathbb{Q}}_\ell$-representations of $H(\mathbb{Q}_p)$,*
- $\Phi(H)$ *the set of conjugacy classes of continuous maps*

$$\mathscr{L}_{\mathbb{Q}_p} = W_{\mathbb{Q}_p} \times \mathrm{SL}(2, \overline{\mathbb{Q}}_\ell) \to {}^L H(\overline{\mathbb{Q}}_\ell)$$

  where $\overline{\mathbb{Q}}_\ell$ has the discrete topology and $\mathrm{SL}(2, \overline{\mathbb{Q}}_\ell)$ acts via an algebraic representation and the map respects the action of $W_{\mathbb{Q}_p}$ on ${}^L H(\overline{\mathbb{Q}}_\ell)$, the L-group of H.

- $\Phi^{\mathrm{ss}}(H)$ *the set of continuous semi-simple homomorphisms*

$$W_{\mathbb{Q}_p} \to {}^L H(\overline{\mathbb{Q}}_\ell)$$

- $(-)^{\mathrm{ss}} : \Phi(H) \to \Phi^{\mathrm{ss}}(H)$ *the map defined by precomposition with*

$$W_{\mathbb{Q}_p} \to W_{\mathbb{Q}_p} \times \mathrm{SL}(2, \overline{\mathbb{Q}}_\ell) = \mathscr{L}_{\mathbb{Q}_p}$$

$$g \mapsto (g, \begin{pmatrix} |g|^{1/2} & 0 \\ 0 & |g|^{-1/2} \end{pmatrix})$$

Then, we assume, for all $b \in B(G)$, that there exists a map

$$\mathrm{LLC}_b : \Pi(J_b) \to \Phi(J_b)$$

$$\rho \mapsto \phi_\rho$$

satisfying the following properties:

1. *The diagram*

$$\begin{tikzcd} \Pi(J_b) \arrow[r, "\mathrm{LLC}_b"] \arrow[dr, "\mathrm{LLC}_b^{\mathrm{FS}}"'] & \Phi(J_b) \arrow[d, "(-)^{\mathrm{ss}}"] \\ & \Phi^{\mathrm{ss}}(J_b) \end{tikzcd}$$

   commutes, where $\mathrm{LLC}_b^{\mathrm{FS}}$ is the Fargues-Scholze local Langlands correspondence for J_b.

2. *Consider ϕ_ρ as an element of $\Phi(G)$ given by composing with the twisted embedding ${}^LJ_b(\overline{\mathbb{Q}}_\ell) \simeq {}^LM_b(\overline{\mathbb{Q}}_\ell) \to {}^LG(\overline{\mathbb{Q}}_\ell)$ (as defined in [FS21, Section IX.7.1]). Then ϕ_ρ factors through the natural embedding ${}^LT \to {}^LG$ if and only if $b \in B(G)_{\mathrm{un}}$.*

3. *If ρ is a representation such that $\mathscr{L}_{\mathbb{Q}_p} \to {}^LJ_b(\overline{\mathbb{Q}}_\ell) \to {}^LG(\overline{\mathbb{Q}}_\ell)$ factors through ${}^LT$, where the last map is the twisted embedding then, by (2), the element b is unramified, and we require that ρ is isomorphic to an irreducible constituent of $i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{-1/2}$ for $w \in W_b$. Here $\chi : T(\mathbb{Q}_p) \to \overline{\mathbb{Q}}_\ell^*$ is the character attached to the induced toral parameter ϕ_T, and δ_{P_b} is the modulus character of $M_b \simeq J_b$ with respect to the standard parabolic P_b with Levi factor M_b.*

Remark 2.7.6. This assumption might seem a bit daunting, but is verifiable in many cases. In particular, the first assumption follows from the compatibility of the Fargues-Scholze correspondence with the Harris-Taylor correspondence for groups of type A_n and its inner forms ([HKW22, Theorem 1.0.3]). Similarly, for groups of type C_2 and their inner forms over a unramified extension L with $p > 2$, this follows from the main theorem of [Ham21b], and, for odd unramified unitary groups over $\mathbb{Q}_p$ this follows from the main theorem of [BHN22]. The methods employed in these two papers should generalize to at least a few other cases.

Assumption (2) is also a standard and verifiable conjecture in the cases where the local Langlands correspondence is known to exist. If ρ is a representation such that ϕ_ρ factors through ${}^LT$ then we are claiming that J_b has a Borel subgroup. For non quasi-split groups, it is conjectured that one should only consider L-parameters coming from the L-groups of the Levi subgroups of the non quasi-split group. These are referred to as relevant L-parameters. In particular, one expects the L-packets of LLC_b over irrelevant ϕ to be empty (See for example [Kal16, Conjecture A.2]), so if ϕ_ρ factors through ${}^LT$ for some ρ under LLC_b it should imply that the group J_b has a Borel B_b. Assumption (3) is just the expectation that, when ϕ_ρ factors through ${}^LT$, the members of the L-packet should be given by the irreducible constituents of the parabolic inductions from T to J_b. The Weyl group twists appear since the Weyl group conjugates of $\phi_\rho : W_{\mathbb{Q}_p} \to {}^LJ_b(\overline{\mathbb{Q}}_\ell) \simeq {}^LM_b(\overline{\mathbb{Q}}_\ell)$ all map to the same parameter when viewed as a parameter valued in ${}^LG$, and the modulus twist by $\delta_{P_b}^{-1/2}$ appears since we are comparing this to an L-parameter of G via the twisted embedding ${}^LJ_b(\overline{\mathbb{Q}}_\ell) \simeq {}^LM_b(\overline{\mathbb{Q}}_\ell) \to {}^LG(\overline{\mathbb{Q}}_\ell)$.

Under this assumption, we will deduce our main Corollary of the filtered eigensheaf property.

Corollary 2.7.7. *Under Assumption 2.7.5, consider $b \in B(G)$ with corresponding locally closed HN-strata $\mathrm{Bun}_G^b \subset \mathrm{Bun}_G$. For ϕ a generic parameter, the following is true.*

1. *If $b \notin B(G)_{\mathrm{un}}$ the restriction*
$$\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^b}$$
vanishes.

2. *If $b \in B(G)_{\mathrm{un}}$ is an unramified element and ρ is a smooth irreducible $\overline{\mathbb{F}}_\ell$-representation occurring as a constituent of $\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^b}$ then ρ is an irreducible constituent of $i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{-1/2}$ for some $w \in W_b$.*

Proof. As noted above, $\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^b}$ will be valued in an unbounded complex of smooth $\overline{\mathbb{F}}_\ell$-representations of the σ-centralizer J_b of b. If we consider a smooth irreducible constituent of this restriction ρ then, as already discussed above, it follows that the Fargues-Scholze parameter $\phi_\rho^{\mathrm{FS}} : W_{\mathbb{Q}_p} \to {}^LJ_b(\overline{\mathbb{F}}_\ell)$ under the twisted embedding
$${}^LJ_b(\overline{\mathbb{F}}_\ell) \to {}^LG(\overline{\mathbb{F}}_\ell)$$
agrees with ϕ. Using [Dat05, Lemma 6.8], we can choose $\tilde{\rho}$ a lift of ρ to a smooth irreducible $\overline{\mathbb{Q}}_\ell$-representation admitting a $J_b(\mathbb{Q}_p)$-stable $\overline{\mathbb{Z}}_\ell$-lattice such that ρ occurs as a subquotient of $\tilde{\rho}$ mod ℓ. Since the Fargues-Scholze correspondence is compatible with reduction mod ℓ [FS21, Section IX.5.2], it follows that the Fargues-Scholze parameter $\phi_{\tilde{\rho}}^{\mathrm{FS}}$ factors through ${}^LG(\overline{\mathbb{Z}}_\ell)$ and that it equals ϕ_ρ^{FS} mod ℓ. Now, since ϕ_ρ^{FS} factors through ${}^LT$ and induces a generic parameter, we claim that the same is true for $\phi_{\tilde{\rho}}^{\mathrm{FS}}$. This follows through standard deformation theory. In particular, if
$$H^1(W_{\mathbb{Q}_p}, \alpha \circ \phi_T)$$
vanishes for all Γ-orbits α of roots, then any lift of ϕ_ρ^{FS} will factor through ${}^LT$, but this vanishing is guaranteed by ϕ_T being generic (cf. [CS17, Lemma 6.2.2]). It is also easy to see that $\phi_{\tilde{\rho}}^{\mathrm{FS}}$ must be generic since its mod ℓ reduction is. Now, by Assumption 2.7.5, we note that $\phi_{\tilde{\rho}}^{\mathrm{FS}}$ is the semi-simplification of $\phi_{\tilde{\rho}}$, the L-parameter attached to $\tilde{\rho}$, but by Lemma 2.3.18 and genericity that implies that $\phi_{\tilde{\rho}}|_{W_{\mathbb{Q}_p}} = \phi_{\tilde{\rho}}^{\mathrm{FS}}$. The two claims now follow from Assumptions 2.7.5 (2) and (3), respectively. $\square$

This statement will allow us to give a complete description of the eigensheaf $\mathrm{nEis}(\mathscr{S}_{\phi_T})$ for ϕ_T satisfying the slightly stronger condition of weak normalized regularity. In particular, as we will see in §9, for the restrictions of the sheaf to Bun_G^b for $b \in B(G)_{\mathrm{un}}$, we will always be able to evaluate the stalks in terms of normalized parabolic inductions of Weyl group translates of the character χ,and, by the previous Corollary, we know these are the only possible non-zero stalks.

Now we turn our attention to studying how the geometric Eisenstein functor interacts with Verdier duality.

2.8 Eisenstein Series and Verdier Duality

We would like to study how the normalized Eisenstein functor interacts with Verdier duality. This will be done assuming the following claim.

Assumption 2.8.1. *We assume that the sheaf* $j_!(\mathrm{IC}_{\mathrm{Bun}_B})$ *is ULA with respect to the morphism defined by* $\overline{\mathfrak{q}}$*, in the sense of [FS21, Definition IV.2.31].*

Remark 2.8.2. A proof of this claim should appear in upcoming work [HHS].

We assume this claim, and use it to show that this implies that our Eisenstein functor commutes with Verdier duality when ϕ_T is weakly generic. In particular, if $\mathbb{D}_{\mathrm{Bun}_G}$ (resp. $\mathbb{D}_{\mathrm{Bun}_T}$) denotes Verdier duality on Bun_G (resp. Bun_T), our main goal is to prove the following.

Theorem 2.8.3. *Assuming 2.8.1 then, for* ϕ_T *a weakly generic toral parameter, there is an isomorphism of objects in* $\mathrm{D}(\mathrm{Bun}_G)$

$$\mathbb{D}_{\mathrm{Bun}_G}(\mathrm{nEis}(\mathscr{S}_{\phi_T})) \simeq \mathrm{nEis}(\mathbb{D}_{\mathrm{Bun}_T}(\mathscr{S}_{\phi_T}))$$

where we note that $\mathbb{D}_{\mathrm{Bun}_T}(\mathscr{S}_{\phi_T}) \simeq \mathscr{S}_{\phi_T^\vee}$*, if* $\phi_T^\vee$ *denotes the parameter dual to* ϕ_T*.*

First, let's record some implications of Assumption 2.8.1.

Lemma 2.8.4. *The sheaf* $j_*(\mathrm{IC}_{\mathrm{Bun}_B})$ *is also ULA with respect to* $\overline{\mathfrak{q}}$*, and we have isomorphisms*

$$\mathbb{D}_{\overline{\mathrm{Bun}}_B}(j_!(\mathrm{IC}_{\mathrm{Bun}_B})) \simeq j_*(\mathrm{IC}_{\mathrm{Bun}_B})$$

and

$$\mathbb{D}_{\overline{\mathrm{Bun}}_B}(j_*(\mathrm{IC}_{\mathrm{Bun}_B})) \simeq j_!(\mathrm{IC}_{\mathrm{Bun}_B})$$

of objects in $\mathrm{D}(\overline{\mathrm{Bun}}_B)$.

Proof. The first claimed isomorphism just follows from Corollary 2.6.2, and the fact that we always have an isomorphism of derived functors $\mathbb{D}_{\overline{\mathrm{Bun}_B}} \circ j_! \simeq j_* \circ \mathbb{D}_{\mathrm{Bun}_B}$, by projection formula. Now, since the dualizing object on Bun_T is just the constant sheaf, by Lemma 2.6.5 and the fact that T is unimodular, it follows that we have a unique (up to fixing Haar measures on $T(\mathbb{Q}_p)$) isomorphism $\mathbb{D}_{\overline{\mathrm{Bun}_B}} \simeq \mathbb{D}_{\overline{\mathrm{Bun}_B}/\mathrm{Bun}_T}$. Therefore, by [FS21, Corollary IV.2.25], we can show that $j_*(\mathbb{D}_{\mathrm{Bun}_B}(\mathrm{IC}_{\mathrm{Bun}_B})) \simeq j_*(\mathrm{IC}_{\mathrm{Bun}_B}) \simeq \mathbb{D}_{\overline{\mathrm{Bun}_B}}(j_!(\mathrm{IC}_{\mathrm{Bun}_B})) \simeq \mathbb{D}_{\overline{\mathrm{Bun}_B}/\mathrm{Bun}_T}(j_!(\mathrm{IC}_{\mathrm{Bun}_B}))$ is also ULA with respect to $\overline{\mathfrak{q}}$, where the first isomorphism follows from Corollary 2.6.2. The second claimed isomorphism now follows from the first and the fact that ULA objects are reflexive with respect to Verdier duality, again by [FS21, Corollary IV.2.25]. □

We will now combine this with the following lemma.

Lemma 2.8.5. *Let $f : X \to S$ be a map of decent v-stacks which are fine over a base $*$. Suppose that A is ULA with respect to f and $B \in \mathrm{D}(S)$, then we have a natural isomorphism*

$$\mathbb{D}_X(A \otimes f^*(B)) \simeq \mathbb{D}_X(A) \otimes f^*(\mathbb{D}_S(B))$$

in $\mathrm{D}(X)$.

Proof. Let $g : S \to *$ denote the structure morphism. It follows, by [FS21, Proposition IV.2.19] and A being ULA, that we can rewrite the RHS of the above isomorphism as

$$\mathbb{D}_X(A) \otimes f^*(\mathbb{D}_S(B)) \simeq R\mathscr{H}om(A, f^!(\mathbb{D}_S(B)))$$

which in turn is equal to

$$R\mathscr{H}om(A, f^!(R\mathscr{H}om(B, g^!(\Lambda))))$$

by definition. Now, by projection formula, we can further rewrite this as

$$R\mathscr{H}om(A, R\mathscr{H}om(f^*(B), f^!g^!(\Lambda)))$$

but, by Hom-Tensor duality, this is just

$$R\mathscr{H}om(A \otimes f^*(B), f^!g^!(\Lambda)) = \mathbb{D}_X(A \otimes f^*(B))$$

as desired. □

Combining the previous two Lemmas, we deduce the following.

Corollary 2.8.6. *There is an isomorphism*

$$\mathbb{D}_{\overline{\mathrm{Bun}}_B}(\overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T})\otimes j_!(\mathrm{IC}_{\mathrm{Bun}_B}))\simeq\overline{\mathfrak{q}}^*(\mathbb{D}_{\mathrm{Bun}_T}(\mathscr{S}_{\phi_T}))\otimes\mathbb{D}_{\overline{\mathrm{Bun}}_B}(j_!(\mathrm{IC}_{\mathrm{Bun}_B}))\simeq\overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T^\vee})\otimes j_*(\mathrm{IC}_{\mathrm{Bun}_B})$$

of objects in $\mathrm{D}(\overline{\mathrm{Bun}}_B)$. *Similarly, using Lemma 2.8.4, we also have an isomorphism*

$$\mathbb{D}_{\overline{\mathrm{Bun}}_B}(\overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T})\otimes j_*(\mathrm{IC}_{\mathrm{Bun}_B}))\simeq\overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T^\vee})\otimes j_!(\mathrm{IC}_{\mathrm{Bun}_B})$$

of objects in $\mathrm{D}(\overline{\mathrm{Bun}}_B)$.

Now let's apply these results to prove Theorem 2.8.3.

Proof. (Theorem 2.8.3) First note that, using projection formula with respect to j, we have an isomorphism:

$$\mathrm{nEis}(\mathscr{S}_{\phi_T})=\mathfrak{p}_!(\mathfrak{q}^*(\mathscr{S}_{\phi_T})\otimes\mathrm{IC}_{\mathrm{Bun}_B})=\overline{\mathfrak{p}}_!j_!(j^*\overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T})\otimes\mathrm{IC}_{\mathrm{Bun}_B})\simeq\overline{\mathfrak{p}}_!(\overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T})\otimes j_!(\mathrm{IC}_{\mathrm{Bun}_B}))$$

Now, by Proposition 2.5.9, we have that $\overline{\mathfrak{p}}_!$ is equivalent to $\overline{\mathfrak{p}}_*$, this means that we have an isomorphism

$$\mathbb{D}_{\mathrm{Bun}_G}(\mathrm{nEis}(\mathscr{S}_{\phi_T}))\simeq\overline{\mathfrak{p}}_!(\mathbb{D}_{\overline{\mathrm{Bun}}_B}(\overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T})\otimes j_!(\mathrm{IC}_{\mathrm{Bun}_B})))$$

but now, by Corollary 2.8.6, this is isomorphic to

$$\overline{\mathfrak{p}}_!(\overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T^\vee})\otimes j_*(\mathrm{IC}_{\mathrm{Bun}_B}))\tag{2.6}$$

Therefore, we need to exhibit an isomorphism between this sheaf and

$$\mathrm{nEis}(\mathbb{D}_{\mathrm{Bun}_T}(\mathscr{S}_{\phi_T}))\simeq\overline{\mathfrak{p}}_!(\overline{\mathfrak{q}}^*(\mathbb{D}_{\mathrm{Bun}_T}(\mathscr{S}_{\phi_T}))\otimes j_!(\mathrm{IC}_{\mathrm{Bun}_B}))\simeq\overline{\mathfrak{p}}_!(\overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T^\vee})\otimes j_!(\mathrm{IC}_{\mathrm{Bun}_B}))\tag{2.7}$$

In other words, we need to show that the cone of the natural map

$$\overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T^\vee})\otimes j_!(\mathrm{IC}_{\mathrm{Bun}_B}))\rightarrow\overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T^\vee})\otimes j_*(\mathrm{IC}_{\mathrm{Bun}_B}))\tag{2.8}$$

is trivial after applying $\overline{\mathfrak{p}}_!$. We will do this by factorizing (8). In particular, note, by applying projection formula to j as above and rewriting $\mathrm{IC}_{\mathrm{Bun}_B}\simeq\mathfrak{q}^*(\Delta_B^{1/2})[\dim(\mathrm{Bun}_B)]\simeq j^*\overline{\mathfrak{q}}^*(\Delta_B^{1/2})[\dim(\mathrm{Bun}_B)]$, we see that (7) is isomorphic to

$$\overline{\mathfrak{p}}_!(j_!j^*\overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T^\vee}\otimes\Delta_B^{1/2}))[\dim(\mathrm{Bun}_B)]$$

and therefore applying Verdier duality and Theorem 2.6.1 it follows that (6) is isomorphic to

$$\overline{\mathfrak{p}}_!(j_*j^*\overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T^\vee}\otimes\Delta_B^{1/2}))[\dim(\mathrm{Bun}_B)]$$

Therefore, we can rewrite the map (8) as

$$j_!j^*\overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T}\otimes\Delta_B^{1/2})\to j_*j^*\overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T}\otimes\Delta_B^{1/2}))$$

Now note that we can factorize this morphism via the adjunction maps as

$$j_!j^*\overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T^\vee}\otimes\Delta_B^{1/2})[\dim(\mathrm{Bun}_B)]\xrightarrow{(1)}\overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T^\vee}\otimes\Delta_B^{1/2})[\dim(\mathrm{Bun}_B)]\xrightarrow{(2)}j_*j^*\overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T^\vee}\otimes\Delta_B^{1/2}))[\dim(\mathrm{Bun}_B)]$$

By the octahedral axiom, it suffices to show the cone of (1) and (2) are trivial after applying $\overline{\mathfrak{p}}_!$. The cone of (1) is relatively easy to get a handle on, but the cone of (2) is more tricky. To do this, we note it suffices to show the claim after applying Verdier duality on $\mathbb{D}_{\overline{\mathrm{Bun}_B}}$. This follows because $\overline{\mathfrak{p}}_!\simeq\overline{\mathfrak{p}}_*$ and Verdier duality on Bun_G can be checked to be a conservative functor. In particular, one can use the semi-orthogonal decomposition of $\mathrm{D}(\mathrm{Bun}_G)$ into the HN-strata $\mathrm{D}(\mathrm{Bun}_G^b)$ to reduce it to the claim that smooth duality on the unbounded derived category of smooth $\overline{\mathbb{F}}_\ell$-representations of a p-adic reductive group is conservative. Now, by the above discussion, the Verdier Dual of the target of (2) is equal to $j_!j^*\overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T}\otimes\Delta_B^{1/2})[\dim(\mathrm{Bun}_B)]$. Hence, we deduce that the Verdier dual of (2) is equal to a map

$$j_!j^*\overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T}\otimes\Delta_B^{1/2}))\to\mathbb{D}_{\overline{\mathrm{Bun}_B}}(\overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T^\vee}\otimes\Delta_B^{1/2}))\simeq\overline{\mathfrak{q}}^!(\mathscr{S}_{\phi_T}\otimes\Delta_B^{-1/2})$$

where $\Delta_B^{-1/2}$ is the sheaf on Bun_T whose restriction to each connected component is given by the character $\delta_B^{-1/2}$. As a quick sanity check, note that, by Theorem 2.6.1, we have an isomorphism $j^*\overline{\mathfrak{q}}^!(\mathscr{S}_{\phi_T}\otimes\Delta_B^{-1/2})\simeq j^!\overline{\mathfrak{q}}^!(\mathscr{S}_{\phi_T}\otimes\Delta_B^{-1/2})\simeq\mathfrak{q}^!(\mathscr{S}_{\phi_T}\otimes\Delta_B^{-1/2}\otimes\Delta_B)[\dim(\mathrm{Bun}_B)]\simeq\mathfrak{q}^*(\mathscr{S}_{\phi_T}\otimes\Delta_B^{1/2})[\dim(\mathrm{Bun}_B)]$, and so we can see that this is the natural map coming from adjunction. Now, observe that ϕ_T is weakly generic if and only if $\phi_T^\vee$ is weakly generic (since taking duals just exchanges the role of positive and negative roots). Therefore, it suffices to show the following.

Lemma 2.8.7. *Assume that ϕ_T is weakly generic toral parameter, then the cone of the morphism*

$$j_!j^*\overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T}\otimes\Delta_B^{1/2})\simeq j_!\mathfrak{q}^*(\mathscr{S}_{\phi_T}\otimes\Delta_B^{1/2})\to\overline{\mathfrak{q}}^*(\mathscr{S}_{\phi_T}\otimes\Delta_B^{1/2})$$

vanishes after applying $\overline{\mathfrak{p}}_!$. *Similarly, the cone of the natural map*

$$j_! j^* \overline{\mathfrak{q}}^* (\mathscr{S}_{\phi_T} \otimes \Delta_B^{1/2}) \simeq j_! \mathfrak{q}^* (\mathscr{S}_{\phi_T} \otimes \Delta_B^{1/2}) \to \overline{\mathfrak{q}}^! (\mathscr{S}_{\phi_T} \otimes \Delta_B^{-1/2})$$

vanishes after applying $\overline{\mathfrak{p}}_!$.

Proof. We start with the first map. By excision, it suffices to show that the restriction to the locally closed stratum ${}_{\overline{\nu}}\overline{\mathrm{Bun}}_B$ vanishes for all $\overline{\nu} \in \Lambda_{G,B}^{\mathrm{pos}} \setminus \{0\}$ after applying $\overline{\mathfrak{p}}_!$. In particular, we are tasked with computing

$$\overline{\mathfrak{p}}_! j_{\overline{\nu}!} j_{\overline{\nu}}^* \overline{\mathfrak{q}}^* (\mathscr{S}_{\phi_T} \otimes \Delta_B^{1/2})$$

for all such $\overline{\nu}$. Note that the composite map

$$\mathrm{Bun}_B \times \mathrm{Div}^{(\overline{\nu})} \xrightarrow{j_{\overline{\nu}}} \overline{\mathrm{Bun}}_B \xrightarrow{\overline{\mathfrak{q}}} \mathrm{Bun}_T$$

can be identified with

$$\mathrm{Bun}_B \times \mathrm{Div}^{(\overline{\nu})} \xrightarrow{\mathfrak{q} \times \mathrm{id}} \mathrm{Bun}_T \times \mathrm{Div}^{(\overline{\nu})} \xrightarrow{h^{\rightarrow}_{(\overline{\nu})}} \mathrm{Bun}_T$$

where the last map is the Hecke operator defined in §2.3.3. Therefore, we are reduced to computing

$$\overline{\mathfrak{p}}_! j_{\overline{\nu}!} (\mathfrak{q} \times \mathrm{id})^* (h^{\rightarrow}_{(\overline{\nu})})^* (\mathscr{S}_{\phi_T} \otimes \Delta_B^{1/2}) \simeq \overline{\mathfrak{p}}_! j_{\overline{\nu}!} (\mathfrak{q} \times \mathrm{id})^* (\mathscr{S}_{\phi_T} \otimes \Delta_B^{1/2} \boxtimes E_{\phi_T}^{(\overline{\nu})}(\langle \hat{\rho}, \overline{\nu} \rangle))$$

However, note that we have an equality: $\overline{\mathfrak{p}} \circ j_{\overline{\nu}} = \mathfrak{p} \times g$, where $g : \mathrm{Div}^{(\overline{\nu})} \to *$ is the structure map. Therefore, by Künneth formula, we obtain that the RHS can be identified with

$$\mathrm{nEis}(\mathscr{S}_{\phi_T}) \otimes R\Gamma_c(\mathrm{Div}^{(\overline{\nu})}, E_{\phi_T}^{(\overline{\nu})}(\langle \hat{\rho}, \overline{\nu} \rangle))$$

but now it follows, by Corollary 2.3.21 and the weak genericity assumption on ϕ_T, that the complex $R\Gamma_c(\mathrm{Div}^{(\overline{\nu})}, E_{\phi_T}^{(\overline{\nu})}(\langle \hat{\rho}, \overline{\nu} \rangle))$ is trivial. Now for the second map we argue similarly. In particular, it suffices to show that $\overline{\mathfrak{q}}^!(\mathscr{S}_{\phi_T} \otimes \Delta_B^{-1/2})$ vanishes after applying $\overline{\mathfrak{p}}_! j_{\overline{\nu}*} j_{\overline{\nu}}^!$, but then, as before, we observe that $j_{\overline{\nu}}^! \circ \overline{\mathfrak{q}}^! = (\mathfrak{q} \times \mathrm{id})^! \circ (h^{\rightarrow}_{\overline{\nu}})^!$. By uniformizing Div^1 by punctured positive Banach-Colmez spaces, the dualizing object on $\mathrm{Div}^{(\overline{\nu})}$ can be identified up to a shift with $\Lambda(\langle \hat{\rho}, \overline{\nu} \rangle)$. It follows that $h_{\overline{\nu}}^!(\Delta_B^{-1/2} \otimes \mathscr{S}_{\phi_T})$ can be identified up to a shift with $E_{\phi_T}^{(\overline{\nu})}(-\langle \hat{\rho}, \overline{\nu} \rangle + \langle \hat{\rho}, \overline{\nu} \rangle) = E_{\phi_T}^{(\overline{\nu})}$. Similarly, we can compute $(\mathfrak{q} \times \mathrm{id})^!$ using Theorem 2.6.1, and this will not effect the $\mathrm{Div}^{(\overline{\nu})}$-factor. Using that $\overline{\mathfrak{p}}_! \circ j_{\overline{\nu}*} \simeq \overline{\mathfrak{p}}_* \circ j_{\overline{\nu}*} = (\mathfrak{p} \times g)_*$ together with the properness of g [FS21, Proposition II.1.21], we can apply Künneth to see the desired vanishing follows from the vanishing of the complex

$$R\Gamma_c(\mathrm{Div}^{(\overline{\nu})}, E_{\phi_T}^{(\overline{\nu})})$$

which again follows from Corollary 2.3.21. □

□

We now conclude by noting the following important corollary of assumption 2.8.1.

Corollary 2.8.8. *The functor*

$$\mathrm{nEis}(-) : \mathrm{D}(\mathrm{Bun}_T) \to \mathrm{D}(\mathrm{Bun}_G)$$

induces a functor

$$\mathrm{nEis}(-) : \mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_T) \to \mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_G)$$

on the full subcategories of ULA objects over $*$.

Proof. By projection formula applied with respect to the open immersion j, we have a natural isomorphism

$$\mathrm{nEis}(-) \simeq \overline{\mathfrak{p}}_*(\overline{\mathfrak{q}}^*(-) \otimes j_!(\mathrm{IC}_{\mathrm{Bun}_B})),$$

as explained above. In particular, given $A \in \mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_T)$, we have, by [FS21, Proposition IV.2.26] and assumption 2.8.1, that $\overline{\mathfrak{q}}^*(A) \otimes j_!(\mathrm{IC}_{\mathrm{Bun}_B})$ is ULA over $*$. Now, since the map $\overline{\mathfrak{p}}$ is proper after restricting to connected components, the claim follows by [FS21, Proposition IV.2.11]. □

For the rest of the paper, we will assume 2.8.1 and thereby the validity of Theorem 2.8.3 and Corollary 2.8.8. In addition, we will assume compatibility with a suitably nice form of the local Langlands correspondence (Assumption 2.7.5).

2.9 Stalks of Geometric Eisenstein Series

Now our aim is to explicitly determine the stalks of the sheaf $\mathrm{nEis}(\mathscr{S}_{\phi_T})$, when ϕ_T is weakly normalized regular. Recall this means that ϕ_T is generic, and, for all $w \in W_G$ non-trivial, $\chi \otimes \delta_B^{1/2} \not\simeq (\chi \otimes \delta_B^{-1/2})^w$. Genericity will allow us to apply the results of Corollary 2.7.7, and the second condition will appear naturally in the computation of the stalks. It is helpful to treat each connected component separately. In particular, using Corollary 2.5.3, we consider the decomposition:

$$\mathrm{nEis}(\mathscr{S}_{\phi_T}) = \bigoplus_{\overline{\nu} \in B(T)} \mathrm{nEis}^{\overline{\nu}}(\mathscr{S}_{\phi_T})$$

The main result of this section is as follows.

Theorem 2.9.1. *Consider ϕ_T a weakly normalized regular parameter with associated character $\chi : T(\mathbb{Q}_p) \to \Lambda^*$. We fix $\overline{\nu} \in B(T)$ with image $b \in B(G)_{\mathrm{un}}$, dominant reduction b_T, and associated Borel B_b. Using Corollary 2.2.11, we can write $\overline{\nu} = w(b_T)$ for a unique $w \in W_b$. Then we have an isomorphism*

$$\mathrm{nEis}^{\overline{\nu}}(\mathscr{S}_{\phi_T}) \simeq j_{b!}(i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{-1/2})[-\langle 2\hat{\rho}, \nu_b \rangle]$$

under the identification $\mathrm{D}(J_b(\mathbb{Q}_p), \Lambda) \simeq \mathrm{D}(\mathrm{Bun}_G^b)$, where $j_b : \mathrm{Bun}_G^b \to \mathrm{Bun}_G$ is the locally closed immersion defined by the HN-stratum corresponding to b, and P_b is the standard parabolic with Levi factor $M_b \simeq J_b$.

By varying $\overline{\nu}$ over all connected components, we obtain the following.

Corollary 2.9.2. *Consider ϕ_T a weakly normalized regular parameter, with associated character $\chi : T(\mathbb{Q}_p) \to \Lambda^*$. For $b \in B(G)$, the stalk $\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^b} \in \mathrm{D}(\mathrm{Bun}_G^b) \simeq \mathrm{D}(J_b(\mathbb{Q}_p), \Lambda)$ is given by*

1. *an isomorphism $\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^b} \simeq \bigoplus_{w \in W_b} i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{-1/2}[-\langle 2\hat{\rho}, \nu_b \rangle]$ if $b \in B(G)_{\mathrm{un}}$,*
2. *an isomorphism $\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^b} \simeq 0$ if $b \notin B(G)_{\mathrm{un}}$.*

In particular, $\mathrm{nEis}(\mathscr{S}_{\phi_T})$ is a perverse sheaf on Bun_G with respect to the standard t-structure defined by the HN-strata using Theorem 2.8.3.

First, consider the following easy Lemma.

Lemma 2.9.3. *For $\overline{\nu} \in B(T)$ with image b in $B(G)$, the restriction*

$$\mathrm{nEis}^{\overline{\nu}}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^{b'}}$$

for $b' \in B(G)$ vanishes unless $b \succeq b'$ in the natural partial ordering on $B(G)$.

Proof. This follows from the observation that the image of $\mathrm{Bun}_B^{\overline{\nu}}$ under $\mathfrak{p}^{\overline{\nu}}$ is contained in the open substack $\mathrm{Bun}_B^{\leq b}$ parametrizing bundles with associated Kottwitz element less than b. In particular, using the Tannakian formalism [Zie15, Theorems 4.42, 4.43], this reduces to the observation that, for GL_n, a bundle $\mathscr{E}$ with a filtration by vector subbundles has Harder-Narasimhan polygon less than or equal to Harder-Narasimhan polygon of the direct sum of the graded pieces of the filtration, which is an easy consequence of the formalism of Harder-Narasimhan reductions (See for example [Ked17, Corollary 3.4.18]). $\square$

Thus, for a fixed $b' \in B(G)$, Lemma 2.9.3 tells us that $\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^{b'}}$ is a direct sum of $\mathrm{nEis}^{\overline{\nu}}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^{b'}}$ for $\overline{\nu}$ whose image $b \in B(G)$ satisfies $b \succeq b'$. Now the key point is that, under the weak normalized regularity assumption, all the contributions will vanish except when $b' = b$. This is one of the many reasons that weak normalized regularity (or at least genericity) is absolutely necessary to get a reasonable eigensheaf. In general, all possible $\overline{\nu}$ contribute to $\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^{b'}}$, and $\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^{b'}}$ will be equal to an infinite direct sum of smooth irreducible representations sitting in infinitely many degrees. We now reduce Theorem 2.9.1 to two propositions. We first have the following proposition describing the contribution of the split reduction in the connected components $\mathrm{Bun}_B^{\overline{\nu}}$.

Proposition 2.9.4. *Let $\overline{\nu} \in B(T)$ be an element mapping to $b \in B(G)_{\mathrm{un}}$. We write $\overline{\nu} = w(b_T)$ as above. If $\xi : T(\mathbb{Q}_p) \to \Lambda^*$ is any smooth character, we have an isomorphism*

$$\mathrm{Eis}^{\overline{\nu}}(\xi)|_{\mathrm{Bun}_G^b} \simeq \mathrm{Ind}_{B_b}^{J_b}(\xi^w \otimes (\delta_B^w)^{-1/2} \otimes \delta_{B_b}^{1/2}) \otimes \delta_{P_b}^{-1/2}[-\langle 2\hat{\rho}, \nu_b \rangle]$$

of complexes of smooth $J_b(\mathbb{Q}_p)$-modules, under the identification $\mathrm{D}(\mathrm{Bun}_G^b) \simeq \mathrm{D}(J_b(\mathbb{Q}_p), \Lambda)$, where $w \in W_b$ is identified with a representative of minimal length and δ_{P_b} is as in assumption 2.7.5. .

In particular, using the isomorphism

$$\mathrm{nEis}(\mathscr{S}_{\phi_T}) \simeq \mathrm{Eis}(\mathscr{S}_{\phi_T} \otimes \Delta_B^{1/2})$$

we have, for all $\overline{\nu} \in B(T)$ mapping to $b \in B(G)_{\mathrm{un}}$ and χ the character attached to ϕ_T, an isomorphism

$$\begin{aligned}\mathrm{nEis}^{\overline{\nu}}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^b} = \mathrm{Eis}(\mathscr{S}_{\phi_T} \otimes \Delta_B^{1/2})|_{\mathrm{Bun}_G^b} &\simeq \mathrm{Ind}_{B_b}^{J_b}(\chi^w \otimes (\delta_B^w)^{1/2} \otimes (\delta_B^w)^{-1/2} \otimes \delta_{B_b}^{1/2}) \otimes \delta_{P_b}^{-1/2}[-\langle 2\hat{\rho}, \nu_b\rangle] \\ &\simeq \mathrm{Ind}_{B_b}^{J_b}(\chi^w \otimes \delta_{B_b}^{1/2}) \otimes \delta_{P_b}^{-1/2}[-\langle 2\hat{\rho}, \nu_b\rangle] \\ &\simeq i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{-1/2}[-\langle 2\hat{\rho}, \nu_b\rangle]\end{aligned}$$

This tells us that all the claimed contributions to the restriction $\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^b}$ appear. All that remains is to show is that there are no additional contributions, and this is precisely what weak normalized regularity will allow us to do.

Proposition 2.9.5. *Assume ϕ_T is weakly normalized regular, then, for all $\overline{\nu} \in B(T)$ mapping to $b \in B(G)_{\mathrm{un}}$, the sheaf $\mathrm{nEis}^{\overline{\nu}}(\mathscr{S}_{\phi_T})$ is only supported on the HN-strata Bun_G^b.*

We dedicate the remainder of this section to the proof of these two propositions.

The Proof of Proposition 2.9.4

We let $\mathrm{Bun}_B^{\overline{\nu}}$ be the connected component defined by $\overline{\nu} \in B(T)$ mapping to $b \in B(G)_{\mathrm{un}}$, and let $\mathrm{Bun}_B^{\overline{\nu},b}$ be the preimage of Bun_G^b along $\mathfrak{p}^{\overline{\nu}} : \mathrm{Bun}_B^{\overline{\nu}} \to \mathrm{Bun}_G$. Topologically, the stack $\mathrm{Bun}_B^{\overline{\nu},b}$ is just a point defined by the split reduction $Q_{\overline{\nu}} = \mathscr{F}_{\overline{\nu}} \times^T B$, as in Definition 2.6.16. We already saw in the proof of Theorem 2.6.1 what the automorphisms of this B-bundle are. In particular, they are given by the group diamond $\mathscr{G}_{\overline{\nu}}$, where $\mathscr{G}_{\overline{\nu}}(S) = Q_{\overline{\nu}}(X_S)$ for $S \in \mathrm{Perf}$, and $Q_{\overline{\nu}}$ is defined as $\mathscr{F}_{\overline{\nu}} \times^T B$. In other words, we have an isomorphism

$$\mathrm{Bun}_B^{\overline{\nu},b} \simeq [*/\mathscr{G}_{\overline{\nu}}]$$

and the map $\mathfrak{p}^{\overline{\nu}} : \mathrm{Bun}_B^{\overline{\nu}} \to \mathrm{Bun}_G$ induces a map of the form

$$[*/\mathscr{G}_{\overline{\nu}}] \to [*/\mathscr{J}_b] \simeq \mathrm{Bun}_G^b$$

which we will abusively denote by $\mathfrak{p}$. This map is given by an injection

$$\mathscr{G}_{\overline{\nu}} \hookrightarrow \mathscr{J}_b$$

of group diamonds, coming from the fact that $Q_{\overline{\nu}}$ defines a reduction of $\mathscr{F}_b$. Moreover, the map $\mathfrak{q}^{\overline{\nu}} : \mathrm{Bun}_B^{\overline{\nu},b} \to \mathrm{Bun}_T$ is identified with a map of the form

$$[*/\mathscr{G}_{\overline{\nu}}] \to [*/\underline{T(\mathbb{Q}_p)}]$$

which we will abusively denote by $\mathfrak{q}$. This map factors as

$$[*/\mathscr{G}_{\overline{\nu}}] \longrightarrow \left[*/\underline{B_b(\mathbb{Q}_p)}\right] \xrightarrow{q^\natural} \left[*/\underline{T(\mathbb{Q}_p)}\right] \simeq^w \left[*/\underline{T(\mathbb{Q}_p)}\right]$$

where the first map is given by the semi-direct decomposition

$$\mathscr{G}_{\overline{\nu}} \simeq \mathscr{G}_{\overline{\nu}}^{>0} \ltimes \mathscr{G}_{\overline{\nu}}^{=0}$$

and the identification $\mathscr{G}_{\overline{\nu}}^{=0} \simeq B_b(\mathbb{Q}_p)$, the second map $q^\natural$ is the natural projection, and the last isomorphism is given by conjugating by the minimal length representative $w \in W_b$. We will explain more below why this conjugation by w appears. For now, it follows by base change that we have an isomorphism

$$\mathrm{Eis}^{\overline{\nu}}(\xi)|_{\mathrm{Bun}_G^b} \simeq \mathfrak{p}_! \mathfrak{q}^*(\xi)[d_{\overline{\nu}}]$$

where $d_{\overline{\nu}} = \dim(\mathrm{Bun}_B^{\overline{\nu}})$. We let $[*/\underline{J_b(\mathbb{Q}_p)}] \to [*/\mathscr{J}_b]$ be the natural map, as in §2.6.1. Our goal is to compute

$$s^* \mathfrak{p}_! \mathfrak{q}^*(\xi)[d_{\overline{\nu}}]$$

as a complex of $J_b(\mathbb{Q}_p)$-representations. To do this, we consider the stack $Y^{\overline{\nu}}$ defined by the Cartesian diagram

$$\begin{tikzcd} Y^{\overline{\nu}} \arrow[r, "\tilde{\mathfrak{p}}"] \arrow[d, "\tilde{s}"] & \left[*/\underline{J_b(\mathbb{Q}_p)}\right] \arrow[d, "s"] \\ {[*/\mathscr{G}_{\overline{\nu}}]} \arrow[r, "\mathfrak{p}"] \arrow[d, "\mathfrak{q}"] & {[*/\mathscr{J}_b]} \\ \left[*/\underline{T(\mathbb{Q}_p)}\right] & \end{tikzcd} \tag{2.9}$$

By base-change, it suffices to compute:

$$\tilde{\mathfrak{p}}_! \tilde{s}^* \mathfrak{q}^*(\xi)[d_{\overline{\nu}}]$$

Let's now describe the stack $Y^{\overline{\nu}}$. We define $\mathscr{G}^{\overline{\nu}}$ to be the cokernel of the natural map of group diamonds:

$$0 \to \mathscr{G}_{\overline{\nu}}^{>0} \to \mathscr{J}_b^{>0} \to \mathscr{G}^{\overline{\nu}} \to 0$$

The diamond $\mathscr{G}_{\overline{\nu}}^{=0} \simeq \underline{B_b(\mathbb{Q}_p)} \subset \underline{J_b(\mathbb{Q}_p)} \simeq \mathscr{J}_b^{=0}$ acts on the right of the first term by the action described in §2.6.1. It acts on the second term by the right conjugation on the opposite parabolic P_b^-. This map is equivariant with respect to these actions and hence $\mathscr{G}^{\overline{\nu}}$ also acquires a right action of $\underline{B_b(\mathbb{Q}_p)}$. We deduce that the space $Y^{\overline{\nu}}$ is isomorphic to the v-stack quotient:

$$[\mathscr{G}^{\overline{\nu}}/\underline{B_b(\mathbb{Q}_p)}]$$

With this in hand, we further refine diagram (9)

$$\begin{tikzcd}
\left[\mathscr{G}^{\overline{\nu}}/\underline{B_b(\mathbb{Q}_p)}\right] \arrow[rr, "\tilde{\mathfrak{p}}"] \arrow[dr, "j"] \arrow[dd, "\tilde{s}"] & & \left[*/\underline{J_b(\mathbb{Q}_p)}\right] \arrow[dd, "s"] \\
& \left[*/\underline{B_b(\mathbb{Q}_p)}\right] \arrow[ur, "i"] \arrow[ddl, "q"] & \\
{[*/\mathscr{G}_{\overline{\nu}}]} \arrow[rr, "\mathfrak{p}"] \arrow[d, "\mathfrak{q}"] & & {[*/\mathscr{J}_b]} \\
\left[*/\underline{T(\mathbb{Q}_p)}\right] & &
\end{tikzcd} \tag{2.10}$$

where i and j are the natural maps, and q is the composition of the natural projection $q^\natural : [*/\underline{B_b(\mathbb{Q}_p)}] \to [*/\underline{T(\mathbb{Q}_p)}]$ followed by conjugation by w. The reason that conjugation by w appears is in order to make this diagram commutative. To illustrate the point, consider an element $b \in B(G)$ which admits a canonical reduction to $b_T \in B(T)_{\text{basic}}$, so that $J_{b_T} \simeq T$ and $W_b = W_G$. We write b_T^- for the conjugate of b_T under the element of longest length w_0. Then there are two distinct ways of presenting Bun_G^b in terms of the moduli space of B-bundles. Namely, one has

$$\text{Bun}_{B^-}^{b_T} \simeq [*/\mathscr{J}_b]$$

and

$$\mathrm{Bun}_B^{b_T^-} \simeq [*/\mathscr{J}_b].$$

However, only the first will identify the projection to Bun_T with the natural map $\mathrm{Bun}_G^b \simeq [*/\mathscr{J}_b] \to [*/\underline{J_b(\mathbb{Q}_p)}]$ considered in [FS21], as is implicit in [FS21, Proposition III.5.1]. In the second identification, one needs to conjugate by the element of longest length in order to make this true. In particular, this is why the action of $T(\mathbb{Q}_p)$ on $\mathscr{J}_b^{>0}$ by right conjugation on the parabolic P_b^- appeared above.

Using the diagram (10), we can further reduce to computing

$$\tilde{\mathfrak{p}}_! \tilde{\mathfrak{s}}^* \mathfrak{q}^*(\xi)[d_{\overline{\nu}}] \simeq i_! j_! \tilde{\mathfrak{s}}^* \mathfrak{q}^*(\xi)[d_{\overline{\nu}}] \simeq i_! j_! j^* q^*(\xi)[d_{\overline{\nu}}]$$

but now, by projection formula, this is isomorphic to

$$i_!(q^*(\xi) \otimes j_!(\Lambda))[d_{\overline{\nu}}].$$

Now recall by Lemma 2.6.17 we have an equality

$$d_{\overline{\nu}} = \langle 2\hat{\rho}, \nu_b \rangle - 2\langle 2\hat{\rho}_G^w, \nu_b \rangle$$

where $\hat{\rho}^w$ is the sum of the positive roots $\hat{\alpha} > 0$ such that $w^{-1}(\hat{\alpha}) < 0$. To proceed further, we consider the character

$$\delta(t) := \delta_{B_b}^{1/2} \otimes (\delta_B^w)^{-1/2} \otimes \delta_{P_b}^{-1/2}|_{T(\mathbb{Q}_p)}(t)$$

This is the unique rational character of $T(\mathbb{Q}_p)$ that, after restricting to $A(\mathbb{Q}_p)$ is given by

$$\prod_{\substack{\hat{\alpha}>0 \\ \langle \hat{\alpha}, \nu_b \rangle = 0}} |\det(\mathrm{Ad}(t|\mathfrak{g}_{\hat{\alpha}})|^{1/2} \prod_{\hat{\alpha}>0} |\det(\mathrm{Ad}(t|\mathfrak{g}_{w(\hat{\alpha})}))|^{-1/2} \prod_{\substack{\hat{\alpha}>0 \\ \langle \hat{\alpha}, \nu_b \rangle \neq 0}} |\det(\mathrm{Ad}(t|\mathfrak{g}_{\hat{\alpha}})|^{-1/2} = \prod_{\substack{\hat{\alpha}>0 \\ \langle w^{-1}(\hat{\alpha}), \nu_b \rangle > 0 \\ \langle \hat{\alpha}, \nu_b \rangle \neq 0}} |\det(\mathrm{Ad}(t|$$

Here we have used that w, as a minimal length representative in W_b, will not send any of the roots lying in B_b to a negative root. We have the following lemma.

Lemma 2.9.6. *$j_!(\Lambda)$ is isomorphic to $(q^\natural)^*(\delta)[2(\langle 2\hat{\rho}_G^w, \nu_b \rangle - \langle 2\hat{\rho}, \nu_b \rangle)]$.*

Let's see why the result follows from this. In particular, using the formula $d_{\overline{\nu}} = \langle 2\hat{\rho}, \nu_b \rangle - 2\langle 2\hat{\rho}_G^w, \nu_b \rangle$, this gives us an isomorphism:

$$\mathrm{Eis}^{\overline{\nu}}(\xi) \simeq i_!(q^*(\xi) \otimes (q^\natural)^*(\delta))[-2\langle 2\hat{\rho}, \nu_b \rangle + 2\langle 2\hat{\rho}_G^w, \nu_b \rangle + \langle 2\hat{\rho}, \nu_b \rangle - 2\langle 2\hat{\rho}_G^w, \nu_b \rangle]$$

$$\simeq i_!((q^\natural)^*(\xi^w \otimes \delta))[\langle -2\hat{\rho}, \nu_b\rangle]$$

However, now $i_!((q^\natural)^*(\xi \otimes \delta)))$ will be identified with compactly supported functions on $J_b(\mathbb{Q}_p)$ which transform under the action of $B_b(\mathbb{Q}_p)$ by the character $\xi^w \otimes \delta$ via the natural projection $B_b(\mathbb{Q}_p) \to T(\mathbb{Q}_p)$. However, this is precisely the parabolic induction $\mathrm{Ind}_{B_b}^{J_b}(\xi^w \otimes \delta)$, and the claim follows. We now prove the lemma.

Proof. (Lemma 2.9.6) We need to determine $j_!(\Lambda)$, where j is the map

$$[\mathscr{G}^{\overline{\nu}}/\underline{B_b(\mathbb{Q}_p)}] \to [*/\underline{B_b(\mathbb{Q}_p)}].$$

Recall that $\mathscr{G}^{\overline{\nu}}$ is defined via the short exact sequence of group diamonds:

$$0 \to \mathscr{G}_{\overline{\nu}}^{>0} \to \mathscr{J}_b^{>0} \to \mathscr{G}^{\overline{\nu}} \to 0$$

However, as in §2.6.1, the map $\mathscr{G}_{\overline{\nu}}^{>0} \to \mathscr{J}_b^{>0}$ respects the filtration by commutator subgroups as well as the action of the slope homomorphism, and we similarly see the cokernel $\mathscr{G}^{\overline{\nu}}$ has an induced filtration by commutator subgroups. Namely, since $\mathscr{G}_{\overline{\nu}}^{>0}$ breaks up in terms of the positive Banach-Colmez spaces $\mathscr{H}^0(Q_{\hat{\alpha}})$, for $\hat{\alpha}$ a positive root such that $w^{-1}(\hat{\alpha}) < 0$ and $\langle \hat{\alpha}, \nu_b\rangle \neq 0$, and $\mathscr{J}_b^{>0}$ breaks up as $\mathscr{H}^0(Q_{\hat{\alpha}})$, for $\hat{\alpha}$ a positive root such that $\langle \hat{\alpha}, \nu_b\rangle \neq 0$, we can write $\mathscr{G}^{\overline{\nu}}$ as an iterated fibration of the positive Banach-Colmez spaces

$$\mathscr{H}^0(Q_{\hat{\alpha}})$$

where $\hat{\alpha}$ is a positive root such that $\langle \hat{\alpha}, \nu_b\rangle \neq 0$ and $w^{-1}(\hat{\alpha}) > 0$. Therefore, we deduce that j is an iterated fibration of positive Banach-Colmez spaces, and we can apply the proof of [FS21, Proposition V.2.1] to deduce that the adjunction

$$j^! j_!(\Lambda) \simeq \Lambda$$

is an isomorphism. The claim is therefore reduced to showing that:

$$j^!(\Lambda) \simeq \delta^{-1}[2(\langle 2\hat{\rho}, \nu_b\rangle - \langle 2\hat{\rho}_G^w, \nu_b\rangle)]$$

As per usual, we consider the Cartesian diagram

$$\begin{tikzcd} \mathscr{G}^{\overline{\nu}} \arrow[r, "\tilde{j}"] \arrow[d, "\tilde{\pi}"] & * \arrow[d, "\pi"] \\ \left[\mathscr{G}^{\overline{\nu}}/\underline{B_b(\mathbb{Q}_p)}\right] \arrow[r, "j"] & \left[*/\underline{B_b(\mathbb{Q}_p)}\right] \end{tikzcd}$$

and, by base-change, obtain an isomorphism:

$$\tilde{j}^!(\Lambda) \simeq \tilde{\pi}^* j^!(\Lambda)$$

By Lemma 2.6.8, the dualizing object on the Banach-Colmez spaces $\mathscr{H}^0(Q_{\hat{\alpha}})$ is isomorphic to $|\cdot|^{-\dim(\hat{\mathfrak{g}}_{\hat{\alpha}})}[2\langle\hat{\alpha}, \nu_b\rangle]$ as a sheaf with right $\mathbb{Q}_p^*$-action. However, as discussed above, in the present situation $A(\mathbb{Q}_p)$ acts on $\mathscr{J}_b^{>0}$ via right conjugation of the opposite parabolic P_b^-. Hence, the induced right action on this Banach-Colmez space is by

$$A(\mathbb{Q}_p) \xrightarrow{\hat{\alpha}^{-1}} \mathbb{Q}_p^*$$

and the right scaling action of $\mathbb{Q}_p^*$ on $\mathscr{H}^0(Q_{\hat{\alpha}})$. This tells us that the dualizing object on $\mathscr{G}^{\overline{\nu}}$ as a sheaf with the relevant right $A(\mathbb{Q}_p)$-action is isomorphic to

$$\prod_{\substack{\hat{\alpha}>0 \\ w^{-1}(\hat{\alpha})>0}} |\det(\mathrm{Ad}(t|\mathfrak{g}_{\hat{\alpha}})|[2 \sum_{\substack{\hat{\alpha}>0 \\ w^{-1}(\hat{\alpha})>0}} \langle\hat{\alpha}, \nu_b\rangle]$$

but this is isomorphic to

$$\delta^{-1}|_{A(\mathbb{Q}_p)}[2(\langle 2\hat{\rho}, \nu_b\rangle - \langle 2\hat{\rho}_G^w, \nu_b\rangle)]$$

as desired. However, now we can also see that it is also isomorphic to this as a sheaf with right $T(\mathbb{Q}_p)$-action, by using that the action of $T(\mathbb{Q}_p)$ on $\mathscr{G}^{\overline{\nu}}$ factorizes over the adjoint action, as in the proof of Proposition 2.6.15. □

The Proof of Proposition 2.9.5

We argue by induction on $b \in B(G)_{\mathrm{un}}$ with respect to the partial ordering on $B(G)$ and the following stronger statement.

"For $b \in B(G)_{\mathrm{un}}$ with dominant reduction b_T, and $\overline{\nu} = w(b_T) \in B(T)$ mapping to b for varying $w \in W_b$, we have an isomorphism

$$\mathrm{nEis}^{w(b_T)}(\mathscr{S}_{\phi_T}) \simeq j_{b!}(i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{-1/2})[-\langle 2\hat{\rho}, \nu_b\rangle]$$

of sheaves in $\mathrm{D}(\mathrm{Bun}_G)$."

The base case will be when b is such that any $b' \in B(G)_{\mathrm{un}}$ satisfying $b \succeq b'$ is equal to b. Since the stalk of $\mathrm{nEis}^{\overline{\nu}}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^{b'}}$ will only be non-trivial for $b' \in B(G)$ such that $b \succeq_{\neq} b'$ by Lemma 2.9.3, the result in this case follows follows from Proposition 2.9.4 and Corollary 2.7.7 (1), where we note that ϕ_T is weakly normalized regular and therefore generic. For the inductive step, assume the claim is true for all $b' \in B(G)_{\mathrm{un}}$ such that $b \succeq_{\neq} b'$. Let $\overline{\nu} \in B(T)$ be an element mapping to b. By Proposition 2.9.4 and Corollary 2.7.7 (1) again, it suffices to show that the restriction of $\mathrm{nEis}^{\overline{\nu}}(\mathscr{S}_{\phi_T})$ to $\mathrm{Bun}_G^{b'}$ vanishes for all such b'. By Corollary 2.7.7 (2), it suffices to show, for all $w \in W_{b'}$, that the complex

$$R\mathscr{H}om(\mathrm{nEis}^{\overline{\nu}}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^{b'}}, i_{B_{b'}}^{J_{b'}}(\chi^w) \otimes \delta_{P_{b'}}^{-1/2}) = R\mathscr{H}om(\mathrm{nEis}^{\overline{\nu}}(\mathscr{S}_{\phi_T}), j_{b*}(i_{B_{b'}}^{J_{b'}}(\chi^w) \otimes \delta_{P_{b'}}^{-1/2}))$$

is trivial. In particular, since, by Corollary 2.8.8, we know that $\mathrm{nEis}^{\overline{\nu}}(\mathscr{S}_{\phi_T})$, is ULA it follows that $\mathrm{nEis}^{\overline{\nu}}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^{b'}}$ is admissible in the sense that, for all compact open $K \subset J_b(\mathbb{Q}_p)$, $\mathrm{nEis}^{\overline{\nu}}(\mathscr{S}_{\phi_T})^K|_{\mathrm{Bun}_G^{b'}}$ is a perfect complex. By Corollary 2.7.7 (2) and [Vig96, p. II.5.13], we know that there are only finitely many possiblities for smooth irreducible constituents of $\mathrm{nEis}^{\overline{\nu}}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^{b'}}$. Therefore, by choosing $K \subset G(\mathbb{Q}_p)$ sufficiently small (so that every such constituent has a non-zero fixed vector), we deduce that $\mathrm{nEis}^{\overline{\nu}}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^{b'}}$ is a compelx with finite length cohomology, which reduces us to showing that the previous complex is trivial. To do this, let $\overline{\nu}' = w(b'_T)$ be the element mapping to $b' \in B(G)$ defined by $w \in W_{b'}$. Our inductive hypothesis tells us that we have an isomorphism

$$j_{b'!}(i_{B_{b'}}^{J_{b'}}(\chi^w) \otimes \delta_{P_{b'}}^{-1/2})[-\langle 2\hat{\rho}, \nu_{b'}\rangle] \simeq \mathrm{nEis}^{\overline{\nu}'}(\mathscr{S}_{\phi_T})$$

varying over $\overline{\nu}'$ mapping to $b' \in B(G)$. If we write $\mathrm{nEis}_*^{\overline{\nu}'}(\mathscr{S}_{\phi_T})$ for the sheaf defined by replacing $\mathfrak{p}_!$ with $\mathfrak{p}_*$ in the definition of $\mathrm{nEis}_*^{\overline{\nu}'}(\mathscr{S}_{\phi_T})$. It follows, using Theorem 2.8.3 and Theorem 2.6.1, that this is isomorphic to $j_{b'*}(i_{B_{b'}}^{J_{b'}}(\chi^w) \otimes \delta_{P_{b'}}^{-1/2})[-\langle 2\hat{\rho}, \nu_b\rangle]$ (See the discussion preceding Proposition 2.11.12 for details). Therefore, it then suffices to show, for all $\overline{\nu}'$ mapping to $b' \in B(G)_{\mathrm{un}}$ and $\overline{\nu}$ mapping to $b \in B(G)_{\mathrm{un}}$, that

$$R\mathscr{H}om(\mathrm{nEis}^{\overline{\nu}}(\mathscr{S}_{\phi_T}), \mathrm{nEis}_*^{\overline{\nu}'}(\mathscr{S}_{\phi_T}))$$

is trivial. To aid our analysis, we consider the following functor

$$\mathrm{CT}^{\overline{\nu}}(-) := \mathfrak{q}_*^{\overline{\nu}} \circ \mathfrak{p}^{\overline{\nu}!}(-) : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_T^{\overline{\nu}})[-\dim(\mathrm{Bun}_B^{\overline{\nu}})]$$

which is in particular the right adjoint of the unnormalized Eisenstein functor $\mathrm{Eis}^{\overline{\nu}}(-) := \mathfrak{p}_!(\mathfrak{q}^*(-)[\dim(\mathrm{Bun}_B^{\overline{\nu}})])$. Writing $\mathrm{nEis}^{\overline{\nu}}(\mathscr{S}_{\phi_T})$ as $\mathrm{Eis}^{\overline{\nu}}(\mathscr{S}_{\phi_T} \otimes \Delta_B^{1/2})$ and using adjunction, it suffices to show that the complex

$$R\mathscr{H}om_{T(\mathbb{Q}_p)}(\chi \otimes \delta_B^{1/2}, \mathrm{CT}^{\overline{\nu}} \circ \mathrm{nEis}_*^{\overline{\nu}'}(\mathscr{S}_{\phi_T}))$$

is trivial in $\mathrm{D}(\mathrm{Bun}_T^{\overline{\nu}}) \simeq \mathrm{D}(T(\mathbb{Q}_p),\Lambda)$. To show this, we first look at the diagram

$$\begin{tikzcd} \mathrm{Bun}_B^{\overline{\nu}} \times_{\mathrm{Bun}_G} \mathrm{Bun}_B^{\overline{\nu}'} \arrow[r, "{'\mathfrak{p}^{\overline{\nu}}}"] \arrow[d, "{'\mathfrak{p}^{\overline{\nu}'}}"] & \mathrm{Bun}_B^{\overline{\nu}'} \arrow[r, "\mathfrak{q}^{\overline{\nu}'}"] \arrow[d, "\mathfrak{p}^{\overline{\nu}'}"] & \mathrm{Bun}_T^{\overline{\nu}'} \\ \mathrm{Bun}_B^{\overline{\nu}} \arrow[r, "\mathfrak{p}^{\overline{\nu}}"] \arrow[d, "\mathfrak{q}^{\overline{\nu}}"] & \mathrm{Bun}_G & \\ \mathrm{Bun}_T^{\overline{\nu}} & & \end{tikzcd}$$

and note, by base-change, that we have a natural isomorphism

$$\mathfrak{q}^{\overline{\nu}}_* \circ \mathfrak{p}^{\overline{\nu}!} \circ \mathfrak{p}^{\overline{\nu}'}_* \circ \mathfrak{q}^{\overline{\nu}'*}(-) \simeq \mathfrak{q}^{\overline{\nu}}_* \circ {'\mathfrak{p}}^{\overline{\nu}'}_* \circ {'\mathfrak{p}}^{\overline{\nu}!} \circ \mathfrak{q}^{\overline{\nu}'*}(-)$$

of derived functors $\mathrm{D}(\mathrm{Bun}_T^{\overline{\nu}'}) \to \mathrm{D}(\mathrm{Bun}_T^{\overline{\nu}})$. This tells us that $\mathrm{CT}^{\overline{\nu}} \circ \mathrm{nEis}^{\overline{\nu}'}_*(\mathscr{S}_{\phi_T})$ is the direct image of the complex

$${'\mathfrak{p}}^{\overline{\nu}!}(\mathfrak{q}^{\overline{\nu}'*}(\mathscr{S}_{\phi_T} \otimes \Delta_B^{1/2}))[\dim(\mathrm{Bun}_B^{\overline{\nu}'}) - \dim(\mathrm{Bun}_B^{\overline{\nu}})]$$

on $\mathrm{Bun}_B^{\overline{\nu}} \times_{\mathrm{Bun}_G} \mathrm{Bun}_B^{\overline{\nu}'}$ onto $\mathrm{Bun}_T^{\overline{\nu}}$. By [Ham21a, Lemma 4.9], the space $\mathrm{Bun}_B^{\overline{\nu}} \times_{\mathrm{Bun}_G} \mathrm{Bun}_B^{\overline{\nu}'}$ has a locally closed stratification given by the generic relative position of the two bundles

$$\bigsqcup_{w \in W_G} (\mathrm{Bun}_B^{\overline{\nu}} \times_{\mathrm{Bun}_G} \mathrm{Bun}_B^{\overline{\nu}'})_w$$

which we denote by $Z_w^{\overline{\nu},\overline{\nu}'}$ for varying $w \in W_G$. Using the excision spectral sequence, this implies that $\mathrm{CT}^{\overline{\nu}} \circ \mathrm{nEis}^{\overline{\nu}'}_*(\mathscr{S}_{\phi_T})$ also admits a filtration whose graded pieces we write as $(\mathrm{CT}^{\overline{\nu}} \circ \mathrm{nEis}^{\overline{\nu}'}_*(\mathscr{S}_{\phi_T}))_w$. Consider the following claim.

Proposition 2.9.7. *Let $\overline{\nu}$ and $\overline{\nu}'$ be two elements mapping to b and b' in $B(G)_{\mathrm{un}}$, respectively.*

1. *Suppose that $b \neq b'$ then the stack $Z_w^{\overline{\nu},\overline{\nu}'}$ is empty if $w = 1$.*
2. *If $w \neq 1$ then $(\mathrm{CT}^{\overline{\nu}} \circ \mathrm{nEis}^{\overline{\nu}'}_*(\mathscr{S}_{\phi_T}))_w$ is an extension of complexes in $\mathrm{D}(T(\mathbb{Q}_p),\Lambda)$ isomorphic to $(\chi \otimes \delta_B^{-1/2})^w$.*

First, let's finish the proof of Proposition 2.9.5 assuming this. By the above discussion, it suffices to show that the complex

$$R\mathscr{H}om_{T(\mathbb{Q}_p)}(\chi \otimes \delta_B^{1/2}, (\mathrm{CT}^{\overline{\nu}} \circ \mathrm{nEis}_*^{\overline{\nu}'}(\mathscr{S}_{\phi_T}))_w)$$

is trivial for all $w \in W_G$. This is trivial if $w = 1$ by point (1). If $w \neq 1$, then it follows from point (2) and the fact that the existence of an isomorphism $\chi \otimes \delta_B^{1/2} \simeq \chi^w \otimes (\delta_B^{-1/2})^w$ would contradict Condition 2.3.7 (3) in the definition of weak normalized regularity. Therefore, since ϕ_T is weak normalized regular by assumption, the claim follows. Let's now finish up by reducing this Proposition to a simpler claim, which we will prove in the next section. Proposition 2.9.7 is an analogue of [BG08, Proposition 10.8] and the idea behind its proof is the same.

Proof. (Proposition 2.9.7) We first begin by elucidating the geometry of the spaces $Z_w^{\overline{\nu},\overline{\nu}'}$ a bit more. For $S \in \mathrm{Perf}$, note that a S-point of $\mathrm{Bun}_B^{\overline{\nu}} \times_{\mathrm{Bun}_G} \mathrm{Bun}_B^{\overline{\nu}'}$ corresponds to a pair of B-structures on a G-bundle $\mathscr{F}_G$ on X_S. Namely, it parametrizes a pair $\mathscr{F}_B^1$ (resp. $\mathscr{F}_B^2$) of two B-structures on a G-bundle $\mathscr{F}_G$ whose reduction to T, denoted $\mathscr{F}_T^1$ (resp. $\mathscr{F}_T^2$) is isomorphic to $\mathscr{F}_{\overline{\nu}}$ (resp. $\mathscr{F}_{\overline{\nu}'}$) after pulling back to any geometric point of S. We can think of it as parameterizing sections

$$X_S \to B\backslash G/B$$

such that the degree is of the specified form. More transparently, we can think of a point of $\mathrm{Bun}_B^{\overline{\nu}} \times_{\mathrm{Bun}_G} \mathrm{Bun}_B^{\overline{\nu}'}$ as the B-bundle $\mathscr{F}_B^2$ together with a section

$$s : X_S \to \mathscr{F}_B^2 \times^B G/B$$

where B acts via conjugation on G/B. For $\hat{\lambda} \in \hat{\Lambda}_G^+$, we recall that by interpreting $\mathscr{V}^{\hat{\lambda}}$ as global sections of the appropriately twisted bundle on G/B corresponding to $\hat{\lambda}$ under Borel-Weil-Bott, every point in G/B gives rise to a line $\ell^{\hat{\lambda}} \subset \mathscr{V}^{\hat{\lambda}}$. We consider the B-stable subspace $\mathscr{V}_{\geq w}^{\hat{\lambda}} \subset \mathscr{V}^{\hat{\lambda}}$ consisting of weights greater than or equal to $w(\hat{\lambda})$ and $\mathscr{V}_{>w}^{\hat{\lambda}} \subset \mathscr{V}_{\geq w}^{\hat{\lambda}}$ the codimension 1 subspace consisting of weights strictly greater than $w(\hat{\lambda})$. We let $(G/B)_w := BwB/B$ be the locally closed Schubert cell attached to $w \in W_G$. We write $(G/B)_{\geq w}$ for its closure. The closure is stratified by the Schubert cells indexed by elements $w' \in W_G$ with length less than or equal to w. Then the line $\ell^{\hat{\lambda}} \subset \mathscr{V}^{\hat{\lambda}}$ will correspond to a point in $(G/B)_{\geq w}$ if and only if it belongs to $\mathscr{V}_{\geq w}^{\hat{\lambda}}$ for all $\hat{\lambda} \in \hat{\Lambda}_G^+$. Moreover, the point belongs to the stratum $(G/B)_w$ if and only the projection to $\mathscr{V}_{\geq w}^{\hat{\lambda}}/\mathscr{V}_{>w}^{\hat{\lambda}}$ is non-zero. This allows us to explain what it means to lie in the locally closed stratum $Z_w^{\overline{\nu},\overline{\nu}'}$. In particular, by definition [Ham21a, Page 26], lying in this stratum is equivalent to the condition that s factors through $\mathscr{F}_B^2 \times^B (G/B)_{\succeq w}$ and is not contained in any closed strata defined by $\mathscr{F}_B^2 \times^B (G/B)_{\succeq w'}$ for any $w' > w$ in the Bruhat order. This implies that the section s determines a set of line subbundles

$$\mathscr{L}^{\hat{\lambda}}_{\mathscr{F}_T^1} \to (\mathscr{V}^{\hat{\lambda}}_{\geq w})_{\mathscr{F}_B^2}$$

for all $\hat{\lambda} \in \hat{\Lambda}_G^+$. Moreover, via the inclusion $\mathscr{V}^{\hat{\lambda}}_{\geq w} \subset \mathscr{V}^{\hat{\lambda}}$, these give the Plücker description of the B-structure $\mathscr{F}_B^1$ such that $\mathscr{F}_G \simeq \mathscr{F}_B^1 \times^B G \simeq \mathscr{F}_B^2 \times^B G$. For this to define a point in $Z_w^{\overline{\nu},\overline{\nu}'}$, these need to satisfy the condition that the induced map

$$\mathscr{L}^{\hat{\lambda}}_{\mathscr{F}_T^1} \to (\mathscr{V}^{\hat{\lambda}}_{\geq w})_{\mathscr{F}_B^2} \to (\mathscr{V}^{\hat{\lambda}}_{\geq w}/\mathscr{V}^{\hat{\lambda}}_{> w})_{\mathscr{F}_B^2} = (\mathscr{L}^{\hat{\lambda}})_{(\mathscr{F}_T^2)^w}$$

is non-zero map of $\mathscr{O}_{X_S}$-modules (cf. [Bra+02a, Page 14],[BG08, Page 48],[Sch15a, Propositions 4.3.2, 4.4.2]). In particular, since $\mathscr{L}^{\hat{\lambda}}_{\mathscr{F}_T^1}$ is a line bundle, the map being non-zero implies it is a fiberwise injective map of line bundles. Now, recalling our choice of Borel, if we define $\theta := \overline{\nu} - w(\overline{\nu}')$ then the support of the torsion of the cokernel of this map of line bundles determines a point in $\mathrm{Div}^{(\theta)}$, by the assumptions on the degrees and Lemma 2.5.20. For this strata to be non-empty, we must have that $\theta \in \Lambda_{G,B}^{pos}$. Therefore, for all non-empty strata, we have a map:

$$\pi_w : Z_w^{\overline{\nu},\overline{\nu}'} \to \mathrm{Div}^{(\theta)}$$

Now, with these preparations out of the way, let's start with the proof. For Point (1), note that if $w = 1$ then we have an injective map of line bundles

$$\mathscr{L}^{\hat{\lambda}}_{\mathscr{F}_T^1} \to \mathscr{L}^{\hat{\lambda}}_{\mathscr{F}_T^2}$$

for all $\hat{\lambda}$, which give rise to the embeddings defined by $\mathscr{F}_B^1$ when composed with the injections of bundles $\mathscr{L}^{\hat{\lambda}}_{\mathscr{F}_T^2} \to \mathscr{V}^{\hat{\lambda}}_{\mathscr{F}_G}$ defined by $\mathscr{F}_B^2$, by construction. However, since the composition $\mathscr{L}^{\hat{\lambda}}_{\mathscr{F}_T^1} \to \mathscr{V}^{\hat{\lambda}}_{\mathscr{F}_G}$ is also a map of vector bundles, this is impossible unless $\mathscr{F}_T^1 \simeq \mathscr{F}_T^2$, which would contradict our assumption that $\overline{\nu}, \overline{\nu}' \in B(T)$ map to $b \neq b'$ in $B(G)$. Therefore, we have established point (1). For point (2), we write $\mathfrak{q}_1$ (resp. $\mathfrak{q}_2$) for the natural projections of $Z_w^{\overline{\nu},\overline{\nu}'}$ to $\mathrm{Bun}_T^{\overline{\nu}}$ (resp. $\mathrm{Bun}_T^{\overline{\nu}'}$). We note that $\mathfrak{q}_2$ is equal to the composition

$$Z_w^{\overline{\nu},\overline{\nu}'} \xrightarrow{\mathfrak{q}_1 \times \pi_w} \mathrm{Bun}_T^{\overline{\nu}} \times \mathrm{Div}^{(\theta)} \xrightarrow{{}^{\mathrm{op}}h^{\rightarrow}_{(\theta)}} \mathrm{Bun}_T^{w(\overline{\nu}')} \xrightarrow{w^{-1}} \mathrm{Bun}_T^{\overline{\nu}'}$$

where ${}^{\mathrm{op}}h_{\overrightarrow{(\theta)}}$ is the map sending $(\mathscr{F}_T, (D_i)_{i\in\mathscr{J}})$ to the bundle $\mathscr{F}_T(\sum_{i\in\mathscr{J}} \alpha_i \cdot D_i)$ and the last map is given by conjugation by w. Recall, $(\mathrm{CT}^{\overline{\nu}} \circ \mathrm{nEis}_*^{\overline{\nu}'}(\mathscr{S}_{\phi_T}))_w$ is given (up to a shift) by the sheaf

$$\mathfrak{q}_{1*} \circ {}'\mathfrak{p}^{\overline{\nu}'!} \circ \mathfrak{q}^{\overline{\nu}'*}(\chi \otimes \delta_B^{1/2})$$

We can write this as the Verdier dual of $\mathfrak{q}_{1!} \circ {}'\mathfrak{p}^{\overline{\nu}'*} \circ \mathfrak{q}^{\overline{\nu}'!}(\chi^{-1} \otimes (\delta_B^{1/2})^{-1}) \simeq \mathfrak{q}_{1!} \circ {}'\mathfrak{p}^{\overline{\nu}*} \circ \mathfrak{q}^{\overline{\nu}*}(\chi^{-1} \otimes \delta_B^{1/2})$, where the last isomorphism is Theorem 2.6.1. Replacing χ by χ^{-1}, this reduces us to showing that $\mathfrak{q}_{1!} \circ {}'\mathfrak{p}^{\overline{\nu}'*} \circ \mathfrak{q}^{\overline{\nu}'*}(\chi \otimes \delta_B^{1/2}) \simeq \mathfrak{q}_{1!} \circ \mathfrak{q}_2^*(\chi \otimes \delta_B^{1/2})$ is an extension of complexes which are isomorphic to $(\delta_B^{1/2})^w \otimes \chi^w$. Using the above factorization of $\mathfrak{q}_2$, we rewrite this as

$$\mathfrak{q}_{1!} \circ (\mathfrak{q}_1 \times \pi_w)^* \circ ({}^{\mathrm{op}}h_{\overrightarrow{(\theta)}})^* \circ (w^{-1})^*(\chi \otimes \delta_B^{1/2}) \simeq \mathfrak{q}_{1!} \circ (\mathfrak{q}_1 \times \pi_w)^*((\chi^w \otimes (\delta_B^{1/2})^w) \boxtimes (E^{(\theta)}_{\phi_T^w \otimes (\hat{\rho}^w \circ |\cdot|)})^{\vee})$$

but now, by projection formula, we are reduced to the following, which is an analogue of [BG08, Proposition 10.10]

Proposition 2.9.8. *The direct image* $(\mathfrak{q}_1 \times \pi_w)_!(\Lambda)$ *is an extension of complexes which are pullbacks of complexes on* $\mathrm{Div}^{(\theta)}$.

□

We will prove Proposition 2.9.8 by relating the spaces $Z_w^{\overline{\nu},\overline{\nu}'}$ to some variants of what are called Zastava or semi-infinite flag spaces in the classical literature, as first studied over function fields by Feign, Finkelberg, Kusnetzov, and Mirković [Fei+99; FM99].

Zastava Spaces

We let $U' \subset U$ be the subgroup defined by the positive root spaces $\hat{\alpha} > 0$ such that $w(\hat{\alpha}) < 0$. We set $B' := TU' \subset B$ to be the subgroup of the Borel defined by these root spaces. We recall that U' acts simply transitively on the closed Schubert cell $(G/B)_{\geq w}$ and use this to define the w-twisted version of the Zastava space.

Definition 2.9.9. For $\theta \in \Lambda_{G,B}^{pos}$, we let $W_w^{\theta} \to \mathrm{Div}^{(\theta)}$ be the v-sheaf parameterizing, for $S \in \mathrm{Perf}$, a triple

$$(\mathscr{F}_{U'}, s, D)$$

of the datum:

- A U'-bundle $\mathscr{F}_{U'}$ on X_S.
- A section $s : X_S \to \mathscr{F}_{U'} \times^{U'} (G/B)_{\geq w}$ that does not lie in $(G/B)_{\geq w'}$ for any $w' > w$ in the Bruhat order.
- A divisor $D \in \mathrm{Div}^{(\theta)}$ such that the induced non-zero ($\implies$ fiberwise injective) maps of line bundles

$$\mathscr{L}^{\hat{\lambda}} \to (\mathscr{V}^{\hat{\lambda}}_{\geq w})_{\mathscr{F}_{U'}} \to (\mathscr{V}^{\hat{\lambda}}_{\geq w}/\mathscr{V}^{\hat{\lambda}}_{>w})_{\mathscr{F}_{U'}} = \mathscr{O}_{X_S}$$

  for all $\hat{\lambda} \in \hat{\Lambda}^+_G$ have cokernel with torsion supported on D.

Classically, the usual Zastava space in the literature is the same datum as above in the case that $w = w_0$ together with a level structure on the bundle $\mathscr{L}^{\hat{\lambda}}$ so that it encodes information about enhanced B-structures on one of the factors. It's importance is that it provides a local model for the singularities of the Drinfeld compactification $\overline{\mathrm{Bun}}_B$. The space we have defined above in the case that $w = w_0$ is the open part of the Zastava space which models just the space Bun_B. As seen in our description of $Z_w^{\overline{\nu},\overline{\nu}'}$ in the previous section, this will clearly have a relationship to the spaces we are interested in describing. Let us first just consider the case of the element of longest length. We claim that the following is true.

Lemma 2.9.10. *For $w = w_0$ the element of longest length and $\theta = \overline{\nu} - w(\overline{\nu}') \in \Lambda^{pos}_{G,B}$ for $\overline{\nu}'$ and $\overline{\nu}$ as above, there exists a commutative diagram*

$$\begin{tikzcd} Z_w^{\overline{\nu},\overline{\nu}'} \arrow[r] \arrow[d, "\mathfrak{q}_1 \times \pi_w"] & W_w^{\theta} \arrow[d] \\ \mathrm{Bun}_T^{\overline{\nu}} \times \mathrm{Div}^{(\theta)} \arrow[r, "p_2"] & \mathrm{Div}^{(\theta)} \end{tikzcd}$$

which is a Cartesian square.

Proof. When $w = w_0$, we have that $U' = U$ and $B' = B$. Given an S-point of W_w^{θ}, the maps

$$\mathscr{L}^{\hat{\lambda}} \to (\mathscr{V}^{\hat{\lambda}}_{\geq w})_{\mathscr{F}_U} \hookrightarrow (\mathscr{V}^{\hat{\lambda}})_{\mathscr{F}_G}$$

of vector bundles on X_S define a B-structure $\mathscr{F}_B$ on the G-bundle $\mathscr{F}_U \times^U G$. We note that the induced map

$$\mathscr{L}^{\hat{\lambda}} \to (\mathscr{V}^{\hat{\lambda}}_{\geq w})_{\mathscr{F}_U} \to (\mathscr{V}^{\hat{\lambda}}_{\geq w}/\mathscr{V}^{\hat{\lambda}}_{>w})_{\mathscr{F}_U} = \mathscr{O}_{X_S}$$

with torsion cokernel of support given by $D \in \mathrm{Div}^{(\theta)}$ induces an identification $\mathscr{L}^{\hat{\lambda}} \simeq \mathscr{O}_{X_S}(-\langle \hat{\lambda}, \theta \rangle \cdot D)$ implying that $\mathscr{F}_B \times^B T$ has Kottwitz invariant given by $\theta = \overline{\nu} - w(\overline{\nu}')$ after pulling back to a geometric point. Given a bundle $\mathscr{F}_T^1 \in \mathrm{Bun}_T^{\overline{\nu}}$ of degree $\overline{\nu}$, we can dualize the above maps to get a fiberwise injection

$$\mathscr{O}_{X_S} \to (\mathscr{L}^{\hat{\lambda}})^\vee \simeq \mathscr{O}_{X_S}(\langle \hat{\lambda}, \theta \rangle \cdot D)$$

of line bundles, and then tensor by $\mathscr{L}^{\hat{\lambda}}_{\mathscr{F}_T^1}$ to get an injection

$$\mathscr{L}^{\hat{\lambda}}_{\mathscr{F}_T^1} \to \mathscr{L}^{\hat{\lambda}}_{\mathscr{F}_T^1}(\langle \hat{\lambda}, \theta \rangle \cdot D)$$

Similarly, by taking duals and twisting the U-torsor $\mathscr{F}_U$ by $\mathscr{F}_T^1$, we obtain $B = T \ltimes U$-bundles $\mathscr{F}_B^1$ and $\mathscr{F}_B^2$ defining points in $Z_w^{\overline{\nu},\overline{\nu}'}$, giving rise to a map

$$(\mathrm{Bun}_T^{\overline{\nu}} \times \mathrm{Div}^{(\theta)}) \times_{\mathrm{Div}^{(\theta)}} W_w^{\theta} \to Z_w^{\overline{\nu},\overline{\nu}'}$$

which we can see is an isomorphism. In particular, given a point in $Z_w^{\overline{\nu},\overline{\nu}'}$ corresponding to B-bundles $\mathscr{F}_B^1$ and $\mathscr{F}_B^2$ then we can define a U-bundle $\mathscr{F}_B^2 \times^B U$, and, as already seen in the previous section, we get a section $s : X_S \to \mathscr{F}_U \times^U (G/B)_{\geq w}$, $D \in \mathrm{Div}^{(\theta)}$, and a T-bundle $\mathscr{F}_T^1$ of the desired form. $\square$

Now this lemma implies Proposition 2.9.8 in the case that $w = w_0$. In particular, under the isomorphism

$$(\mathrm{Bun}_T^{\overline{\nu}} \times \mathrm{Div}^{(\theta)}) \times_{\mathrm{Div}^{(\theta)}} W_w^{\theta} \simeq Z_w^{\overline{\nu},\overline{\nu}'}$$

$\mathrm{Bun}_T^{\overline{\nu}}$ splits off as direct factor, and so, by Künneth, we deduce the claim. Now we would like to apply a similar argument using the spaces W_w^{θ} in the case that w is a general element. However, we run into a problem that, in general, all we get is a map

$$(\mathrm{Bun}_T^{\overline{\nu}} \times \mathrm{Div}^{(\theta)}) \times_{\mathrm{Div}^{(\theta)}} W_w^{\theta} \to Z_w^{\overline{\nu},\overline{\nu}'}$$

where attached to a point in $(\mathrm{Bun}_T^{\overline{\nu}} \times \mathrm{Div}^{(\theta)}) \times_{\mathrm{Div}^{(\theta)}} W_w^{\theta}$ we only get a B'-torsor $\mathscr{F}_{B'}^1$ with T-factor of degree $\overline{\nu}$. The above map is then given by a base-change of the natural map $f_{\overline{\nu}} : \mathrm{Bun}_{B'}^{\overline{\nu}} \to \mathrm{Bun}_B^{\overline{\nu}}$. In particular, if we let $\tilde{Z}_w^{\overline{\nu},\overline{\nu}'} \to Z_w^{\overline{\nu},\overline{\nu}'}$ be the base-change of $Z_w^{\overline{\nu},\overline{\nu}'}$ along the map $f_{\overline{\nu}}$ then the analogue of this Lemma holds with $\tilde{Z}_w^{\overline{\nu},\overline{\nu}'}$ in place of $Z_w^{\overline{\nu},\overline{\nu}'}$ by the same argument. Now let's study the map $f_{\overline{\nu}} : \mathrm{Bun}_{B'}^{\overline{\nu}} \to \mathrm{Bun}_B^{\overline{\nu}}$ in a particular example and see how to prove Proposition 2.9.8 in this case.

Example 2.9.11. Suppose that $G = \mathrm{GL}_3$ and let $\overline{\nu}$ correspond to a tuple of integers $(-e,-f,-g) \in \mathbb{Z}^3 \simeq B(T)$ via the Kottwitz invariant. We suppose w corresponds to the simple reflection exchanging the first and second basis vectors. After rigidifying the T-bundle $\mathscr{F}_T$ to be isomorphic to $(\mathscr{O}(e), \mathscr{O}(f), \mathscr{O}(g))$, we can view $\mathrm{Bun}_B^{\overline{\nu}}$ as the moduli space of torsors under the U-torsor

$$\begin{pmatrix} 1 & \mathscr{O}(e) \otimes \mathscr{O}(f)^\vee & \mathscr{O}(e) \otimes \mathscr{O}(g)^\vee \\ 0 & 1 & \mathscr{O}(f) \otimes \mathscr{O}(g)^\vee \\ 0 & 0 & 1 \end{pmatrix}$$

over X, where the automorphisms (up to rigidification) of a point in $\mathrm{Bun}_B^{\overline{\nu}}$ are given by considering the $\mathscr{H}^0$ Banach-Colmez spaces attached to these bundles. Similarly, after rigidification, we can view $\mathrm{Bun}_{B'}^{\overline{\nu}}$ as the moduli space of torsors under the U'-torsor

$$\begin{pmatrix} 1 & \mathscr{O}(e) \otimes \mathscr{O}(f)^\vee & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

over X, and the map $f_{\overline{\nu}}$ is given by taking direct sums of the extension of $\mathscr{O}(e)$ by $\mathscr{O}(f)$ defined by the point in $\mathrm{Bun}_{B'}^{\overline{\nu}}$ with $\mathscr{O}(g)$. In particular, we can see that the fibers of the map $f_{\overline{\nu}} : \mathrm{Bun}_{B'}^{\overline{\nu}} \to \mathrm{Bun}_B^{\overline{\nu}}$ are an iterated fibration in the Banach-Colmez spaces $\mathscr{H}^0(\mathscr{O}(e) \otimes \mathscr{O}(g)^\vee)$ and $\mathscr{H}^0(\mathscr{O}(f) \otimes \mathscr{O}(g)^\vee)$. If we assume that these Banach-Colmez spaces are positive it follows from the proof of [FS21, Proposition V.2.1] that the adjunction

$$f_{\overline{\nu}!} f_{\overline{\nu}}^! \to \mathrm{id}$$

is an equivalence. In particular, combining this with the above discussion would give us the proof of Proposition 2.9.8 in this case. We now consider $d \in \mathbb{N}_{>0}$ and fix a closed point $\infty \in X$ in the Fargues-Fontaine curve over an algebraically closed complete field F in characteristic p. We look at the short-exact sequence of $\mathscr{O}_X$-modules

$$0 \to \mathscr{O}_X(-d) \to \mathscr{O}_X \to \mathscr{O}_{X,\infty}/t_\infty^d \to 0$$

where $\mathscr{O}_{X,\infty}$ is the completed local ring and t_∞ is the uniformizing parameter corresponding to an untilt C of F. Tensoring by $\mathscr{O}(g)$, we get a short exact sequence:

$$0 \to \mathscr{O}_X(g-d) \to \mathscr{O}_X(g) \to \mathscr{O}_{X,\infty}/t_\infty^d \to 0$$

Let $\overline{\nu}_d$ correspond to the tuple of integers $(e, f, g-d) \in \mathbb{Z}^3$. Then we consider the natural map $f_{\overline{\nu}_d} : \mathrm{Bun}_{B'}^{\overline{\nu}_d} \to \mathrm{Bun}_B^{\overline{\nu}_d}$. If we choose d sufficiently large such that the spaces $\mathscr{H}^0(\mathscr{O}(e)\otimes\mathscr{O}_X(g-d)^\vee)$ and $\mathscr{H}^0(\mathscr{O}(f)\otimes\mathscr{O}_X(g-d)^\vee)$ are positive Banach-Colmez spaces then the fibers of $f_{\overline{\nu}_d}$ will be an iterated fibration in these positive Banach-Colmez spaces, and therefore we can again conclude that the adjunction

$$f_{\overline{\nu}_d}f^!_{\overline{\nu}_d}\to \mathrm{id}$$

is an isomorphism. Now we claim that we have a map:

$$\mathrm{Bun}_B^{\overline{\nu}}\to\mathrm{Bun}_B^{\overline{\nu}_d}$$

Explicitly, given a point $\mathrm{Bun}_{B'}^{\overline{\nu}}$, we have an exact sequence

$$0\to\mathscr{E}_2\to\mathscr{E}\to\mathscr{O}(g)\to 0$$

of bundles, where $\mathscr{E}_2$ is an extension of $\mathscr{O}(e)$ and $\mathscr{O}(f)$. We can then consider the pullback of this exact sequence with respect to the map $\mathscr{O}(g-d)\to\mathscr{O}(g)$ given by the modification, which will give us a point in $\mathrm{Bun}_B^{\overline{\nu}_d}$. We write $f_{\overline{\nu},d\infty}$: $\mathrm{Bun}_B^{\overline{\nu},d\infty}:=\mathrm{Bun}_B^{\overline{\nu}}\times_{\mathrm{Bun}_B^{\overline{\nu}_d}}\mathrm{Bun}_{B'}^{\overline{\nu}_d}\to\mathrm{Bun}_B^{\overline{\nu}}$ for the pullback of $f_{\overline{\nu}_d}$ along this map. We again conclude that the adjunction

$$f_{\overline{\nu},d\infty !}f^!_{\overline{\nu},d\infty}\to\mathrm{id}$$

is an isomorphism. If we could use this map instead of $f_{\overline{\nu}}$, we could prove the claim by arguing as above. Indeed, consider $\mathfrak{q}_1\times\pi_w : Z_w^{\overline{\nu},\overline{\nu}'}\to\mathrm{Bun}_T^{\overline{\nu}}\times\mathrm{Div}^{(\theta)}$. If we base-change all the above spaces to F, and let $(\mathrm{Div}\setminus\infty)^{(\theta)}$ be the partially symmetrized power defined by $\mathrm{Div}^1\setminus\infty$, the open complement of the closed point $\infty\to\mathrm{Div}^1$ defined by the fixed untilt, then, if we consider the map $\mathfrak{q}_1\times\pi_w$: $Z_w^{\overline{\nu},\overline{\nu}'}\to\mathrm{Bun}_T^{\overline{\nu}}\times(\mathrm{Div}\setminus\infty)^{(\theta)}$ restricted to this locus, we can show that the base-change $Z_w^{\overline{\nu},\overline{\nu}'}$ along $f_{\overline{\nu},d\infty}$ sits in a analogous Cartesian square to Lemma 2.9.10, and deduce Proposition 2.9.8 for the restriction of $(\mathfrak{q}_1\times\pi_w)_!f_{\overline{\nu},d\infty !}f^!_{\overline{\nu},d\infty}(\Lambda)\simeq(\mathfrak{q}_1\times\pi_w)_!(\Lambda)$ to the open strata $(\mathrm{Div}\setminus\infty)^{(\theta)}\subset\mathrm{Div}^{(\theta)}$. However, by excision, we can reduce Proposition 2.9.8 to studying this restriction with the claim over the closed complement being trivial. The claim in this case follows.

With this motivating example, all that remains is to formalize the above argument. In particular, first off note, by [FS21, Corollary V.2.3], that for the proof of Proposition 2.9.8 it suffices to consider the base-change of all the above spaces to the base $*=\mathrm{Spd}(F)$ for F an algebraically closed perfectoid field in characteristic

p. We consider such a field with fixed characteristic 0 untilt C, and let $\infty \to \mathrm{Div}^1$ be the closed F-point defined by this untilt. We consider the open complement $\mathrm{Div}^1 \setminus \infty$, and the partially symmetrized powers $(\mathrm{Div}^1 \setminus \infty)^{(\theta)}$ defined by the open subset. By applying excision, it suffices to verify Proposition 2.9.8 over this open subset with the claim over the closed complement being trivial. We abuse notation and write $Z_w^{\overline{\nu},\overline{\nu}'}$ and W_w^{θ} for the base-change to this open subspace for the rest of the section. Now let's consider the spaces defined by the locally pro-finite sets $\underline{B(\mathbb{Q}_p)}$ (resp. $\underline{B'(\mathbb{Q}_p)}$), and the natural map

$$f_0 : \mathrm{Bun}_{B'}^0 \simeq [*/\underline{B'(\mathbb{Q}_p)}] \to \mathrm{Bun}_B^0 \simeq [*/\underline{B(\mathbb{Q}_p)}]$$

of v-stacks. The fibers of this map are an iterated fibration in $\mathscr{H}^0(\mathscr{O}_X) = \underline{\mathbb{Q}_p}$ indexed by the positive roots $\hat{\alpha} > 0$ such that $w(\hat{\alpha}) > 0$. We choose $\overline{\nu}_\infty \in \mathbb{X}_*(\overline{T_{\overline{\mathbb{Q}}_p}})_\Gamma$ to be an element such that $\langle \overline{\nu}_\infty, \hat{\alpha} \rangle < 0$ for all $\hat{\alpha} > 0$ such that $w(\hat{\alpha}) > 0$. Recalling our choice of Borel, this implies that the map

$$f_{\overline{\nu}_\infty} : \mathrm{Bun}_{B'}^{\overline{\nu}_\infty} \to \mathrm{Bun}_B^{\overline{\nu}_\infty}$$

is a fibration in iterated positive Banach-Colmez spaces. We consider a modification $\mathscr{F}_T^0 \dashrightarrow \mathscr{F}_{\overline{\nu}_\infty}$ at ∞ of meromorphy $\overline{\nu}_\infty$. This modification induces a map

$$\mathrm{Bun}_B^0 \to \mathrm{Bun}_B^{\overline{\nu}_\infty}$$

which we precompose with the map

$$\mathrm{Bun}_B^{\overline{\nu}} \to \mathrm{Bun}_B^0$$

also given by an appropriate modification. This allows us to define

$$\mathrm{Bun}_{B'}^{\overline{\nu},\infty} := \mathrm{Bun}_{B'}^{\overline{\nu}_\infty} \times_{\mathrm{Bun}_B^{\overline{\nu}_\infty}} \mathrm{Bun}_B^{\overline{\nu}}$$

by base-changing $f_{\overline{\nu}_\infty}$. We write

$$f_{\overline{\nu},\infty} : \mathrm{Bun}_{B'}^{\overline{\nu},\infty} \to \mathrm{Bun}_B^{\overline{\nu}}$$

for the base-change of $f_{\overline{\nu}_\infty}$. By the proof of [FS21, Proposition V.2.1], we have that the adjunction

$$f_{\overline{\nu},\infty!} f_{\overline{\nu},\infty}^! \to \mathrm{id}$$

is an isomorphism, since $f_{\overline{\nu}_\infty}$ and in turn $f_{\overline{\nu},\infty}$ is an iterated fibration of positive Banach-Colmez spaces. Now we define

$$\tilde{Z}_w^{\overline{\nu},\overline{\nu}'} := Z_w^{\overline{\nu},\overline{\nu}'} \times_{\mathrm{Bun}_B^{\overline{\nu}}} \mathrm{Bun}_{B'}^{\overline{\nu},\infty}$$

By the previous adjunction, it suffices to show the analogue of Proposition 2.9.8 for the composition

$$\tilde{Z}_w^{\overline{\nu},\overline{\nu}'} \to Z_w^{\overline{\nu},\overline{\nu}'} \xrightarrow{\mathfrak{q}_1 \times \pi_w} \mathrm{Bun}_T^{\overline{\nu}} \times (\mathrm{Div} \setminus \infty)^{(\theta)}$$

and the shriek pullback of the constant sheaf along the first map. However, by now arguing exactly as in the proof of Lemma 2.9.10, with B' and its unipotent radical U' replacing B and U, we can deduce that the base-change of $W_w^{\theta} \to (\mathrm{Div} \setminus \infty)^{(\theta)}$ along $p_2 : \mathrm{Bun}_T^{\overline{\nu}} \times (\mathrm{Div} \setminus \infty)^{(\theta)} \to (\mathrm{Div} \setminus \infty)^{(\theta)}$ is precisely the space $\tilde{Z}_w^{\overline{\nu},\overline{\nu}'}$. This concludes the proof of Proposition 2.9.8 by Künneth.

2.10 The Hecke Eigensheaf Property

2.10.1 Tilting Eigensheaves

We would now like to combine our work in the previous sections and use it to construct eigensheaves. We would like to do this in a uniform way for coefficient systems $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$, where Λ has the discrete topology unless otherwise stated. One of the issues is that the representation theory of ${}^LG/\Lambda$ is substantially different in each of these three cases. The structure of the representation theory of $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^LG)$ was described in §2. With $\overline{\mathbb{Q}}_\ell$-coefficients, the category is semisimple with simple objects given by V_{μ^Γ} for $\mu^\Gamma \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+/\Gamma$ a Γ-orbit of a geometric dominant cocharacter μ. If $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell\}$ this is no longer so straightforward; the representation V_{μ^Γ} can fail to be irreducible. To develop a good theory of algebraic representations with these coefficients, we will need to invoke our assumption that ℓ is very good with respect to G and use the theory of tilting modules. We first discuss the general notion of a tilting module. Fix H a split connected reductive group over $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$. We consider the involution

$$\mathbb{D} : \mathrm{Rep}_\Lambda(H) \to \mathrm{Rep}_\Lambda(H)$$

$$V \mapsto (V^*)^\sigma$$

where V^* is the dual representation and σ is the Chevalley involution. For $\lambda \in \mathbb{X}^*(H)^+$ a dominant character, we let V^λ denote the highest weight representation attached to λ by Borel-Weil-Bott, and we write $V_\lambda := \mathbb{D}(V^\lambda)$. We now come to our key definition.

Definition 2.10.1. Given $V \in \mathrm{Rep}_\Lambda(H)$, we say that V has a Weyl (resp. good) filtration if it admits a filtration whose successive quotients are isomorphic to V^λ (resp. V_λ). We say V is tilting if it admits both a good and a Weyl filtration. We write $\mathrm{Tilt}_\Lambda(H) \subset \mathrm{Rep}_\Lambda(H)$ for the full sub-category of tilting modules.

The category of tilting modules is additive, but usually not abelian. If $\Lambda = \overline{\mathbb{Q}}_\ell$ the highest weight modules are simple, and $\mathrm{Tilt}_\Lambda(H) = \mathrm{Rep}_\Lambda(H)$. Therefore, this is only an interesting notion if $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell\}$. The key point of moving to this sub-category is that we have the following generalization of usual highest weight theory due Ringel and Donkin [Rin91], [Don93].

Theorem 2.10.2. *For each $\lambda \in \mathbb{X}^*(H)^+$, there exists a unique indecomposable tilting module $\mathscr{T}_\lambda \in \mathrm{Tilt}_\Lambda(H)$ with highest weight λ. We have that* $\dim(\mathscr{T}_\lambda(\lambda)) = 1$*, and, for varying λ, this parametrizes all indecomposable tilting modules.*

We also get the usual classification of all tilting modules in terms of highest weight tilting modules.

Proposition 2.10.3. *[Jan03, Section E.22],[Mat00, Lemma 7.3] For all $V \in$* $\mathrm{Tilt}_\Lambda(H)$*, there exists unique integers $n(\lambda) \in \mathbb{N}_{\geq 0}$ for all $\lambda \in \mathbb{X}^*(H)^+$ and an isomorphism*

$$V \simeq \bigoplus_{\lambda \in \mathbb{X}^*(H)^+} (\mathscr{T}_\lambda)^{n(\lambda)}$$

of tilting modules.

Now we come to a difficult result which was proven by [Wan82] for groups of type A_n, [Don93] for almost all groups, and [Mat00] in general.

Theorem 2.10.4. *If we have two tilting modules $V, V' \in$* $\mathrm{Tilt}_\Lambda(H)$ *then the tensor product $V \otimes V'$ is tilting.*

We can now extend this to the L-group using our assumption that ℓ is very good with respect to G. By Theorem 2.10.2 applied to $H = \hat{G}$, we deduce that we have a well-defined category $\mathrm{Tilt}_\Lambda(\hat{G})$ of tilting modules, where each object can be written as a direct sum of highest weight tilting modules $\mathscr{T}_\mu$ for $\mu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$

using Proposition 2.10.3. Now, given such a μ, we consider the reflex field E_μ, and extend this to a representation of $W_{E_\mu} \ltimes \hat{G}$, as in [Kot97a, Lemma 2.1.2]. We define the tilting module $\mathscr{T}_{\mu^\Gamma}$ as the induction of this representation from $W_{E_\mu} \ltimes \hat{G}$ to $W_{\mathbb{Q}_p} \ltimes \hat{G}$, and let $\mathrm{Tilt}_\Lambda({}^LG) \subset \mathrm{Rep}_\Lambda({}^LG)$ be the full sub-category given by direct sums of such modules, where we note that $\mathscr{T}_{\mu^\Gamma}$ only depends on the Γ-orbit of $\mu^\Gamma \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+/\Gamma$ of μ. Now, since ℓ is very good, it follows that $W_{\mathbb{Q}_p}$ acts on $\hat{G}$ via a quotient Q that is prime to ℓ by definition [FS21, Page 33], and therefore $W_{\mathbb{Q}_p}/W_{E_\mu}$ is also of order prime to ℓ. Combing this observation, Frobenius Reciprocity/Mackey theory, and Theorem 2.10.4, we conclude that $\mathscr{T}_{\mu^\Gamma}$ is indeed an irreducible representation of ${}^LG$, and that $\mathrm{Tilt}({}^LG)$ is preserved under tensor products. This allows us to define the following.

Definition 2.10.5. Given a continuous L-parameter $\phi : W_{\mathbb{Q}_p} \to {}^LG(\Lambda)$, we say a sheaf $\mathscr{S}_\phi \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ is a tilting eigensheaf with eigenvalue ϕ if, for all $V \in \mathrm{Tilt}_\Lambda({}^LG^I)$, we are given isomorphisms

$$\eta_{V,I} : T_V(\mathscr{S}_\phi) \simeq \mathscr{S}_\phi \boxtimes r_V \circ \phi$$

of sheaves in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{BW^I_{\mathbb{Q}_p}}$, which are natural in I and V, and compatible with compositions and exterior tensor products in V. If $\Lambda = \overline{\mathbb{Q}}_\ell$ this recovers Definition 2.3.1. We similarly say $\mathscr{S}_\phi$ is a weak tilting eigensheaf if only the isomorphisms $\eta_{V,I}$ exist.

Remark 2.10.6. Our discussion of highest weight theory in §2.2.1 for $\hat{G}^\Gamma$ also extends to the tilting modules $\mathscr{T}_\mu$ under our assumption that ℓ is very good. In particular, using Proposition 2.10.2, we can understand the possible weights occurring in $\mathscr{T}_\mu|_{\hat{G}^\Gamma}$ in terms of $\overline{\nu} \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$, which lie in the convex hull of the W_G orbit of $\mu_\Gamma \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+_\Gamma$, and the $\overline{\nu}$ weight space of $\mathscr{T}_\mu|_{\hat{G}^\Gamma}$ will be a direct sum over the weight spaces $\mathscr{T}_\mu(\nu)$ for $\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ mapping to $\overline{\nu}$, as in Lemma 2.2.8. To see this, we note, by [FS21, Proposition VIII.5.11], ℓ being very good implies that we have the following:

1. $\hat{G}^\Gamma$ is a smooth linear algebraic group, and $\hat{G}^{\Gamma,\circ}$ is reductive.
2. $\hat{G}^\Gamma/\hat{G}^{\Gamma,\circ} \simeq \hat{T}^\Gamma/\hat{T}^{\Gamma,\circ}$ is of order prime to ℓ, where $\hat{G}^{\Gamma,\circ}$ (resp. $\hat{T}^{\Gamma,\circ}$) denotes the neutral component of $\hat{G}^\Gamma$ (resp. $\hat{T}^\Gamma$), and the isomorphism follows as in [Zhu15, Lemma 4.6].

These observations allows us the see the highest weight theory of the (possibly disconnected) group $\hat{G}^{\Gamma}$ behaves as expected with modular coefficients by mimicking the proof of Lemma 2.2.7.

We now define our candidate tilting eigensheaf for each of the possible coefficient systems $\Lambda \in \{\overline{\mathbb{F}}_{\ell}, \overline{\mathbb{Z}}_{\ell}, \overline{\mathbb{Q}}_{\ell}\}$.

2.10.2 The Construction of the Eigensheaf

We fix a toral parameter $\phi_T : W_{\mathbb{Q}_p} \to {}^{L}T(\Lambda)$, with induced parameter $\phi : W_{\mathbb{Q}_p} \xrightarrow{\phi_T} {}^{L}T(\Lambda) \to {}^{L}G(\Lambda)$. Our goal is to construct a candidate tilting eigensheaf with respect to the parameter ϕ. If $\Lambda = \overline{\mathbb{F}}_{\ell}$, we have already carried this out. It is simply the sheaf $\mathrm{nEis}(\mathscr{S}_{\phi_T}) \in \mathrm{D}(\mathrm{Bun}_G)$ viewed as a sheaf in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_{\ell})$ via the identification $\mathrm{D}(\mathrm{Bun}_G) \simeq \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_{\ell})$ [FS21, Proposition VII.6.6], obtained by embedding both categories into $\mathrm{D}_{\blacksquare}(\mathrm{Bun}_G, \overline{\mathbb{F}}_{\ell})$. To move beyond this case, we need to invoke the following Lemma.

Lemma 2.10.7. *For $\Lambda = \overline{\mathbb{F}}_{\ell}$ and ϕ_T weakly normalized regular, the sheaf* $\mathrm{nEis}(\mathscr{S}_{\phi_T}) \in \mathrm{D}(\mathrm{Bun}_G)$ *is ULA with respect to the structure map* $\mathrm{Bun}_G \to *$.

Proof. Given $A \in \mathrm{D}(\mathrm{Bun}_G)$, we recall that A being ULA with respect to the map $\mathrm{Bun}_G \to *$ is equivalent to saying that its stalks $A|_{\mathrm{Bun}_G^b}$ are valued in a complex of smooth representations such that $A^K|_{\mathrm{Bun}_G^b}$ is a perfect complex of Λ-modules for all open pro-p subgroups $K \subset J_b(\mathbb{Q}_p)$ [FS21, Theorem V.7.1]. In particular, the result follows from Corollary 2.9.2. □

Now, if $\Lambda = \overline{\mathbb{Z}}_{\ell}$ then, by taking inverse limits with respect to the mod ℓ^n reductions of ϕ_T, and considering the systems of sheaves given by applying the Eisenstein functor to the eigensheaf attached to these reductions, we obtain a sheaf

$$\widehat{\mathrm{nEis}(\mathscr{S}_{\phi_T})} \in \mathrm{D}_{\text{ét}}^{\mathrm{ULA}}(X, \overline{\mathbb{Z}}_{\ell})$$

Now, we have a fully faithful embedding

$$\mathrm{D}_{\text{ét}}^{\mathrm{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_{\ell}) \hookrightarrow \mathrm{D}_{\blacksquare}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_{\ell})$$

given as in [FS21, Page 261]. This embedding is used to define the Hecke operators in the setting of solid sheaves (See [FS21, Page 264]), utilizing that sheaves in the Satake category are ULA over Div^I. In particular, this embedding is Hecke equivariant in the appropriate sense, and so the filtered eigensheaf property transfers to the image of $\widehat{\mathrm{nEis}(\mathscr{S}_{\phi_T})}$ in $\mathrm{D}_{\blacksquare}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$. We also have a natural embedding $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell) \hookrightarrow \mathrm{D}_{\blacksquare}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$, and we can analogously define the set of ULA objects in it [FS21, Definition VII.7.8], denoted $\mathrm{D}_{\mathrm{lis}}^{\mathrm{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$. We have the following claim.

Lemma 2.10.8. *Under the embeddings of* $\mathrm{D}_{\mathrm{lis}}^{\mathrm{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$ *and* $\mathrm{D}_{\acute{e}t}^{\mathrm{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$ *into* $\mathrm{D}_{\blacksquare}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$ *described above, these two full subcategories are isomorphic.*

Proof. We have a semi-orthogonal decomposition of $\mathrm{D}_{\mathrm{lis}}^{\mathrm{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$ ([FS21, Proposition VII.7.3]) and $\mathrm{D}_{\mathrm{\acute{e}t}}^{\mathrm{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$ into $\mathrm{D}(J_b(\mathbb{Q}_p), \overline{\mathbb{Z}}_\ell)_{\mathrm{adm}}$ and $\hat{\mathrm{D}}(J_b(\mathbb{Q}_p), \overline{\mathbb{Z}}_\ell)_{\mathrm{adm}}$ by excision, respectively. Here $\hat{\mathrm{D}}(J_b(\mathbb{Q}_p), \overline{\mathbb{Z}}_\ell)$ denotes the derived category of ℓ-complete smooth representations of $J_b(\mathbb{Q}_p)$, and the subscript adm is used to denote the full subcategory of objects such that its invariants under an open compact $K \subset J_b(\mathbb{Q}_p)$ is a perfect complex. Since the semi-orthogonal decompositions are compatible with the two embeddings into $\mathrm{D}_{\blacksquare}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$ it suffices to show that we have an identification

$$\mathrm{D}(J_b(\mathbb{Q}_p), \overline{\mathbb{Z}}_\ell)_{\mathrm{adm}} \simeq \hat{\mathrm{D}}(J_b(\mathbb{Q}_p), \overline{\mathbb{Z}}_\ell)_{\mathrm{adm}},$$

but this follows from [Han20, Proposition 2.6]. $\square$

Using the isomorphism supplied by the previous Lemma, we can regard $\widehat{\mathrm{nEis}(\mathscr{S}_{\phi_T})} \in \mathrm{D}_{\mathrm{\acute{e}t}}^{\mathrm{ULA}}(X, \overline{\mathbb{Z}}_\ell)$ as an object in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$, which we denote by $\mathrm{nEis}(\mathscr{S}_{\phi_T})$. Since these isomorphisms are compatible with Hecke operators it follows that Corollary 2.7.4 transfers to this sheaf. It now remains to describe the desired sheaf when $\Lambda = \overline{\mathbb{Q}}_\ell$. In this case, we need to assume the parameter $\phi_T : W_{\mathbb{Q}_p} \to {}^LT(\overline{\mathbb{Q}}_\ell)$ is of the form $\overline{\phi}_T \otimes \overline{\mathbb{Q}}_\ell$, where $\overline{\phi}_T : W_{\mathbb{Q}_p} \to {}^LT(\overline{\mathbb{Z}}_\ell)$ is a parameter with weakly normalized regular mod ℓ-reduction. Then we consider the sheaf $\mathrm{nEis}(\mathscr{S}_{\overline{\phi}_T}) \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$ constructed above, and define

$$\mathrm{nEis}(\mathscr{S}_{\phi_T}) := \mathrm{nEis}(\mathscr{S}_{\overline{\phi_T}})[\frac{1}{\ell}] \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$$

by taking the colimit over the multiplication by ℓ maps. Now with the candidate eigensheaf defined, we begin the proof of the eigensheaf property. To capture the necessary integrality conditions, we define the following.

Definition 2.10.9. For $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$ with the discrete topology and a continuous toral parameter $\phi_T : W_{\mathbb{Q}_p} \to {}^LT(\Lambda)$ we say that ϕ_T is integral if it admits a mod ℓ-reduction. In particular, if $\Lambda = \overline{\mathbb{Q}}_\ell$, we assume it is of the form $\overline{\phi}_T \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{Q}}_\ell$ for some continuous parameter $\overline{\phi}_T : W_{\mathbb{Q}_p} \to {}^LT(\overline{\mathbb{Z}}_\ell)$.

2.10.3 The Hecke Eigensheaf Property

We start with the following Theorem.

Theorem 2.10.10. *For $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$ with the discrete topology, we consider $\phi_T : W_{\mathbb{Q}_p} \to {}^LT(\Lambda)$ an integral parameter with weakly normalized regular mod ℓ reduction. There then exists a perverse sheaf $\mathrm{nEis}(\mathscr{S}_{\phi_T}) \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ which is a filtered eigensheaf with eigenvalue ϕ as in Corollary 2.7.4. If V is a direct sum of $\boxtimes_{i\in I}\mathscr{T}_{\mu_i^\Gamma}$ for geometric dominant cocharacters μ_i, and ϕ_T is strongly μ_i-regular (Definition 2.3.14), the filtration on $T_V(\mathrm{nEis}(\mathscr{S}_{\phi_T})$ splits uniquely, and we have a natural isomorphism*

$$T_V(\mathrm{nEis}(\mathscr{S}_{\phi_T})) \simeq \mathrm{nEis}(\mathscr{S}_{\phi_T}) \boxtimes r_V \circ \phi$$

of sheaves in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G)^{BW^I_{\mathbb{Q}_p}}$. In particular, if ϕ_T is strongly μ-regular for all geometric dominant cocharacters μ then $\mathrm{nEis}(\mathscr{S}_{\phi_T})$ is a tilting eigensheaf. For $b \in B(G)$, the stalk $\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^b} \in \mathrm{D}(\mathrm{Bun}_G^b) \simeq \mathrm{D}(J_b(\mathbb{Q}_p), \Lambda)$ is given by

1. *an isomorphism $\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^b} \simeq \bigoplus_{w\in W_b} i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{-1/2}[-\langle 2\hat{\rho}, \nu_b\rangle]$ if $b \in B(G)_{\mathrm{un}}$,*

2. *an isomorphism $\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{\mathrm{Bun}_G^b} \simeq 0$ if $b \notin B(G)_{\mathrm{un}}$.*

Moreover, if $\mathbb{D}_{\mathrm{Bun}_G}$ denotes Verdier duality on Bun_G, we have an isomorphism

$$\mathbb{D}_{\mathrm{Bun}_G}(\mathrm{nEis}(\mathscr{S}_{\phi_T})) \simeq \mathrm{nEis}(\mathscr{S}_{\phi_T^\vee})$$

of sheaves in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$.

Proof. The existence of $\mathrm{nEis}(\mathscr{S}_{\phi_T})$ and the transfer of the filtered Hecke eigensheaf property to this sheaf was discussed in the previous section. The claim on Verdier duality follows from Theorem 2.8.3, and the discussion in [FS21, Section VII.5]. The description of the stalks follows from the construction and Corollary 2.9.2. It remains to show that the filtration on $T_V(\mathrm{nEis}(\mathscr{S}_{\phi_T}))$ splits uniquely for $\boxtimes_{i\in I}\mathscr{T}_{\mu_i^\Gamma} = V \in \mathrm{Rep}_\Lambda({}^LG)$ such that ϕ_T is strongly μ_i-regular for all $i \in I$. To do this, we note that an extension between the graded pieces of the filtration on $T_V(\mathrm{nEis}(\mathscr{S}_{\phi_T}))$ is specified by a cohomology class in the H^1 of $R\Gamma(W^I_{\mathbb{Q}_p}, \boxtimes_{i\in I}(\nu_i - \nu_i')^\Gamma \circ \phi_T) \simeq \bigotimes^{\mathbb{L}}_{i\in I} R\Gamma(W_{\mathbb{Q}_p}, (\nu_i - \nu_i')^\Gamma \circ \phi_T)$ for ν_i and ν_i' defining two distinct Γ-orbits of weights of the representation $\mathscr{T}_\mu$ in $\hat{T}$ for all $i \in I$.

In particular, we note if ϕ_T is strongly μ_i-regular then the H^1 vanishes and the filtration splits, and similarly the H^0 vanishes so the splitting is unique. In particular, since we know the filtration satisfies all the desired compatibilities for the eigensheaf property if we know strong μ-regularity for all μ, it follows from the splitting being unique that $\mathrm{nEis}(\mathscr{S}_{\phi_T})$ is a tilting eigensheaf. $\square$

We now fix an integral parameter ϕ_T with weakly normalized regular mod ℓ-reduction and consider the sheaf $\mathrm{nEis}(\mathscr{S}_{\phi_T}) \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ supplied by the previous theorem. We come to the following definition.

Definition 2.10.11. Given a tuple of geometric dominant cocharacters $(\mu_i)_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+)^I$ for all $i \in I$, we say that $\phi_T : W_{\mathbb{Q}_p} \to {}^LT(\Lambda)$ is $(\mu_i)_{i\in I}$-regular if the filtration on

$$T_V(\mathrm{nEis}(\mathscr{S}_{\phi_T}))$$

splits (but not necessarily uniquely) for the tilting module $V = \boxtimes_{i\in I}\mathscr{T}_{\mu_i^\Gamma} \in \mathrm{Tilt}_\Lambda({}^LG^I)$.

As seen in the proof of Theorem 2.10.10, if the μ_i are such that ϕ_T is strongly μ_i-regular then it follows that ϕ_T is $(\mu_i)_{i\in I}$-regular. For certain μ_i, strong μ_i-regularity will be implied by genericity, and the following Proposition allows us to deduce that the filtration splits in more cases.

Proposition 2.10.12. *Suppose that $(\mu_{1i})_{i\in I}, (\mu_{2i})_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+)^I$ are tuples of characters such that ϕ_T is $(\mu_{1i})_{i\in I}$ and $(\mu_{2i})_{i\in I}$-regular. Then if $V \simeq \boxtimes_{i\in I}\mathscr{T}_{\mu_i^\Gamma} \in \mathrm{Tilt}_\Lambda({}^LG^I)$ occurs as a direct summand of the tensor product $\boxtimes_{i\in I}\mathscr{T}_{\mu_{1i}^\Gamma} \otimes \mathscr{T}_{\mu_{2i}^\Gamma} \in \mathrm{Tilt}({}^LG^I)$ then it follows that ϕ_T is $(\mu_i)_{i\in I}$-regular.*

Proof. It suffices to show that, given $(\mu_{1i})_{i\in I}, (\mu_{2i})_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+)^I$ with $V_1 := \boxtimes_{i\in I}\mathscr{T}_{\mu_{1i}^\Gamma}$ and $V_2 := \boxtimes_{i\in I}\mathscr{T}_{\mu_{2i}^\Gamma}$ such that we know the filtration on $T_{V_1}(\mathrm{nEis}(\mathscr{S}_{\phi_T}))$ and $T_{V_2}(\mathrm{nEis}(\mathscr{S}_{\phi_T}))$ splits, the same is true for the filtration on $T_V(\mathrm{nEis}(\mathscr{S}_{\phi_T}))$, where $V \in \mathrm{Tilt}_\Lambda({}^LG^I)$ is a direct summand of $V_1 \otimes V_2$. To do this, we can use the isomorphism

$$T_{V_1}T_{V_2}(\mathrm{nEis}(\mathscr{S}_{\phi_T}))|_\triangle \simeq T_{V_1\otimes V_2}(\mathrm{nEis}(\mathscr{S}_{\phi_T}))$$

coming from the fusion product, where $\triangle : \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{BW_{\mathbb{Q}_p}^{I\sqcup I}} \to \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{BW_{\mathbb{Q}_p}^{I}}$ is the natural map given by diagonal restriction. By assumption, the diagonal restriction of the filtration on the LHS splits. Moreover, by the compatibilities of the filtration, we know that the filtration on the LHS refines the filtration on the RHS. In particular, we deduce that the filtration on $T_{V_1\otimes V_2}(\mathrm{nEis}(\mathscr{S}_{\phi_T}))$ has a splitting, and since V is a direct summand of $V_1 \otimes V_2$, we know that the filtration also splits on $T_V(\mathrm{nEis}(\mathscr{S}_{\phi_T}))$. $\square$

In particular, by considerations of highest weight, the tilting module $\boxtimes_{i\in I}\mathscr{T}_{(\mu_{1i}+\mu_{2i})^\Gamma}$ is always a direct summand of $\boxtimes_{i\in I}\mathscr{T}_{\mu_{1i}^\Gamma}\otimes\mathscr{T}_{\mu_{2i}^\Gamma}$. Using this, we deduce the following Corollary.

Corollary 2.10.13. *If ϕ_T is $(\mu_{1i})_{i\in I}$ and $(\mu_{2i})_{i\in I}$-regular then it is also $(\mu_{1i}+\mu_{2i})_{i\in I}$-regular.*

This motivates the following definition.

Definition 2.10.14. We say a parameter ϕ_T is normalized regular if it is integral with weakly normalized regular mod ℓ reduction and if there exists a set of of elements μ_k for $k = 1,\dots,n$ such that:

1. The μ_k generate $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ in the sense that any $\mu = \sum_{k=1}^n n_k\mu_k$ as elements in $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ for some $n_k \in \mathbb{Z}$.
2. ϕ_T is strongly μ_k-regular, for all $k = 1,\dots,n$.

This allows us to state the following corollary.

Corollary 2.10.15. *Suppose that ϕ_T is normalized regular then it is $(\mu_i)_{i\in I}$-regular for all finite index sets I and $\mu_i \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$. In particular, the sheaf $\mathrm{nEis}(\mathscr{S}_{\phi_T})$ is a (weak) tilting eigensheaf with eigenvalue ϕ given by the composite of ϕ_T with the natural map ${}^LT \to {}^LG$.*

Proof. We recall that $\mathrm{nEis}(\mathscr{S}_{\phi_T})$ being a (weak) tilting eigensheaf means that, for all $V \in \mathrm{Tilt}_\Lambda({}^LG^I)$, we know that we have an isomorphism

$$T_V(\mathrm{nEis}(\mathscr{S}_{\phi_T})) \simeq \mathrm{nEis}(\mathscr{S}_{\phi_T}) \boxtimes r_V \circ \phi$$

but we do not know that these isomorphisms satisfy the desired compatibilities. To show this, it suffices to show the filtration on $T_V(\mathrm{nEis}(\mathscr{S}_{\phi_T}))$ just splits (not necesarily uniquely) for all highest weight tilting modules $V = \boxtimes_{i\in I}\mathscr{T}_{\mu_i^\Gamma}$ corresponding to the Γ-orbits of varying $(\mu_i)_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+)^I$. Since ϕ_T is strongly μ_k-regular for the $k = 1,\dots,n$ appearing in the definition of normalized regularity, it follows, as in the proof of Theorem 2.10.10, that the filtration on $T_V(\mathrm{nEis}(\mathscr{S}_{\phi_T}))$

splits for any tuple of the generating cocharacters $(\mu_i)_{i\in I}$ such that $\mu_i = \mu_k$ for all $i \in I$ and varying $k = 1,\ldots,n$ appearing in Definition 2.10.14. However, since every tuple of cocharacters μ_i, can be written as a linear combination of μ_k for $k = 1,\ldots,n$, this follows from Corollary 2.10.13. $\square$

By choosing μ_k for $k = 1,\ldots,n$ to be, for example a set of minuscule/quasi-minuscule generators of $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$, this allows to see that μ-regularity for all μ is actually an open condition on the variety of unramified twists of a fixed ϕ_T. This will be sufficient for most applications of the Eisenstein series to descirbing the cohomology of local shtukas and Shimura varieties in the Grothendieck group. However, for more refined applications of our results it becomes important to spell out the precise assumptions needed to deduce μ-regularity for all μ. Ideally, it should be implied by genericity of ϕ_T for all μ, as is suggested by Conjecture 2.1.29. For $G = \mathrm{GL}_n$, we show how to verify this from the results proven above.

Corollary 2.10.16. *For $G = \mathrm{GL}_n$, if ϕ_T is an integral generic L-parameter (In particular, its mod ℓ-reduction is weakly normalized regular by Lemma 2.3.13) then the sheaf* $\mathrm{nEis}(\mathscr{S}_{\phi_T})$ *is a (weak) tilting eigensheaf.*

Proof. We give the proof for $\Lambda = \overline{\mathbb{F}}_\ell$, with the other cases being strictly easier. It suffices to show for all tuples $(\mu_i)_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+)^I$ that ϕ_T is $(\mu_i)_{i\in I}$ regular. By Lemma 2.3.15, we know that if ϕ_T is generic then it is strongly μ-regular for the cocharacter $\mu = (1,0,\ldots,0)$ corresponding to the standard representation V_{std} of $\hat{G} \simeq \mathrm{GL}_n$. By Proposition 2.10.13, it suffices to show that if ϕ_T is generic then μ-regularity holds for the fundamental coweights $\mu = \omega_i = (1^i, 0^{n-i})$ for $i = 1,\ldots,n$. We recall that the ω_i are minuscule in this case and therefore it follows by Lemma A.2.1 that the representation $\Lambda^i(V_{\mathrm{std}}) \simeq V_{\omega_i}$ is irreducible and thereby equal to the tilting module $\mathscr{T}_{\omega_i}$. However, now the result follows from Proposition 2.10.12 and the fact that $\Lambda^i(V_{\mathrm{std}})$ is a direct summand of $V_{\mathrm{std}}^{\otimes n}$. $\square$

With our main results in hand, we can move into applications.

2.11 Applications

Now we will deduce some applications to the cohomology of local shtuka spaces. In §9.1, we will use our eigensheaf to derive an analogue of an averaging formula of Shin for the cohomology of local shtuka spaces. In §9.2 and §9.3, we will discuss a refined version of this averaging formula, and use it to derive a very explicit formula for the isotypic parts of local shtuka spaces with respect to parabolic inductions of characters coming from normalized regular parameters. By combining this with a shtuka analogue of Boyer's trick, we will show that this gives rise to a geometric construction of intertwining operators, and recovers a result analogous to a result of Xiao and Zhu [XZ17] on the irreducible components of affine Deligne-Lusztig varieties, but on the generic fiber.

Let's first recall the key definitions. We say a local shtuka datum is a triple (G,b,μ) for μ a geometric dominant cocharacter of G and $b \in B(G,\mu)$ an element of the μ-admissible locus of the Kottwtiz set of G (Definition 2.2.5). We let E be the reflex field of μ. The triple (G,b,μ) defines a diamond

$$\mathrm{Sht}(G,b,\mu)_{\infty} \to \mathrm{Spd}(\breve{E})$$

parameterizing modifications $\mathscr{F}_b \to \mathscr{F}_G^0$ with meromorphy bounded by μ on X[7]. It carries an action of $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$ and a (non-effective) descent datum from $\breve{E}$ down to E. This allows us to consider the tower of quotients

$$\mathrm{Sht}(G,b,\mu)_{\infty}/\underline{K} =: \mathrm{Sht}(G,b,\mu)_K$$

for varying open compact subgroups $K \subset G(\mathbb{Q}_p)$. We write $\mathscr{S}_{\mu}$ for the Λ-valued sheaf attached to the highest weight tilting module $\mathscr{T}_{\mu}$ of $W_E \ltimes \hat{G}$ as in the previous section. This is given by pulling back the sheaf on $\mathrm{Hck}_{G,E}$ defined by $\mathscr{T}_{\mu}$ and Theorem 2.4.2 along the natural map $\mathrm{Sht}(G,b,\mu) \to \mathrm{Hck}_{G,E}$. In particular, the sheaf $\mathscr{S}_{\mu}$ is equivariant with respect to the actions of $G(\mathbb{Q}_p)$ and $J_b(\mathbb{Q}_p)$ by construction. Letting $\mathrm{Sht}(G,b,\mu)_{K,\mathbb{C}_p}$ denote the base-change of these spaces to $\mathbb{C}_p$, we can now define the complex

$$R\Gamma_c(G,b,\mu) := \mathrm{colim}_{K\to\{1\}} R\Gamma_c(\mathrm{Sht}(G,b,\mu)_{K,\mathbb{C}_p}, \mathscr{S}_{\mu})$$

of $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E$-modules. We now want to disentangle the $G(\mathbb{Q}_p)$ and $J_b(\mathbb{Q}_p)$ action. To do this, for π (resp. ρ) a smooth irreducible representation of $G(\mathbb{Q}_p)$ (resp. $J_b(\mathbb{Q}_p)$) on Λ-modules, we define the π (resp. ρ)-isotypic part. I.e the complexes

$$R\Gamma_c(G,b,\mu)[\pi] := R\Gamma_c(G,b,\mu) \otimes_{\mathscr{H}(G)} \pi$$

and

$$R\Gamma_c(G,b,\mu)[\rho] := R\Gamma_c(G,b,\mu) \otimes_{\mathscr{H}(J_b)} \rho$$

[7]Note that this is the space denoted $\mathrm{Sht}(G,b,\mu^{-1})$ in [SW20b]. We find that our convention simplifies certain formulae.

where $\mathscr{H}(G) := C_c^\infty(G(\mathbb{Q}_p), \Lambda)$ (resp. $\mathscr{H}(J_b)$) is the usual smooth Hecke algebra of G (resp. J_b). We recall that $\mathrm{Sht}(G,b,\mu)_\infty$ has dimension equal to $\langle 2\hat{\rho}, \mu \rangle =: d$. The complexes $R\Gamma_c(G,b,\mu)[\pi]$ (resp. $R\Gamma_c(G,b,\mu)[\rho]$) are concentrated in degrees $-d \le i \le d$ and are valued in admissible $J_b(\mathbb{Q}_p)$ (resp. $G(\mathbb{Q}_p)$) representations, which are moreover of finite length with $\overline{\mathbb{Q}}_\ell$-coefficients [FS21, Page 317]. Similarly, we define the complexes

$$R\Gamma_c^\flat(G,b,\mu)[\pi] := R\mathscr{H}om(R\Gamma_c(G,b,\mu), \pi)$$

and

$$R\Gamma_c^\flat(G,b,\mu)[\rho] := R\mathscr{H}om(R\Gamma_c(G,b,\mu), \rho).$$

We will make regular use of the relationship between these complexes and Hecke operators. For simplicity, we will mostly stick to the isotypic parts $R\Gamma_c^\flat(G,b,\mu)[\rho]$, as it removes various annoying twists and shifts[8]; however, one can easily translate to the isotypic parts $R\Gamma_c^\flat(G,b,\mu)[\rho]$ in the cases we consider, using Corollary 2.11.15. We write

$$T_\mu : \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \to \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{BW_E}$$

for the Hecke operator defined by the representation $\mathscr{T}_\mu$. For $b \in B(G)$, consider the natural map

$$f : \mathscr{J}_b^{>0} \to *,$$

where $\mathrm{Aut}(\mathscr{E}_b) \simeq J_b(\mathbb{Q}_p) \ltimes \mathscr{J}_b^{>0}$, and the semidirect product structures is given by allowing $\mathrm{Aut}(\mathscr{E}_b)$ to act on the canonical reduction of $\mathscr{E}_b$. Since f is an iterated fibration of Banach-Colmez spaces and $\mathscr{J}_b^{>0}$ has an action of $J_b(\mathbb{Q}_p)$ on the right coming from the canonical reduction of $\mathscr{E}_b$. As mentioned in Proposition 2.9.4, this comes from the natural right conjugation action on P_b^-. We have an isomorphism

$$f_!(\Lambda) \simeq \kappa[-2\langle 2\hat{\rho}_G, \nu_b \rangle]$$

for some character $\kappa : J_b(\mathbb{Q}_p) \to \Lambda^*$ (cf. [GI16, Lemma 4.18]). We have the following relationship between isotypic parts of Shtuka spaces and Hecke operators.

Lemma 2.11.1. *[FS21, Section IX.3] Given a local shtuka datum (G,b,μ) as above and π (resp. ρ) a smooth representation of $G(\mathbb{Q}_p)$ (resp. $J_b(\mathbb{Q}_p)$) on Λ-modules, we can consider the associated sheaves $\rho \in \mathrm{D}(J_b(\mathbb{Q}_p), \Lambda) \simeq \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G^b, \Lambda)$ and $\pi \in \mathrm{D}(G(\mathbb{Q}_p), \Lambda) \simeq \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G^{\mathbf{1}}, \Lambda)$ on the HN-strata j_b :*

[8] We thank Naoki Imai for drawing our attention to this.

$\mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$ *and* $j_{\mathbf{1}} : \mathrm{Bun}_G^{\mathbf{1}} \hookrightarrow \mathrm{Bun}_G$*, respectively. There then exists an isomorphism*

$$R\Gamma_c(G,b,\mu)[\rho \otimes \kappa^{-1}][2\langle 2\hat{\rho}_G, \nu_b\rangle] \simeq j_{\mathbf{1}}^* T_\mu j_{b!}(\rho)$$

of complexes of $G(\mathbb{Q}_p) \times W_E$*-modules and an isomorphism*

$$R\Gamma_c(G,b,\mu)[\pi] \simeq j_b^* T_{\mu^{-1}} j_{\mathbf{1}!}(\pi)$$

of complexes of $J_b(\mathbb{Q}_p) \times W_E$*-modules, where* $\mu^{-1} := -w_0(\mu)$ *is a dominant inverse of* μ*. Similarly, we have an isomorphism*

$$R\Gamma_c^\flat(G,b,\mu)[\rho] \simeq j_{\mathbf{1}}^* T_\mu j_{b*}(\rho)$$

of complexes of $G(\mathbb{Q}_p) \times W_E$*-modules, and an isomorphism*

$$R\Gamma_c^\flat(G,b,\mu)[\pi][-2\langle 2\rho_G, \nu_b\rangle] \simeq j_b^! T_{\mu^{-1}} j_{\mathbf{1}*}(\pi) \otimes \kappa^{-1}$$

of complexes of $J_b(\mathbb{Q}_p) \times W_E$*-modules.*

Remark 2.11.2. This comparison comes from the fact that when comparing Hecke operators and isotypic parts of Shtuka spaces, that, after several applications of base-change, the difference will be controlled by the ! pushforward of the constant sheaf along the map

$$\mathrm{Sht}(G,b,\mu)_\infty \to \mathrm{Gr}^b_{G,\mu^{-1}}$$

from the Shtuka space to the adic Newton strata of the B_{dR}^+ affine Grassmannian. This is in particular a $\mathscr{J}_b$-torsor. The third and fourth relationship can be obtained from the first and second by acting via Verdier duality and using Hom-Tensor duality.

We can be more explicit about this character. In particular, we have the following, whose argument is provided in the paper [HI23].

Proposition 2.11.3. *We have an isomorphism* $\kappa \simeq \delta_{P_b}^{-1}$*, where* δ_{P_b} *is the modulus character of the standard parabolic* P_b *with Levi factor* M_b *transferred to* J_b *along the inner twisting. In particular, we have, by Lemma 2.11.1, isomorphisms*

$$R\Gamma_c(G,b,\mu)[\rho \otimes \delta_{P_b}][2\langle 2\hat{\rho}_G, \nu_b\rangle] \simeq j_{\mathbf{1}}^* T_\mu j_{b!}(\rho)$$

and

$$R\Gamma_c^\flat(G,b,\mu)[\rho] \simeq j_{\mathbf{1}}^* T_\mu j_{b*}(\rho)$$

of $G(\mathbb{Q}_p) \times W_E$*-modules.*

We note that, since we used the tilting module $\mathscr{T}_\mu$ in the definition of $R\Gamma_c(G,b,\mu)$ and the Hecke operator T_μ, the complex $R\Gamma_c(G,b,\mu)$ will in general be different from the usual complex defined with respect to the highest weight module V_μ. However, they will agree when we impose the following condition on μ with respect to our coefficient system Λ.

Definition 2.11.4. For $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$, we will say μ is tilting if the representation $V_\mu \in \mathrm{Rep}_\Lambda(\hat{G})$ lies in the full subcategory $\mathrm{Tilt}_\Lambda(\hat{G})$ of tilting modules or equivalently if it is irreducible with coefficients in Λ.

Remark 2.11.5. If $\Lambda = \overline{\mathbb{Q}}_\ell$ this condition always holds. Moreover, by Lemma A.2.1, this always holds if μ is minuscule. In Appendix A.2, we give more insight into this notion.

We now consider a toral parameter $\phi_T : W_{\mathbb{Q}_p} \to {}^LT(\Lambda)$ with associated smooth character $\chi : T(\mathbb{Q}_p) \to \Lambda^*$. Unless otherwise stated, we will assume that ϕ_T is integral with weakly normalized regular mod ℓ-reduction. Given such a ϕ_T, we set ϕ to be the L-parameter of G induced via the natural embedding ${}^LT(\Lambda) \to {}^LG(\Lambda)$ and consider the sheaf $\mathrm{nEis}(\mathscr{S}_{\phi_T}) \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)$ given by Theorem 2.10.10. We begin our analysis by relating our eigensheaves to an averaging formula of Shin.

2.11.1 The Averaging Formula

For μ a geometric dominant cocharacter with reflex field E, we write $r_\mu : W_E \ltimes \hat{G} \to \mathrm{GL}(\mathscr{T}_\mu)$ for the map defined by $\mathscr{T}_\mu$. Since the Hecke operator T_{μ^Γ} attached to $\mathscr{T}_{\mu^\Gamma} \in \mathrm{Rep}_\Lambda({}^LG)$ factors through the Hecke operator T_μ attached to $\mathscr{T}_\mu$ (cf. [FS21, Pages 313-315]) if ϕ_T is μ-regular then, by definition, we have an isomorphism

$$T_\mu(\mathrm{nEis}(\mathscr{S}_{\phi_T})) \simeq r_\mu \circ \phi|_{W_E} \boxtimes \mathrm{nEis}(\mathscr{S}_{\phi_T})$$

of objects in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)^{BW_E}$. Now let's apply the restriction functor $j_{\mathbf{1}}^*(-)$ to both sides of this isomorphism. By the description of the stalks, we know that the RHS is equal to $r_\mu \circ \phi|_{W_E} \boxtimes \pi$, where $\pi := i_B^G(\chi)$ is the normalized parabolic induction of the character χ attached to ϕ_T by class field theory. We can also simplify the RHS. In particular, first off note that, since any G-bundle $\mathscr{F}_G$ on X that occurs as a modification $\mathscr{F}_G \dashrightarrow \mathscr{F}_G^0$ of type μ lies in the set $B(G,\mu)$ by [Rap18, Proposition A.9], we have an isomorphism

$$j_{\mathbf{1}}^*(T_\mu(\mathrm{nEis}(\mathscr{S}_{\phi_T}))) \simeq j_{\mathbf{1}}^* T_\mu(\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{B(G,\mu)})$$

where here we view $B(G,\mu)$ as the open subset of Bun_G defined by the identification $B(G) \simeq |\mathrm{Bun}_G|$ describing the underlying topological space $|\mathrm{Bun}_G|$ of Bun_G [Vie, Theorem 1.1], where $B(G)$ has the natural topology given by the partial ordering. We can further refine this by applying excision with respect to the locally closed stratification by the Harder-Narasimhan strata $\mathrm{Bun}_G^b \subset \mathrm{Bun}_G$ for $b \in B(G,\mu)$, using [FS21, Proposition VII.7.3]. The excision spectral sequence then tells us that the LHS has a filtration whose graded pieces are isomorphic to:

$$j_{\mathbf{1}}^*(T_\mu(j_{b!}j_b^*(\mathrm{nEis}(\mathscr{S}_{\phi_T})))),$$

which, using Proposition 2.11.12, and the description of the stalks of the Eigensheaf one can show is isomorphic to $j_{\mathbf{1}}^*(T_\mu(j_{b*}j_b^*(\mathrm{nEis}(\mathscr{S}_{\phi_T}))))$. In particular, in $K_0(G(\mathbb{Q}_p) \times W_E, \Lambda)$, the Grothendieck group of Λ-valued smooth admissible $G(\mathbb{Q}_p)$-representations with a continuous action of W_E, this tells us that we have an equality:

$$\sum_{b \in B(G,\mu)} [j_{\mathbf{1}}^*(T_\mu(j_{b*}j_b^*(\mathrm{nEis}(\mathscr{S}_{\phi_T}))))] = [r_\mu \circ \phi|_{W_E} \boxtimes \pi] \tag{2.11}$$

Now, using the description of the stalks of $\mathrm{nEis}(\mathscr{S}_{\phi_T})$ and Lemma 2.11.1, we can spell out the LHS more clearly. In particular, we define the following.

Definition 2.11.6. For ϕ_T an arbitrary toral parameter with induced parameter ϕ and $b \in B(G)$, we define the complex of smooth admissible $J_b(\mathbb{Q}_p)$-representations $\mathrm{Red}_{b,\phi}$ as follows. If $b \notin B(G)_{\mathrm{un}}$, we set $\mathrm{Red}_{b,\phi}$ to be equal to 0, and, if $b \in B(G)_{\mathrm{un}}$, we set $\mathrm{Red}_{b,\phi}$ to be equal to

$$\bigoplus_{w \in W_b} i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{-1/2}[-\langle 2\hat{\rho}, \nu_b \rangle] \in \mathrm{D}(J_b(\mathbb{Q}_p), \Lambda)$$

where δ_{P_b} is the modulus character of J_b defined by the standard parabolic $P_b \subset G$ with Levi factor $M_b \simeq J_b$, as before.

This allows us to deduce the following from equation (11).

Theorem 2.11.7. *For $\phi_T : W_{\mathbb{Q}_p} \to {}^LT(\Lambda)$ an integral parameter with weakly normalized regular mod ℓ reduction, if $\pi := i_B^G(\chi)$ is the normalized parabolic induction of the smooth character χ attached to ϕ_T then, for any geometric dominant cocharacter μ such that ϕ_T is μ-regular, we have an equality*

$$\sum_{b \in B(G,\mu)} [R\Gamma_c^\flat(G,b,\mu)[\mathrm{Red}_{b,\phi}]] = [r_\mu \circ \phi|_{W_E} \boxtimes \pi]$$

in $K_0(G(\mathbb{Q}_p) \times W_E, \Lambda)$.

Remark 2.11.8. We note that, in the above analysis, we didn't necessarily have to restrict to the HN-strata $\mathrm{Bun}_G^{\mathbf{1}}$ or even a single cocharacter. In particular, by considering Hecke operators defined by representations in $\mathrm{Tilt}_\Lambda({}^L G^I)$ for a finite index set I, we could have deduced an analogous formula for shtuka spaces with I legs for an arbitrary finite index set I. We could have also restricted to any HN-stratum; however, if the HN-stratum is not defined by a basic element, the answer is not as clean as above. In particular, given a G-bundle $\mathscr{F}_b$ on X corresponding to a general element $b \in B(G)$, it is to the best of our knowledge completely unknown exactly which G-bundles $\mathscr{F}_G$ occur as modifications $\mathscr{F}_G \dashrightarrow \mathscr{F}_b$ of type μ. It would be interesting to understand this question better. We leave it to the reader to work out the precise statements of these more general implications.

We can use this claim to deduce the averaging formula for an arbitrary toral parameter ϕ_T when $\Lambda = \overline{\mathbb{Q}}_\ell$, by viewing both sides as trace forms on $K_0(T(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ and using a continuity argument. We note that, in this case, μ is always tilting so we have that $R\Gamma_c(G,b,\mu)$ and $R\Gamma_c^\flat(G,b,\mu)$ are just the usual complexes. We recall that $f : K_0(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell) \to \overline{\mathbb{Q}}_\ell$ is a trace form if it can be written as $\mathrm{tr}(\delta|-)$ for $\delta \in C_c^\infty(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$. We now fix a $\delta \in C_c^\infty(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ and $\gamma \in W_E$, we define the following functions attached to this datum

$$f_L^{\delta,\gamma} : K_0(T(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell) \to \overline{\mathbb{Q}}_\ell$$

$$\chi \mapsto \mathrm{tr}(\delta \times \gamma | i_B^G(\chi) \boxtimes r_\mu \circ \iota(\chi)|_{W_E})$$

$$f_R^{\delta,\gamma} : K_0(T(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell) \to \overline{\mathbb{Q}}_\ell$$

$$\chi \mapsto \sum_{b \in B(G,\mu)_{\mathrm{un}}} \sum_{w \in W_b} \mathrm{tr}(\delta \times \gamma | R\Gamma_c^\flat(G,b,\mu)[i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{1/2}])(-1)^{\langle 2\hat{\rho}_G, \nu_b \rangle}$$

where $\iota(\chi) \simeq \phi_T$ is the isomorphism given by local class field theory. We have the following lemma.

Lemma 2.11.9. *The functions $f_L^{\delta,\gamma}$ and $f_R^{\delta,\gamma}$ define trace forms on $K_0(T(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$.*

Proof. This follows from the fact that normalized parabolic induction takes trace forms to trace forms as can be checked from the characterization of trace forms in the trace Paley-Wiener theorem [BDK86], and the fact that $\mathrm{tr}(\delta \times \gamma | R\Gamma_c(G,b,\mu)[-])$ defines a trace form on $K_0(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ by [HKW22, Theorem 6.5.4]. $\square$

This gives the following.

Theorem 2.11.10. *For $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\overline{\mathbb{Q}}_\ell)$ an arbitrary toral parameter with associated character χ, and $i_B^G(\chi) =: \pi$, we have an equality*

$$\sum_{b \in B(G,\mu)} [R\Gamma_c^\flat(G,b,\mu)[\mathrm{Red}_{b,\phi}]] = [r_\mu \circ \phi|_{W_E} \boxtimes \pi]$$

in $K_0(G(\mathbb{Q}_p) \times W_E, \overline{\mathbb{Q}}_\ell)$.

Proof. It suffices to show that the trace forms $f_L^{\delta,\gamma}(\chi)$ and $f_R^{\delta,\gamma}(\chi)$ agree for varying δ and γ and all $\chi \in K_0(T(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$. We define the difference $\Delta_{\delta,\gamma} := f_L^{\delta,\gamma}(\chi) - f_R^{\delta,\gamma}(\chi)$. We say that a subset $S \in K_0(T(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ is dense if $\Delta_{\delta,\gamma}(x) = 0$ for all $x \in S$ implies that $\Delta_{\delta,\gamma} = 0$. Using Theorem 2.11.7 and Lemma 2.10.15, we can reduce to showing that the subset S of all characters χ which are normalized regular and admit a $\overline{\mathbb{Z}}_\ell$-lattice is dense. This is relatively easy to show. In particular, if we view $\Delta_{\delta,\gamma}$ as a regular function on the variety of unramified twists of a fixed character χ then the set of characters admitting a $\overline{\mathbb{Z}}_\ell$ lattice is Zariski-dense. Moreover, the locus where χ is normalized regular is also clearly Zariski dense, since it is implied by insisting that χ precomposed with sums of coroots is not the norm or trivial character for the sums of coroots appearing in differences of distinct weights of V_{μ_k}, for a finite list of generating cocharacters μ_k with $k = 1, \ldots, n$ appearing in the definition of μ-regularity, and it is also clear for the condition of weak normalized regularity. Therefore, the claim follows, using the previous Lemma. $\square$

This theorem is compatible with existing results. We recall that Shin [Shi12] and Bertoloni-Meli [Ber21], have described similar averaging formulas. In particular, given a refined endoscopic datum $\mathfrak{e} = (H, \mathscr{H}, s, \eta)$ (Definition A.3.1), Shin and Bertoloni-Meli define maps

$$\mathrm{Red}_b^{\mathfrak{e}}(-) : K_0^{st}(H(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell) \to K_0(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$$

where $K_0^{st}(H(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ denotes the Grothendieck group of stable virtual $\overline{\mathbb{Q}}_\ell$-representations of $H(\mathbb{Q}_p)$. If we are given an L-parameter ϕ which factors as $\mathscr{L}_{\mathbb{Q}_p} \xrightarrow{\phi^H} \mathscr{H} \xrightarrow{{}^L\eta} {}^L G$ then, using the local Langlands correspondence for G, we are able to attach a stable distribution $S\Theta_\phi \in K_0^{st}(H(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ which should satisfy the endoscopic character identities as in [Kal16, Conjecture D]. The averaging formula (Conjecture A.3.2) is a conjectural formula for

$$\sum_{b \in B(G,\mu)} R\Gamma_c^\flat(G,b,\mu)[\mathrm{Red}_b^{\mathfrak{e}}(S\Theta_\phi)]$$

in $K_0(G(\mathbb{Q}_p) \times W_E, \overline{\mathbb{Q}}_\ell)$. Our averaging formula is related to the case when $\mathfrak{e}_{\mathrm{triv}} = (G, 1, {}^L G, \mathrm{id})$ is the trivial endoscopic datum. In particular, if ϕ_T is a generic toral parameter then, by Lemma 2.3.18, ϕ should define an actual L-parameter with trivial monodromy, and we can consider the L-packet $\Pi_\phi(G)$ under the local Langlands correspondence for G appearing in Assumption 2.7.5. By Assumption 2.7.5 (3), the members of the L-packet will be given by the irreducible constituents of $i_B^G(\chi)$. Therefore, we have that $S\Theta_\phi = [\pi]$ in $K_0^{st}(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$, and in the appendix we verify that the following is true.

Proposition 2.11.11. *Let $\chi : W_{\mathbb{Q}_p} \to \overline{\mathbb{Q}}_\ell^*$ be a smooth generic character, so that, using Lemma 2.3.18, we have an equality*

$$S\Theta_\phi = [\pi]$$

in $K_0(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)^{\mathrm{st}}$ under the local Langlands correspondence appearing in Assumption 2.7.5. Then we always have

$$[\mathrm{Red}_{b,\phi}] = \mathrm{Red}^{\mathfrak{e}_{\mathrm{triv}}}([\pi])$$

in the Grothendieck group $K_0(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$, and Conjecture A.3.2 holds true for the L-parameter ϕ attached to χ.

We would now like to refine our averaging formula further. In particular, using Theorem 2.8.3, we can upgrade this equality in the Grothendieck group to a genuine isomorphism of complexes.

2.11.2 The Refined Averaging Formula and Intertwining Operators

Consider an element $b \in B(G)_{\mathrm{un}}$ with anti-dominant reduction $b_T^- \in B(T)$. For $w \in W_b$, we set $\rho_{b,w} := i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{-1/2}$ to be the twisted normalized parabolic induction, and consider $j_b : \mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$, the inclusion of the locally closed HN-strata corresponding to b. Temporarily, we will work with a more general integral toral parameter. As seen in §2.10, the sheaf $\mathrm{nEis}(\mathscr{S}_{\phi_T}) \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ with its desired properties might not be well-defined for $\Lambda \in \{\overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$ if ϕ_T isn't integral with weakly normalized regular mod ℓ-reduction; however, we note that, since b_T^- is antidominant, when $\Lambda = \overline{\mathbb{F}}_\ell$, we always have that $\mathrm{nEis}^{b_T^-}(\mathscr{S}_{\phi_T})$ is only supported on Bun_G^b, since $\mathrm{Bun}_B^{b_T^-}$ will only parametrize split reductions. Moreover,

$\mathrm{nEis}^{b_T^-}(\mathscr{S}_{\phi_T})$ will be isomorphic to $j_{b!}(\rho_{b,w_0})$ by Proposition 2.9.4. Therefore, by the same procedure carried out in §2.10, we always get a well-defined sheaf $\mathrm{nEis}^{b_T^-}(\mathscr{S}_{\phi_T}) \in \mathrm{D_{lis}}(\mathrm{Bun}_G, \Lambda)$ for any integral toral parameter ϕ_T such that we have an isomorphism

$$j_{b!}(\rho_{b,w_0})[-\langle 2\hat{\rho}, \nu_b\rangle] \simeq \mathrm{nEis}^{b_T^-}(\mathscr{S}_{\phi_T}) \tag{2.12}$$

and, if ϕ_T is an integral parameter with weakly generic mod ℓ-reduction, then, as in §2.10, Theorem 2.8.3 extends in a natural way to $\mathrm{nEis}^{b_T^-}(\mathscr{S}_{\phi_T})$. We can act by $\mathbb{D}_{\mathrm{Bun}_G}(-)$ on both sides of (12). By the commutation of Eisenstein series with Verdier duality, the RHS of (12) becomes

$$\mathrm{nEis}^{b_T^-}(\mathscr{S}_{\phi_T^\vee}) \simeq j_{b!}(\rho_{b,w_0}^* \otimes \delta_{P_b}^{-1})[-\langle 2\hat{\rho}, \nu_b\rangle]$$

where $\rho_{b,w_0}^* = (i_{B_b}^{J_b}(\chi^{w_0}) \otimes \delta_{P_b}^{-1/2})^* = i_{B_b}^{J_b}((\chi^{w_0})^{-1}) \otimes \delta_{P_b}^{1/2}$ denotes the contragradient. On the other hand, the LHS of (12) becomes

$$j_{b*}(\mathbb{D}_{\mathrm{Bun}_G^b}(\rho_{b,w_0}))[\langle 2\hat{\rho}, \nu_b\rangle]$$

We now need to be a bit careful. In particular, we recall that $\mathrm{Bun}_G^b \simeq [*/\mathscr{J}_b]$, where $\mathscr{J}_b$ is the group diamond parameterizing automorphisms of $\mathscr{F}_b$, and we are implicitly using the identification $\mathrm{D_{lis}}(\mathrm{Bun}_G^b, \Lambda) \simeq \mathrm{D}(J_b(\mathbb{Q}_p), \Lambda)$ given by pullback along the map $p : [*/\mathscr{J}_b] \to [*/\underline{J_b(\mathbb{Q}_p)}]$, as in [FS21, Proposition V.2.2, VII.7.1]. Therefore, we need to account for the shifts and twists given by $p^!$. We can use that the natural section s of p is an iterated fibration of positive Banach-Colmez space and Proposition 2.11.3, to show that $p^!(-) \simeq p^*(- \otimes \delta_{P_b}^{-1})[-2\langle 2\hat{\rho}, \nu_b\rangle]$, and therefore the LHS of (12) becomes $j_{b*}(\rho_{b,w_0}^* \otimes \delta_{P_b}^{-1})[\langle 2\hat{\rho}, \nu_b\rangle - 2\langle 2\hat{\rho}, \nu_b\rangle] = j_{b*}(\rho_{b,w_0}^* \otimes \delta_{P_b}^{-1})[-\langle 2\hat{\rho}, \nu_b\rangle]$. In conclusion, we have an isomorphism:

$$j_{b*}(\rho_{b,w_0}^* \otimes \delta_{P_b}^{-1})[-\langle 2\hat{\rho}, \nu_b\rangle] \simeq j_{b!}(\rho_{b,w_0}^* \otimes \delta_{P_b}^{-1})[-\langle 2\hat{\rho}, \nu_b\rangle]$$

Relaxing the contragradients and cancelling the shifts, we deduce an isomorphism:

$$j_{b*}(\rho_{b,w_0}) \simeq j_{b!}(\rho_{b,w_0}).$$

Now, given $w \in W_b$, we can replace $\mathscr{S}_{\phi_T}$ by $\mathscr{S}_{\phi_T^w}$ in the above argument, where ϕ_T^w is the conjugate of ϕ_T by w. This tells us that, if ϕ_T^w is integral with weakly generic mod ℓ reduction, we have an isomorphism:

$$j_{b*}(\rho_{b,ww_0}) \simeq j_{b!}(\rho_{b,ww_0})$$

In conclusion, we deduce the following.

Proposition 2.11.12. *For $b \in B(G)_{\mathrm{un}}$, $w \in W_b$, and $\rho_{b,ww_0} = i_{B_b}^{J_b}(\chi^{ww_0}) \otimes \delta_{P_b}^{-1/2}$ as defined above, where χ is the character attached to an integral toral parameter ϕ_T such that its conjugate ϕ_T^w has weakly generic mod ℓ reduction, we have an isomorphism*

$$j_{b!}(\rho_{b,ww_0}) \simeq j_{b*}(\rho_{b,ww_0})$$

of objects in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$.

Remark 2.11.13. Note that, if we want this to hold for all $b \in B(G)_{\mathrm{un}}$ and $w \in W_b$, this is equivalent to assuming that the mod ℓ-reduction of ϕ_T is generic, since W_G acts transitively on the Γ-orbits of coroots.

Remark 2.11.14. If b is basic then this precisely says that the sheaf defined by ρ_{b,ww_0} is inert in the sense of [Han20, Definition 2.19]. In particular, this Proposition, in conjunction with [Han20, Theorem 2.22] and Lemma 2.3.18, seems to suggest that inert sheaves should correspond precisely to the representations whose semi-simplified L-parameter comes from the semi-simplification of a parameter with non-trivial monodromy. For example, if one takes the constant sheaf on $\mathrm{Bun}_G^{\mathbf{1}}$ and considers $j_{\mathbf{1}!}(\Lambda)$ then we have that $\mathbb{D}_{\mathrm{Bun}_G}(j_{\mathbf{1}!}(\Lambda)) \simeq j_{\mathbf{1}*}(\Lambda)$ which one can check is not isomorphic to $j_!(\Lambda)$. Similarly, we see that the L-parameter attached to the trivial representation comes from the semi-simplification of a parameter with non-trivial monodromy (the Steinberg parameter).

Now consider μ a geometric dominant cocharacter of G with reflex field E and an element $b \in B(G,\mu)$. Applying $j_{\mathbf{1}}^{*}T_{\mu}(-)$ to both sides of the previous isomorphism, we conclude, using Proposition 2.11.3, an isomorphism

$$R\Gamma_c(G,b,\mu)[\rho_{b,w} \otimes \delta_{P_b}] \simeq R\Gamma_c^{\flat}(G,b,\mu)[\rho_{b,w}][-2\langle 2\rho_G, \nu_b\rangle].$$

Corollary 2.11.15. *Let (G,b,μ) be a local shtuka datum. For $b \in B(G)_{\mathrm{un}}$, $w \in W_b$, and ϕ_T an integral toral parameter such that ϕ_T^w has weakly generic mod ℓ reduction, there is an isomorphism*

$$R\Gamma_c(G,b,\mu)[\rho_{b,w} \otimes \delta_{P_b}] \simeq R\Gamma_c^{\flat}(G,b,\mu)[\rho_{b,w}][-2\langle 2\rho_G, \nu_b\rangle]$$

of complexes of $G(\mathbb{Q}_p) \times W_E$-modules. In particular, by Remark 2.11.13, if the mod ℓ-reduction of ϕ_T is generic, then this is true for all $w \in W_b$.

We now claim that the cohomology of $R\Gamma_c(G,b,\mu)[\rho_{b,w} \otimes \delta_{P_b}]$ should be concentrated in degree $\langle 2\hat{\rho}, \nu_b\rangle$, for $\rho_{b,w}$ as above. To do this, let's put ourselves back in the position of an integral ϕ_T with weakly normalized regular mod ℓ reduction.

We saw that in the previous section that the excision spectral sequence applied to $\mathrm{nEis}(\mathscr{S}_{\phi_T})|_{B(G,\mu)}$ gives rise to a filtration whose graded pieces are isomorphic to $j_{b!}j_b^*(\mathrm{nEis}(\mathscr{S}_{\phi_T}))$, but Lemma 2.11.12 implies that these graded pieces are also isomorphic to $j_{b*}j_b^*(\mathrm{Eis}(\mathscr{S}_{\phi_T}))$. In particular, this allows us to deduce that the edge maps in the excision spectral sequence split, and therefore the sequence degenerates, giving an isomorphism:

$$\bigoplus_{b\in B(G,\mu)} j_{b*}j_b^*(\mathrm{nEis}(\mathscr{S}_{\phi_T})) \simeq \mathrm{nEis}(\mathscr{S}_{\phi_T})|_{B(G,\mu)}$$

We now would like to apply the eigensheaf property. So fix a geometric dominant cocharacter, and assume that ϕ_T is μ-regular. If $\pi = i_B^G(\chi)$ is the normalized parabolic induction of χ as above then, using our description of the stalks, we deduce the following "refined averaging formula".

Theorem 2.11.16. *For $\phi_T : W_{\mathbb{Q}_p} \to {}^LT(\Lambda)$ an integral toral parameter with weakly normalized regular mod ℓ-reduction and μ a geometric dominant cocharacter such that ϕ_T is μ-regular, we have an isomorphism*

$$\bigoplus_{b\in B(G,\mu)_{\mathrm{un}}} \bigoplus_{w\in W_b} R\Gamma_c^{\flat}(G,b,\mu)[\rho_{b,w}][-\langle 2\hat{\rho}_G, \nu_b\rangle] \simeq (i_B^G(\chi)\boxtimes r_\mu\circ\phi|_{W_E})$$

of complexes of $G(\mathbb{Q}_p)\times W_E$-modules.

Unless otherwise stated, we will from now on assume that ϕ_T is integral with weakly normalized regular mod ℓ reduction. Using the previous formula, we can give a very explicit description of the complexes $R\Gamma_c^{\flat}(G,b,\mu)[\rho_{b,w}]$, for $b\in B(G,\mu)_{\mathrm{un}}$ and $w\in W_b$.

Corollary 2.11.17. *For μ a geometric dominant cocharacter with reflex field E such that ϕ_T is μ-regular, fixed $b\in B(G,\mu)_{\mathrm{un}}$, and varying $w\in W_b$, the complex $R\Gamma_c^{\flat}(G,b,\mu)[\rho_{b,w}]$ is isomorphic to $\phi_{b,w}^{\mu}\boxtimes\sigma[\langle 2\hat{\rho},\nu_b\rangle]$, for $\phi_{b,w}^{\mu}$ a representation of W_E and σ a subrepresentation of $i_B^G(\chi)$. Moreover, we have an isomorphism*

$$\bigoplus_{b\in B(G,\mu)_{\mathrm{un}}} \bigoplus_{w\in W_b} \phi_{b,w}^{\mu} \simeq r_\mu\circ\phi|_{W_E}$$

of W_E-representations.

This leads to a natural question. How can we describe the W_E-representations $\phi_{b,w}^{\mu}$ in terms of the weights appearing in $r_\mu\circ\phi|_{W_E}$. We recall, by Corollary 2.2.9, that the orbit of b_T under the Weyl group W_G can be described as $w(b_T)$ for $w \in W_b$ varying; moreover, using Corollary 2.2.9 and Remark 2.10.6, we see that we have a correspondence between $B(G,\mu)_{\mathrm{un}}$ and the set of Weyl orbits of weights which can occur in the representation $\mathscr{T}_\mu|_{\hat{G}^\Gamma}$. In particular, given $\overline{\nu} \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$, we consider the subspace

$$\bigoplus_{\substack{\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \\ \tilde{\nu}_\Gamma = \overline{\nu}}} \tilde{\nu} \circ \phi_T|_{W_E} \otimes \mathscr{T}_\mu(\nu)$$

of $(r_\mu \circ \phi)|_{W_E}$, where we note that if we forget the Galois action then this identifies with the $\overline{\nu}$ weight space of $\mathscr{T}_\mu|_{\hat{G}^\Gamma}$ by Lemma 2.2.8. Now the refined averaging formula suggests the following.

Conjecture 2.11.18. *For all geometric dominant cocharacters μ such that ϕ_T is μ-regular, an unramified element $b \in B(G,\mu)_{\mathrm{un}}$, and a Weyl group element $w \in W_b$, we have an isomorphism*

$$\bigoplus_{\substack{\widetilde{w(b_T)} \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \\ \widetilde{w(b_T)}_\Gamma = w(b_T)}} \widetilde{w(b_T)} \circ \phi_T|_{W_{E'}} \otimes \mathscr{T}_\mu(\widetilde{w(b_T)}) \simeq \phi^\mu_{b,w}|_{W_{E'}}$$

of $W_{E'}$-representations, where b_T is a dominant reduction of b and $E'|E$ denotes the splitting field of G.

For the rest of this section, let us look at some cases where this can be shown explicitly, using a shtuka analogue of Boyer's trick. To illustrate the idea, we begin with a particularly nice example, where Theorem 2.11.16 and Conjecture 2.11.18 can be checked by hand.

Example 2.11.19. Let $G = \mathrm{GL}_2$ and $\mu = (1,0)$. Write $\phi_T = \phi_1 \oplus \phi_2$, and consider the set $B(G,\mu)$. It consists of two elements: the μ-ordinary element and the basic element. Only the μ-ordinary element lies in $B(G,\mu)_{\mathrm{un}}$; therefore, only this element contributes to the expression in Theorem 2.11.16. Namely, if b_μ denotes the μ-ordinary element, we note that $\langle 2\hat{\rho}, \nu_{b_\mu}\rangle = \langle 2\hat{\rho}, \mu\rangle = 1$. We conclude that Theorem 2.11.16 is an isomorphism

$$R\Gamma^\flat_c(G,b_\mu,\mu)[\chi \otimes \delta_B^{-1/2}] \oplus R\Gamma^\flat_c(G,b_\mu,\mu)[\chi^{w_0} \otimes \delta_B^{-1/2}] \simeq i_B^{GL_2}(\chi) \boxtimes \phi[1]$$

of $G(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$-representations. This can be seen through direct computation. In particular, we have an isomorphism $J_{b_\mu} \simeq T$, and, since μ is minuscule, we have that $\mathscr{S}_\mu \simeq \Lambda[1](\frac{1}{2})$. The space $\mathrm{Sht}(G, b_\mu, \mu)_{\infty,\mathbb{C}_p}$ is the moduli space parameterizing modifications $\mathscr{O}(-1) \oplus \mathscr{O} \dashrightarrow \mathscr{O}^2$ of type $(1,0)$. Every such modification is determined by an injection $\mathscr{O}(-1) \hookrightarrow \mathscr{O}$ of line bundles. Formally, this implies that the space $\mathrm{Sht}(G, b, \mu)_{\infty,\mathbb{C}_p}$ as a space with $\mathrm{GL}_2(\mathbb{Q}_p)$-action is parabolically induced from the space parameterizing such injections as a space with $T(\mathbb{Q}_p)$ action. Here $T(\mathbb{Q}_p)$ acts on the space of injections $\mathscr{O}(-1) \hookrightarrow \mathscr{O}$ via the scaling action precomposed with projection to the first factor of $T(\mathbb{Q}_p)$. This is a manifestation of the fact that $\mathrm{Sht}(G, b_\mu, \mu)_{\infty,\mathbb{C}_p}$ is a $\mathscr{J}_b$-torsor over the flag variety $\underline{(G/B)(\mathbb{Q}_p)} \simeq \mathbb{P}^1(\mathbb{Q}_p) \subset \mathbb{P}^1_{\mathbb{C}_p} \simeq \mathrm{Gr}_{G,\leq\mu^{-1},\mathbb{C}_p}$, where the last isomorphism is the Bialynicki-Birula map. In particular, note that the compactly supported cohomology of $\underline{G/B(\mathbb{Q}_p)}$ is precisely the space of compactly supported functions on $(G/B)(\mathbb{Q}_p)$. All in all, this allows us to conclude isomorphisms

$$R\Gamma^\flat_c(G, b_\mu, \mu)[\chi \otimes \delta_B^{-1/2}] = \mathrm{Ind}_{B^-}^G(\chi \otimes \delta_B^{-1/2}) \boxtimes \phi_1[1] = i_B^G(\chi^{w_0}) \boxtimes \phi_1[1]$$

$$R\Gamma^\flat_c(G, b_\mu, \mu)[\chi^{w_0} \otimes \delta_B^{-1/2}] = \mathrm{Ind}_{B^-}^G(\chi^{w_0} \otimes \delta_B^{-1/2}) \boxtimes \phi_2[1] = i_B^G(\chi) \boxtimes \phi_2[1],$$

where there is a cancellation of the $\frac{1}{2}$ Tate twist in $\mathscr{S}_\mu$ and the Tate twist coming from $\delta_B^{-1/2}$, as in §2.3. the compactly supported cohomology of $\mathscr{J}_b^{>0}$ contribute. Now, if χ is attached to a generic parameter ϕ_T, this implies that $i_B^G(\chi)$ is irreducible as in Example 2.3.9, and it follows that we have an isomorphism $i_B^G(\chi) \simeq i_B^G(\chi^{w_0})$, which allows us to conclude the result.

Now let's generalize this example. In particular, recall that $B(G, \mu)$ has a distinguished μ-ordinary element, denoted b_μ, which is the maximal element with respect to the partial ordering on $B(G, \mu)$, and has the property that $\tilde{\mu} = \nu_{b_\mu}$, where $\tilde{\mu}$ is the weighted average over the Galois orbit of μ as in §2.2.1. If we write μ_T for μ viewed as a geometric cocharacter of T in the negative Weyl chamber defined by the choice of Borel, we can see that b_μ admits a dominant reduction to the unique element $b_{\mu_T} \in B(T, \mu_T)$. In other words, the element b_μ always lies in $B(G, \mu)_{\mathrm{un}} := B(G, \mu) \cap B(G)_{\mathrm{un}}$. Conjecture 2.11.18 suggests to us that this should give rise to the contribution given by the highest weight of $\mathscr{T}_\mu|_{\hat{G}^\Gamma}$, which will have multiplicity one. We now prove the following result using a shtuka analogue of Boyer's trick [Boy99a] proven by Gaisin-Imai [GI16].

Proposition 2.11.20. *For μ any geometric dominant cocharacter with reflex field E, $b_\mu \in B(G, \mu)_{\mathrm{un}}$ the μ-ordinary element with dominant reduction b_{μ_T}, $w \in W_b$ varying, and ϕ_T any toral parameter, we have an isomorphism*

$$R\Gamma^\flat_c(G, b_\mu, \mu)[\rho_{b_\mu,w}] \simeq w(\mu_T) \circ \phi_T|_{W_E} \boxtimes i_B^G(\chi^{ww_0})[\langle 2\hat{\rho}, \nu_{b_\mu} \rangle]$$

of $W_E \times G(\mathbb{Q}_p)$-representations, where $w, w_0 \in W_b$ are representatives of minimal length.

Proof. We note that the element $b_\mu \in B(G,\mu)$ is Hodge-Newton reducible in the sense of [RV14, Definition 4.5]. In particular, b_μ is induced from the unique element $b_{\mu_T} \in B(T,\mu_T)$ via the natural map $B(T) \to B(G)$. Consider a rank k vector bundle of the form $\bigoplus_{i=1}^{k} \mathscr{O}(n_i)$ for $n_i \in \mathbb{Z}$ and suppose we have a modification:

$$\bigoplus_{i=1}^{k} \mathscr{O}(n_i) \dashrightarrow \mathscr{O}^n$$

Then it is easy to see that such a modification will be determined by a tuple of modifications

$$\mathscr{O}(n_i) \dashrightarrow \mathscr{O}$$

for all $i = 1, \dots, k$. If we apply the Tannakian formalism [GI16, Lemma 4.11], this tells us that the space $\mathrm{Sht}(G, b_\mu, \mu)_{\infty,\mathbb{C}_p}$ parameterizing modifications of the form

$$\mathscr{F}_{b_\mu} \dashrightarrow \mathscr{F}_G^0$$

will be determined by the spaces $\mathrm{Sht}(T, w(b_{\mu_T}), w(\mu_T))_{\infty,\mathbb{C}_p}$ parametrizing modifications of the form

$$\mathscr{F}_{w(b_{\mu_T})} \dashrightarrow \mathscr{F}_T^0$$

with meromorphy equal to $w(\mu_T)$ for varying $w \in W_{b_\mu}$. In particular, this tells us that the moduli space $\mathrm{Sht}(G, b_\mu, \mu)_{\infty,\mathbb{C}_p}$ parameterizing modifications of meromorphy $\leq \mu$ is actually equal to the open subspace $\mathrm{Sht}(G, b_\mu, \mu)_\infty^\mu$ parameterizing modifications of meromorphy equal to μ. This is because any modification induced from a modification

$$\mathscr{F}_{b_{\mu_T}} \dashrightarrow \mathscr{F}_T^0$$

of type μ_T will be of type μ, which implies that we have an isomorphism $\mathscr{S}_\mu \simeq \Lambda[d](\frac{d}{2})$, where $d = \langle 2\hat{\rho}, \nu_{b_\mu} \rangle = \langle 2\hat{\rho}, \mu \rangle$ using [FS21, Proposition VI.7.5]. Here we need to be a bit careful since $\mathscr{S}_\mu$ is the pullback of the sheaf associated to the tilting module $\mathscr{T}_\mu$ not V_μ as per usual. However, we note that the above discussion tells us that the Newton strata in the Schubert cell/variety $\mathrm{Gr}_{G,\leq\mu^{-1},\mathbb{C}_p}^{b_\mu} = \mathrm{Gr}_{G,\mu^{-1},\mathbb{C}_p}^{b_\mu}$ has only non-empty intersection with the semi-infinite cells $\mathrm{S}_{G,w(\mu_T),\mathbb{C}_p}$ indexed by the Weyl group orbits of the highest weight, using the Remark proceeding 2.4.7. Since both $\mathscr{T}_\mu$ and V_μ have highest weight with multiplicity one, the discrepancy doesn't matter via Corollary 2.4.9. It remains to describe the complex $R\Gamma_c^\flat(G, b_\mu, \mu)[\rho_{b_\mu,w}]$. Using our above observations, [GI16, Theorem 4.26] (as in the proof [HI23, Theorem 4.21]), it follows that we have an isomorphism

$$R\Gamma_c^\flat(G, b_\mu, \mu)[\rho_{b_\mu,w}] \simeq \mathrm{Ind}_{P_{b_\mu}^-}^G (R\Gamma_c^\flat(M, b_{\mu_M}, \mu_M)[i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_{b_\mu}}^{-1/2}][\langle 2\hat{\rho}, \mu\rangle](\langle\hat{\rho}, \mu\rangle)$$

of complexes of $G(\mathbb{Q}_p) \times W_E$-representation, where $M = M_{b_\mu}$ is the centralizer of the slope homomorphism, μ_M is the G-dominant choice of μ viewed as a cocharacter of M, and $b_{\mu_M} \in B(M, \mu_M)$ is the μ-ordinary element. We note that, since $b_\mu \in B(G)_{\mathrm{un}}$, it follows that $B(M, \mu_M)$ is a singleton and that μ_M is central with respect to M. In particular, $\mathrm{Sht}(M, \mu_M, b_{\mu_M})$ is 0-dimensional and identifies with the profinite set $\underline{M(\mathbb{Q}_p)}$ (recall that $M \simeq J_{b_\mu} \simeq J_{b_{\mu_M}}$ in this case). This allows us to identify

$$R\Gamma_c^\flat(M_b, \mu_M, b_{\mu_M})[i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_{b_\mu}}^{-1/2}] \simeq i_{B\cap M_b}^{M_b}(\chi^w) \otimes \delta_{P_{b_\mu}}^{-1/2} \boxtimes w(\mu_T) \circ \phi_T(-\langle\hat{\rho}, \mu\rangle)|_{W_E},$$

where we can identify the 1-dimensional Weil group action through excursion algebra considerations. Therefore, we get an isomorphism

$$\begin{aligned} R\Gamma_c^\flat(G, b_\mu, \mu)[\rho_{b_\mu,w}] \simeq \mathrm{Ind}_{P_{b_\mu}^-}^G (i_{B\cap M_b}^{M_b}(\chi^w) \otimes \delta_{P_{b_\mu}}^{-1/2}) \boxtimes w(\mu_T) \circ \phi_T|_{W_E})[\langle 2\hat{\rho}, \mu\rangle](\langle -\hat{\rho}, \mu\rangle + \langle\hat{\rho}, \mu\rangle) \simeq \\ i_B^G(\chi^{ww_0}) \boxtimes w(\mu_T) \circ \phi_T|_{W_E}[\langle 2\hat{\rho}, \mu\rangle] \end{aligned}$$

of complexes of $G(\mathbb{Q}_p) \times W_E$-representations which gives the desired result. □

Remark 2.11.21. We note that in the proof we did not use any of our results on geometric Eisenstein series. It would be interesting to generalize some of these computations to some non-principal situations. In particular, if one works with a general parabolic P with Levi factor M and a supercuspidal parameter $\phi_M : W_{\mathbb{Q}_p} \to {}^L M$ and assumes that Fargues' conjecture holds on the Levi subgroup M, then the results of [GI16] guarantee that one has similar formulas relating the cohomology of the shtuka spaces of G to basic local shtuka spaces of M for Hodge-Newton reducible $b \in B(G, \mu)$ admitting a reduction to M. The description of the eigensheaf in Fargues' conjecture, as described for odd unramified unitary groups in [BHN22] and partially for GSp_4 in [Ham21b], gives one a very explicit description of these basic local shtuka spaces, which would in turn give a computational approach to understanding and generalizing these formulas beyond the principal case. We fully anticipate that some of the methods we use in the principal case generalize to the non-principal case; however, the results seem much more technical, and these computations would give a nice foothold into the problem.

This explicit calculation has some interesting consequences. In particular, we already saw in Example 2.11.19 that relating Proposition 2.11.20 to Theorem 2.11.16 required using the existence of an isomorphism $i_B^G(\chi) \simeq i_B^G(\chi^w)$; i.e. intertwining operators. This phenomenon actually persists. In particular, we deduce the following.

Corollary 2.11.22. *Let $\chi : T(\mathbb{Q}_p) \to \Lambda^*$ be a character obtained from an integral toral parameter ϕ_T whose mod ℓ-reduction is weakly normalized regular. For $\mu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$, let W_μ be the stabilizer of the action of W_G on $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ and assume that ϕ_T is μ-regular. Then, for all $w \in W_G/W_\mu$ a minimal length representative, we have an isomorphism*

$$i_B^G(\chi) \simeq i_B^G(\chi^{ww_0})$$

of $G(\mathbb{Q}_p)$-representations.

Proof. Since M_{b_μ} will be by construction the centralizer of μ, we have an isomorphism $W_{M_{b_\mu}} \simeq W_\mu$. The previous proposition then tells us that we have an isomorphism

$$R\Gamma_c^\flat(G, b_\mu, \mu)[i_{B_{b_\mu}}^{J_{b_\mu}}(\chi^w) \otimes \delta_{P_{b_\mu}}^{1/2}] \simeq w(\mu_T) \circ \phi_T|_{W_E} \boxtimes i_B^G(\chi^{ww_0})[\langle 2\hat{\rho}, \nu_{b_\mu} \rangle]$$

for all $w \in W_G/W_\mu$ a minimal length representative. On the other hand, since ϕ_T is μ-regular, Corollary 2.11.17 tells us that the LHS must be isomorphic as a $G(\mathbb{Q}_p)$ representation to some copies of subrepresentations of $i_B^G(\chi)$. To show that $i_B^G(\chi) \simeq i_B^G(\chi^{ww_0})$, it therefore suffices to show that they are equal in the Grothendieck group. However, we claim that $[i_B^G(\chi)] \simeq [i_B^G(\chi^{ww_0})]$ in $K_0(G(\mathbb{Q}_p), \Lambda)$ for any $\Lambda \in \{\overline{\mathbb{Q}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{F}}_\ell\}$ and $w \in W_G$. With $\overline{\mathbb{Q}}_\ell$-coefficients, this is classical [Dij72a, Theorem 4]. It suffices to treat the case of $\overline{\mathbb{F}}_\ell$-coefficients. In this case, after choosing a lift $\tilde{\chi}$ of χ, we have an equality $[i_B^G(\tilde{\chi}) \otimes \overline{\mathbb{Q}}_\ell] = [i_B^G(\tilde{\chi}^{ww_0}) \otimes \overline{\mathbb{Q}}_\ell]$. So we can find $\overline{\mathbb{Z}}_\ell$-lattices in both representations such that this equality is also true in the Grothendieck group $K_0(G(\mathbb{Q}_p), \overline{\mathbb{Z}}_\ell)$. However, the semi-simplification mod ℓ doesn't depend on the choice of $\overline{\mathbb{Z}}_\ell$-lattice, by the strong Brauer-Nesbitt principle of Vignéras [Vig96, Section 2.5]. It follows that the equality $[i_B^G(\chi)] = [i_B^G(\chi^{ww_0})]$ holds in $K_0(G(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)$ as well. □

In particular, the refined averaging formula, together with the direct computation of provided above, gives rise to an isomorphism: $i_{\chi,w} : i_B^G(\chi) \simeq i_B^G(\chi^w)$. If $\Lambda = \overline{\mathbb{Q}}_\ell$ this recovers the following special case of Proposition A.1.3.

Corollary 2.11.23. *Suppose that $\chi : T(\mathbb{Q}_p) \to \overline{\mathbb{Q}}_\ell^*$ is a normalized regular character admitting a $\overline{\mathbb{Z}}_\ell$-lattice then we have isomorphisms $i_B^G(\chi) \simeq i_B^G(\chi^w)$ for all $w \in W_G$.*

We also have the following result which implies that the representations are in fact irreducible under the imposed conditions, reproving cases of Corollary A.1.4.

Corollary 2.11.24. *Let $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\Lambda)$ be an integral weakly normalized regular parameter with associated character χ. Suppose that ϕ_T is μ-regular for some μ which is not fixed under any element in the Weyl group and χ is regular then $i_B^G(\chi)$ is irreducible.*

Proof. First note that, by Corollary 2.11.22, we have an isomorphism $i_B^G(\chi) \simeq i_B^G(\chi^w) = i_{B^w}^G(\chi)$ for all $w \in W_G$. Here B^w is the conjugate of B by w. We write r_B^G for the normalized parabolic restriction functor. We recall that we are working with ℓ-modular coefficients in possibly non-banal characteristic so it is not automatic that all the constituents of $i_B^G(\chi)$ are not cuspidal. In particular, we will need the following lemma.

Lemma 2.11.25. *Let $w_0 \in W_G$ be the element of longest length for a character $\chi : T(\mathbb{Q}_p) \to \Lambda^*$. If we have an isomorphism $i_B^G(\chi) \simeq i_B^G(\chi^{w_0})$ of $G(\mathbb{Q}_p)$-modules then any non-zero quotient σ' of $i_B^G(\chi)$ satisfies $r_B^G(\sigma') \neq 0$*

Proof. We apply second adjointness [Dat+22, Corollary 1.3] to the map

$$i_{B^{w_0}}^G(\chi) \xrightarrow{\simeq} i_B^G(\chi) \to \sigma'$$

to conclude the existence of a non-zero map $\chi \to r_B^G(\sigma')$, which implies the claim. $\square$

Now suppose for the sake of contradiction that $i_B^G(\chi)$ is not irreducible. Then there exists an exact sequence

$$0 \to \sigma \to i_B^G(\chi) \to \sigma' \to 0$$

Since parabolic restriction is exact (for example by using second adjointness), we get an exact sequence

$$0 \to r_B^G(\sigma) \to r_B^G i_B^G(\chi) \to r_B^G(\sigma') \to 0.$$

From here, we conclude an equality of the length of representations.

$$\ell(r_B^G(\sigma)) + \ell(r_B^G(\sigma')) = \ell(r_B^G(i_B^G(\chi))) \leq |W_G|,$$

where the inequality follows from the geometric Lemma [Dat05, Section 2.8][9]. By the previous Lemma, we conclude that $\ell(r_B^G(\sigma)) < W_G$. Now, since we know that $\sigma \subset i_B^G(\chi) \simeq i_B^G(\chi^w)$ for all $w \in W_G$, Frobenius reciprocity implies that we have non-zero maps $r_B^G(\sigma) \to \chi^w$ for all $w \in W_G$. This gives a contradiction by the regularity of χ. $\square$

This suggests an interesting relationship between the theory of geometric Eisenstein series over the Fargues-Fontaine curve and the classical theory of intertwining operators and the Langlands quotient, which we hope is explored more in the future. This prospect becomes even more exciting in the ℓ-modular situation. The theory of intertwining operators in this context has been partially explored by Dat [Dat05, Sections 6-8], and the Langlands quotient theorem does not naively hold in this context, as the following example illustrates.

Example 2.11.26. Suppose that $\ell \neq 2$, $\ell \mid p-1$, and $G = \mathrm{GL}_2$. We let $\chi = \delta_B^{-1/2}$ be the modulus character. We note that, by our assumption that $\ell \mid p-1$, we have an isomorphism $|\cdot| \simeq \mathbf{1}_T$. It follows that we have that $\delta_B^{-1/2} \simeq \delta_B^{1/2}$. We consider the usual short exact sequence

$$0 \to \mathbf{1}_G \to i_B^G(\delta_B^{-1/2}) \to \mathrm{St}_G \to 0$$

where St_G denotes the Steinberg representation. Acting by smooth duality actually gives a splitting of the short exact sequence and in turn a chain of isomorphisms

$$i_B^G(\delta_B^{-1/2}) \simeq \mathrm{St}_G \oplus \mathbf{1}_G \simeq i_B^G(\delta_B^{1/2})$$

of smooth $G(\mathbb{Q}_p)$-representations. In particular, we see that $i_B^G(\delta_B^{1/2})$ does not have a unique irreducible quotient in this case, so that the Langlands quotient theorem cannot naively hold.

Let's now explore a case in which Proposition 2.11.20 can be used to verify Conjecture 2.11.18. Suppose that μ is a geometric dominant cocharacter such that the image $\mu_\Gamma \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma^+$ is quasi-minuscule or minuscule with respect to the pairing with $\mathbb{X}_*(\hat{T}^\Gamma)$. In this case, we recall that the orbit of the highest weight space $\mathscr{T}_\mu|_{\hat{G}^\Gamma}(b_\mu)$ forms a closed orbit under the relative Weyl group W_G, where $b_\mu \in B(G,\mu)_{\mathrm{un}}$ is the μ-ordinary element. It then follows that all the weight

[9] Note that this bound however fails without taking normalized restriction because of the aforementioned cuspidal constituents of $i_B^G(\chi)$ in non-banal characteristic [Dat05, Page 48]

spaces of $\mathcal{T}_\mu|_{\hat{G}^\Gamma}$ in this orbit will be given by the κ-invariants of $w(b_{\mu_T}) \in B(T) \simeq \mathbb{X}^*(\hat{T}^\Gamma)$, for $w \in W_G$ varying. However, by Proposition 2.11.20, we see that all the weight spaces of this form come from the contribution of the μ-ordinary element to the refined averaging formula. If μ_Γ is minuscule with respect to the above pairing this is the only weight space in $\mathcal{T}_\mu|_{\hat{G}^\Gamma}$, and we see that Conjecture 2.11.18 is true. If μ_Γ is quasi-minuscule, the only other element in $B(G,\mu)_{\text{un}}$ is the basic element, denoted $\mu^\flat$. By Corollary 2.2.9, the highest weight representation $\mathcal{T}_\mu|_{\hat{G}^\Gamma}$ admits a central weight space $\mathcal{T}_\mu|_{\hat{G}^\Gamma}(\mu_T^\flat)$ in this case, where $\mu_T^\flat \in B(T) \simeq \mathbb{X}^*(\hat{T}^\Gamma)$ is the (unique) reduction to T of $\mu^\flat \in B(G,\mu)_{\text{un}}$. We deduce the following from the refined averaging formula.

Corollary 2.11.27. *Let μ be a geometric dominant cocharacter such that ϕ_T is strongly μ-regular with reflex field E. Assume that μ_Γ is quasi-minuscule with respect to the pairing with $\mathbb{X}_*(\hat{T}^\Gamma)$. Let $\mu^\flat \in B(G,\mu)$ be the unique basic element. It follows by Corollary 2.2.9 that $\mu^\flat$ is unramified in this case. We let $\mu_T^\flat$ be its unique reduction to $B(T)$. There is an isomorphism*

$$R\Gamma_c(G,b,\mu^\flat)[i_{B_b}^{J_b}(\chi)] \simeq R\Gamma_c^\flat(G,b,\mu^\flat)[i_{B_b}^{J_b}(\chi)] \simeq \bigoplus_{\substack{\tilde{\mu}_T^\flat \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \\ \tilde{\mu}_{T\Gamma}^\flat = \mu_T^\flat}} \tilde{\mu}_T^\flat \circ \phi_T|_{W_{E'}} \otimes \mathcal{T}_\mu(\tilde{\mu}_T^\flat) \boxtimes i_B^G(\chi)$$

of complexes of $W_{E'} \times G(\mathbb{Q}_p)$-modules, where $E'|E$ denotes the splitting field of G.

Proof. The isomorphism of the $\flat$ isotypic part with the non-$\flat$ isotypic part follows from Proposition 2.11.15, noting that for b basic the shifts and twists do not occur. and The rest follows from combining Theorem 2.11.16 and Proposition 2.11.20. Noting that, by the strong μ-regularity assumption on ϕ_T, the Weil group action of the contribution of the highest weight to the refined averaging formula must be distinct from the Weil group action on the contribution of the central weight spaces, by the vanishing of the H^0s for the differences of these weight spaces. $\square$

Remark 2.11.28. If G is split then the condition that the central weight space of $\mathcal{T}_\mu|_{\hat{G}^\Gamma}$ is non-zero cannot occur if μ is minuscule. However, if G is not split then it can occur that the central weight space is non-trivial even if μ is minuscule. For example, if one considers an odd quasi-split unitary group U_n and the cocharacter $(1,0,\ldots,0,0)$ then the σ-centralizer will be isomorphic to U_n, and therefore the basic element $b \in B(G,\mu)$ lies in $B(G)_{\text{un}}$. A more in depth characterization of when this can occur is given in [XZ17, Remark 4.2.11]. This result is in some sense a generic fiber manifestation of some of the results in [XZ17]. Here, in the case that G is unramified, Xiao and Zhu relate the irreducible components of affine Deligne-Luztig varieties to the central weight spaces appearing above, and use these irreducible components to construct cohomological correspondences on the special fibers of certain Shimura varieties using uniformization. These affine Deligne-Luztig varieties are precisely the special fibers of a natural integral model of $\mathrm{Sht}(G,\mu^\flat,\mu)_\infty/\underline{K}$ for a choice of hyperspecial subgroup $K \subset G(\mathbb{Q}_p)$.

Chapter 3

Further Applications and Conjectures

In the first part of this chapter, we explain some of the ideas in joint work with Si Ying Lee [HL23], where we combine the results discussed in chapters 1 and 2 together with the geometry of the Hodge-Tate period morphism to deduce generalizations of the torsion vanishing results proven by Caraiani-Scholze, Koshikawa, and Santos [CS17; CS19; Kos21b; San23]. In the second section of this chapter, we explain a very general torsion vanishing Conjecture generalizing the torsion vanishing results discussed in the first section (Conjecture 3.2.19). Moreover, we explain how these conjectures are motivated by thinking about the analogue of the theory of geometric Eisenstein series described in chapter 2 in the non-principal case.

3.1 Applications to Torsion Vanishing

Let $\mathbf{G}$ be a connected reductive group over $\mathbb{Q}$ admitting a Shimura datum $(\mathbf{G}, X)$, and let $\mathbb{A}$ (resp. $\mathbb{A}_f$) denote the adeles (resp. finite adeles) of $\mathbb{Q}$. Fix a prime number $p > 0$ and let $G := \mathbf{G}_{\mathbb{Q}_p}$ be the base-change to $\mathbb{Q}_p$. We will assume that G is unramified so that there exists a hyperspecial subgroup $K_p := K_p^{\mathrm{hs}} \subset G(\mathbb{Q}_p)$ and a Borel B with maximal torus T, which we now fix. We consider the open compact subgroup $K := K^p K_p \subset \mathbf{G}(\mathbb{A}_f)$, where $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$ denotes a sufficiently small level away from p. Let $\mathrm{Sh}(\mathbf{G}, X)_{K^p K_p}$ denote the attached Shimura variety defined over the reflex field E. We take a prime $\ell \neq p$ and assume that ℓ is very good with respect to G, as in [FS21, Page 33] throughout. We will be interested in understanding the ℓ-torsion cohomology groups

$$R\Gamma_c(\mathrm{Sh}(\mathbf{G},X)_{K,\overline{E}},\overline{\mathbb{F}}_\ell)$$

and

$$R\Gamma(\mathrm{Sh}(\mathbf{G},X)_{K,\overline{E}},\overline{\mathbb{F}}_\ell).$$

In particular, since the level at p is hyperspecial, the unramified Hecke algebra

$$H_{K_p^{\mathrm{hs}}} := \overline{\mathbb{F}}_\ell[K_p^{\mathrm{hs}}\backslash G(\mathbb{Q}_p)/K_p^{\mathrm{hs}}]$$

will act on these complexes on the right. Given a maximal ideal $\mathfrak{m} \subset H_{K_p^{\mathrm{hs}}}$, we can localize both of these cohomology groups at $\mathfrak{m}$. We will be interested in describing this localization. To do this, we recall that, given such a maximal ideal $\mathfrak{m} \subset H_{K_p}$, this defines an unramified L-parameter

$$\phi_{\mathfrak{m}} : W_{\mathbb{Q}_p} \to {}^LG(\overline{\mathbb{F}}_\ell)$$

specified by the semisimple element $\phi_{\mathfrak{m}}(\mathrm{Frob}_{\mathbb{Q}_p})$. In particular, if T denotes the maximal torus of G then $\phi_{\mathfrak{m}}$ is induced from a parameter $\phi_{\mathfrak{m}}^T : W_{\mathbb{Q}_p} \to {}^LT(\overline{\mathbb{F}}_\ell) \subset {}^LG(\overline{\mathbb{F}}_\ell)$ factoring through the L-group of the maximal torus. Now, recall that the irreducible representations of ${}^LT$ correspond to the Γ-orbits $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$ of geometric dominant cocharacters of G, where $\Gamma := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is the absolute Galois group. We then have the following definition.

Definition 3.1.1. (Definition 2.3.8) Given a toral L-parameter $\phi_T : W_{\mathbb{Q}_p} \to {}^LT(\overline{\mathbb{F}}_\ell)$, we say that ϕ_T is generic if, for all $\alpha \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$ corresponding to a Γ-orbit of coroots, we have that the complex $R\Gamma(W_{\mathbb{Q}_p}, \alpha \circ \phi_T)$ is trivial. Similarly, we say that $\mathfrak{m}$ is generic if $\phi_{\mathfrak{m}}^T$ is a generic toral parameter.

If $G = \mathrm{GL}_n$ then this coincides with the notion of generic considered in [CS17, Definition I.9]. We set $d = \dim(\mathrm{Sh}(\mathbf{G},X)_K)$. Motivated by [CS17, Theorem 1.1] and [CS19, Theorem 1.1], we make the following conjecture.

Conjecture 3.1.2. *Let $(\mathbf{G},X)$ be a Shimura datum such that $G = \mathbf{G}_{\mathbb{Q}_p}$ is unramified and $K = K_pK^p$ with $K_p = K_p^{\mathrm{hs}}$ hyperspecial. If $\mathfrak{m} \subset H_{K_p}^{\mathrm{hs}}$ is a generic maximal ideal then the cohomology of $R\Gamma(\mathrm{Sh}(\mathbf{G},X)_{K,\overline{E}},\overline{\mathbb{F}}_\ell)_{\mathfrak{m}}$ (resp. $R\Gamma_c(\mathrm{Sh}(\mathbf{G},X)_{K,\overline{E}},\overline{\mathbb{F}}_\ell)_{\mathfrak{m}}$) is concentrated in degrees $d \leq i \leq 2d$ (resp. $0 \leq i \leq d$).*

We first recall the motivating situation of Caraiani-Scholze [CS17; CS19]. Let $F/\mathbb{Q}$ be a CM field and let $(B,*,V,(\cdot,\cdot))$ be a PEL datum with B a central simple F-algebra and V a non-zero finite type left B-module. Let $(\mathbf{G},X)$ denote the Shimura datum attached to it with reflex field $E \subset \mathbb{C}$, where $\mathbf{G}$ is a connected reductive group over $\mathbb{Q}$ defined by the B-linear automorphisms of V preserving the choice of pairing $(\cdot,\cdot)$. We have the following result.

Theorem 3.1.3. *[CS17; CS19; Kos21b; San23] Assume that $(\mathbf{G},X)$ is a PEL type Shimura datum of type A attached to a PEL datum as above. If the prime p splits completely in F then Conjecture 3.1.2 is true.*

Remark 3.1.4. Caraiani-Scholze and Koshikawa prove this under the assumption that $B = F$ and $V = F^{2n}$, and the global unitary group $\mathbf{G}$ is quasi-split or in the case when p is split in F and the Shimura variety is compact. These additional assumptions were removed in the PhD thesis of Santos [San23].

Remark 3.1.5. We believe that Conjecture 3.1.2 should be true under the weaker hypothesis that $H^2(W_{\mathbb{Q}_p}, \alpha \circ \phi_T)$ is trivial for all Γ-orbits of coroots α, as is shown in [CS17; CS19; Kos21b; San23] in situation of this Theorem. However, the theory of geometric Eisenstein series which we will use to prove these results becomes more complicated in this case (See the discussion around Conjecture 2.1.29); in particular, there should be a non-trivial difference between the sheaf $\mathrm{nEis}(\mathscr{S}_{\phi_T^{\mathfrak{m}}})$ and the true candidate for the eigensheaf $\widetilde{\mathrm{nEis}}(\mathscr{S}_{\phi_T^{\mathfrak{m}}})$ without the full generic assumption), and so a proof under these weaker hypothesis using our methods would require more deeply understanding geometric Eisenstein series when this assumption is dropped (at least in the case when G is not-split).

Caraiani-Scholze [CS17; CS19] proved this result under some small restrictions, which Koshikawa [Kos21b] was able to remove by exhibiting a much more flexible method for proving Theorem 3.1.3 using compatibility of the Fargues-Scholze correspondence with the semi-simplificaiton of the Harris-Taylor correspondence. In [HL23], we expand the scope of Koshikawa's technique motivated by the analysis in chapter 2. We then carry the strategy out in some particular cases using work on local-global compatibility of the Fargues-Scholze local Langlands correspondence as is shown in Theorem 1.1.2 for GSp_4 and [BHN22, Theorem 1.1] for GU_n and U_n. One of the basic ingredients we use is the perspective on Mantovan's product formula provided by the Hodge-Tate period morphism. To explain this, we let $\mu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ denote the minuscule geometric dominant cocharacter of G determined by the Hodge cocharacter of X and an isomorphism $j : \mathbb{C} \simeq \overline{\mathbb{Q}}_p$ which we fix from now on. We consider the Kottwitz set $B(G)$ and with it the subset $B(G,\mu) \subset B(G)$ of μ-admissible elements. Let $\mathfrak{p}|p$ be the prime dividing p in the reflex field E, associated to the embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ given by j. We let $E_{\mathfrak{p}}$ be the completion at $\mathfrak{p}$, $C := \hat{\overline{E}}_{\mathfrak{p}}$ be the completion of the algebraic closure, and $\breve{E}_{\mathfrak{p}}$ be the compositum of $E_{\mathfrak{p}}$ with the completion of the maximal unramified extension of $\mathbb{Q}_p$. We recall that, attached to each element $b \in B(G,\mu)$, there exists a diamond

$$\mathrm{Sht}(G,b,\mu)_\infty \to \mathrm{Spd}(\breve{E}_{\mathfrak{p}})$$

parametrizing modifications

$$\mathscr{E}_b \dashrightarrow \mathscr{E}_0$$

of meromorphy μ between the G-bundle $\mathscr{E}_b$ corresponding to b and the trivial G-bundle. The space has an action by $G(\mathbb{Q}_p) = \mathrm{Aut}(\mathscr{E}_0)$ and $J_b(\mathbb{Q}_p) \subset \mathrm{Aut}(\mathscr{E}_b)$, where J_b is the σ-centralizer of b. This allows us to consider the quotients

$$\mathrm{Sht}(G,b,\mu)_\infty/\underline{K_p} \to \mathrm{Spd}(\breve{E}_{\mathfrak{p}})$$

for varying compact open subgroups $K_p \subset G(\mathbb{Q}_p)$. We can consider the compactly supported cohomology

$$R\Gamma_c(\mathrm{Sht}(G,b,\mu)_{\infty,C}/\underline{K_p^{\mathrm{hs}}}, \overline{\mathbb{F}}_\ell)$$

at hyperspecial level with torsion coefficients, with is action by $W_{E_{\mathfrak{p}}} \times J_b(\mathbb{Q}_p) \times H_{K_p^{\mathrm{hs}}}$. Now, the Mantovan product formula tells us that if we look at the non-compactly supported cohomology $R\Gamma(\mathrm{Sh}(\mathbf{G},X)_{K,\overline{E}}, \overline{\mathbb{F}}_\ell)$ then this should always admit a filtration in the derived category whose graded pieces are

$$R\Gamma(\mathrm{Ig}^b, \overline{\mathbb{F}}_\ell) \otimes^{\mathbb{L}}_{\mathscr{H}(J_b)} R\Gamma_c(\mathrm{Sht}(G,b,\mu)_{\infty,C}/\underline{K_p^{\mathrm{hs}}}, \overline{\mathbb{F}}_\ell(d_b))[2d_b]$$

for varying $b \in B(G,\mu)$, where the objects are as follows.

1. Ig^b is the perfect Igusa variety attached to an element $b \in B(G,\mu)$ in the μ-admissible locus inside $B(G)$ and $d_b := \dim(\mathrm{Ig}^b) = \langle 2\rho_G, \nu_b \rangle$, where ρ_G is the half sum of all positive roots and ν_b is the slope homomorphism of b.

2. $\mathscr{H}(J_b) := C_c^\infty(J_b(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)$ is the usual smooth Hecke algebra.

3. $\overline{\mathbb{F}}_\ell(d_b)$ is the sheaf on $\mathrm{Sht}(G,b,\mu)_{\infty,C}/\underline{K_p^{\mathrm{hs}}}$ with trivial Weil group action and $J_b(\mathbb{Q}_p)$ action as defined in [Kos21b, Lemma 7.4].

Such a filtration should always exist, but isn't currently proven in general. In the case that the Shimura datum $(\mathbf{G}, X)$ is PEL of type A or C, a modern proof of this result can be found in [Kos21b, Theorem 7.1].

This filtration on the complex $R\Gamma_c(\mathrm{Sh}(\mathbf{G}, X)_{K,\overline{E}}, \overline{\mathbb{F}}_\ell)$ allows us to roughly split the verification of Conjecture 3.1.2 into two parts.

1. Controlling the cohomology of the shtuka spaces
$$R\Gamma_c(\mathrm{Sht}(G, b, \mu)_{\infty,C}/\underline{K_p^{\mathrm{hs}}}, \overline{\mathbb{F}}_\ell)_{\mathfrak{m}}.$$
2. Controlling the cohomology of the Igusa varieties $R\Gamma(\mathrm{Ig}^b, \overline{\mathbb{F}}_\ell)$.

We first discuss point (1). One of the key observations underlying Koshikawa's method is that the cohomology of the space $\mathrm{Sht}(G, b, \mu)_\infty$ computes the action of a Hecke operator T_μ corresponding to μ on Bun_G the moduli stack of G-bundles of the Fargues-Fontaine curve. The Hecke operators commute with the action of the excursion algebra on Bun_G, and the action of the excursion algebra on a smooth irreducible representation ρ, viewed as a sheaf on Bun_G, determines the Fargues-Scholze parameter of ρ. It follows that $R\Gamma_c(\mathrm{Sht}(G, b, \mu)_{\infty,C}/\underline{K_p^{\mathrm{hs}}}, \overline{\mathbb{F}}_\ell(d_b))_{\mathfrak{m}}$ as a complex of smooth $J_b(\mathbb{Q}_p)$-modules will have irreducible constituents ρ with Fargues-Scholze parameter ϕ_ρ^{FS} equal to $\phi_{\mathfrak{m}}$ as conjugacy classes of parameters. When $\mathbf{G}_{\mathbb{Q}_p} = G$ is a product of GL_ns as in Theorem 3.1.3, it follows from the work of Hansen-Kaletha-Weinstein that the Fargues-Scholze correspondence with rational coefficients agrees with the semi-simplification of the Harris-Taylor correspondence, where we recall that J_b is a product of inner forms of Levi subgroups if G is quasi-split. In particular, using that $\mathfrak{m}$ is generic, it follows that $\phi_\rho^{\mathrm{FS}} = \phi_{\mathfrak{m}}$ must lift to a $\overline{\mathbb{Z}}_\ell$ parameter which is also generic in the analogous sense, and the condition of generic implies that the lift cannot come from the semi-simplification of a parameter with non-trivial monodromy. Using this, one can deduce that such a non-zero ρ only exists if the group J_b is quasi-split, which can only happen if $b \in B(G, \mu)$ is the maximal μ-ordinary element. In this particular case (G is a product of GL_ns), this can only happen if $b \in B(G, \mu)$ is the μ-ordinary (maximal) element.

This argument of Koshikawa was the inspiration for the proof of Corollary 2.7.7. In particular, there it was noted that, for $\mathfrak{m}$ generic, the cohomology $R\Gamma_c(\mathrm{Sht}(G, b, \mu)_{\infty,C}/\underline{K_p^{\mathrm{hs}}}, \overline{\mathbb{F}}_\ell(d_b))_{\mathfrak{m}}$ should only be non-trivial if $b \in B(G, \mu)_{\mathrm{un}} := B(G)_{\mathrm{un}} \cap B(G, \mu)$, where $B(G)_{\mathrm{un}}$ is as defined in 2.2.1, as long as the Fargues-Scholze local Langlands correspondence has certain expected properties (as in

Assumption 2.7.5). These will be precisely the elements for which J_b is quasi-split with Borel B_b (Lemma 2.2.12).

We recall that the set $B(G,\mu)_{\mathrm{un}}$ corresponds to the Weyl group orbits of weights of the representation V_μ of $\hat{G}$ restricted to $\hat{G}^\Gamma$, by Corollary 2.2.9. In particular, since μ is minuscule, if G is split then $B(G,\mu)_{\mathrm{un}}$ consists of one element corresponding to the unique Weyl group orbit of the highest weight. Moreover, the contribution of the cohomology of this shtuka space is rather easy to compute (as in the proof of Proposition 2.11.20), and the problem reduces to controlling the cohomology of Ig^b when $b \in B(G,\mu)_{\mathrm{un}}$ is the μ-ordinary element. However, if G is not split then the restriction of V_μ may have multiple Weyl group orbits of weights, and one needs to control the cohomology groups

$$R\Gamma_c(\mathrm{Sht}(G,b,\mu)_{\infty,C}/\underline{K_p^{\mathrm{hs}}},\overline{\mathbb{F}}_\ell(d_b))_{\mathfrak{m}}$$

for a general $b \in B(G,\mu)_{\mathrm{un}}$. In particular, for non-split G, the basic element could be unramified, as discussed in Remark 2.11.28, and in this case we have that the attatched Igusa variety is just a profinite set (cf. Definition 1.4.1). Therefore, the problem of torsion vanishing requires completely controlling the generic part of the torsion cohomology of this basic local Shimura variety.

Such control of the cohomology of shtuka spaces with torsion coefficients was also attained in chapter 2. To formulate this properly, given a maximal ideal $\mathfrak{m} \subset H_{K_p^{\mathrm{hs}}}$ we construct in Appendix B.1 a full-subcategory $\mathrm{D}(\mathrm{Bun}_G,\overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}} \subset \mathrm{D}(\mathrm{Bun}_G,\overline{\mathbb{F}}_\ell)$ of the category of étale $\overline{\mathbb{F}}_\ell$-sheaves on Bun_G together with an idempotent localization map $(-)_{\phi_{\mathfrak{m}}} : \mathrm{D}(\mathrm{Bun}_G,\overline{\mathbb{F}}_\ell) \to \mathrm{D}(\mathrm{Bun}_G,\overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}}$ such that, on smooth irreducible representations, the localization map is either an isomorphism or 0 depending on if the representation has Fargues-Scholze parameter conjugate to $\phi_{\mathfrak{m}}$ or not (Lemma B.1.7 (1)). If $\mathfrak{m}$ is generic it follows, by the proof of Corollary 2.7.7, that, assuming 2.7.5 holds for G, an object $A \in \mathrm{D}(\mathrm{Bun}_G,\overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}}$ is only supported on the HN-strata indexed by $b \in B(G)_{\mathrm{un}}$, and its restriction to such a $b \in B(G)_{\mathrm{un}}$ will be valued in constituents of the representations of the form $\rho_{b,w} := i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{-1/2}$, for χ the character attached to ϕ_T by Artin reciprocity. Here we recall $w \in W_b := W_G/W_{M_b}$ ranges over a set of representatives of minimal length, where M_b is the centralizer of the slope homomorphism of b, B_b is the Borel of M_b transferred to J_b via the inner twisting, as in Lemma 2.2.12, and δ_{P_b} is the modulus character of M_b transferred to J_b. For the representations $\rho_{b,w}$, we know that ! and $*$ push-forwards are isomorphic by Proposition 2.11.12 and the generic assumption. We let $\mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_G,\overline{\mathbb{F}}_\ell)$ denote the full subcategory of ULA objects, where we recall by [FS21, Theorem V.7.1], that this is equivalent to insisting that the restrictions to all the HN-strata indexed by $b \in B(G)$ are valued in the full subcategories $\mathrm{D}^{\mathrm{adm}}(J_b(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)$ of admissible complexes (i.e the invariants under all open compacts $K \subset J_b(\mathbb{Q}_p)$ is a perfect complex). Moreover, we can show that the representations $\rho_{b,w}$ are semi-simple. In particular, by combining these facts, one can show that one has a direct sum decomposition

$$\mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}} \simeq \bigoplus_{b \in B(G)_{\mathrm{un}}} \mathrm{D}^{\mathrm{adm}}(J_b(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}}$$

given by the splitting of the excision filtration via the aforementioned equivalence of the ! and $*$-pushforwards. Now the key result that will be important for torsion vanishing is studying how Hecke operators interact with the perverse t-structure on Bun_G on this localized category $\mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}}$.

We recall that $\mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ has an action by Hecke operators. In particular, for each geometric dominant cocharacter μ, we have a correspondence

$$\begin{tikzcd} & \mathrm{Hck}_{G,\leq\mu} \arrow[dl, "h_\mu^{\leftarrow}"'] \arrow[dr, "h_\mu^{\rightarrow}"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times \mathrm{Spd}(C) \end{tikzcd}$$

where $\mathrm{Hck}_{G,\leq\mu}$ is the stack parametrizing modifications $\mathscr{E}_1 \to \mathscr{E}_2$ of a pair of G-bundles with meromorphy bounded by μ at the closed Cartier divisor defined by the fixed untilt over C, and $h_\mu^{\rightarrow}$ (resp. $h_\mu^{\leftarrow}$) remembers $\mathscr{E}_2$ (resp. $\mathscr{E}_1$). We define

$$T_\mu : \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell) \to \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)^{BW_{E_\mu}}$$

$$A \mapsto h^{\rightarrow}_{\mu*}(h_\mu^{\leftarrow*}(A) \otimes^{\mathbb{L}} \mathscr{S}_\mu)$$

where E_μ is the reflex field of μ and $\mathscr{S}_\mu$ is a sheaf on $\mathrm{Hck}_{G,\leq\mu}$ attached to the highest weight tilting module $\mathscr{T}_\mu$ of highest weight μ by geometric Satake. The action of Hecke operators commutes with the action of excursion operators and therefore the action of the spectral Bernstein center and preserves ULA objects. It follows that we have an induced map

$$T_\mu : \mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}} \to \mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}}^{BW_{E_\mu}}$$

on the localized category (See Lemma B.1.7 (2)).

We are almost ready to state the local result we will need. To do this, we recall that $\mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ has a natural perverse t-structure. In particular, Bun_G is cohomologically smooth of ℓ-dimension 0. Moreover, each one of the HN-strata Bun_G^b are isomorphic to $[*/\mathscr{J}_b]$, which is cohomologically smooth of ℓ-dimension $-\dim_\ell(\mathscr{J}_b) = -d_b = -\dim(\mathrm{Ig}^b)$. Therefore, we can define a perverse t-structure ${}^{\mathrm{p}}\mathrm{D}^{\geq 0}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ on $\mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ given by insisting that the ! (resp. $*$) restrictions to Bun_G^b are concentrated in degrees $\geq \langle 2\rho_G, \nu_b \rangle$ (resp. $\leq \langle 2\rho_G, \nu_b \rangle$). We let $\mathrm{Perv}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ be the heart of this t-structure, and, for $\mathfrak{m}$ a maximal ideal, let $\mathrm{Perv}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_{\phi_\mathfrak{m}} := \mathrm{Perv}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell) \cap \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_{\phi_\mathfrak{m}}$.

The key point is now, assuming that $\phi_T^{\mathfrak{m}}$ is weakly normalized regular (Definition 2.3.8) so in particular generic, and that we know something about the structure of the Fargues-Scholze local Langlands correspondence (Assumption 2.7.5), the filtered perverse tilting eigensheaf $\mathrm{nEis}(\mathscr{S}_{\phi_\mathfrak{m}^T})$ supplied by Theorem 2.10.10 has stalk at $b \in B(G)_{\mathrm{un}}$ given by a direct sum of the representations $\rho_{b,w}$, which will be semi-simple, as discussed above. Moreover, we saw that these representations give rise to all possible smooth irreducible representations occurring in $\mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_{\phi_\mathfrak{m}} \simeq \bigoplus_{b \in B(G)_{\mathrm{un}}} \mathrm{D}^{\mathrm{adm}}(J_b(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)_{\phi_\mathfrak{m}}$. Assuming the filtration on $T_\mu(\mathrm{nEis}(\mathscr{S}_{\phi_\mathfrak{m}^T}))$ splits (i.e when ϕ_T is μ-regular (Definition 2.1.15)), this will give us an isomorphism

$$T_\mu(\mathrm{nEis}(\mathscr{S}_{\phi_\mathfrak{m}^T})) \simeq \mathrm{nEis}(\mathscr{S}_{\phi_\mathfrak{m}^T}) \boxtimes r_\mu \circ \phi_\mathfrak{m} \in \mathrm{Perv}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_{\phi_\mathfrak{m}}^{BW_{E_\mu}}$$

where $r_\mu : \hat{G} \to \mathrm{GL}(\mathscr{T}_\mu)$ is the map attached to the tilting module $\mathscr{T}_\mu$. In particular, this allows us to see that, on this localized category, perverse sheaves are sent to perverse sheaves under Hecke operators. So, using local-global compatibiltiy results for the Fargues-Scholze correspondence as shown in Theorem 1.1.2, we can prove the following.

Theorem 3.1.6. *Let μ be a geometric dominant minuscule cocharacter and G a product of groups satisfying the conditions of Table (3.1) with p and ℓ satisfying the corresponding conditions. Then if $\mathfrak{m}$ is generic the restriction of the Hecke operator*

$$j_1^* T_\mu : \mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_{\phi_\mathfrak{m}} \to \mathrm{D}^{\mathrm{adm}}(G(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)_{\phi_\mathfrak{m}}^{BW_{E_\mu}}$$

is perverse t-exact. In particular, it induces maps

$$j_1^* T_\mu : {}^{\mathrm{p}}\mathrm{D}^{\mathrm{ULA},\geq 0}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_{\phi_\mathfrak{m}} \to \mathrm{D}^{\mathrm{adm},\geq 0}(G(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)_{\phi_\mathfrak{m}}^{BW_{E_\mu}}$$

and

$$j_1^* T_\mu : {}^{\mathrm{p}}\mathrm{D}^{\mathrm{ULA},\leq 0}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)_{\phi_\mathfrak{m}} \to \mathrm{D}^{\mathrm{adm},\leq 0}(G(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)_{\phi_\mathfrak{m}}^{BW_{E_\mu}}$$

on the halves of the perverse t-structure, where we note that the perverse t-structure on $\mathrm{D}(\mathrm{Bun}_G^1, \overline{\mathbb{F}}_\ell) \simeq \mathrm{D}(G(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)$ coincides with the usual t-structure.

Here is the table summarizing our local constraints:

G	Constraint on G	ℓ	p
$\mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GL}_n)$	$L/\mathbb{Q}_p$ unramified	$(\ell,[L:\mathbb{Q}_p])=1$	
$\mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GSp}_4)$	$L=\mathbb{Q}_p$	$(\ell,2(p^4-1))=1$	
	$L/\mathbb{Q}_p$ unramified	$(\ell,2[L:\mathbb{Q}_p](p^{4[L:\mathbb{Q}_p]}-1))=1$	$p\neq 2$
$\mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GU}_2)$	$L/\mathbb{Q}_p$ unramified	$(\ell,[L:\mathbb{Q}_p])=1$	
$G=\mathrm{U}_n(L/\mathbb{Q}_p)$	n odd L unramified	$\ell\neq 2$	
$G=\mathrm{GU}_n(L/\mathbb{Q}_p)$	n odd L unramified	$\ell\neq 2$	
$G(\mathrm{SL}_{2,L})$	$L/\mathbb{Q}_p$ unramified	$(\ell,[L:\mathbb{Q}_p])=1$	
$G(\mathrm{Sp}_{4,L})$	$L/\mathbb{Q}_p$ unramified, $L\neq\mathbb{Q}_p$	$(\ell,2[L:\mathbb{Q}_p](p^{4[L:\mathbb{Q}_p]}-1))=1$	$p\neq 2$

(3.1)

Here, the groups $G(\mathrm{SL}_{2,L})$ and $G(\mathrm{Sp}_{4,L})$ are the similitude subgroup of $\mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GL}_2)$ (resp. $\mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GSp}_4)$), i.e. the subgroup of elements such that the similitude factor lies in $\mathbb{Q}_p$.

Remark 3.1.7. As discussed above, assuming 2.7.5 holds for G, the results of chapter 2 allow one to verify this for any μ after imposing some additional conditions such as weak normalized regularity and μ-regularity on the toral parameter $\phi_T^{\mathfrak{m}}$ attached to the maximal ideal $\mathfrak{m}$. However, for the groups considered, we show that these additional conditions are superfluous and all one needs is generic, except for the case where $G=\mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GSp}_4)$ or $G=G(\mathrm{Sp}_{4,L})$ with $L/\mathbb{Q}_p$ nontrivial, where we need an extra banality assumption on the prime ℓ. It should be the case (Conjecture 2.1.29) that the results used to establish this theorem should always be true just under the condition that $\mathfrak{m}$ is generic. The rest of the assumptions come from the very good assumption on ℓ, and the need to work with good reduction Shimura varieties for some of the results describing the behavior of the Fargues-Scholze correspondence, as in Chapter 1.

We should also warn the reader that some of the results of chapter 2 and in particular this consequence, are currently contingent on showing that $j_!(\mathrm{IC}_{\mathrm{Bun}_B})$ is ULA with respect to the map $\overline{\mathfrak{q}}:\overline{\mathrm{Bun}}_B\to\mathrm{Bun}_T$ (Assumption 2.8.1).

These local torsion vanishing results would allow us to prove Conjecture 3.1.2 in several new cases if one could get control over the Igusa varieties Ig^b. In Koshikawa's argument, this is done by using a semi-perversity result proven by Caraiani-Scholze [CS19, Theorem 4.6.1], which was further generalized in work of Santos [San23]. Roughly speaking, we want to show that $R\Gamma(\mathrm{Ig}^b,\overline{\mathbb{F}}_\ell)$ is concentrated in degrees $\geq d_b$, so that the complex of $J_b(\mathbb{Q}_p)$-representations $R\Gamma(\mathrm{Ig}^b,\overline{\mathbb{F}}_\ell)$ defines the stalk of a semi-perverse sheaf on Bun_G at $b \in B(G)$, to which we can apply the previous result. In the case that the Shimura vareities $\mathrm{Sh}(\mathbf{G},X)_K$ are compact, there is a simpler way of seeing this. In particular, Ig^b is known to be a perfect affine scheme in this case, and so the desired semi-perversity just follows from Artin vanishing by using Poincaré duality on the global Shimura variety. It turns out that this style of argument can be made to work even in the non-compact case. In [CS17; CS19; Kos21b; San23], the non-compactly supported cohomology $R\Gamma(\mathrm{Sh}(\mathbf{G},X)_K,\overline{\mathbb{F}}_\ell)_{\mathfrak{m}}$ is studied together with its filtration involving $R\Gamma(\mathrm{Ig}^b,\overline{\mathbb{F}}_\ell)$ coming from Mantovan's formula, and shown to be concentrated in degrees $\geq d$. However, one could also study the compactly supported cohomology $R\Gamma_c(\mathrm{Sh}(\mathbf{G},X)_K,\overline{\mathbb{F}}_\ell)_{\mathfrak{m}}$ and show that it is concentrated in degrees $\leq d$, à la Poincaré duality. To do this, we recall [CS19, Section 3.3] that, in the non-compact case, the perfect scheme Ig^b is not affine, but it admits a partial minimal compactification $g_b : \mathrm{Ig}^b \hookrightarrow \mathrm{Ig}^{b,*}$ which is affine, as proven in the more general setting of PEL type A or C by Santos [San23]. We define

$$V_b := R\Gamma_{c-\partial}(\mathrm{Ig}^{b,*},\overline{\mathbb{F}}_\ell) := R\Gamma(\mathrm{Ig}^{b,*},g_{b!}(\overline{\mathbb{F}}_\ell))$$

the partially compactly supported cohomology, which is supported in degrees $\leq d_b$ by Artin-vanishing and the affineness of $\mathrm{Ig}^{b,*}$. Now, for $K \subset \mathbf{G}(\mathbb{A}_f)$ a sufficiently small open compact, we define $\mathscr{S}(\mathbf{G},X)_K := (\mathrm{Sh}(\mathbf{G},X)_K \otimes_E E_{\mathfrak{p}})^{\mathrm{ad}}$ to be the adic space over $\mathrm{Spa}(E_{\mathfrak{p}})$ attached to the Shimura variety. We can define the infinite level perfectoid Shimura varieties $\mathscr{S}(\mathbf{G},X)_{K^p}$ by taking the inverse limit of the finite level spaces $\mathscr{S}(\mathbf{G},X)_{K^pK_p}$ as $K_p \to \{1\}$. The base-change $\mathscr{S}(\mathbf{G},X)_{K^p,C}$ is representable by a perfectoid space if $(\mathbf{G},X)$ is of pre-abelian type, and in general it is diamond. By the results of [Sch15b; Han20], we have a Hodge-Tate period map

$$\pi_{\mathrm{HT}} : [\mathscr{S}(\mathbf{G},X)_{K^p,C}/\underline{G(\mathbb{Q}_p)}] \to [\mathscr{F}\ell_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}]$$

recording the Hodge-Tate filtration on the abelian varieties with additional structure that $\mathscr{S}(\mathbf{G},X)_{K^p,C}$ parametrizes. Here $\mathscr{F}\ell_{G,\mu^{-1}} := (G_C/P_{\mu^{-1}})^{\mathrm{ad}}$ is the adic flag variety attached to the parabolic in G_C given by a dominant inverse of μ. We recall that the flag variety $[\mathscr{F}\ell_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}]$ admits a locally closed stratification $i_b : [\mathscr{F}\ell^b_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}] \hookrightarrow [\mathscr{F}\ell_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}]$ indexed by $b \in B(G,\mu)$, given by pulling the HN-stratification along the natural map $h^{\leftarrow} : [\mathscr{F}\ell_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}] \to \mathrm{Bun}_G$. We will now impose the following very mild assumption in what follows, which we need to apply Hartogs' principle in our proof.

Assumption 3.1.8. *Write* $\partial \mathrm{Ig}^{b,*} \subset \mathrm{Ig}^{b,*}$ *for the closed complement of* Ig^{b} *in* $\mathrm{Ig}^{b,*}$. *We assume that* $(\mathbf{G},X)$ *is a PEL datum of type A or C such that, for all* $b \in B(G,\mu)$, *the perfect scheme* $\partial \mathrm{Ig}^{b,*}$ *is empty or has codimension in* $\mathrm{Ig}^{b,*}$ *greater than* 2.

Remark 3.1.9. If $\mathbf{G}$ is simple then it is easy to show that this assumption will be satisfied if $\dim(\mathscr{S}(\mathbf{G},X)_{K^p,C}) \geq 2$, by using that the boundary of the partially minimally compactified Igusa varieties is expressible as the Igusa varieties of Shimura varieties attached to Levis of $\mathbb{Q}$-rational parabolics of $\mathbf{G}$. Moreover, if $\mathscr{S}(\mathbf{G},X)_{K^p,C}$ is compact then it is automatic that $\partial \mathrm{Ig}^{b,*}$ is empty. Therefore, if $\mathbf{G}$ is simple, this is excluding the cases where $\dim(\mathscr{S}(\mathbf{G},X)_{K^p,C}) = 1$ and $\mathscr{S}(\mathbf{G},X)_{K^p,C}$ is non-compact. There are two possibilities; either $(\mathbf{G},X)$ is the Shimura datum attached to the modular curve or it is the Shimura datum attached to the unitary Shimura curve. In the latter case, we have that the connected components are given by modular curves. In these cases, the results of [CS19] are sufficient to prove Conjecture 3.1.2.

Now one can show that the stalk of $R\pi_{\mathrm{HT}!}(\overline{\mathbb{F}}_\ell)$ at a geometric point $x : \mathrm{Spa}(C,C^+) \to \mathscr{F}\ell_{G,\mu^{-1}}$ which lies in the adic Newton strata $\mathscr{F}\ell^{b}_{G,\mu^{-1}}$ is given by V_b. Moreover, if we write $h_b^{\leftarrow} : [\mathscr{F}\ell^{b}_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}] \to [\mathrm{Spd}(C)/\mathscr{J}_b] \simeq \mathrm{Bun}_G^b$ for the pullback of $h^{\leftarrow}$ to Bun_G^b then one can deduce that the complex $i_{b!}i_b^*R\pi_{\mathrm{HT}!}(\overline{\mathbb{F}}_\ell)$ is isomorphic $h^{\leftarrow *}j_{b!}(V_b)$. Therefore, by excision, we deduce that the complex of $G(\mathbb{Q}_p) \times W_{E_{\mathfrak{p}}}$-representations

$$h_*^{\rightarrow}R\pi_{\mathrm{HT}!}(\overline{\mathbb{F}}_\ell) \simeq R\Gamma_c(\mathscr{S}_{K^p,C},\overline{\mathbb{F}}_\ell) \simeq \mathrm{colim}_{K_p \to \{1\}} R\Gamma_c(\mathscr{S}_{K^pK_p,C},\overline{\mathbb{F}}_\ell) \simeq \mathrm{colim}_{K_p \to \{1\}} R\Gamma_c(\mathrm{Sh}(\mathbf{G},X)_{K^pK_p,C})$$

has a filtration with graded pieces isomorphic to $h_*^{\rightarrow}h^{\leftarrow *}(j_{b!}(V_b))$ for varying $b \in B(G,\mu)$, where $h^{\rightarrow} : [\mathscr{F}\ell_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}] \to [\mathrm{Spd}(C)/\underline{G(\mathbb{Q}_p)}]$. Here the second isomorphism follows since compactly supported cohomology respects taking limits of space, and the third isomorphism is a standard comparison result due to Huber [Hub96, Theorem 3.5.13].

Now, via the Bialynicki-Birula isomorphism and Beauville-Laszlo gluing, the flag variety $[\mathscr{F}\ell_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}]$ identifies with an open substack of $\mathrm{Hck}_{G,\leq\mu}$ for the fixed minuscule μ. In particular, under this relationship the maps $h_{\mu}^{\rightarrow}$ and $h_{\mu}^{\leftarrow}$ identify with $h^{\rightarrow}$ and $h^{\leftarrow}$, and therefore we can relate the graded pieces of the excision filtration to Hecke operators. We write

$$R\Gamma_c(G,b,\mu) := \mathrm{colim}_{K_p \to \{1\}} R\Gamma_c(\mathrm{Sht}(G,b,\mu)/\underline{K_p},\overline{\mathbb{F}}_\ell(d_b))$$

for the complex of $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_{E_{\mathfrak{p}}}$-modules defined by the cohomology at infinite level, and deduce the following variant of the Mantovan product formula for the compactly supported cohomology.

Theorem 3.1.10. *The complex $R\Gamma_c(\mathscr{S}_{K^p,C},\overline{\mathbb{F}}_\ell)$ has a filtration as a complex of $G(\mathbb{Q}_p)\times W_{E_\mathfrak{p}}$-representations with graded pieces isomorphic to $j_1^*(T_\mu j_{b!}(V_b))[-d](-\frac{d}{2})$. More specifically, the graded pieces are isomorphic to*

$$(R\Gamma_c(G,b,\mu)\otimes^{\mathbb{L}}_{\mathscr{H}(J_b)}V_b)[2d_b].$$

as $G(\mathbb{Q}_p)\times W_{E_\mathfrak{p}}$-modules.

We now apply our localization functor $(-)_{\phi_\mathfrak{m}}:\mathrm{D}(\mathrm{Bun}_G)\to\mathrm{D}(\mathrm{Bun}_G)_{\phi_\mathfrak{m}}$ to $R\Gamma_c(\mathscr{S}_{K^p,C},\overline{\mathbb{F}}_\ell)$ viewed as a sheaf on Bun_G by ! extending along the neutral strata. After applying $R\Gamma(K_p^{\mathrm{hs}},-)$, this agrees with $R\Gamma_c(\mathscr{S}_{K^pK_p^{\mathrm{hs}},C},\overline{\mathbb{F}}_\ell)_\mathfrak{m}$, the usual localization under the unramified Hecke algebra (Lemma B.1.7 (3)), which is the object we want to study. This in turn admits a filtration by $R\Gamma(K_p^{\mathrm{hs}},(j_1^*T_\mu j_{b!}(V_b)))_{\phi_\mathfrak{m}}[-d](\frac{-d}{2})$. However, now we know by the above, that the natural map $j_{b!}(V_b)\to j_{b*}(V_b)$ is an isomorphism after applying $(-)_{\phi_\mathfrak{m}}$. Moreover, it can only have interesting contributions coming from the unramified elements $B(G,\mu)_{\mathrm{un}}$. In particular, we can deduce the following Corollary.

Theorem 3.1.11. *Suppose $(\mathbf{G},X)$ is a PEL datum of type A or C such that $\mathbf{G}_{\mathbb{Q}_p}$ is a product of simple groups as in Table (3.1) with p and ℓ satisfying the corresponding conditions, the complex $R\Gamma_c(\mathscr{S}_{K,C},\overline{\mathbb{F}}_\ell)_\mathfrak{m}\simeq R\Gamma_c(\mathrm{Sh}(\mathbf{G},X)_{K^p,C},\overline{\mathbb{F}}_\ell)_\mathfrak{m}$ breaks up as a direct sum*

$$\bigoplus_{b\in B(G,\mu)_{\mathrm{un}}} R\Gamma_c(\mathrm{Sht}(G,b,\mu)_{\infty,C}/\underline{K_p^{\mathrm{hs}}},\overline{\mathbb{F}}_\ell(d_b))_\mathfrak{m}\otimes^{\mathbb{L}}_{\mathscr{H}(J_b)}R\Gamma_{c-\partial}(\mathrm{Ig}^{b,*},\overline{\mathbb{F}}_\ell)$$

of $H_{K_p^{\mathrm{hs}}}\times W_{E_\mathfrak{p}}$-modules.

As a consequence, we immediately deduce our main Theorem, by combining Theorem 3.1.6 with the fact that $R\Gamma_{c-\partial}(\mathrm{Ig}^{b,*},\overline{\mathbb{F}}_\ell)\in\mathrm{D}^{\leq d_b}(J_b(\mathbb{Q}_p),\overline{\mathbb{F}}_\ell)$, by Artin vanishing.

Theorem 3.1.12. *Suppose $(\mathbf{G},X)$ is a PEL datum of type A or C such that $\mathbf{G}_{\mathbb{Q}_p}$ is a product of simple groups as in Table (3.1) with p and ℓ satisfying the corresponding conditions then Conjecture 3.1.2 is true.*

As we will now explain more, in the case that the unique basic element $b\in B(G,\mu)_{\mathrm{un}}$ is unramified the contribution of the corresponding summand in 3.1.11 to middle degree cohomology should serve as a generic fiber analogue of the description of the middle degree cohomology on the special fiber of the integral model at hyperspecial level, as provided in [XZ17, Theorem 1.1.4].

3.2 Conjectures and Concluding Remarks

3.2.1 Relationship to Xiao-Zhu

Assume that the basic element $b \in B(G,\mu)_{\mathrm{un}}$ is unramified (See [XZ17, Remark 4.2.11] for a classification). Let us look at the middle degree cohomology $H^d(R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p,C},\overline{\mathbb{F}}_\ell)_{\phi_{\mathfrak{m}}})$. By Theorem 3.1.11, it has a summand isomorphic to

$$H^d(R\Gamma_c(G,b,\mu)\otimes^{\mathbb{L}}_{\mathscr{H}(J_b)} R\Gamma_{c-\partial}(\mathrm{Ig}^b,\overline{\mathbb{F}}_\ell)).$$

To describe this, let $\mathbf{G}'$ be the unique $\mathbb{Q}$-inner form of $\mathbf{G}$ such that $\mathbf{G}(\mathbb{A}^{p\infty}) \simeq \mathbf{G}'(\mathbb{A}^{p\infty})$, $\mathbf{G}'(\mathbb{R})$ is compact modulo center, and $\mathbf{G}_{\mathbb{Q}_p} \simeq J_b$ (See [Han20, Proposition 3.1] for the existence). We write $C(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p,\overline{\mathbb{F}}_\ell)$ for the set of all continuous functions on the profinite set $\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p$ It is easy to show that one has an isomorphism

$$C(K^p\backslash\mathbf{G}'(\mathbb{A}_f)/\mathbf{G}'(\mathbb{Q}),\overline{\mathbb{F}}_\ell) \simeq R\Gamma_{c-\partial}(\mathrm{Ig}^{b,*},\overline{\mathbb{F}}_\ell)$$

for example using basic uniformization of PEL type Shimura varieties (See [Han20, Theorem 3.4]), as described in Definition 1.4.1. Here the LHS means compactly supported smooth functions. We let $V_\mu \in \mathrm{Rep}_{\overline{\mathbb{F}}_\ell}(\hat{G})$ be the highest weight representation of highest weight μ, where we recall that this agrees with the highest weight tilting module since μ is minuscule by Proposition A.2.1. Let b_T denote the unique (since b is basic) reduction of $b \in B(G)$ to $B(T)$, and regard it as an element in $B(T) \simeq \mathbb{X}^*(\hat{T}^\Gamma)$. It should be the case that, under possible additional constraints on $\mathfrak{m}$ depending on μ (See for example Conjecture 2.11.18 and [XZ17, Definition 1.4.2]), we have an isomorphism between

$$C(K^p\backslash\mathbf{G}'(\mathbb{A}_f)/\mathbf{G}'(\mathbb{Q}),\overline{\mathbb{F}}_\ell)\otimes^{\mathbb{L}}_{\mathscr{H}(J_b)} R\Gamma_c(\mathrm{Sht}(G,b,\mu)_{\infty,C}/\underline{K_p^{\mathrm{hs}}},\overline{\mathbb{F}}_\ell)_{\mathfrak{m}}$$

and

$$C(K^p\backslash\mathbf{G}'(\mathbb{A}_f)/\mathbf{G}'(\mathbb{Q}),\overline{\mathbb{F}}_\ell)_{\mathfrak{m}}\otimes V_\mu|_{\hat{G}^\Gamma}(b_T)[-d](-\frac{d}{2})$$

of $G(\mathbb{Q}_p)$-representations, where we note that $J_b \simeq G$ if $b \in B(G,\mu)_{\mathrm{un}}$ since b is basic and J_b must be quasi-split since it is unramified[1]. This would follow from Conjecture 2.11.18. In particular, by arguing as in Koshikawa [Kos21b, Page 6], we know that $R\Gamma_c(\mathrm{Sht}(G,b,\mu)_{\infty,C}/\underline{K_p^{\mathrm{hs}}},\overline{\mathbb{F}}_\ell)_{\mathfrak{m}}$ will have irreducible constituents given by the representations of $J_b(\mathbb{Q}_p)$ with Fargues-Scholze parameter equal $\phi_{\mathfrak{m}}$

[1] One should also be able to describe the Weil group action, as in Conjecture 2.11.18.

as conjugacy classes of parameters. Moreover, using that Assumption 2.7.5 holds for the groups appearing in Table 3.1, we know that they have to be constituents of the $i_B^G(\chi)$, which will be irreducible by Corollary 2.11.24 under the generic assumption and the constraints appearing in Table 3.1. Then Conjecture 2.11.18 would imply that $R\Gamma_c(G,b,\mu)[i_B^G(\chi)] \simeq i_B^G(\chi) \otimes V_\mu|_{\hat{G}^\Gamma}(b_T)[-d](\frac{-d}{2})$ as $G(\mathbb{Q}_p)$-modules. Assume ℓ is banal (i.e coprime to the pro-order of K_p^{hs}) then passing to K_p^{hs}-invariants, recalling that it is exact under the banal hypothesis, gives us the claimed description of the generic part of the basic locus.

Remark 3.2.1. If $B(G,\mu)_{\mathrm{un}}$ consists of only the basic element and the μ-ordinary element, and $\phi_{\mathfrak{m}}^T$ is strongly μ-regular (Definition 2.3.14), then Conjecture 2.11.18 holds, by Corollary 2.11.27. In particular, the description of the generic part of the cohomology of the basic locus claimed above can be made unconditional.

We note that this description of the middle degree cohomology on the rigid generic fiber of the Shimura variety at hyperspecial level parallels Theorem [XZ17, Theorem 1.14 (1)], describing the middle degree cohomology on the special fiber of the natural integral model[2].

3.2.2 Non-Principal Geometric Eisenstein Series and a General Torsion Vanishing Conjecture

In this section, we would like to formulate some conjectures on the behavior of non-principal geometric Eisenstein series. These conjectures will lead to a natural generalization of Conjecture 3.2.19 that goes far beyond the scope of just describing the generic localization of the torsion cohomology at hyperspecial level. We fix $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$ a coefficient system. If $\Lambda \in \{\overline{\mathbb{Z}}_\ell, \overline{\mathbb{F}}_\ell\}$ we assume that ℓ is very good with respect to $G/\mathbb{Q}_p$ a fixed connected reductive quasi-split group G. We assume that the derived group of G is simply connected, as in chapter 2, and fix a choice $T \subset B \subset G$ of maximal torus and Borel in G.

We let $P \subset G$ be a parabolic subgroup which is standard with respect to the

[2]For this comparison it would have been more natural to consider an analogue of Theorem 3.1.11 with $\overline{\mathbb{Q}}_\ell$-coefficients. This is indeed doable assuming that $\phi_{\mathfrak{m}}$ admits a $\overline{\mathbb{Z}}_\ell$-lattice as in the statement of Theorem 2.10.10. This integrality condition is however an artifiact of the theory of solid $\overline{\mathbb{Q}}_\ell$-sheaves not being properly understood (e.g excision fails) and should be removable with more technology.

choice of Borel, and denote its Levi factor by M. We look at the diagram

$$\begin{tikzcd} \mathrm{Bun}_P \arrow[r, "\mathfrak{p}_P"] \arrow[d, "\mathfrak{q}_P"] & \mathrm{Bun}_G \\ \mathrm{Bun}_M & \end{tikzcd} \tag{3.2}$$

of v-stacks. We would now like to use this to define a geometric Eisenstein functor that parabolically induces eigensheaves on Bun_M with eigenvalue give by a supercuspidal L-parameter $\phi_M : W_{\mathbb{Q}_p} \to {}^LM(\Lambda)$ to eigensheaves on Bun_G with eigenvalue $\phi : W_{\mathbb{Q}_p} \xrightarrow{\phi_M} {}^LM(\Lambda) \to {}^LG(\Lambda)$. First, let us review what the structure of the eigensheaf $\mathscr{S}_{\phi_M}$ should be for such a supercuspidal L-parameter ϕ_M. When $\Lambda = \overline{\mathbb{Q}}_\ell$, the eigensheaf $\mathscr{S}_{\phi_M}$ should have the form specified by Fargues' Conjecture [Far16, Conjecture 4.4]; namely, the stalks at all the basic elements should be given by the L-packets specified by Kaletha's refined local Langlands correspondence [Kal16]. However, we also expect that some version of this should also be true with ℓ-modular coefficients.

Conjecture 3.2.2. *Let $\phi_M : W_{\mathbb{Q}_p} \to {}^LM(\Lambda)$ a supercuspidal L-parameter. For all $\nu \in B(M)_{basic}$, there should exist a finite set of smooth irreducible representations $\Pi_{\phi_M}(J_\nu)$ only depending on the isomorphism class of J_ν as a connected reductive group such that*

$$\bigoplus_{\nu \in B(M)_{basic}} \bigoplus_{\pi \in \Pi_{\phi_M}(J_\nu)} j_{\nu !}(\pi)$$

is an eigensheaf with eigenvalue ϕ_M.

Remark 3.2.3. It is easy to see that such an eigensheaf must be supported on the basic locus, using that the parameter is supercuspidal and that the σ-centralizers at the non-basic elements are inner forms of *proper* Levi subgroups of G (cf. the discussion proceeding 1.3.21). As mentioned above, with $\overline{\mathbb{Q}}_\ell$-coefficients this is essentially a slightly weaker form of Fargues' conjecture [Far16, Conjecture 4.4], where the key difference is that we don't insist that the eigensheaf property respect the action of the centralizer group S_ϕ, as described in chapter 1 for $G = \mathrm{GSp}_4$ in §1.2. In particular, this weaker form of Fargues' Conjecture for $M = \mathrm{GSp}_4$ follows from generalizing the analysis used in the proof of Theorem 1.8.2, as mentioned in Remark 1.8.4, and should (more or less) follow from showing compatibility of the Fargues-Scholze local Langlands with the refined local Langlands of Kaletha.

Pinning down the S_ϕ-action required for the full form Fargues' conjecture concretely translates into determining which of the two possibilities for the statement of Theorem 1.8.2 occurs, and this should be doable through more refined trace formula analysis. In the case of odd unramified unitary groups, this was carried out by Bertoloni-Meli–Nguyen [MN21], and in turn we have in joint work [BHN22] been able to show the full form of Fargues' Conjecture in this case. For GL_n, this was carried out by Anschütz and Le Bras [AL21a], and for torii with general coefficient systems by Zou [Zou22].

The general case of this conjecture with $\overline{\mathbb{F}}_\ell$-coefficients is essentially completely unknown aside from the case of torii. In particular, the definition of the packets $\Pi_{\phi_M}(J_\nu)$ are much more mysterious, but for general linear groups it will be given by the correspondence of Vigneras [Vig01][3]. By combining the spectral action with the calculations of Dat, specifying how these representations contribute to the Lubin-Tate tower [Dat12] with modular coefficients, one should be able to verify this conjecture in this particular case (cf. the analysis explained in Remark 1.8.4 and [BHN22, Section 4]). However, for general reductive groups, this is currently very mysterious.

We would now like to define a geometric Eisenstein functor which will parabolically induce the eigensheaves described in the above conjecture. Unfortunately, due to the formalism of ℓ-adic sheaves not being as well-behaved when $\Lambda \in \{\overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$, for now we can only really do this properly with $\overline{\mathbb{F}}_\ell$-coefficients, and then use analysis similar to §2.10 to get statements for the other coefficient systems. Here, using [FS21, Proposition VII.6.6], we have an identification $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell) \simeq \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ with the category of étale $\overline{\mathbb{F}}_\ell$-sheaves studied in [Sch18]. We define a functor

$$\mathrm{nEis}_P(-) : \mathrm{D}(\mathrm{Bun}_M, \overline{\mathbb{F}}_\ell) \to \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$$

$$A \mapsto \mathfrak{p}_{P!}(\mathfrak{q}_P^*(A) \otimes^{\mathbb{L}} \mathrm{IC}_{\mathrm{Bun}_P}).$$

Here $\mathrm{IC}_{\mathrm{Bun}_P} := \mathfrak{q}^{\dagger *}(\Delta_P^{1/2})$, where $\mathfrak{q}^\dagger : \mathrm{Bun}_P \to \mathrm{Bun}_M \to \mathrm{Bun}_{M^{\mathrm{ab}}} := \bigsqcup_{\nu \in \mathbb{X}_*(M^{ab}_{\overline{\mathbb{Q}}_p})_\Gamma} [*/\underline{M^{ab}(\mathbb{Q}_p)}]$ denotes the natural map to Bun of the abelianization M^{ab} of M, and $\Delta_P^{1/2} := \bigoplus_{\nu \in \mathbb{X}_*(M^{ab}_{\overline{\mathbb{Q}}_p})} j_{\nu !}(\delta_P^{1/2})[\langle 2\rho_G, \nu \rangle]$ is the sheaf defined by the modulus character $\delta_P^{1/2}$ viewed as a character on $M^{ab}(\mathbb{Q}_p)$. We note

[3] This can be shown by using the compatibility of both Fargues-Scholze and Vigneras construction with mod ℓ-reduction, and the semi-simplification of the Harris-Taylor correspondence.

that, since the derived group of G is simply connected, we have an isomorphism $\mathbb{X}_*(Z(\hat{M})^\Gamma) \simeq B(M)_{basic} \xrightarrow{\simeq} B(M^{ab}) \simeq \mathbb{X}_*(M^{ab}_{\overline{\mathbb{Q}}_p})_\Gamma$, using [Kot97a, Lemma 5.13] for the second isomorphism. Therefore, the natural map $\mathfrak{q} : \mathrm{Bun}_M \to \mathrm{Bun}_{M^{ab}}$ induces an isomorphism of connected components. Moreover, by [Ham21a, Proposition 3.16] and its proof, we know that the preimage Bun_P^ν of the connected component Bun_M^ν indexed by ν under $\mathfrak{q}$ is cohomologically smooth of ℓ-dimension $\langle 2\rho_G, \nu\rangle$, and the induced morphism $\mathfrak{q}^\nu : \mathrm{Bun}_P^\nu \to \mathrm{Bun}_M^\nu$ is open with connected fibers. In particular, as in chapter 2, it follows that we have a decomposition into connected components

$$\mathrm{Bun}_P = \bigsqcup_{\nu \in \mathbb{X}_*(M^{ab}_{\overline{\mathbb{Q}}_p})_\Gamma} \mathrm{Bun}_P^\nu,$$

and the connected component of Bun_P indexed by ν is cohomologically smooth of dimension $\langle 2\rho_G, \nu\rangle$. This motivates the following generalization of Corollary 2.6.2, which should follow through similar methods.

Conjecture 3.2.4. *The sheaf* $\mathrm{IC}_{\mathrm{Bun}_P}$ *is Verdier self-dual on* Bun_P.

We assume the existence of the eigensheaf $\mathscr{S}_{\phi_M}$ with eigenvalue ϕ_M as in Conjecture 3.2.2, and look at the sheaf

$$\mathrm{nEis}(\mathscr{S}_{\phi_M}) \in \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell).$$

We now ask the question: "When is this a Hecke eigensheaf?". As in chapter 2, we cannot expect this to always be true. In particular, we need to impose some condition generalizing the generic condition (Definition 2.3.8) in chapter 2 to a general Levi M.

To do this, we need to introduce some notation. As noted above, we have an isomorphism $\mathbb{X}_*(M^{ab}_{\overline{\mathbb{Q}}_p})_\Gamma \simeq \mathbb{X}^*(Z(\hat{M}^\Gamma))$. We denote this lattice by $\Lambda_{G,P}$ in what follows. If we let $\widetilde{\mathscr{J}}$ (resp. $\widetilde{\mathscr{J}_M}$) denote the set of vertices of the absolute Dynkin diagram of G (resp. M) then the absolute simple coroots $\tilde{\alpha}_i$ for $i \in \tilde{\mathscr{J}} \setminus \tilde{\mathscr{J}}_M$ define elements of $\mathbb{X}_*(M^{ab}_{\overline{\mathbb{Q}}_p})$, and we let $\Lambda_{G,P}^{\mathrm{pos}}$ denote the positive span of the image of these coroots in the coinvariant lattice, as in Definition 2.2.8. Moreover, the absolute Galois group Γ will permute the simple coroots indexed by $\tilde{\mathscr{J}}$ and $\tilde{\mathscr{J}}_M$. We write $\mathscr{J}_{G,P}$ for the orbits of $\tilde{\mathscr{J}} \setminus \tilde{\mathscr{J}}_M$ under this map, and, for $i \in \mathscr{J}_{G,P}$, write $\alpha_i \in \mathbb{X}_*(M^{ab}_{\overline{\mathbb{Q}}_p})_\Gamma$ for the element in the coinvariant lattice that this element maps to. We let N be the unipotent radical of the standard parabolic P, and consider the representation V_{ad}^{N} given by looking at the action of ${}^{L}M$ on the Lie algebra of ${}^{L}N$ via the adjoint action. We write $r_{\mathrm{ad}}^{N}: {}^{L}M \to \mathrm{GL}(V_{\mathrm{ad}})$ for the associated map. Consider $\theta \in \Lambda_{G,P} \simeq \mathbb{X}^*(Z(\hat{M}^{\Gamma}))$, and write $V_{\mathrm{ad}}^{N,\theta}$ for the maximal subrepresentation of V_{ad}^{N} whose restriction to $Z(\hat{M}^{\Gamma})$ is given by the character θ. We let $r_{\mathrm{ad}}^{N,\theta}: {}^{L}M \to \mathrm{GL}(V_{\mathrm{ad}}^{N,\theta})$ denote the corresponding map.

We now come to our key definition.

Definition 3.2.5. We say a supercuspidal L-parameter $\phi_M: W_{\mathbb{Q}_p} \to {}^{L}M(\Lambda)$ is of Langlands-Shahidi type if, for all $\theta \in \Lambda_{G,P}$[4], the Galois cohomology groups

$$R\Gamma(W_{\mathbb{Q}_p}, r_{\mathrm{ad}}^{N,\theta} \circ \phi_M) \simeq 0$$

and

$$R\Gamma(W_{\mathbb{Q}_p}, r_{\mathrm{ad}}^{N,\theta} \circ \phi_M^{\vee}) \simeq 0$$

are trivial for all θ.

Remark 3.2.6. We note, since we enforced this condition on both $r_{\mathrm{ad}}^{N,\theta} \circ \phi_M$ and $r_{\mathrm{ad}}^{N,\theta} \circ \phi_M^{\vee}$, that this is independent of the choice of parabolic P.

Remark 3.2.7. In the case that $M = T$, we claim that this recovers the condition of generic given by Definition 2.3.8. In particular, we consider the natural map $(-)_{\Gamma}: \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma \to \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_{\Gamma}$ from Γ-orbits of geometric dominant cocharacters to the coinvariants. Then, for $\theta \in \Lambda_{G,B}^{\mathrm{pos}}$, one has an isomorphism

$$r_{\mathrm{ad}}^{N,\theta} \circ \phi_T \simeq \bigoplus_{\substack{\tilde{\theta} \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \backslash \Gamma \\ \tilde{\theta}_{\Gamma} = \theta}} \tilde{\theta} \circ \phi_T \otimes V_{\mathrm{ad}}^{N}(\tilde{\theta})$$

of $W_{\mathbb{Q}_p}$-representations (cf. Lemma 2.2.8), where $V_{\mathrm{ad}}^{N}(\tilde{\theta})$ denotes the multiplicity of the representation of ${}^{L}T$ corresponding to $\tilde{\theta}$ in V_{ad}^{N}. Using this, we see that the condition is equivalent to generic, as desired.

The terminology of "Langlands-Shahidi type" comes from the fact that the representation $r_{\mathrm{ad}}^{N} \circ \phi_M$ is precisely the representation which appears in the description of the constant term of usual Eisenstein series via the Langlands-Shahidi method. In a similar vein, this should be a condition guaranteeing that the geometric Eisenstein series over the Fargues-Fontaine curve is as simple as possible; namely, we can formulate the first version of our conjecture.

[4] In fact, it easy to show that $r_{\mathrm{ad}}^{N,\theta} \circ \phi_M$ can only be non-zero for $\theta \in \Lambda_{G,P}^{\mathrm{pos}}$, since P was assumed to be standard with respect to the choice of Borel.

Conjecture 3.2.8. *For $\phi_M : W_{\mathbb{Q}_p} \to {}^LM(\overline{\mathbb{F}}_\ell)$ a L-parameter of Langlands-Shahidi type, we have that*

$$\mathrm{nEis}_P(\mathscr{S}_{\phi_M}) \in \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$$

is a perverse Hecke eigensheaf with eigenvalue $\phi : W_{\mathbb{Q}_p} \xrightarrow{\phi_M} {}^LM(\overline{\mathbb{F}}_\ell) \to {}^LG(\overline{\mathbb{F}}_\ell)$.

Let us explain roughly why one should believe such a statement to be true. The key point is that this should be a condition guaranteeing that the "true" eigensheaf classically defined using a geometric Eisenstein functor attached to a compactificaiton of $\mathfrak{p} : \mathrm{Bun}_P \to \mathrm{Bun}_G$ agrees with $\mathrm{nEis}_P(\mathscr{S}_{\phi_M})$. Let us first explain the structure of the relevant compactifications, as is done in the toral case in §2.5.2. As seen there, one can use the Plücker description of Bun_P to define Drinfeld compactifications, which we denote by $\overline{\mathrm{Bun}}_P$ and $\widetilde{\mathrm{Bun}}_P$. These come equipped with open immersions

$$\overline{j}_P : \mathrm{Bun}_P \hookrightarrow \overline{\mathrm{Bun}}_P$$

$$\widetilde{j}_P : \mathrm{Bun}_P \hookrightarrow \widetilde{\mathrm{Bun}}_P$$

and a map $\mathfrak{t}_P : \widetilde{\mathrm{Bun}}_P \to \overline{\mathrm{Bun}}_P$, which will be the identity on the open subspace Bun_P. The map $\mathfrak{p}_P$ extends to a map $\widetilde{\mathfrak{p}}_P$ (resp. $\overline{\mathfrak{p}}_P$) along $\widetilde{j}_P$ (resp. $\overline{j}_P$). Moreover, the map $\mathfrak{q}_P : \mathrm{Bun}_P \to \mathrm{Bun}_M$ (resp. $\mathfrak{q}_P^\dagger : \mathrm{Bun}_P \to \mathrm{Bun}_{M^{ab}}$) considered above, extends to maps $\widetilde{\mathfrak{q}}_P$ (resp. $\overline{\mathfrak{q}}_P$) along $\widetilde{j}_P$ (resp. $\overline{j}_P$).

We suppress giving the full definition of these v-stacks here, but, to give some more flavor for these compactifications, we note that, for $S \in \mathrm{Perf}$, the Plücker description of the space Bun_P tells us that it will parametrize a set of embeddings

$$\mathscr{F} \hookrightarrow \mathscr{E},$$

where $\mathscr{F}$ is a rank k vector bundle and $\mathscr{E}$ is a rank n vector bundle for $1 \leq k < n$ with cokernel isomorphic to a vector bundle on the Fargues-Fontaine curve X_S over S. Such a map is equivalent to the datum of the map given by its top exterior power

$$\Lambda^k(\mathscr{F}) \hookrightarrow \Lambda^k(\mathscr{E}),$$

and one can either consider fiberwise-injective $\mathscr{O}_{X_S}$-module maps of the form $\mathscr{F} \hookrightarrow \mathscr{E}$ or fiberwise-injective injective $\mathscr{O}_{X_S}$-module maps of the form $\Lambda^k(\mathscr{F}) \hookrightarrow \Lambda^k(\mathscr{E})$. These will define different notions of "enhanced" P-structures which $\widetilde{\mathrm{Bun}}_P$ and $\overline{\mathrm{Bun}}_P$ will parametrize, respectively. From this point of view, we can think of the map $\mathfrak{t}_P$ as given by taking top exterior powers. In particular, when P is a Borel so that $\mathscr{F}$ will always be a line bundle, we have an equality $\overline{\mathrm{Bun}}_B = \widetilde{\mathrm{Bun}}_B$ of the two compactifications.

Similar to the case of Bun_P, one should be able to construct a Verdier self-dual sheaf $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P} \in \mathrm{D}(\widetilde{\mathrm{Bun}}_P, \overline{\mathbb{F}}_\ell)$. Assuming such a sheaf existed, one could consider the functor

$$\begin{aligned} \widetilde{\mathrm{nEis}}_P(-) : \mathrm{D}(\mathrm{Bun}_M, \overline{\mathbb{F}}_\ell) &\to \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell) \\ A &\mapsto \mathfrak{p}_{P!}(\mathfrak{q}_P^*(A) \otimes^{\mathbb{L}} \mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}), \end{aligned}$$

such that it is always the case that $\widetilde{\mathrm{nEis}}_P(\mathscr{S}_{\phi_M})$ is an eigensheaf with eigenvalue ϕ, as in [BG02], without assuming any condition on ϕ_M. Unfortunately, in this context, it is more difficult to define $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$, as the usual formalism of intersection cohomology (even in classical rigid geometry) is not as well-behaved. Nevertheless, inspired by the main results of Braverman-Gaitsgory on geometric Eisenstein series in the global function field setting [BG02], we can still make the following Conjecture.

Conjecture 3.2.9. *There exists a sheaf* $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P} \in \mathrm{D}(\widetilde{\mathrm{Bun}}_P, \overline{\mathbb{F}}_\ell)$ *on* $\widetilde{\mathrm{Bun}}_P$*, with associated functor* $\widetilde{\mathrm{nEis}}_P(-)$ *as defined above, satisfying the following properties.*

1. *The sheaf* $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$ *is Verdier self-dual on* $\widetilde{\mathrm{Bun}}_P$ *and ULA with respect to the natural map* $\widetilde{\mathfrak{q}}_P : \widetilde{\mathrm{Bun}}_P \to \mathrm{Bun}_M$.
2. *We have an isomorphism* $\widetilde{j}_P^*(\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}) \simeq \mathrm{IC}_{\mathrm{Bun}_P}$.
3. *There is a natural isomorphism* $\mathbb{D}_{\mathrm{Bun}_G}(\widetilde{\mathrm{nEis}}_P(-)) \simeq \widetilde{\mathrm{nEis}}_P(\mathbb{D}_{\mathrm{Bun}_M}(-))$, *where* $\mathbb{D}_Z$ *denotes Verdier duality on* Z.
4. *For* $\phi_M : W_{\mathbb{Q}_p} \to {}^L M(\overline{\mathbb{F}}_\ell)$ *any semi-simple L-parameter and* $\mathscr{S}_{\phi_M} \in \mathrm{D}(\mathrm{Bun}_M, \overline{\mathbb{F}}_\ell)$ *any eigensheaf with eigenvalue* ϕ_M*, the sheaf* $\widetilde{\mathrm{nEis}}_P(\mathscr{S}_{\phi_M})$ *is an eigensheaf with eigenvalue* $\phi : W_{\mathbb{Q}_p} \xrightarrow{\phi_M} {}^L M(\overline{\mathbb{F}}_\ell) \to {}^L G(\overline{\mathbb{F}}_\ell)$.

Remark 3.2.10. Using similar arguments to §2.8, we note that (3) would follow from (1) and the properness of $\widetilde{\mathfrak{p}}_P : \widetilde{\mathrm{Bun}}_P \to \mathrm{Bun}_G$ after restricting to the fibers over the connected components of Bun_M. This properness should hold in the Fargues-Fontaine setup; however, is a lot more subtle beyond the case where P is a Borel (Proposition 2.5.9).

We assume such a sheaf exists then, by property (2) and projection formula, we have a natural map

$$\mathrm{nEis}_P(\mathscr{S}_{\phi_M}) \to \widetilde{\mathrm{nEis}}_P(\mathscr{S}_{\phi_M}) \tag{3.3}$$

given by adjunction. The key motivation behind Conjecture 3.2.8 is that, when ϕ_M is of Langlands-Shahidi type, this map should be an isomorphism. Therefore, by Property (4), we have that $\mathrm{nEis}_P(\mathscr{S}_{\phi_M})$ is an eigensheaf with eigenvalue ϕ as desired. The reason this should be true is that the cokernel of this map should admit a filtration with graded pieces isomorphic to $R\Gamma(W_{\mathbb{Q}_p}, r_{\mathrm{ad}}^{N,\theta} \circ \phi_M)$ for varying $\theta \in \Lambda_{G,P}^{\mathrm{pos}}$. We explain this briefly now.

The space $\overline{\mathrm{Bun}}_P$ will behave very similarly to the space $\overline{\mathrm{Bun}}_B$ studied in chapter 2. In particular, for each $\theta \in \Lambda_{G,P}^{\mathrm{pos}} \setminus 0$, we can write $\theta := \sum_{i \in \mathscr{J}_{G,P}} n_i \theta_i$, and consider $\mathrm{Div}^{(\theta)} := \prod_{i \in \mathscr{J}_{G,P}} \mathrm{Div}_{E_i}^{(n_i)}$, where E_i is the reflex field of the Γ-orbit attached to $i \in \mathscr{J}_{G,P}$, as in §2.3.3. For $\theta \in \Lambda_{G,P}^{\mathrm{pos}}$, we will get a locally closed strata

$$_{\theta}\overline{\mathrm{Bun}}_P \subset \overline{\mathrm{Bun}}_P$$

corresponding to the locus where the cokernel of the maps in the Plücker description have torsion specified by θ. This will be isomorphic to

$$_{\theta}\overline{\mathrm{Bun}}_P \simeq \mathrm{Div}^{(\theta)} \times \mathrm{Bun}_P$$

as in Proposition 2.5.19. For varying θ, these form a stratification of $\overline{\mathrm{Bun}}_P$. In turn, we can define a locally closed stratification $_{\theta}\widetilde{\mathrm{Bun}}_P$ of $\widetilde{\mathrm{Bun}}_P$, by pulling back the strata $_{\theta}\overline{\mathrm{Bun}}_P$ along the map $\mathfrak{t}_P$. This should be isomorphic to

$$\mathrm{Hck}_M^{+,(\theta)} \times_{\mathrm{Bun}_M} \mathrm{Bun}_P$$

where $\mathrm{Hck}_M^{+,(\theta)} \subset \mathrm{Hck}_M^{(\theta)}$ is a subspace of the symmetrized version of the Hecke stack fibered over $\mathrm{Bun}_M \times \mathrm{Div}^{(\theta)}$ via the map $h_M^{\rightarrow} \times \mathrm{supp}$ (For a more precise description in the global function-field setting see [Bra+02b, Propositions 1.7, 1.9]). Now, the key point is that the perverse sheaves attached to the representations $V_{\mathrm{ad}}^{N,\theta}$ under geometric Satake should appear in the $*$-pullback of the sheaf $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$ to the strata indexed by θ, as in [Bra+02b, Theorem 1.12] in the global function-field setting. It would follow that the cone of (3.3) is related to the Hecke operator $T_{V_{\mathrm{ad}}^{N,\theta}}(\mathscr{S}_{\phi_M}) \simeq \mathscr{S}_{\phi_M} \boxtimes r_{V_{\mathrm{ad}}^{N,\theta}} \circ \phi_M$, where we have used the eigensheaf property for $\mathscr{S}_{\phi_M}$. From here, it would follow that the cone of the map (3.3) is given by Eisenstein functors tensored by the Galois cohomology groups $R\Gamma(W_{\mathbb{Q}_p}, r_{\mathrm{ad}}^{N,\theta} \circ \phi_M)$ for $\theta \in \Lambda_{G,P} \setminus 0$ varying, and this vanishes by the Langlands-Shahidi condition.

This motivates the belief that $\mathrm{nEis}_P(\mathscr{S}_{\phi_M})$ should satisfy the same good properties as $\widetilde{\mathrm{nEis}}_P(\mathscr{S}_{\phi_M})$ when ϕ_M is of Langlands-Shahidi type. In particular, as discussed in chapter 2, we expect that, under a condition such as the vanishing of the H^0 of the complexes $R\Gamma(W_{\mathbb{Q}_p}, r_{\mathrm{ad}}^{N,\theta} \circ \phi_M)$ for $\theta \in \Lambda_{G,P} \setminus 0$ (See [BG02, Theorem 2.2.4] for this statement in the toral case[5]), we expect that $\widetilde{\mathrm{nEis}}_P(\mathscr{S}_{\phi_M})$ should satisfy a functional equation, and in turn the sheaf $\mathrm{nEis}_P(\mathscr{S}_{\phi_M})$ should also satisfy a similar functional equation if ϕ_M is of Langlands-Shahidi type. With this in place, we conjecture the following.

Conjecture 3.2.11. *If ϕ_M is of Langlands-Shahidi type then we have an isomorphism*

$$\mathrm{nEis}_P(\mathscr{S}_{\phi_M}) \simeq \mathrm{nEis}_Q(\mathscr{S}_{\phi_M})$$

for any two parabolics P and Q with Levi M.

As discussed in chapter 2 for the toral case, this functional equation would have many implications for the stalks of the sheaf $\mathrm{nEis}_P(\mathscr{S}_{\phi_M})$. To describe this, we let $B(G)_M := \mathrm{Im}(B(M)_{basic} \to B(G))$ be the set of elements in $B(G)$ admitting a basic reduction to M, generalizing the set of unramified elements studied in chapter 2. For $b \in B(G)_M$, we consider the set $W[M, M_b]$ as defined in [BM22, Section 5.3]. This will be identified with the set of elements in W_G such that

$$w(M) \subset M_b$$

$$w(M \cap B) \subset B, w^{-1}(M_b \cap B) \subset B$$

One can show the following facts about $B(G)_M$, by similar arguments as in §2.2.1.

Lemma 3.2.12. *Let $b \in B(G)_M$, the following is true*

1. *There is an injection $i^{-1}(b) \hookrightarrow W[M, M_b]$. The image is given by the set of elements w such that $w(M) \subset M_b$ transfers to a Levi of $J_{b_{M_b}}$ under the inner twisting between M_b and $J_{b_{M_b}}$. Namely, for every such element $\nu \in B(M)_{basic}$, there exists a unique $w \in W[M, M_b]$ such that $w(\nu) \in B(w(M))_{basic}$ has G dominant slopes.*

2. *An element $b \in B(G)$ lies in $B(G)_M$ if and only if there exists $w \in W[M, M_b]$ such that the parabolic $w(P) \cap M_b$ of M_b transfers to a parabolic subgroup $Q_{b,w} \subset G_b$ under the inner twisting. More precisely, if ν maps to $b \in B(G)$*

[5] In particular, the analogue of their regularity condition for the eigensheaf $\mathscr{S}_{\phi_T}$ is equivalent to assuming that, for all Γ-orbits of coroots $\alpha \circ \phi_T$ does not contain a copy of the trivial representation, which is precisely what the vanishing of the H^0 implies.

with corresponding Weyl group element w_ν as in (1) then $w_\nu(P)\cap M_b$ transfers to a parabolic subgroup of G. Moreover, for every element $\nu \in B(L)_{\mathrm{basic}}$ mapping to b, the Levi factor of Q_{b,w_ν} is equal to $J_{w_\nu(\nu)}$.

Note that the eigensheaf $\mathscr{S}_{\phi_M}$ should be supported on the basic (or semi-stable) locus $\mathrm{Bun}_M^{\mathrm{ss}} \simeq \bigsqcup_{\nu\in B(M)_{basic}}[*/\underline{J_\nu(\mathbb{Q}_p)}]$; therefore, to compute $\mathrm{nEis}_P(\mathscr{S}_{\phi_M})$ it suffices to consider the pullback of the diagram 3.2 to this locus, giving

$$\begin{tikzcd} \mathrm{Bun}_P^{\mathrm{ss}} \arrow[r, "\mathfrak{p}^{\mathrm{ss}}"] \arrow[d, "\mathfrak{q}^{\mathrm{ss}}"] & \mathrm{Bun}_G \\ \mathrm{Bun}_M^{\mathrm{ss}} & \end{tikzcd} .$$

The fibers of $\mathfrak{q}^{\mathrm{ss}}$ over the elements with anti-dominant isocrystal slopes (= dominant HN-slopes) will parameterize split P-structures, and therefore be relatively easy to compute with. For example, over the connected component $\nu = 1$ the previous diagram becomes

$$\begin{tikzcd} \left[*/\underline{P(\mathbb{Q}_p)}\right] \arrow[r] \arrow[d] & \left[*/\underline{G(\mathbb{Q}_p)}\right] \\ \left[*/\underline{M(\mathbb{Q}_p)}\right] & \end{tikzcd}$$

and we obtain that

$$\mathrm{nEis}^1(\mathscr{S}_{\phi_M}) \simeq \bigoplus_{\pi\in\Pi_{\phi_M}(M)} j_{1!}(i_P^G(\pi)).$$

where $\Pi_{\phi_M}(M)$ is the packet described in Conjecture 3.2.2.

We can perform a similar calculation for any element ν after restricting to Bun_G^b, as in Proposition 2.9.4. For any element $\nu \in B(M)_{basic}$ mapping to $b \in B(G)_M$, we write $w_\nu \in W[M,M_b]$ for the corresponding Weyl group element supplied by 3.2.12 (1). Then we should have

$$\mathrm{nEis}_P^\nu(\mathscr{S}_{\phi_M})|_{\mathrm{Bun}_G^b} \simeq \bigoplus_{\pi\in\Pi_{\phi_M}(J_\nu)} i_{Q_{b,w_\nu}}^{J_b}(\pi^{w_\nu}) \otimes \delta_{P_b}^{-1/2}[-\langle 2\rho_G, \nu_b\rangle]$$

where δ_{P_b} is the modulus character of the standard parabolic in G with Levi factor M_b transferred to J_b along the inner twisting, Q_{b,w_ν} is the parabolic supplied by 3.2.12 (2), and π^{w_ν} is given by applying the isomorphism $J_\nu \simeq J_{w_\nu(\nu)}$ of σ-centralizers induced by w_ν. Then Conjecture 3.2.11 will tell us that this in fact the only relevant contribution, so that we have

$$\mathrm{nEis}_P^\nu(\mathscr{S}_{\phi_M}) \simeq \bigoplus_{\pi \in \Pi_{\phi_M}(J_\nu)} j_{b!}(i_{Q_{b,w_\nu}}^{J_b}(\pi^{w_\nu}) \otimes \delta_{P_b}^{-1/2})[-\langle 2\rho_G, \nu_b\rangle]$$

This motivates the following definition.

Definition 3.2.13. For $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$ and $\phi_M : W_{\mathbb{Q}_p} \to {}^LM(\Lambda)$ a supercuspidal L-parameter with induced parameter $\phi : W_{\mathbb{Q}_p} \to {}^LG(\Lambda)$, assuming the validity of Conjecture 3.2.2, and in particular the existence of the packets $\Pi_{\phi_M}(J_\nu)$ for all $\nu \in B(M)_{basic}$, we define, for fixed $b \in B(G)_M$, the complex

$$\mathrm{Red}^{\mathrm{tw}}_{b,\phi} := \bigoplus_{\pi \in \Pi_{\phi_M}(J_\nu)} \bigoplus_{\nu \in i_M^{-1}(b)} i_{Q_{b,w_\nu}}^{J_b}(\pi^{w_\nu}) \otimes \delta_{P_b}^{-1/2}[-\langle 2\rho_G, \nu_b\rangle]$$

of $J_b(\mathbb{Q}_p)$-representations.

In the case that $\Lambda = \overline{\mathbb{F}}_\ell$, we should have by the above discussion an isomorphism

$$\mathrm{nEis}_P(\mathscr{S}_{\phi_M}) := \bigoplus_{b \in B(G)_M} j_{b!}(\mathrm{Red}^{\mathrm{tw}}_{b,\phi}).$$

With this motivating analysis, we collect the discussion thus far and formulate a general conjecture for arbitrary coefficients $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$. This is a non-principal version of Conjecture 2.1.29.

Conjecture 3.2.14. *For $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$ and $\phi_M : W_{\mathbb{Q}_p} \to {}^LM(\Lambda)$ a supercuspidal L-parameter of Langlands-Shahidi type. There exists a sheaf* $\mathrm{nEis}_P(\mathscr{S}_{\phi_M}) \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_M, \Lambda)$ *satisfying the following properties.*

1. *We have an isomorphism*

$$\mathrm{nEis}_P(\mathscr{S}_{\phi_M}) := \bigoplus_{b \in B(G)_M} j_{b!}(\mathrm{Red}^{\mathrm{tw}}_{b,\phi})$$

   of objects in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$.

2. *The sheaf* $\mathrm{nEis}_P(\mathscr{S}_{\phi_M})$ *is an eigensheaf with eigenvalue* $\phi : W_{\mathbb{Q}_p} \xrightarrow{\phi_M} {}^LM(\Lambda) \to {}^LG(\Lambda)$.

3. *If $\Lambda = \overline{\mathbb{F}}_\ell$ then the sheaf* $\mathrm{nEis}_P(\mathscr{S}_{\phi_M})$ *agrees with the sheaf defined above under the identification* $\mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell) \simeq \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$.

4. *For fixed $b \in B(G)_M$, the natural map*

$$j_{b!}(\mathrm{Red}^{\mathrm{tw}}_{b,\phi}) \to Rj_{b*}(\mathrm{Red}^{\mathrm{tw}}_{b,\phi})$$

is an isomorphism of sheaves in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$[6].

Remark 3.2.15. We note that (4) should follow from (1), since we believe that for $\Lambda = \overline{\mathbb{F}}_\ell$ we should have an isomorphism $\mathbb{D}_{\mathrm{Bun}_G}(\mathrm{nEis}_P(\mathscr{S}_{\phi_M})) \simeq \mathrm{nEis}_P(\mathbb{D}_{\mathrm{Bun}_M}(\mathscr{S}_{\phi_M}))$. This follows, since $\mathrm{nEis}_P(\mathscr{S}_{\phi_M})$ should agree with the conjectural sheaf $\widetilde{\mathrm{nEis}}_P(\mathscr{S}_{\phi_M})$ under the Langlands-Shahidi condition as discussed above, and $\widetilde{\mathrm{Eis}}_P(\mathscr{S}_{\phi_M})$ should satisfy this property by Conjecture 3.2.9 (3).

Remark 3.2.16. When $\Lambda = \overline{\mathbb{Q}}_\ell$, one should be able to compare the description of the stalks with the averaging formula of Shin, as is does in Appendix A.3 for the toral case. Recall that, in the case of rational coefficients, the packets $\Pi_{\phi_M}(M_\nu)$ describing the stalks of the sheaf $\mathscr{S}_{\phi_M}$ should be specified by the refined local Langlands correspondence of Kaletha. This should relate to the description of the local Langlands correspondence for $B(G)$ over the parameter ϕ provided in recent work of Meli-Oi [BM22] (up to modulus character twists).

Remark 3.2.17. As in the proof of Corollary 2.11.22, this should imply that the one has isomorphisms of the form

$$i_P^G(\pi) \simeq i_Q^G(\pi)$$

for all pairs of parabolics Q and P with shared Levi factor M and $\pi \in \Pi_{\phi_M}(M)$. This is an interesting question in representation theory in its own right especially with ℓ-modular coefficients, independent of any kind of consequences for categorical local Langlands.

By carrying out analysis described in the previous section and in §2.11, this should imply several consequences for the cohomology of local and global Shimura varieties, we conclude by explaining the global applications.

Fix a Shimura datum $(\mathbf{G}, X)$ with reflex field E, and let $E_{\mathfrak{p}}$ denote the completion at a place $\mathfrak{p}|p$ determined by an isomorphism $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$ and assume $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Q}}_\ell\}$ in what follows. We can look at the $G(\mathbb{Q}_p) \times W_{E_{\mathfrak{p}}}$-representation

$$R\Gamma_c(\mathscr{S}(\mathbf{G}, X)_{K^p, C}, \Lambda)$$

[6] We note that this also makes sense with coefficients $\Lambda \in \{\overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$ despite being in the realm of solid sheaves, using [FS21, Proposition VII.7.3]

defined by the cohomology at infinite level. By applying Corollary B.1.8, we obtain a $G(\mathbb{Q}_p) \times W_{E_{\mathfrak{p}}}$-equivariant decomposition of this

$$R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p,C},\Lambda) = \bigoplus_{\phi} R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p,C},\Lambda)_{\phi}$$

running over semi-simple L-parameters $\phi : W_{\mathbb{Q}_p} \to {}^L G(\Lambda)$. Each summand has irreducible constituents with Fargues-Scholze parameter equal to ϕ by Lemma B.1.7 (2). For such a ϕ, we let (ϕ_M, M) denote a cuspidal support. I.e $\phi_M : W_{\mathbb{Q}_p} \to {}^L M(\Lambda)$ is a supercuspidal L-parameter such that ϕ is induced by composing with the natural embedding ${}^L M(\Lambda) \to {}^L G(\Lambda)$. We say ϕ is of Langlands-Shahidi type if ϕ_M is. Note that this does not depend on the choice of cuspidal support.

We conjecture the following generalization of the results discussed in §3.1 (e.g. Theorem 3.1.6). This, as in the toral case, should follow from showing forms of Conjecture 3.2.14.

Conjecture 3.2.18. *Let ϕ be a semi-simple L-parameter of Langlands-Shahidi type with cuspidal support (M, ϕ_M). The category $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_{\phi}$ of ϕ-local lisse-étale Λ-sheaves (as defined in Appendix B.1) breaks up as a direct sum*

$$\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_{\phi} \simeq \bigoplus_{b \in B(G)_M} \mathrm{D}(\mathrm{Bun}_G^b, \Lambda)_{\phi}$$

and the $!$ and $*$ pushhforwards agree for any smooth irreducible representation ρ of $J_b(\mathbb{Q}_p)$ lying in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G^b, \Lambda)_{\phi}$ for $b \in B(G)_M$ with respect to the inclusion $\mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$.

Given $V \in \mathrm{Rep}_{\Lambda}({}^L G^I)$, the map induced by associated the Hecke operator

$$T_V : \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_{\phi} \to \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_{\phi}^{BW_{\mathbb{Q}_p}^I}$$

is perverse t-exact, where the fact the Hecke operator preserves this subcategory is Lemma B.1.7 (2).

In particular, by combining this with a generalization of Theorem 3.1.10 and the analysis described in the previous section, we could deduce the following Conjectures as a consequence.

Conjecture 3.2.19. *Let ϕ be a semi-simple L-parameter of Langlands-Shahidi type with cuspidal support (M, ϕ_M). Then the complex $R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p,C},\Lambda)_{\phi}$ (resp. $R\Gamma(\mathscr{S}(\mathbf{G},X)_{K^p,C},\Lambda)_{\phi}$) is concentrated in degrees $0 \leq i \leq d$ (resp. $d \leq i \leq 2d$).*

Remark 3.2.20. We note, if ϕ is supercuspidal (i.e doesn't factor through a proper Levi subgroup) and $\Lambda = \overline{\mathbb{Q}}_\ell$, this essentially reduces to the Kottwitz Conjecture as proven in chapter 1 for GSp_4. In particular, in this case only the basic locus will contribute to the cohomology of the global Shimura variety (using compatibility), and it reduces to the Kottwitz conjecture for the basic local Shimura variety.

Remark 3.2.21. For $(\mathbf{G},X)$ of PEL type A or C satisfying assumption 3.1.8, we would also obtain a $W_{E_\mathfrak{p}} \times G(\mathbb{Q}_p)$-equivariant direct sum decomposition

$$R\Gamma_c(\mathscr{S}(\mathbf{G},X)_{K^p,C},\Lambda(d_b))_\phi \simeq \bigoplus_{b\in B(G)_M} (R\Gamma_c(G,b,\mu)_\phi \otimes^{\mathbb{L}} V_b)[2d_b]$$

where $R\Gamma_c(G,b,\mu) := \mathrm{colim}_{K_p\to\{1\}} R\Gamma_c(\mathrm{Sht}(G,b,\mu)_{\infty,C}/\underline{K_p},\Lambda(d_b))$ and $R\Gamma_c(G,b,\mu)_\phi$ is the projection applied to the complex viewed as a $G(\mathbb{Q}_p)$-representation. This should also generalize once one has appropriate general definitions of Ig^b and $\mathrm{Ig}^{b,*}$ so that one can actually define $V_b := R\Gamma_{c-\partial}(\mathrm{Ig}^{b,*},\Lambda)$. Under possible additional constraints on ϕ, one should also be able to describe the contribution of $R\Gamma_c(G,b,\mu)_\phi$ in terms of the decomposition $V_\mu|_{Z(\hat{M}^\Gamma)} = \mathscr{T}_\mu|_{Z(\hat{M}^\Gamma)}$ for $b \in B(G)_M$, as is explained in the toral case in §3.2.1. It would be interesting to formulate an optimal conjecture of this type.

Appendix A

Appendix to Chapter 2

A.1 Intertwining Operators and the Irreducibility of Principal Series

We want to show the irreducibility of principal series representations obtained from the characters χ attached to parameters $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\overline{\mathbb{Q}}_\ell)$ satisfying the Conditions in 2.3.8. We let $\chi : T(\mathbb{Q}_p) \to \mathbb{C}^*$ be the character attached to ϕ_T via local class field theory and a fixed isomorphism $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$ sending $p^{1/2}$ to the fixed choice of square root in $\overline{\mathbb{Q}}_\ell$. Our goal will be to show the following two facts.

Proposition A.1.1. *Suppose that χ is a regular character in the sense that it is not fixed under any $w \in W_G$ then if we have an isomorphism*

$$i_B^G(\chi) \simeq i_B^G(\chi^{w_0})$$

the representation $i_B^G(\chi)$ is irreducible.

Remark A.1.2. This also follows using Frobenius reciprocity and second adjointness, as in the proof of Corollary 2.11.24. However, the method we exhibit here gives much more insight into how to compute the reducibility points of principal series representations.

Proposition A.1.3. *If $\chi : T(\mathbb{Q}_p) \to \mathbb{C}^*$ is a generic character then, for all $w \in W_G$, we have an isomorphism*

$$i_B^G(\chi) \simeq i_B^G(\chi^w)$$

of smooth $G(\mathbb{Q}_p)$-representations.

By combining these two Propositions, we deduce the following.

Corollary A.1.4. *For χ a generic regular character, the induction $i_B^G(\chi)$ is always irreducible.*

The idea behind proving such results is due to Speh and Vogan [SV80, Theorem 3.14] in the archimedean case. They study the reducibility of principal series using the Langlands classification [BW80, Section IV]. Strictly speaking, their analysis is for the archimedean place, but it is easy to see that it extends to the case of a p-adic group using the analogous Langlands classification there [BW80, Section XI.2]. They (roughly) break the problem of understanding the reducibility points of principal series representations into two parts:

1. ([SV80, Theorem 3.14(a)]) Understanding the reducibility points of non-unitary principal series with respect to the parabolic inductions from T to M_i for $i \in \mathscr{J}$, where M_i is the rank 1 Levi subgroup of G whose relative Dynkin diagram is given by $\{i\} \subset \mathscr{J}$.

2. ([SV80, Theorem 3.14(b)]) Understanding the reducibility points of unitary principal series representations with respect to induction from T to a (not necessarily rank 1) proper Levi subgroup of G.

We will now explain this heuristic in our case. To see analogous analysis worked out more explicitly for specific p-adic reductive groups, we point the reader to [Tad94, Section 7], [Mui97, Section 3], [Mat10, Section 3] for a small sample. For (2), a very definitive answer to such questions can be found in the paper of Keys [Key82]. In particular, by [Key82, Corollary 1] the number of irreducible components of such a unitary parabolic induction can be computed in terms of the Knapp-Stein R-Group [KS72; Kna73a; Kna73b], which Keys determines for all split groups. While this is very interesting, we will not address this here. In particular, we have the following.

Corollary A.1.5. *If χ is a regular unitary character then the normalized parabolic induction $i_B^G(\chi)$ is irreducible.*

Proof. This follows from the Bruhat decomposition and the fact that $i_B^G(\chi)$ is unitary and therefore fully decomposable (See [Cas95, Theorem 6.6.1] or [Bru61]). $\square$

Now consider $\mathbb{X}^*(T_{\overline{\mathbb{Q}}_p})^\Gamma \otimes \mathbb{R} \simeq \mathbb{X}_*(A) \otimes \mathbb{R} \simeq \mathbb{R}^d$ the set of unramified characters. For $s \in \mathbb{R}^d$, we write ν^s for the associated unramified character. We will say that ν^s is positive (resp. strictly positive) if, for all simple (reduced) positive coroots $\alpha_{i,A}$, the precomposition of ν^s with $\alpha_{i,A}$ is positive (resp. strictly positive), or in other words that s lies in the positive Weyl chamber of $\mathbb{X}_*(A)\otimes\mathbb{R}$ defined by the Borel. We write $\chi = \mu_\chi \nu^{s_\chi}$, where μ_χ is a unitary character and ν^{s_χ} is an unramified character of T, for some $s_\chi \in \mathbb{R}^d$. We now consider intertwining operators. Recall that, for $w \in W_G$ and $s \in \mathbb{R}^d$, we have the intertwining operator

$$I_w(\mu_\chi, s) : i_B^G(\mu_\chi \nu^s) \to i_B^G((\mu_\chi \nu^s)^w)$$

$$f(g) \mapsto \int_{U_w} f(w^{-1}ug)du$$

where $U_w := U \cap wU^- w^{-1}$ and U^- is the unipotent radical of the opposite Borel. This integral will converge if ν^s lies sufficiently deep in the dominant Weyl chamber, and admits a meromorphic continuation as a function of s (where one allows s to be complex and imposes this constraint on the real part). Away from these poles, it gives rise to an intertwining operator between $i_B^G(\mu_\chi \nu^s)$ and $i_B^G((\mu_\chi \nu^s)^w)$ in the usual representation theory sense. For our purposes, we will be interested in the intertwining operator $I_{w_0}(\mu_\chi, s)$ for the element of longest length w_0. In this case, one can see that the operator is convergent for all s which are strictly positive, and the image of the operator is the unique irreducible Langlands quotient of $i_B^G(\mu_\chi \nu^s)$ (See [BW80, Sections XI.2.6, XI.2.7]). This quotient has multiplcity one and therefore the intertwining space between $i_B^G(\chi)$ and $i_B^G(\chi^{w_0})$ is one-dimensional. It follows that, if ν^{s_χ} is strictly positive, it suffices to exhibit an isomorphism $i_B^G(\chi) \simeq i_B^G(\chi^{w_0})$ to show irreducibility. We will now use this kind of analysis to prove Proposition A.1.1.

Proof. (Proposition A.1.1) First off note that, for all χ, we have an equality

$$[i_B^G(\chi)] = [i_B^G(\chi^{w_0})]$$

in $K_0(G(\mathbb{Q}_p))$ (See for example [Dij72b, Theorem 4]). This allows us to, without loss of generality, assume that ν^{s_χ} is positive. Now, consider the set of $i \in \mathscr{J}$ such that the precomposition of ν^{s_χ} with $\alpha_{i,A}$ is equal to 0. This defines a parabolic P_χ of G which we decompose as $P_\chi = M_\chi A_\chi N_\chi$, where A_χ is the maximal split torus in the center of M_χ. If $\mathscr{J}_{M_\chi}$ denotes the vertices of the relative Dynkin diagram of M_χ then $\mathscr{J}_{M_\chi} \subset \mathscr{J}$ corresponds to the set of simple positive coroots where this precomposition vanishes. Now, set $\nu_1 := \nu^{s_\chi}|_{A\cap M_\chi}$ and $\nu_2 := \nu^{s_\chi}|_{A_\chi}$. We consider the parabolic induction

$$i_{B\cap M_\chi}^{M_\chi}(\mu_\chi \otimes \nu_1)$$

where we now note that $\mu_\chi \otimes \nu_1$ is unitary by construction. Therefore, $i_{B\cap M_\chi}^{M_\chi}(\mu_\chi \otimes \nu_1)$ is unitary and thereby fully decomposable. It follows that, since $i_{B\cap M_\chi}^{M_\chi}(\mu_\chi \otimes \nu_1)$ and $i_{B\cap M_\chi}^{M_\chi}((\mu_\chi \otimes \nu_1)^{w_0^{M_\chi}})$ are equal in the Grothendieck group, we have an isomorphism $i_{B\cap M_\chi}^{M_\chi}(\mu_\chi \otimes \nu_1) \simeq i_{B\cap M_\chi}^{M_\chi}((\mu_\chi \otimes \nu_1)^{w_0^{M_\chi}})$, where $w_0^{M_\chi}$ is the element of longest length of the Weyl group of M_χ. Now, we have an isomorphism:

$$i_B^G(\chi) \simeq i_{P_\chi}^G((i_{B\cap M_\chi}^{M_\chi}(\mu_\chi \otimes \nu_1)) \otimes \nu_2)$$

Since χ is regular then, it follows by Lemma A.1.5, that the unitary induction $i_{B\cap M_\chi}^{M_\chi}(\mu_\chi \otimes \nu_1)$ is irreducible. Therefore, the RHS is the induction of an irreducible tempered representation times an unramified character ν_2 satisfying the property that $\langle \alpha_{i,A}, \nu_2 \rangle > 0$ for all $i \in \mathscr{J} \setminus \mathscr{J}_{M_\chi}$ by construction. Again applying the Langlands classification [BW80, Sections XI.2.6, XI.2.7], the intertwining operator attached to the parabolic P_χ and the element $w_0 w_0^{M_\chi}$ converges for $s = s_\chi$, and since it maps to a unique quotient of multiplicity one it suffices to exhibit an isomorphism between $i_{P_\chi}^G((i_{B\cap M_\chi}^{M_\chi}(\mu_\chi \otimes \nu_1)) \otimes \nu_2)$ and the induction twisted by $w_0 w_0^{M_\chi}$. However, since we just saw that $i_{B\cap M_\chi}^{M_\chi}((\mu_\chi \otimes \nu_1)^{w_0^{M_\chi}}) \simeq i_{B\cap M_\chi}^{M_\chi}(\mu_\chi \otimes \nu_1)$, it suffices to show we have an isomorphism $i_B^G(\chi) \simeq i_B^G(\chi^{w_0})$. This establishes the claim. □

Now we just need to show Proposition A.1.3.

Proof. We claim that this reduces to the analogous question for G a group of rank 1. In particular, let's consider for $i \in \mathscr{J}$ the simple positive (reduced) coroot $\alpha := \alpha_{i,A}$ and the rank 1 parabolic $P_\alpha = M_\alpha N_\alpha A_\alpha$ attached to it. As before, we write $\nu_1^\alpha := \nu^{s_\chi}|_{A\cap M_\alpha}$ and $\nu_2^\alpha := \nu^{s^\chi}|_{A_\alpha}$. For all simple positive coroots α, we have an isomorphism

$$i_B^G(\chi) \simeq i_{P_\alpha}^G((i_{B\cap M_\alpha}^{M_\alpha}(\mu_\chi \otimes \nu_1^\alpha)) \otimes \nu_2^\alpha)$$

However, if w_α is the simple reflection corresponding to α, we have that

$$i_B^G(\chi^{w_\alpha}) \simeq i_{P_\alpha}^G((i_{B\cap M_\alpha}^{M_\alpha}((\mu_\chi \otimes \nu_1^\alpha)^{w_\alpha}) \otimes \nu_2^\alpha)$$

Therefore, if we can show the existence of an isomorphism:

$$i_{B\cap M_\alpha}^{M_\alpha}(\mu_\chi \otimes \nu_1^\alpha) \simeq i_{B\cap M_\alpha}^{M_\alpha}((\mu_\chi \otimes \nu_1^\alpha)^{w_\alpha})$$

It will imply that we have an isomorphism:

$$i_B^G(\chi) \simeq i_B^G(\chi^{w_\alpha})$$

Now we can proceed by induction on the length of Weyl group elements. We just described the base case, but then, by replacing χ with χ^{w_α} we can proceed by considering another simple reflection attached to another simple positive coroot distinct from α. We note that, since χ being generic is a condition for all coroots (not just simple), at each step of the induction we are tasked with showing the following.

Proposition A.1.6. *Let G be a absolutely simple, simply connected, quasi-split connected reductive group of split rank* 1*. Let α be the unique simple (reduced) positive coroot of G and w_α the corresponding simple reflection. Then, for all χ a generic character of $T(\mathbb{Q}_p)$, we have an isomorphism*

$$i_B^G(\chi) \simeq i_B^G(\chi^{w_\alpha})$$

of smooth $G(\mathbb{Q}_p)$-representations.

It remains to justify the absolutely simple simply connected assumption. To do this, note that given a G not satisfying these conditions, we can find a central isogeny $f : \tilde{G} \to G$, where $\tilde{G}$ is a product of torii and absolutely simple simply connected groups. If we let $\tilde{B}$ be the preimage of the Borel B with maximal torus given by $\tilde{T}$ the preimage of T then, since $\mathrm{Ker}(f)$ is contained in the center, we have an isomorphism $\tilde{G}/\tilde{B} \simeq G/B$. This implies that we have an isomorphism:

$$i_B^G(\chi)|_{\tilde{G}(\mathbb{Q}_p)} \simeq i_{\tilde{B}}^{\tilde{G}}(\chi|_{\tilde{T}})$$

Moreover, since f will induce an isomorphism on the root spaces, it follows that if χ is generic with respect to G then $\chi|_{\tilde{T}}$ is generic with respect to $\tilde{G}$. This reduces us to exhibiting the desired isomorphism for $\tilde{G}$. Now, we prove the Proposition A.1.6 through brute force. In particular, we will use Tits' classification theorem [Tit79] (See also [Car01] for the classification in rank 1). We adopt the same notation as in [Tit79]. Since we are assuming the group to be quasi-split, there are two cases.

${}^1A^1_{1,1}$

In this case, we have that $G = \mathrm{SL}_2$. We saw in Example 2.3.12 that genericity guaranteed irreducibility aside from the case where $\chi^2 \simeq \mathbf{1}$, but since this is a unitary character it still follows that we have the desired isomorphism.

$^2A^1_{2,1}$

In this case, the group cannot be split; in particular, we have that $G = \mathrm{SU}_3$ is a quasi-split special unitary group attached to a quadratic extension $E/\mathbb{Q}_p$. We saw in Example 2.3.17 that χ being generic guaranteed irreducibility and hence the desired isomorphism. $\square$

A.2 Tilting Cocharacters

We consider a general quasi-split connected reductive group $G/\mathbb{Q}_p$, and a geometric dominant cocharacter $\mu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$. For $\Lambda \in \{\overline{\mathbb{Q}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{F}}_\ell\}$, we are interested in understanding the condition of μ being tilting (Definition 2.11.4). Recall that this means that the representation $V_\mu \in \mathrm{Rep}_\Lambda(\hat{G})$ attached to μ lies in the subcategory $\mathrm{Tilt}_\Lambda(\hat{G})$. If $\Lambda = \overline{\mathbb{Q}}_\ell$ this is always true, and so we fix $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell\}$ in what follows. Since this only involves the representation theory $\hat{G}$ we may, without loss of generality, assume G is split in what follows. This is simply the question of when the highest weight module V_μ of $\hat{G}$ is irreducible with Λ-coefficients. This question has been studied extensively (See for example [Jan03, Pages 283-286] for a comprehensive overview). In the first two sections, we discuss some general theory to determine when μ is tilting in this split case, and then provide a table summarizing when μ is tilting in the case that μ is a fundamental coweight.

A.2.1 General Theory

We assume G is a split connected reductive group throughout this section, with Langlands dual group $\hat{G}$. For μ a dominant cocharacter, the condition that μ is tilting is equivalent to showing that the highest weight G-module $V_\mu \in \mathrm{Rep}_\Lambda(\hat{G})$ is simple. We begin with the following lemma.

Lemma A.2.1. *If μ is minuscule then it is tilting.*

Proof. In this case, the weights of V_μ form a closed Weyl group orbit. It follows that V_μ is always irreducible and therefore tilting. $\square$

We would now like to provide a finer criterion for irreducibility. To do this, we will introduce some notation. Given a coroot ν and $r \in \mathbb{Z}$, we consider the affine reflection of $\mathbb{X}_*(T) \otimes \mathbb{R}$ given as

$$s_{\nu,r}(\mu) := s_\nu(\mu) + r\nu$$

where $s_\nu \in W_G$ is the reflection attached to ν. We set W_ℓ to be the subgroup generated by reflections $s_{\nu,n\ell}$, where ν is a coroot and $n \in \mathbb{Z}$, and write ρ for the sum of all coroots of G. Elements $w \in W_\ell$ act on $\mu \in \mathbb{X}_*(T) \otimes \mathbb{R}$, via the standard dot action $w \cdot \mu := w(\mu + \rho) - \rho$. In other words, we regard $s_{\nu,n\ell}$ as a reflection around the hyperplane:

$$\{\mu \in \mathbb{X}_*(T) \otimes \mathbb{R} \mid \langle \mu + \rho, \nu^\vee \rangle = n\ell\}$$

It follows that the standard alcove for this action is given by

$$C = \{\mu \in \mathbb{X}_*(T) \otimes \mathbb{R} \mid 0 < \langle \mu + \rho, \nu^\vee \rangle < \ell\}$$

and we denote the closure by $\overline{C}$. We now have the following slightly more general criterion for the irreducibility of V_μ.

Proposition A.2.2. *[Jan03, Corollary 5.6] Suppose that $\mu \in \overline{C} \cap \mathbb{X}_*(T)^+$ then μ is tilting.*

For a given μ, this will give us a lower bound on the ℓ for which μ is tilting. However, it is only a sufficient condition and not necessary. In particular, note that we have the following.

Theorem A.2.3. *[Mat00, Theorem 2.6] If $\mu = (\ell - 1)(\rho)$ then μ is tilting.*

So $V_{(\ell-1)(\rho)}$ will always be simple, but $(\ell - 1)(\rho)$ will not usually lie in $\overline{C} \cap \mathbb{X}_*(T)^+$. Moreover, if we define the Coxeter number $h = \max_\nu\{\langle \rho, \nu^\vee \rangle + 1\}$ ranging over all coroots ν then it is easy to see that $C \cap \mathbb{X}_*(T) \neq \emptyset$ is equivalent to $\ell \geq h$, and so, for small ℓ, Proposition A.2.2 tells us nothing. To tackle these more general cases, we introduce a sum formula for the characters of the representations. Namely, for $V \in \mathrm{Rep}_\Lambda(\hat{G})$, we write $\mathrm{ch}(V) := \sum_{\nu \in \mathbb{X}_*(T)} \dim(V(\nu))e^\nu$ for the character of V. For μ a dominant cocharacter, we write $\chi(\mu) := \mathrm{ch}(V_\mu)$ for the character of V_μ. Then we have the following.

Proposition A.2.4. *[Jan03, Section 8.19] For each $\mu \in \mathbb{X}_*(T)^+$, there is a filtration of $\hat{G}$-modules*

$$\cdots \subset V_\mu^2 \subset V_\mu^1 \subset V_\mu^0 = V_\mu$$

such that

$$\sum_{i>0} \mathrm{ch}(V_\mu^i) = \sum_\nu \sum_{0 < m\ell < \langle \mu + \rho, \nu^\vee \rangle} \nu_\ell(m\ell)\chi(s_{\nu,m\ell} \cdot \mu)$$

where $\nu_\ell(-)$ is the ℓ-adic valuation and ν ranges over all coroots. Moreover, we have that V_μ/V_μ^1 is isomorphic to the irreducible socle of V_μ. In particular, we see that μ is tilting if and only if

$$\sum_{\nu} \sum_{0<m\ell<\langle \mu+\rho,\nu^\vee\rangle} \nu_\ell(m\ell)\chi(s_{\nu,m\ell}\cdot\mu) = 0$$

This generalizes Proposition A.2.2 and gives a computational method for verifying when μ is tilting. See for example [Jan03, Section 8.20] for this worked out for $G = \mathrm{SL}_4$ and $\ell > 3$, [GS88] for a table answering this question for G an exceptional group and certain μ, and [BW71] for an analogous table for certain exceptional groups and low rank classical groups. In general, a precise classification of when μ is tilting for all G seems to be quite complicated, and to the best of our knowledge is unknown. However, when G is of type A_{n-1}, there exists a complete classification.

Proposition A.2.5. *[Jan91, Page 113] For G of type A_{n-1}, μ is tilting if and only if for each coroot ν of G the following is satisfied. Write $\langle \mu+\rho,\nu^\vee\rangle = a\ell^s + b\ell^{s+1}$, with $a,b,s \in \mathbb{N}$ and $0 < a < \ell$. Then there have to be positive coroots $\beta_0,\beta_1,\dots,\beta_b$ such that $\langle \mu+\rho,\beta_i^\vee\rangle = \ell^{s+1}$ for $1 \leq i \leq b$, $\langle \mu+\rho,\beta_0^\vee\rangle = ap^s$, $\nu = \sum_{i=0}^b \beta_i$, and $\sum_{i=1}^b \beta_i$ is a coroot.*

For general types, we will content ourselves with describing the fundamental coweights, where we can give a full description of the tilting condition.

A.2.2 The Tilting Condition for Fundamental Coweights

We assume that G is a split adjoint group, and let $\hat{\alpha}_j$ denote the simple roots, where we use the enumeration as in [Bou68, Pages 250-275]. We choose fundamental coweights characterized by $\langle \varpi_i, \hat{\alpha}_j\rangle = \delta_{ij}$. We will be interested in the question of when the representation V_{ϖ_i} is irreducible. In this case, we have a complete classification [Jan03, Pages 286-287],[Jan91, Section 4.6]. We note that, if ϖ_i is minuscule, this is automatic by Lemma A.2.1, so in what follows we simply provide a list of ℓ for the non-minuscule ϖ_i (See [LR08, Page 221] for a classification). This namely implies the case of A_n is trivial, since all fundamental weights are minuscule. Additionally, we recall that our results will only apply if ℓ is very good in the sense of [FS21, Page 33], so we have also enumerated the condition that ℓ is very good for the different types[1].

[1] We warn the reader if G is non-split there can also be additional constraints on ℓ.

Type of G	Type of $\hat{G}$	μ	ℓ	ℓ very good
$B_n, n \geq 2$	C_n	$\varpi_i,\ 1 < i \leq n$	$\ell \mid \binom{n+1-(i+j)/2}{(i-j)/2},\ 0 \leq j < i,\ j \cong i \mod 2$	$\ell \neq 2$
$C_n, n \geq 2$	B_n	$\varpi_i,\ 1 \leq i < n$	$\ell = 2$	$\ell \neq 2$
$D_n, n \geq 4$	D_n	$\varpi_i,\ 1 < i < n-1$	$\ell = 2$	$\ell \neq 2$
E_6	E_6	ϖ_1, ϖ_6 ϖ_3, ϖ_5 ϖ_2 ϖ_4	$\emptyset$ $\ell = 2$ $\ell = 3$ $\ell = 2, 3$	$\ell \neq 2, 3$
E_7	E_7	ϖ_1 ϖ_2 ϖ_3, ϖ_5 ϖ_4 ϖ_6 ϖ_7	$\ell = 2$ $\ell = 3$ $\ell = 2, 3$ $\ell = 2, 3, 13$ $\ell = 2, 7$ $\emptyset$	$\ell \neq 2, 3$
E_8	E_8	ϖ_1 ϖ_2 ϖ_3 ϖ_4 ϖ_5 ϖ_6 ϖ_7 ϖ_8	$\ell = 2$ $\ell = 2, 3, 7$ $\ell = 2, 3, 19$ $\ell = 2, 3, 5, 13, 19$ $\ell = 2, 3, 5$ $\ell = 2, 3, 5, 7$ $\ell = 2, 3, 5$ $\emptyset$	$\ell \neq 2, 3, 5$
F_4	F_4	ϖ_1 ϖ_2, ϖ_3 ϖ_4	$\ell = 2$ $\ell = 2, 3$ $\ell = 3$	$\ell \neq 2, 3$
G_2	G_2	ϖ_1 ϖ_2	$\ell = 3$ $\ell = 2$	$\ell \neq 2, 3$

By comparing the fourth and fifth columns, we deduce the following.

Proposition A.2.6. *For G a split adjoint connected reductive group over $\mathbb{Q}_p$, if ℓ is very good then ϖ_i for any $i \in \mathscr{J}$ is tilting for all G of type A_n, C_n, D_n, E_6, F_4, and G_2.*

A.3 Relationship to the Classical Averaging Formula, by Alexander Bertoloni-Meli

In this appendix, we show that the averaging formula proven in Theorem 2.11.10 is compatible with existing formulas and conjectures in the literature.

A.3.1 Averaging Formulas

To begin, we recall the general statement of these averaging formulas. Such formulas first appeared in the book of Harris–Taylor ([HT01]) and are classically deduced by studying the geometry of the mod-p fibers of Shimura varieties. These fibers admit a Newton stratification in terms of the set $B(G,\mu)$ and the strata are uniformized by Rapoport–Zink spaces and Igusa varieties. The cohomological consequence of this is the formula of Mantovan ([Man05]) which up to twists is given as

$$\sum_i \varinjlim_{K\subset G(\mathbb{A}_f)} (-1)^i H^i_c(\mathrm{Sh}(G,X)_K,\overline{\mathbb{Q}}_\ell) = \sum_{b\in B(G,\mu)} R\Gamma^\flat_c(G,b,\mu)[\sum_j (-1)^j \varinjlim_{K^p\subset G(\mathbb{A}^p_f)} H^j_c(\mathrm{Ig}^b_{K^p},\overline{\mathbb{Q}}_\ell)], \tag{A.1}$$

in $K_0(G(\mathbb{A}_f)\times W_{E_\mu})$, where $\mathrm{Sh}(G,X)_K$ (resp. $\mathrm{Ig}^b_{K^p}$) is the Shimura variety (resp. Igusa variety) determined by the associated data. Averaging formulas can then be deduced by studying isotypic pieces of the above formula.

In order to precisely state these averaging formulas, we first recall some facts about stable characters following [Hir04]. Let G be a connected reductive group with Levi subgroup M and parabolic P. Let $K_0(G(\mathbb{Q}_p),\mathbb{C})^{st}\subset K_0(G(\mathbb{Q}_p),\mathbb{C})$ denote the subgroup of virtual representations with stable character. Then the normalized Jacquet module and parabolic induction functors induce morphisms

$$i^G_P : K_0(M(\mathbb{Q}_p),\mathbb{C})\to K_0(G(\mathbb{Q}_p),\mathbb{C}), \qquad r^G_P : K_0(G(\mathbb{Q}_p),\mathbb{C})\to K_0(M(\mathbb{Q}_p),\mathbb{C}).$$

Moreover, one can show these operations preserve stability so that we get homomorphisms

$$i^G_P : K_0(M(\mathbb{Q}_p),\mathbb{C})^{st}\to K_0(G(\mathbb{Q}_p),\mathbb{C})^{st}, \qquad r^G_P : K_0(G(\mathbb{Q}_p),\mathbb{C})^{st}\to K_0(M(\mathbb{Q}_p),\mathbb{C})^{st}.$$

Now let G^* denote the unique quasi-split group that is an inner form of G. We assume that G arises as an extended pure inner twist of G^* and fix this extra structure (G,ρ,z). One can work with more general G using Kaletha's theory of rigid inner twists, but this is not necessary to explore the connections to this paper where G is always quasi-split.

We also need to introduce endoscopy for G. For convenience, we recall:

Definition A.3.1. A *refined endoscopic datum* for a connected reductive group G over a local field F is a tuple $(H, \mathscr{H}, s, \eta)$ which consists of

- a quasi-split group H over F,
- an extension $\mathscr{H}$ of W_F by $\widehat{H}$ such that the map $W_F \to \mathrm{Out}(\widehat{H})$ coincides with the map $\rho_H : W_F \to \mathrm{Out}(\widehat{H})$ induced by the action of W_F on $\widehat{H} \subset {}^L H$,
- an element $s \in Z(\widehat{H})^\Gamma$,
- an L-homomorphism $\eta : \mathscr{H} \to {}^L G$,

satisfying the condition:

- we have $\eta(\widehat{H}) = Z_{\widehat{G}}(s)^\circ$.

Now suppose that $(H, \mathscr{H}, s, \eta)$ is a refined endoscopic datum for G. Then after fixing splittings of $G, H, \widehat{G}, \widehat{H}$, there is a canonical endoscopic transfer of distributions inducing a morphism

$$\mathrm{Trans}_H^G : K_0(H(\mathbb{Q}_p), \mathbb{C})^{st} \to K_0(G(\mathbb{Q}_p), \mathbb{C}).$$

Furthermore, suppose we have a refined endoscopic datum $(H_M, \mathscr{H}_M, s, \eta_M)$ of M such that $\mathscr{H}_M$ is a Levi subgroup of $\mathscr{H}$ and the following diagram commutes:

$$\begin{tikzcd} \mathscr{H} \arrow[r, "\eta"] & {}^L G \\ \mathscr{H}_M \arrow[r, "\eta_M"] \arrow[u, hook] & {}^L M. \arrow[u, hook] \end{tikzcd}$$

The datum $(H_M, \mathscr{H}_M, H, M, s, \eta)$ along with these compatibilities is called an *embedded* endoscopic datum in [Ber21; BS22]. Our fixed choice of splittings of G, H and their duals determines from P a parabolic subgroup P_{H_M} of H with Levi subgroup H_M. We then have an equality

$$\mathrm{Trans}_H^G \circ i_{P_{H_M}}^H = i_P^G \circ \mathrm{Trans}_{H_M}^M. \tag{A.2}$$

There is also a compatibility of Trans and r which we now recall. A refined endoscopic datum $\mathfrak{e} = (H, \mathscr{H}, s, \eta)$ of G and a Levi subgroup $M \subset G$ can be upgraded to the structure of an embedded endoscopic datum in potentially many non-equivalent ways and these are parametrized by a set $D(M, \mathfrak{e}) \cong W(\widehat{H}) \setminus W(M,H)/W(\widehat{M})$, where $W(M,H)$ is defined to be the subset of the Weyl group $W(\widehat{G})$ of $\widehat{G}$ such that $Z_{^L H}((w \circ \eta)^{-1}(Z(\widehat{M})^{\Gamma}))$ surjects onto W_F, and where $W(\widehat{H})$ is identified with a subgroup of $W(\widehat{G})$ via η. Then we have

$$r_P^G \circ \mathrm{Trans}_H^G = \sum_{D(M,\mathfrak{e})} \mathrm{Trans}_{H_M}^M \circ r_{P_{H_M}}^H.$$

We now define the map $\mathrm{Red}_b^{\mathfrak{e}}$ which plays a crucial role in the statement of the averaging formula. We define

$$\mathrm{Red}_b^{\mathfrak{e}} : K_0(H(\mathbb{Q}_p), \mathbb{C})^{st} \to K_0(J_b(\mathbb{Q}_p), \mathbb{C}) \tag{A.3}$$

by

$$\mathrm{Red}_b^{\mathfrak{e}} = (\sum_{D(M,\mathscr{H})} \mathrm{Trans}_{J_b}^{H_M} \circ r_{P_H^{op}}^H) \otimes \overline{\delta}_P^{\frac{1}{2}} e(J_b),$$

where $\overline{\delta}_P$ is the transport of the modulus character for P to J_b, and $e(J_b) \in \{\pm 1\}$ is the Kottwitz sign.

Then the (still largely conjectural) averaging formula gives a relation satisfied by $R\Gamma_c^{\flat}(G,b,\mu)$ at Langlands (or Arthur) parameters ϕ for which there is an associated stable distribution $S\Theta_{\phi,G}$ on G satisfying endoscopic character identities as in [Kal16, Conjecture D]. In particular, this is the case for all tempered L-parameters. To describe the expected formula, fix such a parameter ϕ and suppose that $(H, \mathscr{H}, s, \eta)$ is an endoscopic datum such that ϕ factors as $\mathscr{L}_F \xrightarrow{\phi^H} \mathscr{H} \to {}^L G$. Then we expect

Conjecture A.3.2 (Averaging Formula)**.** *We expect the following equality in $K_0(G(\mathbb{Q}_p) \times W_{E_\mu})$:*

$$\sum_{b \in B(G,\mu)} [R\Gamma_c^{\flat}(G,b,\mu)[\mathrm{Red}_b^{\mathfrak{e}}(S\Theta_{\phi^H,H})]] = \mathrm{Trans}_H^G(S\Theta_{\phi^H,H}) \boxtimes \mathrm{tr}(r_\mu \circ \phi|_{W_{E_\mu}} \mid s).$$

In particular, when $\mathfrak{e}$ is the trivial endoscopic datum given by $\mathfrak{e}_{\mathrm{triv}} = (G^*, {}^L G^*, 1, \mathrm{id})$ then we expect

$$\sum_{b \in B(G,\mu)} [R\Gamma_c^{\flat}(G,b,\mu)[\mathrm{Red}_b^{\mathfrak{e}_{\mathrm{triv}}}(S\Theta_{\phi,G^*})]] = S\Theta_{\phi,G} \boxtimes r_\mu \circ \phi|_{W_{E_\mu}}.$$

For GL_n, this is known in the trivial endoscopic case for all representations by [Shi12]. Because L-packets are singletons for GL_n, the trivial endoscopic case implies the endoscopic versions of the formula. In [MN21], the formula is proven for discrete parameters and elliptic endoscopy of unramified GU_n, for n odd. In [Ber21], a strategy is outlined to prove this formula in the elliptic endoscopic cases using the cohomology of Igusa and Shimura varieties. This strategy should (eventually) yield results comparable to [MN21] whenever adequate global results are known about the Langlands correspondence and the cohomology of Shimura and Igusa varieties.

The averaging formulas imply strong results about $R\Gamma_c^{\flat}(G,b,\mu)$. For instance, in [MN21, §6] it is shown that the averaging formula for each elliptic endoscopic group and for ϕ a supercuspidal parameter implies the Kottwitz conjecture as in [RV14, Conjecture 7.3].

A.3.2 Proof of Proposition 2.11.11

The averaging formula in §2.11.1 corresponds to the case of the trivial endoscopic triple $\mathfrak{e}_{\mathrm{triv}} = (G^*, {}^LG^*, 1, \mathrm{id})$. Hence to check that Theorem 2.11.10 agrees with A.3.2, we just need to check that $\mathrm{Red}_b^{\mathfrak{e}_{\mathrm{triv}}}(S\Theta_{\phi,G^*})$ coincides with $\mathrm{Red}_{b,\phi}$ for ϕ induced from ϕ_T generic. Since ϕ_T is generic, by Lemma 2.3.18 ϕ should give rise to a well-defined L-parameter with trivial monodromy. Therefore, under the LLC_G appearing in Assumption 2.7.5, we are assuming the parameter ϕ has an L-packet given by the irreducible constituents of the multiplicity free representation $i_B^G(\chi)$, by Assumption 2.7.5 (3). Suppose first that $b \in B(G)_{\mathrm{un}}$. Then we have

$$[\mathrm{Red}_{b,\phi}] = \sum_{w \in W_G/W_{M_b}} i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{1/2}(-1)^{\langle 2\hat{\rho}_G, \nu_b\rangle}.$$

in $K_0(G(\mathbb{Q}_p))$. The set $D(M,\mathfrak{e})$ is a singleton and corresponds to the trivial embedded datum where $H_M = M$. Note that $r_{P_b^{op}}^G(i_B^G(\chi)) = r_{P_b}^G(i_B^G(\chi))$ and that the latter term can be simplified by the geometric lemma of [BZ77].

$$\begin{aligned}
\mathrm{Red}_b^{\mathfrak{e}_{\mathrm{triv}}}(S\Theta_{\phi,G^*}) &= (\mathrm{Trans}_{M_b}^{J_b} \circ r_{P_b^{op}}^{G})(i_B^G(\chi)) \otimes \overline{\delta}_P^{\frac{1}{2}} e(J_b) \\
&= \mathrm{Trans}_{M_b}^{J_b} \circ (\sum_{w \in W_G/W_{M_b}} i_{B\cap M_b}^{M_b} \chi^w) \otimes \overline{\delta}_P^{\frac{1}{2}} e(J_b) \\
&= (\sum_{w \in W_G/W_{M_b}} i_{B_b}^{J_b}) \circ \mathrm{Trans}_T^{T_b} \chi^w \otimes \overline{\delta}_P^{\frac{1}{2}} (-1)^{\langle 2\hat{\rho}_G, \nu_b \rangle} \\
&= \mathrm{Red}_{b,\phi}.
\end{aligned}$$

where the third equality is (A.2) combined with an application of [HKW22, Lemma A.2.1].

Now consider the case where $b \notin B(G)_{\mathrm{un}}$. We must show that $\mathrm{Red}_b^{\mathfrak{e}_{\mathrm{triv}}}(i_B^G(\chi)) = 0$, for which it suffices to show that $\mathrm{Trans}_{M_b}^{J_b}(i_{B\cap M_b}^{M_b}(\chi^w)) = 0$ for each $w \in W_G/W_{M_b}$. This follows from the fact that T does not transfer to J_b by assumption and the character of $i_{B\cap M_b}^{M_b}(\chi^w)$ is supported on the conjugates of T as per [Dij72b, Theorem 3].

Appendix B

Appendix to Chapter 3

B.1 Spectral Decomposition of Sheaves on Bun_G, by David Hansen

Let $G/\mathbb{Q}_p$ be a connected reductive group, $k/\mathbf{Z}_\ell$ an algebraically closed field. If $\mathrm{char}(k) \neq 0$ we assume ℓ is very good for G. We write $\mathrm{D}(H(\mathbb{Q}_p), k)$ for the unbounded derived category of smooth k-representations of $H(\mathbb{Q}_p)$.

Set $\mathrm{D}(\mathrm{Bun}_G) = \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, k)$, regarded as a stable ∞-category whenever convenient. Let $\mathfrak{X}_{\hat{G}} = Z^1(W_E, \hat{G})_k/\hat{G}$ be the stack of L-parameters over k, and let $X_{\hat{G}}$ be its coarse moduli space, $q : \mathfrak{X}_{\hat{G}} \to X_{\hat{G}}$ the natural map. We will regard $\mathfrak{X}_{\hat{G}}$ as a disjoint union of finite type algebraic stacks over k, and $X_{\hat{G}}$ as a disjoint union of finite type affine k-schemes. As in [FS21], we have the spectral action of $\mathrm{Perf}(\mathfrak{X}_{\hat{G}})$ on $\mathrm{D}(\mathrm{Bun}_G)$, and there is a natural map $\Psi_G : \mathscr{O}(\mathfrak{X}_{\hat{G}}) = \mathscr{O}(X_{\hat{G}}) \to \mathfrak{Z}(\mathrm{D}(\mathrm{Bun}_G)) := \pi_0(\mathrm{id}_{\mathrm{D}(\mathrm{Bun}_G)})$, where we recall that $Z^1(W_E, \hat{G})_k$ is a disjoint union of affine schemes by [FS21, Theorem VIII.1.3]. These two structures are compatible (as proven by Zou [Zou22, Theorem 5.2.1]).

By [FS21, Prop. VIII.3.8], the set of closed points $X_{\hat{G}}(k)$ is naturally in bijection with the set of isomorphism classes of semisimple L-parameters $\phi : W_E \to {}^LG(k)$. Let $\mathfrak{m}_\phi \subset \mathscr{O}(X_{\hat{G}})$ be the maximal ideal associated with a given ϕ.

Definition B.1.1. Given any ϕ as above, $\mathrm{D}(\mathrm{Bun}_G)_\phi \subset \mathrm{D}(\mathrm{Bun}_G)$ is the full subcategory of $A \in \mathrm{D}(\mathrm{Bun}_G)$ such that for every $f \in \mathscr{O}(X_{\hat{G}}) \smallsetminus \mathfrak{m}_\phi$, $A \xrightarrow{\cdot f} A$ is an isomorphism. Here $\cdot f$ is the endomorphism of A induced by Ψ_G.

We will call objects of $\mathrm{D}(\mathrm{Bun}_G)_\phi$ *ϕ-local* sheaves.

By construction, $\mathrm{D}(\mathrm{Bun}_G)_\phi$ is a full subcategory of $\mathrm{D}(\mathrm{Bun}_G)$ stable under arbitrary limits and colimits, and the tautological inclusion functor $\iota_\phi : \mathrm{D}(\mathrm{Bun}_G)_\phi \hookrightarrow \mathrm{D}(\mathrm{Bun}_G)$ commutes with limits and colimits. By the ∞-categorical adjoint functor theorem [Lur09, Cor. 5.5.2.9.(2)], it therefore admits a left adjoint $\mathscr{L}_\phi : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_G)_\phi$.[1] The unit of the adjunction gives a map $A \to \iota_\phi \mathscr{L}_\phi A =: A_\phi$ functorially in A. Since ι_ϕ is fully faithful, $\mathscr{L}_\phi \iota_\phi = \mathrm{id}$, so $(A_\phi)_\phi = A_\phi$, i.e. the endofunctor $A \rightsquigarrow A_\phi$ is idempotent. We remark that $\mathrm{D}(\mathrm{Bun}_G)_\phi$ is a Bousfield localization of $\mathrm{D}(\mathrm{Bun}_G)$, and the map $A \to A_\phi$ is the initial map from A to a ϕ-local sheaf.

Proposition B.1.2. *The full subcategory* $\mathrm{D}(\mathrm{Bun}_G)_\phi$ *is preserved by the spectral action, and $A \rightsquigarrow A_\phi$ commutes with the spectral action. Moreover,* $\mathrm{supp}(A_\phi) \subseteq \mathrm{supp}(A)$.

Proof. The first claim is clear, since the spectral action commutes with the action of $\mathscr{O}(X_{\hat{G}})$. For the remaining claims (and some later arguments), it is useful to give an explicit formula for A_ϕ. Let $\mathscr{I}_\phi$ be the diagram category whose objects are elements of $\mathscr{O}(X_{\hat{G}}) \smallsetminus \mathfrak{m}_\phi$ and where a morphism $f \to g$ is an element $h \in \mathscr{O}(X_{\hat{G}}) \smallsetminus \mathfrak{m}_\phi$ such that $g = fh$. This is clearly cofiltered. Let $F \in \mathrm{Fun}(\mathscr{I}_\phi, \mathrm{D}(\mathrm{Bun}_G))$ be the functor sending f to A and sending a morphism $h \in \mathrm{Mor}(f,g)$ to $\cdot h \in \mathrm{End}(A)$. Then $A_\phi = \mathrm{colim}_{i \in \mathscr{I}_\phi} F(i)$. The remaining claims are now immediate, since perfect complexes on $\mathfrak{X}_{\hat{G}}$ are in particular $\mathscr{O}_{X_{\hat{G}}}$-modules and the spectral action is an action of ∞-categories. $\square$

To make sense of the next proposition, note that for any $A, B \in \mathrm{D}(\mathrm{Bun}_G)$, $\mathrm{Hom}(B,A)$ is naturally a $\mathfrak{Z}(\mathrm{D}(\mathrm{Bun}_G))$-module, whence a $\mathscr{O}(X_{\hat{G}})$-module.

Proposition B.1.3. *If $C \in \mathrm{D}(\mathrm{Bun}_G)$ is compact, then* $\mathrm{Hom}(C, A_\phi) \cong \mathrm{Hom}(C,A)_{\mathfrak{m}_\phi}$ *functorially in A and C, where the RHS is the usual localization as an $\mathscr{O}(X_{\hat{G}})$-module.*

Proof. Notation as in the previous proof, we have

$$\begin{aligned}\mathrm{Hom}(C, A_\phi) &\cong \mathrm{Hom}(C, \mathrm{colim}_{i \in \mathscr{I}_\phi} F(i)) \\ &\cong \mathrm{colim}_{i \in \mathscr{I}_\phi} \mathrm{Hom}(C, F(i)) \\ &\cong \mathrm{Hom}(C,A)_{\mathfrak{m}_\phi}\end{aligned}$$

where the second isomorphism follows from the compactness of C and the third isomorphism is immediate from the definition of $(-)_{\mathfrak{m}_\phi}$. $\square$

[1] To see that ι_ϕ is accessible, use [Lur09, Prop. 5.4.7.7] together with the fact that ι_ϕ admits a right adjoint, which follows from [Lur09, Cor. 5.5.2.9.(1)].

Proposition B.1.4. *If A is ULA, then also A_ϕ is ULA.*

Proof. Recall from [FS21, Prop. VII.7.9] that $B \in \mathrm{D}(\mathrm{Bun}_G)$ is ULA iff $\mathrm{RHom}(C,B) \in \mathrm{Perf}(k)$ is a perfect complex for all compact objects $C \in \mathrm{D}(\mathrm{Bun}_G)$. Now, if C is compact, $\mathrm{RHom}(C,-)$ commutes with filtered colimits, so

$$\begin{aligned}\mathrm{RHom}(C,A_\phi) &\simeq \mathrm{RHom}(C,\mathrm{colim}_{i\in\mathscr{I}_\phi}F(i))\\ &\simeq \mathrm{colim}_{i\in\mathscr{I}_\phi}\mathrm{RHom}(C,F(i))\end{aligned}$$

with notation as in the proof of Proposition B.1.2. Since $F(i) \simeq A$ for all i, $\mathrm{colim}_{i\in\mathscr{I}_\phi}\mathrm{RHom}(C,F(i))$ is a filtered colimit of perfect complexes P_i which vanish outside a finite interval independent of n, and with $\dim_k(H^j(P_i))$ bounded independently of i. It then easily follows that $\mathrm{colim}_{i\in\mathscr{I}_\phi}\mathrm{RHom}(C,F(i))$ is perfect, whence the claim. □

Proposition B.1.5. *If A is ULA, the natural maps $A \to \prod_\phi A_\phi \leftarrow \oplus_\phi A_\phi$ are isomorphisms, where the direct sum is over all semi-simple L-parameters. In particular, A_ϕ is functorially a direct summand of A for ULA sheaves A, and the functor $(-)_\phi$ on ULA sheaves is perverse t-exact.*

The isomorphism $\oplus_\phi A_\phi \xrightarrow{\sim} \prod_\phi A_\phi$ may be surprising at first glance. To put this in context, we remind the reader that if $(\pi_i)_{i\in I}$ is a collection of admissible smooth $k[G(\mathbb{Q}_p)]$-modules whose product $\prod_i \pi_i$ is admissible, then $\oplus_i \pi_i \xrightarrow{\sim} \prod_i \pi_i$ automatically, because admissibility of $\prod_i \pi_i$ implies that for any given compact open subgroup $K \subset G(\mathbb{Q}_p)$ we have $\pi_i^K = 0$ for all but finitely many i. A similar argument occurs in the following proof, which actually shows that if $(A_i)_{i\in I}$ is any collection of ULA sheaves on Bun_G whose product $\prod_i A_i$ is ULA, then $\oplus_i A_i \xrightarrow{\sim} \prod_i A_i$ automatically.

Proof. We first show that $A \to \prod_\phi A_\phi$ is an isomorphism. Let C be any compact object. It suffices to prove that the natural map

$$\mathrm{Hom}(C,A) \to \prod_\phi \mathrm{Hom}(C,A_\phi) \cong \mathrm{Hom}(C,\prod_\phi A_\phi)$$

is an isomorphism, since $\mathrm{D}(\mathrm{Bun}_G)$ is compactly generated [FS21, Theorem I.5.1 (iii)]. As in the previous proof, $\mathrm{RHom}(C,A)$ is a perfect complex, so $\mathrm{Hom}(C,A)$ is a finite k-vector space. In particular, it is a finite length $\mathscr{O}(X_{\hat{G}})$-module supported at a finite set of closed points $S \subset X_{\hat{G}}(k)$, so if $\phi \notin S$ then $\mathrm{Hom}(C,A_\phi) =$

$\mathrm{Hom}(C,A)_{\mathfrak{m}_\phi} = 0$ using Proposition B.1.3. We then conclude that

$$\begin{aligned}\mathrm{Hom}(C,A) &= \oplus_{\phi\in S}\mathrm{Hom}(C,A)_{\mathfrak{m}_\phi}\\ &= \oplus_{\phi\in S}\mathrm{Hom}(C,A_\phi)\\ &= \prod_\phi \mathrm{Hom}(C,A_\phi)\end{aligned}$$

where the first equality follows from general nonsense about finite length modules over commutative rings, the second equality follows from Proposition B.1.3, and the third equality follows from the vanishing of $\mathrm{Hom}(C,A_\phi)$ for all but finitely many ϕ. This also shows that $\mathrm{Hom}(C,\oplus_\phi A_\phi) \cong \oplus_\phi \mathrm{Hom}(C,A_\phi) \to \prod_\phi \mathrm{Hom}(C,A_\phi)$ is an isomorphism (here again the first isomorphism follows from compactness of C), which implies that $\oplus_\phi A_\phi \xrightarrow{\sim} \prod_\phi A_\phi$ is an isomorphism. □

Next, recall the Verdier duality functor $\mathbb{D}_{\mathrm{Bun}_G}$ on $\mathrm{D}(\mathrm{Bun}_G)$, which induces an involutive anti-equivalence on the subcategory of ULA sheaves. Recall also that, for any A, the diagram

$$\begin{tikzcd} \mathscr{O}(X_{\hat{G}}) \arrow[r, "\Psi_G"] \arrow[d, "f\mapsto f^\vee"] & \mathrm{End}(A) \arrow[d] \\ \mathscr{O}(X_{\hat{G}}) \arrow[r, "\Psi_G"] & \mathrm{End}(\mathbb{D}_{\mathrm{Bun}_G}(A)) \end{tikzcd}$$

commutes, where $f \mapsto f^\vee$ is the involution of $\mathscr{O}(X_{\hat{G}})$ induced by composition with the Chevalley involution at the level of L-parameters. Since $f \in \mathfrak{m}_\phi$ iff $f^\vee \in \mathfrak{m}_{\phi^\vee}$, we deduce that if A is ϕ-local then $\mathbb{D}_{\mathrm{Bun}_G}(A)$ is $\phi^\vee$-local. Using biduality, we also get that if A is ULA then A is ϕ-local if and only if $\mathbb{D}_{\mathrm{Bun}_G}(A)$ is $\phi^\vee$-local.

Corollary B.1.6. *If A is ULA, then* $\mathbb{D}_{\mathrm{Bun}_G}(A_\phi) \cong (\mathbb{D}_{\mathrm{Bun}_G}(A))_{\phi^\vee}$.

Proof. By Proposition B.1.5 and the remarks preceding its proof, the decomposition $A = \oplus_\phi A_\phi$ dualizes to a decomposition

$$\mathbb{D}_{\mathrm{Bun}_G}(A) = \prod_\phi \mathbb{D}_{\mathrm{Bun}_G}(A_\phi) \cong \oplus_\phi \mathbb{D}_{\mathrm{Bun}_G}(A_\phi).$$

On the other hand, applying Proposition B.1.5 directly to $\mathbb{D}_{\mathrm{Bun}_G}(A)$ gives a decomposition

$$\mathbb{D}_{\mathrm{Bun}_G}(A) \cong \oplus_{\phi'} (\mathbb{D}_{\mathrm{Bun}_G}(A))_{\phi'},$$

so comparing these we get a natural isomorphism

$$\oplus_{\phi}\mathbb{D}_{\mathrm{Bun}_G}(A_{\phi}) \cong \oplus_{\phi'}(\mathbb{D}_{\mathrm{Bun}_G}(A))_{\phi'}.$$

Applying $(-)_{\phi^{\vee}}$ to both sides and using that $\mathbb{D}_{\mathrm{Bun}_G}(A_{\phi})$ is $\phi^{\vee}$-local, we get the claim. $\square$

We now conclude with the following consequences of the above discussion.

Lemma B.1.7. *The following is true.*

1. *Any Schur irreducible object $A \in \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_{\ell})_{\phi}$ has Fargues-Scholze parameter equal to ϕ as conjugacy classes of parameters.*

2. *Given $V \in \mathrm{Rep}_k({}^L G^I)$, the Hecke operator $T_V : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_G)^{BW^I_{\mathbb{Q}_p}}$ takes the subcategory $\mathrm{D}(\mathrm{Bun}_G)_{\phi}$ to $\mathrm{D}(\mathrm{Bun}_G)_{\phi}^{BW^I_{\mathbb{Q}_p}}$, and there is a natural isomorphism $T_V((-)_{\phi}) \simeq (T_V(-))_{\phi}$.*

3. *Suppose that G is unramified. Let $K_p^{\mathrm{hs}} \subset G(\mathbb{Q}_p)$ be a choice of hyperspecial level, and $H_{K_p}^{\mathrm{hs}}$ the unramified Hecke algebra with coefficients in k. Consider $\mathfrak{m} \subset H_{K_p^{\mathrm{hs}}}$ a maximal ideal, with associated L-parameter $\phi_{\mathfrak{m}} : W_{\mathbb{Q}_p} \to {}^L G(k)$. Then, given a smooth irreducible representation $A \in \mathrm{D}(G(\mathbb{Q}_p), k) \subset \mathrm{D}(\mathrm{Bun}_G)$, we have an isomorphism*

$$R\Gamma(K_p^{\mathrm{hs}}, A)_{\mathfrak{m}} \simeq R\Gamma(K_p^{\mathrm{hs}}, A_{\phi_{\mathfrak{m}}}),$$

   where the LHS is the usual localization under the smooth Hecke algebra.

4. *If $A \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G)$ is ULA then one has a direct sum decomposition*

$$A \simeq \bigoplus_{\phi} A_{\phi}$$

   ranging over all semi-simple L-parameters.

Proof. Claims (2) and (4) follow from Proposition B.1.2 and Proposition B.1.5, respectively, where for claim (2) we use the relationship between Hecke operators and the spectral action described above.

For (1), this follows since the action of $\mathscr{O}_{\mathfrak{X}_{\hat{G}}}(\mathfrak{X}_{\hat{G}})$ on A will factor through the maximal ideal $\mathfrak{m}_A$ defined by the semi-simple L-parameter ϕ_A^{FS} attached to

A by the above discussion, and therefore $A \in \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ forces an equality of maximal ideals $\mathfrak{m}_A = \mathfrak{m}_\phi$.

For (3), we use the arguments in Koshikawa [Kos21b, Page 6]. In particular, there it it is shown that the map

$$\mathscr{O}_{\mathfrak{X}_{\hat{G}}}(\mathfrak{X}_{\hat{G}}) \to \mathrm{End}_{G(\mathbb{Q}_p)}(\mathrm{cInd}_{K_p^{\mathrm{hs}}}^{G(\mathbb{Q}_p)}(\overline{\mathbb{F}}_\ell)) \simeq H_{K_p}^{\mathrm{hs,op}}$$

given by the spectral action, factors through the usual action by the unramified Hecke algebra composed with the involution $KhK \to Kh^{-1}K$. Moreover, the pullback of the maximal ideal $\mathfrak{m} \subset H_{K_p^{\mathrm{hs}}}$ is given by the maximal ideal $\mathfrak{m}_{\phi_{\mathfrak{m}}} \subset \mathscr{O}_{X_{\hat{G}}}(\mathfrak{X}_{\hat{G}})$. Now, by arguing as in Proposition B.1.3, we have an identification:

$$\mathrm{RHom}(\mathrm{cInd}_{K_p^{\mathrm{hs}}}^{G(\mathbb{Q}_p)}(\overline{\mathbb{F}}_\ell), A_{\phi_{\mathfrak{m}}}) \simeq \mathrm{RHom}(\mathrm{cInd}_{K_p^{\mathrm{hs}}}^{G(\mathbb{Q}_p)}(\overline{\mathbb{F}}_\ell), A)_{\mathfrak{m}_{\phi_{\mathfrak{m}}}}.$$

Using Frobenius reciprocity, this gives an identification:

$$R\Gamma(K_p^{\mathrm{hs}}, A_{\phi_{\mathfrak{m}}}) \simeq R\Gamma(K_p^{\mathrm{hs}}, A)_{\mathfrak{m}_{\phi_{\mathfrak{m}}}}$$

but the RHS identifies with $R\Gamma(K_p^{\mathrm{hs}}, A)_{\mathfrak{m}}$, as explained above. □

We note that we get the following Corollary of this.

Corollary B.1.8. *Let A be a complex of smooth $G(\mathbb{Q}_p)$-representations with coefficients in k which is admissible (i.e A^K is a perfect complex for all compact open $K \subset G(\mathbb{Q}_p)$). We then have a decomposition*

$$A \simeq \bigoplus_{\phi} A_\phi$$

running over semisimple L-parameters, where any irreducible constituent π of A_ϕ has Fargues-Scholze parameter equal to ϕ_π^{FS}, as conjugacy classes of parameters.

Proof. This follows immediately by applying to Lemma B.1.7 (1) and (4) to the full subcategory $\mathrm{D}(G(\mathbb{Q}_p), k) \subset \mathrm{D}(\mathrm{Bun}_G)$ □

Bibliography

[AL21a] Johannes Anschütz and Arthur César Le Bras. “Averaging Functors in Fargues’ Program for GL_n”. In: *Preprint* (2021). arXiv:2104.04701.

[AL21b] Johannes Anschütz and Arthur-César Le Bras. “A Fourier Transform for Banach-Colmez spaces”. In: *Preprint* (2021). arXiv:2111.11116.

[Art02] James Arthur. “A stable trace formula. I. General expansions”. In: *J. Inst. Math. Jussieu* 1.2 (2002), pp. 175–277. ISSN: 1474-7480. DOI: 10.1017/S1474-748002000051. URL: https://doi.org/10.1017/S1474-748002000051.

[Art04] James Arthur. “Automorphic representations of GSp(4)”. In: *Contributions to automorphic forms, geometry, and number theory*. Johns Hopkins Univ. Press, Baltimore, MD, 2004, pp. 65–81.

[Art81] James Arthur. “The trace formula in invariant form”. In: *Ann. of Math. (2)* 114.1 (1981), pp. 1–74. ISSN: 0003-486X. DOI: 10.2307/1971376. URL: https://doi.org/10.2307/1971376.

[BDK86] J. Bernstein, P. Deligne, and D. Kazhdan. “Trace Paley-Wiener theorem for reductive p-adic groups”. In: *J. Analyse Math.* 47 (1986), pp. 180–192. ISSN: 0021-7670. DOI: 10.1007/BF02792538. URL: https://doi.org/10.1007/BF02792538.

[Ber21] Alexander Bertoloni Meli. *An averaging formula for the cohomology of PEL-type Rapoport–Zink spaces*. 2021. DOI: 10.48550/ARXIV.2103.11538. URL: https://arxiv.org/abs/2103.11538.

[BG02] A. Braverman and D. Gaitsgory. “Geometric Eisenstein series”. In: *Invent. Math.* 150.2 (2002), pp. 287–384. ISSN: 0020-9910. DOI: 10.1007/s00222-002-0237-8. URL: https://doi.org/10.1007/s00222-002-0237-8.

[BG08] Alexander Braverman and Dennis Gaitsgory. "Deformations of local systems and Eisenstein series". In: *Geom. Funct. Anal.* 17.6 (2008), pp. 1788–1850. ISSN: 1016-443X. DOI: 10.1007/s00039-007-0645-4. URL: https://doi.org/10.1007/s00039-007-0645-4.

[BHN22] Alexander Bertoloni Meli, Linus Hamann, and Kieu-Hieu Nguyen. "Compatibility of Fargues-Scholze correspondence for unitary groups". In: *Preprint* (2022). arXiv:2207.13193.

[BM22] A. Bertoloni Meli and O. Masao. "The $B(G)$ parametrization of the local Langlands correspondence". In: *Preprint* (2022). arXiv:2211.13864.

[Bor79] A. Borel. "Automorphic L-functions". In: *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*. Proc. Sympos. Pure Math., XXXIII. Amer. Math. Soc., Providence, R.I., 1979, pp. 27–61.

[Bou68] N. Bourbaki. *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*. Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics], No. 1337. Hermann, Paris, 1968, 288 pp. (loose errata).

[Boy99a] P. Boyer. "Mauvaise réduction des variétés de Drinfeld et correspondance de Langlands locale". In: *Invent. Math.* 138.3 (1999), pp. 573–629. ISSN: 0020-9910. DOI: 10.1007/s002220050354. URL: https://doi.org/10.1007/s002220050354.

[Boy99b] P. Boyer. "Mauvaise réduction des variétés de Drinfeld et correspondance de Langlands locale". In: *Invent. Math.* 138.3 (1999), pp. 573–629. ISSN: 0020-9910. DOI: 10.1007/s002220050354. URL: https://doi.org/10.1007/s002220050354.

[Bra+02a] A. Braverman et al. "Intersection cohomology of Drinfeld's compactifications". In: *Selecta Math. (N.S.)* 8.3 (2002), pp. 381–418. ISSN: 1022-1824. DOI: 10.1007/s00029-002-8111-5. URL: https://doi.org/10.1007/s00029-002-8111-5.

[Bra+02b] A. Braverman et al. "Intersection cohomology of Drinfeld's compactifications". In: *Selecta Math. (N.S.)* 8.3 (2002), pp. 381–418. ISSN: 1022-1824. DOI: 10.1007/s00029-002-8111-5. URL: https://doi.org/10.1007/s00029-002-8111-5.

[Bra03] Tom Braden. "Hyperbolic localization of intersection cohomology". In: *Transform. Groups* 8.3 (2003), pp. 209–216. ISSN: 1083-4362. DOI: 10.1007/s00031-003-0606-4. URL: https://doi.org/10.1007/s00031-003-0606-4.

[Bru61] François Bruhat. "Distributions sur un groupe localement compact et applications à l'étude des représentations des groupes $\wp$-adiques". In: *Bull. Soc. Math. France* 89 (1961), pp. 43–75. ISSN: 0037-9484. URL: http://www.numdam.org/item?id=BSMF_1961__89__43_0.

[BS22] Alexander Bertoloni Meli and Sug Woo Shin. *The stable trace formula for Igusa varieties, II*. 2022. DOI: 10.48550/ARXIV.2205.05462. URL: https://arxiv.org/abs/2205.05462.

[BW71] N. Burgyone and C. Williamson. "Some Computations involving simple Lie algebras". In: *Proc. 2nd Symposium of Symbolica and Algebraic Manipulation* (1971).

[BW80] Armand Borel and Nolan R. Wallach. *Continuous cohomology, discrete subgroups, and representations of reductive groups*. Annals of Mathematics Studies, No. 94. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1980, pp. xvii+388. ISBN: 0-691-08248-0; 0-691-08249-9.

[BZ77] I. N. Bernstein and A. V. Zelevinsky. "Induced representations of reductive p-adic groups. I". In: *Ann. Sci. École Norm. Sup. (4)* 10.4 (1977), pp. 441–472. ISSN: 0012-9593. URL: http://www.numdam.org/item?id=ASENS_1977_4_10_4_441_0.

[Car01] L. Carbone. "On the classification of rank 1 groups over non-archimedean local fields". In: *Online Notes* (2001). URL: https://sites.math.rutgers.edu/~carbonel/pdfs/Classification.pdf.

[Cas95] W. Casselman. "Theory of admissible representations". In: *Online Notes* (1995). URL: https://personal.math.ubc.ca/~cass/research/pdf/p-adic-book.pdf.

[CFS21] Miaofen Chen, Laurent Fargues, and Xu Shen. "On the structure of some p-adic period domains". In: *Camb. J. Math.* 9.1 (2021), pp. 213–267. ISSN: 2168-0930. DOI: 10.4310/CJM.2021.v9.n1.a4. URL: https://doi.org/10.4310/CJM.2021.v9.n1.a4.

[CG15] Ping-Shun Chan and Wee Teck Gan. "The local Langlands conjecture for $\mathrm{GSp}(4)$ III: Stability and twisted endoscopy". In: *J. Number Theory* 146 (2015), pp. 69–133. ISSN: 0022-314X. DOI: 10.1016/j.jnt.2013.07.009. URL: https://doi.org/10.1016/j.jnt.2013.07.009.

[Cho17] Kwangho Choiy. "The local Langlands conjecture for the p-adic inner form of Sp_4". In: *Int. Math. Res. Not. IMRN* 6 (2017), pp. 1830–1889. ISSN: 1073-7928. DOI: 10.1093/imrn/rnw043. URL: https://doi.org/10.1093/imrn/rnw043.

[CS17] Ana Caraiani and Peter Scholze. "On the generic part of the cohomology of compact unitary Shimura varieties". In: *Ann. of Math. (2)* 186.3 (2017), pp. 649–766. ISSN: 0003-486X. DOI: 10.4007/annals.2017.186.3.1. URL: https://doi.org/10.4007/annals.2017.186.3.1.

[CS19] Ana Caraiani and Peter Scholze. "On the generic part of the cohomology of non-compact unitary Shimura varieties". In: *Preprint* (2019). arXiv:1909.01898.

[Dat+20] Jean-François Dat et al. "Moduli of Langlands Parameters". In: *Preprint* (2020). arXiv:2009.06708.

[Dat+22] Jean-François Dat et al. "Finiteness for Hecke Algebras of p-adic Groups". In: *Preprint* (2022). arXiv:2203.04929.

[Dat05] J.-F. Dat. "v-tempered representations of p-adic groups. I. l-adic case". In: *Duke Math. J.* 126.3 (2005), pp. 397–469. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-04-12631-4. URL: https://doi.org/10.1215/S0012-7094-04-12631-4.

[Dat12] J.-F. Dat. "Théorie de Lubin-Tate non Abélienne ℓ-entière". In: *Duke Math. J.* 161.6 (2012), pp. 951–1010. ISSN: 0012-7094. DOI: 10.1215/00127094-1548425. URL: https://doi.org/10.1215/00127094-1548425.

[Dij72a] G. van Dijk. "Computation of certain induced characters of p-adic groups". In: *Math. Ann.* 199 (1972), pp. 229–240. ISSN: 0025-5831. DOI: 10.1007/BF01429876. URL: https://doi.org/10.1007/BF01429876.

[Dij72b] G. van Dijk. "Computation of certain induced characters of p-adic groups". In: *Math. Ann.* 199 (1972), pp. 229–240. ISSN: 0025-5831. DOI: 10.1007/BF01429876. URL: https://doi.org/10.1007/BF01429876.

[Don93] Stephen Donkin. "On tilting modules for algebraic groups". In: *Math. Z.* 212.1 (1993), pp. 39–60. ISSN: 0025-5874. DOI: 10.1007/BF02571640. URL: https://doi.org/10.1007/BF02571640.

[Far16] L. Fargues. "Geometrization of the Local Langlands correspondence: An Overview". In: *Preprint* (2016). arXiv. 1602.00999.

[Far20] Laurent Fargues. "Simple connexité des fibres d'une application d'Abel-Jacobi et corps de classes local". In: *Ann. Sci. Éc. Norm. Supér. (4)* 53.1 (2020), pp. 89–124. ISSN: 0012-9593. DOI: 10.24033/asens.2418. URL: https://doi.org/10.24033/asens.2418.

[Fei+99] Boris Feigin et al. "Semi-infinite flags. II. Local and global intersection cohomology of quasimaps' spaces". In: *Differential topology, infinite-dimensional Lie algebras, and applications*. Vol. 194. Amer. Math. Soc. Transl. Ser. 2. Amer. Math. Soc., Providence, RI, 1999, pp. 113–148. DOI: 10.1090/trans2/194/06. URL: https://doi.org/10.1090/trans2/194/06.

[FGV02] E. Frenkel, D. Gaitsgory, and K. Vilonen. "On the geometric Langlands conjecture". In: *J. Amer. Math. Soc.* 15.2 (2002), pp. 367–417. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-01-00388-5. URL: https://doi.org/10.1090/S0894-0347-01-00388-5.

[FM99] Michael Finkelberg and Ivan Mirković. "Semi-infinite flags. I. Case of global curve $\mathbf{P}^1$". In: *Differential topology, infinite-dimensional Lie algebras, and applications*. Vol. 194. Amer. Math. Soc. Transl. Ser. 2. Amer. Math. Soc., Providence, RI, 1999, pp. 81–112. DOI: 10.1090/trans2/194/05. URL: https://doi.org/10.1090/trans2/194/05.

[FS21] L. Fargues and P. Scholze. “Geometrization of the local Langlands Correspondence”. In: *Preprint* (2021). arXiv:2102.13459.

[Gai04] D. Gaitsgory. “On a vanishing conjecture appearing in the geometric Langlands correspondence”. In: *Ann. of Math. (2)* 160.2 (2004), pp. 617–682. ISSN: 0003-486X. DOI: 10.4007/annals.2004.160.617. URL: https://doi.org/10.4007/annals.2004.160.617.

[GHN19] Ulrich Görtz, Xuhua He, and Sian Nie. “Fully Hodge-Newton decomposable Shimura varieties”. In: *Peking Math. J.* 2.2 (2019), pp. 99–154. ISSN: 2096-6075. DOI: 10.1007/s42543-019-00013-2. URL: https://doi.org/10.1007/s42543-019-00013-2.

[GHW22] Daniel Gulotta, David Hansen, and Jared Weinstein. “An enhanced six-functor formalism for diamonds and *v*-stacks”. In: *Preprint* (2022). arXiv:2201.12467.

[GI16] I. Gaisan and N. Imai. “Non-semi-stable loci in Hecke stacks and Fargues’ Conjecture”. In: *Preprint* (2016). arXiv:1608.07446.

[Gro83] F. D. Grosshans. “The invariants of unipotent radicals of parabolic subgroups”. In: *Invent. Math.* 73.1 (1983), pp. 1–9. ISSN: 0020-9910. DOI: 10.1007/BF01393822. URL: https://doi.org/10.1007/BF01393822.

[Gro99] Benedict H. Gross. “Algebraic modular forms”. In: *Israel J. Math.* 113 (1999), pp. 61–93. ISSN: 0021-2172. DOI: 10.1007/BF02780173. URL: https://doi.org/10.1007/BF02780173.

[GS88] Peter B. Gilkey and Gary M. Seitz. “Some representations of exceptional Lie algebras”. In: vol. 25. 1-3. Geometries and groups (Noordwijkerhout, 1986). 1988, pp. 407–416. DOI: 10.1007/BF00191935. URL: https://doi.org/10.1007/BF00191935.

[GT10] Wee Teck Gan and Shuichiro Takeda. “The local Langlands conjecture for Sp(4)”. In: *Int. Math. Res. Not. IMRN* 15 (2010), pp. 2987–3038. ISSN: 1073-7928. DOI: 10.1093/imrn/rnp203. URL: https://doi.org/10.1093/imrn/rnp203.

[GT11] Wee Teck Gan and Shuichiro Takeda. “The local Langlands conjecture for GSp(4)”. In: *Ann. of Math. (2)* 173.3 (2011), pp. 1841–1882. ISSN: 0003-486X. DOI: 10.4007/annals.2011.173.3.12. URL: https://doi.org/10.4007/annals.2011.173.3.12.

[GT14] Wee Teck Gan and Welly Tantono. "The local Langlands conjecture for $\mathrm{GSp}(4)$, II: The case of inner forms". In: *Amer. J. Math.* 136.3 (2014), pp. 761–805. ISSN: 0002-9327. DOI: 10.1353/ajm.2014.0016. URL: https://doi.org/10.1353/ajm.2014.0016.

[GT19] Toby Gee and Olivier Taïbi. "Arthur's multiplicity formula for $\mathbf{GSp}_4$ and restriction to $\mathbf{Sp}_4$". In: *J. Éc. polytech. Math.* 6 (2019), pp. 469–535. ISSN: 2429-7100. DOI: 10.5802/jep.99. URL: https://doi.org/10.5802/jep.99.

[Ham21a] Linus Hamann. "A Jacobian Criterion for Artin v-stacks". In: *Preprint* (2021). arXiv:2209.07495.

[Ham21b] Linus Hamann. "Compatibility of the Gan-Takeda and Fargues-Scholze local Langlands correpsondences". In: *Preprint* (2021). arXiv:2109.01210.

[Han20] D. Hansen. "On the supercuspidal cohomology of basic local Shimura varieties". In: (2020). Available at home page of first named author.

[Han21] David Hansen. "Moduli of local shtukas and Harris's conjecture". In: *Tunis. J. Math.* 3.4 (2021), pp. 749–799. ISSN: 2576-7658. DOI: 10.2140/tunis.2021.3.749. URL: https://doi.org/10.2140/tunis.2021.3.749.

[He14] Xuhua He. "Geometric and homological properties of affine Deligne-Lusztig varieties". In: *Ann. of Math. (2)* 179.1 (2014), pp. 367–404. ISSN: 0003-486X. DOI: 10.4007/annals.2014.179.1.6. URL: https://doi.org/10.4007/annals.2014.179.1.6.

[HHS] Linus Hamann, David Hansen, and Peter Scholze. "Geometric Eisenstein Series and the Fargues-Fontaine Curve," in: (). In Preparation.

[HI23] Linus Hamann and Naoki Imai. "Dualizing Complexes on the Moduli of Parabolic Bundles". In: *preparation* (2023).

[Hir04] Kaoru Hiraga. "On functoriality of Zelevinski involutions". In: *Compos. Math.* 140.6 (2004), pp. 1625–1656. ISSN: 0010-437X. DOI: 10.1112/S0010437X04000892. URL: https://doi.org/10.1112/S0010437X04000892.

[HKW22] David Hansen, Tasho Kaletha, and Jared Weinstein. "On the Kottwitz conjecture for local shtuka spaces". In: *Forum Math. Pi* 10 (2022), Paper No. e13, 79. DOI: 10.1017/fmp.2022.7. URL: https://doi.org/10.1017/fmp.2022.7.

[HL23] Linus Hamann and Si Ying Lee. "Torsion Vanishing for some Shimura Varieties". In: *Preparation* (2023).

[HP79] R. Howe and I. I. Piatetski-Shapiro. "A counterexample to the "generalized Ramanujan conjecture" for (quasi-) split groups". In: *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*. Proc. Sympos. Pure Math., XXXIII. Amer. Math. Soc., Providence, R.I., 1979, pp. 315–322.

[HT01] Michael Harris and Richard Taylor. *The geometry and cohomology of some simple Shimura varieties*. Vol. 151. Annals of Mathematics Studies. With an appendix by Vladimir G. Berkovich. Princeton University Press, Princeton, NJ, 2001, pp. viii+276. ISBN: 0-691-09090-4.

[Hub96] Roland Huber. *Étale cohomology of rigid analytic varieties and adic spaces*. Aspects of Mathematics, E30. Friedr. Vieweg & Sohn, Braunschweig, 1996, pp. x+450. ISBN: 3-528-06794-2. DOI: 10.1007/978-3-663-09991-8. URL: https://doi.org/10.1007/978-3-663-09991-8.

[IM21] Tetsushi Ito and Yoichi Mieda. "Local Saito-Kurokawa A-packets and ℓ-adic cohomology of Rapoport-Zink tower for GSp_4". In: *Preparation* (2021).

[Jan03] Jens Carsten Jantzen. *Representations of algebraic groups*. Second. Vol. 107. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2003, pp. xiv+576. ISBN: 0-8218-3527-0.

[Jan91] Jens C. Jantzen. "First cohomology groups for classical Lie algebras". In: *Representation theory of finite groups and finite-dimensional algebras (Bielefeld, 1991)*. Vol. 95. Progr. Math. Birkhäuser, Basel, 1991, pp. 289–315.

[JS07] Dihua Jiang and David Soudry. "The multiplicity-one theorem for generic automorphic forms of GSp(4)". In: *Pacific J. Math.* 229.2 (2007), pp. 381–388. ISSN: 0030-8730. DOI: 10.2140/pjm.2007.229.381. URL: https://doi.org/10.2140/pjm.2007.229.381.

[Kal16] Tasho Kaletha. "The local Langlands conjectures for non-quasi-split groups". In: *Families of automorphic forms and the trace formula*. Simons Symp. Springer, [Cham], 2016, pp. 217–257.

[Kaz86] David Kazhdan. "Cuspidal geometry of p-adic groups". In: *J. Analyse Math.* 47 (1986), pp. 1–36. ISSN: 0021-7670. DOI: 10.1007/BF02792530. URL: https://doi.org/10.1007/BF02792530.

[Ked17] K. Kedlaya. "Sheaves, stacks, and shtukas". In: *Lecture Notes* (2017). URL: http://swc.math.arizona.edu/aws/2017/2017KedlayaNotes.pdf.

[Key82] Charles David Keys. "On the decomposition of reducible principal series representations of p-adic Chevalley groups". In: *Pacific J. Math.* 101.2 (1982), pp. 351–388. ISSN: 0030-8730. URL: http://projecteuclid.org/euclid.pjm/1102724780.

[Key84] David Keys. "Principal series representations of special unitary groups over local fields". In: *Compositio Math.* 51.1 (1984), pp. 115–130. ISSN: 0010-437X. URL: http://www.numdam.org/item?id=CM_1984__51_1_115_0.

[KM76] Finn Faye Knudsen and David Mumford. "The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div"". In: *Math. Scand.* 39.1 (1976), pp. 19–55. ISSN: 0025-5521. DOI: 10.7146/math.scand.a-11642. URL: https://doi.org/10.7146/math.scand.a-11642.

[Kna73a] A. W. Knapp. "Commutativity of intertwining operators". In: *Bull. Amer. Math. Soc.* 79 (1973), pp. 1016–1018. ISSN: 0002-9904. DOI: 10.1090/S0002-9904-1973-13308-7. URL: https://doi.org/10.1090/S0002-9904-1973-13308-7.

[Kna73b] A. W. Knapp. "Determination of intertwining operators". In: *Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972)*. 1973, pp. 263–268.

[Kos21a] T. Koshikawa. “On Eichler-Shimura relations for local Shimura varieties”. In: *Preprint* (2021). arXiv:2106.10603.

[Kos21b] T. Koshikawa. “On the generic part of the cohomology of local and global Shimura varieties”. In: *Preprint* (2021). arXiv:2106.10602.

[Kot97a] Robert E. Kottwitz. “Isocrystals with additional structure. II”. In: *Compositio Math.* 109.3 (1997), pp. 255–339. ISSN: 0010-437X. DOI: 10.1023/A:1000102604688. URL: https://doi.org/10.1023/A:1000102604688.

[Kot97b] Robert E. Kottwitz. “Isocrystals with additional structure. II”. In: *Compositio Math.* 109.3 (1997), pp. 255–339. ISSN: 0010-437X. DOI: 10.1023/A:1000102604688. URL: https://doi.org/10.1023/A:1000102604688.

[KS16] Arno Kret and Sug-Woo Shin. “Galois representations for general symplectic groups”. In: *Preprint* (2016). arXiv:1609.04223.

[KS72] A. W. Knapp and E. M. Stein. “Irreducibility theorems for the principal series”. In: *Conference on Harmonic Analysis (Univ. Maryland, College Park, Md., 1971)*. 1972, 197–214. Lecture Notes in Math., Vol. 266.

[KS99] Robert E. Kottwitz and Diana Shelstad. “Foundations of twisted endoscopy”. In: *Astérisque* 255 (1999), pp. vi+190. ISSN: 0303-1179.

[KST20] Ju-Lee Kim, Sug Woo Shin, and Nicolas Templier. “Asymptotic behavior of supercuspidal representations and Sato-Tate equidistribution for families”. In: *Adv. Math.* 362 (2020), pp. 106955, 57. ISSN: 0001-8708. DOI: 10.1016/j.aim.2019.106955. URL: https://doi.org/10.1016/j.aim.2019.106955.

[Kur78] Nobushige Kurokawa. “Examples of eigenvalues of Hecke operators on Siegel cusp forms of degree two”. In: *Invent. Math.* 49.2 (1978), pp. 149–165. ISSN: 0020-9910. DOI: 10.1007/BF01403084. URL: https://doi.org/10.1007/BF01403084.

[Laf18] Vincent Lafforgue. “Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale”. In: *J. Amer. Math. Soc.* 31.3 (2018), pp. 719–891. ISSN: 0894-0347. DOI: 10.1090/jams/897. URL: https://doi.org/10.1090/jams/897.

[Lan97] R. P. Langlands. “Representations of abelian algebraic groups”. In: Special Issue. Olga Taussky-Todd: in memoriam. 1997, pp. 231–250. DOI: 10.2140/pjm.1997.181.231. URL: https://doi.org/10.2140/pjm.1997.181.231.

[Lau90] G. Laumon. “Faisceaux automorphes liés aux séries d’Eisenstein”. In: *Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988)*. Vol. 10. Perspect. Math. Academic Press, Boston, MA, 1990, pp. 227–281.

[Le 18] Arthur César Le Bras. “Espaces de Banach-Colmez et faisceaux cohérents sur la courbe de Fargues-Fontaine”. In: *Duke Math. J.* 167.18 (2018), pp. 3455–3532. ISSN: 0012-7094. DOI: 10.1215/00127094-2018-0034. URL: https://doi.org/10.1215/00127094-2018-0034.

[Li-22] Daniel Siyuan Li-Huerta. *The plectic conjecture for local fields*. 2022. URL: https://arxiv.org/abs/2205.05462.

[LR05] Erez M. Lapid and Stephen Rallis. “On the local factors of representations of classical groups”. In: *Automorphic representations, L-functions and applications: progress and prospects*. Vol. 11. Ohio State Univ. Math. Res. Inst. Publ. de Gruyter, Berlin, 2005, pp. 309–359. DOI: 10.1515/9783110892703.309. URL: https://doi.org/10.1515/9783110892703.309.

[LR08] Venkatramani Lakshmibai and Komaranapuram N. Raghavan. *Standard monomial theory*. Vol. 137. Encyclopaedia of Mathematical Sciences. Invariant theoretic approach, Invariant Theory and Algebraic Transformation Groups, 8. Springer-Verlag, Berlin, 2008, pp. xiv+265. ISBN: 978-3-540-76756-5.

[Lur09] Jacob Lurie. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. ISBN: 978-0-691-14049-0; 0-691-14049-9. DOI: 10.1515/9781400830558. URL: https://doi.org/10.1515/9781400830558.

[LW13] Jean-Pierre Labesse and Jean-Loup Waldspurger. *La formule des traces tordue d’après le Friday Morning Seminar*. Vol. 31. CRM Monograph Series. With a foreword by Robert Langlands [dual English/French text]. American Mathematical Society, Providence, RI, 2013, pp. xxvi+234. ISBN: 978-0-8218-9441-5. DOI: 10.1090/crmm/031. URL: https://doi.org/10.1090/crmm/031.

[Man05] Elena Mantovan. "On the cohomology of certain PEL-type Shimura varieties". In: *Duke Math. J.* 129.3 (2005), pp. 573–610. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-05-12935-0. URL: https://doi.org/10.1215/S0012-7094-05-12935-0.

[Mat00] Olivier Mathieu. "Tilting modules and their applications". In: *Analysis on homogeneous spaces and representation theory of Lie groups, Okayama–Kyoto (1997)*. Vol. 26. Adv. Stud. Pure Math. Math. Soc. Japan, Tokyo, 2000, pp. 145–212. DOI: 10.2969/aspm/02610145. URL: https://doi.org/10.2969/aspm/02610145.

[Mat10] Ivan Matić. "The unitary dual of p-adic SO(5)". In: *Proc. Amer. Math. Soc.* 138.2 (2010), pp. 759–767. ISSN: 0002-9939. DOI: 10.1090/S0002-9939-09-10065-5. URL: https://doi.org/10.1090/S0002-9939-09-10065-5.

[MN21] A. Bertoloni Meli and K.H. Nguyen. "The Kottwitz conjecture for unitary PEL-type Rapoport–Zink spaces". In: *Preprint* (2021). arXiv:2104.05912.

[Mui97] Goran Muić. "The unitary dual of p-adic G_2". In: *Duke Math. J.* 90.3 (1997), pp. 465–493. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-97-09012-8. URL: https://doi.org/10.1215/S0012-7094-97-09012-8.

[MV07] I. Mirković and K. Vilonen. "Geometric Langlands duality and representations of algebraic groups over commutative rings". In: *Ann. of Math. (2)* 166.1 (2007), pp. 95–143. ISSN: 0003-486X. DOI: 10.4007/annals.2007.166.95. URL: https://doi.org/10.4007/annals.2007.166.95.

[MW18] Colette Mœglin and J.-L. Waldspurger. "La formule des traces locale tordue". In: *Mem. Amer. Math. Soc.* 251.1198 (2018), pp. v+183. ISSN: 0065-9266. DOI: 10.1090/memo/1198. URL: https://doi.org/10.1090/memo/1198.

[Ngu19] Kieu Hieu Nguyen. "Un cas PEL de la conjecture de Kottwitz". In: *Preprint* (2019). arXiv:1903.11505.

[Rap18] Michael Rapoport. “Appendix to On the p-adic cohomology of the Lubin-Tate tower”. In: *Ann. Sci. Éc. Norm. Supér. (4)* 51.4 (2018), pp. 811–863. ISSN: 0012-9593. DOI: 10.24033/asens.2367. URL: https://doi.org/10.24033/asens.2367.

[Rin91] Claus Michael Ringel. “The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences”. In: *Math. Z.* 208.2 (1991), pp. 209–223. ISSN: 0025-5874. DOI: 10.1007/BF02571521. URL: https://doi.org/10.1007/BF02571521.

[Rog90] Jonathan D. Rogawski. *Automorphic representations of unitary groups in three variables*. Vol. 123. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 1990, pp. xii+259. ISBN: 0-691-08586-2; 0-691-08587-0. DOI: 10.1515/9781400882441. URL: https://doi.org/10.1515/9781400882441.

[RR96] M. Rapoport and M. Richartz. “On the classification and specialization of F-isocrystals with additional structure”. In: *Compositio Math.* 103.2 (1996), pp. 153–181. ISSN: 0010-437X. URL: http://www.numdam.org/item?id=CM_1996__103_2_153_0.

[RV14] Michael Rapoport and Eva Viehmann. “Towards a theory of local Shimura varieties”. In: *Münster J. Math.* 7.1 (2014), pp. 273–326. ISSN: 1867-5778.

[RW18] Simon Riche and Geordie Williamson. “Tilting modules and the p-canonical basis”. In: *Astérisque* 397 (2018), pp. ix+184. ISSN: 0303-1179.

[San23] Mafalda Santos. “Imperial College London PhD Thesis”. In: *Preprint* (2023).

[Sch15a] Simon Schieder. “The Harder-Narasimhan stratification of the moduli stack of G-bundles via Drinfeld’s compactifications”. In: *Selecta Math. (N.S.)* 21.3 (2015), pp. 763–831. ISSN: 1022-1824. DOI: 10.1007/s00029-014-0161-y. URL: https://doi.org/10.1007/s00029-014-0161-y.

[Sch15b] Peter Scholze. “On torsion in the cohomology of locally symmetric varieties”. In: *Ann. of Math. (2)* 182.3 (2015), pp. 945–1066. ISSN: 0003-486X. DOI: 10.4007/annals.2015.182.3.3. URL: https://doi.org/10.4007/annals.2015.182.3.3.

[Sch18] P. Scholze. “Étale Cohomology of Diamonds”. In: *Preprint* (2018). arXiv:1709.07343.

[Sha90] Freydoon Shahidi. “A proof of Langlands’ conjecture on Plancherel measures; complementary series for p-adic groups”. In: *Ann. of Math. (2)* 132.2 (1990), pp. 273–330. ISSN: 0003-486X. DOI: 10.2307/1971524. URL: https://doi.org/10.2307/1971524.

[She17] Xu Shen. “Perfectoid Shimura varieties of abelian type”. In: *Int. Math. Res. Not. IMRN* 21 (2017), pp. 6599–6653. ISSN: 1073-7928. DOI: 10.1093/imrn/rnw202. URL: https://doi.org/10.1093/imrn/rnw202.

[She21] Xu Shen. “Harder-Narasimhan strata and p-adic period domains”. In: *Preprint* (2021). arXiv:1909.02230.

[Shi12] Sug Woo Shin. “On the cohomology of Rapoport-Zink spaces of EL-type”. In: *Amer. J. Math.* 134.2 (2012), pp. 407–452. ISSN: 0002-9327. DOI: 10.1353/ajm.2012.0009. URL: https://doi.org/10.1353/ajm.2012.0009.

[Sor10] Claus M. Sorensen. “Galois representations attached to Hilbert-Siegel modular forms”. In: *Doc. Math.* 15 (2010), pp. 623–670. ISSN: 1431-0635. DOI: 10.1007/s00031-010-9092-7. URL: https://doi.org/10.1007/s00031-010-9092-7.

[SS97] Peter Schneider and Ulrich Stuhler. “Representation theory and sheaves on the Bruhat-Tits building”. In: *Inst. Hautes Études Sci. Publ. Math.* 85 (1997), pp. 97–191. ISSN: 0073-8301. URL: http://www.numdam.org/item?id=PMIHES_1997__85__97_0.

[SV80] Birgit Speh and David A. Vogan Jr. “Reducibility of generalized principal series representations”. In: *Acta Math.* 145.3-4 (1980), pp. 227–299. ISSN: 0001-5962. DOI: 10.1007/BF02414191. URL: https://doi.org/10.1007/BF02414191.

[SW13] Peter Scholze and Jared Weinstein. “Moduli of p-divisible groups”. In: *Camb. J. Math.* 1.2 (2013), pp. 145–237. ISSN: 2168-0930. DOI: 10.4310/CJM.2013.v1.n2.a1. URL: https://doi.org/10.4310/CJM.2013.v1.n2.a1.

[SW20a] P. Scholze and J. Weinstein. *Berkeley lectures on p-adic Geometry*. Vol. 389. Annals of Mathematics Studies. Princeton University Press, 2020.

[SW20b] P. Scholze and J. Weinstein. *Berkeley lectures on p-adic Geometry*. Vol. 389. Annals of Mathematics Studies. Princeton University Press, 2020.

[Tad94] Marko Tadić. "Representations of p-adic symplectic groups". In: *Compositio Math.* 90.2 (1994), pp. 123–181. ISSN: 0010-437X. URL: http://www.numdam.org/item?id=CM_1994__90_2_123_0.

[Tit79] J. Tits. "Reductive groups over local fields". In: *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*. Proc. Sympos. Pure Math., XXXIII. Amer. Math. Soc., Providence, R.I., 1979, pp. 29–69.

[Tow13] Nelson J. Townsend. *Properties of Gamma factors for $GSp(4) \times GL(r)$ with r = 1, 2*. Thesis (Ph.D.)–University of California, San Diego. ProQuest LLC, Ann Arbor, MI, 2013, p. 69. ISBN: 978-1303-62050-8. URL: http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqm&rft_dat=xri:pqdiss:3605550.

[Vie] Eva Viehmann. *On Newton strata in the B_{dR}^{+}-Grassmannian*. DOI: 10.48550/ARXIV.2101.07510. URL: https://arxiv.org/abs/2101.07510.

[Vig01] Marie-France Vignéras. "Correspondance de Langlands semi-simple pour $\mathrm{GL}(n,F)$ modulo $\ell \neq p$". In: *Invent. Math.* 144.1 (2001), pp. 177–223. ISSN: 0020-9910. DOI: 10.1007/s002220100134. URL: https://doi.org/10.1007/s002220100134.

[Vig96] Marie-France Vignéras. *Représentations l-modulaires d'un groupe réductif p-adique avec $l \neq p$*. Vol. 137. Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1996, xviii and 233. ISBN: 0-8176-3929-2.

[VZ84] David A. Vogan Jr. and Gregg J. Zuckerman. "Unitary representations with nonzero cohomology". In: *Compositio Math.* 53.1 (1984), pp. 51–90. ISSN: 0010-437X. URL: http://www.numdam.org/item?id=CM_1984__53_1_51_0.

[Wan82] Jian Pan Wang. "Sheaf cohomology on G/B and tensor products of Weyl modules". In: *J. Algebra* 77.1 (1982), pp. 162–185. ISSN: 0021-8693. DOI: 10.1016/0021-8693(82)90284-8. URL: https://doi.org/10.1016/0021-8693(82)90284-8.

[Wes12] Uwe Weselmann. "A twisted topological trace formula for Hecke operators and liftings from symplectic to general linear groups". In: *Compos. Math.* 148.1 (2012), pp. 65–120. ISSN: 0010-437X. DOI: 10.1112/S0010437X11005641. URL: https://doi.org/10.1112/S0010437X11005641.

[XZ17] Liang Xiao and Xinwen Zhu. "Cycles on Shimura varieties via Geometric Satake". In: *Preprint* (2017). arXiv. 1707.05700.

[Zhu15] Xinwen Zhu. "The geometric Satake correspondence for ramified groups". In: *Ann. Sci. Éc. Norm. Supér. (4)* 48.2 (2015), pp. 409–451. ISSN: 0012-9593. DOI: 10.24033/asens.2248. URL: https://doi.org/10.24033/asens.2248.

[Zhu20] X. Zhu. "Coherent Sheaves on the stack of Langlands Parameters". In: *Preprint* (2020). arXiv:2008.02998.

[Zie15] Paul Ziegler. "Graded and filtered fiber functors on Tannakian categories". In: *J. Inst. Math. Jussieu* 14.1 (2015), pp. 87–130. ISSN: 1474-7480. DOI: 10.1017/S1474748013000376. URL: https://doi.org/10.1017/S1474748013000376.

[Zou22] K. Zou. "The Categorical Form of Fargues' conjecture for Tori". In: *Preprint* (2022). arXiv:2202.13238.