

A JACOBIAN CRITERION FOR ARTIN v -STACKS

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Abstract. We prove a generalization of the Jacobian criterion of Fargues-Scholze for spaces of sections of a smooth quasi-projective variety over the Fargues-Fontaine curve [FS21, Section IV.4]. Namely, we show how to use their criterion to deduce an analogue for spaces of sections of a smooth Artin stack over the (schematic) Fargues-Fontaine curve obtained by taking the stack quotient of a smooth quasi-projective variety by the action of a linear algebraic group. As an application, we show various moduli stacks appearing in the Fargues-Scholze geometric Langlands program are cohomologically smooth Artin v -stacks and compute their ℓ -dimensions.

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NOTATION AND CONVENTIONS

- (1) Let $\ell \neq p$ be distinct primes.
- (2) Let \mathbb{Q}_p denote the p -adic numbers.
- (3) We let Perf denote the category of affinoid perfectoid spaces in characteristic p . For $S \in \text{Perf}$, we let Perf_S denote the category of affinoid perfectoid spaces over S .
- (4) For $S \in \text{Perf}$, we will write X_S for the (schematic) relative Fargues-Fontaine curve over S .
- (5) We will always use F to denote an algebraically closed complete non-archimedean field in characteristic p .
- (6) Given a scheme S , an Artin S -stack X is a stack $X \rightarrow S$ admitting a smooth surjective morphism from a S -scheme U such that the diagonal map $X \rightarrow X \times_S X$ is representable, quasi-separated, and quasi-compact [LM00, Definition 4.1].
- (7) We will freely use the formalism of étale cohomology of diamond and v -stacks, as developed in [Sch18; FS21]. In particular, we will make regular use of the notion of Artin v -stacks. We recall [FS21, Section IV.1] that this is a small v -stack X such that the diagonal map $X \rightarrow X \times X$ is representable in locally spatial diamonds, and there is some surjective map $f : U \rightarrow X$ such that f is separated and cohomologically smooth.

- (8) We recall that a cohomologically smooth map $f : X \rightarrow Y$ of Artin v -stacks is of ℓ -dimension $d \in \frac{1}{2}\mathbb{Z}$ if $Rf^1(\mathbb{F}_\ell)$ sits locally in homological degree $2d$ [FS21, Definition IV.1.17]. Unfortunately, as it stands, this is the only well-behaved notion of dimension for diamonds and v -stacks (see the discussion surrounding [FS21, Problem I.11.1]); however, for most of the key applications to ℓ -adic sheaves it is sufficient. We note that, if $f : X \rightarrow Y$ is cohomologically smooth, the sheaf $Rf^1(\mathbb{F}_\ell)$ will be locally constant so any such map decomposes into a disjoint union of $f_d : Y_d \rightarrow X$ that are pure of ℓ -dimension d . If an Artin v -stack X is cohomologically smooth of pure ℓ -dimension over some specified base we will write $\dim_\ell(X)$ for the ℓ -dimension.

1. INTRODUCTION

For G/\mathbb{Q}_p a connected reductive group, Fargues and Scholze [FS21] have recently been able to develop the geometric tools necessary to make sense of objects like the moduli stack of G -bundles on the Fargues-Fontaine curve and have used it to construct a general candidate for the local Langlands correspondence of G . One of the key new geometric tools that aids their analysis is a kind of Jacobain criterion for certain locally spatial diamonds. In particular, if one has a smooth quasi-projective variety Z over the Fargues-Fontaine curve X_S one can consider the following moduli space attached to it.

Definition 1.1. For Z a smooth quasi-projective scheme over X_S , we define $\mathcal{M}_Z \rightarrow S$ to be the functor on Perf_S , which parametrizes, for $T \in \text{Perf}_S$, sections:

$$\begin{array}{ccc} & & Z \\ & \nearrow s & \downarrow \\ X_T & \longrightarrow & X_S \end{array}$$

We recall classically that, if we have an F -point of \mathcal{M}_Z corresponding to a map $X_F \rightarrow X_S$ and a section $s : X_F \rightarrow Z$ over this the map, the tangent space at this point should be given by $H^0(X_F, s^*T_{Z/X_S})$, and the obstruction space should be given by $H^1(X_F, s^*T_{Z/X_S})$. In particular, we should expect smoothness to hold if this obstruction space vanishes, and that, in this case, the dimension locally around the point defined by s is the dimension of $H^0(X_F, s^*(T_{Z/X_S}))$. Over the Fargues-Fontaine curve, making sense of such a statement is a bit subtle. In particular, $H^0(X_F, s^*(T_{Z/X_S}))$ is far from being finite dimensional as a vector space. However, one can extract a finite dimensional space from it by considering the pro-étale sheaf $\mathcal{H}^0(s^*(T_{Z/X_S})) \rightarrow \text{Spa}(F)$ on Perf_F which sends $T \in \text{Perf}_F$ to the space of sections $H^0(X_T, s^*(T_{Z/X_S})_T)$, where $s^*(T_{Z/X_S})_T$ denotes the base change of $s^*(T_{Z/X_S})$ to X_T . This is representable by a finite dimensional locally spatial diamond called a Banach-Colmez space, as defined in [Le 18], and looks like a perfectoid open unit disc quotiented out by a pro-étale group action (See Proposition 3.18). Roughly speaking, we should expect that, infinitesimally around the point defined by s , the moduli space \mathcal{M}_Z looks like the locally spatial diamond $\mathcal{H}^0(s^*(T_{Z/X_S}))$. This space will define a cohomologically smooth diamond if the pullback $s^*(T_{Z/X_S})$ has positive HN-slopes. Moreover, the positive slope assumption also implies that the obstruction space $H^1(X_F, s^*(T_{Z/X_S}))$ vanishes. Therefore, we should expect smoothness of the space \mathcal{M}_Z around the points defined by such sections. This motivates the following definition.

Definition 1.2. For Z as above, we let $\mathcal{M}_Z^{\text{sm}} \subset \mathcal{M}_Z$ denote the open sub-functor parametrizing sections $s : X_T \rightarrow Z$ such that the pullback of the tangent bundle $s^*(T_{Z/X_S})$ has positive HN-slopes after pulling back to any geometric point of T .

Then the Jacobian criterion of Fargues-Scholze is as follows.

Theorem 1.3. [FS21, Theorem IV.4.2] For $S \in \text{Perf}$ and $Z \rightarrow X_S$ a smooth quasi-projective variety over X_S , the v -sheaf \mathcal{M}_Z defines a locally spatial diamond, the map $\mathcal{M}_Z \rightarrow S$ is compactifiable,

and the map $\mathcal{M}_Z^{sm} \rightarrow S$ is cohomologically smooth.

Moreover, for any geometric point $x : \mathrm{Spa}(F) \rightarrow \mathcal{M}_Z^{sm}$, given by a map $\mathrm{Spa}(F) \rightarrow S$ and a section $s : X \rightarrow Z$, the map $\mathcal{M}_Z^{sm} \rightarrow S$ is at x of ℓ -dimension equal to the degree of $s^*(T_{Z/X_S})$.

Remark 1.4. Fargues and Scholze formulate their result with the adic Fargues-Fontaine curve and adic spaces over it; however, as we will want to later consider Artin stack quotients of the spaces Z , we find it more convenient to stick to the schematic formalism. This has the small caveat that one needs to restrict to affinoid perfectoid spaces, as one cannot make sense of gluing otherwise. Restricting to affinoids is a rather minute restriction, since the category of perfectoid spaces with any reasonable topology is generated by affinoids.

We briefly recall the importance of this result. In particular, Fargues and Scholze use this to show that Bun_G is a cohomologically smooth Artin v -stack of ℓ -dimension 0. Namely, in [FS21, Section V.3], Fargues and Scholze consider moduli spaces of P -bundles, where $P \subset G$ is a proper parabolic subgroup. These give rise to cohomologically smooth locally spatial diamonds, represented by an iterated fibration of negative Banach-Colmez spaces (See Theorem 4.5). One uses these moduli spaces to uniformize Bun_G by sending a parabolic structure to its induced G -bundle. To verify the desired claim, one needs to show that the fibers of this uniformization map are cohomologically smooth and compute their ℓ -dimension. This is where the Jacobian criterion comes in. In particular, the moduli space of P -structures on a fixed G -bundle \mathcal{E} on X_S is a space of sections of the projective variety $\mathcal{E}^{\mathrm{gm}}/P$ over X_S , where $\mathcal{E}^{\mathrm{gm}}$ denotes the geometric realization of \mathcal{E} . These charts and their smoothness properties serve as a key computational tool for establishing many foundational results on the sheaf theory of Bun_G .

While this story is nice, for many applications one wants to consider moduli spaces like Bun_G and be able to efficiently study their smoothness properties without having to rely on explicit charts. Classically, such a formalism already exists. For example, to any Artin stack $X \rightarrow \mathrm{Spec} k$ for k an algebraically closed field, one can associate to it a tangent complex T_X^* (Theorem 2.5). This is a complex of quasi-coherent sheaves on X concentrated in degrees $[-1, \infty)$. The cohomology in degree 0 encodes the tangent vectors at a point and the cohomology in degree -1 encodes the automorphisms at a point, while the higher cohomology encodes information about obstructions to higher order deformations. In particular, if one has a k -point $x : \mathrm{Spec} k \rightarrow X$, such that the cohomology of the pullback $Lx^*(T_{X/k}^*)$ has vanishing cohomology in degrees ≥ 1 , the moduli space X is smooth at x and locally has dimension equal to the Euler characteristic of the complex $Lx^*(T_{X/k}^*)$. More specifically, infinitesimally around x the space X should look like the Picard groupoid

$$[H^0(Lx^*(T_{X/k}^*))/H^{-1}(Lx^*(T_{X/k}^*))]$$

associated to the two term complex $Lx^*(T_{X/k}^*)$ of vector spaces over k , where we regard the cohomology groups as affine spaces over k . If we let $X = \mathrm{Bun}_G$ be the moduli space of G -bundles over a smooth projective curve $Y/\mathrm{Spec} k$ and consider such a point x corresponding to a G -torsor \mathcal{F}_G over Y then the cohomology of $Lx^*(T_{X/k}^*)$ is described by the cohomology of the complex $\mathcal{F}_G \times^{G, \mathrm{Ad}} \mathrm{Lie}(G)[1]$ on Y (See Examples 2.11 and 2.13). We note that, since Y is a curve, the complex has vanishing cohomology in degree ≥ 1 . Hence, the deformations are always unobstructed and infinitesimally it looks like

$$[H^1(Y, \mathcal{F}_G \times^{G, \mathrm{Ad}} \mathrm{Lie}(G))/H^0(Y, \mathcal{F}_G \times^{G, \mathrm{Ad}} \mathrm{Lie}(G))] \rightarrow \mathrm{Spec} k$$

and from this we can see that Bun_G is smooth of dimension $\dim(G)(1 - g)$, where g is the genus of Y .

The main aim of this note is to develop a formalism that would allow similar arguments to work for spaces occurring in the Fargues-Scholze geometric Langlands program. To get at this, let us consider a different way of understanding the previous argument more in line with the type

of analysis used in Theorem 1.3. In particular, recall that Bun_G can be written as the moduli space of sections $[Y/G] \rightarrow Y$, where $[Y/G]$ is the classifying stack of G over Y . Given a section $s : Y \rightarrow [Y/G]$ corresponding to a G -bundle \mathcal{F}_G on Y , we can compute that the pullback of the tangent complex $T_{[Y/G]/Y}^*$ is isomorphic to the complex of vector bundles $\mathcal{F}_G \times^{G, \text{Ad}} \text{Lie}(G)[1]$, and, as seen above, the cohomology of this complex controls the deformation theory of Bun_G around the point corresponding to s , in perfect analogy to Theorem 1.3.

With this in mind, it becomes tempting to consider an analogue of Theorem 1.3 for spaces of sections \mathcal{M}_Z , where Z is a smooth quasi-projective Artin X_S -stack. We show that such an analogue indeed exists for Z which are obtained as stack quotients of a smooth quasi-projective variety by an action of a linear algebraic group. In particular, given a smooth quasi-projective variety $Z \rightarrow X_S$ with an action of a linear algebraic group H/\mathbb{Q}_p , we consider the stack quotient $[Z/H] \rightarrow X_S$ and the v -stack of sections $\mathcal{M}_{[Z/H]} \rightarrow S$. Given a section $s : X_T \rightarrow [Z/H]$, we can consider the complex $s^*(T_{([Z/H])/X_S}^*)$ on X_T . Since Z is smooth, we can realize this as a two term complex of vector bundles on X_T sitting in cohomological degrees -1 and 0 , denoted $\{\mathcal{E}^{-1} \rightarrow \mathcal{E}^0\}$ (See Proposition 2.9). It therefore follows that the cohomology of this complex is concentrated in degrees $[-1, 1]$, since T is affinoid. In analogy with the above, it should follow that, given an F -point of \mathcal{M}_Z corresponding to a map $X_F \rightarrow X_S$ and a section $s : X_F \rightarrow [Z/H]$ over it, if s defines a smooth point then, infinitesimally around the point defined by s , $\mathcal{M}_{[Z/H]}$ should look like the v -stack

$$[\mathcal{H}^0(s^*(T_{([Z/H])/X_S}^*))/\mathcal{H}^{-1}(s^*(T_{([Z/H])/X_S}^*))] \rightarrow \text{Spa}(F)$$

which we call the Picard v -groupoid associated to the complex $s^*(T_{([Z/H])/X_S}^*)$ (See Definition 3.6). We should expect that s defines a smooth point if this Picard v -groupoid defines a cohomologically smooth Artin v -stack and the obstruction space defined by $H^1(X_F, s^*(T_{([Z/H])/X_S}^*))$ vanishes. One can check that, if $\{\mathcal{E}^{-1} \rightarrow \mathcal{E}^0\}$ represents the complex $s^*(T_{([Z/H])/X_S}^*)$, this follows from insisting that \mathcal{E}^0 is a bundle with strictly positive slopes (Proposition 3.20). In this case, one can check that the above v -stack is an Artin v -stack of pure ℓ -dimension equal to $\dim_\ell(\mathcal{H}^0(s^*(T_{([Z/H])/X_S}^*))) - \dim_\ell(\mathcal{H}^{-1}(s^*(T_{([Z/H])/X_S}^*)))$. This leads us to make the following definition.

Definition 1.5. For H and Z as above, we let $\mathcal{M}_{[Z/H]}^{\text{sm}} \subset \mathcal{M}_{[Z/H]}$ denote the open sub-functor parametrizing sections $s : X_T \rightarrow [Z/H]$ such that, $s^*(T_{([Z/H])/X_S}^*)$ is represented by a complex $\{\mathcal{E}^{-1} \rightarrow \mathcal{E}^0\}$ of vector bundles on X_T for \mathcal{E}^0 a bundle with strictly positive slopes after pulling back to any geometric point of T .

Remark 1.6. As we will see later (Lemma 4.3), insisting that a T -point in $\mathcal{M}_{[Z/H]}$ corresponding to a section $s : X_T \rightarrow [Z/H]$ lies in $\mathcal{M}_{[Z/H]}^{\text{sm}}$ is equivalent to insisting that, after pulling back to any geometric point of T , $\mathcal{H}^0(s^*(T_{([Z/H])/X_S}^*)) \rightarrow \text{Spa}(F)$ is cohomologically smooth and $H^1(X_F, s^*(T_{([Z/H])/X_S}^*))$ vanishes. The above definition makes it clear that $\mathcal{M}_{[Z/H]}^{\text{sm}} \subset \mathcal{M}_{[Z/H]}$ is an open sub-functor, via upper semi-continuity of the slope polygon on T [KL15, Theorem 7.4.5]. However, this latter condition is easier to check in practice.

Our main Theorem is as follows.

Theorem 1.7. *For $S \in \text{Perf}$, let H be a linear algebraic group over \mathbb{Q}_p and $Z \rightarrow X_S$ a smooth quasi-projective variety over X_S with an action of H . Then $\mathcal{M}_{[Z/H]} \rightarrow S$ defines an Artin v -stack, and $\mathcal{M}_{[Z/H]}^{\text{sm}} \rightarrow S$ is cohomologically smooth map of Artin v -stacks.*

Moreover, for any geometric point $x : \text{Spa}(F) \rightarrow \mathcal{M}_{[Z/H]}^{\text{sm}}$ given by a map $\text{Spa}(F) \rightarrow S$ and a section $s : X_F \rightarrow Z$, the map $\mathcal{M}_{[Z/H]}^{\text{sm}} \rightarrow S$ is at x of ℓ -dimension equal to the quantity:

$$\dim_\ell(\mathcal{H}^0(s^*(T_{([Z/H])/X_S}^*))) - \dim_\ell(\mathcal{H}^{-1}(s^*(T_{([Z/H])/X_S}^*)))$$

In particular, this is well-defined¹ and equal to the ℓ -dimension of the Picard v -groupoid

$$[\mathcal{H}^0(s^*(T_{([Z/H])/X_S}^*))/\mathcal{H}^{-1}(s^*(T_{([Z/H])/X_S}^*))]$$

over $\mathrm{Spa}(F)$. If one chooses a presentation $s^*(T_{([Z/H])/X_S}^*) = \{\mathcal{E}^{-1} \rightarrow \mathcal{E}^0\}$ of this complex as a two term complex of vector bundles on X_F , then this is equal to:

$$\mathrm{deg}(\mathcal{E}^0) - \mathrm{deg}(\mathcal{E}^{-1})$$

Remark 1.8. One could probably prove an analogue of Theorem 1.7 in the adic formalism, but this would require making sense of stack quotients $[Z/H]$, for H a linear algebraic group and Z a sous-perfectoid space. This is possible by using the fact that H -torsors over a sous-perfectoid space are sous-perfectoid (See the discussion before [SW20, Theorem 19.5.2]). Then one would need to develop a theory of tangent complexes for these objects, using [FS21, Section IV.4.1] as a starting point.

The proof of this theorem is by dévissage to Theorem 1.3. The key point is that, by choosing a closed embedding $H \hookrightarrow \mathrm{GL}_n$ for some n sufficiently large n , one can assume that $H = \mathrm{GL}_n$. Then one has a natural map of v -stacks:

$$\mathcal{M}_{[Z/\mathrm{GL}_n]} \rightarrow \mathcal{M}_{[X_S/\mathrm{GL}_n]} \simeq \mathrm{Bun}_{\mathrm{GL}_n, S}$$

We can then find cohomologically smooth charts for the moduli space $\mathrm{Bun}_{\mathrm{GL}_n, S}$ which can be written as spaces of sections of a suitable quasi-projective varieties Z_i over X_S . By pulling back these charts along this map, we can find charts for $\mathcal{M}_{[Z/\mathrm{GL}_n]}$ which are given as spaces of sections of a fiber product $Z_i \times_{[X_S/\mathrm{GL}_n]} [Z/\mathrm{GL}_n]$, which will also be representable by a smooth quasi-projective variety over X_S , and the condition on the pullback of the tangent bundle of this space needed to apply Theorem 1.3 can be related to the above condition on the tangent complex of $[Z/\mathrm{GL}_n]$ via looking at the distinguished triangle of tangent complexes given by the projection $Z_i \times_{[X_S/\mathrm{GL}_n]} [Z/\mathrm{GL}_n] \rightarrow [Z/\mathrm{GL}_n]$. This will in turn give the desired claim.

As an application of these ideas, we can give an alternative proof that Bun_G is a cohomologically smooth Artin v -stack of ℓ -dimension 0, as well as show that the moduli stack Bun_P for $P \subset G$ a parabolic is cohomologically smooth. Perhaps more interestingly however, we will verify that, if $G = \mathrm{GL}_n$, the analogue of Laumon's compactification $\mathrm{Bun}_P \subset \overline{\mathrm{Bun}}_P^L$ [Lau90] of the moduli stack of P -structures is a cohomologically smooth Artin v -stack. This result plays an important role in the theory of geometric Eisenstein series on the Fargues-Fontaine curve for general linear groups, which we intend to explore in future work.

In §2, we review the theory of the tangent complex for usual Artin stacks. §3 is a review of the theory of Banach-Colmez spaces, where we prove various properties of "stacky" Banach-Colmez like spaces obtained from it. In §4, we conclude by giving the proof of Theorem 1.7 and give some applications to verifying the cohomological smoothness of various moduli stacks appearing in the Fargues-Scholze geometric Langlands correspondence; in particular, we will show smoothness of Laumon's compactification.

2. THE TANGENT COMPLEX OF AN ARTIN STACK

2.1. Review of the Tangent Complex. In this section, we will review the theory of the tangent complex of an Artin stack. Most of this material in the context of schemes can be found in the book of Illusie [Ill71], and for Artin stacks in the book of Laumon-Moret-Bailly [LM00]. We also recommend the interested reader take a look at the paper of Olsson [Ols07], where several mistakes in the book of [LM00] related to the functoriality of the lisse-étale site and the derived pull-back between maps of Artin stacks are remedied.

¹See Proposition 3.19 and 3.20

Let's start with some motivation. Let S be a scheme (affine for simplicity) and $X \rightarrow S$ an Artin S -stack. Let $D = \text{Spec } \mathbb{Z}[\epsilon]/\epsilon^2$ be the ring of dual numbers. Write $S[\epsilon]/\epsilon^2 := S \times_{\text{Spec } \mathbb{Z}} D$ for the dual numbers over S . Given a point $x : S \rightarrow X$, we can define a tangent vector of X at x to be a lift of the map x to a map $S[\epsilon]/\epsilon^2 \rightarrow X$. Let $T_{X/S,x}$ be the set of tangent vectors. This comes equipped with several kinds of additional structure. For starters, the set forms the objects of a groupoid, since they can be realized as a subgroupoid of $X(S[\epsilon]/\epsilon^2)$ given as the fiber over x of the natural map $X(S[\epsilon]/\epsilon^2) \rightarrow X(S)$. Moreover, it has an addition given by $S[\epsilon]/\epsilon^2 \rightarrow S[\epsilon]/\epsilon^2 \amalg_x S[\epsilon]/\epsilon^2$ defined by the map of rings sending $a+b\epsilon_1+c\epsilon_2$ to $a+(b+c)\epsilon$, where $a, b, c \in \mathcal{O}_S$. This gives $T_{X/S,x}$ the structure of a Picard category. In other words, a symmetric monoidal category such that tensoring by a fixed object has an inverse (See [LM00, Section 14.4] for a more precise definition). Moreover, it has a natural scalar multiplication by $\lambda \in \mathcal{O}_S$, given by the map sending $a+b\epsilon \mapsto a+\lambda b\epsilon$. This induces an endomorphism $\lambda : T_{X/S,x} \rightarrow T_{X/S,x}$ satisfying natural compatibilities. This gives it the structure of a Picard S -stack, as in [LM00, Definition 14.4.2]. By [LM00, Theorem 14.4.5], a category with such structures is essentially equivalent to a two step complex in locally-free \mathcal{O}_S -modules $C^{-1} \xrightarrow{d} C^0$. In particular, to such a complex one can consider the category whose objects are given by sections of C^0 and where a morphism from x to y is given by an element $f \in C^{-1}$ such that $df = y - x$. Note, in particular, that H^0 of this complex gives the isomorphism classes of objects in our category and H^{-1} gives the automorphisms of the unit object. Quasi-isomorphisms of complexes give rise to equivalences of the associated category. Therefore, such categories are equivalent to objects in the derived category of perfect complexes of \mathcal{O}_S -modules concentrated in two degrees. Let C^* be such a complex associated to $T_{X/S,x}$. The above analysis suggests if X is smooth that, infinitesimally around x , the stack X should look like the Picard groupoid $[H^0(C^*)/H^{-1}(C^*)] \rightarrow S$, where we are considering these cohomology groups as affine spaces over S .

This leads us to the expectation that there should exist an object $T_{X/S}^*$ in the derived category of quasi-coherent sheaves on X , whose pullback to x gives rise to a complex whose Picard groupoid is precisely $T_{X/S,x}$ if X is smooth. In the general case, we should expect its higher cohomology to carry information about obstructions to higher order deformations.

In preparation for constructing this object, let us briefly recall the definition of quasi-coherent sheaves on an Artin S -stack X . We have the following key definition of Laumon-Moret-Bailly.

Definition 2.1. [LM00, Definition 12.1] For $X \rightarrow S$ an Artin S -stack. We define the lisse-étale site of X , denoted $\text{Lis-ét}(X)$, where objects are (U, u) , for U a scheme over S and $u : U \rightarrow X$ a smooth morphism over S . Coverings are families of jointly surjective étale morphisms $\{U_i \rightarrow U\}_{i \in I}$.

Remark 2.2. It is easily verified [LM00, Proposition 12.2.1] that a sheaf \mathcal{F} on this site is equivalent to a system of sheaves \mathcal{F}_U on $(U)_{\text{ét}}$ and, for every morphism $\{\phi : U \rightarrow V\} \in \text{Lis-ét}(X)$, a map

$$\theta_\phi : \phi^{-1}(\mathcal{F}_V) \rightarrow \mathcal{F}_U$$

satisfying the following properties:

- (1) θ_ϕ is an isomorphism if ϕ is étale.
- (2) For maps $U \xrightarrow{\phi} V \xrightarrow{\psi} W$, we have $\theta_\psi \circ \phi^{-1}(\theta_\phi) = \theta_{\psi \circ \phi}$.

This site has a structure sheaf, denoted $\mathcal{O}_{X_{\text{lis-ét}}}$, sending $U \in \text{Lis-ét}(X)$ to $\Gamma(U, \mathcal{O}_U)$. We can therefore consider $\mathcal{O}_{X_{\text{lis-ét}}}$ -modules on X . This allows us to define the following.

Definition 2.3. [LM00, Definition 13.2.2] We say a $\mathcal{O}_{X_{\text{lis-ét}}}$ -module \mathcal{F} on X is a quasi-coherent sheaf if, for all $U \in \text{Lis-ét}(X)$, \mathcal{F}_U is a quasi-coherent sheaf on $(U)_{\text{ét}}$ and, for all morphisms $\phi : U \rightarrow V$, the map $\phi^*(\mathcal{F}_V) \rightarrow \mathcal{F}_U$ induced by θ_ϕ is an isomorphism. It follows, by [Ols07, Lemma 6.2], that this is equivalent to checking this condition for smooth maps $\phi : U \rightarrow V$. Similarly, we say such \mathcal{F} is a vector bundle on X if all the restrictions \mathcal{F}_U are vector bundles on $(U)_{\text{ét}}$.

We let $\mathrm{QCoh}(X)$ denote the category of quasi-coherent sheaves on X . One can then easily verify that this is an abelian sub-category with enough injectives/projectives [LM00, Lemma 13.1.3]. From here, we can consider $\mathrm{D}_{\mathrm{qcoh}}(X, \mathcal{O}_X)$, the full derived sub-category of the derived category of $\mathcal{O}_{X_{\mathrm{lis-ét}}}$ -modules with quasi-coherent cohomology. We then define $\mathrm{D}_{\mathrm{qcoh}}^+(X, \mathcal{O}_X)$ (resp. $\mathrm{D}_{\mathrm{qcoh}}^-(X, \mathcal{O}_X)$) to be the sub-category with bounded below (resp. above) cohomology. We note, by [LM00, Proposition 13.2.6], we have a natural functor

$$R\mathcal{H}om_{\mathcal{O}_X}(-, -) : \mathrm{D}_{\mathrm{qcoh}}^-(X, \mathcal{O}_X) \times \mathrm{D}_{\mathrm{qcoh}}^+(X, \mathcal{O}_X) \rightarrow \mathrm{D}_{\mathrm{qcoh}}^-(X, \mathcal{O}_X)$$

Now, we will turn to defining the the derived pullback for a map of Artin \mathcal{S} -stacks. Here, one needs to be a bit careful that, given a morphism $f : X \rightarrow Y$ of algebraic stacks, there is an induced functor

$$\mathrm{Lis-ét}(X) \rightarrow \mathrm{Lis-ét}(Y)$$

$$U \mapsto U \times_X Y$$

which induces a pair of adjoint functors (f^{-1}, f_*) on sheaves, but this does not give rise to a morphism of topoi because f^{-1} is not left exact (See [Ols07, Example 3.4]). However, a left adjoint Lf^* to Rf_* exists after taking truncations (See [Ols07, Section 7]). This leads us to consider $\mathrm{D}'_{\mathrm{qcoh}}(X, \mathcal{O}_X)$ the left-completion of $\mathrm{D}_{\mathrm{qcoh}}(X, \mathcal{O}_X)$. Namely, the category of systems

$$K = \{\cdots \rightarrow K_{\geq -n-1} \rightarrow K_{\geq -n} \rightarrow \cdots \rightarrow K_{\geq 0}\}$$

where $K_{\geq -n} \in \mathrm{D}_{\mathrm{qcoh}}^+(X, \mathcal{O}_X)$ and the maps

$$\begin{aligned} K_{\geq -n} &\rightarrow \tau_{\geq -n} K_{\geq -n} \\ \tau_{\geq -n} K_{\geq -n-1} &\rightarrow \tau_{\geq -n} K_{\geq -n} \end{aligned}$$

are isomorphisms, where $\tau_{\geq -n}$ denote the usual truncation functors in degree $\geq -n$. This comes equipped with a notion of distinguished triangle, shift maps, and cohomology functors, as defined in [Ols07, Section 7]. With this in hand, we can define the following.

Proposition 2.4. [Ols07, Section 7] *Let $f : X \rightarrow Y$ be a quasi-compact and quasi-separated map of Artin stacks. Then there exists a derived pullback functor*

$$Lf^* : \mathrm{D}'_{\mathrm{qcoh}}(Y, \mathcal{O}_Y) \rightarrow \mathrm{D}'_{\mathrm{qcoh}}(X, \mathcal{O}_X)$$

obtained by taking the system $\{\tau_{\geq a} Lf^*\}$, where $\tau_{\geq a} Lf^*$ is the left adjoint to the functor $Rf_* : \mathrm{D}_{\mathrm{qcoh}}^{[a, \infty)}(X, \mathcal{O}_X) \rightarrow \mathrm{Perf}_{\mathrm{qcoh}}^{[a, \infty)}(Y, \mathcal{O}_Y)$ on the derived sub-categories with cohomology concentrated in degrees $[a, \infty)$.

With these definitions out of the way, we have our key result on the tangent complex².

Theorem 2.5. [Ols07; LM00, Proposition 8.1, Theorem 17.3] *Let $f : X \rightarrow Y$ be a quasi-compact and quasi-separated morphism of Artin stacks. Then to f one can associate an object $T_{X/Y}^* \in \mathrm{D}'_{\mathrm{qcoh}}(X, \mathcal{O}_X)$ called the tangent complex of f satisfying the following properties:*

- (1) *For any 2-commutative square of Artin stacks*

$$\begin{array}{ccc} X' & \xrightarrow{A} & X \\ \downarrow f' & & \downarrow f \\ Y' & \xrightarrow{B} & Y \end{array}$$

there is a natural functoriality morphism

$$T_{X'/Y'}^* \rightarrow LA^* T_{X/Y}^*$$

²We warn the reader that, for our purposes, it will be more convenient to work with the tangent complex, while Olsson and Laumon-Moret-Bailly work with the cotangent complex. However, the claims we will make can be obtained from their work by applying $R\mathcal{H}om(-, \mathcal{O}_X)$, as defined above.

which is an isomorphism if the square is Cartesian and either f or B is flat. We will usually consider this map when $Y = Y'$ and $B = id_Y$, in which case we will denote it by df_Y .

(2) If $g : Y \rightarrow Z$ is another morphism of Artin stacks then there exists a distinguished triangle

$$\begin{array}{ccc} & Lf^*T_{Y/Z}^* & \\ \swarrow +1 & & \nwarrow df_Z \\ T_{X/Y}^* & \xrightarrow{\quad} & T_{X/Z}^* \end{array}$$

in $D'_{qcoh}(X, \mathcal{O}_X)$.

Remark 2.6. We note that, if f is smooth, then the naive pullback functor f^* coming from the map of sites $(X)_{\text{lis-ét}} \rightarrow (Y)_{\text{lis-ét}}$ is left-exact (See the discussion preceding [Ols07, Equation 3.3.2]). It follows that, if we have a morphism $f : X \rightarrow Y$ which factors through some smooth map from a scheme, and $\mathcal{F}^* \in D'_{qcoh}(Y, \mathcal{O}_Y)$ is represented by a system of complexes of flat sheaves with respect to X , that $Lf^*(\mathcal{F}^*) = f^*(\mathcal{F}^*)$, where $f^*(\mathcal{F}^*)$ is the system of complexes of sheaves defined by naively pulling back the system of complexes of sheaves defining \mathcal{F}^* .

Let us give some flavor for how to prove this theorem in the particular case of the structure morphism of an Artin S -stack $X \rightarrow S$ for a scheme S , describing its pullback to an atlas and showing that satisfies the cocycle condition, as in Remark 2.2. For a morphism of schemes, $Y \rightarrow Z$, we let $L_{Y/Z}^*$ denote the cotangent complex, as defined in [Ill71, Section II.1]. This is a complex of quasi-coherent sheaves on Y with cohomology concentrated in degrees $(\infty, 0]$. For us, it will be more convenient to work the dual notion. In particular, we define the tangent complex $T_{Y/Z}^* := R\mathcal{H}om_{\mathcal{O}_Y}(L_{Y/Z}^*, \mathcal{O}_Y)$, which will in turn be a complex of quasi-coherent sheaves concentrated in degrees $[0, \infty)$ on X . The analogue of Theorem 2.5 in the schematic case follows from the work of Illusie [Ill71, Section II.2] and we will assume this in what follows. Choose an atlas $f : U \rightarrow X$ in the smooth topology with associated equivalence relation given by $R := U \times_X U$. Let $\Delta_{U/X} : U \rightarrow R$ denote the diagonal homomorphism. This is a section of the natural projection

$$f' : R \rightarrow U$$

obtained as the base change of f . Using this, we set $T_{U/X}^* := \Delta_{U/X}^*(T_{R/U}^*)$. From this, we construct the following complex of quasi-coherent sheaves on U

$$\{T_{U/X}^* \rightarrow T_{U/S}^*\} =: (T_{X/S}^*)_U$$

which is the pullback by $\Delta_{U/X}^*$ of the complex given by

$$\{T_{R/U}^* \rightarrow (f')^*(T_{U/S}^*)\}$$

coming from the distinguished triangle associated to the map of S -schemes $f' : R \rightarrow U$, as in Theorem 2.5 (2). Now, let's check that this description has the expected quasi-isomorphisms for maps between charts, as in Remark 2.2. Namely, we want to show that, if we have a S -scheme $V \rightarrow X$ mapping smoothly to X and a map $\phi : U \rightarrow V$ in the smooth topology, there is a natural map $(T_{X/S}^*)_U \rightarrow \phi^*(T_{X/S}^*)_V$ which is a quasi-isomorphism and that these satisfy a cocycle condition. To accomplish this, we note one easily sees that, using the distinguished triangles of Theorem 2.5 (2) in the case of schemes, we have a distinguished triangle

$$\begin{array}{ccc} & \phi^*T_{V/X}^* & \\ \swarrow +1 & & \nwarrow d\phi_X \\ T_{U/V}^* & \xrightarrow{\quad} & T_{U/X}^* \end{array}$$

for the above definition of $T_{U/X}^*$ and $T_{V/X}^*$. Moreover, we have a map of triangles

$$\begin{array}{ccccc}
 & & \phi^* T_{V/X}^* & & \\
 & \swarrow & \downarrow & \nwarrow & \\
 T_{U/V}^* & \xrightarrow{+1} & & \xrightarrow{d\phi_X} & T_{U/X}^* \\
 \downarrow id & & \downarrow & & \downarrow \\
 T_{U/V}^* & \xrightarrow{+1} & \phi^* T_{V/S}^* & \xrightarrow{d\phi_S} & T_{U/S}^*
 \end{array}$$

From this, it follows that the natural map of complexes

$$\begin{array}{ccc}
 T_{U/X}^* & \longrightarrow & T_U^* \\
 \downarrow & & \downarrow \\
 \phi^* T_{V/X}^* & \longrightarrow & \phi^* T_V^*
 \end{array}$$

induced by ϕ gives rise to the desired quasi-isomorphism: $(T_{X/S}^*)_U \xrightarrow{\simeq} \phi^*(T_{X/S}^*)_V$. One can then check that these isomorphism satisfy the cocycle condition. This suggests (See [Ols07] for more details) that we get a well-defined object $T_{X/S}^* \in D_{\text{qcoh}}(X, \mathcal{O}_X)$ sitting in cohomological degrees $[-1, \infty)$, and that this defines the object in $D'_{\text{qcoh}}(X, \mathcal{O}_X)$ described in Theorem 2.5.

Let us now work out this theory more explicitly, and derive some useful consequences for later. In particular, we recall that classically the tangent/cotangent complex controls smoothness of a morphism of schemes.

Proposition 2.7. [Ill71, Proposition 3.1.2] *Let $f : X \rightarrow Y$ be a locally finitely presented morphism and $T_{X/Y}$ be the usual tangent sheaf of X over Y . There is a natural map of complexes*

$$T_{X/Y}[0] \rightarrow T_{X/Y}^*$$

which induces an isomorphism on cohomology in degree 0. Then f is smooth if and only if this map is a quasi-isomorphism and $T_{X/Y}$ is locally free.

Moreover, we recall that, in the situation of the above proposition, the dimension of a fiber is computed by the rank of $T_{X/Y}$ around that fiber.

This reduces the geometric problem of smoothness to the more algebraic problem of computing the cohomology of the tangent complex. We would like to have a similar result for stacks. We will now work this out more explicitly.

2.2. The Smooth Case. Suppose that $X \rightarrow S$ is a smooth Artin S -stack for an affine scheme S . Given a point $x : S \rightarrow X$, we claim that the pullback of the tangent complex $T_{X/S}^*$ along x represents the Picard tangent groupoid described in the beginning. To see this, let \tilde{x} be the a lift of x along some smooth map $U \rightarrow X$. Using this and the above description of the tangent complex, we compute that the pullback is a complex of \mathcal{O}_S -modules $L\tilde{x}^*(T_{U/X}^* \rightarrow T_{U/S}^*)$. Since $U \rightarrow X$ and $U \rightarrow S$ are smooth by assumption. $T_{U/X}^*$ and $T_{U/S}^*$ are both isomorphic to the usual tangent sheaves $T_{U/X}$ and $T_{U/S}$, which both must be locally free \mathcal{O}_S -modules. Therefore, by Remark 2.6, we have an isomorphism $L\tilde{x}^*(T_{U/X}^* \rightarrow T_{U/S}^*) = \tilde{x}^*(T_{U/X}) \rightarrow \tilde{x}^*(T_{U/S})$, and an element of $\tilde{x}^*(T_{U/X})$ is the same as the datum of a tangent vector $S[\epsilon]/\epsilon^2 \rightarrow U$ at \tilde{x} and a trivialization of the projection of this map to X , with differential given by the forgetful map. Since the map $U \rightarrow X$ is smooth and

S is affine it follows that every tangent vector to X at x can be lifted to a tangent vector of U at \tilde{x} . It follows that the pullback of the above complex represents the Picard tangent groupoid $T_{X/S,x}$ considered above. Moreover, since X is smooth, by definition U will also be smooth. Therefore, the dimension of U over S locally around \tilde{x} is computed by the rank of $\tilde{x}^*T_{U/S}$ as an \mathcal{O}_S -module. It in turn follows that the dimension of X over S locally around x will be computed by the Euler characteristic of $x^*T_{X/S}$ as an \mathcal{O}_S -module, with the rank of H^{-1} as an \mathcal{O}_S -module computing the dimension over S of the automorphism group at x . We can summarize this as follows.

Proposition 2.8. *For a scheme S and a smooth Artin S -stack X , the tangent complex $T_{X/S}^*$ is represented by a two term complex of vector bundles on X sitting in cohomological degrees $[-1, 0]$. Its pullback to a point $x : S \rightarrow X$ is a complex whose associated Picard groupoid is the same as the Picard groupoid defined above.*

Moreover, the pullback of the tangent complex $T_{X/S}^$ to x has Euler characteristic as a locally-free \mathcal{O}_S -module equal to the dimension of X locally around x . In particular, the H^{-1} of this pullback has rank as an \mathcal{O}_S -module equal to the dimension of the group of automorphisms at x over S .*

While this is nice, the real power of this theory lies in its ability to detect when X is smooth over S at a point x . As seen in Proposition 2.7, this information will be described by the higher cohomology of the complex $T_{X/S}^*$.

2.3. The Singular Case. Now, given a general Artin stack $X \rightarrow S$, a point $x : S \rightarrow X$, and a lift \tilde{x} to a smooth chart $U \rightarrow X$ as before. The same analysis as above tells us that the pullback of $T_{X/S}^*$ to x is given by $L\tilde{x}^*(T_{U/X}^* \rightarrow T_{U/S}^*)$. We recall that $T_{U/X}^*$ is the pullback by $\Delta_{U/X}^*$ of the complex given by

$$\{T_{R/U}^* \rightarrow (f')^*(T_U^*)\}$$

but, by Proposition 2.7, the scheme U is smooth over S around \tilde{x} if and only if, after pulling back to \tilde{x} , both of these complexes are concentrated in degree 0 and are represented by vector bundles on \mathcal{O}_S . Therefore, using Remark 2.6, we can deduce that U is smooth locally around \tilde{x} if and only if both terms in $L\tilde{x}^*(T_{U/X}^* \rightarrow T_{U/S}^*) \simeq \tilde{x}^*(T_{U/X}^*) \rightarrow \tilde{x}^*(T_{U/S}^*)$ are represented by vector bundles on S in degree 0, which is in turn equivalent to X being smooth over S locally around x . Therefore, we deduce a special case of the following result.

Proposition 2.9. *[LM00, Proposition 17.10] Let X and Y be Artin S -stacks and $X \rightarrow Y$ a 1-morphism over S that is locally of finite presentation. Then $f : X \rightarrow Y$ is smooth if and only if $T_{X/Y}^*$ is represented by a complex of vector bundles on X sitting in cohomological degrees $[-1, 0]$.*

In particular, the cohomology in degree 0 of $T_{X/S}^*$ specifies the space of tangent vectors at a given point, while the cohomology in degree -1 of T_X^* controls the automorphisms at the point x , and the higher cohomology controls the obstructions to higher order deformations.

This theorem reduces the problem of checking smoothness of an Artin stack X to computing the tangent complex. Let us conclude this section by carrying this out in some examples. Set k to be an algebraically closed field.

Example 2.10. Let Y be a variety with an action of a linear algebraic group $G/\text{Spec } k$. We consider the Artin classifying stack $[Y/G] \rightarrow \text{Spec } k$. Let $\text{Lie}(G)$ be the Lie algebra of G . We look at the usual atlas given by the projection $Y \rightarrow [Y/G]$, we claim that the pullback of $T_{[Y/G]}^*$ to Y under this atlas is isomorphic to $\text{Lie}(G) \otimes \mathcal{O}_Y \rightarrow T_Y^*$, where the differential is defined by the action of G extended \mathcal{O}_Y -linearly. In particular, the pullback by definition is given by:

$$T_{Y/[Y/G]}^* \rightarrow T_Y^*$$

We consider the diagonal map $\Delta : Y \rightarrow Y \times_{[Y/G]} Y$, and let $p_1 : Y \times_{[Y/G]} Y \rightarrow Y$ be the natural projection obtained by base-changing the map defining the atlas. If we write $Y \times_{[Y/G]} Y \simeq Y \times G$

then p_1 is projection to the first factor and Δ is the embedding into the first factor. This allows us to see that $T_{Y/[Y/G]}^*$ is canonically identified with the kernel of the map $T_Y^* \oplus \mathrm{Lie}(G) \otimes \mathcal{O}_Y \rightarrow T_Y^*$ given by projection. From here, the claim easily follows.

Example 2.11. Fix a smooth projective curve $Y/\mathrm{Spec} k$ of genus g . For a connected reductive group $G/\mathrm{Spec} k$, let $X = \mathrm{Bun}_G$ denote the moduli stack of G -bundles on Y . If we take a k -point $x : \mathrm{Spec} k \rightarrow X$ corresponding to a G -bundle \mathcal{F}_G on Y , one can compute that the pullback of this complex to x is quasi-isomorphic to $R\Gamma(Y, \mathcal{F}_G \times^{G, \mathrm{Ad}} \mathrm{Lie}(G)[1])$ (cf. [Beh91, Section 4.5]). Therefore, since Y is a curve, by Propositions 2.7 and 2.9, we can see that Bun_G is a smooth Artin stack of dimension equal to the Euler characteristic of this complex locally around x , which by Riemann-Roch is precisely $\dim(G)(1 - g)$.

The appearance of the Lie algebra of G in both of these examples hints at a general phenomenon relating the two, as alluded to in the introduction. To see this, let's suppose that Y is a smooth projective curve over $\mathrm{Spec} k$ and that Z is a smooth Artin Y -stack. We let $\mathcal{M}_Z \rightarrow \mathrm{Spec} k$ be the moduli space parametrizing for schemes S over $\mathrm{Spec} k$ sections $Y \times_{\mathrm{Spec} k} S \rightarrow Z$ over Y . Suppose that $\mathcal{M}_Z \rightarrow \mathrm{Spec} k$ defines an Artin stack over $\mathrm{Spec} k$ ³. If we are given a point $x : \mathrm{Spec} k \rightarrow \mathcal{M}_Z$ corresponding to a section $s_x : Y \rightarrow Z$ then we see that giving a tangent vector $D \rightarrow \mathcal{M}_Z$ at x is equivalent to giving a lift of the section s_x to a section $\tilde{s}_x : Y[\epsilon]/\epsilon^2 \rightarrow Z$. This gives us an isomorphism

$$H^0(Y, s_x^* T_{Z/Y}^*) \simeq H^0(Lx^* T_X^*)$$

of vector spaces over k . Similarly, one has an isomorphism of H^{-1} for automorphisms of the section s_x . By considering the higher order deformations, we obtain an isomorphism of complexes of vector spaces

$$R\Gamma(Y, s_x^* T_{Z/Y}^*) \simeq Lx^* T_X^*$$

over k . We record this as a proposition.

Proposition 2.12. *Let Y be a smooth projective curve over an algebraically closed field $\mathrm{Spec} k$, $Z \rightarrow Y$ a smooth Artin Y -stack, and \mathcal{M}_Z the moduli stack of sections of Z over Y . Let $x : \mathrm{Spec} k \rightarrow \mathcal{M}_Z$ be a point with corresponding section $s_x : Y \rightarrow Z$. Then we have an isomorphism*

$$R\Gamma(Y, s_x^* T_{Z/Y}^*) \simeq Lx^* T_X^*$$

of complexes of vector spaces over k .

With this in hand, let's revisit Example 2.11.

Example 2.13. Let $Y \rightarrow \mathrm{Spec} k$ be a smooth projective curve and G/k a connected reductive group, as before. We note that $\mathrm{Bun}_G = \mathcal{M}_{[Y/G]}$. Suppose that we have a G -torsor \mathcal{F}_G on Y corresponding to a section $s : Y \rightarrow [Y/G]$. Proposition 2.12 tells us that to compute the pullback of the tangent complex of $T_{\mathrm{Bun}_G}^*$ to the point defined by \mathcal{F}_G , it suffices to compute:

$$s^* T_{[Y/G]/Y}^*$$

We can compute this similarly to Example 2.10. In particular, we can regard the section

$$s : Y \rightarrow [Y/G]$$

as an atlas for $[Y/G]$ over Y . We then have an isomorphism: $Y \times_{[Y/G]} Y \simeq \mathcal{F}_G$. Arguing as in Example 2.10, we can see that $T_{Y/[Y/G]}^*$ is identified with $\mathrm{Lie}(G) \times^{G, \mathrm{Ad}} \mathcal{F}_G$. Therefore, the formula computing the pullback $s^* T_{[Y/G]/Y}^*$ becomes

$$\{\mathrm{Lie}(G) \times^{G, \mathrm{Ad}} \mathcal{F}_G \rightarrow T_{Y/Y} \simeq 0\} \simeq \mathrm{Lie}(G) \times^{G, \mathrm{Ad}} \mathcal{F}_G[1]$$

³This will be true in all the examples we will consider. For general results concerning the representability of this space see [HR19, Theorem 1.2].

which is precisely what we expect in light of Example 2.11.

Before concluding this section, we will record one useful consequence of the above discussion that will be important for the proof of Theorem 1.7. From now on, let Y be an arbitrary scheme. Suppose we have a quasi-compact and quasi-separated morphism $f : Z_1 \rightarrow Z_2$ of Artin Y -stacks. Let's suppose we have a section $s_x : Y \rightarrow Z_1$ and let $s_y : Y \rightarrow Z_2$ be the corresponding section induced by f . We let Z denote the fiber of f over s_y . Then s_x induces a section $Y \rightarrow Z$, which we will abusively also denote by s_x . It follows from Theorem 2.5 (2) that the following is true, which we record as a corollary for future use.

Corollary 2.14. *With notation as above, we have the following distinguished triangle of tangent complexes*

$$\begin{array}{ccc} & Ls_y^* T_{Z_2/Y}^* & \\ & \swarrow \quad \searrow & \\ Ls_x^* T_{Z/Y}^* & \xrightarrow{\quad +1 \quad} & Ls_x^* T_{Z_1/Y}^* \end{array}$$

$df_{x/y}$

in $D'_{qcoh}(X, \mathcal{O}_X)$. In particular, if we assume that $f : Z_1 \rightarrow Z_2$ is a smooth map of Artin Y -stacks then, by Remark 2.6, this gives a distinguished triangle

$$\begin{array}{ccc} & s_y^* T_{Z_2/Y}^* & \\ & \swarrow \quad \searrow & \\ s_x^* T_{Z/Y}^* & \xrightarrow{\quad +1 \quad} & s_x^* T_{Z_1/Y}^* \end{array}$$

$df_{x/y}$

relating the naive pullbacks of tangent complexes.

We will now review the other key player in our main Theorem, Banach-Colmez spaces.

3. REVIEW OF BANACH-COLMEZ SPACES

In this section, we will review the theory of Banach-Colmez spaces, as originally introduced in [Le 18] and later refined in [FS21, Chapter II], with some new additions related to the smoothness properties over a geometric point and v -stack quotients of Banach-Colmez space like objects.

Fix $S \in \text{Perf}$ and consider the relative Fargues-Fontaine curve X_S and \mathcal{E} a vector bundle on it. We have the following key definition.

Definition 3.1. We define the v -sheaf $\mathcal{H}^0(\mathcal{E}) \rightarrow S$ (resp. $\mathcal{H}^1(\mathcal{E}) \rightarrow S$) to be the functor on Perf_S sending $T \in \text{Perf}_S$ to $H^0(X_T, \mathcal{E}_T)$ (resp. $H^1(X_T, \mathcal{E}_T)$), where \mathcal{E}_T is the base-change of \mathcal{E} to X_T . We refer to these v -sheaves as Banach-Colmez spaces.

Given two bundles \mathcal{F} and \mathcal{E} on X_S , we can identify $\mathcal{H}^0(\mathcal{F}^\vee \otimes \mathcal{E})$ with the moduli space, denoted $\mathcal{H}om(\mathcal{F}, \mathcal{E})$, parametrizing maps $\mathcal{F}_T \rightarrow \mathcal{E}_T$ of \mathcal{O}_{X_T} -modules. This observation allows us to consider the following open sub-functors.

Definition 3.2. Let \mathcal{F} and \mathcal{E} be two bundles on the Fargues-Fontaine curve. We consider the following open sub-functors of the v -sheaf $\mathcal{H}^0(\mathcal{F}^\vee \otimes \mathcal{E})$.

- (1) We let $\text{Surj}(\mathcal{F}, \mathcal{E}) \subset \mathcal{H}^0(\mathcal{F}^\vee \otimes \mathcal{E})$ be the sub-functor parametrizing surjections $\mathcal{F}_T \rightarrow \mathcal{E}_T$ of \mathcal{O}_{X_T} -modules.
- (2) We let $\text{Inj}(\mathcal{F}, \mathcal{E}) \subset \mathcal{H}^0(\mathcal{F}^\vee \otimes \mathcal{E})$ be the sub-functor parametrizing maps $\mathcal{F}_T \rightarrow \mathcal{E}_T$ whose pullback to any geometric point $\text{Spa}(F)$ of T is an injection of \mathcal{O}_{X_F} -modules.

Remark 3.3. One can check that these both give rise to well-defined open sub-functors of $\mathcal{H}^0(\mathcal{F}^\vee \otimes \mathcal{E})$. For Surj , see [FS21, Lemma IV.1.20], and for Inj the result over a geometric point is [Bir+22, Proposition 3.3.6]. To verify the claim in general, by [Sch18, Proposition 10.11], it suffices to check

the claim after passing to a v -cover of S . Since $\text{rank}(\mathcal{E})$, $\text{rank}(\mathcal{F})$, and $\text{deg}(\mathcal{F})$ are locally constant on S by [KL15, Lemma 7.2.2], we can therefore assume they are constant. Then the proof given in [Bir+22, Proposition 3.3.6] works exactly the same.

Similarly, if we consider $\mathcal{H}^1(\mathcal{F}^\vee \otimes \mathcal{E})$, this corresponds to the v -sheaf $\mathcal{E}xt^1(\mathcal{F}, \mathcal{E})$ parametrizing extensions of \mathcal{O}_{X_T} -modules of the form:

$$0 \rightarrow \mathcal{F}_T \rightarrow \mathcal{G} \rightarrow \mathcal{E}_T \rightarrow 0$$

Using this, we can define the following.

Definition 3.4. Suppose that \mathcal{F} and \mathcal{E} are two vector bundles on X_S of constant rank. For \mathcal{G} a vector bundle on the Fargues-Fontaine curve X_F of rank equal to $\text{rank}(\mathcal{F}) + \text{rank}(\mathcal{E})$, we consider the locally closed sub-functor

$$\mathcal{E}xt^1(\mathcal{F}, \mathcal{E})^{\mathcal{G}} \subset \mathcal{E}xt^1(\mathcal{F}, \mathcal{E}) \simeq \mathcal{H}^1(\mathcal{F}^\vee \otimes \mathcal{E})$$

parametrizing extensions whose central term is isomorphic to \mathcal{G} after pulling back to a geometric point.

Remark 3.5. The claim that this defines a locally closed sub-functor is easy to see by upper semi-continuity of the Harder-Narasimhan polygon [KL15, Thm 7.4.5]. In particular, if \mathcal{G} is a semi-stable vector bundle then this is actually an open sub-functor.

For our intended analysis of diamonds and v -stacks coming from the tangent complex, we will also want to consider v -sheaves and stacks coming from two term complexes of bundles on X_S . So, let $\mathcal{E}^* := \{\mathcal{E}^{-1} \rightarrow \mathcal{E}^0\}$ be a complex of vector bundles \mathcal{E}^i on X_S for $i = -1, 0$. Since S is affinoid, it is easy to check that the cohomology of this complex will be concentrated in degrees $[-1, 1]$ (See the proof of [KL15, Proposition 8.7.13] or [FS21, Section II.2]). We define the following v -sheaves.

Definition 3.6. For $i \in \mathbb{Z}$ and a two term complex $\mathcal{E}^* = \{\mathcal{E}^{-1} \rightarrow \mathcal{E}^0\}$ of vector bundles on X_S as above, we define the v -sheaves $\mathcal{H}^i(\mathcal{E}^*) \rightarrow S$ as the sheaf sending $T \in \text{Perf}_S$ to the hypercohomology $\mathbb{H}^i(X_T, \mathcal{E}_T^*)$, where \mathcal{E}_T^* is the base-change of the complex to X_T . We consider the action of $x \in \mathbb{H}^{-1}(X_T, \mathcal{E}^*)$ on $y \in \mathbb{H}^0(X_T, \mathcal{E}^*)$ via the map $y \mapsto y + d(x)$ and check that this gives a well-defined action on cohomology. This allows us to form the Banach-Colmez space like v -stack

$$\mathcal{P}(\mathcal{E}^*) := [\mathcal{H}^0(\mathcal{E}^*)/\mathcal{H}^{-1}(\mathcal{E}^*)] \rightarrow T$$

which we refer to as the Picard v -groupoid of \mathcal{E}^* .

With these definitions out of the way, let us prove one basic lemma on smoothness that will aid us in our analysis in this section. Let us first recall the basic definition.

Definition 3.7. [FS21, Definition IV.1.11] Let $f : X \rightarrow Y$ be a map of Artin v -stacks. We say f is cohomologically smooth if there exists a cohomologically smooth surjection $V \rightarrow X$ from a locally spatial diamond such that the composite $V \rightarrow X \xrightarrow{f} Y$ is separated, and that for one such map (equivalently any, using [Sch18, Proposition 23.13]) the composite is cohomologically smooth, where we note that it is a separated morphism that is representable in locally spatial diamonds, by the definition of Artin v -stack, so cohomological smoothness is well-defined [Sch18, Definition 23.8].

Now we have the following basic lemma.

Lemma 3.8. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be maps of Artin v -stacks. Assume that f and g are cohomologically smooth then $g \circ f$ is also cohomologically smooth. Conversely, if f and $g \circ f$ are cohomologically smooth, g is surjective, then g is cohomologically smooth.*

Moreover, assuming that f and g are cohomologically smooth of pure ℓ -dimension d and e then $g \circ f$ is cohomologically smooth of pure ℓ -dimension equal to $d + e$.

Proof. Assuming all the maps are representable in locally spatial diamonds this is precisely [Sch18, Proposition 23.13]. For the first part, since f and g are cohomologically smooth, we can choose cohomologically smooth surjections $V \rightarrow X$ and $W \rightarrow Y$ meeting the conditions of Definition 3.7 for f and g , respectively. Now, using [Sch18, Proposition 23.13], one can check that the fiber product $(W \times_Y X) \times_X V \rightarrow X$ is the desired cohomologically smooth surjection for the map $g \circ f$. For the second part, we can choose a cohomologically smooth surjection $V \rightarrow X$ for the map $X \rightarrow Y$, since $g \circ f$ is cohomologically smooth and g is surjective, this also defines a cohomologically smooth surjection for the map $g \circ f : X \rightarrow Z$. Therefore, we get a diagram

$$\begin{array}{ccc} V & \longrightarrow & Y \\ & \searrow & \downarrow g \\ & & Z \end{array}$$

where $V \rightarrow Y$ and $V \rightarrow Z$ are cohomologically smooth surjections. The claim follows. For the claim on ℓ -dimension, it follows easily from the formula

$$R((g \circ f)^!)(\mathbb{F}_\ell) = Rg^!(Rf^!(\mathbb{F}_\ell)) = Rf^!(\mathbb{F}_\ell) \otimes_{\mathbb{F}_\ell} f^* Rg^!(\mathbb{F}_\ell)$$

which follows since $g \circ f$ is cohomologically smooth, by definition of $R(g \circ f)^!$ [FS21, Definition IV.1.15]. \square

With this in hand, we will now study the representability of the Banach-Colmez spaces defined above over an arbitrary base S and then refine it in the case that S is a geometric point.

3.1. Banach-Colmez Spaces in Families. We recall the key smoothness and representability statements for families of Banach-Colmez spaces proven in [FS21].

Proposition 3.9. [FS21, Proposition II.2.6, II.3.5] *Let \mathcal{E} be a vector bundle on X_S . Then the following is true.*

- (1) *The v -sheaf $\mathcal{H}^0(\mathcal{E}) \rightarrow S$ defines a locally spatial partially proper diamond over S , this is cohomologically smooth if \mathcal{E} has strictly positive HN-slopes (See Definition 3.17) after pulling back to any geometric point of S . We will refer to this as a positive Banach-Colmez space.*
- (2) *If \mathcal{E} is a vector bundle with strictly negative HN-slopes after pulling back to any geometric point of S then the functor $\mathcal{H}^1(\mathcal{E}) \rightarrow S$ defines a locally spatial, partially proper, and cohomologically smooth diamond over S . We will refer to this as a negative Banach-Colmez space.*

Remark 3.10. In particular, it is unknown and in general may be untrue that $\mathcal{H}^1(\mathcal{E}) \rightarrow S$ defines a locally spatial diamond over S , for \mathcal{E} any vector bundle on X_S (with not necessarily negative slopes) even though this is always true for the Banach-Colmez space $\mathcal{H}^0(\mathcal{E})$.

As a consequence of this, we can formally deduce the following Corollary from Remarks 3.3 and 3.5.

Corollary 3.11. *Let \mathcal{F} and \mathcal{E} be two bundles on the relative Fargues-Fontaine curve X_S . Then the following is true.*

- (1) *The moduli spaces $\mathcal{I}nj(\mathcal{F}, \mathcal{E}) \rightarrow S$ and $\mathcal{S}urj(\mathcal{F}, \mathcal{E}) \rightarrow S$ define locally spatial partially proper diamonds over S . Moreover, if the slopes of \mathcal{F} are strictly less than the slopes of \mathcal{E} then these diamonds are cohomologically smooth.*
- (2) *If \mathcal{F} and \mathcal{E} are bundles such that the slopes of \mathcal{F} are strictly greater than the slopes of \mathcal{E} then $\mathcal{E}xt^1(\mathcal{F}, \mathcal{E}) \rightarrow S$ defines a locally spatial, partially proper, and cohomologically smooth diamond over S . Moreover, if we assume that \mathcal{F} and \mathcal{E} are of constant rank on X_S and let \mathcal{G} be a semistable vector bundle on X_F of rank equal to $\text{rank}(\mathcal{F}) + \text{rank}(\mathcal{E})$ then the same is true for the open sub-functor $\mathcal{E}xt^1(\mathcal{F}, \mathcal{E})^{\mathcal{G}}$.*

We also have the following result for the hyper-cohomology of a two term complex of bundles on X_S .

Proposition 3.12. *[FS21, Proposition II.3.5] Let $\mathcal{E}^* := \{\mathcal{E}^{-1} \rightarrow \mathcal{E}^0\}$ be a two-term complex of bundles on X_S . Assume that \mathcal{E}^{-1} has negative HN-slopes after pulling back to any geometric point of S then the v -sheaf $\mathcal{H}^0(\mathcal{E}^*) \rightarrow S$ is representable in locally spatial partially proper diamonds. Moreover, if \mathcal{E}^0 has strictly positive HN-slopes then this defines a cohomologically smooth diamond over S .*

Remark 3.13. In fact, this result follows immediately from Proposition 3.9. Namely, by considering the spectral sequence computing hyper-cohomology, we obtain a short exact sequence

$$0 \rightarrow \mathcal{H}^0(\mathcal{E}^0) \rightarrow \mathcal{H}^0(\mathcal{E}^*) \rightarrow \mathcal{H}^1(\mathcal{E}^{-1}) \rightarrow 0$$

of v -sheaves. The claim then follows from [Sch18, Proposition 23.3] and Proposition 3.9.

We now study the geometric properties of the Picard v -groupoids defined above. We first have the following claim.

Proposition 3.14. *Let \mathcal{E} be a vector bundle on X_S then the Picard v -groupoid*

$$\mathcal{P}(\mathcal{E}[1]) = [\mathcal{H}^1(\mathcal{E})/\mathcal{H}^0(\mathcal{E})] \rightarrow S$$

defines an Artin v -stack cohomologically smooth over S .

Proof. By dualizing the statement of [KL15, Proposition 6.2.4], we can find an injection of vector bundles

$$\mathcal{E} \hookrightarrow \mathcal{O}_{X_S}^{\oplus m}(n)$$

for all n sufficiently large and fixed $m > 0$. By choosing n to be strictly positive, we can arrange that the slopes of the cokernel \mathcal{G} are strictly positive after pulling back to any geometric point of S . Now it suffices to check the desired statement étale-locally on S . By [FS21, Proposition II.3.4], we can arrange, after replacing S with an étale covering $\tilde{S} \rightarrow S$, that $\mathcal{H}^1(\mathcal{O}_{X_S}^{\oplus m}(n)) = 0$. Therefore, considering the long exact cohomology sequence attached to the above exact sequence we obtain

$$0 \rightarrow \mathcal{H}^0(\mathcal{E}) \rightarrow \mathcal{H}^0(\mathcal{O}_{X_S}(n)^{\oplus m}) \rightarrow \mathcal{H}^0(\mathcal{G}) \rightarrow \mathcal{H}^1(\mathcal{E}) \rightarrow 0$$

This gives an isomorphism

$$[\mathcal{H}^1(\mathcal{E})/\mathcal{H}^0(\mathcal{E})] \simeq [\mathcal{H}^0(\mathcal{G})/\mathcal{H}^0(\mathcal{O}_{X_S}(n)^{\oplus m})]$$

of v -stacks. However, using Proposition 3.9 (1), it is easy to see that the RHS is an Artin v -stack cohomologically smooth over S . \square

We now bootstrap to a slightly more general claim.

Proposition 3.15. *Let $\mathcal{E}^* = \{\mathcal{E}^{-1} \rightarrow \mathcal{E}^0\}$ be an arbitrary two term complex of vector bundles on X_S . The Picard v -groupoid $\mathcal{P}(\mathcal{E}^*)$ is an Artin v -stack. It is cohomologically smooth if \mathcal{E}^0 has strictly positive slopes after pulling back to any geometric point of S .*

Proof. By using the spectral sequence computing hyper-cohomology of \mathcal{E}^* , we obtain an isomorphism $\mathcal{H}^0(\mathcal{E}^{-1}) \simeq \mathcal{H}^{-1}(\mathcal{E}^*)$ and a short exact sequence

$$0 \rightarrow \mathcal{H}^0(\mathcal{E}^0) \rightarrow \mathcal{H}^0(\mathcal{E}^*) \rightarrow \mathcal{H}^1(\mathcal{E}^{-1}) \rightarrow 0$$

of v -sheaves. This implies that there exists a natural map

$$\mathcal{P}(\mathcal{E}^*) = [\mathcal{H}^0(\mathcal{E}^*)/\mathcal{H}^{-1}(\mathcal{E}^*)] \rightarrow [\mathcal{H}^1(\mathcal{E}^{-1})/\mathcal{H}^0(\mathcal{E}^{-1})]$$

of v -stacks with fibers given by $\mathcal{H}^0(\mathcal{E}^0)$. Therefore, the claim follows by Lemma 3.8, Proposition 3.9, Proposition 3.14, and [FS21, Proposition IV.1.8 (iii)]. \square

While these stacky Banach-Colmez spaces might appear a bit esoteric, they actually appear quite naturally. In particular, using the above smoothness results we can verify the following.

Proposition 3.16. *[AL21, Lemma 4.1] Let G be a connected reductive group over \mathbb{Q}_p with parabolic subgroup $P \subset G$. Let M denote the Levi factor of P . Consider the map $\mathfrak{q} : \text{Bun}_P \rightarrow \text{Bun}_M$ induced by the surjection $P \rightarrow M$. This defines a (non-representable) cohomologically smooth morphism of Artin v -stacks.*

Proof. For $S \in \text{Perf}$, consider an S -point of Bun_M corresponding to an M -bundle \mathcal{F}_M on X_S . It suffices to show that the fiber of \mathfrak{q} over this S -point is a cohomologically smooth Artin v -stack. The fiber has a filtration coming from the filtration of the unipotent radical of P by commutator subgroups. The graded pieces of this filtration are of the form $\mathcal{P}(\mathcal{E}[1])$ for \mathcal{E} a vector bundle on X_S determined by \mathcal{F}_M . Therefore, the claim follows from Proposition 3.14. \square

Now let's move to the situation where S is a geometric point. Here, we will be able to be a lot more explicit. In particular, in the case that the above spaces are cohomologically smooth, we will be able to show that they are pure of ℓ -dimension specified by the degrees of the intervening bundles. This will in particular tell us how to compute the ℓ -dimension of the fibers of the above spaces over geometric points of S in terms of the pullback of the relevant bundles to these geometric points.

3.2. Banach-Colmez Spaces over a Geometric Point. For the desired applications, we will want some more refined analysis in the case that $S = \text{Spa}(F)$. Assume for simplicity that F is the tilt of C , the completed algebraic closure of a finite extension E/\mathbb{Q}_p of the p -adic integers. Here, using the slope classification of vector bundles, one can be a lot more explicit and prove more general smoothness and representability statements. We note that most of the claims also hold over an arbitrary base if one imposes the assumption that the vector bundles involved have locally constant Harder-Narasimhan slopes. In particular, the key advantage of working over a geometric point is that we can talk about the slope filtration on the bundle. Namely, by [FS21, Theorem 2.19], the vector bundle \mathcal{E} on X_F carries a filtration $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \dots \subset \mathcal{E}_k = \mathcal{E}$, such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semi-stable of slope λ_i and $\lambda_1 > \dots > \lambda_k$. Moreover, the extension groups describing this filtration vanish, so the filtration splits. We can also classify the the graded pieces. In particular, for every $\lambda \in \mathbb{Q}$ there exists a unique stable bundle $\mathcal{O}(\lambda)$ on X of slope λ , and the semistable bundles of slope λ are precisely the direct sums $\mathcal{O}(\lambda)^{\oplus n}$. Therefore, we can make the following definition.

Definition 3.17. For \mathcal{E} a bundle on X_F as above, with Harder-Narasimhan filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \dots \subset \mathcal{E}_k = \mathcal{E}$$

for any $\lambda \in \mathbb{Q}$, we define $\mathcal{E}^{\geq \lambda}$ (resp. $\mathcal{E}^{> \lambda}$) to be the subbundle given by \mathcal{E}_i for the largest value of i such that the slope of $\mathcal{E}_i/\mathcal{E}_{i-1}$ is $\geq \lambda$ (resp. $> \lambda$). We set $\mathcal{E}^{< \lambda} = \mathcal{E}/\mathcal{E}^{\geq \lambda}$ and $\mathcal{E}^{\leq \lambda} = \mathcal{E}/\mathcal{E}^{> \lambda}$.

As noted above, the Harder-Narasimhan filtration is split, therefore, for any vector bundle \mathcal{E} on X_F , we have a decomposition $\mathcal{E} \simeq \mathcal{E}^{< 0} \oplus \mathcal{E}^{\geq 0}$, and we can further decompose $\mathcal{E}^{\geq 0}$ as $\mathcal{E}^{=0} \oplus \mathcal{E}^{> 0}$, where $\mathcal{E}^{=0}$ is the semistable direct summand of slope 0. We can see that $H^0(X, \mathcal{E}) \simeq H^0(X, \mathcal{E}^{\geq 0})$ and $H^1(X, \mathcal{E}) \simeq H^1(X, \mathcal{E}^{< 0})$, and we obtain the analogous result for the associated Banach-Colmez spaces over $\text{Spa}(F)$. In particular, it follows, by Proposition 3.9, that $\mathcal{H}^0(\mathcal{E}) \rightarrow \text{Spd}(F)$ and $\mathcal{H}^1(\mathcal{E}) \rightarrow \text{Spd}(F)$ are always locally spatial partially proper diamonds over $\text{Spa}(F)$ for any bundle \mathcal{E} . Moreover, we see that $\mathcal{H}^1(\mathcal{E})$ is always cohomologically smooth over $\text{Spa}(F)$. We can refine this a bit further. For this, consider the perfectoid space

$$\widetilde{\mathbb{D}}^n := \text{Spa}(F^\circ[[T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}]], F^\circ[[T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}]]]) \times_{\text{Spa}(F^\circ, F^\circ)} \text{Spa}(F, F^\circ)$$

given by the perfectoid open unit ball.

Proposition 3.18. *Let \mathcal{E} be any vector bundle on X_F . The following is true.*

- (1) *The Banach-Colmez space $\mathcal{H}^0(\mathcal{E}) \rightarrow \mathrm{Spd}(F)$ is a locally spatial partially proper diamond. If \mathcal{E} has only positive slopes then we can find an isomorphism:*

$$\mathcal{H}^0(\mathcal{E}) \simeq \widetilde{\mathbb{D}}^d / \underline{E}^m$$

where $d = \deg(\mathcal{E})$ and \underline{E}^m is a locally profinite group acting freely on $\widetilde{\mathbb{D}}^d$. In general, there exist isomorphisms: $\mathcal{H}^0(\mathcal{E}) \simeq \mathcal{H}^0(\mathcal{E}^{=0}) \times_{\mathrm{Spd}(F)} \mathcal{H}^0(\mathcal{E}^{>0})$ and $\mathcal{H}^0(\mathcal{E}^{=0}) \simeq \underline{E}^n$, where n is the rank of the semistable summand of slope zero. It follows that $\mathcal{H}^0(\mathcal{E})$ is cohomologically smooth of pure ℓ -dimension equal to $\deg(\mathcal{E})$ if and only if \mathcal{E} has no summand of slope 0.

- (2) *The diamond $\mathcal{H}^1(\mathcal{E}) \simeq \mathcal{H}^1(\mathcal{E}^{<0}) \rightarrow \mathrm{Spd}(F)$ is locally spatial partially proper and cohomologically smooth of pure ℓ -dimension $-\deg(\mathcal{E}^{<0})$.*

Proof. The explicit presentation in part (1) is precisely [Bir+22, Prop 3.3.2]. The if direction of the claim on smoothness follows from Proposition 3.9 (1). The only if direction follows since $f^!(\mathbb{F}_\ell)$ for $f : \underline{E}^n \rightarrow \mathrm{Spd}(F)$ is equal to the sheaf of \mathbb{F}_ℓ -valued distributions on \underline{E}^n , which is far from being constant (cf. [FS21, Remark IV.1.10]). It remains to establish part (2), we have already discussed the first part of the claim above. It remains to show the claim on the ℓ -dimension. We can assume that \mathcal{E} has only negative slopes. Using the identification

$$\mathcal{H}^1(\mathcal{F} \oplus \mathcal{G}) = \mathcal{H}^1(\mathcal{F}) \times_{\mathrm{Spd}(F)} \mathcal{H}^1(\mathcal{G})$$

we may assume that \mathcal{E} is stable of slope $\lambda = \frac{s}{r} \in \mathbb{Q}$ for $s, r \in \mathbb{Z}$ with $(s, r) = 1$. Let $i_\infty : \{\infty\} \rightarrow X_F$ denote the inclusion of the closed point of X_F defined by the untilt C of F . Multiplying the structure sheaf \mathcal{O} by the local parameter corresponding to this point gives rise to an exact sequence:

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(1) \rightarrow (i_\infty)_*(C) \rightarrow 0$$

We then tensor it by $\mathcal{O}(\lambda)$ to obtain an exact sequence:

$$0 \rightarrow \mathcal{O}(\lambda) \rightarrow \mathcal{O}(\lambda + 1) \rightarrow i_{\infty*}(C)^r \rightarrow 0$$

Assume first that $-1 \leq \lambda < 0$ then, taking cohomology, we get a sequence of pro-étale sheaves:

$$0 \rightarrow \mathcal{H}^0(\mathcal{O}(\lambda + 1)) \rightarrow (\mathbb{A}_C^r)^\diamond \rightarrow \mathcal{H}^1(\mathcal{O}(\lambda)) \rightarrow 0$$

Using Part (1), Lemma 3.8, [Sch18, Proposition 27.5], [Sch18, Proposition 23.3] for the representability, and [Sch18, Proposition 24.2] if $\lambda = -1$, this implies $\mathcal{H}^1(\mathcal{O}(\lambda))$ is representable in locally spatial diamonds and cohomologically smooth of the desired ℓ -dimension. Similarly, if $\lambda < -1$, one gets via taking cohomology an exact sequence:

$$0 \rightarrow (\mathbb{A}_C^r)^\diamond \rightarrow \mathcal{H}^1(\mathcal{O}(\lambda)) \rightarrow \mathcal{H}^1(\mathcal{O}(\lambda + 1)) \rightarrow 0$$

whereby the result follows from another application of Lemma 3.8, [Sch18, Proposition 27.5], [Sch18, Proposition 23.3] for the representability, and induction. \square

Now let's revisit some of the other Banach-Colmez space like objects over a point. In particular, we can show the following.

Proposition 3.19. *Let $\mathcal{E}^* = \{\mathcal{E}^{-1} \rightarrow \mathcal{E}^0\}$ be a two term complex of vector bundles on X_F . Then, for all i , the v -sheaves $\mathcal{H}^i(\mathcal{E}^*)$ define locally spatial, partially proper diamonds over $\mathrm{Spd}(F)$, the non-trivial $\mathcal{H}^i(\mathcal{E}^*)$ are cohomologically smooth in the following cases:*

- (1) $\mathcal{H}^1(\mathcal{E}^*)$ is always cohomologically smooth of pure ℓ -dimension equal to $-\deg((\mathcal{E}^0)^{<0})$.
- (2) $\mathcal{H}^0(\mathcal{E}^*)$ is cohomologically smooth if and only if \mathcal{E}^0 does not have a summand of slope 0. In this case, it has pure ℓ -dimension equal to $\deg((\mathcal{E}^0)^{>0}) - \deg((\mathcal{E}^{-1})^{<0})$.
- (3) $\mathcal{H}^{-1}(\mathcal{E}^*)$ is cohomologically smooth if and only if \mathcal{E}^{-1} does not have a summand of slope 0. In this case, it has pure ℓ -dimension equal to $\deg((\mathcal{E}^{-1})^{>0})$.

Proof. Using the spectral sequence computing hyper-cohomology, we deduce isomorphisms $\mathcal{H}^{-1}(\mathcal{E}^*) \simeq \mathcal{H}^0(\mathcal{E}^{-1})$ and $\mathcal{H}^1(\mathcal{E}^*) \simeq \mathcal{H}^1(\mathcal{E}^0)$. Parts (1) and (3) therefore immediately follow from Proposition 3.18. For Part (2), we have a short exact sequence of v -sheaves

$$0 \rightarrow \mathcal{H}^0(\mathcal{E}^0) \rightarrow \mathcal{H}^0(\mathcal{E}^*) \rightarrow \mathcal{H}^1(\mathcal{E}^{-1}) \rightarrow 0$$

as in Remark 3.13. Thus, the claim follows from Lemma 3.8, Proposition 3.18, and [Sch18, Proposition 23.3] for the representability, aside from the only if direction of part (2); but, for this, we can argue as in the proof of Proposition 3.18. \square

With this in hand, we consider the Picard v -groupoids attached to such complexes.

Proposition 3.20. *Let $\mathcal{E}^* := \{\mathcal{E}^{-1} \rightarrow \mathcal{E}^0\}$ be a two term complex of bundles on X_F . The Picard v -groupoid $\mathcal{P}(\mathcal{E}^*) \rightarrow \mathrm{Spd}(F)$ is an Artin v -stack. It is cohomologically smooth over $\mathrm{Spd}(F)$ if \mathcal{E}^0 does not have a summand of slope 0; in which case, it is pure of ℓ -dimension equal to*

$$\deg((\mathcal{E}^0)^{\geq 0}) - \deg(\mathcal{E}^{-1})$$

Proof. This follows from Lemma 3.8, Proposition 3.19, and [Sch18, Proposition 24.2]. \square

We now conclude with a result related to the smoothness of certain Banach-Colmez like spaces coming from torsion sheaves on X_F . These smoothness results will imply smoothness of Laumon's compactification of the moduli space of P -structures. We recall that X_F is a Dedekind scheme, so we can consider a torsion sheaf \mathcal{Q} on X_F . This will break up as a direct sum of $\mathcal{O}_{X_F, x}/t_x^n$, where $n \geq 1$, x is a closed point of X_F , and t_x is a uniformizing parameter at this point. We already saw in the proof of Proposition 3.18, that the functor on Perf_F defined by $\mathcal{H}^0(\mathcal{Q})$ defines a locally spatial partially proper diamond that is cohomologically smooth of dimension $\deg(\mathcal{Q})$. In particular, it breaks up as a direct product of spaces each of which can be written as an iterated fibration of an affine space $(\mathbb{A}_C^1)^\diamond$. We record this as a proposition.

Proposition 3.21. *Let \mathcal{Q} be a torsion sheaf on X . The functor $\mathcal{H}^0(\mathcal{Q})$ on Perf_F defined by sending $T \in \mathrm{Perf}_F$ to $\mathcal{H}^0(X_T, \mathcal{Q}_T)$, where \mathcal{Q}_T denotes the base change of \mathcal{Q} to X_T , is locally spatial and cohomologically smooth over $\mathrm{Spd}(F)$. It is of pure ℓ -dimension equal to $\deg(\mathcal{Q})$.*

We would now like to consider another such space coming from torsion sheaves. In particular, let \mathcal{F} be a vector bundle on X and consider the v -sheaf, denoted $\mathcal{E}xt^1(\mathcal{Q}, \mathcal{F})$ on Perf_F parametrizing, for $T \in \mathrm{Perf}_F$, extensions of the form

$$0 \rightarrow \mathcal{F}_T \rightarrow \mathcal{G} \rightarrow \mathcal{Q}_T \rightarrow 0$$

where \mathcal{G} an \mathcal{O}_{X_T} -module on X_T and \mathcal{F}_T and \mathcal{Q}_T denote the base change of \mathcal{F} and \mathcal{Q} to X_T . We now have the following.

Proposition 3.22. *The space $\mathcal{E}xt^1(\mathcal{Q}, \mathcal{G}) \rightarrow \mathrm{Spd}(F)$ is a locally spatial, partially proper, and cohomologically smooth diamond over $\mathrm{Spd}(F)$ of pure ℓ -dimension equal to $\deg(\mathcal{Q})\mathrm{rank}(\mathcal{G})$.*

Proof. It suffices to treat the case of $\mathcal{Q} = \mathcal{O}_{X_F, x}/t_x^n$ for x a closed point of X with local parameter t_x . Consider the short exact sequence obtained by multiplication by t_x^n on the structure sheaf \mathcal{O}_{X_F} :

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(n) \rightarrow \mathcal{O}_{X, x}/t_x^n \rightarrow 0$$

Now applying $\mathrm{Hom}(-, \mathcal{G})$ and taking cohomology, we obtain a long exact sequence of v -sheaves:

$$0 \rightarrow \mathcal{H}^0(\mathcal{G}(-n)) \rightarrow \mathcal{H}^0(\mathcal{G}) \rightarrow \mathcal{E}xt^1(\mathcal{Q}, \mathcal{G}) \rightarrow \mathcal{H}^1(\mathcal{G}(-n)) \rightarrow \mathcal{H}^1(\mathcal{G}) \rightarrow 0$$

However, note that given a vector bundle \mathcal{H} on X together with a modification $\mathcal{H}|_{X \setminus x} \simeq \mathcal{G}_{X \setminus x}$ that we obtain an isomorphism

$$\mathcal{E}xt^1(\mathcal{Q}, \mathcal{G}) \simeq \mathcal{E}xt^1(\mathcal{Q}, \mathcal{H})$$

of functors on Perf_F . We can therefore assume that \mathcal{G} has slopes $> n$, in which case, we obtain a short exact sequence:

$$0 \rightarrow \mathcal{H}^0(\mathcal{G}(-n)) \rightarrow \mathcal{H}^0(\mathcal{G}) \rightarrow \mathcal{E}xt^1(\mathcal{Q}, \mathcal{G}) \rightarrow 0$$

However, since \mathcal{G} and $\mathcal{G}(-n)$ will by assumption have positive slopes, it follows that they define cohomologically smooth locally spatial diamonds. The claim therefore follows by Lemma 3.8, Proposition 3.18, and [Sch18, Proposition 23.3] for the representability. \square

We now conclude with one last easy proposition concerning these torsion sheaves.

Proposition 3.23. *Let \mathcal{Q}_1 and \mathcal{Q}_2 be two torsion sheaves on X_F . The following is true.*

- (1) *The functor $\text{Hom}(\mathcal{Q}_1, \mathcal{Q}_2)$ sending $S \in \text{Perf}_F$ to the set of homomorphisms of \mathcal{O}_{X_S} -modules $\mathcal{Q}_{1,S} \rightarrow \mathcal{Q}_{2,S}$, where $\mathcal{Q}_{1,S}$ (resp. $\mathcal{Q}_{2,S}$) denote the base changes of \mathcal{Q}_1 and \mathcal{Q}_2 to X_S is representable in locally spatial diamonds and cohomologically smooth over $\text{Spd}(F)$*
- (2) *The functor $\mathcal{E}xt^1(\mathcal{Q}_1, \mathcal{Q}_2)$ sending $S \in \text{Perf}_F$ to the set of extensions of \mathcal{O}_{X_S} -modules*

$$0 \rightarrow \mathcal{Q}_{1,S} \rightarrow \mathcal{E} \rightarrow \mathcal{Q}_{2,S} \rightarrow 0$$

where $\mathcal{Q}_{1,S}$ (resp. $\mathcal{Q}_{2,S}$) denote the base changes of \mathcal{Q}_1 and \mathcal{Q}_2 to X_S and \mathcal{E} is an \mathcal{O}_{X_S} -module on X_S is representable in locally spatial diamonds and cohomologically smooth over $\text{Spd}(F)$.

Moreover, they are both pure of the same ℓ -dimension over $\text{Spd}(F)$.

Proof. We can assume that \mathcal{Q}_1 (resp. \mathcal{Q}_2) is supported at a single closed point x_1 (resp. x_2) and for integers m_1 and m_2 we have a isomorphisms

$$\mathcal{Q}_1 \simeq \mathcal{O}_{X_F, x_1} / t_{x_1}^{m_1}$$

and

$$\mathcal{Q}_2 \simeq \mathcal{O}_{X_F, x_2} / t_{x_2}^{m_2}$$

where t_{x_1} (resp. t_{x_2}) denotes a local parameter for x_1 (resp. x_2). The sheaves $\text{Hom}(\mathcal{Q}_1, \mathcal{Q}_2)$ and $\mathcal{E}xt^1(\mathcal{Q}_1, \mathcal{Q}_2)$ will be trivial unless $x_1 = x_2 = x$, in which case it will be (non-canonically) isomorphic to

$$\mathcal{O}_{X_F, x} / t_x^m$$

where $m = \min\{m_1, m_2\}$. Therefore, part (1) follows from Proposition 3.21. For Part (2), we consider the exact sequence

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(m_1) \rightarrow \mathcal{Q}_1 \rightarrow 0$$

apply the functor $\text{Hom}(-, \mathcal{Q}_2)$, and take the cohomology

$$0 \rightarrow \text{Hom}(\mathcal{Q}_1, \mathcal{Q}_2) \rightarrow \mathcal{H}^0(\mathcal{Q}_2) \rightarrow \mathcal{H}^0(\mathcal{Q}_2) \rightarrow \mathcal{E}xt^1(\mathcal{Q}_1, \mathcal{Q}_2) \rightarrow 0$$

and identify the cokernel of the first map with $\mathcal{H}^0(\mathcal{Q})$ for \mathcal{Q} a torsion sheaf of length $m_2 - m$. Therefore, the claim follows by [Sch18, Proposition 23.13] for the representability and Lemma 3.8 for the claim on ℓ -dimension. \square

4. A JACOBIAN CRITERION FOR ARTIN v -STACKS

4.1. Proof of the Main Theorem. In this section, we will deduce Theorem 1.7 from Theorem 1.3. Before doing this, let us mention a particular nice example where a result similar to Theorem 1.7 can be easily verified.

Example 4.1. Let S be a perfectoid space and $\mathcal{E}^{\text{gm}} \rightarrow X_S$ be the geometric realization of a vector bundle \mathcal{E} of rank n on X_S . We then consider the Artin X_S -stack $[X_S / \mathcal{E}^{\text{gm}}] \rightarrow X_S$. We claim that $\mathcal{M}_{[X_S / \mathcal{E}^{\text{gm}}]}$ is given by $[\mathcal{H}^1(\mathcal{E}) / \mathcal{H}^0(\mathcal{E})]$ the Picard v -groupoid attached to the complex $\mathcal{E}[1]$ of bundles on X_S , as in Definition 3.6. To see this, we note that the datum of a section over X_S is the same as an isomorphism classes of \mathcal{E} -torsors over X_S . Such an \mathcal{E} -torsor is given by an element of $H^1(X_S, \mathcal{E})$; however, elements of this set are themselves torsors under $H^0(X_S, \mathcal{E})$, and these parametrize the

automorphisms of the associated \mathcal{E} -torsors defined by elements of $H^1(X_S, \mathcal{E})$. From this, the claim easily follows. It follows by Proposition 3.14 that this is a cohomologically smooth Artin v -stack. Let's see what the pullback of the tangent complex looks like in this case. In particular, suppose we have an S -point of \mathcal{M}_Z . This determines an atlas

$$X_S \rightarrow [X_S/\mathcal{E}^{\text{gm}}]$$

such that $R = \mathcal{E}^{\text{gm}}$. Then computing as in Examples 2.10 and 2.10, we see that the pullback of the tangent complex $s_x^* T_{[X_S/\mathcal{E}^{\text{gm}}]/X_S}$ is given by

$$\{\mathcal{E} \rightarrow 0\} \simeq \mathcal{E}[1]$$

In particular, assuming S is affinoid, this complex has cohomology concentrated only in degrees -1 and 0 and its cohomology in these degrees is given by $H^0(X_S, \mathcal{E})$ and $H^1(X_S, \mathcal{E})$, respectively. Therefore, $\mathcal{M}_{[X_S/\mathcal{E}^{\text{gm}}]}$ is isomorphic to the Picard v -groupoid of its associated tangent complex, so we can think of this example as the "linear" case of Theorem 1.7.

Now to prove the claim we will use the following lemma.

Lemma 4.2. *For $S \in \text{Perf}$, let Z be a smooth quasi-projective scheme over X_S with an action of GL_n . We consider the natural morphism of Artin X_S -stacks $p : [Z/\text{GL}_n] \rightarrow [X_S/\text{GL}_n]$. Suppose, for some countable index set $i \in I$, there exists a family of smooth quasi-projective X_S -schemes Z_i and a morphism of X_S -stacks*

$$g_i : Z_i \rightarrow [X_S/\text{GL}_n]$$

satisfying the following:

- (1) *The induced map on v -stacks of sections $\mathcal{M}_{g_i} : \mathcal{M}_{Z_i} \rightarrow \mathcal{M}_{[X_S/\text{GL}_n]}$ is a cohomologically smooth map of Artin v -stacks representable in locally spatial diamonds.*
- (2) *For any quasi-compact open subset U of the v -stack $\mathcal{M}_{[X_S/\text{GL}_n]}$ there exists finitely many $i \in I$ such that U is contained in the image of \mathcal{M}_{g_i} .*
- (3) *$[Z/\text{GL}_n] \times_{[X_S/\text{GL}_n]} Z_i$ is representable by a smooth quasi-projective variety over X_S .*

Then $\mathcal{M}_{[Z/\text{GL}_n]}$ defines an Artin v -stack over S .

Moreover, if we consider the image of $\mathcal{M}_{[Z/\text{GL}_n] \times_{[X_S/\text{GL}_n]} Z_i}^{\text{sm}}$ in $\mathcal{M}_{[Z/\text{GL}_n]}$ for varying $i \in I$, denoted $\mathcal{M}'_{[Z/\text{GL}_n]}$ ⁴, then, $\mathcal{M}'_{[Z/\text{GL}_n]} \rightarrow S$ is cohomologically smooth.

Proof. We first note that the Cartesian diagram of X_S -stacks

$$\begin{array}{ccc} [Z/\text{GL}_n] \times_{[X_S/\text{GL}_n]} Z_i & \xrightarrow{\tilde{g}_i} & [Z/\text{GL}_n] \\ \downarrow & & \downarrow p \\ Z_i & \xrightarrow{g_i} & [X_S/\text{GL}_n] \end{array}$$

induces a Cartesian diagram on spaces of sections

$$\begin{array}{ccc} \mathcal{M}_{[Z/\text{GL}_n] \times_{[X_S/\text{GL}_n]} Z_i} & \xrightarrow{\mathcal{M}_{\tilde{g}_i}} & \mathcal{M}_{[Z/\text{GL}_n]} \\ \downarrow & & \downarrow \mathcal{M}_p \\ \mathcal{M}_{Z_i} & \xrightarrow{\mathcal{M}_{g_i}} & \mathcal{M}_{[X_S/\text{GL}_n]} \end{array}$$

⁴We note, by [Sch18, Proposition 23.11], that this defines an open sub-functor of $\mathcal{M}_{[Z/\text{GL}_n]}$ since these maps are cohomologically smooth by assumption.

Since this diagram is Cartesian, by (1) the map \mathcal{M}_{g_i} is a cohomologically smooth separated, and representable map; hence, the same is true for $\mathcal{M}_{\tilde{g}_i}$. Therefore, we see that

$$\bigsqcup_{i \in I} \mathcal{M}_{[Z/\mathrm{GL}_n] \times_{[X_S/\mathrm{GL}_n]} Z_i} \xrightarrow{\bigsqcup_{i \in I} \mathcal{M}_{\tilde{g}_i}} \mathcal{M}_{[Z/\mathrm{GL}_n]}$$

defines a cohomologically smooth, representable, and surjective map of v -stacks by (2). Lastly, by (3) and Theorem 1.3, we know that the spaces $\mathcal{M}_{[Z/\mathrm{GL}_n] \times_{[Z/\mathrm{GL}_n]} Z_i} \rightarrow S$ are representable in locally spatial diamonds. It follows that $\mathcal{M}_{[Z/\mathrm{GL}_n]}$ is an Artin v -stack. We also know by Theorem 1.3 that the spaces $\mathcal{M}_{[Z/\mathrm{GL}_n] \times_{[X_S/\mathrm{GL}_n]} Z_i}^{\mathrm{sm}}$ are cohomologically smooth; therefore $\mathcal{M}'_{[Z/\mathrm{GL}_n] \times_{[X_S/\mathrm{GL}_n]} Z_i}$ is cohomologically smooth. \square

Now, let us deduce Theorem 1.7. We first prove the following lemma.

Lemma 4.3. *For $S \in \mathrm{Perf}$, let Z be a smooth quasi-projective variety over X_S with an action of a linear algebraic group H , with $\mathcal{M}_{[Z/H]}$ the v -stack of sections. Then, for $T \in \mathrm{Perf}_S$, a section $s : X_T \rightarrow [Z/H]$ defining a point in $\mathcal{M}_{[Z/H]}^{\mathrm{sm}}$ (Definition 1.2) is equivalent to insisting that, after pulling back to any geometric point of T , $\mathcal{H}^0(s^*(T^*_{([Z/H])/X_S})) \rightarrow \mathrm{Spa}(F)$ is cohomologically smooth over $\mathrm{Spd}(F)$ and $H^1(X_F, s^*(T^*_{([Z/H])/X_S}))$ vanishes.*

Proof. We note that, if we choose a presentation $s^*(T^*_{([Z/H])/X_S}) = \{\mathcal{E}^{-1} \rightarrow \mathcal{E}^0\}$ of the tangent complex as a two-term complex of vector bundles on X_F , the vanishing of the H^1 implies \mathcal{E}^0 has positive slopes. Then the claim follows from Proposition 3.19. \square

With this in hand, we can prove our main theorem.

Theorem 4.4. *For $S \in \mathrm{Perf}$, let H be a linear algebraic group over \mathbb{Q}_p and $Z \rightarrow X_S$ a smooth quasi-projective variety over X_S with an action of H . Then $\mathcal{M}_{[Z/H]} \rightarrow S$ defines an Artin v -stack, and $\mathcal{M}_{[Z/H]}^{\mathrm{sm}} \rightarrow S$ is cohomologically smooth map of Artin v -stacks.*

Moreover, for any geometric point $x : \mathrm{Spa}(F) \rightarrow \mathcal{M}_{[Z/H]}^{\mathrm{sm}}$ given by a map $\mathrm{Spa}(F) \rightarrow S$ and a section $s : X_F \rightarrow Z$, the map $\mathcal{M}_{[Z/H]}^{\mathrm{sm}} \rightarrow S$ is at x of ℓ -dimension equal to the quantity:

$$\dim_{\ell}(\mathcal{H}^0(s^*(T^*_{([Z/H])/X_S}))) - \dim_{\ell}(\mathcal{H}^{-1}(s^*(T^*_{([Z/H])/X_S})))$$

In particular, by Propositions 2.5, 3.19, and 3.20, this is well-defined and equal to the ℓ -dimension of the Picard v -groupoid

$$[\mathcal{H}^0(s^*(T^*_{([Z/H])/X_S})) / \mathcal{H}^{-1}(s^*(T^*_{([Z/H])/X_S}))]$$

over $\mathrm{Spa}(F)$. If one chooses a presentation $s^*(T^*_{([Z/H])/X_S}) = \{\mathcal{E}^{-1} \rightarrow \mathcal{E}^0\}$ using Proposition 2.7 of this complex as a two term complex of vector bundles on X_F , then this is equal to:

$$\deg(\mathcal{E}^0) - \deg(\mathcal{E}^{-1})$$

Proof. First note that we can choose a closed embedding for $H \hookrightarrow \mathrm{GL}_n$ for some sufficiently large n . Using this, we have an isomorphism of X_S -stacks: $[Z/H] \simeq [Z \times^H \mathrm{GL}_n / \mathrm{GL}_n]$. Thus, without loss of generality, we can assume that $H = \mathrm{GL}_n$. We now wish to apply Lemma 4.2. Set $i = (N, r, \mathcal{G}) \in I$ to be the countable index set parameterizing $N \in \mathbb{Z}$, $r \in \mathbb{Z}$ such that $r \geq n$, and \mathcal{G} a vector bundle on X_S of rank equal to $r - n$ which has slopes strictly greater than $\mathcal{O}(N)^{\oplus r}$ after pulling back to any geometric point of S . Then, for such an $i \in I$, we let \mathcal{M}_{Z_i} be equal to the locally spatial diamond $\mathrm{Surj}(\mathcal{O}(N)^{\oplus r}, \mathcal{G}) / \mathrm{Aut}(\mathcal{O}(N)^{\oplus r})$ parametrizing surjections of vector bundles from $\mathcal{O}(N)^{\oplus r}$ to \mathcal{G} on X_S . This is a space of sections of $Z_i \rightarrow X_S$, where Z_i is an open subset of the moduli space $[((\mathcal{O}(N)^{\oplus r})^{\vee} \otimes \mathcal{G})^{\mathrm{gm}} \setminus \{0\} / \mathrm{GL}(\mathcal{O}(N)^{\oplus r})]$ defined as the complement of the vanishing locus of the

appropriate matrix minors. Note that this is in particular represented by a quasi-projective variety. We claim that the map

$$[\mathcal{S}urj(\mathcal{O}(N)^{\oplus r}, \mathcal{G})/\mathcal{A}ut(\mathcal{O}(N)^{\oplus r})] \rightarrow \text{Bun}_{\text{GL}_n, S}$$

is induced by a map of X_S -stacks $Z_i \rightarrow [X_S/\text{GL}_n]$. This can be seen as follows. The space Z_i has a universal GL_n -torsor whose fiber over a section $X_S \rightarrow Z_i$ is precisely the geometric realization of the kernel of the associated surjection. Let us write $\widetilde{Ker} \rightarrow Z_i$ for this torsor. This torsor corresponds to a map $Z_i \rightarrow [Z_i/\text{GL}_n]$ and the desired map is induced by the natural composition:

$$g_i : Z_i \rightarrow [Z_i/\text{GL}_n] \rightarrow [X_S/\text{GL}_n]$$

We can see that the maps \mathcal{M}_{g_i} are cohomologically smooth and representable in locally spatial diamonds since its fibers are given by open subsets of cohomologically smooth Banach-Colmez spaces by the assumptions on slopes, as in the second part of Corollary 3.11 (2), using the semi-stability of $\mathcal{O}(N)^{\oplus r}$. Moreover, for varying $i \in I$, the image contains any quasi-compact open of $\text{Bun}_{\text{GL}_n, S}$ (cf. [FS21, Example IV.4.7]); therefore, we have checked (1) and (2) of Lemma 4.2. It remains to check (3). For this, we need to show that the fiber product $[Z/\text{GL}_n] \times_{[X_S/\text{GL}_n]} Z_i$ is representable by a smooth quasi-projective variety over X_S . However, this follows since the projection

$$[Z/\text{GL}_n] \times_{[X_S/\text{GL}_n]} Z_i \rightarrow Z_i$$

is the base-change of $[Z/\text{GL}_n] \rightarrow [X_S/\text{GL}_n]$ and Z and Z_i are both smooth quasi-projective over X_S . Now, consider the projection:

$$pr_1 : [Z/\text{GL}_n] \times_{[X_S/\text{GL}_n]} Z_i \rightarrow [Z/\text{GL}_n]$$

We can compute fibers of this map in terms of fibers of the map $g_i : Z_i \rightarrow [X_S/\text{GL}_n]$. In particular, assume we have a section $s_y : X_T \rightarrow [Z/\text{GL}_n]$ for $T \in \text{Perf}_S$ and, by abuse of notation, write s_y for the induced section $X_T \rightarrow [X_S/\text{GL}_n]$ via $[Z/\text{GL}_n] \rightarrow [X_S/\text{GL}_n]$. Let \mathcal{G}_y denote the corresponding vector bundle of rank n over X_T defined by s_y . The fiber over this section is given by an open subset of the moduli stack:

$$[X_T/\mathcal{G}_T^\vee \otimes \mathcal{G}_y]$$

where \mathcal{G}_T is the base change of \mathcal{G} to X_T . Recalling that the fibers of the map \mathcal{M}_{g_i} were open subsets of negative Banach-Colmez spaces, this corresponds precisely with what we expect, via Example 4.1. In particular, g_i is a smooth morphism of Artin X_S -stacks. To show the desired smoothness statement, we need to compute the pullback of the tangent bundle under a section $s_x : X_F \rightarrow [Z/\text{GL}_n] \times_{[X_S/\text{GL}_n]} Z_i$ defined with respect to any geometric point $\text{Spa}(F) \rightarrow S$. Assume that $T = \text{Spa}(F)$ and let s_y now denote the section given by composing s_x with the map the map p_1 . By abuse of notation, let us also write $s_x : X \rightarrow [X_S/\mathcal{G}^\vee \otimes \mathcal{G}_y]$ for the section of the fiber defined by s_x over s_y . It follows, by Corollary 2.14, that we get a distinguished triangle of tangent complexes:

$$\begin{array}{ccc} & s_y^* T_{([Z/\text{GL}_n])/X_S}^* & \\ & \swarrow \quad \quad \quad \searrow & \\ s_x^* T_{([X_S/\mathcal{G}^\vee \otimes \mathcal{G}_y])/X_S}^* & \xrightarrow{\quad +1 \quad} & s_x^* T_{([Z/\text{GL}_n] \times_{[X_S/\text{GL}_n]} Z_i)/X_S}^* \\ & \nwarrow \quad \quad \quad \swarrow & \\ & & dpr_{1,x/y} \end{array}$$

Now, by the assumptions on slopes, it follows that $s_x^* T_{([X_S/\mathcal{G}^\vee \otimes \mathcal{G}_y])/X_S}^*$ will be concentrated in degree 0 and have cohomology in this degree isomorphic to $H^1(X_S, \mathcal{G}^\vee \otimes \mathcal{G}_y)$, as in Example 4.1. Since we want to show that $\mathcal{M}_{[Z/\text{GL}_n]}^{\text{sm}}$ is cohomologically smooth, we can assume that s_y defines a section in this moduli space. We need to show that $s_x^* T_{([Z/\text{GL}_n] \times_{[X_S/\text{GL}_n]} Z_i)/X_S}^* = s_x^* T_{([Z/\text{GL}_n] \times_{[X_S/\text{GL}_n]} Z_i)/X_S}$ is a vector bundle with strictly positive slopes, where the equality follows from the representability statement. It follows that, since s_y defines a point in $\mathcal{M}_{[Z/H]}^{\text{sm}}$, that $H^1(X_S, s_y^* T_{([Z/\text{GL}_n])/X_S}^*)$ is trivial. Thus, using the distinguished triangle, the bundle $s_x^* T_{([Z/\text{GL}_n] \times_{[X_S/\text{GL}_n]} Z_i)/X_S}$ also has

trivial cohomology in degree 1 and we see that $s_x^*(T_{([Z/\mathrm{GL}_n] \times_{[X_S/\mathrm{GL}_n]} Z_i)/X_S})$ has positive slopes. We need to now show it has strictly positive slopes; but, by Proposition 3.18 (1), it suffices to show \mathcal{H}^0 defines a cohomologically smooth space. To do this, we consider the long exact sequence of v -sheaves over $\mathrm{Spa}(F)$ coming from this distinguished triangle:

$$\begin{array}{ccccc} & & 0 & \longrightarrow & \mathcal{H}^{-1}(s_y^* T_{([Z/\mathrm{GL}_n] \times_{[X_S/\mathrm{GL}_n]} Z_i)/X_S}^*) \\ & & & \searrow & \\ \mathcal{H}^1(\mathcal{G}^\vee \otimes \mathcal{G}_y) & \longleftarrow & \mathcal{H}^0(s_x^*(T_{([Z/\mathrm{GL}_n] \times_{[X_S/\mathrm{GL}_n]} Z_i)/X_S})) & \longrightarrow & \mathcal{H}^0(s_y^* T_{([Z/\mathrm{GL}_n] \times_{[X_S/\mathrm{GL}_n]} Z_i)/X_S}^*) \\ & & & \searrow & \\ & & 0 & \longleftarrow & \end{array}$$

We can see that $[\mathcal{H}^1(\mathcal{G}^\vee \otimes \mathcal{G}_y)/\mathcal{H}^{-1}(s_y^* T_{([Z/\mathrm{GL}_n] \times_{[X_S/\mathrm{GL}_n]} Z_i)/X_S}^*)]$ is a cohomologically smooth Artin v -stack over F , by Proposition 3.20 and [Sch18, Proposition 24.12]. Moreover, by Lemma 4.3, the same is true for $\mathcal{H}^0(s_y^* T_{([Z/\mathrm{GL}_n] \times_{[X_S/\mathrm{GL}_n]} Z_i)/X_S}^*)$; thus, it follows by Lemma 3.8 that $\mathcal{H}^0(s_x^*(T_{([Z/\mathrm{GL}_n] \times_{[X_S/\mathrm{GL}_n]} Z_i)/X_S}))$ is cohomologically smooth over $\mathrm{Spd}(F)$, giving the desired claim. To verify the claim on ℓ -dimension, we can use Theorem 1.3 to compute the ℓ -dimension of $\mathcal{M}_{[Z/\mathrm{GL}_n] \times_{[X_S/\mathrm{GL}_n]} Z_i}^{\mathrm{sm}}$ in terms of the ℓ -dimension of the Banach-Colmez space $\mathcal{H}^0(s_x^*(T_{([Z/\mathrm{GL}_n] \times_{[X_S/\mathrm{GL}_n]} Z_i)/X_S}))$. Moreover, the ℓ -dimension of the fibers of

$$\mathcal{M}_{[Z/\mathrm{GL}_n] \times_{[X_S/\mathrm{GL}_n]} Z_i}^{\mathrm{sm}} \rightarrow \mathcal{M}_{[Z/\mathrm{GL}_n]}^{\mathrm{sm}}$$

are equal to $\dim_\ell(\mathcal{H}^1(\mathcal{G}^\vee \otimes \mathcal{G}_y))$. Therefore, the claim on the ℓ -dimension follows from the above long exact sequence and Lemma 3.8. \square

Now, let's show the utility of this result in some applications.

4.2. Applications. First, let's start by reproving a result shown in Fargues-Scholze.

Theorem 4.5. [FS21, Theorem IV.1.18] *Let G/\mathbb{Q}_p be a connected reductive group. Let $\mathrm{Bun}_G \rightarrow \mathrm{Spd}(F)$ be the moduli stack parametrizing G -bundles on the Fargues-Fontaine curve X_F . Then Bun_G is an Artin v -stack cohomologically smooth over $\mathrm{Spd}(F)$ of pure ℓ -dimension 0.*

Proof. We apply Theorem 1.7 with $Z = X_F$ and $H = G$. In particular, $\mathcal{M}_{[Z/H]} \simeq \mathrm{Bun}_G$ in this case. Given a section $s : X_F \rightarrow [X_F/\mathrm{GL}_n]$ corresponding to a G -torsor \mathcal{F}_G , we compute as in Example 2.11, that $s^* T_{[Z/H]/X_F}^* \simeq \mathrm{Lie}(G) \times^{G, \mathrm{Ad}} \mathcal{F}_G[1] = \{\mathrm{Lie}(G) \times^{G, \mathrm{Ad}} \mathcal{F}_G \rightarrow 0\}$. Moreover, we see that $\deg(\mathrm{Lie}(G) \times^{G, \mathrm{Ad}} \mathcal{F}_G) = 0$ because G is reductive and therefore the bundle $\mathrm{Lie}(G) \times^{G, \mathrm{Ad}} \mathcal{F}_G$ is self-dual. This gives the desired claim. \square

Let's now consider a slightly more interesting example. Let G/\mathbb{Q}_p be a connected reductive group as before and assume that G is quasi-split for simplicity. Fix a maximal split torus, maximal torus, and Borel $A \subset T \subset B \subset G$. Let $P \subset G$ be a proper parabolic subgroup with Levi factor M . We recall [Vie21] that the underlying topological space of Bun_M , denoted $|\mathrm{Bun}_M|$, is isomorphic to the Kottwitz set $B(M)$ of M , equipped with the partial ordering given by the maps

$$\begin{aligned} \nu \times \kappa : B(M) &\rightarrow (X_*(M_{\overline{\mathbb{Q}}_p}) \otimes \mathbb{Q})^{\Gamma, +} \times \pi_1(M)_\Gamma \\ b &\mapsto (\nu_b, \kappa(b)) \end{aligned}$$

where $\Gamma := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is the absolute Galois group, $(X_*(M_{\overline{\mathbb{Q}}_p}) \otimes \mathbb{Q})^{\Gamma, +}$ is the set of Galois-invariant dominant rational cocharacters of M , and $\pi_1(M)$ is Borovoi's fundamental group of $M_{\overline{\mathbb{Q}}_p}$. Namely, we say that $b \succeq b'$ if $\nu_b - \nu_{b'}$ is a linear combination of positive roots and $\kappa(b) = \kappa(b')$. Therefore, the connected components of Bun_M , the moduli stack of M -bundles on X_F are parametrized by

the minimal elements $B(M)_{basic} \xrightarrow{\simeq} \pi_1(M)_\Gamma$ in this partial ordering on $B(M)$. In particular, each basic element defines an open and dense Harder-Narasimhan strata in the associated connected component. Given $\nu \in B(M)_{basic}$, we write Bun_M^ν for the associated connected component. We let Bun_P denote the moduli space of P -bundles on X_F , and let Bun_P^ν denote the pre-image of this connected component under the natural map $\mathfrak{q} : \text{Bun}_P \rightarrow \text{Bun}_M$. Using the proof of Proposition 3.16 and [Sch18, Proposition 23.11], we can show that this morphism is open with connected fibers, which implies that Bun_P^ν are the connected components of Bun_P . Given a P -bundle \mathcal{G}_P on X_F and a character $\chi \in X^*(P)$ in the character lattice of P we have an associated line bundle $\chi_*(\mathcal{G}_P)$. The mapping

$$\begin{aligned} X^*(P) &\rightarrow \mathbb{Q} \\ \chi &\mapsto \deg(\chi_*(\mathcal{G}_P)) \end{aligned}$$

defines an element $\mu(\mathcal{G}_P) \in X_*(A)_\mathbb{Q}$ the set of rational cocharacters of A . We refer to this as the slope cocharacter of \mathcal{G}_P . For all $\nu \in B(M)_{basic}$, we can consider the corresponding M -bundle \mathcal{F}_M^ν then we define the integer

$$d_\nu := \langle 2\rho_G, \mu(\mathcal{F}_M^\nu \times^M P) \rangle$$

where $2\rho_G \in X_*(A)$ is the sum of all positive roots of G defined by the choice of Borel.

Example 4.6. Suppose that $G = \text{GL}_n$, B is the upper triangular Borel, h_i are positive integers such that $\sum_{i=1}^k h_i = n$, and P is the upper triangular parabolic with Levi factor $M = \text{GL}_{h_1} \times \cdots \times \text{GL}_{h_k}$. Then $\nu \in B(M)_{basic}$ is specified by a tuple of integers (d_1, \dots, d_k) , where d_i are the degrees of the semistable bundles of rank h_i defining the M -bundle \mathcal{F}_M^ν . Then we have an equality:

$$d_\nu = \sum_{j>i} h_i d_j - h_j d_i$$

In particular, if the slopes of the M -bundle are increasing d_ν is positive, and if they are decreasing d_ν is negative.

Proposition 4.7. *Let G/\mathbb{Q}_p be a quasi-split connected reductive group with $P \subset G$ a parabolic of G . Write $\text{Bun}_P \rightarrow \text{Spd}(F)$ for the moduli stack parameterizing P -bundles on X_F . Then Bun_P is an Artin v -stack cohomologically smooth over $\text{Spd}(F)$. The connected components Bun_P^ν are of pure ℓ -dimension equal to d_ν .*

Proof. We apply Theorem 1.7 with $Z = X_F$ and $H = P$, so we have $\mathcal{M}_{[X_F/P]} \simeq \text{Bun}_P$. Then, as in the proof of Theorem 4.5, we can see that, given a section $s : X_F \rightarrow [X_F/P]$ corresponding to a P -bundle \mathcal{G}_P , the pullback $s^*T_{[X_F/P]}^*$ is isomorphic to $\text{Lie}(P) \times^{P,Ad} \mathcal{F}_P[1]$. From here, one easily verifies the desired cohomological smoothness. It remains to show the claim on ℓ -dimension, so suppose that $\mathcal{G}_P \in \text{Bun}_P^\nu$. We need to compute the quantity:

$$\deg(\text{Lie}(P) \times^{P,Ad} \mathcal{F}_P)$$

To do this, we first note, by breaking up $\text{Lie}(P)$ into root spaces, that this is equal to the quantity:

$$\sum_{\chi \in X^*(P)} \deg(\chi_*(\mathcal{F}_P))$$

Note, since M is a reductive group, the contributions from $X^*(M)$ all cancel out, as in the proof of Theorem 4.5. Therefore, this is equal to

$$\sum_{\chi \in X^*(U)} \deg(\chi_*(\mathcal{F}_P))$$

where U is the unipotent radical of P . However, by assumption, $\mathcal{F}_P \times^P M$ will be an M -bundle lying in the same connected component of Bun_M as \mathcal{F}_M^ν ; therefore, κ applied to their associated Kottwitz elements will be the same. This implies we have an equality⁵:

$$\sum_{\chi \in X^*(U)} \deg(\chi^*(\mathcal{F}_P)) = \sum_{\chi \in X^*(U)} \deg(\chi^*(\mathcal{F}_M^\nu))$$

However, we note that, by definition of the slope cocharacter this is the same as

$$\langle 2\rho_U, \mu(\mathcal{G}_P^\nu) \rangle$$

where $2\rho_U := 2\rho_G - 2\rho_M$, and $2\rho_M$ is the sum of all positive roots of M with respect to the choice of Borel. However, since $\mathcal{G}_P^\nu \times^P M$ is a semistable M -bundle by definition $\langle 2\rho_M, \mu(\mathcal{G}_P) \rangle = 0$. Thus, this is equal to

$$d_\nu = \langle 2\rho_G, \mu(\mathcal{G}_P^\nu) \rangle$$

□

Remark 4.8. (1) We could have also easily seen this more directly using Lemma 3.8, the proof of Proposition 3.16, Proposition 3.20, and Theorem 2.5.

(2) We note that if $G = \text{GL}_n$ so that a P -bundle \mathcal{G}_P corresponds to a flag of vector bundles $\mathcal{L}_\bullet = \{0 = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_k = \mathcal{L}\}$ then we have an identification

$$(F_k/F_0)\mathcal{R}\mathcal{H}om_{\text{fil}}(\mathcal{L}_\bullet, \mathcal{L}_\bullet)[1] \simeq \text{Lie}(P) \times^P \mathcal{G}_P[1]$$

where $\mathcal{R}\mathcal{H}om_{\text{fil}}(-, -)$ denotes the filtered $\mathcal{R}\mathcal{H}om$ with filtration F_i . The natural filtration on the LHS corresponds to the filtration by root spaces on the RHS.

We will now move into more interesting applications of the Theorem. Let us consider two parabolics $P_1, P_2 \subset G$ of a fixed quasi-split G . Regard P_1, P_2 , and G as constant group schemes over X_F . We will consider the Artin X_F -stack:

$$P_1 \backslash G / P_2 \rightarrow X_F$$

Sections of this moduli space corresponds to a P_1 -bundle \mathcal{G}_{P_1} and a P_2 -bundle \mathcal{G}_{P_2} on X_F together with an isomorphism

$$\mathcal{G}_{P_1} \times^{P_1} G \simeq \mathcal{G}_{P_2} \times^{P_2} G$$

of their induced G -bundles (See [Sch15, Section 4]). In particular, we have an isomorphism:

$$\text{Bun}_{P_1} \times_{\text{Bun}_G} \text{Bun}_{P_2} \simeq \mathcal{M}_{P_1 \backslash G / P_2}$$

We let M_1 and M_2 denote the Levi factors of P_1 and P_2 , respectively. We consider the double cosets

$$W_{M_1} \backslash W_G / W_{M_2}$$

where W_{M_1}, W_{M_2} , and W_G denote the relative Weyl group of M_1, M_2 , and G , respectively. Let Φ_0 be the set of relative roots of G , Φ_0^+ the set of positive roots defined by B , and Δ_0 the set of simple (reduced) roots. For $\alpha \in \Delta_0$, we let r_α denote the simple reflection corresponding to it. We then have the function

$$\ell(-) : W_G \rightarrow \mathbb{N}_{\geq 0}$$

given by the length of the element relative to the generating set $\{r_\alpha\}_{\alpha \in \Delta_0}$. We can define a set of representatives of the above double cosets by considering the representatives of minimal length. Namely

$$W := \{w \in W_G \mid \ell(w) \leq \ell(w_1 w w_2) \text{ for all } w_1 \in W_{M_1}, w_2 \in W_{M_2}\}$$

⁵More precisely, if M^{ab} denotes the abelianization of M the assumption implies that the induced M^{ab} -bundles of these two M -bundles agree, and from here the equality follows.

where $\ell : W_G \rightarrow \mathbb{Z}$ is the function measuring the length. We recall that we have a partial ordering on $w \in W_G$, where we say that $w' \succeq w$ if given a reduced expression $w = r_{\alpha_1} \cdots r_{\alpha_{\ell(w)}}$, for $\{\alpha_j \in \Delta_0\}$, that $w' = r_{\alpha_{i_1}} r_{\alpha_{i_2}} \cdots r_{\alpha_{i_n}}$ with $1 \leq i_1 < \cdots < i_n \leq \ell(w)$.

We now consider the above set of representatives W equipped with this partial ordering. We recall that we have a Bruhat decomposition

$$G = \bigcup_{w \in W} P_1 w P_2$$

which induces a stratification of $P_1 \backslash G / P_2$ into locally closed substacks, denoted $(P_1 \backslash G / P_2)_w$. The closure of this substack, denoted $(P_1 \backslash G / P_2)_{\succeq w}$, is stratified by $(P_1 \backslash G / P_2)_{w'}$ for $w' \succeq w$. We write $(\text{Bun}_{P_1} \times_{\text{Bun}_G} \text{Bun}_{P_2})_{\succeq w} \subset \text{Bun}_{P_1} \times_{\text{Bun}_G} \text{Bun}_{P_2}$ for the substack defined by sections $s : X_S \rightarrow P_1 \backslash G / P_2$ over X_F which factor over this closed substack. We have the following Lemma.

Lemma 4.9. $(\text{Bun}_{P_1} \times_{\text{Bun}_G} \text{Bun}_{P_2})_{\succeq w} \subset \text{Bun}_{P_1} \times_{\text{Bun}_G} \text{Bun}_{P_2}$ defines a closed substack of $\text{Bun}_{P_1} \times_{\text{Bun}_G} \text{Bun}_{P_2}$.

Proof. For $T \in \text{Perf}_F$, consider a map $T \rightarrow \text{Bun}_{P_1} \times_{\text{Bun}_G} \text{Bun}_{P_2}$ corresponding to a section $s : X_T \rightarrow P_1 \backslash G / P_2$. We consider the closed subspace $Z_T \subset X_T$ defined by the pre-image of the closed substack $(P_1 \backslash G / P_2)_{\succeq w}$ under s . Now write \mathcal{X}_T for the adic Fargues-Fontaine curve associated to T . There is a map of locally ringed spaces $\mathcal{X}_T \rightarrow X_T$ (See [FS21, Propositions II.2.7, II.2.9]) and we write \mathcal{Z}_T for the preimage of Z_T under this map. There is then a natural map of topological spaces

$$|\mathcal{X}_T| \rightarrow |T|$$

given by the Teichmüller character which is closed. In particular, it is a specializing quasi-compact map of spectral spaces (cf. the proofs of [Bir+22, Proposition 3.3.6] and [FS21, Proposition IV.4.22]). From here, it is easy to see that the fiber of T over the substack $(\text{Bun}_{P_1} \times_{\text{Bun}_G} \text{Bun}_{P_2})_{\succeq w}$ is precisely the closed subspace given by the image of \mathcal{Z}_T under this map, so this is indeed defines a closed substack. \square

For $w \in W$, we can use this to define a locally closed stratification $\bigcup_{w \in W} (\text{Bun}_{P_1} \times_{\text{Bun}_G} \text{Bun}_{P_2})_w$ of $\text{Bun}_{P_1} \times_{\text{Bun}_G} \text{Bun}_{P_2}$. In particular, for fixed $w \in W$, we define the substack

$$(\text{Bun}_{P_1} \times_{\text{Bun}_G} \text{Bun}_{P_2})_w := (\text{Bun}_{P_1} \times_{\text{Bun}_G} \text{Bun}_{P_2})_{\succeq w} \setminus \bigcup_{\substack{w' \\ w' \succeq \neq w}} (\text{Bun}_{P_1} \times_{\text{Bun}_G} \text{Bun}_{P_2})_{\succeq w'}$$

which we see is locally closed by Lemma 4.9. To illuminate this a bit more, given a section $s : X_F \rightarrow P_1 \backslash G / P_2$, this will define a F -point lying in the strata $(\text{Bun}_{P_1} \times_{\text{Bun}_G} \text{Bun}_{P_2})_w$ if, over an open and dense subset $U \subset X_F$, s factors through $(P_1 \backslash G / P_2)_w$. If such a s correspond to a pair of bundles \mathcal{G}_{P_1} and \mathcal{G}_{P_2} this is the same as saying they are in generic relative position w , as in [Sch15, Definition 4.1.3]. We recall that the element w determines parabolic subgroups Q_1 and Q_2 of P_1 and P_2 respectively such that their Levi factors are conjugate under w . It follows, by [Sch15, Corollary 4.2.12], that the parabolic structures lying in relative position w implies that \mathcal{G}_{P_1} and \mathcal{G}_{P_2} admit further reductions to \mathcal{G}_{Q_1} and \mathcal{G}_{Q_2} such that the Levi factors are isomorphic under conjugation by w . We now have the following result.

Proposition 4.10. *For P_1, P_2 two proper parabolic subgroups of a connected reductive group G/\mathbb{Q}_p the moduli space*

$$\text{Bun}_{P_1} \times_{\text{Bun}_G} \text{Bun}_{P_2}$$

is an Artin v -stack. The locally closed strata

$$(\text{Bun}_{P_1} \times_{\text{Bun}_G} \text{Bun}_{P_2})_{w \in W}$$

are cohomologically smooth over $\text{Spd}(F)$.

Proof. The claim that it is an Artin v -stack follows immediately from Theorem 1.7 applied to $Z = G/P_2$ and $H = P_1$. It remains to verify the smoothness statement. Let us first explain the case of GL_n . In general, it will reduce to this case. Similar analysis can be found in the proof of [Lau90, Proposition 4.3.1]. We suppose that P_1 is a parabolic with Levi factor $M_1 := \mathrm{GL}_{h_1} \times \cdots \times \mathrm{GL}_{h_k}$ and P_2 is a parabolic with Levi factor $M_2 := \mathrm{GL}_{h'_1} \times \cdots \times \mathrm{GL}_{h'_{k'}}$ where $\sum_{i=1}^k h_i = n = \sum_{i=1}^{k'} h'_i$. Suppose that we have a section $s : X_F \rightarrow P_1 \backslash G/P_2$ defining parabolic structures \mathcal{G}_{P_1} and \mathcal{G}_{P_2} . These correspond to a flag of vector bundles $\mathcal{L} = \{0 = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_k = \mathcal{L}\}$ and $\mathcal{L}' = \{0 = \mathcal{L}'_0 \subset \mathcal{L}'_1 \subset \cdots \subset \mathcal{L}'_{k'} = \mathcal{L}\}$, where \mathcal{L} is some fixed bundle of rank n . We let d_i and d'_j for $i = 1, \dots, k$ and $j = 1, \dots, k'$ denote the degrees of the M_1 and M_2 -bundle induced by \mathcal{G}_{P_1} and \mathcal{G}_{P_2} , respectively. We fix tuples $((h_{ij})_{\substack{1 \leq i \leq k \\ 1 \leq j \leq k'}})$ and $((d_{ij})_{\substack{1 \leq i \leq k \\ 1 \leq j \leq k'}})$ satisfying:

$$\begin{cases} h_{i1} + \cdots + h_{ik'} = h_i & (i = 1, \dots, k) \\ h_{1j} + \cdots + h_{kj} = h_j & (j = 1, \dots, k') \end{cases}$$

and

$$\begin{cases} d_{i1} + \cdots + d_{ik'} = d_i & (i = 1, \dots, k) \\ d_{1j} + \cdots + d_{kj} = d_j & (j = 1, \dots, k') \end{cases}$$

such that the integers h_{ij} (resp. d_{ij}) correspond to the ranks (resp. degrees) of the graded pieces of the bi-filtration induced on \mathcal{L} . The \mathcal{O}_{X_F} -modules

$$\mathcal{L}_{ij} := \mathcal{L}_i \cap \mathcal{L}'_j / (\mathcal{L}_{i-1} \cap \mathcal{L}'_j + \mathcal{L}_i \cap \mathcal{L}'_{j-1})$$

away from the torsion summands give rise to the further reductions of \mathcal{G}_{P_1} and \mathcal{G}_{P_2} to parabolic subgroups Q_1 and Q_2 described above. In particular, fixing the ν_{ij} determines the generic relative position w and fixing the d_{ij} defines an open and closed substack in $(\mathrm{Bun}_{P_1} \times_{\mathrm{Bun}_G} \mathrm{Bun}_{P_2})_w$ as the degree is locally constant. As in Remark 4.8, we can compute that the pullback of the tangent complex $s^*T_{[Z/H]/X_F}$ is quasi-isomorphic to the complex

$$(F_k F'_{k'} / (F_0 + F'_0)) \mathcal{R}\mathcal{H}om_{\mathrm{bifil}}((\mathcal{L}, F, F'), (\mathcal{L}, F, F'))$$

a graded piece of the bi-filtered $\mathcal{R}\mathcal{H}om$. This admits a filtration whose graded pieces are isomorphic to

$$\mathcal{R}\mathcal{H}om(\mathcal{L}_{ij}, \mathcal{L}_{pq})[1]$$

for $1 \leq i \leq p \leq k$ and $1 \leq j \leq q \leq k'$. Using Lemmas 3.8 and 4.3, we reduce to showing that H^1 of the graded pieces vanishes and that H^0 defines a cohomologically smooth space. By writing \mathcal{L}_{ij} and \mathcal{L}_{pq} as a direct sum of vector bundles and torsion sheaves, it is easy to see that the H^1 of this complex vanishes and, by Propositions 3.18, 3.21, 3.22, and 3.23, that H^0 defines a cohomologically smooth space. We note that in this case we can explicitly show that the open and closed substack of $(\mathrm{Bun}_{P_1} \times_{\mathrm{Bun}_G} \times \mathrm{Bun}_{P_2})_w$ defined by fixing the degree of the bi-graded pieces is, by Theorem 1.7, pure of ℓ -dimension equal to

$$\sum_{\substack{1 \leq i \leq p \leq k \\ 1 \leq j \leq q \leq k'}} h_{ij} d_{pq} - h_{pq} d_{ij}$$

The case of a general reductive group G follows from this one. In particular, if we consider $\mathcal{G}_{P_1} \times^{P_1, Ad} \mathrm{Lie}(P_1)$ and $\mathcal{G}_{P_2} \times^{P_2, Ad} \mathrm{Lie}(P_2)$ and let \mathcal{F}_G be the induced G -bundle. Then these induce parabolic structures on the vector bundle $\mathrm{Lie}(G) \times^{G, Ad} \mathcal{F}_G$, and we can proceed as above to compute the pullback of the tangent complex and show it has the right properties⁶. \square

⁶It would be nice to have a general formula for the ℓ -dimension; unfortunately, we were unable to come up with one.

Remark 4.11. These spaces appear very naturally when studying Geometric Eisenstein series over the Fargues-Fontaine curve. In particular, we consider the natural diagrams of v -stacks

$$\begin{array}{ccc} \mathrm{Bun}_{P_1} & & \\ \downarrow \mathfrak{q}_{P_1} & \searrow \mathfrak{p}_{P_1} & \\ \mathrm{Bun}_{M_1} & & \mathrm{Bun}_G \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Bun}_{P_2} & & \\ \downarrow \mathfrak{q}_{P_2} & \searrow \mathfrak{p}_{P_2} & \\ \mathrm{Bun}_{M_2} & & \mathrm{Bun}_G \end{array}$$

Using this, we can define the Eisenstein functor

$$\mathrm{Eis}_{P_1}(-) := \mathfrak{p}_{P_1,!}(\mathfrak{q}_{P_2})^*(-)$$

and the constant term functor:

$$\mathrm{CT}_{P_2}(-) := \mathfrak{q}_{P_2,!}(\mathfrak{p}_{P_2})^*(-)$$

The compositions $\mathrm{Eis}_{P_1} \circ \mathrm{CT}_{P_2}(-)$ and $\mathrm{CT}_{P_2} \circ \mathrm{Eis}_{P_1}(-)$ can be computed in terms of the cohomology of these spaces of simultaneous reductions.

We will now conclude with the most important application. The smoothness of Laumon's compactification, defined in the function field case in [Lau90]. From now on, we will let $G = \mathrm{GL}_n$ and P be a parabolic subgroup. As discussed above, Bun_P in this case parametrizes a flag of vector bundles $\mathcal{L} = \{0 = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \dots \subset \mathcal{L}_k = \mathcal{L}\}$. Laumon's compactification is a relative compactification of the natural map $\mathfrak{p} : \mathrm{Bun}_P \rightarrow \mathrm{Bun}_G$, which we denote by $\bar{\mathfrak{p}}^L : \overline{\mathrm{Bun}}_P^L \rightarrow \mathrm{Bun}_G$. It parametrizes "quasi-flags". Namely, a flag \mathcal{L} , but where the inclusions $\mathcal{L}_i \rightarrow \mathcal{L}_{i+1}$ are not inclusions of vector bundles, rather a map of \mathcal{O}_{X_S} -modules whose pullback to any geometric point F of S is an injection of \mathcal{O}_{X_F} -modules. There exists a natural open immersion $\mathrm{Bun}_P \subset \overline{\mathrm{Bun}}_P^L$ extending the map \mathfrak{p} to a map $\bar{\mathfrak{p}}^L$, and, using Corollary 3.11, we can see that $\bar{\mathfrak{p}}^L$ is representable in locally spatial diamonds and should be proper after fixing the degree of the graded pieces of a quasi-flag. Classically, the analogue of this compactification in the function field setting defines a smooth Artin stack [Lau90, Lemma 4.2.3]. We show that the same is true in this context. We let M denote the Levi factor of P and fix $\nu \in B(M)_{\text{basic}}$ corresponding to a tuple of integers (d_1, \dots, d_k) and write d_ν for the associated integer, as above. We let $\overline{\mathrm{Bun}}_P^{L,\nu}$ be the open and closed substack of $\overline{\mathrm{Bun}}_P^L$ defined by fixing the degree of the graded pieces. Our main result is as follows.

Proposition 4.12. $\overline{\mathrm{Bun}}_P^L$ is an Artin v -stack cohomologically smooth over $\mathrm{Spa}(F)$. The substacks $\overline{\mathrm{Bun}}_P^{L,\nu}$ are of pure ℓ -dimension d_ν over $\mathrm{Spd}(F)$.

Proof. If we write the Levi factor as $\mathrm{GL}_{h_1} \times \mathrm{GL}_{h_2} \times \dots \times \mathrm{GL}_{h_k}$ for positive integers h_i . Then we define $h^i = \sum_{j=1}^i h_j$ for $i = 1, \dots, k$. Let V_i be the trivial vector bundle of rank i on X_F . In what follows, we will abuse notation and use this to denote both the bundle and its geometric realization over X_F .

We consider the affine spaces $V_{h^i}^\vee \otimes V_{h^{i+1}} \rightarrow X_F$ for $i = 1, \dots, k-1$. Sections of this will parametrize maps $V_{h^i} \rightarrow V_{h^{i+1}}$ of \mathcal{O}_{X_F} -modules. We pass to the stack quotient

$$[V_{h^i}^\vee \otimes V_{h^{i+1}} / \mathrm{GL}_{h^i} \times \mathrm{GL}_{h^{i+1}}] \rightarrow X_F$$

where the GL s act via linear transformations. Sections of this map will correspond to a pair of vector bundles \mathcal{L}_i and \mathcal{L}_{i+1} of rank h^i and h^{i+1} , respectively, together with a map

$$\mathcal{L}_i \rightarrow \mathcal{L}_{i+1}$$

of \mathcal{O}_{X_F} -modules. We can then form the fiber product

$$Z := V_{h^1}^\vee \otimes V_{h^2} \times_{V_{h^2}} V_{h^2}^\vee \otimes V_{h^3} \times_{V_{h^3}} \cdots \times_{V_{h^{k-1}}} V_{h^{k-1}}^\vee \otimes V_{h^k} \rightarrow X_F$$

and consider the stack quotient by $H := \mathrm{GL}_{h^1} \times \mathrm{GL}_{h^2} \cdots \times \mathrm{GL}_{h^k}$, where GL_{h^i} acts on V_{h^i} and $V_{h^i}^\vee$ by linear transformations. In this case, $\mathcal{M}_{[Z/H]}$ will parametrize a tuple of vector bundles $(\mathcal{L}_1, \dots, \mathcal{L}_k)$ and a set of \mathcal{O}_{X_F} -module maps $\mathcal{L}_i \rightarrow \mathcal{L}_{i+1}$ for all $i = 1, \dots, k-1$, where \mathcal{L}_i has rank h^i . $\overline{\mathrm{Bun}}_P^L$ is the open sub-functor of $\mathcal{M}_{[Z/H]}$ corresponding to the subspace where the maps $\mathcal{L}_i \rightarrow \mathcal{L}_{i+1}$ define injections of \mathcal{O}_{X_F} -modules (cf. Remark 3.3). Therefore, we can apply Theorem 1.7 and this reduces to computing the pullback $s^*T_{[Z/H]}^*$ of the tangent complex for a section $s : X_F \rightarrow [Z/H]$ corresponding to a quasi-flag:

$$\mathcal{L} = \{0 = \mathcal{L}_0 \subset \cdots \subset \mathcal{L}_k = \mathcal{L}\}$$

One can verify (cf. [Lau90, Lemma 4.1] and Remark 4.8 (2)) that this pullback is given by:

$$F_k/F_0(\mathcal{R}\mathcal{H}om_{\mathrm{fil}}(\mathcal{L}_\bullet, \mathcal{L}_\bullet))[1]$$

We note that this has a filtration by graded pieces given by

$$\mathcal{R}\mathcal{H}om(\mathcal{L}_i/\mathcal{L}_{i-1}, \mathcal{L}_j/\mathcal{L}_{j-1})[1]$$

for $i \leq j$. It therefore suffices, by Lemmas 3.8 and 4.3, to show that H^1 of these complexes vanish and H^0 defines a cohomologically smooth diamond. As in the proof of Proposition 4.10, by writing $\mathcal{L}_i/\mathcal{L}_{i-1}$ and $\mathcal{L}_j/\mathcal{L}_{j-1}$ as a direct sum of locally free and torsion sheaves, this follows from Proposition 3.18 (2), Proposition 3.21, Proposition 3.22, and Proposition 3.23. Thus, we have proven the desired claim. It is also easy to see that if one fixes the degree of the graded pieces to be given by ν that one can compute the ℓ -dimension to be equal to d_ν . \square

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