GEOMETRIC EISENSTEIN SERIES, INTERTWINING OPERATORS, AND SHIN'S AVERAGING FORMULA

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ABSTRACT. In the geometric Langlands program over function fields, Braverman-Gaitsgory [BG02] and Laumon [Lau90] constructed geometric Eisenstein functors which geometrize the classical construction of Eisenstein series over function fields by replacing the Eisenstein series with an automorphic sheaf on the moduli stack Bun_G of G-bundles of the smooth projective curve defined by the function field. These sheaves are Hecke Eigensheaves, encode the completed Eisenstein series under the function-sheaf dictionary, and satisfy a functional equation with respect to the action of the Weyl group. Fargues and Scholze [FS21] very recently constructed a general candidate for the local Langlands correspondence, via a geometric Langlands correspondence occurring over the Fargues-Fontaine curve. In this note, we carry some of the theory of geometric Eisenstein series over to the Fargues-Fontaine setting. Namely, given a quasi-split connected reductive group $G/\mathbb{Q}_p$ with simply connected derived group and maximal torus T, we construct a geometric Eisenstein functor $\mathrm{nEis}(-)$, which takes sheaves on the moduli stack Bun_T to sheaves on the moduli stack Bun_G. We show that, given a sufficiently nice L-parameter $\phi_T : W_{\mathbb{Q}_p} \to {}^L T$ with induced parameter ϕ given by composing with the natural embedding ${}^L T \to {}^L G$, there is a Hecke eigensheaf on Bun_G with eigenvalue ϕ, given by applying $\mathrm{nEis}(-)$ to the Hecke eigensheaf $\mathcal{S}_{\phi_T}$ on Bun_T attached to ϕ_T constructed by Fargues [Far16; Far20] and Zou [Zou22]. We show that the object $\mathrm{nEis}(\mathcal{S}_{\phi_T})$ interacts well with Verdier duality, and, assuming compatibility of the Fargues-Scholze correspondence with a suitably nice form of the local Langlands correspondence, provide an explicit formula for the stalks of the eigensheaf in terms of parabolic inductions of the character χ attached to ϕ_T. This explicit description of the eigensheaf has several surprising consequences. First, it recovers special cases of an averaging formula of Shin [Shi12; Ber21] for the cohomology of local Shimura varieties with rational coefficients, and generalizes it to the non-minuscule case. Moreover, it refines the averaging formula in the cases where the parameter ϕ is sufficiently nice, giving an explicit formula for the degrees of cohomology that certain parabolic inductions sit in, and this refined formula holds even with torsion coefficients. The sufficiently nice condition on ϕ_T is related to zeros and poles of the intertwining operators attached to the normalized parabolic induction $i_B^G(\chi)$, and we analogously see that the refined averaging formula actually gives a geometric construction of such intertwiners.

CONTENTS

1. INTRODUCTION

1.1. **Geometric Eisenstein Series over Function Fields.** In the Langlands program, Eisenstein series are a way of describing the non-cuspidal automorphic spectrum of a group in terms of the cuspidal automorphic spectrum of its proper Levi subgroups. Over function fields these objects have several geometric incarnations, as first studied extensively by Laumon [Lau90] in the case of GL_n and later refined by Braverman-Gaitsgory [BG02] for general reductive groups. In particular, if one is interested in the function field of a curve Y over a finite field k then one replaces the functions defining Eisenstein series by certain automorphic sheaves. Namely, let G/k be a split connected reductive group; then one wishes to construct "Eisenstein sheaves" on Bun_G the moduli stack of G-bundles on Y. To do this, for $P \subset G$ a proper parabolic subgroup with Levi factor M one considers the following diagram of moduli stacks of bundles

$$
\begin{array}{ccc}
& \mathrm{Bun}_P & \\
{\scriptstyle \mathsf{q}_P} \downarrow & \searrow {\scriptstyle \mathsf{p}_P} & \\
\mathrm{Bun}_M & & \mathrm{Bun}_G
\end{array}
$$

where Bun_P is the moduli stack of P-bundles $\mathcal{G}_P$ on Y and the maps p_P (resp. q_P) send $\mathcal{G}_P$ to the G-bundle (resp. M-bundle) $\mathcal{G}_P \times^P G$ (resp. $\mathcal{G}_P \times^P M$). Using this, one can define an Eisenstein functor which (up to shifts and twists) is given by

$$
\mathrm{Eis}_P(-) := \mathsf{p}_{P!}\mathsf{q}_P^*(-),
$$

and takes ℓ-adic sheaves on Bun_M to ℓ-adic sheaves on Bun_G. Under the function-sheaf dictionary, the values of this functor give rise to the classical Eisenstein series.

In this geometric context, one can ask for even more. Namely, in the geometric Langlands correspondence one is interested in constructing Hecke eigensheaves on Bun_G, and it is natural to ask whether one could upgrade Eis_P to a functor that is well-behaved with respect to the eigensheaf property. In particular, if $\hat{M}$ (resp. $\hat{G}$) denotes the Langlands dual group of M (resp. G) one can consider a $\hat{M}$-local system $E_{\hat{M}}$ and a Hecke eigensheaf $S_{E_{\hat{M}}}$ with eigenvalue $E_{\hat{M}}$. One then considers the induced $\hat{G}$ local system $E_{\hat{G}}$ given by the natural embedding $\hat{M} \hookrightarrow \hat{G}$, and one would like to construct a functor that produces a eigensheaf with eigenvalue $E_{\hat{G}}$ from the Hecke eigensheaf $S_{E_{\hat{M}}}$. One might hope that $\mathrm{Eis}_P(S_{E_{\hat{M}}})$ works; however, this is too naive. Namely, one expects such sheaves to be well-behaved under Verdier duality, and one can easily check that $\mathrm{Eis}_P(\ )$ will not commute with Verdier duality, since the map p_P is not proper. To remedy this, one considers relative Drinfeld compactifications of the map p_P, denoted $\widetilde{\mathrm{Bun}}_P$ and $\overline{\mathrm{Bun}}_P$, respectively. These compactifications are equipped with open immersions $\widetilde{j} : \mathrm{Bun}_P \hookrightarrow \widetilde{\mathrm{Bun}}_P$ and $j : \mathrm{Bun}_P \hookrightarrow \overline{\mathrm{Bun}}_P$, which realize Bun_P as an open and dense subspace, and are defined by considering parabolic structures with torsion at finitely many Cartier divisors. Moreover, they both have maps

$$\overline{\mathsf{p}}_P : \overline{\mathrm{Bun}}_P \to \mathrm{Bun}_G$$

and

$$\widetilde{\mathsf{p}}_P : \widetilde{\mathrm{Bun}}_P \to \mathrm{Bun}_G$$

which are proper after restricting to a connected component and extend p_P, as well as maps $\widetilde{\mathsf{q}}_P : \widetilde{\mathrm{Bun}}_P \to \mathrm{Bun}_M$ and $\overline{\mathsf{q}}_P : \overline{\mathrm{Bun}}_P \to \mathrm{Bun}_{M^{ab}}$ extending the natural maps $\mathsf{q}_P : \mathrm{Bun}_P \to \mathrm{Bun}_M$ and $\mathsf{q}_P^{\vee} : \mathrm{Bun}_P \xrightarrow{\mathsf{q}_P} \mathrm{Bun}_M \to \mathrm{Bun}_{M^{ab}}$, respectively.

To obtain a sheaf that interacts well with Verdier duality, one needs to take account for the singularities of the compactification. Namely, if one considers the intersection cohomology sheaf $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$ of $\widetilde{\mathrm{Bun}}_P$ then the desired functor is given by

$$\widetilde{\mathrm{Eis}}_P(\ ) := \widetilde{\mathsf{p}}_P{}_{!} (\widetilde{\mathsf{q}}_P^*(\ ) \otimes \mathrm{IC}_{\widetilde{\mathrm{Bun}}_P})$$

One of the main results of Braverman-Gaitsgory [BG02] is that this satisfies the desired Hecke eigensheaf property when applied to $S_{E_{\hat{M}}}$. One may also wonder what this corresponds to at the level of functions. We recall that the classical Eisenstein series is known to satisfy a functional equation after multiplying by an appropriate ratio of L-values. The compactified Eisenstein series corresponds to this completed Eisenstein series under the function-sheaf dictionary. In fact, in certain cases one can see the usual functional equation at the sheaf-theoretic level. If $P = B$ is the Borel and we consider the appropriately normalized Hecke eigensheaf $S_{E_{\hat{T}}}$ associated to $E_{\hat{T}}$ via geometric class field theory then $\widetilde{\mathrm{Eis}}(S_{\phi_T})$ satisfies a functional equation under a regularity hypothesis on the local system $E_{\hat{T}}$. Now let $w \in W_G$ be an element of the Weyl group of G with $\widetilde{w} \in N(T)$ a choice of representative. Then $\widetilde{w}$ acts on Bun_T, and, if we write $S_{E_{\hat{T}}}^w$ for the pullback of $S_{E_{\hat{T}}}$ along this automorphism, we have the following result.

Theorem 1.1. *[BG02, Theorem 2.2.4] For $E_{\hat{T}}$ a regular $\hat{T}$-local system on Y and a choice of representative $\widetilde{w} \in N(\hat{T})$ of $w \in W_G$, we have an isomorphism*

$$\widetilde{\mathrm{Eis}}_B(S_{E_{\hat{T}}}) \simeq \widetilde{\mathrm{Eis}}_B(S_{E_{\hat{T}}}^w)$$

of ℓ-adic sheaves on Bun_G.

As alluded to above, one can see that, after passing to functions, this gives precisely the well-known functional equation for the Eisenstein series multiplied by the appropriate ratio of L-values

([BG02, Section 2.2]). Moreover, by [BG08, Theorem 1.5], under the regularity assumption the sheaf $\widetilde{\mathrm{Eis}}_B(S_{E_{\hat{T}}})$ is perverse.

The main goal of this note is to explore what this theory of geometric Eisenstein series has to tell us in the context of the recent geometric Langlands correspondence constructed by Fargues and Scholze.

1.2. **Hecke Eigensheaves over the Fargues-Fontaine Curve.** Consider two distinct primes $\ell \neq p$. Let $G/\mathbb{Q}_p$ be a quasi-split connected reductive group with simply connected derived group over the p-adic numbers, and set $\breve{\mathbb{Q}}_p$ to be the completion of the maximal unramified extension of $\mathbb{Q}_p$ with Frobenius σ. Let $W_{\mathbb{Q}_p} \subset \Gamma := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ denote the Weil group. In [FS21], Fargues and Scholze developed the geometric framework required to make sense of objects like Bun_G, the moduli stack of G-bundles on the Fargues-Fontaine curve X, and show that the local Langlands correspondence for G can be viewed as a geometric Langlands correspondence over X. Namely, they prove a version of Geometric Satake in this setup, allowing them to construct Hecke operators and in turn excursion operators. Their Hecke operators take sheaves on Bun_G to sheaves on Bun_G with a continuous $W_{\mathbb{Q}_p}$-action, via some version of Drinfeld's Lemma. As a consequence, they were able to construct semi-simple Langlands parameters $\phi_\pi^{\mathsf{FS}} : W_{\mathbb{Q}_p} \to {}^L G(\overline{\mathbb{Q}}_\ell)$ attached to any smooth irreducible representation π of $G(\mathbb{Q}_p)$.

Predating the construction of the Fargues-Scholze local Langlands correspondence was Fargues' conjecture as formulated in [Far16]. This asserted the existence of Hecke Eigensheaves attached to supercuspidal L-parameters $\phi : W_{\mathbb{Q}_p} \to {}^L G(\overline{\mathbb{Q}}_\ell)$, where ${}^L G = \hat{G} \rtimes W_{\mathbb{Q}_p}$ is the L-group of G. More precisely, given such a ϕ, Fargues conjectured the existence of a $\overline{\mathbb{Q}}_\ell$-sheaf S_ϕ on Bun_G such that, if one acts via a Hecke operator T_V corresponding to a representation $V \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L G)$ then there is an isomorphism
$$T_V(S_\phi) \simeq S_\phi \boxtimes r_V \circ \phi$$
of sheaves on Bun_G with continuous $W_{\mathbb{Q}_p}$-action satisfying natural compatiblities. The moduli stack Bun_G is stratified by elements of the Kottwitz set $B(G) := G(\breve{\mathbb{Q}}_p)/(b \sim gb\sigma(g)^{-1})$, giving rise to Harder-Narasimhan (abbv. HN)-strata Bun_G^b for all $b \in B(G)$. It was conjectured that the sheaf S_ϕ must be supported on the basic locus $\bigsqcup_{b \in B(G)_{basic}} \mathrm{Bun}_G^b$ or, in bundle-theoretic terms, the locus defined by semistable bundles. Each of the basic strata Bun_G^b are isomorphic to the classifying stack $[*/J_b(\mathbb{Q}_p)]$, where J_b is the σ-centralizer attached to $b \in B(G)$. The σ-centralizers of the basic elements parameterize extended pure inner forms of G in the sense of Kottwitz [Kot97], and the restrictions of the sheaf to these classifying stacks can be interpreted as smooth representations of $J_b(\mathbb{Q}_p)$. Using this, Fargues also gave a conjectural description for what the restrictions of the sheaf should be given by. In particular, for $b \in B(G)_{basic}$ the restriction to Bun_G^b should be a direct sum (up to multiplicities)[1] $\bigoplus_{\pi \in \Pi_\phi(J_b)} \pi$, where $\Pi_\phi(J_b)$ is the L-packet over ϕ as conjectured by Kaletha's refined local Langlands correspondence for G [Kal16]. Assuming the refined local Langlands, the verification of the Hecke eigensheaf property ultimately reduces to a strong form of the Kottwitz conjecture for the cohomology of a space of shtukas parameterizing modifications
$$\mathcal{F}_b \to \mathcal{F}_{b'}$$
on X bounded by a geometric dominant cocharacter μ, where $b, b' \in B(G)_{basic}$ are appropriately chosen basic elements with respect to μ, and $\mathcal{F}_b$ and $\mathcal{F}_{b'}$ denote the bundles on X corresponding to $b, b' \in B(G)$.

Since the work of Fargues-Scholze, the construction of this eigensheaf has been carried out in several cases. For tori, it follows from the work of Fargues [Far16], [Far20], and Zou [Zou22]. For $G = \mathrm{GL}_n$, this is a result of Anschütz and Le-Bras [AL21b]. For general reductive groups, a somewhat

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[1]There are no higher multiplicities if the centralizer of ϕ is abelian.

general strategy for constructing this eigensheaf in the particular case of the group GSp_4 is layed out in [Ham21b], by showing compatibility of the Fargues-Scholze correspondence with the refined local Langlands correspondence of Kaletha [Kal16], and then using this to deduce the non-minuscule cases of the Kottwitz conjecture required for the verification of the Hecke eigensheaf property via the spectral action [FS21, Section X.2]. This strategy is carried out for odd unitary groups in the paper [BHN22]. We recall that the fibers of the local Langlands correspondence over supercuspidal parameters should solely consist of supercuspidal representations. Therefore, the above eigensheaves can be thought of as analogous to supercuspidal representations in the classification of smooth irreducible representations. To have a more definitive connection between the theory of smooth representations and Fargues' eigensheaves, it becomes desirable to construct eigensheaves that serve as the analogues of parabolic inductions of supercuspidals, which will analogously be "parabolically induced" from the eigensheaves attached to supercuspidal parameters. This will be precisely what carrying over the theory of the previous section to the Fargues-Fontaine setting gives us.

1.3. Geometric Eisenstein Series over the Fargues-Fontaine Curve.

Let $A \subset T \subset B \subset G$ be a choice of maximal split torus, maximal torus, and Borel, respectively. In this paper, we will be concerned with studying the geometric Eisenstein functor over the Fargues-Fontaine curve in the principal case (i.e where the parabolic $P \subset G$ is the Borel). Restricting to the principal case has several advantages. For one, one can unconditionally speak about the Hecke eigensheaf attached to a parameter $\phi_T : W_{\mathbb{Q}_p} \to {}^L T$ valued in the maximal torus. Additionally, in this case there exists only one Drinfeld compactification $\overline{\mathrm{Bun}}_B$ of the moduli space of B-structures Bun_B, which has a fairly manageable geometry. As mentioned in §1.1, there are in general two compactifications $\overline{\mathrm{Bun}}_P$ and $\widetilde{\mathrm{Bun}}_P$. The compactification $\overline{\mathrm{Bun}}_P$ has a relatively simple geometry and can be understood more or less the same as $\overline{\mathrm{Bun}}_B$. The problem is that $\overline{\mathrm{Bun}}_P$ admits only a map to $\mathrm{Bun}_{M^{ab}}$ and not to Bun_M. This means we can prove analogous results for the Hecke eigensheaves attached to characters induced from $\phi : W_{\mathbb{Q}_p} \to {}^L M^{ab}$ which correspond to generalized principal series representations induced from M. We have chosen not to do this in this note for simplicity. However, if one wants to consider inductions of Hecke eigensheaves attached to a general supercuspidal parameter $\phi_M : W_{\mathbb{Q}_p} \to {}^L M$ then one is forced to understand the much more complicated geometry of the space $\widetilde{\mathrm{Bun}}_P$. Certainly, some analogues of the results proven in this paper should be possible, but there are many technical hurdles that need to be overcome.

1.3.1. *Geometric Eisenstein Series.* Throughout, we will let $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\Lambda)$ be a continuous parameter valued in the L-group of T, where $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$ has the discrete topology. Our aim is to construct an eigensheaf (Definition 3.1) with eigenvalue ϕ, the composition of ϕ_T with the natural embedding ${}^L T(\Lambda) \to {}^L G(\Lambda)$. This will be an object in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ the category of lisse-étale solid Λ-sheaves, as defined in [FS21, Chapter VII]. We do not work directly with this category of solid sheaves in our argument, as the usual six functors are not as well behaved in this case. Instead, we will first restrict to the case where $\Lambda = \overline{\mathbb{F}}_\ell$ and one has an isomorphism $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell) \simeq \mathrm{D}_{\text{ét}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ with the usual category of étale $\overline{\mathbb{F}}_\ell$ sheaves as defined in [Sch18]. We then construct the lisse-étale sheaves with $\overline{\mathbb{Z}}_\ell$ and $\overline{\mathbb{Q}}_\ell$ coefficients from this sheaf. For the first part of this section, we will always assume that $\Lambda = \overline{\mathbb{F}}_\ell$ unless otherwise stated, and to simplify the notation, we adopt the convention that, when working with $\Lambda = \overline{\mathbb{F}}_\ell$, we will denote the derived category of étale sheaves on a v-stack or diamond Z by simply writing $\mathrm{D}(Z)$. We will assume that the prime ℓ is very good with respect to the group G ([FS21, Page 33]) throughout, to avoid complications in this ℓ-modular setting.

We let S_{ϕ_T} be the eigensheaf on the moduli stack Bun_T parameterizing T-bundles on X attached to ϕ_T by Fargues [Far20; Far17] and Zou [Zou22]. Our aim is to construct an eigensheaf with respect to the parameter ϕ by applying a geometric Eisenstein functor to S_{ϕ_T}. To do this, one needs to show that the relevant geometric objects used in defining geometric Eisenstein series are well-behaved in

this framework. Namely, one can show that the moduli stack of B-bundles on X, denoted Bun_B, gives rise to an Artin v-stack and the maps in the natural diagram

$$
(1) \qquad \begin{array}{ccc}
 & \mathrm{Bun}_B & \\
 & \llap{\mathsf{q}}\downarrow \quad \searrow^{\mathsf{p}} & \\
\mathrm{Bun}_T & & \mathrm{Bun}_G
\end{array}
$$

have good geometric properties; q is a cohomologically smooth (non-representable) map of Artin v-stacks, and p is compactifiable and representable in locally spatial diamonds (See §5.1). It follows that one has a well-defined functor given by $\mathsf{p}_!\mathsf{q}^* : D(\mathrm{Bun}_T) \to D(\mathrm{Bun}_G)$ using the functors constructed in [Sch18].

In order to make the functor $\mathsf{p}_!\mathsf{q}^*$ behave well with respect to Verdier duality, one needs to take into account the appropriate shifts and twists coming from the dualizing object on Bun_B. Namely, using the cohomological smoothness of q and Bun_T, it is easy to see that the moduli stack Bun_B is cohomologically smooth of some ℓ-dimension given by a locally constant function $\dim(\mathrm{Bun}_B) : |\mathrm{Bun}_B| \to \mathbb{Z}$, where $|\mathrm{Bun}_B|$ is the underlying topological space of Bun_B. It follows that, v-locally on Bun_B, the dualizing object is given by $\Lambda[2\dim(\mathrm{Bun}_B)]$. This leads us to our first attempt to define the Eisenstein functor as $\mathrm{Eis}(-) := \mathsf{p}_!(\mathsf{q}^*(-)[\dim(\mathrm{Bun}_B)])$. While this definition is closer to what we want, it has one key flaw; even though the dualizing object on Bun_B is v-locally isomorphic to $\Lambda[2\dim(\mathrm{Bun}_B)]$ it is not equal to this sheaf on the nose. In particular, usually one would need to include some Tate twists. However, all these spaces are defined over the base $* = \mathrm{Spd}\,\overline{\mathbb{F}}_p$, so one cannot naively make sense of this. Typically, this would be encoded via some kind of Frobenius descent datum, but here the answer is even more interesting. Understanding this requires one to explicitly compute the dualizing object on Bun_B. This can be accomplished by noting that each of the connected components of Bun_B are related to Banach-Colmez spaces, the diamonds parameterizing global sections $H^0(X, \mathcal{E})$ and $H^1(X, \mathcal{E})$ for $\mathcal{E}$ a bundle on the Fargues-Fontaine curve X, as studied in [FS21, Chapter 2] and [Le 18]. These objects have a relatively explicit description in terms of pro-étale quotients of perfectoid open unit discs and also have absolute versions defined over $\mathrm{Spd}\,\mathbb{F}_{p^s}$, for some $s \geq 1$. Using this explicit description, one can compute that the dualizing object on these absolute spaces is given by the constant sheaf with the appropriate shift and Tate twist by the dimension. However, these Tate twists do not disappear over the algebraic closure $\overline{\mathbb{F}}_p$; namely, the action of geometric Frobenius is manifestly related to the action of $p^{\mathbb{Z}} \in H^0(X, \mathcal{O}_X)^* = \mathbb{Q}_p^*$ by scaling of global sections, essentially by definition of the Fargues-Fontaine curve. This allows one to see that the dualizing object on the Banach-Colmez space over $\overline{\mathbb{F}}_p$ is isomorphic to a shift of the constant sheaf together with a descent datum with respect to the scaling action by $\mathbb{Q}_p^*$ given by the norm character $|\cdot| : \mathbb{Q}_p^* \to \Lambda^*$, which serves the role of a Frobenius descent datum. Unravelling this all, this will ultimately tell us that the dualizing object on Bun_B is related to the modulus character δ_B. One way of nicely encoding these modulus character twists is to consider the sheaf $\Delta_B^{1/2}$ on Bun_T, which will be the Hecke eigensheaf attached to the L-parameter

$$
\hat{\rho} \cdot |\cdot| : W_{\mathbb{Q}_p} \to {}^L T(\Lambda)
$$

where $\hat{\rho}$ denotes the half sum of all positive roots of G and we abusively write $|\cdot| : W_{\mathbb{Q}_p} \to \Lambda^*$ for the norm character of $W_{\mathbb{Q}_p}$. As a sheaf on Bun_T, the stalks on the connected components $\mathrm{Bun}_T^\nu \simeq [*/\underline{T(\mathbb{Q}_p)}]$ will just be given by the character $\delta_B^{1/2} : T(\mathbb{Q}_p) \to \Lambda^*$. Similarly, we write Δ_B for the sheaf given by the L-parameter $2\hat{\rho} \cdot |\cdot|$. We can now state our first Theorem.

Theorem 1.2. *(Theorem 6.1) The dualizing object on* Bun_B *is isomorphic to*

$$
\mathsf{q}^*(\Delta_B)[2\dim(\mathrm{Bun}_B)]
$$

In particular, the sheaf

$$\mathrm{IC}_{\mathrm{Bun}_B} := \mathsf{q}^*(\Delta_B^{1/2})[\dim(\mathrm{Bun}_B)]$$

is Verdier self-dual on Bun_B.

With this in hand, we can refine the previous definition of the Eisenstein functor. We define the normalized Eisenstein functor to be

$$\mathrm{nEis}(\ ) := \mathsf{p}_!(\mathsf{q}^*(\ ) \otimes \mathrm{IC}_{\mathrm{Bun}_B})$$

This is already very suggestive. Indeed, the (unnormalized) Eisenstein functor will have stalks related to the unnormalized parabolic induction of the characters χ, and the normalized Eisenstein series will have stalks related to the normalized parabolic induction. Moreover, just as smooth duality interacts nicely with normalized parabolic induction so too does Verdier duality with the normalized Eisenstein functor. In order to study how the normalized Eisenstein functor interacts with Verdier duality, it becomes very natural to want a nice compactification of the morphism p, as Eis involves the functor $\mathsf{p}_!$. As seen in §1.1, this can be accomplished by considering an analogue of the Drinfeld compactification $\overline{\mathrm{Bun}}_B$. We show that such a compactification exists and has the right properties.

Theorem 1.3. *(Proposition 5.8, Proposition 5.22) There exists an Artin v-stack* $\overline{\mathrm{Bun}}_B$ *together with an inclusion*

$$j : \mathrm{Bun}_B \hookrightarrow \overline{\mathrm{Bun}}_B$$

which realizes Bun_B *as an open and dense substack. Moreover,* $\overline{\mathrm{Bun}}_B$ *has natural maps* $\overline{\mathsf{q}} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_T$ *(resp.* $\overline{\mathsf{p}} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_G$*) extending the map* q *(resp.* p*) along* j*. The map* $\overline{\mathsf{p}}$ *is representable in nice diamonds and proper after restricting to connected components.*

Now, we would like to claim that the sheaf $\mathrm{nEis}(S_{\phi_T})$ is a Hecke eigensheaf with respect to the parameter ϕ given by composing ϕ_T with the natural embedding $^LT \to {}^LG$; however, in analogy with §1.1, the right object to consider in this case is not $\mathrm{nEis}(S_{\phi_T})$, but rather a compactified version $\overline{\mathrm{Eis}}(S_{\phi_T})$. Unfortunately, there is currently no well-behaved formalism for intersection cohomology in the context of diamonds and v-stacks. This prevents us from even defining the kernel sheaf $\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ typically used in the definition of $\overline{\mathrm{Eis}}(\ )$ in any naive way. There is however a way out if we impose some conditions on our parameter ϕ_T. We write $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$ for the set of Γ-orbits of geometric cocharacters. Given an element $\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$, we can attach to it a representation of ^LT by inducing the representation of $\hat{T}$ defined by a representative of the orbit ν in $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$. We consider the composition $\nu \circ \phi_T$, and we will say that ϕ_T is generic if the Galois cohomology complexes

$$R\Gamma(W_{\mathbb{Q}_p}, \alpha \circ \phi_T)$$

are trivial for all $\alpha \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$ defined by the Γ-orbits of coroots in $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$. This condition may appear mysterious; but there are several ways to see why it is the morally correct condition. Perhaps the most compelling comes from local representation theory. As mentioned in §1.1, the compactified Eisenstein series in the function field setting corresponds to the completed Eisenstein series under the function sheaf dictionary, while the non-compactified Eisenstein series corresponds to just the usual Eisenstein series. In particular, the sheaf $\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ encodes the zeros and poles of the meromorphic continuation of the Eisenstein series. If χ denotes the character of $T(\mathbb{Q}_p)$ attached to ϕ_T via local class field theory then we recall that, if $w \in W_G$ is an element of the relative Weyl group, the local analogue of the meromorphic continuation of Eisenstein series is the theory of intertwining operators. In particular, suppose that $\Lambda = \overline{\mathbb{Q}}_\ell$, then we have maps

$$i_{\chi,w} : i_B^G(\chi) \to i_B^G(\chi^w)$$

of smooth $G(\mathbb{Q}_p)$-representations, which can be viewed as meromorphic functions on the set of complex unramified characters. Now, using local Tate-duality, it is easy to see that the vanishing of the above complexes will imply that χ pre-composed with coroots is not trivial or isomorphic to a power of the norm character. These are precisely the type of conditions one expects to guarantee that the intertwining operators are holomorphic and give rise to an isomorphism. In fact, we show that, for χ attached to a generic parameter ϕ_T, we always have an isomorphism $i_B^G(\chi) \simeq i_B^G(\chi^w)$ for any $w \in W_G$ (Proposition A.2).

This suggests that, at least heuristically, we should always have an isomorphism $\mathrm{nEis}(S_{\phi_T}) \simeq \overline{\mathrm{Eis}}(S_{\phi_T})$, for ϕ_T satisfying some version of genericity and any reasonable definition of $\overline{\mathrm{Eis}}(S_{\phi_T})$. Indeed, this makes sense when we look at the geometry of $\overline{\mathrm{Bun}}_B$; in particular, we will show that the closed complement of Bun_B in $\overline{\mathrm{Bun}}_B$ admits a locally closed stratification given by $\mathrm{Div}^{(\overline{\nu})} \times \mathrm{Bun}_B$, where $\mathrm{Div}^{(\overline{\nu})}$ is a certain partially symmetrized version of the mirror curve Div^1 parameterizing effective Cartier divisors in X attached to $\overline{\nu}$ inside the coinvariant lattice $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$. We recall that there is a natural map $(\;) : \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma \to \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$, from Γ-orbits to coinvariants. This map defines an injection on the Γ-orbits of the simple positive coroots. In particular, for each vertex $i \in J$ of the relative Dynkin diagram of G, we get an element α_i in the coinvariant lattice, which corresponds to a Γ-orbit of positive simple roots. The natural strata of $\overline{\mathrm{Bun}}_B$ are specified by elements $\overline{\nu} \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$ lying in the positive span of these α_i, and each strata corresponds to the locus of B-bundles with torsion specified by $\overline{\nu}$. Now, the restriction of $\overline{\mathfrak{q}}_*(S_{\phi_T})$ to this strata is given by pulling back a Hecke operator on Bun_T applied to S_{ϕ_T} along $\mathfrak{q}$. One can deduce that the factor appearing on the divisor curve $\mathrm{Div}^{(\overline{\nu})}$ will be related to $\alpha_i \circ \phi_T$, via the Hecke eigensheaf property $T_{\alpha_i}(S_{\phi_T}) \simeq \alpha_i \circ \phi_T \otimes S_{\phi_T}$ for S_{ϕ_T}, where we have identified α_i with its associated Γ-orbit via $(\;)$. This implies that the complex $R\Gamma(W_{\mathbb{Q}_p}, \alpha_i \circ \phi_T)$ appears in $\overline{\mathsf{p}}_!$ applied to this restriction as a tensor factor, and will in turn vanish for ϕ_T generic, suggesting an isomorphism of the form $\mathrm{nEis}(S_{\phi_T}) \simeq \overline{\mathrm{Eis}}(S_{\phi_T})$ in this case.

We can turn these heuristics into actual math. In particular, since we expect an isomorphism of the form $\mathrm{nEis}(S_{\phi_T}) \simeq \overline{\mathrm{Eis}}(S_{\phi_T})$ under some verison of genericity, we should expect for such parameters that $\mathrm{nEis}(S_{\phi_T})$ behaves well under Verdier duality, is a perverse Hecke eigensheaf with eigenvalue ϕ, and satisfies the analogue of the functional equation seen in Theorem 1.1 in this case. The precise conditions on ϕ_T that we will need to prove our results will depend on the result and the particular group G. For this reason, we break up our condition into several parts.

Condition/Definition 1.4. *(Condition/Definition 3.7) Given a parameter $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\Lambda)$, we impose the following conditions on ϕ_T in what follows.*

(1) *For all $\alpha \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$ defined by the Γ-orbits of simple coroots in $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$, the Galois cohomology complex $R\Gamma(W_{\mathbb{Q}_p}, \alpha \circ \phi_T)$ is trivial.*

(2) *For all $\alpha \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$ defined by the Γ-orbits of coroots in $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$, the Galois cohomology complex $R\Gamma(W_{\mathbb{Q}_p}, \alpha \circ \phi_T)$ is trivial.*

(3) *If χ is the character attached to ϕ_T under local class field theory. We have, for all $w \neq 1$ in the relative Weyl group W_G of G, that*

$$\chi \cdot \delta_B \neq (\chi \cdot \delta_B^{1/2})^w$$

If ϕ_T satisfies (1) we say that it is weakly generic, and if it satisfies (2) then we say it is generic. If it satisfies (2)-(3) we say it is weakly normalized regular.

Remark 1.5. The relationship between these various conditions appears to be complicated in general, and is related to the behavior of the principal series representations $i_B^G(\chi)$ of G. Roughly speaking, Condition (2) guarantees the irreducibility of non-unitary principal series representations and that the intertwining operators for $i_B^G(\chi)$ are isomorphisms, while Condition (3) guarantees the irreducibility of certain unitary principal series representations. As we will see in Appendix A,

genericity is enough to guarantee that one has an isomorphism $i_B^G(\chi) \simeq i_B^G(\chi^w)$ for all $w \in W_G$. However, one could still have an isomorphism of this form if $i_B^G(\chi)$ is the reducible induction of a unitary character χ. This can happen if the character χ is not regular (i.e it is fixed by some $w \in W_G$). Condition (3) is like such a regularity condition. To illustrate this, note that, for $G = \mathrm{GL}_n$, if we write $\phi_T = \bigoplus_{i=1}^n \phi_i$ as a sum of characters then genericity is equivalent to supposing that

$$R\Gamma(W_{\mathbb{Q}_p}, \phi_{\overline{i}} \quad \phi_j)$$

is trivial for all $i \neq j$. We note that, by local Tate-duality and using that the Euler-Poincaré characteristic of this complex is 0, genericity is equivalent to assuming that ϕ_i is not isomorphic to ϕ_j or $\phi_j(1)$. If we write $\chi = \chi_1 \ldots \chi_n$ as a product of characters of $\mathbb{Q}_p$ then this implies that $\chi_i{}^1\chi_j \neq |\cdot|^{\pm 1}$ for all $i > j$, which is precisely the condition guaranteeing irreducibility. On the other hand, if $G = \mathrm{SL}_2$, and we write χ for the character of $\mathbb{Q}_p$ attached to ϕ_T via local class field theory, we need that $\chi \neq |\cdot|^{\pm 1}$ and $\chi^2 \neq \mathbf{1}$ to guarantee irreducibility of $i_B^G(\chi)$. The condition $\chi \neq |\cdot|^{\pm 1}$ is guaranteed by Condition (2), and Condition (3) is equivalent to $\chi^2 \neq \mathbf{1}$.

Remark 1.6. The choice of calling a parameter satisfying Condition (2) generic is motivated by the analogous notion of decomposed generic considered by Caraiani and Scholze [CS17, Definition 1.9]. In particular, we note that if ϕ_T is unramified and $G = \mathrm{GL}_n$ then, by the previous remark, we see that Condition (2) is precisely equivalent to ϕ_T being decomposed generic.

Weak genericity will be required to show our Eisenstein functor commutes with Verdier duality, while weak normalized regularity will be needed to compute the stalks of our Eisenstein series and show that it satisfies the functional equation. Unfortunately, in general to get the Hecke eigensheaf property for nEis(S_{ϕ_T}) we still need more. Write $(-) : \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \to \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$ for the natural map from geometric cocharacters to their Galois orbits. Given a geometric dominant cocharacter $\mu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ with Galois orbit $\overline{\mu} \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+/\Gamma$, we have an associated representation $V_{\overline{\mu}} \in \mathrm{Rep}(^LG)$. The weights of $V_{\overline{\mu}}|_{^LT}$ can be interpreted in terms of the representations corresponding to the Galois orbits of weights in the usual highest weight representation V_μ of $\hat{G}$. The following condition will guarantee the Hecke eigensheaf property for nEis(S_{ϕ_T}), and the Hecke operator defined by $V_{\overline{\mu}}$.

Definition 1.7. (Definition 3.14) For a toral parameter $\phi_T : W_{\mathbb{Q}_p} \to {}^LT(\Lambda)$ and a geometric dominant cocharacter μ, we say ϕ_T is strongly μ-regular if the Galois cohomology complexes

$$R\Gamma(W_{\mathbb{Q}_p}, (\nu \quad \nu^\emptyset) \quad \phi_T)$$

are trivial for $\nu, \nu^\emptyset$ defining distinct Γ-orbits of weights the highest weight representation of $\hat{G}$ of highest weight μ.

Remark 1.8. We note that genericity usually implies strong μ-regularity for some suitably chosen cocharacters $\mu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$. In particular, note that if $\mu = (1,0,\ldots,0,0)$ and $G = \mathrm{GL}_n$ then, since μ is minuscule, the possible weights of V_μ in $\hat{T}$ are all given by Weyl group orbits of μ, and the possible differences between the weights will consequently be precisely the coroots of GL_n.

Let's now see how these conditions manifest in our results on Eisenstein series. We begin with studying how Verdier duality interacts with Eisenstein series. To do this, we need to make the following assumption.

Assumption 1.9. *(Assumption 8.1) If $j : \mathrm{Bun}_B \hookrightarrow \overline{\mathrm{Bun}}_B$ is the open inclusion into the Drinfeld compactification the sheaf $j_!(\mathrm{IC}_{\mathrm{Bun}_B})$ is ULA with respect to $\overline{\mathsf{q}}$.*

Remark 1.10. This is a precise analogue of [BG02, Theorem 5.1.5]. The proof in this case seems a bit subtle, since being ULA over a point in this case is not a trivial condition, but should appear

in upcoming work of Hansen-Scholze [HS] on geometric Eisenstein series and the Harris-Viehmann conjecture.

The relevance for studying how Verdier duality interacts with nEis( ) is as follows. It follows that $j_!(\mathrm{IC}_{\mathsf{Bun}_B})$ is reflexive with respect to Verdier duality on Bun_B. In particular, using the Verdier self-duality of $\mathrm{IC}_{\mathsf{Bun}_B}$, this allows us to see that the Verdier dual of $j_*(\mathrm{IC}_{\mathsf{Bun}_B})$ is isomorphic to $j_!(\mathrm{IC}_{\mathsf{Bun}_B})$. This reduces the problem of how Eisenstein series interact with Verdier duality to the problem of describing the cone of the map

$$j_!(\mathsf{q}^*(S_{\phi_T})) \to \overline{\mathsf{q}}^*(S_{\phi_T})$$

after applying $\overline{\mathsf{p}}_!$. As already mentioned above, if we look at the restriction of $\overline{\mathsf{q}}^*(S_{\phi_T})$ to the strata $\mathrm{Div}^{(\overline{\nu})} \times \mathsf{Bun}_B$ described above, this vanishes for weakly generic ϕ_T after applying $\overline{\mathsf{p}}_!$, as the Galois cohomology complexes $R\Gamma(W_{\mathbb{Q}_p}, \alpha * \phi_T)$ will appear for $\alpha \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$ corresponding to a Γ-orbit of simple coroots. In particular, if $\mathbb{D}_Z$ denotes Verdier duality on a v-stack or diamond Z, we can show the following.

Theorem 1.11. *(Theorem 8.3) For ϕ_T a weakly generic toral parameter, there is an isomorphism of objects in $\mathrm{D}(\mathsf{Bun}_G)$*

$$\mathbb{D}_{\mathsf{Bun}_G}(\mathrm{nEis}(S_{\phi_T})) \simeq \mathrm{nEis}(\mathbb{D}_{\mathsf{Bun}_T}(S_{\phi_T}))$$

where we note that $\mathbb{D}_{\mathsf{Bun}_T}(S_{\phi_T}) \simeq S_{\phi_{\overline{T}}}$, if $\phi_{\overline{T}}$ is the parameter dual to ϕ_T (i.e the Chevalley involution applied to ϕ_T).

We will assume the validity of the ULA Theorem and thereby the validity of this theorem for the rest of the section. We now turn our attention to the Hecke eigensheaf property. In particular, consider a finite index set I and a representation $V \in \mathrm{Rep}(^LG^I)$. Given $(\nu_i)_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma)^I$, we write $V((\nu_i)_{i \in I})$ for the multiplicity of the weight space of the corresponding representation of $^LT^I$ in $V|_{^LT^I}$, and write $T_{(\nu_i)_{i \in I}}$ for the associated Hecke operator. By applying excision to the aforementioned locally closed stratification of $\overline{\mathsf{Bun}}_B$ and combining it with the geometric Satake correspondence of Fargues-Scholze [FS21, Chapter VI], we can show the following result.

Theorem 1.12. *(Theorem 7.1) For $F \in \mathrm{D}(\mathsf{Bun}_T)$, I a finite index set, and $V \in \mathrm{Rep}(^LG^I)$ with associated Hecke operator T_V, the sheaf $T_V(\mathrm{nEis}(F))$ on Bun_G with continuous $W_{\mathbb{Q}_p}^I$-action has a $W_{\mathbb{Q}_p}^I$-equivariant filtration indexed by $(\nu_i)_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma)^I$. The filtration's graded pieces are isomorphic to*

$$\bigoplus_{(\nu_i)_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma)^I} \mathrm{nEis}(T_{(\nu_i)_{i \in I}}(F)) \otimes V((\nu_i)_{i \in I})$$

Moreover, the filtration is natural in I and V, as well as compatible with compositions and exterior tensor products.

The argument for proving this "filtered eigensheaf property" is very similar to that given by [BG02] in their proof of the Hecke eigensheaf property for the compactified geometric Eisenstein functor in the function field setting. However, there Braverman and Gaitsgory rely on the decomposition theorem applied to the perverse sheaf $\mathrm{IC}_{\overline{\mathsf{Bun}}_B}$, which would not make sense in this context. Nevertheless, we still have access to the excision spectral sequence one usually uses in proving the decomposition theorem. In particular, in the proof of the decomposition theorem one uses the excision spectral sequence and then invokes the theory of weights to show that it degenerates. Something similar happens here. Namely, if we apply this to the sheaf $F = S_{\phi_T}$ then we see that $\mathrm{nEis}(S_{\phi_T})$ is a filtered eigensheaf in the sense that, up to passing to the graded pieces of this filtration, it is an eigensheaf with eigenvalue ϕ, and later we can see, by looking at the Weil group action, this filtration must always split under some version of genericity on ϕ_T.

Even without knowing that the filtration on $\mathrm{nEis}(S_{\phi_T})$ splits, the filtered eigensheaf property can already be used to tell us a lot about the structure of $\mathrm{nEis}(S_{\phi_T})$. In particular, given $b \in B(G)$, the restriction $\mathrm{nEis}(S_{\phi_T})j_{\mathrm{Bun}_G^b}$ to the HN-strata Bun_G^b defines a complex of smooth $J_b(\mathbb{Q}_p)$-representations, and we are interested in describing this restriction. Here J_b is the σ-centralizer of b and it is an extended pure inner form of a Levi subgroup M_b of G. The fact that $\mathrm{nEis}(S_{\phi_T})$ is a filtered Hecke eigensheaf with eigenvalue ϕ implies that, if ρ is an irreducible constituent of this restriction then the Fargues-Scholze parameter ϕ_ρ^{FS} must be equal to ϕ under the twisted embedding ${}^L J_b \to {}^L G$. If we believe that the Fargues-Scholze local Langlands correspondence is the true local Langlands correspondence then this seems to suggest that ρ should be given by a normalized parabolic induction of the character χ attached to ϕ_T via local class field theory. In particular, we note, by deformation theory, that a toral parameter being generic implies that every lift of ϕ_T to $\overline{\mathbb{Z}}_\ell$ factors through ${}^L T$ and that the induced $\overline{\mathbb{Q}}_\ell$-parameter cannot come from the semi-simplification of a parameter with non-trivial monodromy (Lemma 3.18). This imposes a very rigid constraint on J_b. In particular, the Borel $B \cap M_b$ of M_b should transfer to a Borel B_b of J_b. The elements where this occurs are the elements in the image of the map $B(T) \to B(G)$, called the unramified elements $B(G)_{\mathsf{un}}$, as studied in the work of Xiao-Zhu [XZ17].

Corollary 1.13. (Corollary 7.7) *For ϕ_T a generic parameter and $b \in B(G)$, assuming compatibility of the Fargues-Scholze and the conjectural local Langlands correspondence (Assumption 7.5), the restriction $\mathrm{nEis}(S_{\phi_T})j_{\mathrm{Bun}_G^b}$ vanishes unless $b \in B(G)_{\mathsf{un}}$.*

We now assume compatibility (Assumption 7.5) in addition to validity of the ULA theorem (Assumption 8.1) for the rest of the section. The previous corollary tells us that if we are interested in understanding the complex of $J_b(\mathbb{Q}_p)$-representations defined by the stalks $\mathrm{nEis}(S_{\phi_T})j_{\mathrm{Bun}_G^b}$ then we can restrict to the case where $b \in B(G)_{\mathsf{un}}$, and here we expect the stalks to be given by the inductions $i_{B_b}^{J_b}(\chi) \quad \delta_{P_b}^{1/2}$, where δ_{P_b} is the modulus character of the standard parabolc P_b with Levi factor M_b transferred to J_b[2]. This is indeed the case. To understand this, we use that each element $b \in B(G)_{\mathsf{un}}$ has a unique element b_T with anti-dominant slope homomorphism mapping to it. We call such an element HN-dominant, where we recall that the Harder-Narasimhan slopes are the negatives of the isocrystal slopes (See [Far17, Proposition 6.6]). The set of elements in $B(T)$ mapping to b can be described as $w(b_T)$, where we identify $w \in W_b := W_G/W_{M_b}$ with a set of representatives of minimal length in W_G. The connected components of Bun_B and Bun_T are indexed by elements in $B(T) \simeq \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$, giving a direct sum decomposition:

$$\mathrm{nEis}(S_{\phi_T}) = \bigoplus_{\overline{\nu} \in B(T)} \mathrm{nEis}^{\overline{\nu}}(S_{\phi_T})$$

After restricting to Bun_G^b, the point is that only the summands indexed by $\overline{\nu} = w(b_T)$ for $w \in W_b$ survive. It is fairly easy to see this when $\overline{\nu} = b_T$. In particular, the connected component $\mathrm{Bun}_B^{b_T}$ will parametrize split B-structures since the HN-slopes are dominant, and the diagram (1) (essentially) becomes

$$
\begin{array}{ccc}
\left[\,/\underline{B_b(\mathbb{Q}_p)}\right] & & \\
\Big\downarrow{\scriptstyle q} & \searrow^{\ p} & \\
\left[\,/\underline{T(\mathbb{Q}_p)}\right] & & \left[\,/\underline{J_b(\mathbb{Q}_p)}\right]
\end{array}
$$

---

[2]These twists by the modulus character come from the fact that the excursion algebra on Bun_G acts on a smooth irreducible representation $\rho \in \mathrm{D}(J_b(\mathbb{Q}_p), \Lambda) \simeq \mathrm{D}(\mathrm{Bun}_G^b) \subset \mathrm{D}(\mathrm{Bun}_G)$ via the Fargues-Scholze parameter $\phi_\rho^{FS} : W_{\mathbb{Q}_p} \to {}^L J_b(\Lambda)$ of a smooth irreducible representation ρ of J_b, composed with the *twisted* embedding ${}^L J_b(\Lambda) \to {}^L G(\Lambda)$, as in [FS21, Section IX.7.1].

Using this, it is easy to see that $p_!q^*(\chi)$ will be given by compactly supported functions of $J_b/B_b(\mathbb{Q}_p)$ which transform under $B_b(\mathbb{Q}_p)$ via χ. In other words, the unnormalized induction $\text{Ind}_{B_b}^{J_b}(\chi)$. When one accounts for the twists coming from the dualizing object, one finds that the exact formula becomes

$$\text{nEis}^{b_T}(S_{\phi_T}) \simeq j_{b!}(i_{B_b}^{J_b}(\chi) \otimes \delta_{P_b}^{1/2})[-\langle 2\hat{\rho}, \nu_b \rangle]$$

where $j_b : \text{Bun}_G^b \to \text{Bun}_G$ is the inclusion of the HN-strata corresponding to b. Now, what about the connected components $\bar{\nu} = w(b_T)$ with w non-trivial? Here the HN-slopes of $\bar{\nu}$ are at least partially anti-dominant, and therefore $\text{Bun}_B^{w(b_T)}$ will parameterize some non-split extensions. Nonetheless, one finds that $\text{nEis}^{w(b_T)}(S_{\phi_T})$ behaves similarly to the contribution of the connected component given by the HN-dominant reduction. Note that, a priori, the complex $\text{nEis}^{w(b_T)}(S_{\phi_T})$ could be supported on all $b' \in B(G)$ with $b \leq b'$ in the natural partial ordering on $B(G)$. If one imposes the previous compatibility assumption and assumes ϕ_T is generic then one can use the previous corollary to assume that $b' \in B(G)_{\text{un}}$. In this case, the complex $\text{nEis}^{w(b_T)}(S_{\phi_T})|_{\text{Bun}_G^{b'}}$ can be computed in terms of the cohomology of the space of simulatenous reductions of a G-bundle $\mathcal{F}_G$ to two B-bundles with underlying T-bundles given by $w(b_T)$ and a Weyl group translate of b_T', where b_T' is the HN-dominant reduction of b'. This space admits a locally closed stratification by the generic relative position of these two reductions coming from the Bruhat decomposition of $B\backslash G/B$. If $b' \neq b$ then each of the non-empty strata admit a map to a positive symmetric power of the mirror curve Div^1, and are locally modelled by a semi-infinite flag space called a Zastava space, as studied in the function field setting by Feign, Finkelberg, Kusnetzov, and Mirković [Fei+99]. By combining a study of these Zastava spaces with Condition (3) on ϕ_T and induction on b, we can show that the restriction of $\text{nEis}^{w(b_T)}(S_{\phi_T})|_{\text{Bun}_G^{b'}}$ to each of these locally closed strata vanishes, from which we can conclude that the restriction $\text{nEis}^{w(b_T)}(S_{\phi_T})|_{\text{Bun}_G^{b'}}$ vanishes unless $b' = b$, where again only the contribution of the split B-structure matters. All in all, we conclude an isomorphism:

$$\text{nEis}^{w(b_T)}(S_{\phi_T}) \simeq j_{b!}(i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{1/2})[-\langle 2\hat{\rho}, \nu_b \rangle]$$

This parallel behavior between the HN-dominant connected component and the connected components in its Weyl group orbit is no accident. In analogy with §1.1, we expect, for a choice of representative $\tilde{w} \in N(T)$ of $w \in W_G$ in the relative Weyl group, to have an isomorphism

$$\text{nEis}(S_{\phi_T}) \simeq \text{nEis}(S_{\phi_T}^{\tilde{w}})$$

where $S_{\phi_T}^{\tilde{w}}$ is the pullback of S_{ϕ_T} along the map $\text{Bun}_T \to \text{Bun}_T$ induced by $\tilde{w}$. This involution sends the connected component $\text{Bun}_T^{b_T}$ to $\text{Bun}_T^{w(b_T)}$ and sends the character χ to χ^w. In particular, we see that, by our previous description of stalks, this gives the precise analogue of Theorem 1.1. We summarize the above discussion as follows.

Theorem 1.14. *(Corollary 9.2) Consider ϕ_T a weakly normalized regular parameter with associated character $\chi : T(\mathbb{Q}_p) \to \Lambda^*$. Given $b \in B(G)_{\text{un}}$, we consider J_b, M_b, B_b, and W_b as defined above. For $b \in B(G)$, the stalk $\text{nEis}(S_{\phi_T})|_{\text{Bun}_G^b} \in D(\text{Bun}_G^b) \simeq D(J_b(\mathbb{Q}_p), \Lambda)$ is given by*

(1) *an isomorphism $\text{nEis}(S_{\phi_T})|_{\text{Bun}_G^b} \simeq \bigoplus_{w \in W_b} i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{1/2}[-\langle 2\hat{\rho}, \nu_b \rangle]$ if $b \in B(G)_{\text{un}}$,*

(2) *an isomorphism $\text{nEis}(S_{\phi_T})|_{\text{Bun}_G^b} \simeq 0$ if $b \notin B(G)_{\text{un}}$.*

In particular, $\text{nEis}(S_{\phi_T})$ is a perverse sheaf on Bun_G with respect to the standard t-structure defined by the HN-strata.

We note that the previous Corollaries imply that the stalks of the sheaf $\text{nEis}(S_{\phi_T})$ are valued in smooth admissible representation when ϕ_T is weakly normalized regular. This implies that the sheaf $\text{nEis}(S_{\phi_T})$ is ULA with respect to the structure map $\text{Bun}_G \to *$, using the characterization given in [FS21, Theorem V.7.1]. This ULA property allows us to extend the construction of $\text{nEis}(S_{\phi_T})$

to $\overline{\mathbb{Z}}_\ell$ and $\overline{\mathbb{Q}}_\ell$-coefficients, where passing to the world of lisse-étale solid sheaves is a non-trivial matter because of the difference between ℓ-adic and discrete topologies. We will need to work with ϕ_T that is integral in the sense that, if $\Lambda = \overline{\mathbb{Q}}_\ell$, it is of the form $\overline{\phi}_T \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{Q}}_\ell$ for a $\overline{\mathbb{Z}}_\ell$-valued parameter $\overline{\phi}_T$. Given an integral parameter ϕ_T such that the mod ℓ-reduction is weakly normalized regular, we get a sheaf $\mathrm{nEis}(S_{\phi_T}) \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ for all $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$. The description of the stalks, the filtered eigensheaf property with eigenvalue ϕ, and the commutation with Verdier duality all extend in a natural way to these coefficient systems, and we would now like to say that the filtered eigensheaf property implies it is a genuine eigensheaf under the conditions on ϕ_T. For a representation $V \in \mathrm{Rep}\ (^LG)$, the filtered eigensheaf property tells us that $T_V(\mathrm{nEis}(S_{\phi_T}))$ has a filtration whose graded pieces have Weil group action given by $\nu \circ \phi_T$, for $\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ a non-zero weight of V in $\hat{T}$. In order to see this splits, it suffices to show for ν, ν' defining distinct Galois orbits of weights of V in $\hat{T}$ that the extension group $H^1(W_{\mathbb{Q}_p}, (\nu - \nu') \circ \phi_T)$ vanishes for the Γ-orbit $(\nu - \nu')$ defined by $\nu - \nu'$. However, this is equivalent to saying that the entire complex

$$R\Gamma(W_{\mathbb{Q}_p}, (\nu - \nu') \circ \phi_T)$$

is trivial, and this was precisely the kind of vanishing result that strong μ-regularity of ϕ_T guaranteed. Moreover, by the vanishing of the H^0 the splitting will be unique. In particular, if $V = V_{\mu^\Gamma}$ for $\mu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ such that ϕ_T is strongly μ-regular then this allows us to see that we get a unique splitting for $V = V_{\mu^\Gamma}$. Now, as noted in Remark 1.8, we see that strong μ-regularity is usually guaranteed under generic for a sufficiently nicely chosen cocharacter. We would like to use this to conclude that the filtration on $T_{V_{\mu^\Gamma}}$ splits in more generality. Suppose we are given cocharacters $\mu_1, \ldots, \mu_k$ and we know that the filtration splits for the V_{μ_i} then, using the compatibilities of the filtration, we can show the splitting for any representation V realized as a direct summand of $\bigotimes_{k=1}^m V_{\mu_k}^{\otimes n_i}$, but in this case we cannot guarantee that this splitting is unique.

We need to be a bit careful when running the argument sketched above. In particular, if $\Lambda = \overline{\mathbb{Q}}_\ell$, then the category $\mathrm{Rep}\ (^LG)$ is semi-simple with irreducible objects parametrized by Γ-orbits of dominant cocharacters $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+/\Gamma$ and the above argument goes through. If $\Lambda \in \{\overline{\mathbb{Z}}_\ell, \overline{\mathbb{F}}_\ell\}$ this is no longer true. However, in these cases, we can replace $\mathrm{Rep}\ (^LG)$ by a sub-category of tilting modules $\mathrm{Tilt}\ (^LG)$ ([Mat00],[Jan03, Appendix E]), which will be semi-simple with indecomposable objects parameterized by $\mu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+/\Gamma$, denoted $\mathcal{T}_{\mu^\Gamma} \in \mathrm{Tilt}\ (^LG)$. Extending the theory of tilting modules to the full L-group ^LG is a bit subtle. However, this is precisely what our assumption that ℓ is very good with respect to G will allow us to do. This category is preserved under taking tensor products, and therefore we can define the notion of a "tilting eigensheaf" (Definition 10.5) by replacing $\mathrm{Rep}\ (^LG)$ with $\mathrm{Tilt}\ (^LG)$ in the usual definition. For $V \in \mathrm{Rep}\ (^LG^I)$, we write $r_V : {}^LG^I \to \mathrm{GL}(V)$ for the associated map. This allows us to define the following.

Definition 1.15. For a finite index set I, we say a tuple of cocharacters $(\mu_i)_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+)^I$ is $(\mu_i)_{i \in I}$-regular if the filtration on $T_V(\mathrm{nEis}(S_{\phi_T})$ splits for the tilting module $V = \boxtimes_{i \in I} \mathcal{T}_{\mu_i^\Gamma}$.

Remark 1.16. The argument sketched above allows us to show if ϕ_T is $(\mu_{1i})_{i \in I}$ and $(\mu_{2i})_{i \in I}$ regular then this implies that ϕ_T is regular for the cocharacter attached to any highest weight tilting module occurring in the tensor product $\boxtimes_{i \in I} \mathcal{T}_{\mu_{1i}^\Gamma} \otimes \mathcal{T}_{\mu_{2i}^\Gamma}$ (Proposition 10.12). In particular, this implies that ϕ_T is $(\mu_{1i} + \mu_{2i})_{i \in I}$-regular by considerations of highest weight. This property often allows us to see that we get μ-regularity just under the generic hypothesis on ϕ_T. For example, for $G = \mathrm{GL}_n$, and $\Lambda = \overline{\mathbb{Q}}_\ell$ then, as observed in Remark 1.8, genericity will imply strong μ-regularity for $\mu = (1, 0, \ldots, 0)$, and this will imply the filtration splits uniquely for this cocharacter. This allows us to see that the filtration splits uniquely for the standard representation, and this in turn allows us to see that we get a splitting for the representations corresponding to the other fundamental coweights $\omega_i = (1_i, 0_{n-i})$ using the decomposition $V_{\mathrm{std}}^{\otimes i} = \mathrm{Sym}^i(V) \bigoplus \cdots \bigoplus \Lambda^i(V)$ of the tensor powers of the

standard representation V_{std} of GL_n. From here, we can show μ-regularity for all μ using the fact that we can realize T_μ as a direct summand of a tensor product of the representations corresponding to fundamental weights. This argument also works with torsion coefficients since $\Lambda^i(V) = V_{\omega_i}$ will always be tilting by virtue of ω_i being minuscule for all $i = 1, \ldots, n$ when $G = \mathrm{GL}_n$ (Corollary 10.16).

Our main theorem is then as follows.

Theorem 1.17. *(Theorem 10.10) For $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$, we consider $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\Lambda)$ an integral parameter such that its mod ℓ-reduction is weakly normalized regular. There then exists a perverse sheaf $\mathrm{nEis}(S_{\phi_T}) \in \mathrm{D}_{\mathsf{lis}}(\mathrm{Bun}_G, \Lambda)$ which is a filtered eigensheaf with eigenvalue ϕ. If $V \in \mathrm{Tilt}\,({}^L G)$ is a direct sum of tilting modules $\otimes_{i \in I} T_{\mu_i^\Gamma}$ for geometric dominant cocharacters μ_i, and ϕ_T is μ_i-regular (resp. strongly μ_i-regular), the filtration on $T_V(\mathrm{nEis}(S_{\phi_T}))$ splits (resp. splits uniquely), and we have a natural isomorphism*

$$T_V(\mathrm{nEis}(S_{\phi_T})) \simeq \mathrm{nEis}(S_{\phi_T}) \otimes r_V \circ \phi$$

of sheaves in $\mathrm{D}_{\mathsf{lis}}(\mathrm{Bun}_G)^{BW_{\mathbb{Q}_p}^I}$. In particular, if ϕ_T is μ-regular (resp. strongly μ-regular) for all geometric dominant cocharacters μ then $\mathrm{nEis}(S_{\phi_T})$ is a weak tilting eigensheaf (resp. tilting eigensheaf). For $b \in B(G)$, the stalk $\mathrm{nEis}(S_{\phi_T})_{|\mathrm{Bun}_G^b} \in \mathrm{D}(\mathrm{Bun}_G^b) \simeq \mathrm{D}(J_b(\mathbb{Q}_p), \Lambda)$ is given by

   (1) *an isomorphism $\mathrm{nEis}(S_{\phi_T})_{|\mathrm{Bun}_G^b} \simeq \bigoplus_{w \in W_b} i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{1/2}[\langle 2\hat{\rho}, \nu_b \rangle]$ if $b \in B(G)_{\mathsf{un}}$,*
   (2) *an isomorphism $\mathrm{nEis}(S_{\phi_T})_{|\mathrm{Bun}_G^b} \simeq 0$ if $b \notin B(G)_{\mathsf{un}}$.*

Moreover, if $\mathbb{D}_{\mathrm{Bun}_G}$ denotes Verdier duality on Bun_G, we have an isomorphism

$$\mathbb{D}_{\mathrm{Bun}_G}(\mathrm{nEis}(S_{\phi_T})) \simeq \mathrm{nEis}(S_{\phi_{\overline{T}}})$$

of sheaves in $\mathrm{D}_{\mathsf{lis}}(\mathrm{Bun}_G, \Lambda)$.

Remark 1.18. The notion of a weak tilting eigensheaf means that we always have isomorphisms

$$\eta_{V,I} : T_V(\mathrm{nEis}(S_{\phi_T})) \simeq \mathrm{nEis}(S_{\phi_T}) \otimes r_V \circ \phi$$

for $V \in \mathrm{Tilt}({}^L G^I)$ and a finite index set I, but do not necessarily know that the desired compatibilities with respect to I and V. Even though we know these compatibilities for the filtration, it is not necessarily clear that the splitting we produce through our argument respects these compatibilities without assuming strong μ-regularity. Only knowing the compatibilities of the splittings under such restrictive conditions is a bit unfortunate; fortunately, for most of the applications to local Shtuka spaces with one leg it suffices to only know a splitting exists.

This eigensheaf has several suprising applications to the cohomology of local Shimura varieties and shtuka spaces. To formalize this, we define, for $b \in B(G)$, a complex of $J_b(\mathbb{Q}_p)$-representations denoted $\mathrm{Red}_{b,\phi}$. If $b \notin B(G)_{\mathsf{un}}$ we set this to be equal to 0 and if $b \in B(G)_{\mathsf{un}}$ to be equal to $\bigoplus_{w \in W_b} i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{1/2}[\langle 2\hat{\rho}, \nu_b \rangle]$. Now let's consider μ a geometric dominant cocharacter of G with reflex field E and set $B(G, \mu) \subseteq B(G)$ to be the subset of μ-admissible elements (Definition 2.5). We let T_μ be the associated highest weight tilting module of $\hat{G}$. This defines a representation of $W_E \ltimes \hat{G}$ with associated Hecke operator T_μ. We write $r_\mu : W_E \ltimes \hat{G} \to \mathrm{GL}(T_\mu)$ for the associated map. We consider the cohomology of the local shtuka spaces $\mathrm{Sht}(G, b, \mu)_\infty$, as defined in [SW20]. In particular, the representation T_μ defines a sheaf S_μ on $\mathrm{Sht}(G, b, \mu)_\infty$ via geometric Satake, and we can consider the complex $R\Gamma_c(G, b, \mu)$ of $J_b(\mathbb{Q}_p) \times G(\mathbb{Q}_p) \times W_E$-modules attached to cohomology valued in this sheaf.

Remark 1.19. We note that, since we have used the tilting module T_μ in the definition of $R\Gamma_c(G, b, \mu)$ instead of the usual highest weight representation V_μ this is slightly different than the usual definition appearing in the literature. The two definitions will coincide when the representation V_μ defines a

tilting module, which is equivalent to V_μ being irreducible with coefficients in Λ. We say such a μ is tilting if this holds. This will always hold if $\Lambda = \overline{\mathbb{Q}}_\ell$ or if μ is minuscule, and we study this notion more carefully in Appendix B.

We can use $\mathrm{nEis}(S_{\phi_T})$ to describe the cohomology of $R\Gamma_c(G, b, \mu)$. Assume that ϕ_T is μ-regular, the Hecke eigensheaf property then tells us that we have an isomorphism

$$T_\mu(\mathrm{nEis}(S_{\phi_T})) \simeq \mathrm{nEis}(S_{\phi_T}) \otimes r_\mu \circ \phi|_{W_E}$$

of sheaves with continuous W_E-action. If we restrict to the open HN-strata $j_\mathbf{1} : \mathrm{Bun}_G^\mathbf{1} \to \mathrm{Bun}_G$ defined by the trivial element $\mathbf{1} \in B(G)$ then this gives an isomorphism

$$j_\mathbf{1}T_\mu(\mathrm{nEis}(S_{\phi_T})) \simeq i_B^G(\chi) \otimes r_\mu \circ \phi|_{W_E}$$

of complexes of $G(\mathbb{Q}_p) \times W_E$-modules. Now the point is that only the elements $b \in B(G, \mu)$ occur as a modifications $\mathcal{F}_b \dashrightarrow \mathcal{F}_G^0$ of type μ, where $\mathcal{F}_G^0$ is the trivial G-bundle. Therefore, only these stalks contribute to the LHS. By applying excision to the locally closed stratification given by Bun_G^b for $b \in B(G, \mu)$, we find that the LHS has a filtration with graded pieces isomorphic to $j_\mathbf{1}T_\mu(j_{b!}(\mathrm{Red}_{b,\phi}))$, but these are isomorphic to $R\Gamma_c(G, b, \mu)[\mathrm{Red}_{b,\phi}]$. From the above analysis, we deduce the following.

Theorem 1.20. *(Theorem 11.5) For $\phi_T : W_{\mathbb{Q}_p} \to {}^LT(\Lambda)$ an integral toral parameter such that its mod ℓ-reduction is weakly normalized regular and any geometric dominant cocharacter μ such that ϕ_T is μ-regular, we have an equality*

$$\sum_{b \in B(G,\mu)} [R\Gamma_c(G, b, \mu)[\mathrm{Red}_{b,\phi}]] = [r_\mu \circ \phi|_{W_E} \otimes i_B^G(\chi)]$$

in the Grothendieck group $K_0(G(\mathbb{Q}_p) \times W_E, \Lambda)$ of smooth admissible $G(\mathbb{Q}_p)$-representations with a continuous action of W_E.

If we now consider the case where $\Lambda = \overline{\mathbb{Q}}_\ell$ then it follows that the averaging formula is valid for all weakly normalized regular and μ-regular parameters $\phi_T : W_{\mathbb{Q}_p} \to {}^LT(\overline{\mathbb{Q}}_\ell)$, which admit a $\overline{\mathbb{Z}}_\ell$-lattice. Moreover, with $\overline{\mathbb{Q}}_\ell$-coefficients, we can interpret both sides as trace forms on $K_0(T(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$, and use that the set of characters obtained from such parameters is Zariski dense in the variety of unramified characters to conclude the following more general claim.

Theorem 1.21. *(Theorem 11.8) For $\phi_T : W_{\mathbb{Q}_p} \to {}^LT(\overline{\mathbb{Q}}_\ell)$ any toral parameter and μ any geometric dominant cocharacter of G, we have an equality*

$$\sum_{b \in B(G,\mu)} [R\Gamma_c(G, b, \mu)[\mathrm{Red}_{b,\phi}]] = [r_\mu \circ \phi|_{W_E} \otimes i_B^G(\chi)]$$

in $K_0(G(\mathbb{Q}_p) \times W_E, \overline{\mathbb{Q}}_\ell)$.

If μ is minuscule and $G = \mathrm{GL}_n$ then this recovers special cases of an averaging formula of Shin [Shi12], which was formalized for more general reductive groups by Alexander Bertoloni-Meli [Ber21]. In particular, for all χ the induction $i_B^G(\chi)$ defines a class $[i_B^G(\chi)] \in K_0^{st}(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ in the subgroup of the Grothendieck group with stable character sum. To such a class, the averaging formula gives a description of the RHS in terms of an average over $B(G, \mu)$ of the isotypic parts of $R\Gamma_c(G, b, \mu)$ with respect to $\mathrm{Red}_b^c(i_B^G(\chi))$, where $\mathtt{c}$ is a refined endoscopic datum (Definition C.1). In Appendix C, we verify that this indeed agrees with the conjectured averaging formula when $\mathtt{c}$ is the trivial endoscopic datum. This is rather remarkable. Such formulae are typically proven in the minuscule case by stabilizing the trace formula on the Igusa varieties indexed by $b \in B(G, \mu)$, and our analysis gives a more conceptual explanation for them. By combining our work here with the compatibility results proven in [Ham21b] and [BHN22], this gives a proof of this averaging formula

in cases where the non-basic Igusa varieties haven't even been properly defined yet![3] We recall that, in the proof of the averaging formula, we used excision to produce a filtration whose graded pieces were isomorphic to

$$j_{\mathbf{1}} T_\mu(j_{b!}(\mathrm{Red}_{b,\phi})) \simeq j_{\mathbf{1}} T_\mu(j_{b!}(\mathrm{nEis}(S_{\phi_T})j_{\mathrm{Bun}_G^b}))$$

By using the isomorphism $\mathbb{D}_{\mathsf{Bun}_G}(\mathrm{nEis}(S_{\phi_T})) \simeq \mathrm{nEis}(S_{\phi_{\overline{T}}})$, we can show (See Corollary 11.13) that we have an isomorphism: $j_{\mathbf{1}} T_\mu(j_{b!}(\mathrm{Red}_{b,\phi})) \simeq j_{\mathbf{1}} T_\mu(j_{b*}(\mathrm{Red}_{b,\phi}))$ of objects in $\mathrm{D}_{\mathsf{lis}}(\mathrm{Bun}_G, \Lambda)$. This implies that the excision spectral sequence degenerates, allowing us to conclude the following refined averaging formula.

Theorem 1.22. *(Theorem 11.14) For ϕ_T an integral parameter with weakly normalized regular mod ℓ-reduction, and μ any geometric dominant cocharacter such that ϕ_T is μ-regular, we have an isomorphism*

$$\bigoplus_{b \in B(G,\mu)_{\mathsf{un}}} \bigoplus_{w \in W_b} R\Gamma_c(G, b, \mu)[\rho_{b,w}][\langle 2\hat\rho, \nu_b \rangle] = \bigoplus_{b \in B(G,\mu)} R\Gamma_c(G, b, \mu)[\mathrm{Red}_{b,\phi}] \simeq i_B^G(\chi) \otimes r_\mu \circ \phi|_{W_E}$$

of complexes of $G(\mathbb{Q}_p) \times W_E$-modules, where $\rho_{b,w} := i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{1/2}$.

We now assume that ϕ_T is an integral parameter with weakly normalized regular mod ℓ reduction in all that follows. The previous Theorem leads to a very explicit descriptions of the complexes $R\Gamma_c(G, b, \mu)[\rho_{b,w}]$ and the degrees of cohomology they sit in.

Corollary 1.23. *(Corollary 11.15) For μ a geometric dominant cocharacter with reflex field E such that ϕ_T is μ-regular, fixed $b \in B(G,\mu)_{\mathsf{un}}$, and varying $w \in W_b$, the complex $R\Gamma_c(G, b, \mu)[\rho_{b,w}]$ is isomorphic to $\phi_{b,w}^\mu \otimes i_B^G(\chi)[-\langle 2\hat\rho, \nu_b \rangle]$, for $\phi_{b,w}^\mu$ a representation of W_E. Moreover, we have an isomorphism*

$$\bigoplus_{b \in B(G,\mu)_{\mathsf{un}}} \bigoplus_{w \in W_b} \phi_{b,w}^\mu \simeq r_\mu \circ \phi|_{W_E}$$

of W_E-representations.

Remark 1.24. When $G = \mathrm{GL}_n$, we can deduce these consequences for all μ under the assumption that ϕ_T is generic (cf. Remark 1.5). We anticipate that, by combining this statement with the approach to torsion vanishing taken by Koshikawa [Kos21] via using compatibility of the Fargues-Scholze and the usual local Langlands correspondence, it should lead to generalizations of Caraiani and Scholze's results.

It is now natural to wonder what the representations $\phi_{b,w}^\mu$ exactly are. It was already observed by Xiao-Zhu [XZ17] that the elements of the set $B(G, \mu)_{\mathsf{un}}$ correspond to Weyl group orbits of weights of the highest weight module T_μ or rather its restriction $T_\mu|_{\hat G^\Gamma}$ (Corollary 2.9). If we let $b_T \in B(T)$ be the HN-dominant reduction of an element $b \in B(G, \mu)_{\mathsf{un}}$ then the orbit of the character b_T under the Weyl group W_G can be described as $w(b_T)$ for $w \in W_b$ varying. For varying $b \in B(G, \mu)_{\mathsf{un}}$, this describes the set of non-zero weights which can occur in the representation $T_\mu|_{\hat G^\Gamma}$. In particular, given such a $\overline\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$, we can look at the direct sum of weight spaces

$$\bigoplus_{\substack{\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \\ \nu_\Gamma = \overline\nu}} T_\mu(\nu)$$

and this coincides with the weight space of $T_\mu|_{\hat G^\Gamma}(\overline\nu)$ via the isomorphism $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \simeq \mathbb{X}^*(\hat T)$. The refined averaging formula suggests the following relationship.

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[3]However, in the case where G is split and μ is minuscule this formula can actually be checked by hand (See Proposition 11.18), but in the case of unitary groups or restrictions of scalars and μ minuscule this already gives new information.

Conjecture 1.25. *(Conjecture 11.16) For all geometric dominant cocharacters μ such that ϕ_T is μ-regular, an unramified element $b \in B(G,\mu)_{\mathrm{un}}$, and a Weyl group element $w \in W_b$, we have an isomorphism*

$$\bigoplus_{\substack{\widetilde{w(b_T)} \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \\ \widetilde{w(b_T)}_\Gamma = w(b_T)}} \widetilde{w(b_T)} \otimes \phi_T j_{W_E} \otimes T_\mu(\widetilde{w(b_T)}) \simeq \phi_{b,w}^\mu$$

of W_E-representations.

We verify this conjecture in some particular cases, by noting that the contribution from the μ-ordinary locus can be explicitly computed using a shtuka analogue of Boyer's trick [Boy99], as studied by Gaisin-Imai [GI16]. To do this, we note that we have a distinguished element in $B(G,\mu)_{\mathrm{un}}$ called the μ-ordinary element, which we denote by b_μ. It is the maximal element in the partial ordering on $B(G,\mu)$, and we let b_{μ_T} be its HN-dominant reduction. The conjecture suggests that this should correspond to the Weyl group orbit of the highest weight of T_μ. In this case, the space $\mathrm{Sht}(G, b_\mu, \mu)_1$ with its $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$-action is parabolically induced from the space $\mathrm{Sht}(T, b_{\mu_T}, \mu)_1$ with its $T(\mathbb{Q}_p) \times T(\mathbb{Q}_p)$-action. In particular, using this we can deduce the following isomorphism (See Proposition 11.18):

$$R\Gamma_c(G, b_\mu, \mu)[\rho_{b_\mu,w}] \simeq i_B^G(\chi^w) \otimes w(\mu) \otimes \phi_T j_{W_E}[\langle 2\hat\rho, \nu_b \rangle]$$

This calculation has a very interesting consequence. In particular, when combined with the refined averaging formula, we see that we must have an isomorphism $i_B^G(\chi^w) \simeq i_B^G(\chi)$. So, by choosing μ to be sufficiently regular so that $W_{b_\mu} = W_G$, we can deduce the following.

Theorem 1.26. *(Corollary 11.20) For ϕ_T an integral parameter with weakly normalized regular mod ℓ-reduction such that there exists a μ which is not fixed under W_G and ϕ_T is μ-regular, we have an isomorphism*

$$i_{\chi,w} : i_B^G(\chi) \simeq i_B^G(\chi^w)$$

of smooth $G(\mathbb{Q}_p)$-representations for all $w \in W_G$.

This showcases the strong connection between the theory of geometric Eisenstein series and the theory of intertwining operators and the Langlands quotient that has been our philosophical guide throughout. A relation that holds even with mod ℓ-coefficients! With mod ℓ coefficients, there is no good theory of intertwining operators or the Langlands quotient (See however [Dat05], for the current state of the art), and we suspect that further developing the theory of geometric Eisenstein series should provide some insights into these notions in the ℓ-modular setting.

We saw above that our previous conjecture on $\phi_{b,w}^\mu$ can be completely verified using Boyer's trick in the case that the only weights of $T_\mu j_{\hat{G}^\Gamma}$ are orbits of the highest weight. This will be the case when the image $\mu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+ \simeq \mathbb{X}^*(\hat{T})^+$ of μ is minuscule with respect to the pairing with $\mathbb{X}_*(\hat{T})$. If we combine this with the refined averaging formula then we can also deduce the claim when $B(G,\mu)$ has two elements. I.e the case where $T_\mu j_{\hat{G}^\Gamma}$ has two weight spaces; one corresponding to the μ-ordinary element and the other corresponding to the basic element. This will prove the previous conjecture in all cases where $\mu \in \mathbb{X}_*(\hat{T})^+$ is minuscule or quasi-minuscule with respect to the pairing with cocharacters $\mathbb{X}_*(\hat{T})$.

Theorem 1.27. *(Corollary 11.23) For μ a geometric dominant cocharacter and ϕ_T strongly μ-regular such that $\mu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ is minuscule or quasi-minuscule with respect to the pairing with $\mathbb{X}_*(\hat{T})$, the previous conjecture is true.*

Remark 1.28. Even for μ minuscule it can be the case that the image $\mu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ is no longer minuscule with respect to the above pairing, as this corresponds to restricting the highest weight

representation of $\hat{G}$ defined by μ to $\hat{G}$. Therefore, even for μ minuscule, we can still have that the basic element $b \in B(G, \mu)$ is unramified (See [XZ17, Remark 4.2.11]) In these cases, a very analogous result was proven by [XZ17], where they describe the irreducible components of affine Deligne-Lusztig varieties in terms of the weight space defined by the basic element. These affine Deligne-Lusztig varieties describe the special fibers of the local shtuka spaces $\mathrm{Sht}(G, b, \mu)_\infty / \underline{K}$ in the case that G is unramified, and K is a hyperspecial level. Moreover, we suspect that, by using nearby cycles, one could deduce some special cases of our result from theirs.

Throughout our results, we have introduced various technical conditions on ϕ_T. We suspect that some of these conditions are artifacts of the proofs we have used to overcome the technical geometry of Bun_B and its compacitifications in this diamond world. While the conditions are manageable for specific applications to specific groups it leaves one wanting for a more conceptually clear picture. In particular, we conjecture that the following is true, which (modulo checking the compatibilities of the isomorphisms in eigensheaf property) our methods show for GL_n and integral parameters (Corollary 10.16).

Conjecture 1.29. *For $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$ and $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\Lambda)$ a generic toral L-parameter, there exists a sheaf $\mathrm{nEis}(S_{\phi_T}) \in \mathrm{D}_{\mathsf{lis}}(\mathrm{Bun}_G, \Lambda)$ which is a perverse Hecke eigensheaf with eigenvalue ϕ such that one has an isomorphism $\mathbb{D}_{\mathsf{Bun}_G}(\mathrm{nEis}(S_{\phi_T})) \simeq \mathrm{nEis}(S_{\phi_{\overline{T}}})$, and its stalk at all $b \in B(G)$ is isomorphic to $\mathrm{Red}_{b,\phi}$.*

This conjecture would follow from knowing the existence of $\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ and in turn the compactified Eisenstein functor $\overline{\mathrm{Eis}}$ with all the various desiderata proven by Braverman-Gaitsgory [BG02] in the function field setting. In particular, we expect that $\overline{\mathrm{Eis}}(S_{\phi_T})$ should always commute with Verdier duality, and satisfy the functional equation if $\alpha \cdot \phi_T$ is non-trivial for all Γ-orbits of roots. Moreover, by the analogue of the results of [BG08], there should be a natural map

$$\mathrm{nEis}(S_{\phi_T}) \to \overline{\mathrm{Eis}}(S_{\phi_T})$$

whose cone should be given by Eisenstein functors tensored with complexes admitting a filtration isomorphic to $R\Gamma(W_{\mathbb{Q}_p}, \alpha \cdot \phi_T)$ for α a Γ-orbit of coroots of G. In particular, we should have an isomorphism $\overline{\mathrm{Eis}}(S_{\phi_T}) \simeq \mathrm{nEis}(S_{\phi_T})$ precisely when ϕ_T is generic. It follows by our above analysis that this would imply an isomorphism $i_B^G(\chi) \simeq i_B^G(\chi^w)$ for all generic χ, which is precisely what we show in the appendix (Proposition A.2).

In §2, we start by defining the set of unramified elements in $B(G)$ and discussing their relationship with highest weight theory, as in [XZ17]. In §3, we review the construction of eigensheaves on Bun_T attached to parameters ϕ_T, introducing the conditions on our parameter ϕ_T and working through some useful lemmas and examples related to them. In §4, we review the geometric Satake correspondence of Fargues-Scholze, recalling the key results and relating the highest weight theory of ${}^L G$ to the cohomology of semi-infinite Schubert cells. In §5, we introduce Drinfeld's compactifications over the Fargues-Fontaine curve and establish Theorem 1.3. We also introduce a locally closed stratification of $\overline{\mathrm{Bun}}_B$ and show it is well-behaved. In §6, we move into the sheaf theory introducing the normalized Eisenstein functor and establishing Theorem 1.2. In §7, we will study how the Eisenstein functor interacts with Hecke operators, establishing Theorem 1.12. This will ultimately be done via a key diagram relating the action of Hecke operators of Bun_G base-changed along the map $\mathfrak{p} : \mathrm{Bun}_B \to \mathrm{Bun}_G$ to semi-infinite Schubert cells, where it reduces to the results in §4. In §8, we study Verdier duality and show Theorem 1.11. In §9, we will carry out the computation of the stalks of the Eisenstein series $\mathrm{nEis}(S_{\phi_T})$ establishing Theorem 1.14. In §10, we describe the theory of tilting modules for the L-group ${}^L G$, constructing $\mathrm{nEis}(S_{\phi_T})$ with $\overline{\mathbb{Z}}_\ell$ and $\overline{\mathbb{Q}}_\ell$-coefficients and showing Theorem 1.15. Finally, in §11, we deduce the applications to the cohomology of local shtuka spaces showing Theorems 1.18, 1.20, 1.24, and 1.25.

2. Notation

Let $\ell \neq p$ be distinct primes.

Let G be a quasi-split connected reductive group over the p-adic numbers $\mathbb{Q}_p$ with simply connected derived group.

We let $\overline{\mathbb{Q}}_\ell$ denote the algebraic closure of the ℓ-adic numbers, with residue field $\overline{\mathbb{F}}_\ell$ and ring of integers $\overline{\mathbb{Z}}_\ell$, endowed with the discrete topology. Throughout, we will assume that, for our fixed G, ℓ is very good in the sense of [FS21, Page 33].

Let Γ be the absolute Galois group of $\mathbb{Q}_p$, and let $W_{\mathbb{Q}_p} \subset \Gamma$ be the Weil group of $\mathbb{Q}_p$.

We set $L_{\mathbb{Q}_p} := W_{\mathbb{Q}_p} \times \mathrm{SL}_2(\overline{\mathbb{Q}}_\ell)$ to be the Weil-Deligne group.

Fix choices $A \subset T \subset B \subset G$ of maximal split torus, maximal non-split torus, and Borel. We use U to denote the unipotent radical of B.

We let W_G be the relative Weyl group of G and w_0 be the element of longest length.

We write $\mathrm{Ind}_B^G(\ )$ for the unnormalized parabolic induction functor from B to G. We let $\delta_B^{1/2}$ be the modulus character defined by B and set $i_B^G(\ ) := \mathrm{Ind}_B^G(\ \delta_B^{1/2})$ to be the normalized induction.

Let $\breve{\mathbb{Q}}_p$ be the completion of the maximal unramified extension of $\mathbb{Q}_p$ with Frobenius σ. For $E/\mathbb{Q}_p$ a finite extension, we set $\breve{E}$ to be the compositum $E\breve{\mathbb{Q}}_p$.

Set $\mathbb{C}_p$ to be the completion of the algebraic closure of $\mathbb{Q}_p$.

Let $B(G) = G(\breve{\mathbb{Q}}_p)/(g \sim hg\sigma(h)^{-1})$ denote the Kottwitz set of G.

For $b \in B(G)$, we write J_b for the σ-centralizer of b.

We will always work over the fixed base $* := \mathrm{Spd}\,\overline{\mathbb{F}}_p$, unless otherwise stated.

Let Perf denote the category of (affinoid) perfectoid spaces in characteristic p over $*$ endowed with the v-topology. For $S \in \mathrm{Perf}$, let Perf_S denote the category of affinoid perfectoid spaces over it.

For $S \in \mathrm{Perf}$, let X_S denote the relative (schematic) Fargues-Fontaine curve over S.

For $\mathrm{Spa}\,(F, O_F) \in \mathrm{Perf}$ a geometric point, we will often drop the subscript on X_F and just write X for the associated Fargues-Fontaine curve.

For $b \in B(G)$, we write $\mathcal{F}_b$ for the associated G-bundle on X.

For $S \in \mathrm{Perf}$, we let $\mathcal{F}_G^0$ denote the trivial G-bundle on X_S.

We consider coefficient systems $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$, with a fixed choice of square root of $p \in \Lambda$. We define all half Tate twists with respect to this choice.

For an Artin v-stack X, we write $\mathrm{D}_{\blacksquare}(X, \Lambda)$ for the condensed ∞-category of solid Λ-valued sheaves on X, and write $\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \subset \mathrm{D}_{\blacksquare}(X, \Lambda)$ for the full sub-category of Λ-valued lisse-étale sheaves, as defined in [FS21, Chapter VII].

For a v-stack or diamond X, when working with torsion coefficients, we will indicate this by just writing $\mathrm{D}(X) := \mathrm{D}_{\text{ét}}(X, \Lambda)$ for the category of étale Λ-sheaves on X, as defined [Sch18]. If X is an Artin v-stack ([FS21, Definition IV.V.1]) then we will regard it as a condensed

∞-category via the identification $D_{\mathsf{lis}}(X, \Lambda) \simeq D(X)$ when viewed as objects in $D(X, \Lambda)$ [FS21, Proposition VII.6.6], as described in [FS21, Page 312].

We let $\hat{G}$ denote the Langlands dual group of G with fixed splitting $(\hat{T}, \hat{B}, \{X_\alpha\})$.

If E denotes the splitting field of G then the action of $W_{\mathbb{Q}_p}$ factors through $Q := W_{\mathbb{Q}_p}/W_E$. We let ${}^LG := \hat{G} \rtimes Q$ denote the L-group.

For I a finite index set, we let $\mathrm{Rep}({}^LG^I)$ denote the category of finite-dimensional algebraic representations of ${}^LG^I$.

To any condensed ∞-category $\mathcal{C}$, we write $\mathcal{C}^{BW_{\mathbb{Q}_p}^I}$ for the category of objects with continuous $W_{\mathbb{Q}_p}^I$-action, as defined in [FS21, Section IX.1].

We will let $\mathrm{Div}^1 := \mathrm{Spd}\,\breve{\mathbb{Q}}_p/\mathrm{Frob}^{\mathbb{Z}}$ denote the mirror curve, and, for a finite extension $E/\mathbb{Q}_p$, we write Div^1_E for the base-change to E.

For I a finite index set, we let Div^I denote $|I|$-copies of the mirror curve. For $n \in \mathbb{Z}$, we let $\mathrm{Div}^{(n)} = (\mathrm{Div}^1)^n/S_n$, denote the nth symmetric power of the mirror curve, where S_n is the symmetric group on n letters.

For a reductive group $H/\mathbb{Q}_p$, we write $D(H(\mathbb{Q}_p), \Lambda)$ for the unbounded derived category of smooth Λ-representations.

We say a map of v-stacks $f : X \to Y$ is representable in nice diamonds if it is representable in locally spatial diamonds, is compactifiable, and (locally) $\mathrm{tr.deg}(f) < \infty$.

All 6-functors will be implicitly derived unless otherwise stated.

For a locally pro-p group H, we write $\underline{H}$ for the functor sending $S \in \mathrm{Perf}$ to $\mathrm{Cont}(|S|, H)$, the set of continuous maps from the underlying topological space of S to H.

Remark 2.1. At various points, we will need to consider the functors $f_! : D(X) \to D(Y)$ and $f^! : D(Y) \to D(X)$ for certain "stacky" morphisms of Artin v-stacks $f : X \to Y$. The correct definitions of these functors in this case are given in the work of [GHW22]. In particular, they extend the 6-functors studied in [Sch18; FS21] to fine maps [GHW22, Definition 1.3] of decent v-stacks [GHW22, Definition 1.2]. In general being a decent v-stack is stronger than being Artin. However, it is easy to check that all the stacks (resp. morphisms) we consider these functors for will be decent (resp. fine). To see this, one can use [GHW22, Proposition 4.11] which states that if $f : X \to Y$ is a map of v-stacks which is representable in nice diamonds and Y is decent then X is also decent and f is fine. In the cases we consider, one can apply this if one takes $Y = \mathrm{Bun}_G$. To see that Bun_G is decent, one can use the charts studied in [FS21, Section V.3], and take advantage of the fact that the maps defining the charts are formally smooth [FS21, Definition IV.3.1] by [FS21, Proposition IV.4.24]. This in particular allows one to see that these charts map strictly surjectively [GHW22, Defition 4.1] to Bun_G. It remains to explain why the maps appearing in our context are fine, to do this one can combine the previous analysis with [GHW22, Proposition 4.10 (iii)], which says that fine morphisms satisfy the 2 out of 3 property.

When speaking about such fine maps of decent v-stacks we will often just cite theorems that only apply to the setting where f is representable in nice diamonds, and leave it to the reader to check that one can deduce the analogous results from the cited result and the formal properties of the 6-functors defined in [GHW22].

Given a decent v-stack $X \to *$ such that X is fine over $*$, we let $K_X := f^!(\Lambda) \in D(X)$ denote the dualizing object of X. Similarly, for $\mathcal{F} \in D(X)$, we will write $R\Gamma_c(X, \mathcal{F}) := f_!(\mathcal{F}) \in D(\Lambda)$. We write $\mathbb{D}_X(-) := RHom(-, K_X)$ for the Verdier duality functor. For a fine map $f : X \to S$ of decent v-stacks, we write $\mathbb{D}_{X/S} := RHom(-, f^!(\Lambda))$ for relative Verdier duality.

We will use the geometric normalization of local class field theory. For $n \in \mathbb{Z}$, we write (n) for the nth power of the ℓ-adic cyclotomic character of $W_{\mathbb{Q}_p}$. We note that, under this

normalization, (1) is sent to the norm character j $j : \mathbb{Q}_p \to \Lambda$, which acts trivially on $\mathbb{Z}_p$ and sends p to $p^{-1} \in \Lambda$.

Before introducing the rest of the notation, we discuss the relationship between unramified elements in $B(G)$ and the representation theory of the dual group.

2.1. **Unramified Elements in $B(G)$ and Highest Weight Theory.** In this section, we will study the set of unramified elements in the Kottwitz set of G. As we will show, these elements are connected to the highest weight theory of the Langlands dual group $\hat{G}$, as discussed in [XZ17, Section 4.2.1]. First, we recall that the Kottwitz set $B(G)$ of a connected reductive group $G/\mathbb{Q}_p$ is equipped with two maps:

The slope homomorphism

$$\nu : B(G) \to \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_{\mathbb{Q}}^{+,\Gamma}$$

$$b \mapsto \nu_b$$

where $\Gamma := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_{\mathbb{Q}}^+$ is the set of rational dominant cocharacters of G.

The Kottwitz invariant

$$\kappa : B(G) \to \pi_1(G)_\Gamma$$

where $\pi_1(G)$ denotes the algebraic fundamental group of Borovoi.

Now, given a geometric cocharacter μ of G with reflex field E, we can define the element:

$$\tilde{\mu} := \frac{1}{[E : \mathbb{Q}_p]} \sum_{\gamma \in \mathsf{Gal}(E/\mathbb{Q}_p)} \gamma(\mu) \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_{\mathbb{Q}}^{+,\Gamma}$$

We let $\mu^\flat$ be the image of μ in $\pi_1(G)_\Gamma$ ($\simeq \mathbb{X}^*(Z(\hat{G})^\Gamma)$). Via the isomorphim $B(G)_{\mathsf{basic}} \simeq \pi_1(G)_\Gamma$, we regard it as a basic element of $B(G)$, which are the minimal elements in the natural partial ordering on $B(G)$. Now we recall that, for a torus T, we have an isomorphism $B(T) \simeq \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$. We can use this isomorphism to give a nice description of a certain piece of $B(G)$.

Definition 2.2. [XZ17, Section 4.2.1] We let $B(G)_{\mathsf{un}} \subseteq B(G)$ denote the image of the natural map $B(T) \to B(G)$. We refer to this as the set of unramified elements.

We now have the following Lemma. We write $(\ )_\Gamma$ for the natural quotient map $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \to \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$.

Lemma 2.3. *[XZ17, Lemma 4.2.2] Let $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma \simeq B(T) \to B(G)$ be the natural map. Then this induces an isomorphism:*

$$\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma / W_G \simeq B(G)_{\mathsf{un}}$$

Proof. Strictly speaking, the proof given by Xiao-Zhu is only in the case that G is unramified. We remedy this now. Note that it is clear that this map is surjective, so it suffices to check injectivity. Let $\mu_1, \mu_2 \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ be two elements with b_1, b_2 their images in $B(G)$ under the natural composite

$$\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \to \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma \simeq B(T) \to B(G),$$

and suppose that $b_1 = b_2$. Since $\kappa_G(b_1) = \kappa_G(b_2)$, it follows that we have $\mu_1 - \mu_2 = (\gamma - 1)\nu + \alpha$ for some coroot $\alpha \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ and $\gamma \in \Gamma$. We may, without loss of generality, replace μ_1 by $\mu_1 + (\gamma - 1)\nu$, and therefore assume that $\mu_1 - \mu_2 = \alpha$. Since the slope homomorphisms of ν_{b_1} and ν_{b_2} are equal by assumption, we can assume, after conjugating by an element of W_G, that $\tilde{\mu}_1 = \tilde{\mu}_2$. Therefore, it follows that, if E_α denotes the reflex field of α, we have an equality

$$\sum_{g \in \mathsf{Gal}(E_\alpha/\mathbb{Q}_p)} g(\alpha) = 0$$

which in turn implies that

$$\sum_{g \in \mathrm{Gal}(E_\alpha/\mathbb{Q}_p)} (1 - g)(\alpha) = |\mathrm{Gal}(E_\alpha/\mathbb{Q}_p)|\alpha$$

This would imply that $\alpha^\vee$ vanishes in $\mathbb{X}_\bullet(T_{\overline{\mathbb{Q}}_p})_\Gamma$ assuming that $\alpha^\vee$ isn't torsion. However, Γ permutes the simple coroots, which form a basis of all coroots. Therefore, it follows that $\alpha^\vee$ is not torsion.

Now we would like to describe $\mathbb{X}_\bullet(T_{\overline{\mathbb{Q}}_p})_\Gamma/W_G$ slightly differently. To do this, we consider the natural pairing

$$\langle \ , \ \rangle : \mathbb{X}_\bullet(T_{\overline{\mathbb{Q}}_p})_\Gamma \times \mathbb{X}^\bullet(T_{\overline{\mathbb{Q}}_p})^\Gamma \to \mathbb{Z}$$

induced by the usual pairing between cocharacters and characters. We let $\hat{\Delta} \subset \mathbb{X}^\bullet(T_{\overline{\mathbb{Q}}_p})^\Gamma$ (resp. $\Delta \subset \mathbb{X}_\bullet(T_{\overline{\mathbb{Q}}_p})^\Gamma)$) be the set of (absolute) simple roots (resp. coroots) of G. Then we define $\mathbb{X}_\bullet(T_{\overline{\mathbb{Q}}_p})_\Gamma^+$ to be the set of elements in $\mathbb{X}_\bullet(T_{\overline{\mathbb{Q}}_p})_\Gamma$ whose inner product with $\mathrm{Im}(\hat{\Delta} \to \mathbb{X}^\bullet(T_{\overline{\mathbb{Q}}_p})_\Gamma)$ under the natural averaging map is positive. The natural map

$$\mathbb{X}_\bullet(T_{\overline{\mathbb{Q}}_p})_\Gamma^+ \to \mathbb{X}_\bullet(T_{\overline{\mathbb{Q}}_p})_\Gamma/W_G$$

is an isomorphism. We also note that we have a natural partial ordering on $\mathbb{X}_\bullet(T_{\overline{\mathbb{Q}}_p})_\Gamma$. In particular, given $\overline{\nu}, \overline{\nu}' \in \mathbb{X}_\bullet(T_{\overline{\mathbb{Q}}_p})_\Gamma$ we say that $\overline{\nu} \leq \overline{\nu}'$ if $\overline{\nu} - \overline{\nu}'$ is a positive integral combination of $\alpha^\vee$ for $\alpha \in \Delta$. We note that we have a natural injective order preserving map:

$$\mathbb{X}_\bullet(T_{\overline{\mathbb{Q}}_p})_\Gamma^+ \to \mathbb{X}_\bullet(T_{\overline{\mathbb{Q}}_p})_{\mathbb{Q},\Gamma}^{+} \times \pi_1(G)$$

With this, we can reformulate the previous lemma as follows.

Lemma 2.4. *[XZ17, Lemma 4.2.3] The following diagram is commutative and respects the partial ordering*

$$\begin{array}{ccc} B(G)_{\mathrm{un}} & \xrightarrow{\ \sim\ } & \mathbb{X}_\bullet(T_{\overline{\mathbb{Q}}_p})_\Gamma^+ \\ \downarrow & & \downarrow \\ B(G) & \xrightarrow{\ \nu \times \kappa\ } & \mathbb{X}_\bullet(T_{\overline{\mathbb{Q}}_p})_{\mathbb{Q},\Gamma}^{+} \times \pi_1(G) \end{array}$$

Now recall that, for μ a geometric dominant cocharacter of G, we have the following.

Definition 2.5. We define $B(G,\mu) \subset B(G)$ to be subset of $b \in B(G)$ for which $\nu_b \leq \tilde{\mu}$ with respect to the Bruhat ordering and $\kappa(b) = \mu^\flat$.

The previous lemma allows us to interpret the unramified elements in this set as follows.

Corollary 2.6. *[XZ17, Corollary 4.2.4] Under the identifications $\mathbb{X}_\bullet(T_{\overline{\mathbb{Q}}_p})_\Gamma^+ \simeq \mathbb{X}_\bullet(T_{\overline{\mathbb{Q}}_p})_\Gamma/W_G \simeq B(G)_{\mathrm{un}}$, we have an equality:*

$$B(G,\mu)_{\mathrm{un}} := B(G)_{\mathrm{un}} \cap B(G,\mu) = \{\lambda \in \mathbb{X}_\bullet(T_{\overline{\mathbb{Q}}_p})_\Gamma^+ \mid \lambda \leq \mu\}$$

We now would like to connect this set with the highest weight theory for $\hat{G}$. If the group is not split then the unramified elements are naturally connected with the highest weight theory of the subgroup $\hat{G}^\Gamma$. Even though $\hat{G}^\Gamma$ is possibly disconnected its representation theory behaves like a connected reductive group. To see this, first we note that the subgroup $\hat{T}^\Gamma$ defined by the maximal torus has character group isomorphic to $\mathbb{X}_\bullet(T_{\overline{\mathbb{Q}}_p})_\Gamma$, and the partial order described above allows one to talk about the highest weight of a representation. In particular, if we let $\hat{T}^{\Gamma,\circ}$ (resp. $\hat{G}^{\Gamma,\circ}$) denote the neutral component of $\hat{T}^\Gamma$ (resp. $\hat{G}^\Gamma$). Then one can use that the natural map $\hat{T}^\Gamma/\hat{T}^{\Gamma,\circ} \to \hat{G}^\Gamma/\hat{G}^{\Gamma,\circ}$ is an isomorphism ([Zhu15, Lemma 4.6]) to see that usual highest weight theory extends to $\hat{G}^\Gamma$. In particular, we have the following.

Lemma 2.7. *[Zhu15, Lemma 4.10] For $\overline{\mu} \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$, there is a unique up to isomorphism irreducible representation of $V_{\overline{\mu}} \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\hat{G})$ of highest weight $\overline{\mu}$, which give rise to all the irreducible representations in $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\hat{G})$ for varying $\overline{\mu}$. Moreover, the multiplicity of the $\overline{\mu}$ weight space in $V_{\overline{\mu}}$ is 1, and the non-zero weights $\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ of $V_{\overline{\mu}}$ lie in the convex hull of the W_G-orbit of $\overline{\mu}$.*

To a geometric dominant cocharacter μ, we can attach an irreducible representation $V_\mu \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\hat{G})$. This defines a natural representation of $\hat{G} \rtimes W_{E_\mu}$ as in [Kot84, Lemma 2.1.2], where E_μ is the reflex field of μ. An element $\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ defines a representation of $\hat{T}$, and we write $V_\mu(\nu)$ for the corresponding weight space of V_μ. If we consider the restriction $V_\mu|_{\hat{G}^\Gamma}$ then the weight space $V_\mu(\nu)$ gives rise to a ν weight space, where we write $(\ )_\Gamma : \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \to \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$ for the map given by taking coinvariants. Using this, it is easy to see we have the following relationship.

Lemma 2.8. *For $\mu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ and $\overline{\nu} \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_\Gamma$, we have the following equality:*

$$\dim(V_\mu(\overline{\nu})) = \sum_{\substack{\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \\ \nu_\Gamma = \overline{\nu}}} \dim(V_\mu(\nu))$$

We will often combine this lemma with the following, which follows from the above discussion.

Corollary 2.9. *([XZ17, Lemma 4.26]) For μ a geometric dominant cocharacter, under the identification $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+ \simeq B(G)_{\mathrm{un}}$ the elements $\overline{\nu} \in B(G, \mu)_{\mathrm{un}}$ correspond to W_G-orbits of the possible non-zero weights in $V_\mu|_{\hat{G}^\Gamma}$.*

Let's study this now more carefully. For $b \in B(G)$, we want to use the above discussion to understand the fiber of the map:

$$i : B(T) \to B(G)_{\mathrm{un}} \subset B(G)$$

Recall that, given $b \in B(G)$, since G is quasi-split the σ-centralizer J_b is an extended pure inner form (in the sense of [Kot97, Section 5.2]) of a Levi subgroup M_b of G [Kot97, Section 6.2], which is the centralizer of the slope homomorphism ν_b of b. We make the following definition.

Definition 2.10. *For $b \in B(G)$, we let W_{M_b} denote the relative Weyl group of M_b and set $W_b := W_G/W_{M_b}$. We will fix a set of representatives $w \in W_G$ of minimal length, as in [BZ77, Section 2.11], and abuse notation by writing w for both the representative and the corresponding element.*

When combining this discussion with Lemma 2.3, we can deduce the following Corollary.

Corollary 2.11. *For fixed $b \in B(G)_{\mathrm{un}}$, the fiber $i^{-1}(b)$ has a unique element, denoted b_T, whose κ-invariant lies in $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^-$ the conjugate of $\mathbb{X}^+_*(T_{\overline{\mathbb{Q}}_p})$ by the element of longest length. Moreover, we have an equality $i^{-1}(b) = \{w(b_T) \mid w \in W_b\}$.*

Now, given a parabolic P with Levi factor M, the element $b \in B(G)$ admits a reduction to a Levi subgroup M if and only if the parabolic $P \cap M_b$ of M_b transfers to a parabolic of J_b under the inner twisting (apply [CFS21, Pages 13, 28] to the basic reduction of b to M_b). We record this specialized to the case of the Borel for future use.

Lemma 2.12. *An element $b \in B(G)$ lies in $B(G)_{\mathrm{un}}$ if and only if $B \cap M_b$ defines, via the inner twisting, a Borel subgroup of J_b.*

Remark 2.13. We note that, when $b \in B(G)_{\mathrm{un}}$, we have an isomorphism $M_b \simeq J_b$ because J_b will be a quasi-split inner form of M_b.

We leave off with an important observation about the map $(\cdot) : \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma \to \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ from orbits to coinvariants. Let $\tilde{J}$ (resp. J) denote the vertices of the absolute (resp. relative) Dynkin diagram of G. For $i \in \tilde{J}$, we write $\tilde{\alpha}_i \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ for the corresponding simple absolute coroot. We recall that Γ permutes the $\tilde{\alpha}_i$, and the orbits under Γ are in bijection with elements of J; namely, the average over the orbit is the (reduced) positive coroot corresponding to $i \in J$. Therefore, for each $i \in J$, we obtain an element $\alpha_i \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ given by the common image of the elements in the orbit corresponding to $i \in J$ under the map $(\cdot)$. This allows us to make the following definition.

Definition 2.14. We denote the group of coinvariants by $\Lambda_{G,B} := \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$, and set $\Lambda_{G,B}^{pos}$ to be the semi-group spanned by the elements $\alpha_i \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ corresponding to the Γ-orbit of coroots indexed by $i \in J$.

Now we introduce the rest of the notation motivated by the discussion above.

We let $\Lambda_G^+ := \mathbb{X}_*(A)^+$ (resp. $\hat{\Lambda}_G^+ := \mathbb{X}^*(A)^+$) be the semi-group of dominant cocharacters (resp. characters), viewed as elements of the positive Weyl chamber in $\Lambda_G := \mathbb{X}_*(A) = \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ (resp. $\hat{\Lambda}_G = \mathbb{X}^*(A) = \mathbb{X}^*(T_{\overline{\mathbb{Q}}_p})$) defined by the choice of Borel.

To a dominant character $\hat{\lambda} \in \hat{\Lambda}_G^+$, we get an associated highest weight Weyl G-module, denoted $V^{\hat{\lambda}}$. It has a fixed highest weight vector $v^{\hat{\lambda}} \in V^{\hat{\lambda}}$, and, given a pair of such weights $\hat{\lambda}_1, \hat{\lambda}_2$, there is a canonical map

$$V^{\hat{\lambda}_1 + \hat{\lambda}_2} \to V^{\hat{\lambda}_1} \otimes V^{\hat{\lambda}}$$

which takes $v^{\hat{\lambda}_1 + \hat{\lambda}_2}$ to $v^{\hat{\lambda}_1} \otimes v^{\hat{\lambda}_2}$.

Let J be the set of vertices of the relative Dynkin diagram of $G/\mathbb{Q}_p$. For each $i \in J$, we denote the corresponding element in the coinvariant lattice by $\alpha_i \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$, as in Definition 2.14. We warn the reader that this is different then the Γ-orbits defined by the (reduced) simple roots corresponding to $i \in J$, these we will denote by $\alpha_{i,A} \in \mathbb{X}^*(A) \subset \mathbb{X}^*(T_{\overline{\mathbb{Q}}_p})$, and will be sum over the elements in the Galois orbit associated to α_i. On the other hand, in the root lattice $\hat{\Lambda}_G$ we will only be interested in the (reduced) simple positive roots corresponding to $i \in J$, and so we just write this as $\hat{\alpha}_i \in \hat{\Lambda}_G$.

We consider the natural pairing

$$\langle \cdot , \cdot \rangle : \hat{\Lambda}_G \times \Lambda_{G,B} \to \mathbb{Z}$$

given by the identifications $\hat{\Lambda}_G = \mathbb{X}^*(A) \simeq \mathbb{X}^*(T_{\overline{\mathbb{Q}}_p})$ and $\Lambda_{G,B} = \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$.

Using the assumption on the derived group of G being simply connected, we define a set of fundamental weights $\hat{\varpi}_i \in \hat{\Lambda}_G$, non-uniquely characterized by the property that $\langle \hat{\varpi}_i, \alpha_j \rangle = \delta_{ij}$. Namely, this can be defined by taking the sum over the Galois orbits of fundamental weights in $\mathbb{X}^*(T_{\overline{\mathbb{Q}}_p})$.

We will regularly use the natural quotient map

$$(\cdot) : \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \to \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$$

as well as the map

$$(\cdot) : \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \to \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$$

from cocharacters to their Galois orbits. We note that $(\cdot)$ factorizes over $(\cdot)$ as a map of sets.

For a geometric dominant cocharacter $\mu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ with reflex field E, we write V_μ for the natural representation of $W_E \ltimes \hat{G}$ of highest weight μ, as in [Kot84, Lemma 2.1.2]. We also

write $V_{\mu^\Gamma} \in \mathrm{Rep}\ (^LG)$ for the induction of this representation to ^LG, which only depends on the associated Γ-orbit μ of μ.

For $b \in B(G)$, we let J_b be the extended pure inner form of M_b considered above. If $b \in B(G)_{\mathsf{un}}$, we write B_b for the Borel defined by $B \setminus M_b$ under the inner twisting, as in Lemma 2.12.

Given $b \in B(G)_{\mathsf{un}}$, we note that, by Lemma 2.3, there exists a unique element $b_T \in B(T)$ with anti-dominant slope homomorphism with respect to the choice of Borel. We will refer to this as the HN-dominant reduction of b. Recalling that there is a minus sign when passing between isocrystals and bundles (cf. [Far17, Proposition 6.6]), this will correspond to the case where the Harder-Narasimhan slopes are dominant.

We set $W_b := W_G/W_{M_b}$, where W_{M_b} (resp. W_G) is the relative Weyl group of M_b (resp. G). We will identify w with a representative of minimal length in W_G throughout.

We will write $\hat{\rho}$ for the half sum of all positive roots.

3. Geometric Local Class Field Theory

3.1. **Hecke Eigensheaves for Tori.**
In this section, we want to talk about geometric local class field theory. Namely, given a torus T with L-parameter

$$\phi_T : W_{\mathbb{Q}_p} \to\ ^LT(\Lambda)$$

where ^LT denotes the Langlands dual group of T, we want to construct a Hecke eigensheaf, denoted S_{ϕ_T}, on the moduli stack Bun_T. We recall that, for a representation $V \in \mathrm{Rep}\ (^LG^I)$ of I-copies of the L-group of G for some finite index set I, a Hecke operator is a map defined by the correspondence

$$
\begin{array}{ccc}
 & \mathrm{Hck}^I_G & \\
{}^{h\ \pi}\swarrow & & \searrow^{h^!} \\
\mathrm{Bun}_G\ \mathrm{Div}^I & & \mathrm{Bun}_G
\end{array}
$$

where Hck^I_G is the functor that parametrizes, for $S \to \mathrm{Div}^I$ defining a tuple of Cartier divisors in the relative Fargues-Fontaine X_S over S, corresponding to characteristic 0 untilts $S_i^\sharp$ for $i \in I$ of S, a pair of G-torsors $\mathcal{E}_1$, $\mathcal{E}_2$ together with an isomorphism

$$\beta : \mathcal{E}_1|_{X_S \setminus \bigcup_{i \in I} S_i^\sharp} \xrightarrow{\ \sim\ } \mathcal{E}_2|_{X_S \setminus \bigcup_{i \in I} S_i^\sharp}$$

where $h^!\ ((\mathcal{E}_1, \mathcal{E}_2, (S_i^\sharp)_{i \in I})) = \mathcal{E}_1$ and $(h\ \ \pi)((\mathcal{E}_1, \mathcal{E}_2, \beta, (S_i^\sharp)_{i \in I})) = (\mathcal{E}_2, (S_i^\sharp)_{i \in I})$. For an algebraic representation $V \in \mathrm{Rep}\ (^LG^I)$, the geometric Satake correspondence of Fargues-Scholze [FS21, Chapters VI, IX.2] furnishes a sheaf $S_V \in \mathrm{D}\ (\mathrm{Hck}, \Lambda)$. Using this, we can define the Hecke operator as

$$T_V : \mathrm{D}_{\mathsf{lis}}(\mathrm{Bun}_G, \Lambda) \to \mathrm{D}\ (\mathrm{Bun}_G\ \ \mathrm{Div}^I, \Lambda)$$

$$\mathcal{F} \mapsto (h\ \ \pi)_\natural((h^!)\ (\mathcal{F})\ \ S_V)$$

where $(h\ \ \pi)_\natural$ is the natural push-forward. I.e the left adjoint to the restriction functor in the category of solid Λ-sheaves [FS21, Proposition VII.3.1]. This induces a functor

$$T_V : \mathrm{D}_{\mathsf{lis}}(\mathrm{Bun}_G, \Lambda) \to \mathrm{D}_{\mathsf{lis}}(\mathrm{Bun}_G, \Lambda)^{BW^I_{\mathbb{Q}_p}}$$

via a version of Drinfeld's Lemma [FS21, Theorem I.7.2, Proposition IX.2.1, Corollary IX.2.3]. When $\Lambda = \overline{\mathbb{F}}_\ell$, this is essentially the statement that Λ-valued local systems on Div^I are equivalent to continuous representations of $W^I_{\mathbb{Q}_p}$ on finite projective Λ-modules [FS21, Proposition VI.9.2]. In this case, we will freely pass between this perspective of local systems and $W^I_{\mathbb{Q}_p}$-representations.

With this in hand, we can define what it means for a sheaf on Bun_G to be a Hecke eigensheaf.

Definition 3.1. Given a continuous L-parameter $\phi : W_{\mathbb{Q}_p} \to {}^L G(\Lambda)$, we say a sheaf $S_\phi \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ is a Hecke eigensheaf with eigenvalue ϕ if, for all $V \in \mathrm{Rep}\ ({}^L G^I)$ with associated map $r_V : {}^L G^I \to \mathrm{GL}(V)$, we are given isomorphisms

$$\eta_{V,I} : T_V(S_\phi) \simeq S_\phi \boxtimes r_V \circ \phi$$

of sheaves in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{BW_{\mathbb{Q}_p}^I}$, that are natural in I and V, and compatible with compositions and exterior tensor products in V. We will similarly say that S_ϕ is a weak eigensheaf with eigenvalue ϕ if we only know the existence of these isomorphisms.

Remark 3.2. We recall that Hecke operators are monoidal and functorial in (V, I). In particular, given two representations $V, W \in \mathrm{Rep}\ ({}^L G)$, we have a natural isomorphism

$$(T_V \boxtimes \mathrm{id})(T_W)(\ )_\Delta \simeq T_{V \boxtimes W}(\ )$$

where $\Delta : \mathrm{Div}^1 \to (\mathrm{Div}^1)^2$ is the diagonal map. The compatibilities for the isomorphisms $\eta_{V,I}$ are defined with respect to such isomorphisms.

Now let's elucidate what this means for tori. Recall that an irreducible representation of ${}^L T^I$ is parametrized by a tuple of Galois orbits $(\nu_i)_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma)^I$. Similarly, one has a decomposition

$$\mathrm{Hck}_T^I = \bigsqcup_{(\nu_i)_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma)^I} \mathrm{Hck}_{T,(\nu_i)_{i \in I}}^I$$

of Hck_T^I into open and closed substacks, where $\mathrm{Hck}_{T,(\nu_i)_{i \in I}}^I$ parametrizes a modification $E_1 \dashrightarrow E_2$ of meromorphy given by the Galois orbit ν_i over the I Cartier divisors in Div^I. If one lets $S_{(\nu_i)_{i \in I}}$ be the sheaf defined by the representation of ${}^L T^I$ corresponding to $(\nu_i)_{i \in I}$, then this sheaf is simply the constant sheaf supported on the component $\mathrm{Hck}_{T,(\nu_i)_{i \in I}}^I$. Therefore, for studying the Hecke operator $T_{(\nu_i)_{i \in I}}$, we can restrict the Hecke correspondence to the diagram:

$$
\begin{array}{ccc}
& \mathrm{Hck}_{T,(\nu_i)_{i \in I}}^I & \\
{}^{h_{(\nu_i)_{i \in I}} \times \pi} \swarrow & & \searrow {}^{h_{(\nu_i)_{i \in I}}^!} \\
\mathrm{Bun}_T \times \mathrm{Div}^I & & \mathrm{Bun}_T
\end{array}
$$

Let E_{ν_i} denote the reflex field of the Γ-orbit $\nu_i \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$. We can consider the following base-change of Div^I:

$$\mathrm{Div}_{E_{(\nu_i)_{i \in I}}}^I := \prod_{i \in I} \mathrm{Div}_{E_{\nu_i}}^1$$

We note that, since a modification of T-bundles is uniquely determined by the locus of meromorphy, we have an isomorphism $\mathrm{Hck}_{T,(\nu_i)_{i \in I}}^I \simeq \mathrm{Bun}_T \times \mathrm{Div}_{E_{(\nu_i)_{i \in I}}}^I$. Under this identification, we have a map

$$h_{(\nu_i)_{i \in I}}^! : \mathrm{Bun}_T \times \mathrm{Div}_{E_{(\nu_i)_{i \in I}}}^I \to \mathrm{Bun}_T$$

where, given $(\mathcal{F}_T, (D_i)_{i \in I})$, we denote the resulting T-bundle under applying this map as $\mathcal{F}_T(\sum_{i \in I} \nu_i D_i)$. We note that the isomorphism class of this bundle is only determined by the image of ν_i in the coinvariant lattice, and this will be important in the next section. The map $h_{(\nu_i)_{i \in I}} \times \pi$ is defined by the natural finite étale morphism $q_{(\nu_i)_{i \in I}} : \mathrm{Div}_{E_{(\nu_i)_{i \in I}}}^I \to \mathrm{Div}^I$, and pushing forward corresponds to inducing the $\prod_{i \in I} W_{E_{\nu_i}}$ action to $\prod_{i \in I} W_{\mathbb{Q}_p}$.

Now, via local class field theory, there is a character $\chi : T(\mathbb{Q}_p) \to \Lambda$ attached to ϕ_T. Moreover, each connected component Bun_T^ν for varying $\nu \in B(T) \simeq \Lambda_{G,B}$ is isomorphic to the classifying

stack $[\ast/T(\mathbb{Q}_p)]$. As a consequence, we may interpret χ as a sheaf on the connected components $j_{\bar{\nu}} : \mathrm{Bun}_T^{\bar{\nu}} \to \mathrm{Bun}_T$ for $\bar{\nu} \in B(T)$. One might hope that considering

$$S_{\phi_T} := \bigoplus_{\bar{\nu} \in B(T)} j_{\bar{\nu}!}(\chi)$$

the sheaf on Bun_T whose restriction to each connected component is equal to χ gives rise to the desired Hecke eigensheaf. This is indeed the case. In particular, via the realization of local class field theory in the torsion of Lubin-Tate formal groups, we have the following proposition.

Proposition 3.3. *[Far20, Section 9.2],[Zou22] The sheaf S_{ϕ_T} is an eigensheaf with eigenvalue ϕ_T. In particular, for all $(\nu_i)_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma)^I$, we have an isomorphism*

$$T_{(\nu_i)_{i \in I}}(S_{\phi_T}) \simeq \boxtimes_{i \in I} \nu_i \circ \phi_T \boxtimes S_{\phi_T}$$

of objects in $\mathrm{D}_{\mathsf{lis}}(\mathrm{Bun}_T, \Lambda)^{BW_{\mathbb{Q}_p}^I}$. More precisely, if $\tilde{\nu}_i$ is a representative of the Γ-orbit of ν_i for all $i \in I$, we have an isomorphism

$$(h^!_{(\nu_i)_{i \in I}})^*(S_{\phi_T}) \simeq \boxtimes_{i \in I} \tilde{\nu}_i \circ \phi_T|_{W_{E_{\nu_i}}} \boxtimes S_{\phi_T}$$

which after applying $q_{(\nu_i)_{i \in I}}$ gives rise to the previous identification of induced representations.

A special role will be played by the eigensheaf attached to the parameter

$$\hat{\rho} \circ j \circ \jmath : W_{\mathbb{Q}_p} \to {}^L T(\Lambda)$$

where we recall that $\hat{\rho}$ denotes the half sum of all positive roots with respect to the choice of Borel and $j \circ \jmath : W_{\mathbb{Q}_p} \to \Lambda^*$ is the norm character. We note that the value of this sheaf on each connected component is given by the representation $\delta_B^{1/2}$, where δ_B denotes the modulus character defined by B. This leads to the following definition.

Definition 3.4. We let $\Delta_B^{1/2}$ be the eigensheaf on Bun_T attached to the parameter

$$\hat{\rho} \circ j \circ \jmath : W_{\mathbb{Q}_p} \to {}^L T(\Lambda)$$

via Proposition 3.3. Similarly, we write Δ_B for the eigensheaf attached to $2\hat{\rho} \circ j \circ \jmath$, where the stalks of this sheaf are given by δ_B.

The key point is that (up to shifts) the pullback of this eigensheaf to the moduli stack Bun_B gives rise to a sheaf which we will denote by $\mathrm{IC}_{\mathrm{Bun}_B}$. We will see later that this sheaf is Verdier self-dual on Bun_B and therefore tensoring by it will give rise to the morally correct definition of the Eisenstein functor. We note that, given a parameter ϕ_T with associated eigensheaf S_{ϕ_T}, the tensor product $S_{\phi_T} \otimes \Delta_B^{1/2}$ will be the eigensheaf attached to the tensor product $\phi_T \otimes \hat{\rho} \circ j \circ \jmath$ of L-parameters (the L-parameter whose value is equal to the product of the parameters). It therefore follows from Proposition 3.3 that the following is true.

Corollary 3.5. *For all $(\nu_i)_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma)^I$, we have an isomorphism*

$$T_{(\nu_i)_{i \in I}}(S_{\phi_T} \otimes \Delta_B^{1/2}) \simeq \boxtimes_{i \in I}(\nu_i \circ \phi_T)(\langle \hat{\rho}, \nu_i \rangle) \boxtimes (S_{\phi_T} \otimes \Delta_B^{1/2})$$

of objects in $\mathrm{D}_{\mathsf{lis}}(\mathrm{Bun}_T, \Lambda)^{BW_{\mathbb{Q}_p}^I}$.

Remark 3.6. Note that, for any representative $\tilde{\nu} \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ of the orbit $\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$, we have an equality: $\langle \hat{\rho}, \tilde{\nu} \rangle = \langle \hat{\rho}, \nu \rangle$.

Now we discuss the various conditions that we will impose on our parameter ϕ_T, as well as discuss their relationship with the irreducibility of principal series through various examples.

3.2. **Genericity, Weak Normalized Regularity, and the Irreducibility of Principal Series.**
Consider the functor

$$R\Gamma(W_{\mathbb{Q}_p}, \ ) : D_{\mathsf{lis}}(\mathrm{Bun}_T, \Lambda)^{BW_{\mathbb{Q}_p}} \to D_{\mathsf{lis}}(\mathrm{Bun}_T, \Lambda)$$

given by taking continuous cohomology with respect to $W_{\mathbb{Q}_p}$. As we will see later, comput-
ing the Eisenstein functor applied to the eigensheaf S_{ϕ_T} will reduce to computing the values of
$R\Gamma(W_{\mathbb{Q}_p}, (h_\nu) \ (S_{\phi_T}))$ and $R\Gamma(W_{\mathbb{Q}_p}, (h_\nu) \ (S_{\phi_T} \ \Delta_B^{1/2}))$ for $\nu \ 2 \ \mathbb{X} \ (T_{\overline{\mathbb{Q}}_p})/\Gamma$. In this note, we will
want to restrict to the simplest case where these contributions all vanish. The exact conditions we
will need are as follows.

Condition/Definition 3.7. *Given a parameter $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\Lambda)$, we impose the following
conditions on ϕ_T in what follows.*
   1. *For all Γ-orbits $\alpha \ 2 \ \mathbb{X} \ (T_{\overline{\mathbb{Q}}_p})/\Gamma$ of simple coroots in $\mathbb{X} \ (T_{\overline{\mathbb{Q}}_p})$, the Galois cohomology complex
      $R\Gamma(W_{\mathbb{Q}_p}, \alpha \ \phi_T)$ is trivial.*
   2. *For all Γ-orbits $\alpha \ 2 \ \mathbb{X} \ (T_{\overline{\mathbb{Q}}_p})/\Gamma$ of coroots in $\mathbb{X} \ (T_{\overline{\mathbb{Q}}_p})$, the Galois cohomology complex
      $R\Gamma(W_{\mathbb{Q}_p}, \alpha \ \phi_T)$ is trivial.*
   3. *If χ is the character attached to ϕ_T under local class field theory. We have, for all $w \neq 1$ in
      the relative Weyl group W_G of G, that*

$$\chi \quad \delta_B \not\cong (\chi \quad \delta_B^{1/2})^w$$

*If ϕ_T satisfies (1) we say that it is weakly generic, and if it satisfies (2) then we say it is generic.
If it satisfies (2)-(3) we say that it is weakly normalized regular.*

These conditions are related to the irreducibility of the induction $i_B^G(\chi)$. To explain this, let's
translate all these conditions to the character χ. Let $E/\mathbb{Q}_p$ denote the splitting field of G. Then
the action of $W_{\mathbb{Q}_p}$ on $\hat{G}$ factors through $W_{\mathbb{Q}_p}/W_E$. Given $\alpha \ 2 \ \mathbb{X} \ (T_{\overline{\mathbb{Q}}_p})/\Gamma$ a Γ-orbit of coroots with
reflex field E_α, local class field theory [Lan97] gives us a map

$$E_\alpha \to T(E_\alpha)$$

attached to $\tilde{\alpha} \ \phi_T j_{W_{E_\alpha}}$, for a representative $\tilde{\alpha}$ in the Γ-orbit of α. If we post-compose with the
norm map $\mathrm{Nm}_{E_\alpha/\mathbb{Q}_p}$ then we get a map

$$E_\alpha \to T(\mathbb{Q}_p)$$

which only depends on the Galois orbit α. We further precompose with the norm map $\mathrm{Nm}_{E/E_\alpha} :
E \to E_\alpha$, giving a character:

$$E \to T(\mathbb{Q}_p)$$

We write $\chi_\alpha : E \to \overline{\mathbb{Q}}_\ell$ for the precomposition of χ with this map. Now, consider the complex
$R\Gamma(W_{\mathbb{Q}_p}, \alpha \ \phi_T)$, where ϕ_T is the parameter attached to χ. It follows by Schapiro's lemma that we
have an isomorphism:

$$R\Gamma(W_{E_\alpha}, \tilde{\alpha} \ \phi_T j_{W_{E_\alpha}}) \ ' \ R\Gamma(W_{\mathbb{Q}_p}, \alpha \ \phi_T)$$

Applying local Tate duality and using that the Euler-Poincaré characteristic of $R\Gamma(W_{E_\alpha}, \tilde{\alpha} \ \phi_T j_{W_{E_\alpha}})$
is 0, we see that this is equivalent to insisting that $\tilde{\alpha} \ \phi_T j_{W_{E_\alpha}}$ is not the trivial representation or
the cyclotomic character (1). From here, by using compatibility of local class field theory with
restriction and the fact that $\tilde{\alpha} \ \phi_T$ is an extension from W_E, we can see that the above conditions
on ϕ_T translate to the following conditions on χ.

Condition/Definition 3.8. *Given a smooth character $\chi : T(\mathbb{Q}_p) \to \Lambda$ consider the following
conditions on χ.*

(1) *For all Γ-orbits $\alpha \in \mathbb{X}_\bullet(T_{\overline{\mathbb{Q}}_p})/\Gamma$ of positive coroots in $\mathbb{X}_\bullet(T_{\overline{\mathbb{Q}}_p})$, the character χ_α is not isomorphic to the trivial representation $\mathbf{1}$ or $|\cdot|_E^{\pm 1}$, where $|\cdot|_E$ is the norm character on E the splitting field of G.*

(2) *For all $w \in W_G$ non-trivial, we have that*
$$\chi \cdot \delta_B^{1/2} \not\cong (\chi \cdot \delta_B^{1/2})^w$$

We say that χ is generic if (1) holds, and that it is weakly normalized regular if (1)-(2) hold.

We now illustrate how this condition is related to irreducibility of $i_B^G(\chi)$ in some examples. We will assume that $\Lambda = \overline{\mathbb{Q}}_\ell$ in all of the examples for simplicity.

Example 3.9. ($G = \mathrm{GL}_2$) We can write $\chi := \chi_1 \otimes \chi_2$ for $\chi_i : \mathbb{Q}_p \to \overline{\mathbb{Q}}_\ell$ smooth characters and $i = 1, 2$. We see that χ being generic implies that $\chi_1\chi_2^{-1} \not\cong \mathbf{1}$ and $\chi_1\chi_2^{-1} \not\cong |\cdot|^{\pm 1}$, and this latter condition guarantees that the normalized parabolic induction $i_B^{\mathsf{GL}_2}(\chi_1 \otimes \chi_2)$ is irreducible. Let's also look at Condition (2) in this case. Suppose it fails, then we have an isomorphism:
$$\chi_1(t_1)\chi_2(t_2)|t_1t_2^{-1}|^{1/2} \cong \chi_1(t_2)\chi_2(t_1)|t_1t_2^{-1}|^{1/2}$$
Evaluating at $(t_1, t_2) = (t, 1)$ this would imply that $\chi_1\chi_2^{-1} \cong \mathbf{1}$, which would contradict χ being generic.

In particular, we see that χ being generic is enough to guarantee irreducibility and weak normalized regularity for GL_2. In fact, this is a more general phenomenon.

Lemma 3.10. *If $G = \mathrm{GL}_n$ then χ being generic implies that it is weakly normalized regular.*

Proof. (Sketch) We write the character as a product $\chi := \chi_1 \otimes \chi_2 \otimes \cdots \otimes \chi_n$ of characters $\chi_i : \mathbb{Q}_p \to \Lambda$, and write $(t_1, \ldots, t_n)$ for the natural coordinates on $T(\mathbb{Q}_p) \cong (\mathbb{Q}_p)^n$. We have an equality:
$$\chi \cdot \delta_B^{1/2} = \prod_{i=1}^n \chi_i(t_i) \cdot |t_i|^{\frac{n-1}{2}-(i-1)}$$
We visualize the coordinates as a set of vertices of a graph:
$$(t_1) \cdot (t_2) \cdot \quad \cdot (t_{n-1}) \cdot (t_n)$$
Then this has an axis of symmetry between $t_{n/2}$ and $t_{n/2+1}$ if n is even, and an axis of symmetry going through $t_{(n+1)/2}$ if n is odd. The element w corresponds to a permutation σ of the vertices of the graph. Now suppose that σ crosses the line of symmetry. If n is odd we assume that it sends $t_{(n-1)/2}$ to $t_{(n+1)/2}$ then by evaluating the equality
$$\chi \cdot \delta_B^{1/2} \cong \chi \cdot (\delta_B^{1/2})^w$$
at $(1, \ldots, 1, t_{(n-1)/2} = t, 1, \ldots, 1)$ it reduces to
$$\chi_{t_{(n-1)/2}}(t)|t| \cong \chi_{t_{(n+1)/2}}(t)$$
which implies
$$\chi_{t_{(n-1)/2}}(t)^{-1}\chi_{t_{(n+1)/2}}(t) \cong |t|$$
contradicting χ being generic. Similarly, if n is even and the permutation sends $t_{n/2}$ to $t_{n/2+1}$ then evaluating at $(1, \ldots, 1, t_{n/2} = t, 1, \ldots, 1)$ the equality becomes
$$\chi_{t_{n/2}}\chi_{t_{n/2+1}}^{-1}(t) \cong \mathbf{1}$$
which again contradicts χ being generic. Here the point is that the power of the norm character appearing when evaluating at $(1, \ldots, 1, t = t_i, 1, \ldots, 1)$ is given by the distance of t_i from the reflection of t_{i+1} across the line of symmetry. In general, we can consider the cycle which is closest to the line of symmetry, suppose it ends in a permutation $t_i \to t_{i+1}$. If we just evaluated at

$(1, \ldots, 1, t = t_i, 1, \ldots, 1)$ then we would get that $\chi_i(t)\chi_{i+1}(t)^{-1}$ is equal to a very large power of the norm character. However, since we choose the permutation to be as close as possible to the axis of symmetry, the permutation will fix all the vertices from the axis of symmetry to t_{i+1} reflected across it. Therefore, we are free to also evaluate at a power of t on these invariant coordinates, since the characters χ_j corresponding to an invariant coordinate t_j on either side of the equation will cancel. This allows us to reduce the power of the norm character appearing to 1 or 0, and then we see that this contradicts χ being generic again.

Remark 3.11. Note that if $\Lambda = \overline{\mathbb{Q}}_\ell$, we could have replaced the second part of the above argument by simply evaluating the permutation σ on one of the coordinates t_i that σ leaves fixed. This would give a relationship of the form $|\cdot|^n = \mathbf{1}$, which is impossible with rational coefficients. With modular coefficients, this is impossible under a banal hypothesis on the prime ℓ.

In general, genericity does not always imply weak normalized regularity. In particular, Condition 3.7 (3) seems to be related to the irreducibility of some unitary principal series representations.

Example 3.12. ($G = \mathrm{SL}_2$) In this case, $\chi : \mathbb{Q}_p \to \overline{\mathbb{Q}}_\ell$ is a character of $\mathbb{Q}_p$. The induction $i_B^G(\chi)$ will be irreducible if and only if $\chi \neq |\cdot|^{-1}$ and $\chi^2 \neq \mathbf{1}$. The condition that $\chi \neq |\cdot|^{-1}$ is guaranteed by χ being generic but the condition $\chi^2 \neq \mathbf{1}$ is not. However, we note that χ being weakly normalized regular enforces the additional condition that $\chi^2 \neq \mathbf{1}$ guaranteeing irreducibility in this case.

We notice in the previous example that the Condition that

$$\chi \cdot \delta_B^{1/2} \neq (\chi \cdot \delta_B^{1/2})^w$$

for $w \in W_G$ non-trivial guaranteed that the character χ was regular in the sense that $\chi \neq \chi^w$ for all $w \in W_G$. This is the type of condition that guarantees irreducibility of the unitary principal series representations. This is always true for $w_0 \in W_G$ the element of longest length.

Lemma 3.13. *If χ is weakly normalized regular then $\chi \neq \chi^{w_0}$ for the element of longest length $w_0 \in W_G$.*

Proof. We note that the relationship

$$\chi \cdot \delta_B^{1/2} = (\chi \cdot \delta_B^{1/2})^w$$

becomes precisely the relationship

$$\chi = \chi^w$$

for $w = w_0$, using the product expansion of the modulus character and the fact that w_0 sends each (reduced) root $\hat\alpha$ to $-\hat\alpha$.

We now come to the last condition on our parameter ϕ_T. In particular, the exact condition we will need is as follows.

Definition 3.14. For a toral parameter $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\Lambda)$ and a geometric dominant cocharacter μ, we say ϕ_T is strongly μ-regular if the Galois cohomology complexes

$$R\Gamma(W_{\mathbb{Q}_p}, (\nu - \nu') \circ \phi_T)$$

are trivial for ν, ν' distinct weights of the highest weight representation of $\hat{G}$ of highest weight μ_k for all $k = 1, \ldots, n$.

To give some flavor for this condition, we prove the following Proposition.

Lemma 3.15. *Suppose that $G = \mathrm{GL}_n$ and let $\mu = (1, \ldots, 0)$ denote the standard character then strong μ-regularity is implied by generic.*

Proof. We note that since μ is minuscule the standard representation V_μ has weights given by Weyl group orbits of the highest weight. From here, it easily follows that the difference of distinct weights of V_μ in $\hat{T}$ are given by coroots of G and the claim follows.

As mentioned in the introduction, weak normalized regularity and μ-regularity for a geometric dominant cocharacter which isn't fixed under any element of W_G, will imply the existence of isomorphisms $i_{\chi,w} : i_B^G(\chi) \xrightarrow{\sim} i_B^G(\chi^w)$ for all $w \in W_G$ once the theory of geometric Eisenstein series has been developed. Similarly, we will show the following.

Proposition 3.16. *(Proposition A.2) If $\chi : T(\mathbb{Q}_p) \to \overline{\mathbb{Q}}_\ell$ is a generic character then, for all $w \in W_G$, we have an isomorphism $i_B^G(\chi) \cong i_B^G(\chi^w)$.*

In fact, for regular characters χ, the existence of such isomorphisms is equivalent to irreducibility. More generally, we show that such isomorphisms exist assuming χ is generic (Proposition A.2), but this does not guarantee irreducibility of certain unitary principal series as seen when $G = \mathrm{SL}_2$.

Using the Langlands classification, the proof of this proposition will essentially reduce to a calculation of reducibility points in rank 1, where it reduces to Example 3.12 and the following example, which illustrates the behavior of our conditions in the non-split case.

Example 3.17. $(G = \mathrm{U}_3/E)$ Let $E/\mathbb{Q}_p$ be a quadratic extension. We write $\overline{(\ )}$ for the non-trivial automorphism of E over $\mathbb{Q}_p$. If e_1, e_2, e_3 is the standard basis for the cocharacter lattice $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ then $\overline{(\ )}$ acts by

$$e_1 \mapsto e_3$$
$$e_2 \mapsto e_2$$

It follows that the simple coroot $\alpha_1 := e_1 - e_2$ is sent to the simple coroot $\alpha_2 := e_2 - e_3$ under $\overline{(\ )}$. Thus, the Γ-orbits of positive coroots in $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ are given by $\{\alpha_1, \alpha_2\}$ with reflex field E and $\alpha_1 + \alpha_2$ with reflex field $\mathbb{Q}_p$. Now recall that the maximal torus $T(\mathbb{Q}_p) \subset \mathrm{U}_3(\mathbb{Q}_p)$ is isomorphic to $E \times E^1$, via the embedding

$$E \times E^1 \to \mathrm{U}_3(\mathbb{Q}_p)$$
$$t \mapsto \begin{pmatrix} t & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & \bar{t}^{-1} \end{pmatrix}$$

where E^1 denotes the set of norm 1 elements. Then if we write the character $\chi(t,s) : E \times E^1 \to \Lambda$ as $\chi(t,s) = \chi_1(t)\chi_2(ts\bar{t}^{-1})$ the reducibility of $i_B^G(\chi)$ depends solely on χ_1, as in [Rog90, Page 173], where here it reduces to the analogous question for SU_3, and there the reducibility points were studied in [Key84, Section 7]. The induction $i_B^G(\chi)$ is reducible if and only if one of the following hold:

(1) $\chi_1 = \eta| \cdot |_E^{1/2}$, where $\eta|_{\mathbb{Q}_p} = \eta_{E/\mathbb{Q}_p}$,
(2) $\chi_1 = | \cdot |_E^{1}$,
(3) $\chi_1|_{\mathbb{Q}_p}$ is trivial, but χ is not.

Here $| \cdot |_E$ is the norm character of E which is in particular the splitting field of G, and $\eta_{E/\mathbb{Q}_p} : \mathbb{Q}_p \to \Lambda$ is the unique quadratic character with kernel given by $\mathrm{Nm}_{E/\mathbb{Q}_p}(E^\times)$. We note that that the cocharacters of $T(\mathbb{Q}_p)$ given by the Γ-orbits of positive roots are

$$\{\alpha_1, \alpha_2\} : E \to T(\mathbb{Q}_p) = E \times E^1$$
$$t \mapsto (t, t^{-1}\bar{t})$$

and

$$\{\alpha_1 + \alpha_2\} : E \to E \times E^1$$
$$t \mapsto (\mathrm{Nm}_{E/\mathbb{Q}_p}(t), 1)$$

By precomposing χ with the first character, we see that χ being generic implies $\chi_1 \not\cong 1$ and $\chi_1 \not\cong j \ j_E^1$, which implies reducibility point (2) cannot occur. By precomposing χ with the second character, we see that χ being generic implies that $\chi(\mathrm{Nm}_{E/\mathbb{Q}_p}(t)) \cong \chi_1(\mathrm{Nm}_{E/\mathbb{Q}_p}(t)) \not\cong 1$ and $\chi_1(\mathrm{Nm}_{E/\mathbb{Q}_p}(t)) \not\cong j \ j_E^1$. Note that if $\chi \cong \eta j \ j_E^{1/2}$ then we have, for all $t \in E$, an isomorphism:

$$\chi(\mathrm{Nm}_{E/\mathbb{Q}_p}(t)) = \chi_1(t\bar{t}) \cong \eta(\mathrm{Nm}_{E/\mathbb{Q}_p}(t)) j \mathrm{Nm}_{E/\mathbb{Q}_p}(t) j_E^{1/2} \cong j t j_E^1$$

Summarizing, we see again that χ being generic guarantees irreducibility of the two non-unitary inductions. Now, if $\chi j_{\mathbb{Q}_p}$ is trivial then we have that

$$\chi(\mathrm{Nm}_{E/\mathbb{Q}_p}(t)) = \chi_1(t\bar{t}) \cong 1$$

Thus, we see χ being generic guarantees the irreducibility of all principal series. Moreover, χ being weakly normalized regular enforces the additional constraint that

$$\chi(t\bar{t}) \not\cong 1$$

which we just saw follows from χ being generic, so weak normalized regularity follows from generic in this case.

The connection between genericity and irreducibility of non-unitary principal series fits in nicely with the general Langlands philosophy. In particular, since we are inducing from a Borel, we expect that a parameter should have monodromy if it arises as a constituent of the reducible induction of a non-unitary character. We saw in the above examples that this shouldn't occur when the parameter ϕ_T attached to χ is generic. We can analyze when a parameter comes from the semi-simplification of a parameter with non-trivial monodromy and relate this to genericity.

Lemma 3.18. *We let $\phi : L_{\mathbb{Q}_p} \to {}^L G(\overline{\mathbb{Q}}_\ell)$ be an L-parameter. Suppose that $\phi : L_{\mathbb{Q}_p} \to {}^L G(\overline{\mathbb{Q}}_\ell)$ has non-trivial monodromy, and that the semi-simplification ϕ^{ss} (See Assumption 7.5) factors through ${}^L T \to {}^L G$ via the natural embedding. If we write $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\overline{\mathbb{Q}}_\ell)$ for the parameter induced by ϕ^{ss} then ϕ_T is not generic.*

Proof. If ϕ has non-trivial monodromy and the semi-simplification factors through ${}^L T$, there exists a lift

$$
\begin{array}{ccc}
 & & {}^L B(\overline{\mathbb{Q}}_\ell) \\
 & \overset{\tilde{\phi}}{\nearrow} & \downarrow \\
W_{\mathbb{Q}_p} & \xrightarrow{\ \phi_T\ } & {}^L T(\overline{\mathbb{Q}}_\ell)
\end{array}
$$

of ϕ_T, which is not given by the inclusion ${}^L T(\overline{\mathbb{Q}}_\ell) \to {}^L B(\overline{\mathbb{Q}}_\ell)$. Therefore, there exists some Galois orbit of positive coroots $\alpha \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$ such that $\alpha \circ \tilde{\phi} = \alpha \circ \phi_T$ is non-trivial. This implies that there exists a non-trivial class in the H^0 of $R\Gamma(W_{\mathbb{Q}_p}, \alpha \circ \phi_T)$, which would in turn imply ϕ_T is not generic. $\qquad$

We finish this section by deducing a geometric consequence of weak genericity that will be useful for studying how Eisenstein series interact with Verdier duality.

3.3. A Geometric Consequence of Weak Genericity.
We work with torsion coefficents $\Lambda = \overline{\mathbb{F}}_\ell$. Consider the following easy lemma.

Lemma 3.19. *Assume ϕ_T is weakly generic then it follows that, for all the Γ-orbits of simple positive coroots α, the complex $R\Gamma(W_{\mathbb{Q}_p}, \alpha \circ \phi_T(\hbar \hat{\rho}, \alpha \ i))$ is trivial.*

Proof. We note that, since α is a Γ-orbit of simple positive coroots, we have that $\langle \hat{\rho}, \alpha \rangle = 1$; therefore, we want to check that
$$R\Gamma(W_{\mathbb{Q}_p}, (\alpha \circ \phi_T)(1))$$
is trivial. By using local Tate-duality and that the Euler-Poincaré characteristic of this complex is 0, this is equivalent to checking that $(\alpha \circ \phi_T)(1)$ is not the trivial representation or the cyclotomic twist by (-1), which is equivalent to insisting that $(\alpha \circ \phi_T)^{-1}$ is not the trivial representation or the cyclotomic twist by (1). This is the same condition guaranteeing that the complex
$$R\Gamma(W_{\mathbb{Q}_p}, (\alpha \circ \phi_T)^{-1})$$
is trivial, which follows from weak genericity applied to the Γ-orbit of negative simple coroots $-\alpha$.

This has an important geometric consequence, related to the vanishing of certain Galois cohomology groups appearing in the moduli space of B-bundles and its compactifications. To explain this, let $\bar{\nu}$ be an element of $\Lambda_{G,B}^{pos} \cap \{0\}$. We can write this as a linear combination $\sum_{i \in J} n_i \alpha_i$ for positive integers n_i, where the α_i correspond to the Galois orbits of simple absolute roots as in Definition 2.14. Given such an α_i, we can consider the reflex field E_i of the associated Galois orbit, and define the following partially symmetrized curve:
$$\mathrm{Div}^{(\bar{\nu})} := \prod_{i \in J} \mathrm{Div}_{E_i}^{(n_i)}$$

Points of this curve correspond to tuples of Cartier divisors D_i over E_i of degree n_i for all $i \in J$. We can consider the map
$$h_{(\bar{\nu})}^{\rightarrow} : \mathrm{Bun}_T \times \mathrm{Div}^{(\bar{\nu})} \to \mathrm{Bun}_T$$
given by sending $(\mathcal{F}_T, (D_i)_{i \in J})$ to $\mathcal{F}_T(\sum \alpha_i \cdot D_i)$, where we are identifying α_i with its corresponding Γ-orbit.

This partially symmetrized mirror curve $\mathrm{Div}^{(\bar{\nu})}$ behaves a bit strangely if G is not split. To illustrate this, consider the following example.

Example 3.20. Let $G = U_3$ be a unitary group in 3 variables attached to a quadratic extension $E/\mathbb{Q}_p$ and write α_1 and α_2 for the two absolute positive simple roots. We recall, as in Example 3.17, that the Galois group exchanges α_1 and α_2. Therefore, they both map to a unique element $\alpha \in \mathbb{X}_*(T_{\overline{\mathbb{Q}_p}})$ which spans the lattice $\Lambda_{G,B}^{pos}$ by Definition. Consider the element $\bar{\nu} := 2\alpha \in \mathbb{X}_*(T_{\overline{\mathbb{Q}_p}})$. We note that we have an equality $\mathrm{Div}^{(\bar{\nu})} = \mathrm{Div}_E^{(2)}$ in this case. The pre-image of 2α under the natural map $(-) : \mathbb{X}_*(T_{\overline{\mathbb{Q}_p}})/\Gamma \to \mathbb{X}_*(T_{\overline{\mathbb{Q}_p}})$ consists of two elements: the Γ-orbit $\{2\alpha_1, 2\alpha_2\}$ with reflex field E and the Γ-orbit of $\{\alpha_1 + \alpha_2\}$ with reflex field $\mathbb{Q}_p$. We saw in the previous section that the space of modifications defined by the Γ-orbit $\{2\alpha_1, 2\alpha_2\}$ is given by $\mathrm{Bun}_T \times \mathrm{Div}_E^1$, correspondingly we have a natural map
$$\Delta_{\{2\alpha_1, 2\alpha_2\}} : \mathrm{Div}_E^1 \hookrightarrow \mathrm{Div}_E^2 \to \mathrm{Div}_E^{(2)}$$
given by the diagonal embedding composed with the quotient map. It is easy to check we have an equality $h_{\{2\alpha_1, 2\alpha_2\}}^{\rightarrow}(-) = h_{\bar{\nu}}^{\rightarrow} \circ (\mathrm{id} \times \Delta_{\{2\alpha_1, 2\alpha_2\}})$. Perhaps more interestingly, attached to the Γ-orbit $\{\alpha_1 + \alpha_2\}$, we have a twisted diagonal map
$$\Delta_{\{\alpha_1 + \alpha_2\}} : \mathrm{Div}_{\mathbb{Q}_p}^1 \to \mathrm{Div}_E^{(2)}$$
given by sending a Cartier divisor D to its pre-image under the natural finite-étale covering $X_{S,E} \to X_S$ of Fargues-Fontaine curves induced by the extension $E/\mathbb{Q}_p$. By [Li-22, Proposition 1.5], this map defines a closed embedding whose image lies in the complement of the image of $\Delta_{\{2\alpha_1, 2\alpha_2\}}$ in $\mathrm{Div}_E^{(2)}$, and we similarly see that we have a relationship $h_{(\bar{\nu})}^{\rightarrow} \circ (\mathrm{id} \times \Delta_{\{\alpha_1 + \alpha_2\}})(-) = h_{\{\alpha_1 + \alpha_2\}}^{\rightarrow}(-)$.

The previous example illustrates that we can can understand $\mathrm{Bun}_T \times \mathrm{Div}^{(\overline{\nu})}$ as realizing the set of all modifications specified by Galois orbits $\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})/\Gamma$ such that $|\nu| = \overline{\nu}$. This is indeed a general phenomenon. In particular, using that Γ permutes the simple absolute coroots of G, given any such ν with reflex field E_ν, we can define a twisted diagonal embedding

$$\Delta_\nu : \mathrm{Div}^1_{E_\nu} \to \mathrm{Div}^{(\overline{\nu})}$$

such that we have a relationship $h^!_\nu(-) := h^!_{\overline{\nu}}\circ (\mathrm{id} \times \Delta_\nu)(-)$. If $\overline{\nu} = \alpha_i$ for $i \in J$ the map Δ_ν is an isomorphism for the unique Γ-orbit of simple coroots corresponding to α_i. Therefore, the pullback of S_{ϕ_T} along $h_{(\overline{\nu})}$ is isomorphic to $S_{\phi_T \times \tilde{\alpha}_i \circ \phi_T}$ for a choice of representative $\tilde{\alpha}_i$ of the Γ-orbit corresponding to α_i. In general, recall that given a local system $\mathbb{L}$ and n a positive integer, we can consider the symmetric powers

$$\mathbb{L}^{(n)} := \pi_* (\boxtimes_{i=1}^n \mathbb{L})^{S_n}$$

where π_* denotes the push-forward along the S_n-torsor:

$$\pi : (\mathrm{Div}^1)^n \to \mathrm{Div}^{(n)}$$

Using this, we can define a local system on $\mathrm{Div}^{(\overline{\nu})}$ given by

$$E^{(\overline{\nu})}_{\phi_T} := \boxtimes_{i \in J} E^{(n_i)}_{\phi_i}$$

where ϕ_i is the local system on $\mathrm{Div}^1_{E_i}$ corresponding to the character $\tilde{\alpha}_i \circ \phi_T|_{W_{E_i}}$ of W_{E_i} for E_i the reflex field of the Galois orbit corresponding to α_i. With this in hand, we can describe the pullback of S_{ϕ_T} along $h_{(\overline{\nu})}$ as the sheaf $E^{(\overline{\nu})}_{\phi_T} \otimes S_{\phi_T}$ by using that Hecke operators are monoidal, and the natural compatibilities of the eigensheaf. We now state a key vanishing result that will be important for studying how geometric Eisenstein series interact with Verdier duality.

Lemma 3.21. *If ϕ_T is weakly generic then, for all $\overline{\nu} \in \Lambda^{pos}_{G,B} \setminus \{0\}$, the complexes*

$$R\Gamma_c(\mathrm{Div}^{(\overline{\nu})}, E^{(\overline{\nu})}_{\phi_T}(\langle \hat{\rho}, \overline{\nu}\rangle))$$

and

$$R\Gamma_c(\mathrm{Div}^{(\overline{\nu})}, E^{(\overline{\nu})}_{\phi_T})$$

are trivial.

Proof. By Künneth formula, this easily reduces to showing that $R\Gamma_c(\mathrm{Div}^{(n_i)}, E^{(n_i)}_{\phi_i}(n_i))$ is trivial for all $i \in J$ and $n_i \in \mathbb{N}_{>0}$. However, $E^{(n_i)}_{\phi_i}(n_i)$ is given by taking the n_ith symmetric power of $E_{\phi_i}(1)$. Therefore, by Künneth formula again, this reduces us to showing that

$$R\Gamma(W_{\mathbb{Q}_p}, \alpha_i \circ \phi_T(1)) \simeq R\Gamma_c(W_{E_i}, \tilde{\alpha}_i \circ \phi_T|_{W_{E_i}}(1)) \simeq R\Gamma_c(\mathrm{Div}^1_{E_i}, E_{\phi_i}(1))$$

vanishes, where the first isomorphism follows from Schapiro's lemma and the second isomorphism follows from the correspondence between Λ-valued local systems on $\mathrm{Div}^1_{E_i}$ and representations of W_{E_i} [FS21, Proposition VI.9.2]. The vanishing of the LHS follows from weak genericity and Lemma 3.19. The second vanishing statement follows from the same argument and weak genericity.

We will now review the next ingredient in our calculations of geometric Eisenstein series, the Geometric Satake correspondence.

4. Geometric Satake and the Affine Grassmannian

4.1. **The Geometric Satake Correspondence.** We will now recall some facts about the geometric Satake correspondence for the B_{dR}^+ Grassmannian, as proven in [FS21, Chapter VI]. For any finite set I, we consider the Hecke stack

$$\pi_G : \mathrm{Hck}_G^I \to \mathrm{Div}^I$$

as in §3. For a point $S \to \mathrm{Div}^I$, we can consider the completion of the structure sheaf $\mathcal{O}_{X_S}$ at the union of the I Cartier divisors in X_S defined by S. This defines a ring which we denote by B^+, and inverting D, we get a ring which we denote by B. The mapping sending $S \in \mathrm{Div}^I$ to $G(B^+)$ and $G(B)$ defines étale sheaves on Perf, which we denote by $L_{\mathrm{Div}^I}^+ G$ and $L_{\mathrm{Div}^I} G$, respectively. We note that, for $I = \{*\}$ and $S = \mathrm{Spa}(F, \mathcal{O}_F) \to \mathrm{Div}^I$ a geometric point with associated untilt $(C, \mathcal{O}_C)$, we have $B^+ = B_{dR}^+(C, \mathcal{O}_C)$ and $B = B_{dR}(C, \mathcal{O}_C)$, the usual de-Rham period rings attached to the untilt. For simplicity, we will often just drop the subscript Div^I and just write $L^+ G$ and LG for these étale sheaves. By [SW20, Proposition 19.1.2], the Hecke stack can be described as the quotient:

$$[L^+ G \backslash LG / L^+ G] \to \mathrm{Div}^I$$

To study this, we can uniformize this by the quotient

$$\mathrm{Gr}_G^I := LG / L^+ G \to \mathrm{Div}^I$$

which is the Fargues-Fontaine analogue of the Beilinson-Drinfeld Grassmannian. It follows by the results of [SW20, Lecture XX] that the Beilinson-Drinfeld Grassmannian is a well-behaved geometric object; it can be written as a closed union of subsheaves that are proper and representable in spatial diamonds over Div^I, given by bounding the meromorphy of the modifications. We can consider the category

$$\mathrm{D}(\mathrm{Hck}_G^I)^{bd}$$

of objects with support in one of the aforementioned quasi-compact subsets. This carries a monoidal structure coming from the diagram:

$$\mathrm{Hck}_G^I \times_{\mathrm{Div}^I} \mathrm{Hck}_G^I \xleftarrow{(p_1, p_2)} L^+ G \backslash LG \times^{L^+ G} LG / L^+ G \xrightarrow{m} \mathrm{Hck}_G^I$$

Here the middle space can be interpreted as parameterizing a pair of G-bundles $\mathcal{E}_1, \mathcal{E}_2$ together with a pair of modifications $\beta_1 : \mathcal{E}_1 \dashrightarrow \mathcal{E}_0$ and $\beta_2 : \mathcal{E}_0 \dashrightarrow \mathcal{E}_2$ to/from the trivial bundle with the same locus of meromorphy. The maps p_i are the natural projections remembering only the data $(\mathcal{E}_i, \beta_i)$ for $i = 1, 2$, and m is defined by sending $(\mathcal{E}_i, \beta_i)_{i=1,2}$ to $\beta_2 \circ \beta_1 : \mathcal{E}_1 \dashrightarrow \mathcal{E}_2$. Given $A, B \in \mathrm{D}(\mathrm{Hck}_G^I)^{bd}$, we define

$$A \star B := Rm_* (p_1^*(A) \otimes p_2^*(B)) \in \mathrm{D}(\mathrm{Hck}_G^I)^{bd}$$

the convolution of A and B [FS21, Section VI.8]. Since the map m is a fibration in Gr_G^I it is proper over any quasi-compact subset, so by proper base-change it gives a well-defined associative monoidal structure. One can further refine our category of sheaves as follows. In particular, we note that the locus of Hck_G^I where the meromorphy is equal to the Galois orbit of $\mu_i \in \mathbb{X}_* (T_{\overline{\mathbb{Q}}_p})^+$ at the ith Cartier divisor for $i \in I$ is uniformized by a cohomologically smooth diamond of relative dimension $\sum_{i \in I} \langle 2\hat{\rho}, \mu_i \rangle$ over Div^I by [FS21, Proposition VI.2.4]. One can check that this gives rise to a well-defined perverse t-structure on $\mathrm{D}(\mathrm{Hck}_G^I)^{bd}$ over Div^I given by insisting that !-restriction (resp. $*$-restriction) to these strata sit in cohomological degrees $\leq$ (resp. $\geq$) $\sum_{i \in I} \langle 2\hat{\rho}, \mu_i \rangle$, and that convolution preserves perversity [FS21, Proposition VI.8.1]. With this in hand, we arrive at the key definition.

Definition 4.1. [FS21, Definition I.6.2] We define the Satake category

$$\mathrm{Sat}_G^I \subset \mathrm{D}(\mathrm{Hck}_G^I)^{bd}$$

as the category of all $A \in D(\mathrm{Hck}_G^I)^{bd}$ which are perverse, flat (i.e for all Λ-modules M $A \otimes M$ is also perverse), and ULA over Div^I, as defined in [FS21, Chapter V.7].

The ULA and flatness property in the above definition has the key consequence that the pullback of $A \in \mathrm{Sat}_G^I$ to Gr_G^I composed with the push-forward to Div^I has cohomology sheaves valued in Λ-valued local systems on Div^I. In particular, using an analogue of Drinfeld's lemma [FS21, Proposition VI.9.2], this gives rise to a fiber functor

$$F_G^I : \mathrm{Sat}_G^I \to \mathrm{Rep}_{W_{\mathbb{Q}_p}^I}(\Lambda)$$

$$A \mapsto \bigoplus_i H^i(R\pi_{G^I}(A))$$

where $\mathrm{Rep}_{W_{\mathbb{Q}_p}^I}(\Lambda)$ denotes the category of continuous representations of $W_{\mathbb{Q}_p}^I$ on finite projective Λ-modules, and $R\pi_{G^I}$ is the functor given by pulling back to Gr_G^I and taking the push-forward to Div^I, as in [FS21, Definition/Proposition VI.7.10]. Now, by using the factorization structure on these Grassmannians, one can also construct an analogue of the fusion product [FS21, Section VI.9]. Let us recall its construction. We consider a partition $I = I_1 \sqcup \ldots \sqcup I_k$ of this index set. We consider the open subspace

$$\mathrm{Div}^{I;I_1,\ldots,I_k} \subseteq \mathrm{Div}^I$$

parameterizing a tuple of Cartier divisors $(D_i)_{i \in I}$ such that D_i and $D_{i'}$ are disjoint whenever $i, i' \in I = I_1 \sqcup \ldots \sqcup I_k$ lie in different I_js, and let $\mathrm{Hck}_G^{I;I_1,\ldots,I_k}$ be the base-change of Hck_G^I to this open subspace. We can define the subcategory $\mathrm{Sat}_G^{I;I_1,\ldots,I_k} \subseteq D(\mathrm{Hck}_G^{I;I_1,\ldots,I_k})^{bd}$ in an analogous manner to Definition 4.1. We have a natural restriction map

$$\mathrm{Sat}_G^I \to \mathrm{Sat}_G^{I;I_1,\ldots,I_k}$$

which, by [FS21, Proposition VI.9.3], defines a fully faithful embedding. There is also an identification

$$\mathrm{Hck}_G^I \times_{\mathrm{Div}^I} \mathrm{Div}^{I;I_1,\ldots,I_k} \simeq \prod_{j=1}^k \mathrm{Hck}_G^{I_j} \times_{\prod_j \mathrm{Div}^{I_j}} \mathrm{Div}^{I;I_1,\ldots,I_k}$$

giving a natural map

$$\mathrm{Sat}_G^{I_1} \boxtimes \ldots \boxtimes \mathrm{Sat}_G^{I_k} \to \mathrm{Sat}_G^{I;I_1,\ldots,I_k}$$

via taking exterior products. Then, by [FS21, Definition/Proposition VI.9.4], this lies in the full subcategory Sat_G^I, and the resulting map

$$\mathrm{Sat}_G^{I_1} \boxtimes \ldots \boxtimes \mathrm{Sat}_G^{I_k} \to \mathrm{Sat}_G^I$$

is called the fusion product. Now consider the composite

$$\mathrm{Sat}_G^I \boxtimes \mathrm{Sat}_G^I \to \mathrm{Sat}_G^{I \sqcup I} \to \mathrm{Sat}_G^I$$

where the first map is the fusion product, and the last map is given by restricting along the diagonal embedding $\mathrm{Hck}^I \to \mathrm{Hck}^{I \sqcup I}$. Then this is naturally isomorphic to the convolution product. By this comparison between fusion and convolution, one can deduce that the convolution product is in fact symmetric monoidal, and that the functor F_G^I takes this monoidal structure to the usual tensor product on $\mathrm{Rep}_{W_{\mathbb{Q}_p}^I}(\Lambda)$. Now, using Tannakian duality, one deduces the following.

Theorem 4.2. *[FS21, Theorem I.6.3] For a finite index set I, the category Sat_G^I is naturally in I identified with $\mathrm{Rep}(^L G^I)$ the category of representations of $^L G^I$ on finite projective Λ-modules.*

Remark 4.3. One needs to be a bit careful here. In particular, $^L G$ as defined here differs from the usual definition of the L-group; namely, the usual action of $W_{\mathbb{Q}_p}$ on $\hat{G}$ is twisted by a cyclotomic character (See [FS21, Section VI.11] for more details).

One of the key points that also plays an important role in the proof of this theorem is that this construction respects the natural map

$$\mathrm{res}_T^{I,G} : \mathrm{Rep}\ (^LG^I) \to \mathrm{Rep}\ (^LT^I)$$

given by restricting a representation of $^LG^I$ to $^LT^I$ along the natural embedding:

$$^LT^I \to {}^LG^I$$

To explain this, we need to explain how this functor is realized in the geometry of Beilinson-Drinfeld Grassmannians. First note that, as in §3, we have an identification

$$\mathrm{Gr}_T^I = \bigsqcup_{(\nu_i)_{i\in I}\in 2(\mathbb{X}\ (T_{\overline{\mathbb{Q}}_p})/\ )^I} \mathrm{Gr}_{T,(\nu_i)_{i\in I}}^I \simeq \bigsqcup_{(\nu_i)_{i\in I}\in 2(\mathbb{X}\ (T_{\overline{\mathbb{Q}}_p})/\ )^I} \mathrm{Div}_{E_{(\nu_i)_{i\in I}}}^I$$

where $\mathrm{Gr}_{T,(\nu_i)_{i\in I}}$ parametrizes modifications of T-bundles with meromorphy equal to the Galois orbit defined by ν_i at a Cartier divisor D_i for all $i \in I$, and $\mathrm{Div}^I := \prod_{i\in I} \mathrm{Div}_{E_{\nu_i}}^1$, where E_{ν_i} denotes the reflex field of ν_i. In particular, we see that Theorem 4.2 is trivial in this case, as F_T^I will induce an equivalence between Sat_T^I and $(\mathbb{X}\ (T_{\overline{\mathbb{Q}}_p})/\Gamma)^I$-graded objects in $\mathrm{Rep}\ (W_{\mathbb{Q}_p}^I)$ under this isomorphism. We consider the diagram

$$\begin{array}{ccc} & \mathrm{Gr}_B^I & \\ & \swarrow{}_p \quad {}^q\searrow & \\ \mathrm{Gr}_T^I & & \mathrm{Gr}_G^I \end{array}$$

and define the functor:

$$p_!q\ : \mathrm{D}(\mathrm{Gr}_G^I)^{bd} \to \mathrm{D}(\mathrm{Gr}_T^I)^{bd}$$

As we will discuss further in the next section, the fibers of the morphism p over the connected components $\mathrm{Gr}_{T,(\nu_i)_{i\in I}}^I$ give rise to a locally closed stratification of Gr_G^I which embed via the morphism p (cf. [FS21, Example VI.3.4]). These are the so called semi-infinite Schubert cells. If one considers $\mathbb{G}_m$ acting on Gr_G^I via a suitably chosen cocharacter μ composed with the L^+G action on Gr_G^I then one can identify these semi-infinite cells with a union $\mathbb{G}_m$-orbits (the attractor of the $\mathbb{G}_m$-action), and the fixed points will be precisely the connected components:

$$\mathrm{Gr}_T^I = \bigsqcup_{(\nu_i)_{i\in I}\in 2(\mathbb{X}\ (T_{\overline{\mathbb{Q}}_p})/\ )^I} \mathrm{Gr}_{T,(\nu_i)_{i\in I}}^I$$

This allows one to apply a diamond analogue [FS21, Theorem IV.6.5] of Braden's hyperbolic localization theorem [Bra03]. In particular, since sheaves in Sat_G^I pullback to L^+G-equivariant sheaves on Gr_G^I, they will be $\mathbb{G}_m$-equivariant. From this, one can deduce that $p_!q$ is a hyperbolic localization with respect to this $\mathbb{G}_m$-action and will therefore preserve perverse, flat, and ULA objects over Div^I. We therefore get an induced functor

$$\mathrm{CT} : \mathrm{Sat}_G^I \to \mathrm{Sat}_T^I$$

called the constant term functor, as in [FS21, Proposition VI.7.13]. We now consider the following function

$$\deg : j\mathrm{Gr}_T^Ij \to \mathbb{Z}$$

which has value $h2\hat{\rho}, j(\nu_i)_{i\in I}ji$ on the connected component indexed by $(\nu_i)_{i\in I}$, where $j(\nu_i)_{i\in I}j := \sum_{i\in I} \overline{\nu}_i \in \Lambda_{G,B}$. Now, by applying excision with respect to the stratatification by semi-infinite cells one can show that, for all $A \in \mathrm{Sat}_G$, one has an isomorphism $\bigoplus_i H^i(\pi_G\ (A)) \simeq H^0(\pi_T\ (\mathrm{CT}(A)[\deg]))$ (See the proof of [FS21, Proposition VI.7.10]). This in particular gives us the following Proposition.

Proposition 4.4. *For all finite index sets I, there exists a commutative diagram*

$$
\begin{array}{ccc}
\mathrm{Sat}_G^I & \xrightarrow{\mathrm{CT[deg]}} & \mathrm{Sat}_T^I \\
\downarrow{\scriptstyle F_G^I} & & \downarrow{\scriptstyle F_T^I} \\
\mathrm{Rep}\ (^L G^I) & \xrightarrow{\mathrm{res}_T^{I,G}} & \mathrm{Rep}\ (^L T^I)
\end{array}
$$

where here F_G^I (resp. F_T^I) is the equivalence given by Theorem 4.2 for G (resp. T).

Moreover, it follows by [FS21, Proposition VI.9.6], that the fusion product respects the constant term functors.

Proposition 4.5. *For all finite index sets I, with a partition $I_1 \sqcup \ldots \sqcup I_k$, we have a commutative diagram*

$$
\begin{array}{ccccc}
\mathrm{Sat}_G^{I_1} & \ldots & \mathrm{Sat}_G^{I_k} & \longrightarrow & \mathrm{Sat}_G^I \\
& \downarrow{\scriptstyle \mathrm{CT}^{I_1}[deg]} & {\scriptstyle \ldots}\ {\scriptstyle \mathrm{CT}^{I_k}[deg]} & \downarrow{\scriptstyle \mathrm{CT}^I[deg]} & \\
\mathrm{Sat}_T^{I_1} & \ldots & \mathrm{Sat}_T^{I_k} & \longrightarrow & \mathrm{Sat}_T^I
\end{array}
$$

which commutes functorially in I, where the top (resp. bottom) vertical arrow is given by the fusion product for G (resp. T).

We now turn our attention to the semi-infinite cells.

4.2. **The Cohomology of Semi-Infinite Cells.** Theorem 4.2, Proposition 4.4, and Proposition 4.5 have implications for the cohomology of spaces related to moduli spaces of B-structures. We will record this now. We let $E/\mathbb{Q}_p$ be the splitting field of G. Then we have an identification $\mathrm{Gr}_{G,E}^I \simeq \mathrm{Gr}_{G_E}^I$ over the base change Div_E^I of Div^I to E. Sheaves on this space will be equivalent algebraic representations of I copies of the dual group $\hat{G}$. First, we recall that we have the following natural stratification of $\mathrm{Gr}_{G,E}^I$.

Definition 4.6. For $(\lambda_i)_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+)^I$, we let $\mathrm{Gr}_{G,(\lambda_i)_{i \in I},E}^I$ (resp. $\mathrm{Gr}_{G,\leq (\lambda_i)_{i \in I},E}^I$) be the locally closed (resp. closed) subfunctor of Gr_G parameterizing modifications

$$
\mathcal{F}_G^0 \dashrightarrow \mathcal{F}_G
$$

between the trivial G-bundle $\mathcal{F}_G^0$ and a G-bundle $\mathcal{F}_G$ of meromorphy equal to (resp. less than) then $\sum_{D_i = D_j} \lambda_i$ at a Cartier divisor D_j, for some fixed $j \in I$.

As mentioned in the previous section, $\mathrm{Gr}_{G,(\lambda_i)_{i \in I},E}^I$ is representable in nice diamonds and is cohomologically smooth of dimension equal to $\sum_{i \in I} \langle 2\hat{\rho}, \lambda_i \rangle$ over Div_E^I, by [FS21, Proposition VI.2.4]. Similarly, by [SW20, Proposition 19.2.3], $\mathrm{Gr}_{G,\leq (\lambda_i)_{i \in I},E}$ is representable in nice diamonds and proper over Div_E^I. Fix $\boxtimes_{i \in I} V_i = V \in \mathrm{Rep}\ (\hat{G}^I)$ with fixed central character, and suppose the highest weight of V_i is given by λ_i for $\lambda_i \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$. Attached to this, we get a Λ-valued perverse sheaf S_V supported on $\mathrm{Gr}_{G,\leq (\lambda_i)_{i \in I},E}^I$ by Theorem 4.2 and [FS21, Proposition VI.7.5]. We now fix a geometric point $x \to \mathrm{Div}_E^I$. In what follows, for a space $?$ over Div_E^I we write $_x?$ for the base-change to x. Since a local system on Div_E^I will be determined by the W_E^I-representation given by its pullback to this geometric point, looking at this pullback will be sufficient for most calculations. We now consider another stratification of $\mathrm{Gr}_{G,E}^I$.

Definition 4.7. Consider the natural map:

$$
p : \mathrm{Gr}_{B,E}^I \to \mathrm{Gr}_{T,E}^I
$$

For $(\nu_i)_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I$, we set $\mathrm{S}^I_{G,(\nu_i)_{i \in I},E}$ to be the fiber of $\mathrm{Gr}^I_{B,E}$ over the connected component $\mathrm{Gr}^I_{T,(\nu_i)_{i \in I},E}$ in $\mathrm{Gr}^I_{T,E}$ parameterizing modifications of type $(\nu_i)_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I$. We note, by [FS21, Proposition VI.3.1], that the natural map

$$q : \mathrm{Gr}^I_{B,E} \to \mathrm{Gr}^I_{G,E}$$

is a bijection on geometric points and it defines a locally closed embedding on the $\mathrm{S}^I_{G,(\nu_i)_{i \in I},E}$. Therefore, the spaces $\mathrm{S}^I_{G,(\nu_i)_{i \in I},E}$ for varying $(\nu_i)_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I$, form a locally-closed stratification of $\mathrm{Gr}^I_{G,E}$.

Remark 4.8. In particular, given a modification $\beta : \mathcal{F}^0_G \dashrightarrow \mathcal{F}_G$ of G-bundles, by the Tannakian formalism this is equivalent to specifying a set of meromorphic maps

$$V^{\hat{\lambda}}_{\mathcal{F}^0_G} \dashrightarrow V^{\hat{\lambda}}_{\mathcal{F}_G}$$

for all dominant characters $\hat{\lambda} \in \hat{\Lambda}^+_G$ satisfying the Plücker relations (cf. Definition 5.6). The trivial G-bundle $\mathcal{F}^0_G$ admits a natural split B-structure whose associated T-bundle is the trivial T-bundle $\mathcal{F}^0_T$, and this defines a set of maps

$$L^{\hat{\lambda}}_{\mathcal{F}^0_T} \hookleftarrow V^{\hat{\lambda}}_{\mathcal{F}^0_G}$$

where $L^{\hat{\lambda}} := (V^{\hat{\lambda}})^U$ and U is the unipotent radical of B. For a set of divisors $(D_i)_{i \in I} \in \mathrm{Div}^I_E$, we can also consider the meromorphic map

$$L^{\hat{\lambda}}_{\mathcal{F}^0_T}(\sum_{i \in I} \langle \hat{\lambda}, \nu_i \rangle D_i) \dashrightarrow L^{\hat{\lambda}}_{\mathcal{F}^0_T} \to V^{\hat{\lambda}}_{\mathcal{F}^0_G} \dashrightarrow V^{\hat{\lambda}}_{\mathcal{F}_G}$$

defined by modifying the T-bundles by ν_i at D_i for all $i \in I$. We claim that β defines a point in $\mathrm{S}^I_{G,(\nu_i)_{i \in I},E}$ if and only if this map does not have a zero or pole for all $\hat{\lambda}$. This is easy to see. In particular, if this map does not have a zero or a pole then the maps

$$L^{\hat{\lambda}}_{\mathcal{F}^0_T}(\sum_{i \in I} \langle \hat{\lambda}, \nu_i \rangle D_i) \to V^{\hat{\lambda}}_{\mathcal{F}_G}$$

define a B-structure $\mathcal{F}_B$ on $\mathcal{F}_G$ whose underlying T-bundle is given by $\mathcal{F}^0_T(\sum_{i \in I} \langle \hat{\lambda}, \nu_i \rangle D_i)$. Moreover, the construction induces a map of B-bundles

$$\mathcal{F}^0_B \dashrightarrow \mathcal{F}_B$$

which when applying $-^B G$ gives the modification β and when applying $-^B T$ gives rise to a modification defining a point in the union of connected components $\mathrm{Gr}^I_{T,(\nu_i)_{i \in I},E}$ over the point attached to $(D_i)_{i \in I}$ in Div^I_E. In other words, $\mathcal{F}^0_B \dashrightarrow \mathcal{F}_B$ defines an element of the locally closed stratum $\mathrm{S}^I_{G,(\nu_i)_{i \in I},E}$.

For $V \in \mathrm{Rep}(\hat{G}^I)$, we again consider the sheaf $\mathcal{S}_V$ on $\mathrm{Gr}^I_{G,(\lambda_i)_{i \in I},E}$, and pullback to a fixed geometric point $x \to \mathrm{Div}^I_E$ defined by $\mathrm{Spa}(C)$ for C an algebraically closed perfectoid field. If we write $p_{(\nu_i)_{i \in I}} : {}_x\mathrm{S}^I_{G,(\nu_i)_{i \in I},E} \to {}_x\mathrm{Gr}^I_{T,(\nu_i)_{i \in I},E} \times \mathrm{Spa}(C)$ for the induced map on connected components indexed by $(\nu_i)_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I$, we note that we have an isomorphism

$$p_! q^*(\mathcal{S}_V) \simeq \bigoplus_{(\nu_i)_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I} p_{(\nu_i)_{i \in I}!}(\mathcal{S}_V j^*_{x\mathrm{S}^I_{G,(\nu_i)_{i \in I},E}}) = \bigoplus_{(\nu_i)_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I} \mathbb{H}_c({}_x\mathrm{S}^I_{G,(\nu_i)_{i \in I}}, \mathcal{S}_V j^*_{x\mathrm{S}_{G,(\nu_i)_{i \in I}}})$$

(cf. [FS21, Example VI.3.6]). However, this is simply the constant term functor in the previous section. In particular, by Proposition 4.4, we deduce the following.

Corollary 4.9. *For* $V = \boxtimes_{i \in I} V_i \in \mathrm{Rep}(\hat{G}^I)$, *a geometric point* $x \to \mathrm{Div}_E^I$, *and all tuples* $(\nu_i)_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I$, *we have an isomorphism*

$$\mathbb{H}_c^{-\langle 2\hat{\rho}, j(\nu_i)_{i \in I} j \rangle}({}_x\mathrm{S}_{G,(\nu_i)_{i \in I},E}^I, \mathcal{S}_V j_{x}\mathrm{S}_{G,(\nu_i)_{i \in I},E}^I) \simeq \bigotimes_{i \in I} V_i(\nu_i)(\langle -\hat{\rho}, \nu_i \rangle/i)$$

of W_E^I-*modules.*

Remark 4.10. The Tate twists appearing here are due to the difference between the standard definition of ${}^LG$ and the one used in the geometric Satake equivalence, as in the remark proceeding Theorem 4.2.

This will be the key proposition required for the proof of the filtered Hecke eigensheaf property. More specifically, to show the compatibilities of the filtered eigensheaf property, we need to show that this isomorphism is functorial in I. In particular, consider a map of finite index sets $\pi : I \to J$. For $j \in J$, we set $I_j := \pi^{-1}(j)$ and consider the natural map $\Delta_{IJ} : \mathrm{Div}_E^J \to \mathrm{Div}_E^I$, which diagonally embeds the jth copy of Div_E^1 in Div_E^J into $\mathrm{Div}_E^{I_j}$. Then, by the relationship between fusion product and tensor product under Theorem 4.2, we have a identification

$$\Delta_{IJ}(\mathcal{S}_V) \simeq \mathcal{S}_{\boxtimes_{IJ}(V)}$$

of sheaves on $\mathrm{Gr}_{G,E}^J$, where $\boxtimes_{IJ}(V)$ is given by restriction along the corresponding map $\hat{G}^J \to \hat{G}^I$. Now, by Proposition 4.5, we have the following.

Corollary 4.11. *For all finite index sets* I, J *with a map* $f : I \to J$, *a tuple* $(\nu_j)_{j \in J} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^J$, *a representation* $V \in \mathrm{Rep}(\hat{G}^I)$, *and a geometric point* $x \to \mathrm{Div}_E^J$, *the identification* $\Delta_{IJ}(\mathcal{S}_V) \simeq \mathcal{S}_{\boxtimes_{IJ}(V)}$ *induces an isomorphism*

$$\mathbb{H}_c({}_x\mathrm{S}_{G,(\nu_j)_{j \in J},E}^J, \mathcal{S}_{\boxtimes_{IJ}(V)} j_{x}\mathrm{S}_{G,(\nu_j)_{j \in J},E}^J) \simeq \bigoplus_{\substack{(\nu_i)_{i \in I} \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \\ \sum_{i \in I_j} \nu_i = \nu_j}} \mathbb{H}_c({}_x\mathrm{S}_{G,(\nu_i)_{i \in I},E}, \mathcal{S}_V j_{x}\mathrm{S}_{G,(\nu_i)_{i \in I},E}^I)$$

of W_E^J-*modules, where the action on the RHS is via the natural map* $\Delta_{IJ} : W_E^J \to W_E^I$. *This is compatible with the identification*

$$\Delta_{IJ}(V((\nu_j)_{j \in J})) \simeq \bigoplus_{\substack{(\nu_i)_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I \\ \sum_{i \in I_j} \nu_i = \nu_j}} V((\nu_i)_{i \in I})$$

under the isomorphisms of Corollary 4.9.

We note that the previous result has some very useful geometric consequences. Let's explain this in the case that $I = f \cdot g$ is a singleton for a fixed geometric point $x \to \mathrm{Div}_E^1$. Fix $\lambda \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$, and consider the highest weight representation $V_\lambda \in \mathrm{Rep}(\hat{G})$ defined by λ. Since the sheaf $\mathcal{S}_\lambda$ is supported on ${}_x\mathrm{Gr}_{G,\leq\lambda}$, we can deduce the following.

Corollary 4.12. *For* $\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ *and* $\lambda \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ *with associated highest weight representation* $V_\lambda \in \mathrm{Rep}(\hat{G})$, *if the weight space* $V_\lambda(\nu)$ *is non-trivial then the intersection* ${}_x\mathrm{S}_{G,\nu,E} \setminus {}_x\mathrm{Gr}_{G,\leq\lambda,E}$ *is non-empty.*

Remark 4.13. One can also see this by using the Iwasawa decomposition of G and working explicitly with the loop group of G (See the analysis proceeding [She21, Proposition 6.4] and [She21, Remark 6.5]). For example, one can show that the intersection ${}_x\mathrm{Gr}_{G,\leq\lambda,E} \setminus {}_x\mathrm{S}_{G,w_0(\lambda),E}$ is simply the point given by ξ^λ, where $\xi \in B_{dR}^+(C, O_C)$ is the uniformizing parameter defined by the geometric point x. This corresponds to the lowest weight space $V_\lambda(w_0(\lambda))$.

We will now finish our analysis by recording some facts about the closure relationships for these strata.

Proposition 4.14. *[She21, Proposition 6.4] For $\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$ and $\lambda \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$, the closure of the intersection*

$$_x\mathrm{S}_{G,\nu,E} \cap {_x\mathrm{Gr}_{G,\leq\lambda,E}}$$

in $\mathrm{Gr}_{G,\leq\lambda,E}$ is equal to the disjoint union

$$\bigsqcup_{\nu' \leq \nu} {_x\mathrm{S}_{G,\nu',E}} \cap {_x\mathrm{Gr}_{G,\leq\lambda,E}}$$

where this defines a closed subspace in $_x\mathrm{Gr}_{G,\leq\lambda,E}$ by [FS21, Proposition VI.3.1]. In particular, using the previous Corollary, we deduce that $_x\mathrm{S}_{G,w_0(\lambda),E} \cap {_x\mathrm{Gr}_{G,\leq\lambda,E}}$ is a closed subspace and $_x\mathrm{S}_{G,\lambda,E} \cap {_x\mathrm{Gr}_{G,\leq\lambda,E}}$ is an open subspace.

5. Moduli stacks of B-structures

In this section, we will study the moduli stack of B-structures Bun_B and its basic geometric properties. This will allow us to define the geometric Eisenstein functor. For understanding many of the finer properties of this functor, it is important to consider a compactification of the natural morphism $\mathsf{p} : \mathrm{Bun}_B \to \mathrm{Bun}_G$ taking B-bundles to their induced G-bundles. This compactification will be an analogue of Drinfeld's compactification in the function-field setting, denoted $\overline{\mathrm{Bun}}_B$. We will show that this gives rise to an Artin v-stack, which admits Bun_B as an open and dense substack, and that the natural map $\overline{\mathsf{p}} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_G$ extending p is indeed proper after restricting to connected components. We will also define a locally closed stratification of $\overline{\mathrm{Bun}}_B$. These strata will play an important role in the proof of the Hecke eigensheaf property and understanding how the Eisenstein functor interacts with Verdier duality.

5.1. **The Geometry of Bun_B.** We will start by collecting some basic facts about the moduli stack Bun_B parameterizing, for $S \in \mathrm{Perf}$, the groupoid of B-bundles on X_S. Given a B-bundle $\mathcal{G}_B$, we can send it to the induced T-bundle and G-bundle via the natural maps

$$
\begin{array}{ccc}
B & \longrightarrow & G \\
\downarrow & & \\
T & &
\end{array}
$$

which induces a diagram of v-stacks:

$$
\begin{array}{ccc}
\mathrm{Bun}_B & \overset{\mathsf{p}}{\longrightarrow} & \mathrm{Bun}_G \\
\downarrow{\scriptstyle\mathsf{q}} & & \\
\mathrm{Bun}_T & &
\end{array}
$$

Let's first start by breaking this up into connected components. As seen in §3, the connected components of Bun_T are indexed by elements $\overline{\nu} \in B(T) \simeq \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})_{\Gamma} = \Lambda_{G,B}$. This allows us to define the following.

Definition 5.1. For $\overline{\nu} \in B(T)$, we write $\mathrm{Bun}_B^{\overline{\nu}}$ for the pre-image of the connected component $\mathrm{Bun}_T^{\overline{\nu}}$ defined by $\overline{\nu}$ under the map $\mathsf{q}^{\overline{\nu}}$. We write $\mathsf{p}^{\overline{\nu}} : \mathrm{Bun}_B^{\overline{\nu}} \to \mathrm{Bun}_G$ and $\mathsf{q}^{\overline{\nu}} : \mathrm{Bun}_B^{\overline{\nu}} \to \mathrm{Bun}_T^{\overline{\nu}}$ for the restriction of p and q to $\mathrm{Bun}_B^{\overline{\nu}}$, respectively.

We claim that this induces a decomposition of the moduli stack Bun_B into connected components. To each element $\overline{\nu}$, we define the integer $d_{\overline{\nu}} := \langle 2\hat{\rho}, \overline{\nu} \rangle$, where $2\hat{\rho}$ is the sum of all positive roots with respect to the choice of Borel. We note that, if $\overline{\nu}$ is anti-dominant with respect to the choice of Borel, $d_{\overline{\nu}}$ is negative. This will be the case where the HN-slopes are dominant so the B-bundles will

split, and the negative dimension comes from quotienting out by the torsor of splittings. On the other hand, if $\overline{\nu}$ is dominant then the connected component will parametrize non-split B-structures and we analogously see that the dimension will be positive. We have the following claim.

Proposition 5.2. *The map* q *is a cohomologically smooth (non-representable) morphism of Artin v-stacks in the sense of [FS21, Definition IV.1.11]. In particular, for $\overline{\nu} \in \Lambda_{G,B}$, the map $\mathsf{q}^{\overline{\nu}}$ is pure of ℓ-dimension equal to $d_{\overline{\nu}}$, in the sense of [FS21, Definition IV.1.17].*

Proof. This follows from [Ham21a, Proposition 4.7], where we note that Bun_T is an Artin v-stack that is cohomologically smooth of ℓ-dimension 0 (See also [AL21b, Lemma 4.1 (ii)]).

In particular, this implies using [Sch18, Proposition 23.11], that q is a universally open morphism of Artin v-stacks. Moreover, one can check that the fibers of this morphism are connected (See the proof of [AL21b, Lemma 4.1 (ii)] and [Ham21a, Proposition 3.16]). As a consequence, we can deduce that, since $\bigsqcup_{\overline{\nu} \in \Lambda_{G,B}} \mathrm{Bun}_T^{\overline{\nu}}$ is a decomposition of Bun_T into connected components, the following is true.

Corollary 5.3. *The connected components of Bun_B are given by $\mathrm{Bun}_B^{\overline{\nu}}$, for varying $\overline{\nu} \in B(T)$.*

We now comment briefly on the geometry of the map p. In particular, we have the following.

Lemma 5.4. *[AL21b, Lemma 4.1 (ii)] The map p is representable in nice diamonds.*

These properties allow us to define the Eisenstein functor using the derived functors defined in [Sch18].

Definition 5.5. We define a locally constant function
$$\dim(\mathrm{Bun}_B) : |\mathrm{Bun}_B| \to \mathbb{Z}$$
$$x \in |\mathrm{Bun}_B^{\overline{\nu}}| \mapsto d_{\overline{\nu}}$$
and with it the unnormalized Eisenstein functor
$$\mathrm{Eis} : D(\mathrm{Bun}_T) \to D(\mathrm{Bun}_G)$$
$$F \mapsto \mathsf{p}_!(\mathsf{q}^*(F)[\dim(\mathrm{Bun}_B)])$$

In particular, since this definition involves the functor $\mathsf{p}_!$ it is natural to consider a compactification of the morphism p to understand the finer properties of $\mathsf{p}_!$. This leads us to our study of the Drinfeld compactification.

5.2. **The Drinfeld Compactification.**

5.2.1. *The Definition and Basic Properties.* We recall that classically (curve over a finite or complex field) there is a rather straight-forward way of compactifying the map:
$$\mathsf{p} : \mathrm{Bun}_B \to \mathrm{Bun}_G$$
This is called a Drinfeld Compactification of p, denoted $\overline{\mathrm{Bun}}_B$. Its main property is that there exists an open immersion $\mathrm{Bun}_B \to \overline{\mathrm{Bun}}_B$, with topologically dense image, and it has a map $\overline{\mathsf{p}} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_G$ extending p, which is proper after restricting to a connected component. First, as a warm up, let us explain the construction when $G = \mathrm{GL}_2$. For this, we recall that Bun_B can be viewed as parameterizing tuples $(M, L, \kappa : L \hookrightarrow M)$, where E is a rank 2 vector bundle, L is a rank 1 vector bundle, and κ is an injective bundle map. To compactify this space, we will allow κ to be a map of O_{X_S}-modules whose pullback to each geometric point is an injective map of coherent sheaves on X. In other words, we allow M/L to have torsion. For a general G, the idea is to apply the Tannakian formalism. In particular, given a G-bundle F_G we get, for all $\hat{\lambda} \in \hat{\Lambda}_G^+$, an induced highest weight bundle, denoted $V_{F_G}^{\hat{\lambda}}$. A point of Bun_B mapping to F_G via p then defines a set of line subbundles $\kappa^{\hat{\lambda}} : L^{\hat{\lambda}} \hookrightarrow V_{F_G}^{\hat{\lambda}}$ which satisfy some Plücker relations. Using this interpretation, $\overline{\mathrm{Bun}}_B$ can then be

defined as classifying G-bundles $\mathcal{F}_G$ together with a system of maps $\overline{\kappa}^{\hat{\lambda}} : L^{\hat{\lambda}} \to V_{\mathcal{F}_G}^{\hat{\lambda}}$ for all $\hat{\lambda} \in \Lambda_G^+$, which are injective after pulling back to a geometric point and satisfy the same Plücker relations.

We now explain how to construct the aforementioned compactification $\overline{\mathrm{Bun}}_B$ of Bun_B over Bun_G in the Fargues-Fontaine setting. In order to describe the Drinfeld compactification, we note that, for $S \in \mathrm{Perf}$, Bun_B can be viewed as a stack parameterizing triples:

(1) A G-bundle $\mathcal{F}_G$ on X_S.
(2) A T-bundle $\mathcal{F}_T$ on X_S.
(3) A G-equivariant map $\kappa : \mathcal{F}_G \to G/U \times^T \mathcal{F}_T$.

By the Tannakian formalism, (3) can be described as a collection of injective bundle maps on X_S, $\kappa^V : (V^U)_{\mathcal{F}_T} \to V_{\mathcal{F}_G}$ for every G-module V satisfying the following Plücker relations:

(1) For the trivial representation V, κ^V must be the identity map $\mathcal{O}_{X_S} \to \mathcal{O}_{X_S}$.
(2) For a G-module map $V^1 \to V^2$, the induced square

$$
\begin{array}{ccc}
((V^1)^U)_{\mathcal{F}_T} & \xrightarrow{\kappa^{V^1}} & V_{\mathcal{F}_G}^1 \\
\downarrow & & \downarrow \\
((V^2)^U)_{\mathcal{F}_T} & \xrightarrow{\kappa^{V^2}} & V_{\mathcal{F}_G}^2
\end{array}
$$

   commutes.
(3) For two G-modules V^1 and V^2, we have that the diagram

$$
\begin{array}{ccc}
((V^1)^U \otimes (V^2)^U)_{\mathcal{F}_T} & \xrightarrow{\kappa^{V^1} \otimes \kappa^{V^2}} & V_{\mathcal{F}_G}^1 \otimes V_{\mathcal{F}_G}^2 \\
\downarrow & & \downarrow id \\
((V^1 \otimes V^2)^U)_{\mathcal{F}_T} & \xrightarrow{\kappa^{V^1 \otimes V^2}} & V_{\mathcal{F}_G}^1 \otimes V_{\mathcal{F}_G}^2
\end{array}
$$

   commutes.

As mentioned above, the idea will now be to introduce torsion in the above definition. In particular, we have the following definition for $\overline{\mathrm{Bun}}_B$.

Definition 5.6. We define $\overline{\mathrm{Bun}}_B$ to be the v-stack parameterizing, for $S \in \mathrm{Perf}$, triples $(\mathcal{F}_G, \mathcal{F}_T, \overline{\kappa}^V)$, where $\overline{\kappa}^V$ is a map of $\mathcal{O}_{X_S}$-modules defined for every G-module V

$$
(V^U)_{\mathcal{F}_T} \to V_{\mathcal{F}_G}
$$

satisfying the following conditions:

   For every geometric point $s \to S$, the pullback of $\overline{\kappa}^V$ to the Fargues-Fontaine curve over s is an injection of coherent sheaves.
   The Plücker relations hold in the following sense:
   (1) For the trivial representation V, $\overline{\kappa}^V$ is the identity map $\mathcal{O} \to \mathcal{O}$.
   (2) For a G-module map $V^1 \to V^2$, the induced square

$$
\begin{array}{ccc}
((V^1)^U)_{\mathcal{F}_T} & \xrightarrow{\overline{\kappa}^{V^1}} & V_{\mathcal{F}_G}^1 \\
\downarrow & & \downarrow \\
((V^2)^U)_{\mathcal{F}_T} & \xrightarrow{\overline{\kappa}^{V^2}} & V_{\mathcal{F}_G}^2
\end{array}
$$

   commutes.

(3) For two G-modules V^1 and V^2, we have that the diagram

$$
\begin{array}{ccc}
((V^1)^U \quad (V^2)^U)_{\mathcal{F}_T} & \xrightarrow{\overline{\kappa}^{V^1} \quad \overline{\kappa}^{V^2}} & V^1_{\mathcal{F}_G} \quad V^2_{\mathcal{F}_G} \\
\downarrow & & \downarrow id \\
((V^1 \quad V^2)^U)_{\mathcal{F}_T} & \xrightarrow{\overline{\kappa}^{V^1 \quad V^2}} & V^1_{\mathcal{F}_G} \quad V^2_{\mathcal{F}_G}
\end{array}
$$

commutes.

Remark 5.7. To simplify the notation, we will write $L^{\hat{\lambda}} := (V^{\hat{\lambda}})^U$ and $\overline{\kappa}^{\hat{\lambda}}$ for the embedding attached to the highest weight module of G of highest weight $\hat{\lambda} \in \Lambda^+_G$. It follows by construction that it suffices to consider only the embeddings induced by the highest weight Weyl G-modules attached to the fundamental weights $\hat{\varpi}_i \in \hat{\Lambda}^+_G$ for $i \in J$, using our assumption that the derived group of G is simply connected.

This gives rise to a well-defined v-stack and, using this description, we get well-defined morphisms $\overline{p} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_G$ and $\overline{q} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_T$ via projecting the data $(\mathcal{F}_G, \mathcal{F}_T, \overline{\kappa})$ to the first and second factor, respectively. We also get a natural map $j : \mathrm{Bun}_B \to \overline{\mathrm{Bun}}_B$. Now, to conclude this section, we prove some basic things about its geometry.

Proposition 5.8. *The v-stack $\overline{\mathrm{Bun}}_B$ is an Artin v-stack, and the map $j : \mathrm{Bun}_B \to \overline{\mathrm{Bun}}_B$ is an open immersion.*

Proof. It suffices to show the claim after base-change to an algebraically closed perfectoid field $\mathrm{Spa}(F, \mathcal{O}_F)$. We write X for the associated Fargues-Fontaine curve. Recall that, given a scheme Y, one defines the affine closure to be $\overline{Y} = \mathrm{Spec}\,\Gamma(Y, \mathcal{O}_Y)$. We let $\overline{G/U}$ be the affine closure of G/U. Viewing this as a constant scheme over X, we consider the stack:

$$
Z := [(\overline{G/U})/(T \quad G)] \to X
$$

Now, it follows by [BG02, Theorem 1.12], that, for $S \in \mathrm{Perf}$, a section

$$
\begin{array}{ccc}
 & & Z \\
 & \overset{s}{\nearrow} & \downarrow \\
X_S & \longrightarrow & X
\end{array}
$$

is equivalent to the datum of a T-bundle (resp. G-bundle) $\mathcal{F}_T$ (resp. $\mathcal{F}_G$) on X_S together with a family of maps $\overline{\kappa}^V$ of $\mathcal{O}_{X_S}$-modules, satisfying the Plücker conditions of Definition 5.6. Therefore, if we consider $\mathcal{M}_Z$, the moduli stack parameterizing such sections, then $\overline{\mathrm{Bun}}_B$ is the sub-functor corresponding to the locus where these maps are injective after pulling back to a geometric point. By [Ham21a, Remark 3.3], this is an open subfunctor. By [Ham21a, Theorem 1.7], $\mathcal{M}_Z$ is an Artin v-stack; therefore, the same is true for $\overline{\mathrm{Bun}}_B$. It remains to see that Bun_B is an open sub-functor. Now, by the work of [Gro83], it follows that $\overline{G/U}$ is strongly quasi-affine in the sense that $G/U \hookrightarrow \overline{G/U}$ is an open immersion. This induces an open immersion of stacks

$$
[(G/U)/(T \quad G)] = [X/B] \hookrightarrow Z
$$

which, after passing to moduli stacks of sections, gives a natural map $\mathrm{Bun}_B \to \mathcal{M}_Z$ factoring through the open immersion $\overline{\mathrm{Bun}}_B \hookrightarrow \mathcal{M}_Z$. The map $\mathrm{Bun}_B \to \mathcal{M}_Z$ is an open immersion, since $[X/B] \hookrightarrow Z$ is an open immersion, by arguing as in [FS21, Proposition IV.4.22].

5.2.2. *Properness of Compactifications.* We now seek to show that the morphism $\overline{\mathsf{p}} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_G$ is indeed a compactification of the map p. In particular, we write $\overline{\mathrm{Bun}}_B^{\overline{\nu}}$ for the pre-image of the connected component $\mathrm{Bun}_T^{\overline{\nu}} \subset \mathrm{Bun}_T$. Later we will see that $\overline{\mathrm{Bun}}_B^{\overline{\nu}}$ defines the connected components of $\overline{\mathrm{Bun}}_B$. This will follow from Corollary 5.3 and showing that $\mathrm{Bun}_B^{\overline{\nu}}$ is dense inside $\overline{\mathrm{Bun}}_B^{\overline{\nu}}$ (Proposition 5.22). We write $\overline{\mathsf{q}}^{\overline{\nu}} : \overline{\mathrm{Bun}}_B^{\overline{\nu}} \to \mathrm{Bun}_T^{\overline{\nu}}$ and $\overline{\mathsf{p}}^{\overline{\nu}} : \overline{\mathrm{Bun}}_B^{\overline{\nu}} \to \mathrm{Bun}_G$ for the maps induced by $\overline{\mathsf{q}}$ and $\overline{\mathsf{p}}$, respectively. This section will be dedicated to proving the following.

Proposition 5.9. *For all $\overline{\nu} \in \Lambda_{G,B}$, the map $\overline{\mathsf{p}}^{\overline{\nu}}$ is representable in nice diamonds and proper.*

Proof. Consider $S \in \mathrm{Perf}$ and an S-point of Bun_G corresponding to a G-bundle $\mathcal{F}_G$. We let Z denote the fiber of $\overline{\mathsf{p}}$ over the S-point defined by $\mathcal{F}_G$. We need to show $Z \to S$ is representable in nice diamonds and proper. To this end, let $\hat{\varpi}_i$ for $i \in J$ be the set of fundamental weights. The G-bundle $\mathcal{F}_G$ then defines a finite set of vector bundles $(V^{\hat{\varpi}_i})_{\mathcal{F}_G} =: V_i$ for $i \in J$. For $\overline{\nu} \in \Lambda_{G,B}$, we set $d_i := \langle \hat{\varpi}_i, \overline{\nu} \rangle$. We consider the space

$$P_i := (H^0(V_i(-d_i)) \setminus \{0\})/\underline{\mathbb{Q}}_p \to S$$

where $H^0(V_i(-d_i))$ is the Banach-Colmez space parameterizing global sections of the vector bundle V_i twisted by $-d_i$ and $\{0\}$ denotes the 0-section. It follows, by [FS21, Proposition II.2.16], that $P_i \to S$ is representable in nice diamonds and proper. Therefore, the same is true for the product:

$$\prod_{i \in J} P_i \to S$$

This will parametrize line bundles L_i of degree d_i together with a map

$$L_i \to V_i$$

of $\mathcal{O}_{X_S}$-modules whose pullback to each geometric point is an injection of coherent sheaves. Therefore, we get a natural map

$$X \to \prod_{i=1}^r P_i$$

remembering only the embeddings defined by the fundamental weights $\hat{\varpi}_i$. Now, by the Remark proceeding 5.6, it follows that this is an injective map into the subspace where the Plücker relations are satisfied. The desired claim would therefore follow from showing that this locus is closed. To see this, note that the commutativity conditions describing the Plücker relations can be expressed in terms of the vanishing of some linear combination of morphisms of $\mathcal{O}_{X_S}$-modules. Using this, one reduces to checking that, given a vector bundle T on X_S, the point in $H^0(T)$ defined by the zero section of T is closed. By choosing an injection $T \hookrightarrow \mathcal{O}(m)^N$ for sufficiently large m and N, one obtains an injective map $H^0(T) \to H^0(\mathcal{O}(m)^N)$ of diamonds compatible with the zero section. Thus, one reduces to checking the claim for $T = \mathcal{O}(m)^N$. In this case, it is nothing more than [Far20, Lemma 2.10]. $\square$

Now, we seek to describe the finer structure of these compactifications. In particular, a key role will be played by their stratifications. To do this, we will need to take a brief detour to discuss some properties of some $\mathcal{O}_{X_S}$-modules on the Fargues-Fontaine that occur as cokernels of fiberwise injective maps, as considered in the definition of the Drinfeld compactification.

5.2.3. *$\mathcal{O}_S$-flat coherent sheaves on the Fargues-Fontaine Curve.* Throughout this section, we fix $S \in \mathrm{Perf}$ and note that, in Definition 5.6, we imposed the condition that the intervening maps of vector bundles $\mathcal{F} \to \mathcal{F}'$ on X_S satisfy the property that their pullback to each geometric point of S is an injection of coherent sheaves on X. The stratification on the Drinfeld compactification will be given by fixing the length of the torsion of the cokernel of such morphisms. Normally, one could appeal to classical results on flat coherent sheaves in families to get a handle on the structure of such

strata; however, as the Fargues-Fontaine curve X_S is non-Noetherian unless S is a field, one cannot naively define a category of coherent sheaves on it. Nonetheless, we can still define the following.

Definition 5.10. [AL21a, Definition 2.8] For $S \in \mathrm{Perf}$, a flat coherent O_{X_S}-module on X_S is an O_{X_S}-module F which can, locally for the analytic topology on X_S, be written as the cokernel of a fiberwise injective map of bundles on X_S. Equivalently, the map remains an injection after pulling back to any $T \in \mathrm{Perf}_S$.

Remark 5.11. By [AL21a, Proposition 2.6], it follows that we can always find a global presentation of an S-flat coherent sheaf F as a two term complex of vector bundles on X_S.

Remark 5.12. We note that in the classical context of a relative projective curve $X \to S$ over a reduced Noetherian scheme S the analogue of this condition for a coherent sheaf F is equivalent to insisting that F is O_S-flat (See [AL21a, Remark 2.9]). However, as discussed above, the notion of coherent sheaves make no sense in this context, and the notion of flatness also does not make sense as O_{X_S} is not a module over O_S.

We have the following easy lemma, which gives a homological criterion characterizing flat coherent O_{X_S}-modules.

Lemma 5.13. *Let F be an O_{X_S}-module. For integers $a \leq b$ write $\mathrm{Perf}_{[a,b]}(X_S)$ for the derived category of perfect complexes of Tor-amplitude $[a,b]$, as in [AL21a, Section 2.1]. The following conditions are equivalent.*

(1) *F is a flat coherent O_{X_S}-module.*
(2) *F is represented by an object in $\mathrm{Perf}_{[-1,0]}(X_S)$ and $\mathrm{Tor}_{1,O_{X_S}}(O_{X_T}, F)$ is trivial for all $T \in \mathrm{Perf}_S$.*

Proof. For the forward direction, by definition F is represented by an object in $\mathrm{Perf}_{[-1,0]}(X_S)$. For the other condition, we choose a presentation
$$0 \to F_{-1} \to F_0 \to F \to 0$$
of F as the cokernel of a fiberwise injective map of vector bundles. Now, since the first map is injective after tensoring by $- \otimes_{O_{X_S}} O_{X_T}$ for any $T \in \mathrm{Perf}_S$, it follows easily by the associated long exact sequence that $\mathrm{Tor}_{1,O_{X_S}}(O_{X_T}, F) = 0$. For the converse direction, it follows from the proof of [AL21a, Proposition 2.6] that if F is represented by an object in $\mathrm{Perf}_{[-1,0]}(X_S)$ then it can be globally represented by a two term complex of vector bundles on X_S. Choosing such a presentation $F_{-1} \to F_0$, it follows that the defining map must be fiberwise injective by the vanishing of $\mathrm{Tor}_{1,O_{X_S}}(O_{X_T}, F)$ for all $T \in \mathrm{Perf}_S$.

This simple lemma allows us to see the following.

Lemma 5.14. *Flat coherent sheaves on X_S are stable under taking kernels of surjections and cokernels of fiberwise injective O_{X_S}-module maps.*

Proof. We explain the case of taking cokernels of injections with the case of surjections being similar. We consider two flat coherent sheaves F and F' on X_S and a short exact sequence
$$0 \to F \to F' \to F'/F \to 0$$
of O_{X_S}-modules where the first map is fiberwise injective. It easily follows from the corresponding long exact sequences of Tors, the previous lemma applied to F and F', and the fiberwise injectivity of the first map that $\mathrm{Tor}_{i,O_{X_S}}(O_{X_T}, F'/F)$ is trivial for all $T \in \mathrm{Perf}_S$ and $i \geq 1$. Therefore, by the previous lemma, it suffices to show that $F'/F \in \mathrm{Perf}_{[-1,0]}(X_S)$. Since $F, F' \in \mathrm{Perf}_{[-1,0]}(X_S)$ it follows that $F'/F \in \mathrm{Perf}_{[-2,0]}(X_S)$, and, by the vanishing of $\mathrm{Tor}_{i,O_{X_S}}(O_{X_T}, F'/F)$ for $i \geq 2$, it must lie in $\mathrm{Perf}_{[-1,0]}(X_S)$.

Now, consider the underlying topological space of S, denoted $|S|$. We can consider a geometric point $s \in |S|$ and the pullback of such a flat coherent $\mathcal{O}_{X_S}$-module F to X_s. The scheme X_s will just be the usual Fargues-Fontaine curve over a geometric point so it is in particular a Dedekind scheme, and $F|_{X_s}$ will just be a coherent sheaf. Therefore, we have a decomposition

$$F|_{X_s} \simeq F^{\mathrm{tors}}|_{X_s} \oplus F^{\mathrm{vb}}|_{X_s}$$

where $F^{\mathrm{tors}}|_{X_s}$ (resp. $F^{\mathrm{vb}}|_{X_s}$) is a torsion sheaf (resp. vector bundle) on X_s. Given such a torsion sheaf, we write $\lambda(F|_{X_s}) := \ell(F^{\mathrm{tors}}|_{X_s})$ for the length of this torsion sheaf. Our main aim is to prove the following proposition. We would like to thank David Hansen for supplying the idea behind its proof.

Proposition 5.15. *If F is a flat coherent sheaf on X_S then the function*

$$|S| \to \mathbb{N}_0$$
$$s \mapsto \lambda(F|_{X_s})$$

is upper semi-continuous. Moreover, if this function is locally constant on S, then there exists a unique short exact sequence of flat coherent sheaves on X_S

$$0 \to F^{\mathrm{tors}} \to F \to F^{\mathrm{vb}} \to 0$$

where F^{vb} is a vector bundle on X_S, and F^{tors} is a torsion sheaf in the sense that its pullback to each geometric point is a torsion sheaf.

Proof. To do this, we will need to prove the following lemma.

Lemma 5.16. *Let F be a flat coherent sheaf on X_S. We consider the v-sheaf on Perf_S, denoted $H^0(F)$, which sends $T \in \mathrm{Perf}_S$ to the set of global sections of F. The following is true.*
  (1) *$H^0(F) \to S$ is separated and representable in nice diamonds.*
  (2) *The v-sheaf $(H^0(F) \setminus \{0\})/\underline{\mathbb{Q}_p^\times} \to S$ given by deleting the 0-section and quotienting out by the scaling action by $\underline{\mathbb{Q}_p^\times}$ is proper and representable in nice diamonds over S.*

Proof. We can check all claims analytically locally on S. First note, by [AL21a, Lemma 2.12], that analytically locally on S we can find an exact sequence

$$0 \to E \to \mathcal{O}_{X_S}(-n)^m \to F \to 0$$

where E is a vector bundle, for all n sufficiently large and fixed m. Passing to cohomology, this gives us an exact sequence of v-sheaves:

$$0 \to H^0(F) \hookrightarrow H^1(E) \to H^1(\mathcal{O}(-n)^m)$$

Now note that the slopes of E are necessarily negative after pulling back to a geometric point for n sufficiently large. Therefore, we can assume that $H^1(E)$ is a nice diamond by [FS21, Proposition II.3.5]. It follows by [FS21, Proposition 2.5 (ii)] that $H^1(\mathcal{O}_{X_S}(-n)^m)$ is separated which implies that the injective map $H^0(F) \to H^1(E)$ is a closed embedding. Therefore, $H^0(F)$ is also nice. For the second part, we note that the closed embedding

$$H^0(F) \hookrightarrow H^1(E)$$

is compatible with the 0-section and the scaling action. This reduces us to checking that $(H^1(E) \setminus \{0\})/\underline{\mathbb{Q}_p^\times} \to S$ is proper, which follows from [FS21, Proposition II.3.5].

In particular, we will need the following Corollary.

Corollary 5.17. *Let F and F' be two flat coherent sheaves on X_S. We consider the v-sheaf on Perf_S, denoted $\mathcal{H}om(F, F')$, which sends $T \in \mathrm{Perf}_S$ to the set of $\mathcal{O}_{X_S}$-module homomorphisms $F \to F'$. The following is true.*
  (1) *$\mathcal{H}om(F, F') \to S$ is separated and representable in nice diamonds.*

(2) *The diamond* $(\mathrm{Hom}(F, F^\flat) \setminus \{0\})/\underline{\mathbb{Q}_p} \to S$ *given by deleting the* 0*-map and quotienting out by the scaling action by* $\underline{\mathbb{Q}_p}$ *is proper over* S.

Proof. By applying [AL21a, Lemma 2.12], we have analytically locally on S a short exact sequence

$$0 \to E \to \mathcal{O}_{X_S}(-n)^m \to F \to 0$$

for all n sufficiently large. Applying $\mathrm{Hom}(-, F^\flat)$, we get an exact sequence of v-sheaves:

$$0 \to \mathrm{Hom}(F, F^\flat) \to \mathrm{Hom}(\mathcal{O}_{X_S}(-n)^m, F^\flat) \to \mathrm{Hom}(E, F)$$

In other words, $\mathrm{Hom}(F, F^\flat)$ is the fiber of the last map over the 0-section, and in turn the first map is a closed immersion by Lemma 5.16 (1) applied to $H^0(E^- \otimes F^\flat)$. This allows us to replace $F^\flat$ with a vector bundle, and the claim then follows by Lemma 5.16 applied to $F^- \otimes F^\flat$.

Now that we have gotten this out of the way we can finally turn to the proof of our claim. We consider the v-sheaf

$$S_F \to S$$

of fiberwise non-zero global sections s of F which are annihilated by $I_D \subset \mathcal{O}_{X_S}$ for D a degree 1 relative Cartier divisor $D \subset X_S$. Alternatively, we can view this as parameterizing pairs of (D, f) of a degree 1 Cartier divisor in X_S and a point of $\mathrm{Hom}(\mathcal{O}_{X_S}/I_D, F) \setminus \{0\}$. Using this description, we can factorize the map $S_F \to S$ as $S_F \to \mathrm{Div}^1_S \to S$. We let $S_F/\underline{\mathbb{Q}_p}$ be the quotient of this space by the scaling action on the section f. Now the projection $\mathrm{Div}^1_S \to S$ is proper and representable in nice diamonds by combining [FS21, Corollary II.2.4] and [FS21, Proposition II.2.16]. Moreover, by the previous Corollary, we know that $S_F/\underline{\mathbb{Q}_p} \to \mathrm{Div}^1_S$ is proper and representable in nice diamonds because it is a fibration in the spaces $\mathrm{Hom}(\mathcal{O}_{X_T}/I_D, F_T) \setminus \{0\}/\underline{\mathbb{Q}_p}$ for $T \in \mathrm{Perf}_S$ and D a degree 1 relative Cartier divisor in X_T. It follows that the image of $S_F \to S$ is closed. We note that this coincides with the locus where $\lambda(F) > 0$, so we have deduced a special case of the upper semi-continuity result[4]. Now we argue by induction. In particular, by replacing S by its image, it follows by properness that $S_F \to S$ is a v-cover. Therefore, after v-localization, we can assume that $S_F \to S$ admits a section, which implies that there exists a degree 1 Cartier divisor $D \subset X_S$ and a fiberwise non-zero section $s : \mathcal{O}_{X_S}/I_D \to F$. Using Lemma 5.14, we can replace F with $F^\flat = F/s(\mathcal{O}_{X_S})$, noting that the locus where $\lambda(F^\flat) > 0$ coincides with the locus where $\lambda(F) > 1$. Therefore, the claimed upper semi-continuity result follows. For the second claim, uniqueness is clear, and, by [AL21a, Theorem 2.11], the category of flat coherent sheaves on X_S satisfies v-descent, so it suffices to show the claim up to a v-localization. However, now via induction we can just argue as above, using the sections s produced above and Lemma 5.14 to give rise to the map $F^{\mathrm{tors}} \to F$.

Now we can reap the fruit of our labor in this section, and use it to show that the compactification $\overline{\mathrm{Bun}}_B$ has a well-behaved stratification.

5.2.4. *Stratifications.* We now turn our attention to stratifying $\overline{\mathrm{Bun}}_B$. In particular, for an element $\overline{\nu} \in \Lambda^{pos}_{G,B} \setminus \{0\}$, we write $\overline{\nu} = \sum_{i \in J} n_i \alpha_i$ as a positive linear combination of the elements corresponding to Γ-orbits of simple positive coroots α_i. We let $\mathrm{Div}^{(\overline{\nu})}$ be the partially symmetrized power of the mirror curve attached to it, as in §3.3. We have a map of Artin v-stacks

$$j_{\overline{\nu}} : \mathrm{Div}^{(\overline{\nu})} \times \mathrm{Bun}_B \to \overline{\mathrm{Bun}}_B$$

sending a tuple $((f(D_i)_{i \in J}, F_G, F_T, \kappa^{\hat{\lambda}})$, to the tuple

$$(F_G, F_T(-\sum_{i \in J} \alpha_i \cdot D_i), \overline{\kappa}^{\hat{\lambda}})$$

---

[4] We could have also deduced this special case by arguing as in the proof of Proposition 5.8.

where $\overline{\kappa}^{\hat{\lambda}}$ is the natural composition

$$(L^{\hat{\lambda}})_{F_T}\left(\sum_{i \in J}\langle\alpha_i, \hat{\lambda}\rangle \cdot D_i\right) \to L^{\hat{\lambda}}_{F_T} \xrightarrow{\kappa^{\hat{\lambda}}} (V^{\hat{\lambda}})_{F_G}$$

defined by the unique effective modification of T-bundles of the specified meremorphy and support. We now make the following definition.

Definition 5.18. For $\overline{\nu} \in \Lambda_{G,B} \setminus \{0\}$, we define the v-stack $_{\overline{\nu}}\overline{\mathrm{Bun}}_B$ (resp. $_{\leq\overline{\nu}}\overline{\mathrm{Bun}}_B$) to be the locus where, for all $\hat{\lambda} \in \hat{\Lambda}_G^+$, the cokernels $V^{\hat{\lambda}}/\mathrm{Im}(\overline{\kappa}^{\hat{\lambda}})$ have torsion of length equal to (resp. greater than) $\langle\hat{\lambda}, \overline{\nu}\rangle$ after pulling back to any geometric point. By Proposition 5.15 (i), $_{\overline{\nu}}\overline{\mathrm{Bun}}_B$ is a locally closed substack of $\overline{\mathrm{Bun}}_B$, and the closure of $_{\overline{\nu}}\overline{\mathrm{Bun}}_B$ in $\overline{\mathrm{Bun}}_B$ is contained in $_{\leq\overline{\nu}}\overline{\mathrm{Bun}}_B$ (In fact, is equal to it, as will follow from Proposition 5.22).

To work with these strata, we will need the following.

Proposition 5.19. For $\overline{\nu} \in \Lambda_{G,B}^{pos} \setminus \{0\}$, the map $j_{\overline{\nu}}$ induces an isomorphism $\mathrm{Bun}_B \times \mathrm{Div}^{(\overline{\nu})} \simeq {}_{\overline{\nu}}\overline{\mathrm{Bun}}_B$ In particular, $j_{\overline{\nu}}$ is a locally closed embedding.

Proof. It is clear that $j_{\overline{\nu}}$ induces a map into the locally closed stratum $_{\overline{\nu}}\overline{\mathrm{Bun}}_B$. We need to exhibit an inverse of this map. We first begin with the following lemma.

Lemma 5.20. Let Coh be the v-stack parameterizing, for $S \in \mathrm{Perf}$, flat coherent sheaves on X_S, as in [AL21a, Theorem 2.11]. For $k, n \in \mathbb{N}_{\geq 0}$, we set Coh_n^k to be the locally closed (by Proposition 5.15 (i)) substack parameterizing flat coherent sheaves whose torsion length (resp. vector bundle rank) is equal to k (resp. n) after pulling back to a geometric point. There is a well-defined map

$$\mathrm{Coh}_n^k \to \mathrm{Div}^{(k)}$$

of v-stacks, sending a S-flat coherent sheaf F with attached short exact sequence $0 \to F^{\mathrm{tor}} \to F \to F^{\mathrm{vb}} \to 0$, as in Proposition 5.15 (ii), to the support of F^{tor}.

Proof. We note that, given F a S-flat coherent sheaf in Coh_n^k, we have, by Proposition 5.15 (ii), a unique short exact sequence

$$0 \to F^{\mathrm{tor}} \to F \to F^{\mathrm{vb}} \to 0$$

where we note that F^{tor} will be a S-flat coherent sheaf of generic rank 0, using Lemma 5.14. Now we can choose a presentation of this

$$0 \to F_{-1} \to F_0 \to F^{\mathrm{tor}} \to 0$$

for two vector bundles F_{-1} and F_0. Since F^{tor} is of constant rank 0 by construction, the first map must be a fiberwise injective map of vector bundles locally of the same rank. Therefore, analytically locally on S, we can then take the top exterior power of the map $F_{-1} \to F_0$, and this will give rise to a fiberwise injective map of line bundles, which in turn gives rise to a relative Cartier divisor D in X_S (cf. [KM76] to see that this is independent of the choice of presentation). If F defines a point in Coh_n^k then D must be of degree k, and we get the desired map.

We need to exhibit an inverse to the natural map $\mathrm{Bun}_B \times \mathrm{Div}^{(\overline{\nu})} \to {}_{\overline{\nu}}\mathrm{Bun}_B$. To do this, for $S \in \mathrm{Perf}$, consider a short exact sequence of

$$0 \to V_1 \to V_{n+1} \to V_n \to 0$$

of $\mathcal{O}_{X_S}$-modules, where V_1 (resp. V_{n+1}) is a line bundle (resp. rank $n+1$ vector bundle), and the first map is a fiberwise-injective map, so that V_n is S-flat. Assume that V_n defines a point in Coh_n^k for some k, and let D be the degree k Cartier divisor in X_S defined by the previous Lemma. It follows by an application of Proposition 5.15 (ii) that we have a short exact sequence

$$0 \to V_n^{\mathrm{tors}} \to V_n \to V_n^{\mathrm{vb}} \to 0$$

where V_n^{tors} will define a point in Coh_0^k and V_n^{vb} is a rank n vector bundle. Let $\widetilde{V}_1$ denote the preimage of V_n^{tors} in V_{n+1}. It is then easy to see that $V_1 \hookrightarrow V_{n+1}$ gives rise to an isomorphism $V_1(D) \simeq \widetilde{V}_1$. Now, given a S-point of $\overline{\nu}\mathrm{Bun}_B$, we can construct the desired inverse by applying the above argument to the short exact sequences coming from the embeddings $\overline{\kappa}^{\hat{\omega}_i}$, for the fundamental weights $\hat{\omega}_i \in \hat{\Lambda}_G^+$.

With this locally closed stratification in hand, we can now study how Hecke correspondences base-changed to $\overline{\mathrm{Bun}}_B$ interact with it. This will play a key role in the proof of the Hecke eigensheaf property, and reduce showing the density of $\mathrm{Bun}_B \hookrightarrow \overline{\mathrm{Bun}}_B$ to studying the usual closure relationships for semi-infinite Schubert cells.

5.2.5. *The Key diagram and Density of the Compactification.* We would now like to describe how Hecke correspondences on Bun_G interact with pullback along the map $\overline{\mathsf{p}} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_G$. This will be used to show the filtered Hecke eigensheaf property for the geometric Eisenstein series, analogous to the analysis carried out in [BG02, Section 3]. In this section, we will just study the geometry of the relevant diagram and use it to deduce that the open inclusion $\mathrm{Bun}_B \hookrightarrow \overline{\mathrm{Bun}}_B$ defines a dense subset in the underlying topological space of $\overline{\mathrm{Bun}}_B$. We fix a finite index set I, and consider the Hecke stacks $\mathrm{Hck}_{G,E}^I$ base-changed to the field E over which G splits. We consider the usual diagram

$$\mathrm{Bun}_G \times \mathrm{Div}_E^I \xleftarrow{h_G \times \pi} \mathrm{Hck}_{G,E}^I \xrightarrow{h_G'} \mathrm{Bun}_G$$

as in §3 and §4. We now fix a tuple of geometric dominant characters $(\lambda_i)_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+)^I$, and restrict to the locus $\mathrm{Hck}_{G,\,(\lambda_i)_{i \in I},E}$ where the meromorphy of this modification is bounded by $(\lambda_i)_{i \in I}$. We define the v-stack $\overline{Z}_{(\lambda_i)_{i \in I}}^I$ by the Cartesian diagram:

$$
\begin{array}{ccc}
\overline{Z}_{(\lambda_i)_{i \in I}}^I & \xrightarrow{\;^0 h_G'\;} & \overline{\mathrm{Bun}}_B \\
\downarrow{^0\overline{\mathsf{p}}} & & \downarrow{\overline{\mathsf{p}}} \\
\mathrm{Hck}_{G,\,(\lambda_i)_{i \in I},E}^I & \xrightarrow{\;h_G'\;} & \mathrm{Bun}_G
\end{array}
$$

By definition, $\overline{Z}_{(\lambda_i)_{i \in I}}^I$ parametrizes pairs of G-bundles $(\mathcal{F}_G, \mathcal{F}_G^0)$ together with a modification $\mathcal{F}_G \dashrightarrow \mathcal{F}_G^0$ with meromorphy bounded by λ_i at Cartier divisors D_i for $i \in I$ and an enhanced B-structure on $\mathcal{F}_G^0$ specified by maps $\overline{\kappa}^{\hat{\lambda}}$ for $\hat{\lambda} \in \hat{\Lambda}_G^+$. The fact that the modification $\mathcal{F}_G \dashrightarrow \mathcal{F}_G^0$ has meromorphy bounded by $(\lambda_i)_{i \in I}$ implies that, for all $\hat{\lambda} \in \hat{\Lambda}_G^+$, we have an inclusion:

$$V_{\mathcal{F}_G^0}^{\hat{\lambda}} \subset V_{\mathcal{F}_G}^{\hat{\lambda}}(\sum_{i \in I} \langle \hat{\lambda}, w_0(\lambda_i)\rangle \cdot D_i)$$

Therefore, the embeddings

$$\overline{\kappa}^{0\hat{\lambda}} : L_{\mathcal{F}_T^0}^{\hat{\lambda}} \hookrightarrow V_{\mathcal{F}_G^0}^{\hat{\lambda}}$$

give rise to a map:

$$\overline{\kappa}^{\hat{\lambda}} : L_{\mathcal{F}_T^0}^{\hat{\lambda}}(\sum_{i \in I} \langle \hat{\lambda}, w_0(\lambda_i)\rangle \cdot D_i) \hookrightarrow V_{\mathcal{F}_G}^{\hat{\lambda}}$$

This defines for us a morphism

$$\phi_{(\lambda_i)_{i \in I}} : \overline{Z}_{(\lambda_i)_{i \in I}}^I \to \overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I$$

which records the point in $\overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I$ defined by the pair $(\overline{\kappa}^{\hat{\lambda}}, (D_i)_{i \in I})$. This sits in a commutative diagram

$$
(2) \qquad
\begin{array}{ccccc}
\overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I & \xleftarrow{\phi_{(\lambda_i)_{i \in I}}} & \overline{Z}_{(\lambda_i)_{i \in I}}^I & \xrightarrow{^0 h_G^{\prime}} & \overline{\mathrm{Bun}}_B \\
\downarrow {\scriptstyle \overline{\mathsf{p}} \times \mathrm{id}} & & \downarrow {\scriptstyle ^0\mathsf{p}} & & \downarrow {\scriptstyle \overline{\mathsf{p}}} \\
\mathrm{Bun}_G \times \mathrm{Div}_E^I & \xleftarrow[h_G \times \pi]{} & \mathrm{Hck}_{G,\ (\lambda_i)_{i \in I}, E}^I & \xrightarrow{h_G^{\prime}} & \mathrm{Bun}_G
\end{array}
$$

where we note that left square is not Cartesian. We will not use the space $\overline{Z}_{(\lambda_i)_{i \in I}}^I$ at all in our arguments, but it should be important for future applications. For our purposes, we consider $Z_{(\lambda_i)_{i \in I}}^I$, the space obtained by replacing $\overline{\mathrm{Bun}}_B$ with Bun_B on the right hand side of the diagram. This sits in an analogous diagram

$$
(3) \qquad
\begin{array}{ccccc}
\overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I & \xleftarrow{\phi_{(\lambda_i)_{i \in I}}} & Z_{(\lambda_i)_{i \in I}}^I & \xrightarrow{^0 h_G^{\prime}} & \mathrm{Bun}_B \\
\downarrow {\scriptstyle \overline{\mathsf{p}} \times \mathrm{id}} & & \downarrow {\scriptstyle ^0\mathsf{p}} & & \downarrow {\scriptstyle \mathsf{p}} \\
\mathrm{Bun}_G \times \mathrm{Div}_E^I & \xleftarrow[h_G \times \pi]{} & \mathrm{Hck}_{G,\ (\lambda_i)_{i \in I}, E}^I & \xrightarrow{h_G^{\prime}} & \mathrm{Bun}_G
\end{array}
$$

As we will see, the proof of the filtered Hecke eigensheaf Property will ultimately reduce to contemplating the fibers of the morphism $\phi_{(\lambda_i)_{i \in I}}$. For our purposes, it will suffice to consider the pullback of this diagram to a geometric point $\mathrm{Spa}(F, \mathcal{O}_F) \to \mathrm{Div}_E^I$. We denote the resulting space by ${}_x\overline{Z}_{(\lambda_i)_{i \in I}}^I$. It sits in a diagram of the form

$$
\begin{array}{ccccc}
{}_x\overline{\mathrm{Bun}}_B & \xleftarrow{\phi_{(\lambda_i)_{i \in I}}} & {}_x\overline{Z}_{(\lambda_i)_{i \in I}}^I & \xrightarrow{^0 h_G^{\prime}} & \overline{\mathrm{Bun}}_B \\
\downarrow {\scriptstyle \overline{\mathsf{p}}} & & \downarrow {\scriptstyle ^0\overline{\mathsf{p}}} & & \downarrow {\scriptstyle \overline{\mathsf{p}}} \\
{}_x\mathrm{Bun}_G & \xleftarrow[h_G \times \pi]{} & {}_x\mathrm{Hck}_{G,\ (\lambda_i)_{i \in I}, E}^I & \xrightarrow{h_G^{\prime}} & \mathrm{Bun}_G
\end{array}
$$

where ${}_x\mathrm{Hck}_{G,\ (\lambda_i)_{i \in I}, E}^I$ is the Hecke stack parameterizing modifications at the tuple of Cartier divisors $(D_i)_{i \in I}$ corresponding to x and ${}_x\overline{\mathrm{Bun}}_B$ (resp. ${}_x\mathrm{Bun}_G$) denotes the base change of $\overline{\mathrm{Bun}}_B$ (resp. Bun_G) to $\mathrm{Spa}(F, \mathcal{O}_F)$. Consider a tuple $(\nu_i)_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I$ and write $\overline{\nu} := \sum_{i \in I} \nu_i$. Let E_{ν_i} denote the reflex field of ν_i. We view the geometric point $x \to \mathrm{Div}^I$ as a geometric point of $\mathrm{Div}^{(\overline{\nu})}$ via composing with the map

$$
\Delta_{(\nu_i)_{i \in I}} : \mathrm{Div}_E^I \to \prod_{i \in I} \mathrm{Div}_{E_{\nu_i}}^1 \xrightarrow{\prod_{i \in I} \Delta_{\nu_i}} \prod_{i \in I} \mathrm{Div}^{(\nu_i \Gamma)} \to \mathrm{Div}^{(\overline{\nu})}
$$

where the last map is given by taking the union of Cartier divisors and Δ_{ν_i} is the twisted diagonal embedding described in §3.3. We set ${}_{x,(\nu_i)_{i \in I}}\overline{\mathrm{Bun}}_B$ to be the pullback of the locally closed stratum ${}_{\overline{\nu}}\overline{\mathrm{Bun}}_B \hookrightarrow \mathrm{Div}^{(\overline{\nu})} \times \mathrm{Bun}_B$ to this geometric point. The substack ${}_{x,(\nu_i)_{i \in I}}\overline{\mathrm{Bun}}_B$ corresponds to the locus where the embeddings

$$
\mathcal{L}_{\mathcal{F}_T}^{\hat{\lambda}} \to \mathcal{V}_{\mathcal{F}_G}^{\hat{\lambda}}
$$

have a zero of order given by ν_i at D_i for all dominant characters $\hat{\lambda}$ of G and all $i \in I$, and a zero nowhere else. We now consider the open substack ${}_{x,0}\overline{\mathrm{Bun}}_B = {}_x\mathrm{Bun}_B$. Since the maps κ have no zero at the Cartier divisors corresponding to x, it follows that they define an $L^+ B$ torsor over ${}_{x,0}\overline{\mathrm{Bun}}_B$, which we denote by ${}_x B$. We then consider the map

$$
i_{(\nu_i)_{i \in I}} : \overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I \to \overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I
$$

given by sending $(\mathcal{F}_G, \mathcal{F}_T, \overline{\kappa}^{\check{\lambda}}, (D_i)_{i \in I})$ to the object $(\mathcal{F}_G, \mathcal{F}_T(-\sum_{i \in I} \overline{\nu}_i \cdot D_i), L_{\mathcal{F}_T}^{\check{\lambda}}(-\sum_{i \in I} \langle \overline{\nu}_i, \hat{\lambda} \rangle \cdot$

$D_i) \hookrightarrow L_{\mathcal{F}_T}^{\check{\lambda}} \hookrightarrow V_{\mathcal{F}_G}^{\check{\lambda}}, (D_i)_{i \in I})$, and note that the map $i_{(\nu_i)_{i \in I}}$ defines an isomorphism between the pullbacks $_{x,0}\overline{\mathrm{Bun}}_B$ and $_{x,(\nu_i)_{i \in I}}\overline{\mathrm{Bun}}_B$. Therefore, by transport of structure, we get a L^+B-torsor, denoted $_xB^{(\nu_i)_{i \in I}}$, over $_{x,(\nu_i)_{i \in I}}\overline{\mathrm{Bun}}_B$. For $(\nu_i)_{i \in I} \in (\mathbb{X}_\bullet(T_{\overline{\mathbb{Q}}_p}))^I$, we let $_xZ^{I, \cdot, (\nu_i)_{i \in I}}_{(\lambda_i)_{i \in I}}$ (resp. $_xZ^{I,(\lambda_i)_{i \in I}, \cdot}_{(\lambda_i)_{i \in I}}$) be the fibers of $^0h_G^\rightarrow$ (resp. $\phi_{(\lambda_i)_{i \in I}}$) over $_{x,(\nu_i)_{i \in I}}\overline{\mathrm{Bun}}_B$. We now have the following Lemma describing these subspaces, which is an analogue of [BG02, Lemma 3.3.6].

Lemma 5.21. *For tuples $(\nu_i)_{i \in I}, (\nu_i^0)_{i \in I} \in (\mathbb{X}_\bullet(T_{\overline{\mathbb{Q}}_p}))^I$, geometric dominant cocharacters $(\lambda_i)_{i \in I} \in (\mathbb{X}_\bullet(T_{\overline{\mathbb{Q}}_p})^+)^I$, and a geometric point $x \to \mathrm{Div}_E^I$, the following is true.*

(1) *There is an isomorphism*

$$_xZ^{I, \cdot, (\nu_i^0)_{i \in I}}_{(\lambda_i)_{i \in I}} \simeq {}_x\mathrm{Gr}^I_{G, (-w_0(\lambda_i))_{i \in I}} \overset{L^+B}{\times} {}_xB^{(\nu_i^0)_{i \in I}}$$

where the L^+B action on $_x\mathrm{Gr}^I_{G, (-w_0(\lambda_i))_{i \in I}}$ is given by the inclusion $L^+B \hookrightarrow L^+G$.

(2) *Under the identification in (1), the substack $_xZ^{I,(\nu_i)_{i \in I},(\nu_i^0)_{i \in I}}_{(\lambda_i)_{i \in I}} \hookrightarrow {}_xZ^{I, \cdot, (\nu_i^0)_{i \in I}}_{(\lambda_i)_{i \in I}}$ identifies with the substack*

$$_x\mathrm{Gr}^I_{G, (-w_0(\lambda_i))_{i \in I},E} \setminus {}_x\mathrm{S}^I_{G,(-w_0(\lambda_i)-\nu_i+\nu_i^0)_{i \in I},E} \overset{L^+B}{\times} {}_xB^{(\nu_i^0)_{i \in I}} \subset {}_x\mathrm{Gr}^I_{G, (-w_0(\lambda_i))_{i \in I},E} \overset{L^+B}{\times} {}_xB^{(\nu_i^0)_{i \in I}}$$

(3) *When viewed as a stack projecting to $_{x,(\nu_i)_{i \in I}}\overline{\mathrm{Bun}}_B$, the stack $_xZ^{I,(\nu_i)_{i \in I},(\nu_i^0)_{i \in I}}_{(\lambda_i)_{i \in I}}$ identifies with*

$$_x\mathrm{Gr}^I_{G, (\lambda_i)_{i \in I},E} \setminus {}_x\mathrm{S}^I_{G,(\nu_i-\nu_i^0+w_0(\lambda_i))_{i \in I},E} \overset{L^+B}{\times} {}_xB^{(\nu_i)_{i \in I}}$$

Proof. Follows from the definitions and the description of the semi-infinite cells mentioned in the remark proceeding Definition 4.7. ∎

As mentioned earlier, this description of the fibers will serve a key role in the proof of the Hecke eigensheaf property. For now, we content ourselves by using it to prove density.

Proposition 5.22. Bun_B *defines a substack of $\overline{\mathrm{Bun}}_B$ which is topologically dense.*

Proof. Consider a geometric point of $s \to \overline{\mathrm{Bun}}_B$ defined by a triple $(\mathcal{F}_G, \mathcal{F}_T, \overline{\kappa})$, and an open substack of $U \subset \overline{\mathrm{Bun}}_B$ containing s. It suffices to show that U contains a point in Bun_B. We assume that $\overline{\kappa}$ has a singularity ν_i defined at distinct Cartier divisors D_i for $i \in I$ corresponding to geometric points $x_i \to \mathrm{Div}_E^1$. We write $x \to \mathrm{Div}_E^I$ for the associated geometric point given by the product of the x_i, and apply the previous Lemma in the situation that $\nu_i^0 = 0$ for all $i \in I$. By Lemma 5.21 (1), we have that $_x\mathrm{Gr}^I_{G, (-w_0(\lambda_i))_{i \in I},E} \overset{L^+B}{\times} {}_xB \simeq \prod_{i \in I} {}_{x_i}\mathrm{Gr}_{G, -w_0(\lambda_i),E} \overset{L^+B}{\times} {}_{x_i}B$ maps to $\overline{\mathrm{Bun}}_B$ via the morphism $\phi_{(\lambda_i)_{i \in I}}$. Now, for the given $(\nu_i)_{i \in I}$, we can choose λ_i for all $i \in I$ such that the weight space $V_{\lambda_i}(w_0(\lambda_i) + \nu_i)$ is non-zero. Then, by Lemma 5.21 (2) and Corollary 4.12, the fiber of $\phi_{(\lambda_i)_{i \in I}}$ over s is non-empty. Therefore, pulling back U, we get a non-empty open subset of $\prod_{i \in I} {}_{x_i}\mathrm{Gr}_{G, -w_0(\lambda_i),E} \overset{L^+B}{\times} {}_{x_i}B$. By the closure relations of Proposition 4.14, we get that this open subset must have non-empty intersection with the open subspace $\prod_{i \in I} {}_{x_i}\mathrm{Gr}_{G, -w_0(\lambda_i),E} \setminus {}_{x_i}\mathrm{S}_{G, -w_0(\lambda_i),E} \overset{L^+B}{\times} {}_{x_i}B$, but, by another application of Lemma 5.21 (2), this tells us that U must have non-trivial intersection with Bun_B. ∎

6. The Normalized Eisenstein Functor and Verdier Duality on Bun_B

Now that we have finished our geometric preparations, we can start to understand the sheaf theory on the moduli stack of B-structures. Our first order of business is to refine our definition of the Eisenstein functor given in the previous section to better respect Verdier duality on the moduli stack Bun_B.

6.1. **The Normalized Eisenstein Functor.** Before proceeding with our analysis of Eisenstein series, we refine the definition of the Eisenstein functor. There is one key problem with our definition, the sheaf $\Lambda[\dim(\mathrm{Bun}_B)]$ is not Verdier self-dual. In particular, the dualizing object K_{Bun_B} on Bun_B is not isomorphic to $\Lambda[2\dim(\mathrm{Bun}_B)]$; this is only v-locally true on Bun_B. To elucidate the problem, note that, given $\overline{\nu} \in \Lambda_{G,B}$, we can consider the natural map $\mathfrak{q}^{\overline{\nu}} : \mathrm{Bun}_B^{\overline{\nu}} \to \mathrm{Bun}_T^{\overline{\nu}} = [\ast/T(\mathbb{Q}_p)]$. Given a character $\kappa_{\overline{\nu}}$ of $T(\mathbb{Q}_p)$, we can pull this character back along $\mathfrak{q}^{\overline{\nu}}$ to get a sheaf on $\mathrm{Bun}_B^{\overline{\nu}}$. These characters give us sheaves on $\mathrm{Bun}_B^{\overline{\nu}}$, which v-locally will be constant, but are not constant on the nose. Our main Theorem is as follows.

Theorem 6.1. *The dualizing object on* Bun_B *is isomorphic to* $\mathfrak{q}^\ast(\Delta_B)[2\dim(\mathrm{Bun}_B)]$, *with* $\Delta_B \in$ $\mathrm{D}(\mathrm{Bun}_T)$ *as in Definition 3.4.*

Before tackling the proof, we record the key consequence of this theorem for us.

Corollary 6.2. *The sheaf* $\mathfrak{q}^\ast(\Delta_B^{1/2})[\dim(\mathrm{Bun}_B)]$ *on* Bun_B *is Verdier self-dual.*

This motivates the definition of the normalized Eisenstein functor.

Definition 6.3. We let $\mathrm{IC}_{\mathrm{Bun}_B} := \mathfrak{q}^\ast(\Delta_B^{1/2})[\dim(\mathrm{Bun}_B)]$. We define the normalized Eisenstein functor:

$$\mathrm{nEis} : \mathrm{D}(\mathrm{Bun}_T) \to \mathrm{D}(\mathrm{Bun}_G)$$

$$F \mapsto \mathfrak{p}_!(\mathfrak{q}^\ast(F) \otimes \mathrm{IC}_{\mathrm{Bun}_B})$$

In particular, we note that we have a natural isomorphism:

$$\mathrm{nEis}(-) \simeq \mathrm{Eis}(- \otimes \Delta_B^{1/2})$$

We now tackle the proof of Theorem 6.1.

6.2. **The Proof of Theorem 6.1.**

Definition 6.4. Let $b \in B(G)_{\mathsf{un}}$ be an unramified element. We recall, by Corollary 2.11, this admits a unique HN-dominant reduction b_T with anti-dominant slope homomorphism, and we can describe the set of all elements mapping to $b \in B(G)$ as $w(b_T)$, for varying $w \in W_b$. Given $\overline{\nu} \in B(T) \simeq \Lambda_{G,B}$ mapping to b, we write $w_{\overline{\nu}} \in W_b$ for the unique element satisfying the relation $\overline{\nu} = w_{\overline{\nu}}(b_T)$.

As a warm up, we explain the proof in the case that $G = \mathrm{GL}_2$ and B is the upper triangular Borel, proving some key lemmas along the way. We recall that in this case $B(T) \simeq \Lambda_{G,B} \simeq \mathbb{Z}^2$ via the Kottwitz invariant, and so we can index the connected components of Bun_B by a pair of integers. We write $d_{\overline{\nu}}$ for the ℓ-cohomological dimension of the connected component $\mathrm{Bun}_B^{\overline{\nu}}$.

Example 6.5. We first consider the case that $\overline{\nu} = (d,d)$ for $d \in \mathbb{Z}$, so $\overline{\nu}$ is already HN-dominant and $w_{\overline{\nu}} = 1$. Note that the connected component $\mathrm{Bun}_B^{\overline{\nu}}$ parametrizes split reductions, and is isomorphic to $[\ast/B(\mathbb{Q}_p)]$. Now we have the following lemma, which is [HKW22, Example 4.4.3].

Lemma 6.6. *Let H be a locally pro-p group then $K_{[\ast/H]} \simeq Haar(H, \Lambda)$, where $Haar(H, \Lambda)$ is the space of Λ-valued Haar measures on H.*

Therefore, the dualizing object on $\mathrm{Bun}_B^{\overline{\nu}}$ is canonically isomorphic to $Haar(B(\mathbb{Q}_p), \Lambda)$, the space of Haar measures on $B(\mathbb{Q}_p)$. We note that, as a $B(\mathbb{Q}_p)$-representation, this is isomorphic to δ_B by definition of the modulus character. Thus, Theorem 6.1 is true in this case.

Given a diamond or v-stack $X \to \mathrm{Spd}\,\mathbb{F}_p$, we define the local systems $\Lambda(d)$ by pulling back the local system on $\mathrm{Spd}\,\mathbb{F}_p$ given by the representation of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ for which the geometric Frobenius acts via scaling by $p^{-d} \in \Lambda^\ast$. We consider the following key example.

Example 6.7. We let $Z_{\mathbb{F}_p} := H^0(O(1))_{\mathbb{F}_p} \to \operatorname{Spd}\mathbb{F}_p$ be the absolute positive Banach-Colmez space parmeterizing sections of the line bundle $O(1)$ on X_S for $S \in \operatorname{Perf}_{\mathbb{F}_p}$, as in [FS21, Section II.2.2]. This is isomorphic to $\operatorname{Spd}\mathbb{F}_p[\![x^{1/p^\infty}]\!]$ [FS21, Proposition II.2.5 (iv)]. We let $\operatorname{Frob}_{Z_{\mathbb{F}_p}}$ denote the geometric Frobenius sending $x \mapsto x^p$. Then, by [Han21, Proposition 4.8], we have a natural isomorphism

$$K_{Z_{\mathbb{F}_p}} \simeq \Lambda[2](1)$$

of étale sheaves on $Z_{\mathbb{F}_p}$. Now, $K_{Z_{\mathbb{F}_p}}$ has an action of $\mathbb{Q}_p$ coming from the action of $\mathbb{Q}_p$ on $Z_{\mathbb{F}_p} = H^0(O(1))_{\mathbb{F}_p}$ by scaling of global sections. This defines for us a character $\kappa : \mathbb{Q}_p \to \Lambda$. We claim that $\kappa(\mathbb{Z}_p)$ is trivial. To see this, consider the v-stack quotient by the $\mathbb{Z}_p$ action:

$$q : Z_{\mathbb{F}_p} \to [Z_{\mathbb{F}_p}/\underline{\mathbb{Z}_p}]$$

Now, by [Han21, Proposition 4.6][5], fixing a Haar measure on $\mathbb{Z}_p$ determines a unique isomorphism:

$$(q_*(K_{Z_{\mathbb{F}_p}}))^{\mathbb{Z}_p} \simeq K_{[K_{Z_{\mathbb{F}_p}}/\underline{\mathbb{Z}_p}]}$$

The only way $\mathbb{Z}_p$ could act non-trivially on $\Lambda(1)$ is if it acted on $Z_{\mathbb{F}_p}$ by a power of geometric Frobenius. It follows that the action is trivial, since there are no non-trivial homomorphisms from $\mathbb{Z}_p$ to $\mathbb{Z}$. Therefore, we conclude that

$$K_{[K_{Z_{\mathbb{F}_p}}/\underline{\mathbb{Z}_p}]} \simeq \Lambda[2](1)$$

as étale sheaves on $[Z_{\mathbb{F}_p}/\underline{\mathbb{Z}_p}]$, which in particular implies that the action of $\mathbb{Z}_p$ is trivial on $q^*(K_{[Z_{\mathbb{F}_p}/\underline{\mathbb{Z}_p}]}) \simeq K_{Z_{\mathbb{F}_p}}$, where this isomorphism follows from [Sch18, Proposition 24.2]. Now, it remains to determine the value of $\kappa(p)$. To elucidate this, we note that, under the identification $Z_{\mathbb{F}_p} \simeq \operatorname{Spd}\mathbb{F}_p[\![x^{1/p^\infty}]\!]$, the element $p \in \mathbb{Q}_p$ acts via the geometric Frobenius $\operatorname{Frob}_{Z_{\mathbb{F}_p}}$ on $Z_{\mathbb{F}_p}$. It follows by the previous isomorphism that the value of the character $\kappa(p)$ is determined by the action of $\operatorname{Frob}_{Z_{\mathbb{F}_p}}$ on $\Lambda(1)$, which is just the multiplication by p^{-1} map on Λ by Definition. In summary, we have concluded an identification $K_{Z_{\mathbb{F}_p}} \simeq j_!j^*[2]$ of sheaves with $\mathbb{Q}_p$-action, where $j_!j^*$ is the rank 1 local system on $Z_{\mathbb{F}_p}$ with $\mathbb{Q}_p$-action given by the norm character. From here, we conclude the analogous isomorphism

$$K_{Z_{\overline{\mathbb{F}}_p}} \simeq j_!j^*[2]$$

of sheaves with $\mathbb{Q}_p$-action over the algebraic closure.

Let's now push this a bit further and consider the case of a general positive absolute Banach-Colmez space $Z_{\mathbb{F}_p} := H^0(O(d))_{\mathbb{F}_p} \to \operatorname{Spd}\mathbb{F}_p$ for $d \geq 1$. Now we don't have such a simple presentation as in the case that $d = 1$; however, we claim that we still have a similar relationship between the geometric Frobenius and the scaling action by $p^{\mathbb{Z}} \in \mathbb{Q}_p$ on $Z_{\mathbb{F}_p}$. To understand this, we invoke the following explicit description. Recall that, for $S \in \operatorname{Perf}$, we can view the adic Fargues-Fontaine curve X_S as given by gluing the open Fargues-Fontaine curve $Y_{S,[1,p]}$ along the map $\phi : Y_{S,[1,1]} \simeq Y_{S,[p,p]}$ induced by the geometric Frobenius, with notation as in [FS21, Section II.1]. We write $B_{S,[1,p]}$ (resp. $B_{S,[1,1]}$) for the ring of functions of $Y_{S,[1,p]}$ (resp. $Y_{S,[1,1]}$). We make use of the following lemma, which follows from the analysis in [FS21, Section II.2].

Lemma 6.8. *For $d \in \mathbb{Z}$, let $O(d)_{X_S}$ be the natural line bundle of degree d on X_S. We have an isomorphism:*

$$R\Gamma(X_S, O(d)) \simeq \{B_{S,[1,p]} \xrightarrow{\phi - p^d} B_{S,[1,1]}\}$$

---

[5]Strictly speaking, this Proposition only works for the maximal pro-p subgroup $\mathbb{Z}_p \subset \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z} \simeq \mathbb{Z}_p$, but it is easy to see the proof extends to further quotienting out by a finite group.

Therefore, if we write $\mathbf{B}_{[1,p]}$ for the sheaf of rings on $\mathrm{Perf}_{\mathbb{F}_p}$ defined by sending $S \in \mathrm{Perf}_{\mathbb{F}_p}$ to the global sections of $Y_{S,[1,p]}$ then we have an isomorphism

$$H^0(\mathcal{O}(d)) \simeq \mathbf{B}_{[1,p]}^{\phi = p^d}$$

of diamonds over $\mathrm{Spd}(\mathbb{F}_p)$. As in the above example, this identification tells us that action of the geometric Frobenius $\mathrm{Frob}_{Z_{\mathbb{F}_p}}$ on $Z_{\mathbb{F}_p}$ is the same as the scaling action by $p^d \in \mathbb{Q}_p$ on $Z_{\mathbb{F}_p}$. Again, by [Han21, Proposition 4.8], we have an isomorphism

$$K_{Z_{\mathbb{F}_p}} \simeq \Lambda[2d](d)$$

and, it follows that $p^d \in \mathbb{Q}_p$ acts on the sheaf Λ by $p^{-d} \in \Lambda$, which, arguing as above, allows us to conclude an isomorphism

$$K_{Z_{\mathbb{F}_p}} \simeq j_! j^*[2d]$$

of sheaves with $\mathbb{Q}_p$-action.

We record the content of the above example as a Lemma for future use.

Lemma 6.9. *Let $d \in \mathbb{N}_{\geq 1}$ be a positive integer and consider the absolute positive Banach-Colmez space $H^0(\mathcal{O}(d))_{\mathbb{F}_p} \to \mathrm{Spd}\,\mathbb{F}_p$. Then we have an isomorphism $K_{H^0(\mathcal{O}(d))_{\mathbb{F}_p}} \simeq j_! j^*[2d]$ (resp. $K_{H^0(\mathcal{O}(d))_{\overline{\mathbb{F}}_p}} \simeq j_! j^*[2d]$) as sheaves with $\mathbb{Q}_p$-action, where $j_! j^*$ denotes the rank 1 sheaf on $H^0(\mathcal{O}(d))_{\mathbb{F}_p}$ (resp. $H^0(\mathcal{O}(d))_{\overline{\mathbb{F}}_p}$) that transforms under the scaling action by $\mathbb{Q}_p$ via the norm character.*

Remark 6.10. In what follows, it will be important to formalize this in terms of v-stacks. If we consider the v-stack quotient

$$[H^0(\mathcal{O}(d))_{\overline{\mathbb{F}}_p}/\underline{\mathbb{Q}}_p] \to \mathrm{Spd}\,\overline{\mathbb{F}}_p$$

then this admits a natural map $q : [H^0(\mathcal{O}(d))/\underline{\mathbb{Q}}_p] \to [\mathrm{Spd}\,\overline{\mathbb{F}}_p/\underline{\mathbb{Q}}_p]$ to a classifying stack. Then the above isomorphism descends to an identification $K_{[H^0(\mathcal{O}(d))_{\overline{\mathbb{F}}_p}/\underline{\mathbb{Q}}_p]} \simeq q^*(j_! j^*)[2d]$, where we recall that there is an identification $\mathrm{D}([\mathrm{Spd}\,\overline{\mathbb{F}}_p/\underline{\mathbb{Q}}_p]) \simeq \mathrm{D}(\mathbb{Q}_p, \Lambda)$ of the derived category of sheaves on the classifying stack $[\mathrm{Spd}\,\overline{\mathbb{F}}_p/\underline{\mathbb{Q}}_p]$ with the derived category of smooth $\mathbb{Q}_p$-representations on Λ-modules.

Let's now push this a bit further and prove an analogous claim for negative Banach-Colmez spaces.

Lemma 6.11. *Let $d \in \mathbb{N}_{\geq 1}$ and consider the negative absolute Banach-Colmez space $H^1(\mathcal{O}(-d)) \to \mathrm{Spd}\,\mathbb{F}_p$. Then we have isomorphisms $K_{H^1(\mathcal{O}(d))_{\mathbb{F}_p}} \simeq j_! j^{-1}[2d]$ (resp. $K_{H^1(\mathcal{O}(d))_{\overline{\mathbb{F}}_p}} \simeq j_! j^{-1}[2d]$) as sheaves with $\mathbb{Q}_p$-action.*

Proof. Using the explicit description above, we know that, if we consider the sheaf of rings $\mathbf{B}_{[1,p]}$ and $\mathbf{B}_{[1,1]}$ on $\mathrm{Perf}_{\mathbb{F}_p}$ given by taking global sections of $Y_{S,[1,p]}$ (resp. $Y_{S,[1,1]}$), there is an isomorphism

$$H^1(\mathcal{O}(-d))_{\mathbb{F}_p} \simeq \mathrm{Coker}(\mathbf{B}_{[1,p]} \xrightarrow{\phi - p^d} \mathbf{B}_{[1,1]})$$

of v-sheaves on $\mathrm{Perf}_{\mathbb{F}_p}$. As a consequence of this, we can deduce that the geometric Frobenius $\mathrm{Frob}_{Z_{\mathbb{F}_p}}$ on $Z_{\mathbb{F}_p} = H^0(\mathcal{O}(-d))_{\mathbb{F}_p}$ agrees with the scaling action by $p^{-d} \in \mathbb{Q}_p$. Moreover, as in the positive case, we have an isomorphism

$$K_{H^1(\mathcal{O}(d))_{\mathbb{F}_p}} \simeq \Lambda[2d](d)$$

of étale sheaves. One can deduce this from the positive case [Han21, Proposition 4.8(ii)] and the proof of [FS21, Proposition II.2.5 (ii)]. Now, arguing as in the case of positive Banach-Colmez spaces, we deduce that

$$K_{H^1(\mathcal{O}(d))_{\mathbb{F}_p}} \simeq j_! j^{-1}[2d]$$

as sheaves with $\mathbb{Q}_p$-action. The claim over $\overline{\mathbb{F}}_p$ follows.

With these two lemmas in hand, let's continue our calculation of the dualizing object on Bun_B in the case that $G = \mathrm{GL}_2$. We will return to working over the base $* = \mathrm{Spd}\,\overline{\mathbb{F}}_p$ in all that follows.

Example 6.12. Consider the case where $\bar{\nu} = (-d, -e)$ is HN-dominant, so that $d > e$. In this case, $d_{\bar{\nu}} = e - d$ and $w_{\bar{\nu}} = 1$. We note that $d_{\bar{\nu}}$ is negative, so to keep track of the sign change, we consider the absolute value $|d_{\bar{\nu}}|$. The connected component $\mathrm{Bun}_B^{\bar{\nu}}$ just parametrizes the split extensions of $\mathcal{O}(d)$ by $\mathcal{O}(e)$. Therefore, its topological space is just a point. More precisely, if b corresponds to the bundle $\mathcal{O}(d) \oplus \mathcal{O}(e)$ then it is isomorphic $[*/J_b]$, where J_b parametrizes automorphisms of $\mathcal{O}(d) \oplus \mathcal{O}(e)$. The group diamond J_b is isomorphic to

$$\begin{pmatrix} \mathrm{Aut}(\mathcal{O}(d)) & Hom(\mathcal{O}(e), \mathcal{O}(d)) \\ 0 & \mathrm{Aut}(\mathcal{O}(e)) \end{pmatrix} \simeq \begin{pmatrix} \underline{\mathbb{Q}}_p & H^0(\mathcal{O}(|d_{\bar{\nu}}|)) \\ 0 & \underline{\mathbb{Q}}_p \end{pmatrix}$$

In particular, note that in this case $J_b(\mathbb{Q}_p) = T(\mathbb{Q}_p) \simeq \mathbb{Q}_p \times \mathbb{Q}_p$. We need to compute the dualizing object of $[*/J_b]$. To do this, we consider the natural map

$$p : [*/J_b] \to [*/\underline{T(\mathbb{Q}_p)}]$$

induced via the map $J_b \to \underline{T(\mathbb{Q}_p)}$ given by quotienting out by the unipotent part. This map has a section given by the inclusion $\underline{T(\mathbb{Q}_p)} \hookrightarrow J_b$, and we denote this by $s : [*/\underline{T(\mathbb{Q}_p)}] \to [*/J_b]$. First off note, by Lemma 6.6, that the dualizing object on $[*/\underline{T(\mathbb{Q}_p)}]$ can be identified with the set of Haar measures on $T(\mathbb{Q}_p)$. Now, since $T(\mathbb{Q}_p)$ is unimodular, this implies that, as a $T(\mathbb{Q}_p)$-representation, the sheaf is trivial. Therefore, we are reduced to computing $p^!(\Lambda)$. To do this, note that the section s is a fibration in the positive Banach-Colmez space $H^0(\mathcal{O}(|d_{\bar{\nu}}|))$. It therefore follows by the proof of [FS21, Proposition V.2.1] that adjunction induces an isomorphism:

$$s^! s_!(\Lambda) \simeq \Lambda$$

However, since s is a section of p, that gives us a natural isomorphism:

$$s_!(\Lambda) \simeq p^!(\Lambda)$$

Therefore, we are reduced to computing $s_!(\Lambda)$. We now have the following.

Lemma 6.13. *There is a natural isomorphism*

$$s^!(\Lambda) \simeq \delta_B[2|d_{\bar{\nu}}|]$$

of sheaves on $[*/J_b(\mathbb{Q}_p)] \simeq [*/\underline{T(\mathbb{Q}_p)}]$, where $s : [*/\underline{T(\mathbb{Q}_p)}] \to [*/J_b]$ is the natural map.

Proof. We consider the Cartesian diagram:

$$\begin{array}{ccc} H^0(|d_{\bar{\nu}}|) & \xrightarrow{\tilde{s}} & * \\ \downarrow{\tilde{q}} & & \downarrow{q} \\ \left[*/\underline{T(\mathbb{Q}_p)}\right] & \xrightarrow{s} & [*/J_b] \end{array}$$

By base-change, we have a natural isomorphism $\tilde{q}^* s^!(\Lambda) \simeq \tilde{s}^!(\Lambda)$ and, by Lemma 6.9, the RHS is isomorphic to $j_! j^*[2|d_{\bar{\nu}}|]$ as a sheaf with $\mathbb{Q}_p$-action. However, we want to understand how this transforms as a $T(\mathbb{Q}_p)$ representation. To do this, we note that the $T(\mathbb{Q}_p)$ action on $H^0(\mathcal{O}(|d_{\bar{\nu}}|)) \simeq H^0(\mathcal{O}(e) - \mathcal{O}(d))$ comes from the semi-direct product structure on J_b

$$J_b = \begin{pmatrix} \mathrm{Aut}(\mathcal{O}(d)) & Hom(\mathcal{O}(e), \mathcal{O}(d)) \\ 0 & \mathrm{Aut}(\mathcal{O}(e)) \end{pmatrix} = \begin{pmatrix} \underline{\mathbb{Q}}_p & H^0(\mathcal{O}(e) - \mathcal{O}(d)) \\ 0 & \underline{\mathbb{Q}}_p \end{pmatrix} \simeq T(\mathbb{Q}_p) \ltimes H^0(\mathcal{O}(d - e))$$

It follows that if we consider the map

$$T(\mathbb{Q}_p) \to \mathbb{Q}_p$$

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mapsto ad^{-1}$$

then $T(\mathbb{Q}_p)$ acts via composing this map with the scaling action on $H^0(O(jd_{\overline{\nu}}j))$. This implies that $\tilde{q}^* s^!(\Lambda) \simeq \tilde{s}^!(\Lambda) \simeq \delta_B^{-1}[2jd_{\overline{\nu}}j]$ as a sheaf with $T(\mathbb{Q}_p)$-action. This was the desired claim.

Using the previous adjunction, the lemma tells us that $p^!(\Lambda) \simeq s_!(\Lambda) \simeq \delta_B^{-1}[-2jd_{\overline{\nu}}j]$ as a sheaf with $T(\mathbb{Q}_p)$ action. This is seemingly the inverse of the character we want. We need to be a bit careful. As we will see below, due to the minus sign when passing between isocrystals and bundles, the natural map

$$\mathsf{q}^{\overline{\nu}} : \mathrm{Bun}_B^{\overline{\nu}} \to \mathrm{Bun}_T^{\overline{\nu}}$$

identifies with the projection $p : [*/\mathcal{J}_b] \to [*/T(\mathbb{Q}_p)]$ composed with the involution of $T(\mathbb{Q}_p)$ given by conjugation by the element of longest length. Therefore, the pullback $\mathsf{q}^{\overline{\nu}\,*}(\delta_B)$ naturally identifies with $\delta_B^{w_0} = \delta_B^{-1}$ under the $T(\mathbb{Q}_p)$ action described above.

Now let's consider the anti-dominant case.

Example 6.14. Let $\overline{\nu} = (-d, -e)$ be anti HN-dominant so that $d < e$. Then $d_{\overline{\nu}} = e - d = jd_{\overline{\nu}}j$ is positive, and $w_{\overline{\nu}} = w_0$ is the element of longest length. The connected component $\mathrm{Bun}_B^{\overline{\nu}}$ is isomorphic to $[H^1(O(-d_{\overline{\nu}}))/\underline{T(\mathbb{Q}_p)}]$, where $T(\mathbb{Q}_p)$ acts via interpreting $H^1(O(-d_{\overline{\nu}}))$ as a space parameterizing extensions

$$0 \to O(d) \to E \to O(e) \to 0$$

on X_S, and $T(\mathbb{Q}_p)$ acts via the identification $T(\mathbb{Q}_p) \simeq (\mathrm{Aut}(O(d)), \mathrm{Aut}(O(e)))$. Now we can factor the structure map as

$$[H^1(O(-d_{\overline{\nu}}))/\underline{T(\mathbb{Q}_p)}] \xrightarrow{f} [*/\underline{T(\mathbb{Q}_p)}] \to *$$

and, using that $T(\mathbb{Q}_p)$ is unimodular, we reduce to computing $f^!(\Lambda)$ as before. We consider the Cartesian diagram

$$\begin{array}{ccc} H^1(O(-d_{\overline{\nu}})) & \xrightarrow{\ f\ } & * \\ \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle q} \\ \left[H^1(O(-d_{\overline{\nu}}))/\underline{T(\mathbb{Q}_p)}\right] & \xrightarrow{\ f\ } & \left[*/\underline{T(\mathbb{Q}_p)}\right] \end{array}$$

and again, by base-change, this gives us an isomorphism $\tilde{q}^* f^!(\Lambda) \simeq \tilde{f}^!(\Lambda)$. Now, applying Lemma 6.11, we deduce that $\tilde{q}^* f^!(\Lambda)$ is isomorphic to $j \cdot j^{-1}[2d_{\overline{\nu}}]$ as a sheaf with $\mathbb{Q}_p$ action. We note that the $T(\mathbb{Q}_p)$-action comes from the identification $H^1(O(d) \otimes O(e)-) \simeq H^1(O(-d_{\overline{\nu}}))$. Therefore, $T(\mathbb{Q}_p)$ acts via the map

$$T(\mathbb{Q}_p) \to \mathbb{Q}_p$$

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \to ad^{-1}$$

composed with the action of $\mathbb{Q}_p$ on $H^1(O(-d_{\overline{\nu}}))$ by scaling of global sections. As a consequence, we deduce that, as a $T(\mathbb{Q}_p)$-representation, $\Lambda[2d_{\overline{\nu}}](d_{\overline{\nu}})$ is isomorphic to $\delta_B^{-1}[2d_{\overline{\nu}}]$. However, as seen in the previous example, the pullback $\mathsf{q}^{\overline{\nu}}$ twists the above $T(\mathbb{Q}_p)$ action by the element of longest length, so this gives the desired claim.

We will need some formalism to move beyond the case of GL_2. Let G now be a general quasi-split connected reductive group. We let $b \in B(G)_{\mathsf{un}}$ be an unramified element, and set b_T to be the HN-dominant reduction as before. We let $\mathcal{J}_b$ be the group diamond parameterizing automorphisms

of the bundle $\mathcal{F}_b$ associated to b. Now, we can write $\mathcal{J}_b{}^{\geq\lambda}$ for the sub-diamond of automorphisms γ of $\mathcal{F}_b$ such that

$$(\gamma - 1)(\rho_*(\mathcal{F}_b))^{\geq\lambda^0} \subset (\rho_*\mathcal{F}_b)^{\geq\lambda^0+\lambda}$$

for all representations ρ of G. We set $\mathcal{J}_b^{>\lambda} = \bigcap_{\lambda^0>\lambda}\mathcal{J}_b{}^{\geq\lambda^0}$. We note that, since $J_b(\mathbb{Q}_p)$ is the automorphism group of the isocrystal defined by b, and, for $S \in \mathrm{Perf}$, there is an identification: $H^0(X_S, \mathcal{O}_{X_S}) = \underline{\mathbb{Q}_p}(S)$, we have a natural injection

$$\underline{J_b(\mathbb{Q}_p)} \hookrightarrow \mathcal{J}_b$$

which has a natural section given by letting $\mathcal{J}_b$ act on the graded-pieces of the HN-filtration. This gives us a semi-direct product decomposition

$$\mathcal{J}_b \simeq \underline{J_b(\mathbb{Q}_p)} \ltimes \mathcal{J}_b^{>0}$$

of group diamonds (See [FS21, Proposition III.5.1]). We can relate this to B-bundles as follows. We consider the B-torsor $Q := \mathcal{F}_{b_T} \times^T B$ over X_S for $S \in \mathrm{Perf}$. We suppose now that b_T is the element corresponding to Levi factor of the HN-reduction of $\mathcal{F}_b$. Then, as in [FS21, Proposition III.5.1.1][6], we have equalities

$$\mathcal{J}_b(S) = Q(X_S)$$
$$\mathcal{J}_b^U(S) = (E_{b_T} \times^T U)(X_S) = R_u Q(X_S)$$

where U is the unipotent radical of B.

Remark 6.15. This identification is a manifestation of the following easy fact, that we will use implicitly throughout. Given a bundle E on X_S, the set of E-torsors on X_S is parametrized by $H^1(X_S, E)$, and the automorphisms of such E-torsors are parametrized by $H^0(X_S, E)$.

In particular, since Q defines a B-structure on $\mathcal{F}_b$, every such global section induces an automorphism of $\mathcal{F}_b$. Moreover, since b_T corresponds to the Levi factor of an HN-reduction of $\mathcal{F}_b$ this will imply that every element of $\mathcal{J}_b(S)$ can be obtained in this way. Let's now consider the case where b_T is HN-dominant, but does not correspond to an HN-reduction of $\mathcal{F}_b$. In other words, there exist some positive roots $\hat{\alpha}$ such that $\langle\hat{\alpha}, \nu_{b_T}\rangle = 0$ (cf. Example 6.5). In this case, all we can conclude is a strict containment $Q(X_S) \subsetneq \mathcal{J}_b(S)$. In particular, if we write $Q^{>0}(X_S)$ for the subset mapping to $\mathcal{J}_b^{>0}$ then we still have an isomorphism

$$\mathcal{J}_b^{>0}(S) \simeq Q^{>0}(X_S)$$

but, on the slope 0 part, $Q^{=0}(X_S)$ is only properly contained in $\underline{J_b(\mathbb{Q}_p)}(S)$. The points of $Q^{=0}(S)$ will be identified with $\underline{B_b(\mathbb{Q}_p)}(S)$, where $B_b \subset J_b$ is the Borel subgroup defined in Lemma 2.12, and $R_u Q^{=0}$ is identified with the unipotent radical of this Borel. More precisely, if J_b is an inner twisting of $M_b \subset G$ then M_b is the Levi subgroup in G corresponding to the positive simple roots $\hat{\alpha}_i$ such that $\langle\hat{\alpha}_i, \nu_{b_T}\rangle = 0$, by definition of M_b as the maximal Levi admitting a basic reduction of b. Now let's refine this further, recall that U has a filtration by commutator subgroups

$$(1)\qquad U_{i+1} \subset U_i \subset \cdots \subset U_1 \subset U_0 = U$$

where U_i is generated by the root subgroups $U_{\hat{\alpha}}$ for $\hat{\alpha} = \sum_{i\in J} n_i\hat{\alpha}_i$, where $\sum_{i\in J} n_i > i$. As a consequence, we note that

$$U_i/U_{i-1} = \bigoplus_{\hat{\alpha}} U_{\hat{\alpha}}$$

where the direct sum runs over all roots $\hat{\alpha} = \sum_{i\in J} n_i\hat{\alpha}_i$ such that $\sum_{i\in J} n_i = i$. This gives rise to a filtration on $R_u Q$ whose graded pieces will be a direct sum of the line bundles

$$\hat{\alpha}_*(Q)$$

---

[6]Here Fargues-Scholze consider instead the opposite Borel B^-; however, since we have defined the HN-dominant reduction b_T to have anti-dominant slope homomorphism this agrees with their definition.

of degree equal to $\langle \hat{\alpha}, \nu_b \rangle$. This allows us to write $\mathcal{J}_b^{>0}$ as an iterated fibration of positive Banach-Colmez spaces

$$H^0(\hat{\alpha}\ (Q))$$

for $\hat{\alpha}$ such that $\deg(\hat{\alpha}\ (Q)) = \langle \hat{\alpha}, \nu_b \rangle > 0$. The subgroup $T(\mathbb{Q}_p) \quad J_b(\mathbb{Q}_p)$ will act on this positive Banach-Colmez space via the previous semi-direct product structure. The action will be the composite of the character

$$T(\mathbb{Q}_p) \ !\ \mathbb{Q}_p$$

$$t \ \mapsto\ \mathrm{Ad}(t/\mathfrak{g}_{w_0(\hat{\alpha})}) = \mathrm{Ad}(t/\mathfrak{g}_{\hat{\alpha}})^{-1}$$

and the scaling action on $\mathbb{Q}_p$, where $\mathfrak{g}_{\hat{\alpha}}$ is the $\hat{\alpha}$-isotypic root space of $\mathrm{Lie}(G)$. This action will respect the projection $\mathrm{Bun}_B^{b_T} \ !\ \mathrm{Bun}_T^{b_T} = [\ /T(\mathbb{Q}_p)]$, and the minus sign occurs because of the passage between isocrystals and bundles. These observations combined with the above Lemmas relating the dualizing objects on positive Banach-Colmez to norm characters give us everything we need to pin down the dualizing object on $\mathrm{Bun}_B^{\overline{\nu}}$ in the case that $\overline{\nu}$ is HN-dominant.

Proposition 6.16. *Let $b_T\ 2\ B(T)$ be an element with anti-dominant slope homomorphism with respect to B. The dualizing object $K_{\mathrm{Bun}_B^{b_T}}$ on $\mathrm{Bun}_B^{b_T}$ is isomorphic to $(\mathfrak{q}^{b_T})\ (\delta_B)[2d_{b_T}]$.*

Proof. Since b_T has anti-dominant slope homomorphism its HN-slopes are dominant, and therefore $\mathrm{Bun}_B^{b_T}$ parametrizes split reductions. In particular, its underlying topological space is just a point. More specifically, it is isomorphic to $[\ /G_{b_T}]$, where $G_{b_T}(S) := Q(X_S)$, and Q is the torsor defined above. The semi-direct product structure on $\mathcal{J}_b$ and the above discussion imply that we have a semi-direct product structure

$$G_{b_T} \ '\ \underline{B_b(\mathbb{Q}_p)} \ltimes \mathcal{J}_b^{>0}$$

of G_{b_T}. Therefore, we have a natural map

$$[\ /G_{b_T}] \ !\ [\ /\underline{B_b(\mathbb{Q}_p)}]$$

which when composed with $[\ /\underline{B_b(\mathbb{Q}_p)}] \ !\ [\ /\underline{T(\mathbb{Q}_p)}]$ identifies with the map $\mathrm{Bun}_B^{b_T} \ !\ \mathrm{Bun}_T^{b_T}$. Now, by Lemma 6.6, the dualizing object on $[\ /\underline{B_b(\mathbb{Q}_p)}]$ is defined by the modulus character $\delta^{=0} := \delta_{B_b} : B_b(\mathbb{Q}_p) \ !\ T(\mathbb{Q}_p) \ !\ \Lambda$ where the first map is the natural projection and the last map, by definition of B_b, will be given by

$$t \ \mapsto\ \prod_{\substack{\hat{\alpha}>0 \\ \langle\hat{\alpha},\nu_b\rangle=0}} j\mathrm{Ad}(t/\mathfrak{g}_{\hat{\alpha}})j$$

Therefore, we reduce to computing the dualizing object with respect to the map:

$$p : [\ /G_{b_T}] \ !\ [\ /\underline{B_b(\mathbb{Q}_p)}]$$

This map has a natural section

$$s : [\ /\underline{B_b(\mathbb{Q}_p)}] \ !\ [\ /G_{b_T}]$$

the fibers of which are given by $\mathcal{J}_b^{>0}$, which, as seen above, is an iterated fibration of positive Banach-Colmez spaces. Therefore, by [FS21, Proposition V.2.1], we deduce that the adjunction

$$s^! s_!(\Lambda) \ '\ \Lambda$$

is an isomorphism. Arguing as in Example 6.12, this reduces us to showing that

$$s^!(\Lambda) \ '\ (\delta^{>0})^{-1}[\ 2d_{b_T}]$$

where $\delta^{>0} : T(\mathbb{Q}_p) \ !\ \Lambda$ is the character

$$t \ \mapsto\ \prod_{\substack{\hat{\alpha}>0 \\ \langle\hat{\alpha},\nu_b\rangle \not= 0}} j\mathrm{Ad}(t/\mathfrak{g}_{\hat{\alpha}})j$$

so that we have $\delta_B = \delta^{=0}\delta^{>0}$. To see this, we consider the Cartesian diagram

$$
\begin{array}{ccc}
\mathcal{J}_b^{>0} & \xrightarrow{\ s\ } & \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\
[\ /G_{b_T}] & \xrightarrow{\ s\ } & \left[\ /\underline{B_b(\mathbb{Q}_p)}\right]
\end{array}
$$

which, by base-change, gives us an isomorphism $\tilde{q}\ s^!(\Lambda) \simeq \tilde{s}^!(\Lambda)$. As above, we can write $\mathcal{J}_b^{>0}$ as an iterated fibration of the positive Banach-Colmez spaces $H^0(\hat{\alpha}\ (Q))$ for $\hat{\alpha} > 0$ such that $\langle\hat{\alpha}, \nu_b\rangle > 0$. By Lemma 6.9, the dualizing object on $H^0(\hat{\alpha}\ (Q))$ is isomorphic to $j\ j[2\langle\hat{\alpha},\nu_b\rangle]$ as a $\mathbb{Q}_p$ representation. However, as a $B_b(\mathbb{Q}_p)$ representation, this is acted on by the natural surjection $B_b(\mathbb{Q}_p) \to T(\mathbb{Q}_p)$ composed with the character

$$
T(\mathbb{Q}_p) \to \mathbb{Q}_p
$$
$$
t \mapsto \mathrm{Ad}(t/\mathfrak{g}_{\hat{\alpha}})^{-1}
$$

and the $\mathbb{Q}_p$ action via scaling the global sections of $\hat{\alpha}\ (Q)$. Hence, we obtain that the dualizing object on $H^0(\hat{\alpha}\ (Q))$ as a $B_b(\mathbb{Q}_p)$-representation is given by $|\mathrm{Ad}(t/\mathfrak{g}_{\hat{\alpha}})|^{-1}$. Therefore, by using the formula

$$
(g\ f)^!(\Lambda) = g^!(\Lambda)\ g\ (f^!(\Lambda))
$$

for cohomologically smooth morphisms f and g, we deduce that the dualizing object on $\mathcal{J}_b^{>0}$ is isomorphic to

$$
\Lambda\Big[\sum_{\substack{\hat{\alpha}>0 \\ \langle\hat{\alpha},\nu_b\rangle\neq 0}} \langle\hat{\alpha}, \nu_b\rangle\Big]\Big(\prod_{\substack{\hat{\alpha}>0 \\ \langle\hat{\alpha},\nu_b\rangle\neq 0}} |\mathrm{Ad}(t/\mathfrak{g}_{\hat{\alpha}})|^{-1}\Big) = \Lambda[2\langle 2\hat\rho, \nu_b\rangle](\delta^{>0})^{-1} = \Lambda[\ 2d_{b_T}](\delta^{>0})^{-1}
$$

as a sheaf with $B_b(\mathbb{Q}_p)$-action.

We now push this further. Let $\bar{\nu} \in B(T)$ be an element mapping to $b \in B(G)_{\mathrm{un}}$ with HN-dominant reduction b_T. We write $\bar{\nu} = w(b_T)$ for $w := w_{\bar{\nu}} \in W_b$ identified with a representative of minimal length in W_G. We consider the bundle

$$
Q_{\bar{\nu}} = \mathcal{F}_{b_T}\ ^T B^w = \mathcal{F}_{w(b_T)}\ ^T B
$$

and

$$
R_u Q_{\bar{\nu}} = \mathcal{F}_{b_T}\ ^T U^w
$$

Now, since $Q_{\bar{\nu}}$ defines a reduction of $\mathcal{F}_b$, we have as before an inclusion

$$
Q_{\bar{\nu}}(X_S) \hookleftarrow \mathcal{J}_b(S)
$$

giving rise to a slope filtration on $Q_{\bar{\nu}}$. This gives the following definition.

Definition 6.17. For $\bar{\nu} = w(b_T) \in B(T)$ mapping to $b \in B(G)_{\mathrm{un}}$ as above, we let $G_{\bar{\nu}}$ be the group diamond on Perf whose S-valued points are equal to $Q_{\bar{\nu}}(X_S)$. For any $\lambda \in \mathbb{Q}$, we define subsets $G_{\bar{\nu}}^{\lambda}$ and $G_{\bar{\nu}}^{>\lambda}$, as above. We get a semi-direct product structure

$$
G_{\bar{\nu}}^{>0} \rtimes G_{\bar{\nu}}^{=0}
$$

where $G_{\bar{\nu}}^{>0}\ \mathcal{J}_b^{>0}$ and $G_{\bar{\nu}}^{=0}\ \underline{J_b(\mathbb{Q}_p)}$.

In particular, $G_{\bar{\nu}}^{=0}$ will be identified with the Borel subgroup $\underline{B_b^w(\mathbb{Q}_p)} = \underline{B_b(\mathbb{Q}_p)}$, the constant group diamond defined by the standard Borel B_b conjugated by w. Here the equality follows since we are identifying $w \in W_b$ with a representative of minimal length. Now, as above, $R_u Q_w$ has a filtration given by the natural filtration on U_w

$$
(1) \qquad U_{i+1}\ U_i \qquad U_1\ U_0 = U_w
$$

where $U_{i-1}/U_i \simeq \bigoplus_{\hat{\alpha}>0} U_{w(\hat{\alpha})}$, and $\hat{\alpha}$ runs over positive roots $\hat{\alpha} = \sum_{i \in J} n_i \hat{\alpha}_i$ satisfying $\sum_{i \in J} n_i = i$. We have a containment $G_{\overline{\nu}}^{\geq 0}(S) \subset \mathcal{J}_b^{>0}(X_S)$, which allows us to write $G^{>0}$ as an iterated fibration of positive Banach-Colmez spaces $H^0((w(\hat{\alpha}))(Q))$ for $\hat{\alpha}$ such that $\langle w(\hat{\alpha}), \nu_b \rangle > 0$, which is equivalent to insisting that $\langle \hat{\alpha}, \nu_b \rangle \not\leq 0$ and $w(\hat{\alpha}) > 0$. Now, with this in hand, let's prove Theorem 6.1.

Proof. (Theorem 6.1) We begin with the following Lemma.

Lemma 6.18. *If* $\overline{\nu} = w(b_T) \in B(T)$ *is an element mapping to* $b \in B(G)_{\mathsf{un}}$ *with HN-dominant reduction* b_T *then the ℓ-dimension $d_{\overline{\nu}}$ of the connected component* $\mathrm{Bun}_B^{\overline{\nu}}$ *is equal to*

$$\langle 2\hat{\rho}, \nu_b \rangle - 2\langle 2\hat{\rho}^w, \nu_{b_T} \rangle$$

where $2\hat{\rho}^w$ is the sum of the positive roots $\hat{\alpha}$ such that $w(\hat{\alpha}) > 0$.

Proof. By definition $d_{\overline{\nu}} = \langle 2\hat{\rho}, w(b_T) \rangle$, and this is easily identified with the above quantity, recalling that $w_0(\nu_{b_T}) = \nu_b$ by construction.

Write $\overline{\nu} = w(b_T)$ as in the lemma. The key point is that we can factor the structure morphism $\mathrm{Bun}_B^{\overline{\nu}} \to *$ as

$$\mathrm{Bun}_B^{\overline{\nu}} \xrightarrow{f} [*/G_{\overline{\nu}}] \to [*/\underline{B_b(\mathbb{Q}_p)}] \to *$$

We can fit the projection $\mathfrak{q}^{\overline{\nu}} : \mathrm{Bun}_B^{\overline{\nu}} \to \mathrm{Bun}_T^{\overline{\nu}} = [*/\underline{T(\mathbb{Q}_p)}]$ into this picture as follows. In particular, this is equal to the map $\mathrm{Bun}_B^{\overline{\nu}} \to [*/\underline{B_b(\mathbb{Q}_p)}] \to [*/\underline{T(\mathbb{Q}_p)}] \simeq [*/\underline{T(\mathbb{Q}_p)}]$, where the second to last map is the natural projection and the isomorphism is given by conjugating by w. It follows that we want to show that $K_{\mathrm{Bun}_B} \simeq \delta_B^w[2(\langle 2\hat{\rho}, \nu_b \rangle - 2\langle 2\hat{\rho}_G^w, \nu_{b_T} \rangle)]$ with respect to the $T(\mathbb{Q}_p)$-action coming from the natural projection. Let us look at the modulus character $\delta_B^w : T(\mathbb{Q}_p) \to \mathbb{Q}_p$, this is given by

$$t \mapsto |\det(\mathrm{Ad}(t|\mathrm{Lie}(U^w))| = \prod_{\hat{\alpha}>0} |\mathrm{Ad}(t|\mathfrak{g}_{w(\hat{\alpha})})| = \prod_{\substack{\hat{\alpha}>0 \\ w(\hat{\alpha})<0}} |\det(\mathrm{Ad}(t|\mathfrak{g}_{ww_0(\hat{\alpha})}))|^{-1} \prod_{\substack{\hat{\alpha}>0 \\ w(\hat{\alpha})>0}} |\det(\mathrm{Ad}(t|\mathfrak{g}_{w(\hat{\alpha})}))|$$

where the product runs over all positive roots $\hat{\alpha} > 0$. The dualizing object on $[*/\underline{B_b(\mathbb{Q}_p)}]$ is given by the character defined by composing the surjection $B_b(\mathbb{Q}_p) \to T(\mathbb{Q}_p)$ with the map

$$t \mapsto \prod_{\substack{\hat{\alpha}>0 \\ \langle \hat{\alpha}, \nu_b \rangle = 0}} |\mathrm{Ad}(t|\mathfrak{g}_{\hat{\alpha}})| = \prod_{\substack{\hat{\alpha}>0 \\ \langle \hat{\alpha}, \nu_b \rangle = 0}} |\mathrm{Ad}(t|\mathfrak{g}_{w(\hat{\alpha})})|$$

which we denote by $\delta^{=0}$. Here equality follows since we are identifying $w \in W_G/W_{M_b}$ with a representative $w \in W_G$ of minimal length. Analogously, by arguing as in the proof of Proposition 6.16, we can see that the dualizing complex with respect to the map $[*/G_{\overline{\nu}}] \to [*/\underline{B_b(\mathbb{Q}_p)}]$ is given by $\delta^{>0}[-2\langle 2\hat{\rho}_G^w, \nu_b \rangle]$. Here $\delta^{>0}$ is defined by

$$t \mapsto \prod_{\substack{\hat{\alpha}>0 \\ w(\hat{\alpha})>0 \\ \langle \hat{\alpha}, \nu_b \rangle \not\leq 0}} |\det(\mathrm{Ad}(t|\mathfrak{g}_{w(\hat{\alpha})}))|$$

Thus, it suffices to show that

$$f^!(\Lambda) \simeq \Lambda[2(\langle 2\hat{\rho}, \nu_b \rangle - \langle 2\hat{\rho}_G^w, \nu_{b_T} \rangle)]\delta^{<0}$$

where $\delta^{<0}$ is the character defined by

$$t \mapsto \prod_{\substack{\hat{\alpha}>0 \\ w(\hat{\alpha})<0}} |\det(\mathrm{Ad}(t|\mathfrak{g}_{ww_0(\hat{\alpha})}))|^{-1}$$

so that $\delta_B^w = \delta^{>0}\delta^{=0}\delta^{<0}$, where we note that if $w(\hat{\alpha}) < 0$ it is automatic that $\langle\hat{\alpha}, \nu_b\rangle \not\le 0$ since w is a representative of minimal length. To do this, we should proceed analogously to Example 6.14. Namely, define the space $X^{\overline{\nu}}$ by the Cartesian diagram

$$
\begin{array}{ccc}
X^{\overline{\nu}} & \xrightarrow{\ f\ } & \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\
\mathrm{Bun}_B^{\overline{\nu}} & \xrightarrow{\ f\ } & [\ /G_{\overline{\nu}}]
\end{array}
$$

and we need to elucidate the space $X^{\overline{\nu}}$. We claim it is an iterated fibration of negative Banach-Colmez spaces. To see this, we consider $Q_{\overline{\nu}}$ as above, and look at the negative slope part $Q_{\overline{\nu}}^{\le 0}$. We can identify $X^{\overline{\nu}}(S)$ with the set of $Q_{\overline{\nu}}^{\le 0}$-torsors over X_S (cf. the proof of [FS21, Proposition V.3.5]). By considering the filtration of U^w by commutator subgroups, we can write $X^{\overline{\nu}}$ as an iterated fibration of the negative Banach-Colmez spaces

$$H^1((w(\hat{\alpha}))\ (Q))$$

for positive roots $\hat{\alpha} > 0$ such that $\langle w(\hat{\alpha}), \nu_b\rangle < 0$, which since ν_b is dominant and w is a representative of minimal length is equivalent to insisting that $w(\hat{\alpha}) < 0$. Then, using Lemma 6.11, we deduce that the dualizing object on $H^1((w(\hat{\alpha}))\ (Q))$ is equal to $j\ j^{-1}[\langle\hat{\alpha}, \nu_b\rangle]$ as a sheaf with $\mathbb{Q}_p$-action. Now $B_b(\mathbb{Q}_p)$ acts on $H^1((w(\hat{\alpha}))\ (Q))$ via the scaling action on $H^1((w(\hat{\alpha}))\ (Q))$ pre-composed with the natural projection $B_b(\mathbb{Q}_p) \to T(\mathbb{Q}_p)$ and the character

$$T(\mathbb{Q}_p) \to \mathbb{Q}_p$$

$$t \mapsto \mathrm{Ad}(t/\mathsf{g}_{ww_0(\hat{\alpha})}) = \mathrm{Ad}(t/\mathsf{g}_{w(\hat{\alpha})})^{-1}$$

for varying $\hat{\alpha} > 0$ such that $w(\hat{\alpha}) < 0$, where twist by w_0 comes from the minus sign when passing between isocrystals and bundles as above. This tells us that we have an isomorphism

$$\tilde{f}^!(\Lambda) \simeq \Lambda[2(\langle 2\hat{\rho}, \nu_b\rangle\ \ \langle 2\hat{\rho}_G^w, \nu_{b_T}\rangle)]\delta^{<0}$$

as a sheaf with $B_b(\mathbb{Q}_p)$ action, but, by base change, we have an isomorphism $q\ f^!(\Lambda) \simeq \tilde{f}^!(\Lambda)$, and the result follows.

7. The Filtered Eigensheaf Property

7.1. **Proof of the Filtered Eigensheaf Property.** We would now like the describe how Hecke correspondences on Bun_G interact with the Eisenstein functor. This will be used to show the Hecke eigensheaf property. Our analysis is heavily inspired by [BG02, Section 3], where an analogous claim is proven in the classical case. However, unlike the arguments there, we cannot appeal to the decomposition theorem, as the usual formalism of weights doesn't exist in this context. Nonetheless, we still have the excision spectral sequence, which will give us a filtration on the Eisenstein series. In §10, we will show that, under the condition of μ-regularity (Definition 3.14), this filtration splits for the Hecke operator defined by $V_\mu\Gamma \in \mathrm{Rep}\ (^L G)$ when applied to $F = S_{\phi_T}$.

Our goal is the following Theorem.

Theorem 7.1. *For $F \in D(\mathrm{Bun}_T)$, I a finite index set, and $V \in \mathrm{Rep}\ (^L G^I)$, the sheaf $T_V(\mathrm{Eis}(F))$ has a $W_{\mathbb{Q}_p}^I$-equivariant filtration indexed by $(\nu_i)_{i \in I} \in (\mathbb{X}\ (T_{\overline{\mathbb{Q}_p}})/\Gamma)^I$. The filtration's graded pieces are isomorphic to*

$$\mathrm{Eis}(T_{(\nu_i)_{i \in I}}(F))\quad V((\nu_i)_{i \in I})(\ \langle\hat{\rho}, \sum_{i \in I}\nu_i\ \rangle)$$

as sheaves in $D(\mathrm{Bun}_G)^{BW_{\mathbb{Q}_p}^I}$. Moreover, the filtration is natural in I and V, as well as compatible with compositions and exterior tensor products in V.

Let $E/\mathbb{Q}_p$ be an extension over which G splits. The value of $T_V(\mathrm{nEis}(S_{\phi_T}))$ is determined by the value of the Hecke correspondence base-changed to E, using [FS21, Corollary V.2.3] (See [FS21, Page 314]). Here the correspondence is determined by a representation $V = \boxtimes_{i \in I} V_i \in \mathrm{Rep}(\hat{G}^I)$ of I-copies of the dual group. We let $\lambda_i \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ be the highest weight of V_i. By taking direct sums, we can assume WLOG that V has a fixed central character. We let S_V be the Λ-valued sheaf on $\mathrm{Hck}^I_{G,(\lambda_i)_{i \in I},E}$ defined via Theorem 4.2. Our aim is to construct a W_E^I-equivariant filtration on the sheaf:

$$T_V(\mathrm{Eis}(F)) = (h_G^{\prime} \circ \pi)_!(h_G^{\prime *}(p_! q^*(F)[\dim(\mathrm{Bun}_B)]) \otimes S_V) \in \mathrm{D}(\mathrm{Bun}_G)^{BW_E^I}$$

This will be accomplished by contemplating the diagram

$$
\begin{array}{ccccc}
\overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I & \xleftarrow{\phi_{(\lambda_i)_{i \in I}}} & Z^I_{(\lambda_i)_{i \in I}} & \xrightarrow{{}^0 h_G^{\prime}} & \mathrm{Bun}_B \\
\downarrow{\overline{p} \times \mathrm{id}} & & \downarrow{{}^0 p} & & \downarrow{p} \\
\mathrm{Bun}_G \times \mathrm{Div}_E^I & \xleftarrow{h_G \circ \pi} & \mathrm{Hck}^I_{G,(\lambda_i)_{i \in I},E} & \xrightarrow{h_G^{\prime}} & \mathrm{Bun}_G
\end{array}
$$

as defined in §5.2.5 (3). Using base-change on the right Cartesian square, we get an isomorphism

$$T_V(\mathrm{Eis}(F)) \simeq (h_G^{\prime} \circ \pi)_!({}^0 p_! \, {}^0 h_G^{\prime *} q^*(F)[\dim(\mathrm{Bun}_B)] \otimes S_V)$$

but, applying the projection formula with respect to p^0, this becomes

$$T_V(\mathrm{Eis}(F)) \simeq (h_G^{\prime} \circ \pi)_! \, {}^0 p_!({}^0 h_G^{\prime *}(q^*(F)[\dim(\mathrm{Bun}_B)]) \otimes {}^0 p^*(S_V))$$

We define

$$K_V := {}^0 h_G^{\prime *}(\Lambda[\dim(\mathrm{Bun}_B)]) \otimes {}^0 p^*(S_V)$$

allowing us to rewrite our formula nicely as

(4) $T_V(\mathrm{Eis}(F)) \simeq (h_G^{\prime} \circ \pi)_! \, {}^0 p_!({}^0 h_G^{\prime *} q^*(F) \otimes K_V) = (\overline{p} \times \mathrm{id})_! \phi_{(\lambda_i)_{i \in I}!}({}^0 h_G^{\prime *} q^*(F) \otimes K_V)$

Now we would like to reduce the claim to applying excision to $\phi_{(\lambda_i)_{i \in I}!}(K_V)$ with respect to a locally closed stratification of $\overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I$. The claim should then follow from Corollary 4.9 and Lemma 5.21. In order to do this, let's further rewrite the formula. We recall that, for $(\nu_i)_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I$, we have a map

$$i_{(\nu_i)_{i \in I}} : \overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I \to \overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I$$

sending the tuple $(F_G, F_T, \overline{\kappa}^{\hat\lambda}, (D_i)_{i \in I})$ to the tuple $(F_G, F_T(-\sum_{i \in I} \nu_i \cdot D_i), L_{F_T}^{\hat\lambda}(-\langle \sum_{i \in I} \hat{\nu}_i, \hat\lambda\rangle \cdot D_i) \hookrightarrow L_{F_T}^{\hat\lambda} \hookrightarrow V_{F_G}^{\hat\lambda}, (D_i)_{i \in I})$. We also have the map

$$h^{\prime}_{(\nu_i)_{i \in I}} : \mathrm{Bun}_T \times \mathrm{Div}_E^I \simeq \mathrm{Hck}^I_{T,(\nu_i)_{i \in I}} \to \mathrm{Bun}_T$$

as in §3 which is given by modifying a T-bundle by $(\nu_i)_{i \in I}$ at a tuple of divisors $(D_i)_{i \in I}$ defining a point in Div_E^I. Then, for $F \in \mathrm{D}(\mathrm{Bun}_T)$, we recall that we have an identification $(h^{\prime}_{(\nu_i)_{i \in I}})_*(F) = T_{(\nu_i)_{i \in I}}(F)$ of sheaves in $\mathrm{D}(\mathrm{Bun}_T)^{BW_E^I}$. Now, we can verify the following easy Lemma, which follows from the definition of $\phi_{(\lambda_i)_{i \in I}}$.

Lemma 7.2. *The following is true.*
  (1) *The maps $h^{\prime}_{(w_0(\lambda_i))_{i \in I}} \circ (\overline{q} \times \mathrm{id}) \circ \phi_{(\lambda_i)_{i \in I}}$ and $q \circ {}^0 h_G^{\prime}$ from $Z^I_{(\lambda_i)_{i \in I}}$ to Bun_T coincide.*
  (2) *For every $(\nu_i)_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I$, the maps $(\overline{q} \times \mathrm{id}) \circ i_{(\nu_i)_{i \in I}}$ and $(h^{\prime}_{(\nu_i)_{i \in I}} \times \mathrm{id}) \circ (\overline{q} \times \mathrm{id})$ from $\overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I$ to $\mathrm{Bun}_T \times \mathrm{Div}_E^I$ coincide.*

Now, with this in hand, let's revisit equation (4):

$$T_V(\mathrm{Eis}(F)) \simeq (\overline{\mathsf{p}} \times \mathrm{id})_! \phi_{(\lambda_i)_{i \in I}!}({}^0 h_G^! \circ \mathsf{q}^*(F) \otimes K_V)$$

Using Lemma 7.2 (1), we have that

$${}^0 h_G^! \circ \mathsf{q}^*(F) \simeq \phi_{(\lambda_i)_{i \in I}}(\overline{\mathsf{q}} \times \mathrm{id})^*(h_{(w_0(\lambda_i))_{i \in I}}^!)(F)$$

substituting this in and applying projection formula with respect to $\phi_{(\lambda_i)_{i \in I}}$, we can rewrite the RHS as

$$(\overline{\mathsf{p}} \times \mathrm{id})_!((\overline{\mathsf{q}} \times \mathrm{id})^*(h_{(w_0(\lambda_i))_{i \in I}}^!)(F) \otimes \phi_{(\lambda_i)_{i \in I}!}(K_V))$$

Now we claim that we have the following description of $\phi_{(\lambda_i)_{i \in I}!}(K_V)$.

Theorem 7.3. *The sheaf $\phi_{(\lambda_i)_{i \in I}!}(K_V) \in \mathrm{D}(\overline{\mathrm{Bun}}_B)^{BW_E^I}$ has a W_E^I-equivariant filtration indexed by $(\nu_i)_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I$. The graded pieces of this filtration are given by*

$$\bigotimes_{i \in I}(i_{\nu_i !}(j \times \mathrm{id})_!(\Lambda[\dim(\mathrm{Bun}_B)])) \otimes V_i(w_0(\lambda_i) + \nu_i)(\langle \hat\rho, w_0(\lambda_i) + \nu_i \rangle)$$

Assuming this for now, we get that $T_V(\mathrm{Eis}(F))$ has a W_E^I-equivariant filtration with graded pieces given by

$$\bigotimes_{i \in I}(\overline{\mathsf{p}} \times \mathrm{id})_!((\overline{\mathsf{q}} \times \mathrm{id})^* h_{w_0(\lambda_i)}^!(F) \otimes (i_{\nu_i !}(j \times \mathrm{id})_!(\Lambda[\dim(\mathrm{Bun}_B)])) \otimes V_i(w_0(\lambda_i) + \nu_i)(\langle \hat\rho, w_0(\lambda_i) + \nu_i \rangle)))$$

Applying projection formula with respect to i_{ν_i}, we obtain

$$\bigotimes_{i \in I}(\overline{\mathsf{p}} \times \mathrm{id})_! i_{\nu_i !}(i_{\nu_i}^*(\overline{\mathsf{q}} \times \mathrm{id})^* h_{w_0(\lambda_i)}^!(F) \otimes (j \times \mathrm{id})_!(\Lambda[\dim(\mathrm{Bun}_B)])) \otimes V_i(w_0(\lambda_i) + \nu_i)(\langle \hat\rho, w_0(\lambda_i) + \nu_i \rangle)$$

but now, by Lemma 7.2 (2), we have

$$i_{\nu_i}^*(\overline{\mathsf{q}} \times \mathrm{id})^* h_{w_0(\lambda_i)}^!(F) \simeq (\overline{\mathsf{q}} \times \mathrm{id})^* h_{w_0(\lambda_i)+\nu_i}^!(F) \simeq (\overline{\mathsf{q}} \times \mathrm{id})^*(T_{w_0(\lambda_i)+\nu_i}(F))$$

so substituting this into the previous formula we get

$$\bigotimes_{i \in I}(\overline{\mathsf{p}} \times \mathrm{id})_! i_{\nu_i !}((\overline{\mathsf{q}} \times \mathrm{id})^*(T_{w_0(\lambda_i)+\nu_i}(F)) \otimes (j \times \mathrm{id})_!(\Lambda[\dim(\mathrm{Bun}_B)]))) \otimes V_i(w_0(\lambda_i)+\nu_i)(\langle \hat\rho, w_0(\lambda_i)+\nu_i \rangle)$$

Now i_{ν_i} does nothing to the G-bundle $\mathcal{F}_G$ and the copy of Div_E^I. Therefore, this becomes

$$\bigotimes_{i \in I}(\overline{\mathsf{p}} \times \mathrm{id})_!((\overline{\mathsf{q}} \times \mathrm{id})^*(T_{w_0(\lambda_i)+\nu_i}(F)) \otimes (j \times \mathrm{id})_!(\Lambda[\dim(\mathrm{Bun}_B)])) \otimes V_i(w_0(\lambda_i)+\nu_i)(\langle \hat\rho, w_0(\lambda_i)+\nu_i \rangle)$$

which is just

$$\bigotimes_{i \in I}(\mathrm{Eis} \times \mathrm{id})(T_{w_0(\lambda_i)+\nu_i}(F)) \otimes V_i(w_0(\lambda_i) + \nu_i)(\langle \hat\rho, w_0(\lambda_i) + \nu_i \rangle)$$

by an application of projection formula to $j \times \mathrm{id}$. Since $w_0(\lambda_i)$ is the lowest weight of V_i this implies the desired result. Thus, to construct the filtration all we have to do is prove Theorem 7.3.

Proof. (Theorem 7.3) For $(\nu_i)_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I$, let $\overline{\nu} := \sum_{i \in I} \nu_i$. Consider the locally closed stratum $_{\overline{\nu}}\overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I \simeq \mathrm{Bun}_B \times \mathrm{Div}^{(\overline{\nu})} \times \mathrm{Div}_E^I \subset \overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I$. We have the natural projection

$$p_1 : {}_{\overline{\nu}}\overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I \simeq \mathrm{Bun}_B \times \mathrm{Div}^{(\overline{\nu})} \times \mathrm{Div}_E^I \to \mathrm{Div}^{(\overline{\nu})}$$

as well as the map

$$p_2 : {}_{\overline{\nu}}\overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I \to \mathrm{Div}_E^I \xrightarrow{(\nu_i)_{i \in I}} \mathrm{Div}^{(\overline{\nu})}$$

where the first map is the natural projection and $\Delta_{(\nu_i)_{i \in I}}$ is as defined in §5.2.5. Since $\mathrm{Div}^{(\overline{\nu})}$ is proper using [FS21, Proposition II.1.21] and in particular separated, it follows that if we let $_{(\nu_i)_{i \in I}}(\overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I) \to {}_{\overline{\nu}}\overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I$ be the pullback of the diagonal morphism $\Delta_{\mathrm{Div}^{(\overline{\nu})}}$ along (p_1, p_2) that this is a closed immersion. Therefore, we see that the composite $_{(\nu_i)_{i \in I}}(\overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I) \to {}_{\overline{\nu}}\overline{\mathrm{Bun}}_B \to \overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I$ is a locally closed immersion parameterizing $(\overline{\kappa}^{\hat\lambda} : (L^{\hat\lambda})_{\mathcal{F}_T} \to (V^{\hat\lambda})_{\mathcal{F}_G}, \hat\lambda \in \hat\Lambda_G^+, (D_i)_{i \in I})$ such that $\mathrm{Coker}(\overline{\kappa}^{\hat\lambda})$ has torsion of length $\langle \hat\lambda, \nu_i \rangle$ supported at D_i, in the sense of Lemma 5.20, and a 0 nowhere else for all $\hat\lambda \in \hat\Lambda_G^+$ and $i \in I$. If we let $j_{(\nu_i)_{i \in I}} := i_{(\nu_i)_{i \in I}} \circ (j \times \mathrm{id})$ then

we see that this maps isomorphically onto $_{(\nu_i)_{i \in 2I}}(\overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I)$, and for varying $(\nu_i)_{i \in 2I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}))^I$ these form a locally closed stratification of $\overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I$. Now, by applying the excision with respect to this locally closed stratification, we obtain a filtration on $\phi_{(\lambda_i)_{i \in 2I}!}(K_Z)$ whose graded pieces are isomorphic to:

$$j_{(\nu_i)_{i \in 2I}!}j_{(\nu_i)_{i \in 2I}}^*\phi_{(\lambda_i)_{i \in 2I}!}(K_V)$$

Moreover, since the maps $j_{(\nu_i)_{i \in 2I}}$ are defined over the projection to Div_E^I, it follows that this filtration is W_E^I-equivariant. It remains to determine the W_E^I-action on the graded pieces. To do this, we can consider the pullback to a geometric point $x = \mathrm{Spa}(C, O_C) \to \mathrm{Div}_E^I$, where we can regard x as a point of $\mathrm{Div}^{(\overline{\nu})}$, as in §5.2.5. We recall that by definition

$$K_V := {}^0\mathrm{p}_*(S_V) \otimes ({}^0h_G^!)(\Lambda[\dim(\mathrm{Bun}_B)])$$

so, by Lemma 5.21 (3), this identifies with

$$R\Gamma_c({}_x\mathrm{Gr}_{G,\ (\lambda_i)_{i \in 2I},E} \backslash_x S_{G,(\nu_i + w_0(\lambda_i))_{i \in 2I},E}, S_V|_{{}_x\mathrm{Gr}_{G,\ (\lambda_i)_{i \in 2I},E} \backslash_x S_{G,(\nu_i + w_0(\lambda_i))_{i \in 2I},E}})[\langle 2\hat\rho, \sum_{i \in 2I}(\nu_i + w_0(\lambda_i))\rangle + \dim(\mathrm{Bun}_B)]$$

We need to justify that the contribution of the L^+B torsor $_xB$ in Lemma 5.21 (3) is isomorphic to $\Lambda[\langle 2\hat\rho, \sum_{i \in 2I}(\nu_i + w_0(\lambda_i))\rangle]$. To see this, consider the case where $I = \{*\}$ is a singleton, and we have elements $\lambda \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ and $\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$. We consider the L^+B-action on $_x\mathrm{Gr}_{G,\ \lambda,E} \backslash_x S_{G,\nu + w_0(\lambda),E}$. We note that L^+T will act trivially on this space, as can be seen from the definition of the semi-infinite cells. Consider the remaining unipotent part L^+U. Using the filtration on U by commutator subgroups, we can write L^+U as an iterated fibration of $L^+\mathbb{G}_{a,\hat\alpha}$ indexed by the positive roots $\hat\alpha$ of G. One can check that the $L^+\mathbb{G}_{a,\hat\alpha}$ action on $_x\mathrm{Gr}_{G,\ \lambda,E} \backslash_x S_{G,\nu + w_0(\lambda),E}$ factors through the truncated loop group $L_{n_{\hat\alpha}}^+\mathbb{G}_{a,\hat\alpha}$, where $n_{\hat\alpha} = \langle\hat\alpha, \nu + w_0(\lambda)\rangle$ (See for example the proof of [FS21, Proposition VI.2.4]). If we let $O_{X,x}$ denote the completed local ring at the fixed untilt x, with uniformizing parameter t_x then, by writing $O_{X,x}/t_x^{n_{\hat\alpha}}$ as an iterated extension of $O_{X,x}/t_x \simeq C$, we can describe $L_{n_{\hat\alpha}}^+\mathbb{G}_{a,\hat\alpha}$ as a fibration of $(\mathbb{A}_C^1)$ iterated $n_{\hat\alpha}$ times. This tells us that the compactly supported cohomology of $L_{n_{\hat\alpha}}^+\mathbb{G}_{a,\hat\alpha}$ over $\mathrm{Spa}(C, O_C)$ is isomorphic to $\Lambda[-2n_{\hat\alpha}]$. As a consequence, we deduce that the contribution of $_xB$ to the above formula is $\Lambda[-\sum_{\hat\alpha > 0} n_{\hat\alpha}] = \Lambda[-\langle 2\hat\rho, \nu + w_0(\lambda)\rangle]$, by Künneth. Now, by Corollary 4.9, $j_{(\nu_i)_{i \in 2I}}^*\phi_{(\lambda_i)_{i \in 2I}!}(K_V)$ identifies with

$$\boxtimes_{i \in 2I} V_i(w_0(\lambda_i) + \nu_i)(-\langle\hat\rho, (w_0(\lambda_i) + \nu_i)\rangle)[-\langle 2\hat\rho, \sum_{i \in 2I}(w_0(\lambda_i) + \nu_i)\rangle + \langle 2\hat\rho, \sum_{i \in 2I}(w_0(\lambda_i) + \nu_i)\rangle + \dim(\mathrm{Bun}_B)]$$

or rather

$$\boxtimes_{i \in 2I} V_i(w_0(\lambda_i) + \nu_i)(-\langle\hat\rho, w_0(\lambda_i) + \nu_i\rangle)[\dim(\mathrm{Bun}_B)]$$

and so we get the desired result.

It remains to see that this filtration satisfies the desired compatibilities. Consider a map of finite index sets $\pi : I \to J$. For $j \in J$, we set $I_j := \pi^{-1}(j)$ and consider the natural map $\Delta_{IJ} : \mathrm{Div}_E^J \to \mathrm{Div}_E^I$, which diagonally embeds the jth copy of Div_E^1 in Div_E^J into $\mathrm{Div}_E^{I_j}$. Attached to this, we have a Cartesian diagram

$$\begin{array}{ccc}
\overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^J & \xleftarrow{\phi_{(\lambda_j)_{j \in 2J}}} & Z_{(\lambda_j)_{j \in 2J}}^J \\
\downarrow{\mathrm{id} \times \Delta_{IJ}} & & \downarrow{\tilde\Delta_{IJ}} \\
\overline{\mathrm{Bun}}_B \times \mathrm{Div}_E^I & \xleftarrow{\phi_{(\lambda_i)_{i \in 2I}}} & Z_{(\lambda_i)_{i \in 2I}}^I
\end{array}$$

where $\lambda_j := \sum_{i \in 2I_j} \lambda_i$ for all $j \in J$. Base change gives us a natural isomorphism:

(5) $$(\mathrm{id} \times \Delta_{IJ})^*\phi_{(\lambda_i)_{i \in 2I}!}(K_{\boxtimes_{i \in 2I} V_i}) \simeq \phi_{(\lambda_j)_{j \in 2J}!}\tilde\Delta_{IJ}^*(K_{\boxtimes_{i \in 2I} V_i})$$

However, by the relationship between fusion product and tensor product under Theorem 4.2, we deduce that $\tilde{\Delta}_{IJ}(K_{i2IV_i}) \simeq K_{j2JV_j}$, where $V_j := {}_{i2I_j}V_i$. We now compare the two filtrations on the LHS and the RHS of this isomorphism. To do this, we define $(\nu_j)_{j2J}$ by $\nu_j := \sum_{i2I_j}\nu_i$. We note that we have a natural Cartesian diagram

$$\overline{\mathrm{Bun}}_B \quad \mathrm{Div}_E^J \xrightarrow{i_{(\nu_j)_{j2J}}} \overline{\mathrm{Bun}}_B \quad \mathrm{Div}_E^J$$
$$\downarrow \mathrm{id} \quad {}_{IJ} \qquad\qquad \downarrow \mathrm{id} \quad {}_{IJ}$$
$$\overline{\mathrm{Bun}}_B \quad \mathrm{Div}_E^I \xrightarrow{i_{(\nu_i)_{i2I}}} \overline{\mathrm{Bun}}_B \quad \mathrm{Div}_E^I$$

On the LHS of (5), we have a filtration with graded pieces isomorphic to

$$(\mathrm{id} \quad \Delta_{IJ}) \; i_{(\nu_i)_{i2I}!}(j \quad \mathrm{id})_!(\Lambda[\dim(\mathrm{Bun}_B)]) \qquad {}_{i2I}V_i(w_0(\lambda_i)+\nu_i)(\quad \hbar\hat{\rho}, w_0(\lambda_i)+\nu_i i)$$

which is naturally isomorphic to

$$i_{(\nu_j)_{j2J}!}(\mathrm{id} \quad \Delta_{IJ}) \; (j \quad \mathrm{id})_!(\Lambda[\dim(\mathrm{Bun}_B)]) \qquad {}_{i2I}V_i(w_0(\lambda_i)+\nu_i)(\quad \hbar\hat{\rho}, w_0(\lambda_i)+\nu_i i)$$

by base-change applied to the previous Cartesian square. We can further rewrite this as

$$i_{(\nu_j)_{j2J}!} \quad (j \quad \mathrm{id})_!(\Lambda[\dim(\mathrm{Bun}_B)]) \qquad {}_{j2J} \quad {}_{i2I_j} V_i(w_0(\lambda_i)+\nu_i)(\quad \hbar\hat{\rho}, w_0(\lambda_i)+\nu_i i))$$

On the other hand, for such a $(\nu_j)_{j2J}$, the RHS of (5) has a filtration with graded pieces isomorphic to

$$i_{(\nu_j)_{j2J}!}(j \quad \mathrm{id})_!(\Lambda[\dim(\mathrm{Bun}_B)]) \qquad {}_{j2J}V_j(w_0(\lambda_j)+\nu_j)(\quad \hbar\hat{\rho}, w_0(\lambda_j)+\nu_j i))$$

but now note that

$$V_j(w_0(\lambda_j)+\nu_j)(\quad \hbar\hat{\rho}, w_0(\lambda_j)+\nu_j i)) = \bigoplus_{\substack{(\nu_i)_{i2I} 2 \; {}^I_{G,B} \; i2I_j \\ \sum_{i2I_j}\nu_i=\nu_j}} \bigotimes V_i(w_0(\lambda_i)+\nu_i)(\quad \hbar\hat{\rho}, w_0(\lambda_i)+\nu_i i))$$

for all $j \; 2 \; J$. Therefore, the graded piece indexed by $(\nu_j)_{j2J}$ on the RHS have a split filtration with graded pieces isomorphic to the graded pieces coming from the filtration on the LHS. The compatibility of these two filtrations now follows from Corollary 4.11, and the fact that the filtration came from restricting the sheaf S_V to semi-infinite cells. Now, we can reap the fruit of this section using the filtered eigensheaf property to get some control on the stalks of $\mathrm{nEis}(S_{\phi_T})$.

7.2. Consequences of the Filtered Eigensheaf Property.
First, we note, by applying Theorem 7.1 when $F = S_{\phi_T} \quad \Delta_B^{1/2}$ together with Corollary 3.5, we obtain the following.

Corollary 7.4. *For all finite index sets I and $V = {}_{i2I}V_i \; 2 \, \mathrm{Rep} \; ({}^LG^I)$, the sheaf $T_V(\mathrm{nEis}(S_{\phi_T}))$ admits a $W_{\overline{\mathbb{Q}}_p}^I$-equivariant filtration indexed by $(\nu_i)_{i2I} \; 2 \, (\mathbb{X} \; (T_{\overline{\mathbb{Q}}_p})/\Gamma)^I$. The filtration's graded pieces are isomorphic to $\mathrm{nEis}(S_{\phi_T}) \quad {}_{i2I}(\nu_i \quad \phi_T) \quad V_i(\nu_i)$. The filtration is natural in I and V, as well as compatible with compositions and exterior tensor products in V.*

In particular, we note that the direct sum of the graded pieces of the filtration on $T_V(\mathrm{nEis}(S_{\phi_T}))$ is isomorphic to

$$\mathrm{nEis}(S_{\phi_T}) \bigoplus_{(\nu_i)_{i2I} 2(\mathbb{X} \; (T_{\overline{\mathbb{Q}}_p})/ \;)^I} {}_{i2I}\nu_i \quad \phi_T \quad V_i(\nu_i) \simeq \mathrm{nEis}(S_{\phi_T}) \quad r_V \quad \phi$$

as sheaves in $\mathrm{D}(\mathrm{Bun}_G)^{BW_{\mathbb{Q}_p}^I}$, where ϕ is the parameter ϕ_T composed with the natural embedding ${}^LT \; ! \; {}^LG$. In other words, $T_V(\mathrm{nEis}(S_{\phi_T}))$ is a filtered eigensheaf with eigenvalue ϕ. We now would like to use this to deduce some consequences about the stalks of the Eisenstein series $\mathrm{nEis}(S_{\phi_T})$. In particular, let's consider some Schur irreducible constituent A of $\mathrm{nEis}(S_{\phi_T})$. We will now need our assumption that the prime ℓ is very good. Under this assumption, the excursion algebra will

define endomorphims of A which determine and are determined by the parameter ϕ_A^{FS}, as in [FS21, Propositions I.9.1, I.9.3]. Since the excursion algebra is determined by natural transformations of Hecke operators, the filtered Hecke eigensheaf property tells us that these scalars must be specified by ϕ, so that we have an equality: $\phi = \phi_A^{\mathsf{FS}}$, as conjugacy classes of semi-simple parameters. In particular, if we consider $b \in B(G)$ and look at the restriction $\mathrm{nEis}(S_{\phi_T})|_{\mathrm{Bun}_G^b}$ to the locally closed HN-strata $\mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$ indexed by b. Then, by [FS21, Proposition V.2.2], we have a canonical isomorphism $D(\mathrm{Bun}_G^b) \simeq D(J_b(\mathbb{Q}_p), \Lambda)$. By the previous discussion and compatibility of the Fargues-Scholze correspondence with restriction to J_b [FS21, Section IX.7.1], we deduce that any irreducible constituent ρ of the restriction $\mathrm{nEis}(S_{\phi_T})|_{\mathrm{Bun}_G^b}$ has Fargues-Scholze parameter $\phi_\rho^{\mathsf{FS}} : W_{\mathbb{Q}_p} \to {}^L J_b(\Lambda)$ equal to ϕ under the appropriately Tate twisted embedding:

$$ {}^L J_b(\Lambda) \to {}^L G(\Lambda) $$

Now, since our parameter ϕ is induced from the maximal torus, we would like to say that this is impossible unless J_b itself admits a maximal torus, which is in turn equivalent to assuming that b is unramified. Here we need to be a bit careful. In particular, if we consider $G = \mathrm{GL}_2$ and b the element of slope $\frac{1}{2}$ then $J_b = D_{\frac{1}{2}}$ the units in the quaternion division algebra. The trivial representation $\mathbf{1}$ of $D_{\frac{1}{2}}$ has Fargues-Scholze parameter given by

$$ W_{\mathbb{Q}_p} \to \mathrm{GL}_2(\Lambda) $$

$$ g \mapsto \begin{pmatrix} |g|^{1/2} & 0 \\ 0 & |g|^{-1/2} \end{pmatrix} $$

This parameter is induced from a maximal torus of GL_2; however, it is not generic. In particular, the composite of this parameter with the unique simple root defined by the upper triangular Borel gives a Galois representation isomorphic to the norm character $|\cdot|$. Therefore, one might hope that assuming compatibility of some suitably nice form of the local Langlands correspondence for G with the Fargues-Scholze correspondence together with genericity of ϕ_T is enough to give us the desired description of the stalks. This is indeed the case. The assumption we need is as follows.

Assumption 7.5. *For a connected reductive group $H/\mathbb{Q}_p$, we have:*

$\Pi(H)$ *the set of smooth irreducible $\overline{\mathbb{Q}}_\ell$-representations of $H(\mathbb{Q}_p)$,*
$\Phi(H)$ *the set of conjugacy classes of continuous maps*

$$ L_{\mathbb{Q}_p} = W_{\mathbb{Q}_p} \times \mathrm{SL}(2, \overline{\mathbb{Q}}_\ell) \to {}^L H(\overline{\mathbb{Q}}_\ell) $$

where $\overline{\mathbb{Q}}_\ell$ has the discrete topology and $\mathrm{SL}(2, \overline{\mathbb{Q}}_\ell)$ acts via an algebraic representation and the map respects the action of $W_{\mathbb{Q}_p}$ on ${}^L H(\overline{\mathbb{Q}}_\ell)$, the L-group of H.
$\Phi^{\mathsf{ss}}(H)$ *the set of continuous semi-simple homomorphisms*

$$ W_{\mathbb{Q}_p} \to {}^L H(\overline{\mathbb{Q}}_\ell) $$

$(\ )^{\mathsf{ss}} : \Phi(H) \to \Phi^{\mathsf{ss}}(H)$ *the map defined by precomposition with*

$$ W_{\mathbb{Q}_p} \to W_{\mathbb{Q}_p} \times \mathrm{SL}(2, \overline{\mathbb{Q}}_\ell) = L_{\mathbb{Q}_p} $$

$$ g \mapsto \left(g, \begin{pmatrix} |g|^{1/2} & 0 \\ 0 & |g|^{-1/2} \end{pmatrix}\right) $$

Then, we assume, for all $b \in B(G)$, that there exists a map

$$ \mathrm{LLC}_b : \Pi(J_b) \to \Phi(J_b) $$

$$ \rho \mapsto \phi_\rho $$

satisfying the following properties:

(1) *The diagram*

$$\Pi(J_b) \xrightarrow{\ \mathrm{LLC}_b\ } \Phi(J_b)$$

with $\mathrm{LLC}_b^{\mathsf{FS}}$ going diagonally to $\Phi^{\mathsf{SS}}(J_b)$, and $(\ )^{\mathrm{ss}}$ the vertical map.

commutes, where $\mathrm{LLC}_b^{\mathsf{FS}}$ *is the Fargues-Scholze local Langlands correspondence for* J_b.

(2) *Consider* ϕ_ρ *as an element of* $\Phi(G)$ *given by composing with the twisted embedding* $^L J_b(\overline{\mathbb{Q}}_\ell) \simeq {}^L M_b(\overline{\mathbb{Q}}_\ell) \hookrightarrow {}^L G(\overline{\mathbb{Q}}_\ell)$ *(as defined in [FS21, Section IX.7.1]). Then* ϕ_ρ *factors through the natural embedding* $^L T \hookrightarrow {}^L G$ *if and only if* $b \in B(G)_{\mathsf{un}}$.

(3) *If* ρ *is a representation such that* $L_{\mathbb{Q}_p} \to {}^L J_b(\overline{\mathbb{Q}}_\ell) \to {}^L G(\overline{\mathbb{Q}}_\ell)$ *factors through* $^L T$, *where the last map is the twisted embedding then, by (2), the element* b *is unramified, and we require that* ρ *is isomorphic to an irreducible constituent of* $i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{1/2}$ *for* $w \in W_b$. *Here* $\chi : T(\mathbb{Q}_p) \to \overline{\mathbb{Q}}_\ell$ *is the character attached to the induced toral parameter* ϕ_T, *and* $\delta_{P_b}^{1/2}$ *is the modulus character of* $M_b \simeq J_b$.

Remark 7.6. This assumption might seem a bit daunting, but is verifiable in many cases. In particular, the first assumption follows from the compatibility of the Fargues-Scholze correspondence with the Harris-Taylor correspondence for groups of type A_n and its inner forms ([HKW22, Theorem 1.0.3]). Similarly, for groups of type C_2 and their inner forms over a unramified extension L with $p > 2$, this follows from the main theorem of [Ham21b], and, for odd unramified unitary groups over $\mathbb{Q}_p$ this follows from the main theorem of [BHN22]. The methods employed in these two papers should generalize to at least a few other cases.

Assumption (2) is also a standard and verifiable conjecture in the cases where the local Langlands correspondence is known to exist. If ρ is a representation such that ϕ_ρ factors through $^L T$ then we are claiming that J_b has a Borel subgroup. For non quasi-split groups, it is conjectured that one should only consider L-parameters coming from the L-groups of the Levi subgroups of the non quasi-split group. These are referred to as relevant L-parameters. In particular, one expects the L-packets of LLC_b over irrelevant ϕ to be empty (See for example [Kal16, Conjecture A.2]), so if ϕ_ρ factors through $^L T$ for some ρ under LLC_b it should imply that the group J_b has a Borel B_b. Assumption (3) is just the expectation that, when ϕ_ρ factors through $^L T$, the members of the L-packet should be given by the irreducible constituents of the parabolic inductions from T to J_b. The Weyl group twists appear since the Weyl group conjugates of $\phi_\rho : W_{\mathbb{Q}_p} \to {}^L J_b(\overline{\mathbb{Q}}_\ell) \simeq {}^L M_b(\overline{\mathbb{Q}}_\ell)$ all map to the same parameter when viewed as a parameter valued in $^L G$, and the modulus twist by $\delta_{P_b}^{1/2}$ appears since we are comparing this to an L-parameter of G via the twisted embedding $^L J_b(\overline{\mathbb{Q}}_\ell) \simeq {}^L M_b(\overline{\mathbb{Q}}_\ell) \hookrightarrow {}^L G(\overline{\mathbb{Q}}_\ell)$.

Under this assumption, we will deduce our main Corollary of the filtered eigensheaf property.

Corollary 7.7. *Under Assumption 7.5, consider* $b \in B(G)$ *with corresponding locally closed HN-strata* $\mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$. *For* ϕ *a generic parameter, the following is true.*

(1) *If* $b \notin B(G)_{\mathsf{un}}$ *the restriction*

$$\mathrm{nEis}(S_{\phi_T})|_{\mathrm{Bun}_G^b}$$

vanishes.

(2) *If* $b \in B(G)_{\mathsf{un}}$ *is an unramified element and* ρ *is a smooth irreducible* $\overline{\mathbb{F}}_\ell$-*representation occurring as a constituent of* $\mathrm{nEis}(S_{\phi_T})|_{\mathrm{Bun}_G^b}$ *then* ρ *is an irreducible constituent of* $i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{1/2}$ *for some* $w \in W_b$.

Proof. As noted above, $\mathrm{nEis}(S_{\phi_T})|_{\mathrm{Bun}_G^b}$ will be valued in an unbounded complex of smooth $\overline{\mathbb{F}}_\ell$-representations of the σ-centralizer J_b of b. If we consider a smooth irreducible constituent of

this restriction ρ then, as already discussed above, it follows that the Fargues-Scholze parameter $\phi_\rho^{\mathsf{FS}} : W_{\mathbb{Q}_p} \to {}^L J_b(\overline{\mathbb{F}}_\ell)$ under the twisted embedding

$$ {}^L J_b(\overline{\mathbb{F}}_\ell) \to {}^L G(\overline{\mathbb{F}}_\ell) $$

agrees with ϕ. Using [Dat05, Lemma 6.8], we can choose $\tilde{\rho}$ a lift of ρ to a smooth irreducible $\overline{\mathbb{Q}}_\ell$-representation admitting a $J_b(\mathbb{Q}_p)$-stable $\overline{\mathbb{Z}}_\ell$-lattice such that ρ occurs as a subquotient of $\tilde{\rho}$ mod ℓ. Since the Fargues-Scholze correspondence is compatible with reduction mod ℓ [FS21, Section IX.5.2], it follows that the Fargues-Scholze parameter $\phi_{\tilde{\rho}}^{\mathsf{FS}}$ factors through ${}^L G(\overline{\mathbb{Z}}_\ell)$ and that it equals ϕ_ρ^{FS} mod ℓ. Now, since ϕ_ρ^{FS} factors through ${}^L T$ and induces a generic parameter, we claim that the same is true for $\phi_{\tilde{\rho}}^{\mathsf{FS}}$. This follows through standard deformation theory. In particular, if

$$ H^1(W_{\mathbb{Q}_p}, \alpha \circ \phi_T) $$

vanishes for all Γ-orbits α of roots, then any lift of ϕ_ρ^{FS} will factor through ${}^L T$, but this vanishing is guaranteed by ϕ_T being generic (cf. [CS17, Lemma 6.2.2]). It is also easy to see that $\phi_{\tilde{\rho}}^{\mathsf{FS}}$ must be generic since its mod ℓ reduction is. Now, by Assumption 7.5, we note that $\phi_{\tilde{\rho}}^{\mathsf{FS}}$ is the semi-simplification of $\phi_{\tilde{\rho}}$, the L-parameter attached to $\tilde{\rho}$, but by Lemma 3.18 and genericity that implies that $\phi_{\tilde{\rho}}|_{W_{\mathbb{Q}_p}} = \phi_{\tilde{\rho}}^{\mathsf{FS}}$. The two claims now follow from Assumptions 7.5 (2) and (3), respectively.

This statement will allow us to give a complete description of the eigensheaf $\mathrm{nEis}(S_{\phi_T})$ for ϕ_T satisfying the slightly stronger condition of weak normalized regularity. In particular, as we will see in §9, for the restrictions of the sheaf to Bun_G^b for $b \in B(G)_{\mathsf{un}}$, we will always be able to evaluate the stalks in terms of normalized parabolic inductions of Weyl group translates of the character χ, and, by the previous Lemma, we know these are the only possible non-zero stalks.

Now we turn our attention to studying how the geometric Eisenstein functor interacts with Verdier duality.

8. Eisenstein Series and Verdier Duality

We would like to study how the normalized Eisenstein functor interacts with Verdier duality. This will be done assuming the following claim.

Assumption 8.1. *We assume that the sheaf $j_!(\mathrm{IC}_{\mathrm{Bun}_B})$ is ULA with respect to the morphism defined by $\overline{\mathsf{q}}$, in the sense of [FS21, Definition IV.2.31].*

Remark 8.2. A proof of this claim should appear in upcoming work of Hansen-Scholze [HS], where this statement is used in their proof of the Harris-Viehmann conjecture.

We assume this claim, and use it to show that this implies that our Eisenstein functor commutes with Verdier duality when ϕ_T is weakly generic. In particular, if $\mathbb{D}_{\mathrm{Bun}_G}$ (resp. $\mathbb{D}_{\mathrm{Bun}_T}$) denotes Verdier duality on Bun_G (resp. Bun_T), our main goal is to prove the following.

Theorem 8.3. *Assuming 8.1 then, for ϕ_T a weakly generic toral parameter, there is an isomorphism of objects in $\mathrm{D}(\mathrm{Bun}_G)$*

$$ \mathbb{D}_{\mathrm{Bun}_G}(\mathrm{nEis}(S_{\phi_T})) \simeq \mathrm{nEis}(\mathbb{D}_{\mathrm{Bun}_T}(S_{\phi_T})) $$

where we note that $\mathbb{D}_{\mathrm{Bun}_T}(S_{\phi_T}) \simeq S_{\phi_{\overline{T}}}$, if $\phi_{\overline{T}}$ denotes the parameter dual to ϕ_T.

First, let's record some implications of Assumption 8.1.

Lemma 8.4. *The sheaf $j_*(\mathrm{IC}_{\mathrm{Bun}_B})$ is also ULA with respect to $\overline{\mathsf{q}}$, and we have isomorphisms*

$$ \mathbb{D}_{\overline{\mathrm{Bun}}_B}(j_!(\mathrm{IC}_{\mathrm{Bun}_B})) \simeq j_*(\mathrm{IC}_{\mathrm{Bun}_B}) $$

and

$$ \mathbb{D}_{\overline{\mathrm{Bun}}_B}(j_*(\mathrm{IC}_{\mathrm{Bun}_B})) \simeq j_!(\mathrm{IC}_{\mathrm{Bun}_B}) $$

of objects in $\mathrm{D}(\overline{\mathrm{Bun}}_B)$.

Proof. The first claimed isomorphism just follows from Corollary 6.2, and the fact that we always have an isomorphism of derived functors $\mathbb{D}_{\overline{\mathrm{Bun}}_B} \circ j_! \simeq j_* \circ \mathbb{D}_{\mathrm{Bun}_B}$, by projection formula. Now, since the dualizing object on Bun_T is just the constant sheaf, by Lemma 6.6 and the fact that T is unimodular, it follows that we have a unique (up to fixing Haar measures on $T(\mathbb{Q}_p)$) isomorphism $\mathbb{D}_{\overline{\mathrm{Bun}}_B} \simeq \mathbb{D}_{\overline{\mathrm{Bun}}_B/\mathrm{Bun}_T}$. Therefore, by [FS21, Corollary IV.2.25], we can show that $j_*(\mathbb{D}_{\mathrm{Bun}_B}(\mathrm{IC}_{\mathrm{Bun}_B})) \simeq j_*(\mathrm{IC}_{\mathrm{Bun}_B}) \simeq \mathbb{D}_{\overline{\mathrm{Bun}}_B}(j_!(\mathrm{IC}_{\mathrm{Bun}_B})) \simeq \mathbb{D}_{\overline{\mathrm{Bun}}_B/\mathrm{Bun}_T}(j_!(\mathrm{IC}_{\mathrm{Bun}_B}))$ is also ULA with respect to $\overline{q}$, where the first isomorphism follows from Corollary 6.2. The second claimed isomorphism now follows from the first and the fact that ULA objects are reflexive with respect to Verdier duality, again by [FS21, Corollary IV.2.25]. $\square$

We will now combine this with the following lemma.

Lemma 8.5. *Let $f : X \to S$ be a map of decent v-stacks which are fine over a base $*$. Suppose that A is ULA with respect to f and $B \in \mathrm{D}(S)$, then we have a natural isomorphism*

$$\mathbb{D}_X(A \otimes f^*(B)) \simeq \mathbb{D}_X(A) \otimes f^*(\mathbb{D}_S(B))$$

in $\mathrm{D}(X)$.

Proof. Let $g : S \to *$ denote the structure morphism. It follows, by [FS21, Proposition IV.2.19] and A being ULA, that we can rewrite the RHS of the above isomorphism as

$$\mathbb{D}_X(A) \otimes f^*(\mathbb{D}_S(B)) \simeq R\mathcal{H}om(A, f^!(\mathbb{D}_S(B)))$$

which in turn is equal to

$$R\mathcal{H}om(A, f^!(R\mathcal{H}om(B, g^!(\Lambda))))$$

by definition. Now, by projection formula, we can further rewrite this as

$$R\mathcal{H}om(A, R\mathcal{H}om(f^*(B), f^!g^!(\Lambda)))$$

but, by Hom-Tensor duality, this is just

$$R\mathcal{H}om(A \otimes f^*(B), f^!g^!(\Lambda)) = \mathbb{D}_X(A \otimes f^*(B))$$

as desired. $\square$

Combining the previous two Lemmas, we deduce the following.

Corollary 8.6. *There is an isomorphism*

$$\mathbb{D}_{\overline{\mathrm{Bun}}_B}(\overline{q}^*(S_{\phi_T}) \otimes j_!(\mathrm{IC}_{\mathrm{Bun}_B})) \simeq \overline{q}^*(\mathbb{D}_{\mathrm{Bun}_T}(S_{\phi_T})) \otimes \mathbb{D}_{\overline{\mathrm{Bun}}_B}(j_!(\mathrm{IC}_{\mathrm{Bun}_B})) \simeq \overline{q}^*(S_{\phi_{\overline{T}}}) \otimes j_*(\mathrm{IC}_{\mathrm{Bun}_B})$$

of objects in $\mathrm{D}(\overline{\mathrm{Bun}}_B)$. Similarly, using Lemma 8.4, we also have an isomorphism

$$\mathbb{D}_{\overline{\mathrm{Bun}}_B}(\overline{q}^*(S_{\phi_T}) \otimes j_*(\mathrm{IC}_{\mathrm{Bun}_B})) \simeq \overline{q}^*(S_{\phi_{\overline{T}}}) \otimes j_!(\mathrm{IC}_{\mathrm{Bun}_B})$$

of objects in $\mathrm{D}(\overline{\mathrm{Bun}}_B)$.

Now let's apply these results to prove Theorem 8.3.

Proof. (Theorem 8.3) First note that, using projection formula with respect to j, we have an isomorphism:

$$\mathrm{nEis}(S_{\phi_T}) = \mathfrak{p}_!(\mathfrak{q}^*(S_{\phi_T}) \otimes \mathrm{IC}_{\mathrm{Bun}_B}) = \overline{\mathfrak{p}}_!j_!(j^*\overline{\mathfrak{q}}^*(S_{\phi_T}) \otimes \mathrm{IC}_{\mathrm{Bun}_B}) \simeq \overline{\mathfrak{p}}_!(\overline{\mathfrak{q}}^*(S_{\phi_T}) \otimes j_!(\mathrm{IC}_{\mathrm{Bun}_B}))$$

Now, by Proposition 5.9, we have that $\overline{\mathfrak{p}}_!$ is equivalent to $\overline{\mathfrak{p}}_*$, this means that we have an isomorphism

$$\mathbb{D}_{\mathrm{Bun}_G}(\mathrm{nEis}(S_{\phi_T})) \simeq \overline{\mathfrak{p}}_!(\mathbb{D}_{\overline{\mathrm{Bun}}_B}(\overline{\mathfrak{q}}^*(S_{\phi_T}) \otimes j_!(\mathrm{IC}_{\mathrm{Bun}_B})))$$

but now, by Corollary 8.6, this is isomorphic to

$$(6) \qquad\qquad\qquad \overline{\mathfrak{p}}_!(\overline{\mathfrak{q}}^*(S_{\phi_{\overline{T}}}) \otimes j_*(\mathrm{IC}_{\mathrm{Bun}_B}))$$

Therefore, we need to exhibit an isomorphism between this sheaf and

$$(7) \qquad \mathrm{nEis}(\mathbb{D}_{\mathrm{Bun}_T}(S_{\phi_T})) \simeq \bar{p}_!(\bar{q}^*(\mathbb{D}_{\mathrm{Bun}_T}(S_{\phi_T})) \otimes j_!(\mathrm{IC}_{\mathrm{Bun}_B})) \simeq \bar{p}_!(\bar{q}^*(S_{\phi_{\bar{T}}}) \otimes j_!(\mathrm{IC}_{\mathrm{Bun}_B}))$$

In other words, we need to show that the cone of the natural map

$$(8) \qquad \bar{q}^*(S_{\phi_{\bar{T}}}) \otimes j_!(\mathrm{IC}_{\mathrm{Bun}_B})) \to \bar{q}^*(S_{\phi_{\bar{T}}}) \otimes j_*(\mathrm{IC}_{\mathrm{Bun}_B}))$$

is trivial after applying $\bar{p}_!$. We will do this by factorizing (8). In particular, note, by applying projection formula to j as above and rewriting $\mathrm{IC}_{\mathrm{Bun}_B} \simeq q^*(\Delta_B^{1/2})[\dim(\mathrm{Bun}_B)] \simeq j^*\bar{q}^*(\Delta_B^{1/2})[\dim(\mathrm{Bun}_B)]$, we see that (7) is isomorphic to

$$\bar{p}_!(j_!j^*\bar{q}^*(S_{\phi_{\bar{T}}} \otimes \Delta_B^{1/2}))[\dim(\mathrm{Bun}_B)]$$

and therefore applying Verdier duality and Theorem 6.1 it follows that (6) is isomorphic to

$$\bar{p}_!(j_*j^*\bar{q}^*(S_{\phi_{\bar{T}}} \otimes \Delta_B^{1/2}))[\dim(\mathrm{Bun}_B)]$$

Therefore, we can rewrite the map (8) as

$$j_!j^*\bar{q}^*(S_{\phi_T} \otimes \Delta_B^{1/2}) \to j_*j^*\bar{q}^*(S_{\phi_T} \otimes \Delta_B^{1/2}))$$

Now note that we can factorize this morphism via the adjunction maps as

$$j_!j^*\bar{q}^*(S_{\phi_{\bar{T}}} \otimes \Delta_B^{1/2})[\dim(\mathrm{Bun}_B)] \overset{(1)}{\to} \bar{q}^*(S_{\phi_{\bar{T}}} \otimes \Delta_B^{1/2})[\dim(\mathrm{Bun}_B)] \overset{(2)}{\to} j_*j^*\bar{q}^*(S_{\phi_{\bar{T}}} \otimes \Delta_B^{1/2}))[\dim(\mathrm{Bun}_B)]$$

By the octahedral axiom, it suffices to show the cone of (1) and (2) are trivial after applying $\bar{p}_!$. The cone of (1) is relatively easy to get a handle on, but the cone of (2) is more tricky. To do this, we note it suffices to show the claim after applying Verdier duality on $\mathbb{D}_{\overline{\mathrm{Bun}_B}}$. This follows because $\bar{p}_! \simeq \bar{p}_*$ and Verdier duality on Bun_G can be checked to be a conservative functor. In particular, one can use the semi-orthogonal decomposition of $\mathrm{D}(\mathrm{Bun}_G)$ into the HN-strata $\mathrm{D}(\mathrm{Bun}_G^b)$ to reduce it to the claim that smooth duality on the unbounded derived category of smooth $\overline{\mathbb{F}}_\ell$-representations of a p-adic reductive group is conservative. Now, by the above discussion, the Verdier Dual of the target of (2) is equal to $j_!j^*\bar{q}^*(S_{\phi_T} \otimes \Delta_B^{1/2})[\dim(\mathrm{Bun}_B)]$. Hence, we deduce that the Verdier dual of (2) is equal to a map

$$j_!j^*\bar{q}^*(S_{\phi_T} \otimes \Delta_B^{1/2})) \to \mathbb{D}_{\overline{\mathrm{Bun}_B}}(\bar{q}^*(S_{\phi_{\bar{T}}} \otimes \Delta_B^{1/2})) \simeq \bar{q}^!(S_{\phi_T} \otimes \Delta_B^{1/2})$$

where $\Delta_B^{1/2}$ is the sheaf on Bun_T whose restriction to each connected component is given by the character $\delta_B^{1/2}$. As a quick sanity check, note that, by Theorem 6.1, we have an isomorphism $j_*\bar{q}^!(S_{\phi_T} \otimes \Delta_B^{1/2}) \simeq j^!\bar{q}^!(S_{\phi_T} \otimes \Delta_B^{1/2}) \simeq q^!(S_{\phi_T} \otimes \Delta_B^{1/2} \otimes \Delta_B)[\dim(\mathrm{Bun}_B)] \simeq q^*(S_{\phi_T} \otimes \Delta_B^{1/2})[\dim(\mathrm{Bun}_B)]$, and so we can see that this is the natural map coming from adjunction. Now, observe that ϕ_T is weakly generic if and only if $\phi_{\bar{T}}$ is weakly generic (since taking duals just exchanges the role of positive and negative roots). Therefore, it suffices to show the following.

Lemma 8.7. *Assume that ϕ_T is weakly generic toral parameter, then the cone of the morphism*

$$j_!j^*\bar{q}^*(S_{\phi_T} \otimes \Delta_B^{1/2}) \simeq j_!q^*(S_{\phi_T} \otimes \Delta_B^{1/2}) \to \bar{q}^*(S_{\phi_T} \otimes \Delta_B^{1/2})$$

vanishes after applying $\bar{p}_!$. Similarly, the cone of the natural map

$$j_!j^*\bar{q}^*(S_{\phi_T} \otimes \Delta_B^{1/2}) \simeq j_!q^*(S_{\phi_T} \otimes \Delta_B^{1/2}) \to \bar{q}^!(S_{\phi_T} \otimes \Delta_B^{1/2})$$

vanishes after applying $\bar{p}_!$.

Proof. We start with the first map. By excision, it suffices to show that the restriction to the locally closed stratum $_{\overline{\nu}}\overline{\mathrm{Bun}}_B$ vanishes for all $\overline{\nu} \in \Lambda_{G,B}^{\mathrm{pos}} \setminus \{0\}$ after applying $\overline{\mathsf{p}}_!$. In particular, we are tasked with computing

$$\overline{\mathsf{p}}_! j_{\overline{\nu}!} j_{\overline{\nu}}^* \overline{\mathsf{q}}^* (S_{\phi_T} \otimes \Delta_B^{1/2})$$

for all such $\overline{\nu}$. Note that the composite map

$$\mathrm{Bun}_B \times \mathrm{Div}^{(\overline{\nu})} \xrightarrow{j_{\overline{\nu}}} \overline{\mathrm{Bun}}_B \xrightarrow{\overline{\mathsf{p}}} \mathrm{Bun}_T$$

can be identified with

$$\mathrm{Bun}_B \times \mathrm{Div}^{(\overline{\nu})} \xrightarrow{\mathsf{q} \times \mathrm{id}} \mathrm{Bun}_T \times \mathrm{Div}^{(\overline{\nu})} \xrightarrow{h_{(\overline{\nu})}^\rightarrow} \mathrm{Bun}_T$$

where the last map is the Hecke operator defined in §3.3. Therefore, we are reduced to computing

$$\overline{\mathsf{p}}_! j_{\overline{\nu}!} (\mathsf{q} \times \mathrm{id})^* (h_{(\overline{\nu})}^\rightarrow)^* (S_{\phi_T} \otimes \Delta_B^{1/2}) \simeq \overline{\mathsf{p}}_! j_{\overline{\nu}!} (\mathsf{q} \times \mathrm{id})^* (S_{\phi_T} \otimes \Delta_B^{1/2} \otimes E_{\phi_T}^{(\overline{\nu})}(\hbar\hat{\rho}, \overline{\nu}\rangle))$$

However, note that we have an equality: $\overline{\mathsf{p}} \circ j_{\overline{\nu}} = \mathsf{p} \times g$, where $g : \mathrm{Div}^{(\overline{\nu})} \to *$ is the structure map. Therefore, by Künneth formula, we obtain that the RHS can be identified with

$$\mathrm{nEis}(S_{\phi_T}) \otimes R\Gamma_c(\mathrm{Div}^{(\overline{\nu})}, E_{\phi_T}^{(\overline{\nu})}(\hbar\hat{\rho}, \overline{\nu}\rangle))$$

but now it follows, by Corollary 3.21 and the weak genericity assumption on ϕ_T, that the complex $R\Gamma_c(\mathrm{Div}^{(\overline{\nu})}, E_{\phi_T}^{(\overline{\nu})}(\hbar\hat{\rho}, \overline{\nu}\rangle))$ is trivial. Now for the second map we argue similarly. In particular, it suffices to show that $\overline{\mathsf{q}}^!(S_{\phi_T} \otimes \Delta_B^{-1/2})$ vanishes after applying $\overline{\mathsf{p}}_! j_{\overline{\nu}} j_{\overline{\nu}}^!$, but then, as before, we observe that $j_{\overline{\nu}}^! \circ \overline{\mathsf{q}}^! = (\mathsf{q} \times \mathrm{id})^! \circ (h_{\overline{\nu}}^\rightarrow)^!$. By uniformizing Div^1 by punctured positive Banach-Colmez spaces, the dualizing object on $\mathrm{Div}^{(\overline{\nu})}$ can be identified up to a shift with $\Lambda(\hbar\hat{\rho}, \overline{\nu}\rangle)$. It follows that $h_{\overline{\nu}}^!(\Delta_B^{-1/2} \otimes S_{\phi_T})$ can be identified up to a shift with $E_{\phi_T}^{(\overline{\nu})}(-\hbar\hat{\rho}, \overline{\nu}\rangle + \hbar\hat{\rho}, \overline{\nu}\rangle) = E_{\phi_T}^{(\overline{\nu})}$. Similarly, we can compute $(\mathsf{q} \times \mathrm{id})^!$ using Theorem 6.1, and this will not effect the $\mathrm{Div}^{(\overline{\nu})}$-factor. Using that $\overline{\mathsf{p}}_! \circ j_{\overline{\nu}} \simeq \overline{\mathsf{p}} \circ j_{\overline{\nu}} = (\mathsf{p} \times g)$ together with the properness of g [FS21, Proposition II.1.21], we can apply Künneth to see the desired vanishing follows from the vanishing of the complex

$$R\Gamma_c(\mathrm{Div}^{(\overline{\nu})}, E_{\phi_T}^{(\overline{\nu})})$$

which again follows from Corollary 3.21.

For the rest of the paper, we will assume 8.1 and thereby the validity of Theorem 8.3. In addition, we will assume compatibility with a suitably nice form of the local Langlands correspondence (Assumption 7.5).

9. Stalks of Geometric Eisenstein Series

Now our aim is to explicitly determine the stalks of the sheaf $\mathrm{nEis}(S_{\phi_T})$, when ϕ_T is weakly normalized regular. Recall this means that ϕ_T is generic, and, for all $w \in W_G$ non-trivial, $\chi \delta_B^{1/2} \not\simeq (\chi \otimes \delta_B^{1/2})^w$. Genericity will allow us to apply the results of Corollary 7.7, and the second condition will appear naturally in the computation of the stalks. It is helpful to treat each connected component separately. In particular, using Corollary 5.3, we consider the decomposition:

$$\mathrm{nEis}(S_{\phi_T}) = \bigoplus_{\overline{\nu} \in B(T)} \mathrm{nEis}^{\overline{\nu}}(S_{\phi_T})$$

The main result of this section is as follows.

Theorem 9.1. *Consider ϕ_T a weakly normalized regular parameter with associated character $\chi :$ $T(\mathbb{Q}_p) \to \Lambda$. We fix $\bar{\nu} \in B(T)$ with image $b \in B(G)_{\mathrm{un}}$, HN-dominant reduction b_T, and associated Borel B_b. Using Corollary 2.11, we can write $\bar{\nu} = w(b_T)$ for a unique $w \in W_b$. Then we have an isomorphism*

$$\mathrm{nEis}^{\bar{\nu}}(S_{\phi_T}) \simeq j_{b!}(i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{1/2})[\langle 2\hat{\rho}, \nu_b \rangle]$$

under the identification $\mathrm{D}(J_b(\mathbb{Q}_p), \Lambda) \simeq \mathrm{D}(\mathrm{Bun}_G^b)$, where $j_b : \mathrm{Bun}_G^b \to \mathrm{Bun}_G$ is the locally closed immersion defined by the HN-stratum corresponding to b, and P_b is the standard parabolic with Levi factor $M_b \simeq J_b$.

By varying $\bar{\nu}$ over all connected components, we obtain the following.

Corollary 9.2. *Consider ϕ_T a weakly normalized regular parameter, with associated character $\chi :$ $T(\mathbb{Q}_p) \to \Lambda$. For $b \in B(G)$, the stalk $\mathrm{nEis}(S_{\phi_T})|_{\mathrm{Bun}_G^b} \in \mathrm{D}(\mathrm{Bun}_G^b) \simeq \mathrm{D}(J_b(\mathbb{Q}_p), \Lambda)$ is given by*

(1) *an isomorphism $\mathrm{nEis}(S_{\phi_T})|_{\mathrm{Bun}_G^b} \simeq \bigoplus_{w \in W_b} i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{1/2}[\langle 2\hat{\rho}, \nu_b \rangle]$ if $b \in B(G)_{\mathrm{un}}$,*
(2) *an isomorphism $\mathrm{nEis}(S_{\phi_T})|_{\mathrm{Bun}_G^b} \simeq 0$ if $b \notin B(G)_{\mathrm{un}}$.*

In particular, $\mathrm{nEis}(S_{\phi_T})$ is a perverse sheaf on Bun_G with respect to the standard t-structure defined by the HN-strata.

First, consider the following easy Lemma.

Lemma 9.3. *For $\bar{\nu} \in B(T)$ with image b in $B(G)$, the restriction*

$$\mathrm{nEis}^{\bar{\nu}}(S_{\phi_T})|_{\mathrm{Bun}_G^{b'}}$$

for $b' \in B(G)$ vanishes unless $b \leq b'$ in the natural partial ordering on $B(G)$.

Proof. This follows from the observation that the image of $\mathrm{Bun}_B^{\bar{\nu}}$ under $\mathfrak{p}^{\bar{\nu}}$ is contained in the open substack $\mathrm{Bun}_B^{\leq b}$ parametrizing bundles with associated Kottwitz element less than b. In particular, using the Tannakian formalism [Zie15, Theorems 4.42, 4.43], this reduces to the observation that, for GL_n, a bundle E with a filtration by vector subbundles has Harder-Narasimhan polygon less than or equal to Harder-Narasimhan polygon of the direct sum of the graded pieces of the filtration, which is an easy consequence of the formalism of Harder-Narasimhan reductions (See for example [Ked17, Corollary 3.4.18]). $\square$

Thus, for a fixed $b' \in B(G)$, Lemma 9.3 tells us that $\mathrm{nEis}(S_{\phi_T})|_{\mathrm{Bun}_G^{b'}}$ is a direct sum of $\mathrm{nEis}^{\bar{\nu}}(S_{\phi_T})|_{\mathrm{Bun}_G^{b'}}$ for $\bar{\nu}$ whose image $b \in B(G)$ satisfies $b \leq b'$. Now the key point is that, under the weak normalized regularity assumption, all the contributions will vanish except when $b' = b$. This is one of the many reasons that weak normalized regularity (or at least genericity) is absolutely necessary to get a reasonable eigensheaf. In general, all possible $\bar{\nu}$ contribute to $\mathrm{nEis}(S_{\phi_T})|_{\mathrm{Bun}_G^{b'}}$, and $\mathrm{nEis}(S_{\phi_T})|_{\mathrm{Bun}_G^{b'}}$ will be equal to an infinite direct sum of smooth irreducible representations sitting in infinitely many degrees. We now reduce Theorem 9.1 to two propositions. We first have the following proposition describing the contribution of the split reduction in the connected components $\mathrm{Bun}_B^{\bar{\nu}}$.

Proposition 9.4. *Let $\bar{\nu} \in B(T)$ be an element mapping to $b \in B(G)_{\mathrm{un}}$. We write $\bar{\nu} = w(b_T)$ as above. Then, if $\xi : T(\mathbb{Q}_p) \to \Lambda$ is any smooth character, we have an isomorphism*

$$\mathrm{Eis}^{\bar{\nu}}(\xi)|_{\mathrm{Bun}_G^b} \simeq \mathrm{Ind}_{B_b}^{J_b}(\xi^w \otimes \delta_B^{1/2} \otimes (\delta_B^w)^{-1/2})[\langle 2\hat{\rho}, \nu_b \rangle]$$

of complexes of smooth $J_b(\mathbb{Q}_p)$-modules, under the identification $\mathrm{D}(\mathrm{Bun}_G^b) \simeq \mathrm{D}(J_b(\mathbb{Q}_p), \Lambda)$.

In particular, using the isomorphism

$$\mathrm{nEis}(S_{\phi_T}) \simeq \mathrm{Eis}(S_{\phi_T} \otimes \Delta_B^{1/2})$$

we have, for all $\overline{\nu} \in B(T)$ mapping to $b \in B(G)_{\mathsf{un}}$ and χ the character attached to ϕ_T, an isomorphism

$$\mathrm{nEis}^{\overline{\nu}}(S_{\phi_T})j_{\mathsf{Bun}_G^b} = \mathrm{Eis}(S_{\phi_T} \otimes \Delta_B^{1/2})j_{\mathsf{Bun}_G^b} \simeq \mathrm{Ind}_{B_b}^{J_b}(\chi^w \otimes (\delta_B^w)^{1/2} \otimes \delta_B^{1/2} \otimes (\delta_B^w)^{-1/2})[\langle 2\hat{\rho}, \nu_b \rangle]$$

$$\simeq \mathrm{Ind}_{B_b}^{J_b}(\chi^w \otimes \delta_B^{1/2})[\langle 2\hat{\rho}, \nu_b \rangle]$$

$$\simeq i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{1/2}[\langle 2\hat{\rho}, \nu_b \rangle]$$

This tells us that all the claimed contributions to the restriction $\mathrm{nEis}(S_{\phi_T})j_{\mathsf{Bun}_G^b}$ appear. All that remains is to show is that there are no additional contributions, and this is precisely what weak normalized regularity will allow us to do.

Proposition 9.5. *Assume ϕ_T is weakly normalized regular, then, for all $\overline{\nu} \in B(T)$ mapping to $b \in B(G)_{\mathsf{un}}$, the sheaf $\mathrm{nEis}^{\overline{\nu}}(S_{\phi_T})$ is only supported on the HN-strata Bun_G^b.*

We dedicate the remainder of this section to the proof of these two propositions.

9.0.1. *The Proof of Proposition 9.4.* We let $\mathsf{Bun}_B^{\overline{\nu}}$ be the connected component defined by $\overline{\nu} \in B(T)$ mapping to $b \in B(G)_{\mathsf{un}}$, and let $\mathsf{Bun}_B^{\overline{\nu},b}$ be the preimage of Bun_G^b along $\mathsf{p}^{\overline{\nu}} : \mathsf{Bun}_B^{\overline{\nu}} \to \mathsf{Bun}_G$. Topologically, the stack $\mathsf{Bun}_B^{\overline{\nu},b}$ is just a point defined by the split reduction $\mathcal{F}_{\overline{\nu}} \times^T B$. We already saw in the proof of Theorem 6.1 what the automorphisms of this B-bundle are. In particular, they are given by the group diamond $G_{\overline{\nu}}$, where $G_{\overline{\nu}}(S) = Q_{\overline{\nu}}(X_S)$ for $S \in \mathrm{Perf}$, and $Q_{\overline{\nu}}$ is defined as $\mathcal{F}_{w(b_T)} \times^T B = \mathcal{F}_{b_T} \times^T B^w$, as in Definition 6.17. In other words, we have an isomorphism

$$\mathsf{Bun}_B^{\overline{\nu},b} \simeq [\ast/G_{\overline{\nu}}]$$

and the map $\mathsf{p}^{\overline{\nu}} : \mathsf{Bun}_B^{\overline{\nu}} \to \mathsf{Bun}_G$ induces a map of the form

$$[\ast/G_{\overline{\nu}}] \to [\ast/J_b] \simeq \mathsf{Bun}_G^b$$

which we will abusively denote by p. This map is given by the natural inclusion

$$G_{\overline{\nu}} \hookrightarrow J_b$$

of group diamonds, coming from the fact that $Q_{\overline{\nu}}$ defines a reduction of $\mathcal{F}_b$, as in §6.1. Moreover, the map $\mathsf{q}^{\overline{\nu}} : \mathsf{Bun}_B^{\overline{\nu},b} \to \mathsf{Bun}_T$ is identified with a map of the form

$$[\ast/G_{\overline{\nu}}] \to [\ast/\underline{T(\mathbb{Q}_p)}]$$

which we will abusively denote by q. This map factors as

$$[\ast/G_{\overline{\nu}}] \longrightarrow \left[\ast/\underline{B_b(\mathbb{Q}_p)}\right] \xrightarrow{q^\natural} \left[\ast/\underline{T(\mathbb{Q}_p)}\right] \simeq^w \left[\ast/\underline{T(\mathbb{Q}_p)}\right]$$

where the first map is given by the semi-direct decomposition

$$G_{\overline{\nu}} \simeq G_{\overline{\nu}}^{>0} \ltimes G_{\overline{\nu}}^{=0}$$

and the identification $G_{\overline{\nu}}^{=0} \simeq B_b(\mathbb{Q}_p)$, the second map $q^\natural$ is the natural projection, and the last isomorphism is given by conjugating by w. It follows by base change that we have an isomorphism

$$\mathrm{Eis}^{\overline{\nu}}(\xi)j_{\mathsf{Bun}_G^b} \simeq \mathsf{p}_!\mathsf{q}^*(\xi)[d_{\overline{\nu}}]$$

where $d_{\overline{\nu}} = \dim(\mathsf{Bun}_B^{\overline{\nu}})$. We let $s : [\ast/\underline{J_b(\mathbb{Q}_p)}] \to [\ast/J_b]$ be the natural map, as in §6.1. Our goal is to compute

$$s^*\mathsf{p}_!\mathsf{q}^*(\xi)[d_{\overline{\nu}}]$$

as a complex of $J_b(\mathbb{Q}_p)$-representations. To do this, we consider the stack $Y^{\bar\nu}$ defined by the Cartesian diagram

(9)
$$
\begin{array}{ccc}
Y^{\bar\nu} & \xrightarrow{\;\;\mathfrak{p}\;\;} & \left[\,*/\underline{J_b(\mathbb{Q}_p)}\right] \\
\downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} \\
[\,*/G_{\bar\nu}] & \xrightarrow{\;\;\mathfrak{p}\;\;} & [\,*/J_b] \\
\downarrow{\scriptstyle q} & & \\
\left[\,*/\underline{T(\mathbb{Q}_p)}\right] & &
\end{array}
$$

By base-change, it suffices to compute:

$$\tilde{\mathfrak{p}}_! \tilde{s}^* \mathsf{q}^*(\xi)[d_{\bar\nu}]$$

Let's now describe the stack $Y^{\bar\nu}$. We define $G^{\bar\nu}$ to be the cokernel of the natural map of group diamonds:

$$0 \to G_{\bar\nu}^{\geq 0} \to J_b^{>0} \to G^{\bar\nu} \to 0$$

We recall that $G_{\bar\nu}^{=0} \simeq \underline{B_b(\mathbb{Q}_p)} \subset \underline{J_b(\mathbb{Q}_p)} \simeq J_b^{=0}$ acts on the first two terms by the semi-direct product structure. It follows that this is inclusion is equivariant with respect to this action and hence $G^{\bar\nu}$ also acquires an action of $\underline{B_b(\mathbb{Q}_p)}$. We deduce that the space $Y^{\bar\nu}$ is isomorphic to the v-stack quotient:

$$[G^{\bar\nu}/\underline{B_b(\mathbb{Q}_p)}]$$

With this in hand, we further refine diagram (9)

(10)
$$
\begin{array}{ccccc}
\left[G^{\bar\nu}/\underline{B_b(\mathbb{Q}_p)}\right] & & \xrightarrow{\quad\mathfrak{p}\quad} & & \left[\,*/\underline{J_b(\mathbb{Q}_p)}\right] \\
\downarrow & \searrow{\scriptstyle j} & & \nearrow{\scriptstyle i} & \downarrow \\
{\scriptstyle s}\downarrow & & \left[\,*/\underline{B_b(\mathbb{Q}_p)}\right] & & \downarrow{\scriptstyle s} \\
[\,*/G_{\bar\nu}] & \xrightarrow[\quad q\quad]{\;\;\mathfrak{p}\;\;} & & & [\,*/J_b] \\
\downarrow{\scriptstyle q} & & & & \\
\left[\,*/\underline{T(\mathbb{Q}_p)}\right] & & & &
\end{array}
$$

where i and j are the natural maps, and q is the composition of the natural projection $q^{\natural} : [\,*/\underline{B_b(\mathbb{Q}_p)}] \to [\,*/\underline{T(\mathbb{Q}_p)}]$ followed by conjugation by w. This allows us to further reduce to computing

$$\tilde{\mathfrak{p}}_! \tilde{s}^* \mathsf{q}^*(\xi)[d_{\bar\nu}] \simeq i_! j_! \tilde{s}^* \mathsf{q}^*(\xi)[d_{\bar\nu}] \simeq i_! j_! j^* q^*(\xi)[d_{\bar\nu}]$$

but now, by projection formula, this is isomorphic to

$$i_!(q^*(\xi) \otimes j_!(\Lambda))[d_{\bar\nu}].$$

Now recall by Lemma 6.18 we have an equality

$$d_{\bar\nu} = \langle 2\hat\rho, \nu_b \rangle - 2\langle 2\hat\rho_G^w, \nu_b \rangle$$

where $\hat{\rho}^w$ is the sum of the positive roots $\hat{\alpha} > 0$ such that $w(\hat{\alpha}) > 0$. To proceed further, we consider the character

$$\delta(t) := (\delta_B^w)^{-1/2}\,\delta_B^{1/2}(t) = \prod_{\hat{\alpha}>0}|\det(\mathrm{Ad}(t|\mathfrak{g}_{w(\hat{\alpha})}))|^{-1/2}\prod_{\hat{\alpha}>0}|\det(\mathrm{Ad}(t|\mathfrak{g}_{\hat{\alpha}}))|^{1/2} = \prod_{\substack{\hat{\alpha}>0\\ w(\hat{\alpha})<0}}|\det(\mathrm{Ad}(t|\mathfrak{g}_{w(\hat{\alpha})}))|$$

of $T(\mathbb{Q}_p)$. We have the following lemma.

Lemma 9.6. $j_!(\Lambda)$ *is isomorphic to* $(q^\natural)^*(\delta)[2(\langle 2\hat{\rho}_G^w, \nu_b\rangle - \langle 2\hat{\rho}, \nu_b\rangle)]$.

Let's see why the result follows from this. In particular, using the formula $d_{\overline{\nu}} = \langle 2\hat{\rho}, \nu_b\rangle - 2\langle 2\hat{\rho}_G^w, \nu_b\rangle$, this gives us an isomorphism:

$$\mathrm{Eis}^{\overline{\nu}}(\xi) \simeq i_!(q^*(\xi) \otimes (q^\natural)^*(\delta))[-2\langle 2\hat{\rho}, \nu_b\rangle + 2\langle 2\hat{\rho}_G^w, \nu_b\rangle + \langle 2\hat{\rho}, \nu_b\rangle - 2\langle 2\hat{\rho}_G^w, \nu_b\rangle]$$

$$\simeq i_!((q^\natural)^*(\xi^w \otimes \delta))[-\langle 2\hat{\rho}, \nu_b\rangle]$$

However, now $i_!((q^\natural)^*(\xi \otimes \delta)))$ will be identified with compactly supported functions on $J_b(\mathbb{Q}_p)$ which transform under the action of $B_b(\mathbb{Q}_p)$ by the character $\xi^w \otimes \delta$ via the natural projection $B_b(\mathbb{Q}_p) \to T(\mathbb{Q}_p)$. However, this is precisely the parabolic induction $\mathrm{Ind}_{B_b}^{J_b}(\xi^w \otimes \delta)$, and the claim follows. We now prove the lemma.

Proof. (Lemma 9.6) We need to determine $j_!(\Lambda)$, where j is the map:

$$[G^{\overline{\nu}}/\underline{B_b(\mathbb{Q}_p)}] \to [*/\underline{B_b(\mathbb{Q}_p)}]$$

Recall that $G^{\overline{\nu}}$ is defined via the short exact sequence of group diamonds:

$$0 \to G_{\overline{\nu}}^{>0} \to J_b^{>0} \to G^{\overline{\nu}} \to 0$$

However, as in §6.1, the map $G_{\overline{\nu}}^{>0} \to J_b^{>0}$ respects the filtration by commutator subgroups, and we similarly see the cokernel $G^{\overline{\nu}}$ has an induced filtration by commutator subgroups. Namely, since $G_{\overline{\nu}}^{>0}$ breaks up in terms of the positive Banach-Colmez spaces $H^0((w(\hat{\alpha}))^*(\mathcal{Q}))$, for $\hat{\alpha}$ a positive root such that $w(\hat{\alpha}) > 0$ and $\langle \hat{\alpha}, \nu_b\rangle \neq 0$, and $J_b^{>0}$ breaks up as $H^0(\hat{\alpha}^*(\mathcal{Q}))$, for $\hat{\alpha}$ a positive root such that $\langle \hat{\alpha}, \nu_b\rangle \neq 0$, we can write $G^{\overline{\nu}}$ as an iterated fibration of the positive Banach-Colmez spaces

$$H^0(\hat{\alpha}^*(\mathcal{Q}))$$

where $\hat{\alpha}$ is a positive root such that $w(\hat{\alpha}) < 0$. Here we note that it automatically follows that $\langle \hat{\alpha}, \nu_b\rangle \neq 0$, since $w \in W_G/W_{M_b}$ is identified with a representative of minimal length in W_G, by definition of M_b. Therefore, we deduce that j is an iterated fibration of positive Banach-Colmez spaces, and we can apply the proof of [FS21, Proposition V.2.1] to deduce that the adjunction

$$j^!j_!(\Lambda) \simeq \Lambda$$

is an isomorphism. The claim is therefore reduced to showing that:

$$j^!(\Lambda) \simeq \delta^{-1}[2(\langle 2\hat{\rho}, \nu_b\rangle - \langle 2\hat{\rho}_G^w, \nu_b\rangle)]$$

As per usual, we consider the Cartesian diagram

$$
\begin{array}{ccc}
G^{\overline{\nu}} & \xrightarrow{\;\tilde{j}\;} & * \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
\left[G^{\overline{\nu}}/\underline{B_b(\mathbb{Q}_p)}\right] & \xrightarrow{\;j\;} & \left[*/\underline{B_b(\mathbb{Q}_p)}\right]
\end{array}
$$

and, by base-change, obtain an isomorphism:

$$\tilde{j}^!(\Lambda) \simeq \tilde{\pi}^*j^!(\Lambda)$$

By Lemma 6.9, the dualizing object on the Banach-Colmez spaces $H^0(\hat{\alpha}(Q))$ is isomorphic to $j_!j^*[2\langle\hat{\alpha},\nu_b\rangle]$ as a sheaf with $\mathbb{Q}_p$-action. However, $B_b(\mathbb{Q}_p)$ acts on this positive Banach-Colmez spaces via the natural surjection

$$B_b(\mathbb{Q}_p) \to T(\mathbb{Q}_p)$$

composed with the character given by

$$T(\mathbb{Q}_p) \to \mathbb{Q}_p$$
$$t \mapsto \mathrm{Ad}(t|\mathfrak{g}_{w(\hat{\alpha})})^{-1}$$

and the scaling action of $\mathbb{Q}_p$ on $H^0(\hat{\alpha}(Q))$, where the inverse comes from the minus sign when passing between bundles and isocrystals. This tells us that the dualizing object on $G^{\bar{\nu}}$ as a sheaf with $B_b(\mathbb{Q}_p)$-action is isomorphic to

$$\prod_{\substack{\hat{\alpha}>0 \\ w(\hat{\alpha})<0}} |\det(\mathrm{Ad}(t|\mathfrak{g}_{w(\hat{\alpha})})|^{-1}[2\sum_{\substack{\hat{\alpha}>0 \\ w(\hat{\alpha})<0}} \langle\hat{\alpha},\nu_b\rangle]$$

but this is isomorphic to

$$\delta^{-1}[2(\langle 2\hat{\rho},\nu_b\rangle - \langle 2\hat{\rho}_G^w,\nu_b\rangle)]$$

as desired.

9.0.2. *The Proof of Proposition 9.5.* We argue by induction on $b \in B(G)_{\mathrm{un}}$ with respect to the partial ordering on $B(G)$ and the following stronger statement.

"For $b \in B(G)_{\mathrm{un}}$ with HN-dominant reduction b_T, and $\bar{\nu} = w(b_T) \in B(T)$ mapping to b for varying $w \in W_b$, we have an isomorphism

$$\mathrm{nEis}^{w(b_T)}(S_{\phi_T}) \simeq j_{b!}(i^{J_b}_{B_b}(\chi^w) \otimes \delta^{1/2}_{P_b})[-\langle 2\hat{\rho},\nu_b\rangle]$$

of sheaves in $\mathrm{D}(\mathrm{Bun}_G)$."

The base case will be when b is such that any $b' \in B(G)_{\mathrm{un}}$ satisfying $b \preceq b'$ is equal to b. Since the stalk of $\mathrm{nEis}^{\bar{\nu}}(S_{\phi_T})|_{\mathrm{Bun}_G^{b'}}$ will only be non-trivial for $b' \in B(G)$ such that $b \preceq b'$ by Lemma 9.3, the result in this case follows follows from Proposition 9.4 and Corollary 7.7 (1), where we note that ϕ_T is weakly normalized regular and therefore generic. For the inductive step, assume the claim is true for all $b' \in B(G)_{\mathrm{un}}$ such that $b \precneqq b'$. Let $\bar{\nu} \in B(T)$ be an element mapping to b. By Proposition 9.4 and Corollary 7.7 (1) again, it suffices to show that the restriction of $\mathrm{nEis}^{\bar{\nu}}(S_{\phi_T})$ to $\mathrm{Bun}_G^{b'}$ vanishes for all such b'. By Corollary 7.7 (2), it suffices to show, for all $w \in W_{b'}$, that the complex

$$R\mathcal{H}om(\mathrm{nEis}^{\bar{\nu}}(S_{\phi_T})|_{\mathrm{Bun}_G^{b'}}, i^{J_{b'}}_{B_{b'}}(\chi^w) \otimes \delta^{1/2}_{P_b}) = R\mathcal{H}om(\mathrm{nEis}^{\bar{\nu}}(S_{\phi_T}), j_{b'*}(i^{J_{b'}}_{B_{b'}}(\chi^w) \otimes \delta^{1/2}_{P_b}))$$

is trivial. To do this, let $\bar{\nu}' = w(b'_T)$ be the element mapping to $b' \in B(G)$ defined by $w \in W_{b'}$. Our inductive hypothesis tells us that we have an isomorphism

$$j_{b'!}(i^{J_{b'}}_{B_{b'}}(\chi^w) \otimes \delta^{1/2}_{P_b})[-\langle 2\hat{\rho},\nu_{b'}\rangle] \simeq \mathrm{nEis}^{\bar{\nu}'}(S_{\phi_T})$$

varying over $\bar{\nu}'$ mapping to $b' \in B(G)$. If we write $\mathrm{nEis}^{\bar{\nu}'}_*(S_{\phi_T})$ for the sheaf defined by replacing $\mathfrak{p}_!$ with $\mathfrak{p}_*$ in the definition of $\mathrm{nEis}^{\bar{\nu}'}(S_{\phi_T})$. It follows, using Theorem 8.3 and Theorem 6.1, that this is isomorphic to $j_{b'*}(i^{J_{b'}}_{B_{b'}}(\chi^w)) \otimes \delta^{1/2}_{P_b}[-\langle 2\hat{\rho},\nu_b\rangle]$ (See the discussion preceding Proposition 11.10 for details). Therefore, it then suffices to show, for all $\bar{\nu}'$ mapping to $b' \in B(G)_{\mathrm{un}}$ and $\bar{\nu}$ mapping to $b \in B(G)_{\mathrm{un}}$, that

$$R\mathcal{H}om(\mathrm{nEis}^{\bar{\nu}}(S_{\phi_T}), \mathrm{nEis}^{\bar{\nu}'}_*(S_{\phi_T}))$$

is trivial. To aid our analysis, we consider the following functor

$$\mathrm{CT}^{\bar{\nu}}(-) := \mathfrak{q}^{\bar{\nu}}_* \mathfrak{p}^{\bar{\nu}!}(-) : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_T^{\bar{\nu}})[-\dim(\mathrm{Bun}_B^{\bar{\nu}})]$$

which is in particular the right adjoint of the unnormalized Eisenstein functor $\mathrm{Eis}^{\overline{\nu}}(\ ) :=$ $\mathsf{p}_!(\mathsf{q}^*(\ )[\dim(\mathrm{Bun}_B^{\overline{\nu}})])$. Writing $\mathrm{nEis}^{\overline{\nu}}(S_{\phi_T})$ as $\mathrm{Eis}^{\overline{\nu}}(S_{\phi_T} \otimes \Delta_B^{1/2})$ and using adjunction, it suffices to show that the complex

$$R\mathcal{H}om_{T(\mathbb{Q}_p)}(\chi \otimes \delta_B^{1/2}, \mathrm{CT}^{\overline{\nu}} \circ \mathrm{nEis}^{\overline{\nu}^0}(S_{\phi_T}))$$

is trivial in $\mathrm{D}(\mathrm{Bun}_T^{\overline{\nu}}) \simeq \mathrm{D}(T(\mathbb{Q}_p), \Lambda)$. To show this, we first look at the diagram

$$\begin{array}{ccccc}
\mathrm{Bun}_B^{\overline{\nu}} \times_{\mathrm{Bun}_G} \mathrm{Bun}_B^{\overline{\nu}^0} & \xrightarrow{^0\mathsf{p}^{\overline{\nu}}} & \mathrm{Bun}_B^{\overline{\nu}^0} & \xrightarrow{\mathsf{q}^{\overline{\nu}^0}} & \mathrm{Bun}_T^{\overline{\nu}^0} \\
\downarrow{^0\mathsf{p}^{\overline{\nu}^0}} & & \downarrow{\mathsf{p}^{\overline{\nu}^0}} & & \\
\mathrm{Bun}_B^{\overline{\nu}} & \xrightarrow{\quad\mathsf{p}^{\overline{\nu}}\quad} & \mathrm{Bun}_G & & \\
\downarrow{\mathsf{q}^{\overline{\nu}}} & & & & \\
\mathrm{Bun}_T^{\overline{\nu}} & & & &
\end{array}$$

and note, by base-change, that we have a natural isomorphism

$$\mathsf{q}^{\overline{\nu}}_* \mathsf{p}^{\overline{\nu}!} \mathsf{p}^{\overline{\nu}^0}_* \mathsf{q}^{\overline{\nu}^0*}(\ ) \simeq \mathsf{q}^{\overline{\nu}}_* {}^0\mathsf{p}^{\overline{\nu}^0}_* {}^0\mathsf{p}^{\overline{\nu}!} \mathsf{q}^{\overline{\nu}^0*}(\ )$$

of derived functors $\mathrm{D}(\mathrm{Bun}_T^{\overline{\nu}^0}) \to \mathrm{D}(\mathrm{Bun}_T^{\overline{\nu}})$. This tells us that $\mathrm{CT}^{\overline{\nu}} \circ \mathrm{nEis}^{\overline{\nu}^0}(S_{\phi_T})$ is the direct image of the complex

$${}^0\mathsf{p}^{\overline{\nu}!}(\mathsf{q}^{\overline{\nu}^0*}(S_{\phi_T} \otimes \Delta_B^{1/2}))[\dim(\mathrm{Bun}_B^{\overline{\nu}^0}) \otimes \dim(\mathrm{Bun}_B^{\overline{\nu}})]$$

on $\mathrm{Bun}_B^{\overline{\nu}} \times_{\mathrm{Bun}_G} \mathrm{Bun}_B^{\overline{\nu}^0}$ onto $\mathrm{Bun}_T^{\overline{\nu}}$. By [Ham21a, Lemma 4.9], the space $\mathrm{Bun}_B^{\overline{\nu}} \times_{\mathrm{Bun}_G} \mathrm{Bun}_B^{\overline{\nu}^0}$ has a locally closed stratification given by the generic relative position of the two bundles

$$\bigsqcup_{w \in W_G} (\mathrm{Bun}_B^{\overline{\nu}} \times_{\mathrm{Bun}_G} \mathrm{Bun}_B^{\overline{\nu}^0})_w$$

which we denote by $Z_w^{\overline{\nu},\overline{\nu}^0}$ for varying $w \in W_G$. Using the excision spectral sequence, this implies that $\mathrm{CT}^{\overline{\nu}} \circ \mathrm{nEis}^{\overline{\nu}^0}(S_{\phi_T})$ also admits a filtration whose graded pieces we write as $(\mathrm{CT}^{\overline{\nu}} \circ \mathrm{nEis}^{\overline{\nu}^0}(S_{\phi_T}))_w$. Consider the following claim.

Proposition 9.7. *Let $\overline{\nu}$ and $\overline{\nu}^0$ be two elements mapping to b and b^0 in $B(G)_{\mathsf{un}}$, respectively.*

(1) *Suppose that $b \neq b^0$ then the stack $Z_w^{\overline{\nu},\overline{\nu}^0}$ is empty if $w = 1$.*

(2) *If $w \neq 1$ then $(\mathrm{CT}^{\overline{\nu}} \circ \mathrm{nEis}^{\overline{\nu}^0}(S_{\phi_T}))_w$ is an extension of complexes in $\mathrm{D}(T(\mathbb{Q}_p), \Lambda)$ isomorphic to $(\chi \otimes \delta_B^{1/2})^w$.*

First, let's finish the proof of Proposition 9.5 assuming this. By the above discussion, it suffices to show that the complex

$$R\mathcal{H}om_{T(\mathbb{Q}_p)}(\chi \otimes \delta_B^{1/2}, (\mathrm{CT}^{\overline{\nu}} \circ \mathrm{nEis}^{\overline{\nu}^0}(S_{\phi_T}))_w)$$

is trivial for all $w \in W_G$. This is trivial if $w = 1$ by Point (1). If $w \neq 1$, then it follows from Point (2) and the fact that the existence of a map $\chi \otimes \delta_B^{1/2} \to \chi^w \otimes (\delta_B^{1/2})^w$ would contradict Condition 3.7 (3) in the definition of weak normalized regularity by Schur's Lemma. Therefore, since ϕ_T is weak normalized regular by assumption, the claim follows. Let's now finish up by reducing this Proposition to a simpler claim, which we will prove in the next section. Proposition 9.7 is an analogue of [BG08, Proposition 10.8] and the idea behind its proof is the same.

Proof. (Proposition 9.7) We first begin by elucidating the geometry of the spaces $Z_w^{\overline{\nu},\overline{\nu}^0}$ a bit more. For $S \in \mathrm{Perf}$, note that a S-point of $\mathrm{Bun}_B^{\overline{\nu}} \times_{\mathrm{Bun}_G} \mathrm{Bun}_B^{\overline{\nu}^0}$ corresponds to a pair of B-structures on a G-bundle $\mathcal{F}_G$ on X_S. Namely, it parametrizes a pair $\mathcal{F}_B^1$ (resp. $\mathcal{F}_B^2$) of two B-structures on a

G-bundle $\mathcal{F}_G$ whose reduction to T, denoted $\mathcal{F}_T^1$ (resp. $\mathcal{F}_T^2$) is isomorphic to $\mathcal{F}_{\bar{\nu}}$ (resp. $\mathcal{F}_{\bar{\nu}'}$) after pulling back to any geometric point of S. We can think of it as parameterizing sections

$$X_S \to B \backslash G/B$$

such that the degree is of the specified form. More transparently, we can think of a point of $\mathrm{Bun}_B^{\bar{\nu}} \times_{\mathrm{Bun}_G} \mathrm{Bun}_B^{\bar{\nu}'}$ as the B-bundle $\mathcal{F}_B^2$ together with a section

$$s : X_S \to \mathcal{F}_B^2 \times^B G/B$$

where B acts via conjugation on G/B. For $\hat{\lambda} \in \hat{\Lambda}_G^+$, we recall that by interpreting $V^{\hat{\lambda}}$ as global sections of the appropriately twisted bundle on G/B corresponding to $\hat{\lambda}$ under Borel-Weil-Bott, every point in G/B gives rise to a line $\ell^{\hat{\lambda}} \subset V^{\hat{\lambda}}$. We consider the B-stable subspace $V_{\geq w}^{\hat{\lambda}} \subset V^{\hat{\lambda}}$ consisting of weights greater than or equal to $w(\hat{\lambda})$ and $V_{>w}^{\hat{\lambda}} \subset V_{\geq w}^{\hat{\lambda}}$ the codimension 1 subspace consisting of weights strictly greater than $w(\hat{\lambda})$. We let $(G/B)_w := BwB/B$ be the locally closed Schubert cell attached to $w \in W_G$. We write $\overline{(G/B)}_w$ for its closure. The closure is stratified by the Schubert cells indexed by elements $w' \in W_G$ with length less than or equal to w. Then the line $\ell^{\hat{\lambda}} \subset V^{\hat{\lambda}}$ will correspond to a point in $\overline{(G/B)}_w$ if and only if it belongs to $V_{\geq w}^{\hat{\lambda}}$ for all $\hat{\lambda} \in \hat{\Lambda}_G^+$. Moreover, the point belongs to the stratum $(G/B)_w$ if and only the projection to $V_{\geq w}^{\hat{\lambda}}/V_{>w}^{\hat{\lambda}}$ is non-zero. This allows us to explain what it means to lie in the locally closed stratum $Z_w^{\bar{\nu},\bar{\nu}'}$. In particular, by definition [Ham21a, Page 26], lying in this stratum is equivalent to the condition that s factors through $\mathcal{F}_B^2 \times^B \overline{(G/B)}_w$ and is not contained in any closed strata defined by $\mathcal{F}_B^2 \times^B \overline{(G/B)}_{w'}$ for any $w' > w$ in the Bruhat order. This implies that the section s determines a set of line subbundles

$$L_{\mathcal{F}_T^1}^{\hat{\lambda}} \hookrightarrow (V_{\geq w}^{\hat{\lambda}})_{\mathcal{F}_B^2}$$

for all $\hat{\lambda} \in \hat{\Lambda}_G^+$. Moreover, via the inclusion $V_{\geq w}^{\hat{\lambda}} \subset V^{\hat{\lambda}}$, these give the Plücker description of the B-structure $\mathcal{F}_B^1$ such that $\mathcal{F}_G \simeq \mathcal{F}_B^1 \times^B G \simeq \mathcal{F}_B^2 \times^B G$. For this to define a point in $Z_w^{\bar{\nu},\bar{\nu}'}$, these need to satisfy the condition that the induced map

$$L_{\mathcal{F}_T^1}^{\hat{\lambda}} \hookrightarrow (V_{\geq w}^{\hat{\lambda}})_{\mathcal{F}_B^2} \to (V_{\geq w}^{\hat{\lambda}}/V_{>w}^{\hat{\lambda}})_{\mathcal{F}_B^2} = (L^{\hat{\lambda}})_{(\mathcal{F}_T^2)^w}$$

is non-zero map of $\mathcal{O}_{X_S}$-modules (cf. [Bra+02, Page 14],[BG08, Page 48],[Sch15, Propositions 4.3.2, 4.4.2]). In particular, since $L_{\mathcal{F}_T^1}^{\hat{\lambda}}$ is a line bundle, the map being non-zero implies it is a fiberwise injective map of line bundles. Now, recalling the minus sign when passing between isocrystals and bundles, if we define $\theta := \bar{\nu} - w(\bar{\nu}')$ then the support of the torsion of the cokernel of this map of line bundles determines a point in $\mathrm{Div}^{(\theta)}$, by the assumptions on the degrees and Lemma 5.20. For this strata to be non-empty, we must have that $\theta \in \Lambda_{G,B}^{pos}$. Therefore, for all non-empty strata, we have a map:

$$\pi_w : Z_w^{\bar{\nu},\bar{\nu}'} \to \mathrm{Div}^{(\theta)}$$

Now, with these preparations out of the way, let's start with the proof. For Point (1), note that if $w = 1$ then we have an injective map of line bundles

$$L_{\mathcal{F}_T^1}^{\hat{\lambda}} \to L_{\mathcal{F}_T^2}^{\hat{\lambda}}$$

for all $\hat{\lambda}$, which give rise to the embeddings defined by $\mathcal{F}_B^1$ when composed with the injections of bundles $L_{\mathcal{F}_T^2}^{\hat{\lambda}} \to V_{\mathcal{F}_G}^{\hat{\lambda}}$ defined by $\mathcal{F}_B^2$, by construction. However, since the composition $L_{\mathcal{F}_T^1}^{\hat{\lambda}} \to V_{\mathcal{F}_G}^{\hat{\lambda}}$ is also a map of vector bundles, this is impossible unless $\mathcal{F}_T^1 \simeq \mathcal{F}_T^2$, which would contradict our assumption that $\bar{\nu}, \bar{\nu}' \in B(T)$ map to $b \neq b'$ in $B(G)$. Therefore, we have established point (1). For

point (2), we write q_1 (resp. q_2) for the natural projections of $Z_w^{\overline{\nu},\overline{\nu}^0}$ to $\mathrm{Bun}_T^{\overline{\nu}}$ (resp. $\mathrm{Bun}_T^{\overline{\nu}^0}$). We note that q_2 is equal to the composition

$$Z_w^{\overline{\nu},\overline{\nu}^0} \overset{\mathsf{q}_1}{\to} \overset{\pi_w}{\to} \mathrm{Bun}_T^{\overline{\nu}} \times \mathrm{Div}^{(\theta)} \overset{{}^{\mathrm{op}}h_{(\theta)}^!}{\to} \mathrm{Bun}_T^{w(\overline{\nu}^0)} \overset{w}{\to} \mathrm{Bun}_T^{\overline{\nu}^0}$$

where ${}^{\mathrm{op}}h_{(\theta)}^!$ is the map sending $(\mathcal{F}_T, (D_i)_{i \in J})$ to the bundle $\mathcal{F}_T(\sum_{i \in J} \alpha_i \cdot D_i)$ and the last map is given by conjugation by w. Recall, $(\mathrm{CT}^{\overline{\nu}} \circ \mathrm{nEis}^{\overline{\nu}^0}(S_{\phi_T}))_w$ is given (up to a shift) by the sheaf

$$\mathsf{q}_{1 *} \, {}^0\mathsf{p}^{\overline{\nu}^0 !} \, \mathsf{q}^{\overline{\nu}^0}(\chi \boxtimes \delta_B^{1/2})$$

We can write this as the Verdier dual of $\mathsf{q}_{1 !} \, {}^0\mathsf{p}^{\overline{\nu}^0} \, \mathsf{q}^{\overline{\nu}^0 !}(\chi^{-1} \boxtimes (\delta_B^{1/2})^{-1}) \simeq \mathsf{q}_{1 !} \, {}^0\mathsf{p}^{\overline{\nu}} \, \mathsf{q}^{\overline{\nu}}(\chi^{-1} \boxtimes \delta_B^{1/2})$, where the last isomorphism is Theorem 6.1. Replacing χ by χ^{-1}, this reduces us to showing that $\mathsf{q}_{1 !} \, {}^0\mathsf{p}^{\overline{\nu}^0} \, \mathsf{q}^{\overline{\nu}^0}(\chi \boxtimes \delta_B^{1/2}) \simeq \mathsf{q}_{1 !} \, \mathsf{q}_2(\chi \boxtimes \delta_B^{1/2})$ is an extension of complexes which are isomorphic to $(\delta_B^{1/2})^w \boxtimes \chi^w$. Using the above factorization of q_2, we rewrite this as

$$\mathsf{q}_{1 !} \, (\mathsf{q}_1 \times \pi_w) \, ({}^{\mathrm{op}}h_{(\theta)}^!) \, (w^{-1}) \, (\chi \boxtimes \delta_B^{1/2}) \simeq \mathsf{q}_{1 !} \, (\mathsf{q}_1 \times \pi_w) \, ((\chi^w \boxtimes (\delta_B^{1/2})^w) \boxtimes (E_{\phi_T^w \cdot (\check{\rho}^w \cdot j J)}^{(\theta)})^-)$$

but now, by projection formula, we are reduced to the following, which is an analogue of [BG08, Proposition 10.10]

Proposition 9.8. *The direct image* $(\mathsf{q}_1 \times \pi_w)_!(\Lambda)$ *is an extension of complexes which are pullbacks of complexes on* $\mathrm{Div}^{(\theta)}$.

We will prove Proposition 9.8 by relating the spaces $Z_w^{\overline{\nu},\overline{\nu}^0}$ to some variants of what are called Zastava or semi-infinite flag spaces in the classical literature, as first studied over function fields by Feign, Finkelberg, Kusnetzov, and Mirkovic [Fei+99; FM99].

9.0.3. *Zastava Spaces.* We let $U^0 \subset U$ be the subgroup defined by the positive root spaces $\hat{\alpha} > 0$ such that $w(\hat{\alpha}) < 0$. We set $B^0 := TU^0 \subset B$ to be the subgroup of the Borel defined by these root spaces. We recall that U^0 acts simply transitively on the closed Schubert cell $(G/B)_{\leq w}$ and use this to define the w-twisted version of the Zastava space.

Definition 9.9. For $\theta \in \Lambda_{G,B}^{pos}$, we let $W_w^\theta \to \mathrm{Div}^{(\theta)}$ be the v-sheaf parameterizing, for $S \in \mathrm{Perf}$, a triple

$$(\mathcal{F}_{U^0}, s, D)$$

of the datum:

     A U^0-bundle $\mathcal{F}_{U^0}$ on X_S.
     A section $s: X_S \to \mathcal{F}_{U^0} \overset{U^0}{\times} (G/B)_{\leq w}$ that does not lie in $(G/B)_{\leq w^0}$ for any $w^0 > w$ in the Bruhat order.
     A divisor $D \in \mathrm{Div}^{(\theta)}$ such that the induced non-zero ($=)$ fiberwise injective) maps of line bundles

$$L^{\hat{\lambda}} \to (V_{\leq w}^{\hat{\lambda}})_{\mathcal{F}_{U^0}} \to (V_{\leq w}^{\hat{\lambda}}/V_{>w}^{\hat{\lambda}})_{\mathcal{F}_{U^0}} = O_{X_S}$$

     for all $\hat{\lambda} \in \hat{\Lambda}_G^+$ have cokernel with torsion supported on D.

Classically, the usual Zastava space in the literature is the same datum as above in the case that $w = w_0$ together with a level structure on the bundle $L^{\hat{\lambda}}$ so that it encodes information about enhanced B-structures on one of the factors. It's importance is that it provides a local model for the singularities of the Drinfeld compactification $\overline{\mathrm{Bun}}_B$. The space we have defined above in the case that $w = w_0$ is the open part of the Zastava space which models just the space Bun_B. As seen in our description of $Z_w^{\overline{\nu},\overline{\nu}^0}$ in the previous section, this will clearly have a relationship to the spaces we are interested in describing. Let us first just consider the case of the element of longest length. We claim that the following is true.

Lemma 9.10. *For $w = w_0$ the element of longest length and $\theta = \bar\nu - w(\bar\nu^\theta) \in \Lambda_{G,B}^{pos}$ for $\bar\nu^\theta$ and $\bar\nu$ as above, there exists a commutative diagram*

$$
\begin{array}{ccc}
Z_w^{\bar\nu,\bar\nu^\theta} & \longrightarrow & W_w^\theta \\
{\scriptstyle q_1}\,\downarrow\,{\scriptstyle \pi_w} & & \downarrow \\
\mathrm{Bun}_T^{\bar\nu} \times \mathrm{Div}^{(\theta)} & \xrightarrow{\ p_2\ } & \mathrm{Div}^{(\theta)}
\end{array}
$$

which is a Cartesian square.

Proof. When $w = w_0$, we have that $U^\theta = U$ and $B^\theta = B$. Given an S-point of W_w^θ, the maps

$$
L^{\hat\lambda} \hookrightarrow (V_w^{\hat\lambda})_{F_U} \hookleftarrow (V^{\hat\lambda})_{F_G}
$$

of vector bundles on X_S define a B-structure F_B on the G-bundle $F_U \times^U G$. We note that the induced map

$$
L^{\hat\lambda} \hookrightarrow (V_w^{\hat\lambda})_{F_U} \twoheadrightarrow (V_w^{\hat\lambda}/V_{>w}^{\hat\lambda})_{F_U} = O_{X_S}
$$

with torsion cokernel of support given by $D \in \mathrm{Div}^{(\theta)}$ induces an identification $L^{\hat\lambda} \simeq O_{X_S}(-\langle \hat\lambda, \theta \rangle D)$ implying that $F_B \times^B T$ has Kottwitz invariant given by $\theta = \bar\nu - w(\bar\nu^\theta)$ after pulling back to a geometric point. Given a bundle $F_T^1 \in \mathrm{Bun}_T^{\bar\nu}$ of degree $\bar\nu$, we can dualize the above maps to get a fiberwise injection

$$
O_{X_S} \hookrightarrow (L^{\hat\lambda})^- \simeq O_{X_S}(\langle \hat\lambda, \theta \rangle D)
$$

of line bundles, and then tensor by $L_{F_T^1}^{\hat\lambda}$ to get an injection

$$
L_{F_T^1}^{\hat\lambda} \hookrightarrow L_{F_T^1}^{\hat\lambda}(\langle \hat\lambda, \theta \rangle D)
$$

Similarly, by taking duals and twisting the U-torsor F_U by F_T^1, we obtain $B = T \ltimes U$-bundles F_B^1 and F_B^2 defining points in $Z_w^{\bar\nu,\bar\nu^\theta}$, giving rise to a map

$$
(\mathrm{Bun}_T^{\bar\nu} \times \mathrm{Div}^{(\theta)}) \times_{\mathrm{Div}^{(\theta)}} W_w^\theta \hookrightarrow Z_w^{\bar\nu,\bar\nu^\theta}
$$

which we can see is an isomorphism. In particular, given a point in $Z_w^{\bar\nu,\bar\nu^\theta}$ corresponding to B-bundles F_B^1 and F_B^2 then we can define a U-bundle $F_B^2 \times^B U$, and, as already seen in the previous section, we get a section $s : X_S \to F_U \times^U (G/B)_w$, $D \in \mathrm{Div}^{(\theta)}$, and a T-bundle F_T^1 of the desired form.

Now this lemma implies Proposition 9.8 in the case that $w = w_0$. In particular, under the isomorphism

$$
(\mathrm{Bun}_T^{\bar\nu} \times \mathrm{Div}^{(\theta)}) \times_{\mathrm{Div}^{(\theta)}} W_w^\theta \simeq Z_w^{\bar\nu,\bar\nu^\theta}
$$

$\mathrm{Bun}_T^{\bar\nu}$ splits off as direct factor, and so, by Künneth, we deduce the claim. Now we would like to apply a similar argument using the spaces W_w^θ in the case that w is a general element. However, we run into a problem that, in general, all we get is a map

$$
(\mathrm{Bun}_T^{\bar\nu} \times \mathrm{Div}^{(\theta)}) \times_{\mathrm{Div}^{(\theta)}} W_w^\theta \to Z_w^{\bar\nu,\bar\nu^\theta}
$$

where attached to a point in $(\mathrm{Bun}_T^{\bar\nu} \times \mathrm{Div}^{(\theta)}) \times_{\mathrm{Div}^{(\theta)}} W_w^\theta$ we only get a B^θ-torsor $F_{B^\theta}^1$ with T-factor of degree $\bar\nu$. The above map is then given by a base-change of the natural map $f_{\bar\nu} : \mathrm{Bun}_{B^\theta}^{\bar\nu} \to \mathrm{Bun}_B^{\bar\nu}$. In particular, if we let $\tilde{Z}_w^{\bar\nu,\bar\nu^\theta} \to Z_w^{\bar\nu,\bar\nu^\theta}$ be the base-change of $Z_w^{\bar\nu,\bar\nu^\theta}$ along the map $f_{\bar\nu}$ then the analogue of this Lemma holds with $\tilde{Z}_w^{\bar\nu,\bar\nu^\theta}$ in place of $Z_w^{\bar\nu,\bar\nu^\theta}$ by the same argument. Now let's study the map $f_{\bar\nu} : \mathrm{Bun}_{B^\theta}^{\bar\nu} \to \mathrm{Bun}_B^{\bar\nu}$ in a particular example and see how to prove Proposition 9.8 in this case.

Example 9.11. Suppose that $G = \mathrm{GL}_3$ and let $\bar{\nu}$ correspond to a tuple of integers $(e, f, g) \in \mathbb{Z}^3 \simeq B(T)$ via the Kottwitz invariant. We suppose w corresponds to the simple reflection exchanging the first and second basis vectors. After rigidifying the T-bundle $\mathcal{F}_T$ to be isomorphic to $(\mathcal{O}(e), \mathcal{O}(f), \mathcal{O}(g))$, we can view $\mathrm{Bun}_B^{\bar{\nu}}$ as the moduli space of torsors under the U-torsor

$$\begin{pmatrix} 1 & \mathcal{O}(e) & \mathcal{O}(f)- & \mathcal{O}(e) & \mathcal{O}(g)- \\ 0 & & 1 & \mathcal{O}(f) & \mathcal{O}(g)- \\ 0 & & 0 & & 1 \end{pmatrix}$$

over X, where the automorphisms (up to rigidification) of a point in $\mathrm{Bun}_B^{\bar{\nu}}$ are given by considering the H^0 Banach-Colmez spaces attached to these bundles. Similarly, after rigidification, we can view $\mathrm{Bun}_{B^\theta}^{\bar{\nu}}$ as the moduli space of torsors under the U^θ-torsor

$$\begin{pmatrix} 1 & \mathcal{O}(e) & \mathcal{O}(f)- & 0 \\ 0 & 1 & & 0 \\ 0 & & 0 & 1 \end{pmatrix}$$

over X, and the map $f_{\bar{\nu}}$ is given by taking direct sums of the extension of $\mathcal{O}(e)$ by $\mathcal{O}(f)$ defined by the point in $\mathrm{Bun}_{B^\theta}^{\bar{\nu}}$ with $\mathcal{O}(g)$. In particular, we can see that the fibers of the map $f_{\bar{\nu}} : \mathrm{Bun}_{B^\theta}^{\bar{\nu}} \to \mathrm{Bun}_B^{\bar{\nu}}$ are an iterated fibration in the Banach-Colmez spaces $H^0(\mathcal{O}(e) \quad \mathcal{O}(g)-)$ and $H^0(\mathcal{O}(f) \quad \mathcal{O}(g)-)$. If we assume that these Banach-Colmez spaces are positive it follows from the proof of [FS21, Proposition V.2.1] that the adjunction

$$f_{\bar{\nu}!} f_{\bar{\nu}}^! \to \mathrm{id}$$

is an equivalence. In particular, combining this with the above discussion would give us the proof of Proposition 9.8 in this case. We now consider $d \in \mathbb{N}_{>0}$ and fix a closed point $\gamma \in X$ in the Fargues-Fontaine curve over an algebraically closed complete field F in characteristic p. We look at the short-exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{O}_X(-d) \to \mathcal{O}_X \to \mathcal{O}_{X,\gamma}/t_\gamma^d \to 0$$

where $\mathcal{O}_{X,\gamma}$ is the completed local ring and t_γ is the uniformizing parameter corresponding to an untilt C of F. Tensoring by $\mathcal{O}(g)$, we get a short exact sequence:

$$0 \to \mathcal{O}_X(g-d) \to \mathcal{O}_X(g) \to \mathcal{O}_{X,\gamma}/t_\gamma^d \to 0$$

Let $\bar{\nu}_d$ correspond to the tuple of integers $(e, f, g-d) \in \mathbb{Z}^3$. Then we consider the natural map $f_{\bar{\nu}_d} : \mathrm{Bun}_{B^\theta}^{\bar{\nu}_d} \to \mathrm{Bun}_B^{\bar{\nu}_d}$. If we choose d sufficiently large such that the spaces $H^0(\mathcal{O}(e) \quad \mathcal{O}_X(g-d)-)$ and $H^0(\mathcal{O}(f) \quad \mathcal{O}_X(g-d)-)$ are positive Banach-Colmez spaces then the fibers of $f_{\bar{\nu}_d}$ will be an iterated fibration in these positive Banach-Colmez spaces, and therefore we can again conclude that the adjunction

$$f_{\bar{\nu}_d} f_{\bar{\nu}_d}^! \to \mathrm{id}$$

is an isomorphism. Now we claim that we have a map:

$$\mathrm{Bun}_B^{\bar{\nu}} \to \mathrm{Bun}_B^{\bar{\nu}_d}$$

Explicitly, given a point $\mathrm{Bun}_{B^\theta}^{\bar{\nu}}$, we have an exact sequence

$$0 \to E_2 \to E \to \mathcal{O}(g) \to 0$$

of bundles, where E_2 is an extension of $\mathcal{O}(e)$ and $\mathcal{O}(f)$. We can then consider the pullback of this exact sequence with respect to the map $\mathcal{O}(g-d) \to \mathcal{O}(g)$ given by the modification, which will give us a point in $\mathrm{Bun}_B^{\bar{\nu}_d}$. We write $f_{\bar{\nu},d\gamma} : \mathrm{Bun}_B^{\bar{\nu},d\gamma} := \mathrm{Bun}_B^{\bar{\nu}} \times_{\mathrm{Bun}_B^{\bar{\nu}_d}} \mathrm{Bun}_{B^\theta}^{\bar{\nu}_d} \to \mathrm{Bun}_B^{\bar{\nu}}$ for the pullback of $f_{\bar{\nu}_d}$ along this map. We again conclude that the adjunction

$$f_{\bar{\nu},d\gamma\,!} f_{\bar{\nu},d\gamma}^! \to \mathrm{id}$$

is an isomorphism. If we could use this map instead of $f_{\overline{\nu}}$, we could prove the claim by arguing as above. Indeed, consider $\mathsf{q}_1 \times \pi_w : Z_w^{\overline{\nu},\overline{\nu}^0} \to \operatorname{Bun}_T^{\overline{\nu}} \times \operatorname{Div}^{(\theta)}$. If we base-change all the above spaces to F, and let $(\operatorname{Div} \cap \overline{1})^{(\theta)}$ be the partially symmetrized power defined by $\operatorname{Div}^1 \cap \overline{1}$, the open complement of the closed point $\overline{1} \to \operatorname{Div}^1$ defined by the fixed untilt, then, if we consider the map $\mathsf{q}_1 \times \pi_w : Z_w^{\overline{\nu},\overline{\nu}^0} \to \operatorname{Bun}_T^{\overline{\nu}} \times (\operatorname{Div} \cap \overline{1})^{(\theta)}$ restricted to this locus, we can show that the base-change $Z_w^{\overline{\nu},\overline{\nu}^0}$ along $f_{\overline{\nu},d\overline{1}}$ sits in a analogous Cartesian square to Lemma 9.10, and deduce Proposition 9.8 for the restriction of $(\mathsf{q}_1 \times \pi_w)_! f_{\overline{\nu},d\overline{1}\,!} f_{\overline{\nu},d\overline{1}}^! (\Lambda) \simeq (\mathsf{q}_1 \times \pi_w)_!(\Lambda)$ to the open strata $(\operatorname{Div} \cap \overline{1})^{(\theta)} \subset \operatorname{Div}^{(\theta)}$. However, by excision, we can reduce Proposition 9.8 to studying this restriction with the claim over the closed complement being trivial. The claim in this case follows.

With this motivating example, all that remains is to formalize the above argument. In particular, first off note, by [FS21, Corollary V.2.3], that for the proof of Proposition 9.8 it suffices to consider the base-change of all the above spaces to the base $* = \operatorname{Spd}(F)$ for F an algebraically closed perfectoid field in characteristic p. We consider such a field with fixed characteristic 0 untilt C, and let $\overline{1} \to \operatorname{Div}^1$ be the closed F-point defined by this untilt. We consider the open complement $\operatorname{Div}^1 \cap \overline{1}$, and the partially symmetrized powers $(\operatorname{Div}^1 \cap \overline{1})^{(\theta)}$ defined by the open subset. By applying excision, it suffices to verify Proposition 9.8 over this open subset with the claim over the closed complement being trivial. We abuse notation and write $Z_w^{\overline{\nu},\overline{\nu}^0}$ and W_w^θ for the base-change to this open subspace for the rest of the section. Now let's consider the spaces defined by the locally pro-finite sets $\underline{B(\mathbb{Q}_p)}$ (resp. $\underline{B^0(\mathbb{Q}_p)}$), and the natural map

$$f_0 : \operatorname{Bun}_{B^0}^0 \simeq [* / \underline{B^0(\mathbb{Q}_p)}] \to \operatorname{Bun}_B^0 \simeq [* / \underline{B(\mathbb{Q}_p)}]$$

of v-stacks. The fibers of this map are an iterated fibration in $H^0(\mathcal{O}_X) = \underline{\mathbb{Q}_p}$ indexed by the positive roots $\hat{\alpha} > 0$ such that $w(\hat{\alpha}) > 0$. We choose $\overline{\nu}_{\overline{1}} \in \mathbb{X}_*(T_{\overline{\mathbb{Q}_p}})$ to be an element such that $\langle \overline{\nu}_{\overline{1}}, \hat{\alpha} \rangle < 0$ for all $\hat{\alpha} > 0$ such that $w(\hat{\alpha}) > 0$. Recalling the minus sign when passing between isocrystals and bundles, this implies that the map

$$f_{\overline{\nu}_{\overline{1}}} : \operatorname{Bun}_{B^0}^{\overline{\nu}_{\overline{1}}} \to \operatorname{Bun}_B^{\overline{\nu}_{\overline{1}}}$$

is a fibration in iterated positive Banach-Colmez spaces. We consider a modification $F_T^0 \dashrightarrow F_{\overline{\nu}_{\overline{1}}}$ at $\overline{1}$ of meromorphy $\overline{\nu}_{\overline{1}}$. This modification induces a map

$$\operatorname{Bun}_B^0 \to \operatorname{Bun}_B^{\overline{\nu}_{\overline{1}}}$$

which we precompose with the map

$$\operatorname{Bun}_B^{\overline{\nu}} \to \operatorname{Bun}_B^0$$

also given by an appropriate modification. This allows us to define

$$\operatorname{Bun}_{B^0}^{\overline{\nu},\overline{1}} := \operatorname{Bun}_{B^0}^{\overline{\nu}_{\overline{1}}} \times_{\operatorname{Bun}_B^{\overline{\nu}_{\overline{1}}}} \operatorname{Bun}_B^{\overline{\nu}}$$

by base-changing $f_{\overline{\nu}_{\overline{1}}}$. We write

$$f_{\overline{\nu},\overline{1}} : \operatorname{Bun}_{B^0}^{\overline{\nu},\overline{1}} \to \operatorname{Bun}_B^{\overline{\nu}}$$

for the base-change of $f_{\overline{\nu}_{\overline{1}}}$. By the proof of [FS21, Proposition V.2.1], we have that the adjunction

$$f_{\overline{\nu},\overline{1}\,!} f_{\overline{\nu},\overline{1}}^!$$

is an isomorphism, since $f_{\overline{\nu}_{\overline{1}}}$ and in turn $f_{\overline{\nu},\overline{1}}$ is an iterated fibration of positive Banach-Colmez spaces. Now we define

$$\tilde{Z}_w^{\overline{\nu},\overline{\nu}^0} := Z_w^{\overline{\nu},\overline{\nu}^0} \times_{\operatorname{Bun}_B^{\overline{\nu}}} \operatorname{Bun}_{B^0}^{\overline{\nu},\overline{1}}$$

By the previous adjunction, it suffices to show the analogue of Proposition 9.8 for the composition

$$\tilde{Z}_w^{\overline{\nu},\overline{\nu}^0} \to Z_w^{\overline{\nu},\overline{\nu}^0} \xrightarrow{\mathsf{q}_1 \times \pi_w} \operatorname{Bun}_T^{\overline{\nu}} \times (\operatorname{Div} \cap \overline{1})^{(\theta)}$$

and the shriek pullback of the constant sheaf along the first map. However, by now arguing exactly as in the proof of Lemma 9.10, with B^θ and its unipotent radical U^θ replacing B and U, we can deduce that the base-change of W_w^θ ! $(\mathrm{Div}\,n\,1\,)^{(\theta)}$ along $p_2 : \mathrm{Bun}_T^{\overline{\nu}}$ $(\mathrm{Div}\,n\,1\,)^{(\theta)}$! $(\mathrm{Div}\,n\,1\,)^{(\theta)}$ is precisely the space $\tilde{Z}_w^{\overline{\nu},\overline{\nu}^\theta}$. This concludes the proof of Proposition 9.8 by Künneth.

10. The Hecke Eigensheaf Property

10.1. **Tilting Eigensheaves.** We would now like to combine our work in the previous sections and use it to construct eigensheaves. We would like to do this in a uniform way for coefficient systems $\Lambda\ 2\ f\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell g$, where Λ has the discrete topology unless otherwise stated. One of the issues is that the representation theory of $^L G/\Lambda$ is substantially different in each of these three cases. The structure of the representation theory of $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(^L G)$ was described in §2. With $\overline{\mathbb{Q}}_\ell$-coefficients, the category is semisimple with simple objects given by V_{μ^Γ} for $\mu\ \ 2\ \mathbb{X}\ (T_{\overline{\mathbb{Q}}_p})^+/\Gamma$ a Γ-orbit of a geometric dominant cocharacter μ. If $\Lambda\ 2\ f\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell g$ this is no longer so straightforward; the representation V_{μ^Γ} can fail to be irreducible. To develop a good theory of algebraic representations with these coefficients, we will need to invoke our assumption that ℓ is very good with respect to G and use the theory of tilting modules. We first discuss the general notion of a tilting module. Fix H a split connected reductive group over $\Lambda\ 2\ f\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell g$. We consider the involution

$$\mathbb{D} : \mathrm{Rep}\ (H)\ !\ \mathrm{Rep}\ (H)$$

$$V\ \overline{V}\ (V\ )^\sigma$$

where V is the dual representation and σ is the Chevalley involution. For $\lambda\ 2\ \mathbb{X}\ (H)^+$ a dominant character, we let V^λ denote the highest weight representation attached to λ by Borel-Weil-Bott, and we write $V_\lambda := \mathbb{D}(V^\lambda)$. We now come to our key definition.

Definition 10.1. Given $V\ 2\ \mathrm{Rep}\ (H)$, we say that V has a Weyl (resp. good) filtration if it admits a filtration whose successive quotients are isomorphic to V^λ (resp. V_λ). We say V is tilting if it admits both a good and a Weyl filtration. We write $\mathrm{Tilt}\ (H)\ \ \mathrm{Rep}\ (H)$ for the full sub-category of tilting modules.

The category of tilting modules is additive, but usually not abelian. If $\Lambda = \overline{\mathbb{Q}}_\ell$ the highest weight modules are simple, and $\mathrm{Tilt}\ (H) = \mathrm{Rep}\ (H)$. Therefore, this is only an interesting notion if $\Lambda\ 2\ f\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell g$. The key point of moving to this sub-category is that we have the following generalization of usual highest weight theory due Ringel and Donkin [Rin91], [Don93].

Theorem 10.2. *For each $\lambda\ 2\ \mathbb{X}\ (H)^+$, there exists a unique indecomposable tilting module $T_\lambda\ 2$ $\mathrm{Tilt}\ (H)$ with highest weight λ. We have that $\dim(T_\lambda(\lambda)) = 1$, and, for varying λ, this parametrizes all indecomposable tilting modules.*

We also get the usual classification of all tilting modules in terms of highest weight tilting modules.

Proposition 10.3. *[Jan03, Section E.22],[Mat00, Lemma 7.3] For all $V\ 2\ \mathrm{Tilt}\ (H)$, there exists unique integers $n(\lambda)\ 2\ \mathbb{N}\ _0$ for all $\lambda\ 2\ \mathbb{X}\ (H)^+$ and an isomorphism*

$$V\ '\ \bigoplus_{\lambda\, 2\mathbb{X}\ (H)^+} (T_\lambda)^{n(\lambda)}$$

of tilting modules.

Now we come to a difficult result which was proven by [Wan82] for groups of type A_n, [Don93] for almost all groups, and [Mat00] in general.

Theorem 10.4. *If we have two tilting modules $V, V^\theta\ 2\ \mathrm{Tilt}\ (H)$ then the tensor product $V\ \ V^\theta$ is tilting.*

We can now extend this to the L-group using our assumption that ℓ is very good with respect to G. By Theorem 10.2 applied to $H = \hat{G}$, we deduce that we have a well-defined category Tilt $(\hat{G})$ of tilting modules, where each object can be written as a direct sum of highest weight tilting modules T_μ for $\mu \in \mathbb{X}^*(T_{\overline{\mathbb{Q}}_p})^+$ using Proposition 10.3. Now, given such a μ, we consider the reflex field E_μ, and extend this to a representation of $W_{E_\mu} \ltimes \hat{G}$, as in [Kot84, Lemma 2.1.2]. We define the tilting module T_{μ^Γ} as the induction of this representation from $W_{E_\mu} \ltimes \hat{G}$ to $W_{\mathbb{Q}_p} \ltimes \hat{G}$, and let Tilt $({}^L G) \subset$ Rep $({}^L G)$ be the full sub-category given by direct sums of such modules, where we note that T_{μ^Γ} only depends on the Γ-orbit of $\mu \in \mathbb{X}^*(T_{\overline{\mathbb{Q}}_p})^+/\Gamma$ of μ. Now, since ℓ is very good, it follows that $W_{\mathbb{Q}_p}$ acts on $\hat{G}$ via a quotient Q that is prime to ℓ by definition [FS21, Page 33], and therefore $W_{\mathbb{Q}_p}/W_{E_\mu}$ is also of order prime to ℓ. Combing this observation, Frobenius Reciprocity/Mackey theory, and Theorem 10.4, we conclude that T_{μ^Γ} is indeed an irreducible representation of ${}^L G$, and that Tilt$({}^L G)$ is preserved under tensor products. This allows us to define the following.

Definition 10.5. Given a continuous L-parameter $\phi : W_{\mathbb{Q}_p} \to {}^L G(\Lambda)$, we say a sheaf $S_\phi \in$ $\mathrm{D}_{\mathsf{lis}}(\mathrm{Bun}_G, \Lambda)$ is a tilting eigensheaf with eigenvalue ϕ if, for all $V \in$ Tilt $({}^L G^I)$, we are given isomorphisms

$$\eta_{V,I} : T_V(S_\phi) \simeq S_\phi \boxtimes r_V \circ \phi$$

of sheaves in $\mathrm{D}_{\mathsf{lis}}(\mathrm{Bun}_G, \Lambda)^{BW_{\mathbb{Q}_p}^I}$, which are natural in I and V, and compatible with compositions and exterior tensor products in V. If $\Lambda = \overline{\mathbb{Q}}_\ell$ this recovers Definition 3.1. We similarly say S_ϕ is a weak tilting eigensheaf if only the isomorphisms $\eta_{V,I}$ exist.

Remark 10.6. Our discussion of highest weight theory in §2.1 for $\hat{G}$ also extends to the tilting modules T_μ under our assumption that ℓ is very good. In particular, using Proposition 10.2, we can understand the possible weights occurring in $T_\mu|_{\hat{G}^\Gamma}$ in terms of $\overline{\nu} \in \mathbb{X}^*(T_{\overline{\mathbb{Q}}_p})$, which lie in the convex hull of the W_G orbit of $\mu \in \mathbb{X}^*(T_{\overline{\mathbb{Q}}_p})^+$, and the $\overline{\nu}$ weight space of $T_\mu|_{\hat{G}^\Gamma}$ will be a direct sum over the weight spaces $T_\mu(\nu)$ for $\nu \in \mathbb{X}^*(T_{\overline{\mathbb{Q}}_p})$ mapping to $\overline{\nu}$, as in Lemma 2.8. To see this, we note, by [FS21, Proposition VIII.5.11], ℓ being very good implies that we have the following:

  (1) $\hat{G}$ is a smooth linear algebraic group, and $\hat{G}^\circ$ is reductive.
  (2) $\hat{G}/\hat{G}^\circ \simeq \hat{T}/\hat{T}^\circ$ is of order prime to ℓ, where $\hat{G}^\circ$ (resp. $\hat{T}^\circ$) denotes the neutral component of $\hat{G}$ (resp. $\hat{T}$), and the isomorphism follows as in [Zhu15, Lemma 4.6].

These observations allows us the see the highest weight theory of the (possibly disconnected) group $\hat{G}$ behaves as expected with modular coefficients by mimicking the proof of Lemma 2.7.

We now define our candidate tilting eigensheaf for each of the possible coefficient systems $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$.

10.2. The Construction of the Eigensheaf.

We fix a toral parameter $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\Lambda)$, with induced parameter $\phi : W_{\mathbb{Q}_p} \xrightarrow{\phi_T} {}^L T(\Lambda) \to {}^L G(\Lambda)$. Our goal is to construct a candidate tilting eigensheaf with respect to the parameter ϕ. If $\Lambda = \overline{\mathbb{F}}_\ell$, we have already carried this out. It is simply the sheaf $\mathrm{nEis}(S_{\phi_T}) \in \mathrm{D}(\mathrm{Bun}_G)$ viewed as a sheaf in $\mathrm{D}_{\mathsf{lis}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ via the identification $\mathrm{D}(\mathrm{Bun}_G) \simeq \mathrm{D}_{\mathsf{lis}}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ [FS21, Proposition VII.6.6], obtained by embedding both categories into $\mathrm{D}_{\blacksquare}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$. To move beyond this case, we need to invoke the following Lemma.

Lemma 10.7. *For $\Lambda = \overline{\mathbb{F}}_\ell$ and ϕ_T weakly normalized regular, the sheaf $\mathrm{nEis}(S_{\phi_T}) \in \mathrm{D}(\mathrm{Bun}_G)$ is ULA with respect to the structure map $\mathrm{Bun}_G \to *$.*

Proof. Given $A \in \mathrm{D}(\mathrm{Bun}_G)$, we recall that A being ULA with respect to the map $\mathrm{Bun}_G \to *$ is equivalent to saying that its stalks $A|_{\mathrm{Bun}_G^b}$ are valued in a complex of smooth representations such

that $A^K|_{\mathrm{Bun}_G^b}$ is a perfect complex of Λ-modules for all open pro-p subgroups $K \subseteq J_b(\mathbb{Q}_p)$ [FS21, Theorem V.7.1]. In particular, the result follows from Corollary 9.2.

Now, if $\Lambda = \overline{\mathbb{Z}}_\ell$ then, by taking inverse limits with respect to the mod ℓ^n reductions of ϕ_T, and considering the systems of sheaves given by applying the Eisenstein functor to the eigensheaf attached to these reductions, we obtain a sheaf

$$\widehat{\mathrm{nEis}(S_{\phi_T})} \in \mathrm{D}_{\mathrm{\acute{e}t}}^{\mathsf{ULA}}(X, \overline{\mathbb{Z}}_\ell)$$

Now, we have a fully faithful embedding

$$\mathrm{D}_{\mathrm{\acute{e}t}}^{\mathsf{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell) \hookrightarrow \mathrm{D}_\blacksquare(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$$

given as in [FS21, Page 261]. This embedding is used to define the Hecke operators in the setting of solid sheaves (See [FS21, Page 264]), utilizing that sheaves in the Satake category are ULA over Div^I. In particular, this embedding is Hecke equivariant in the appropriate sense, and so the filtered eigensheaf property transfers to the image of $\widehat{\mathrm{nEis}(S_{\phi_T})}$ in $\mathrm{D}_\blacksquare(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$. We also have a natural embedding $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell) \hookrightarrow \mathrm{D}_\blacksquare(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$, and we can analogously define the set of ULA objects in it [FS21, Definition VII.7.8], denoted $\mathrm{D}_{\mathrm{lis}}^{\mathsf{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$. We have the following claim.

Lemma 10.8. *Under the embeddings of $\mathrm{D}_{\mathrm{lis}}^{\mathsf{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$ and $\mathrm{D}_{\mathrm{\acute{e}t}}^{\mathsf{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$ into $\mathrm{D}_\blacksquare(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$ described above, these two full subcategories are isomorphic.*

Proof. We have a semi-orthogonal decomposition of $\mathrm{D}_{\mathrm{lis}}^{\mathsf{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$ ([FS21, Proposition VII.7.3]) and $\mathrm{D}_{\mathrm{\acute{e}t}}^{\mathsf{ULA}}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$ into $\mathrm{D}(J_b(\mathbb{Q}_p), \overline{\mathbb{Z}}_\ell)_{\mathsf{adm}}$ and $\hat{\mathrm{D}}(J_b(\mathbb{Q}_p), \overline{\mathbb{Z}}_\ell)_{\mathsf{adm}}$ by excision, respectively. Here $\hat{\mathrm{D}}(J_b(\mathbb{Q}_p), \overline{\mathbb{Z}}_\ell)$ denotes the derived category of ℓ-complete smooth representations of $J_b(\mathbb{Q}_p)$, and the subscript adm is used to denote the full subcategory of objects such that its invariants under an open compact $K \subseteq J_b(\mathbb{Q}_p)$ is a perfect complex. Since the semi-orthogonal decompositions are compatible with the two embeddings into $\mathrm{D}_\blacksquare(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$ it suffices to show that we have an identification

$$\mathrm{D}(J_b(\mathbb{Q}_p), \overline{\mathbb{Z}}_\ell)_{\mathsf{adm}} \simeq \hat{\mathrm{D}}(J_b(\mathbb{Q}_p), \overline{\mathbb{Z}}_\ell)_{\mathsf{adm}},$$

but this follows from [Han20, Proposition 2.6]. $\square$

Using the isomorphism supplied by the previous Lemma, we can regard $\widehat{\mathrm{nEis}(S_{\phi_T})} \in \mathrm{D}_{\mathrm{\acute{e}t}}^{\mathsf{ULA}}(X, \overline{\mathbb{Z}}_\ell)$ as an object in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$, which we denote by $\mathrm{nEis}(S_{\phi_T})$. Since these isomorphisms are compatible with Hecke operators it follows that Corollary 7.4 transfers to this sheaf. It now remains to describe the desired sheaf when $\Lambda = \overline{\mathbb{Q}}_\ell$. In this case, we need to assume the parameter $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\overline{\mathbb{Q}}_\ell)$ is of the form $\overline{\phi}_T \otimes \overline{\mathbb{Q}}_\ell$, where $\overline{\phi}_T : W_{\mathbb{Q}_p} \to {}^L T(\overline{\mathbb{Z}}_\ell)$ is a parameter with weakly normalized regular mod ℓ-reduction. Then we consider the sheaf $\mathrm{nEis}(S_{\overline{\phi}_T}) \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$ constructed above, and define

$$\mathrm{nEis}(S_{\phi_T}) := \mathrm{nEis}(S_{\overline{\phi}_T})[\tfrac{1}{\ell}] \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$$

by taking the colimit over the multiplication by ℓ maps. Now with the candidate eigensheaf defined, we begin the proof of the eigensheaf property. To capture the necessary integrality conditions, we define the following.

Definition 10.9. For $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$ with the discrete topology and a continuous toral parameter $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\Lambda)$ we say that ϕ_T is integral if it admits a mod ℓ-reduction. In particular, if $\Lambda = \overline{\mathbb{Q}}_\ell$, we assume it is of the form $\overline{\phi}_T \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{Q}}_\ell$ for some continuous parameter $\overline{\phi}_T : W_{\mathbb{Q}_p} \to {}^L T(\overline{\mathbb{Z}}_\ell)$.

10.3. **The Hecke Eigensheaf Property.** We start with the following Theorem.

Theorem 10.10. *For $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$ with the discrete topology, we consider $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\Lambda)$ an integral parameter with weakly normalized regular mod ℓ reduction. There then exists a perverse sheaf $\mathrm{nEis}(S_{\phi_T}) \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ which is a filtered eigensheaf with eigenvalue ϕ as in Corollary 7.4. If V is a direct sum of $\boxtimes_{i \in I} T_{\mu_i^\Gamma}$ for geometric dominant cocharacters μ_i, and ϕ_T is strongly μ_i-regular (Definition 3.14), the filtration on $T_V(\mathrm{nEis}(S_{\phi_T}))$ splits uniquely, and we have a natural isomorphism*

$$T_V(\mathrm{nEis}(S_{\phi_T})) \simeq \mathrm{nEis}(S_{\phi_T}) \otimes r_V \circ \phi$$

of sheaves in $D_{\mathrm{lis}}(\mathrm{Bun}_G)^{BW_{\mathbb{Q}_p}^I}$. In particular, if ϕ_T is strongly μ-regular for all geometric dominant cocharacters μ then $\mathrm{nEis}(S_{\phi_T})$ is a tilting eigensheaf. For $b \in B(G)$, the stalk $\mathrm{nEis}(S_{\phi_T})|_{\mathrm{Bun}_G^b} \in D(\mathrm{Bun}_G^b) \simeq D(J_b(\mathbb{Q}_p), \Lambda)$ is given by

(1) *an isomorphism $\mathrm{nEis}(S_{\phi_T})|_{\mathrm{Bun}_G^b} \simeq \bigoplus_{w \in W_b} i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{1/2}[\langle 2\hat{\rho}, \nu_b \rangle]$ if $b \in B(G)_{\mathrm{un}}$,*

(2) *an isomorphism $\mathrm{nEis}(S_{\phi_T})|_{\mathrm{Bun}_G^b} \simeq 0$ if $b \notin B(G)_{\mathrm{un}}$.*

Moreover, if $\mathbb{D}_{\mathrm{Bun}_G}$ denotes Verdier duality on Bun_G, we have an isomorphism

$$\mathbb{D}_{\mathrm{Bun}_G}(\mathrm{nEis}(S_{\phi_T})) \simeq \mathrm{nEis}(S_{\phi_{\overline{T}}})$$

of sheaves in $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$.

Proof. The existence of $\mathrm{nEis}(S_{\phi_T})$ and the transfer of the filtered Hecke eigensheaf property to this sheaf was discussed in the previous section. The claim on Verdier duality follows from Theorem 8.3, and the discussion in [FS21, Section VII.5]. The description of the stalks follows from the construction and Corollary 9.2. It remains to show that the filtration on $T_V(\mathrm{nEis}(S_{\phi_T}))$ splits uniquely for $\boxtimes_{i \in I} T_{\mu_i^\Gamma} = V \in \mathrm{Rep}({}^L G)$ such that ϕ_T is strongly μ_i-regular for all $i \in I$. To do this, we note that an extension between the graded pieces of the filtration on $T_V(\mathrm{nEis}(S_{\phi_T}))$ is specified by a cohomology class in the H^1 of $R\Gamma(W_{\mathbb{Q}_p}^I, \boxtimes_{i \in I}(\nu_i - \nu_i^\emptyset) \circ \phi_T) \simeq \bigotimes_{i \in I}^{\mathbb{L}} R\Gamma(W_{\mathbb{Q}_p}, (\nu_i - \nu_i^\emptyset) \circ \phi_T)$ for ν_i and $\nu_i^\emptyset$ defining two distinct Γ-orbits of weights of the representation T_μ in $\hat{T}$ for all $i \in I$. In particular, we note if ϕ_T is strongly μ_i-regular then the H^1 vanishes and the filtration splits, and similarly the H^0 vanishes so the splitting is unique. In particular, since we know the filtration satisfies all the desired compatibilities for the eigensheaf property if we know strong μ-regularity for all μ, it follows from the splitting being unique that $\mathrm{nEis}(S_{\phi_T})$ is a tilting eigensheaf.

We now fix an integral parameter ϕ_T with weakly normalized regular mod ℓ-reduction and consider the sheaf $\mathrm{nEis}(S_{\phi_T}) \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ supplied by the previous theorem. We come to the following definition.

Definition 10.11. Given a tuple of geometric dominant cocharacters $(\mu_i)_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+)^I$ for all $i \in I$, we say that $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\Lambda)$ is $(\mu_i)_{i \in I}$-regular if the filtration on

$$T_V(\mathrm{nEis}(S_{\phi_T}))$$

splits (but not necessarily uniquely) for the tilting module $V = \boxtimes_{i \in I} T_{\mu_i^\Gamma} \in \mathrm{Tilt}({}^L G^I)$.

As seen in the proof of Theorem 10.10, if the μ_i are such that ϕ_T is strongly μ_i-regular then it follows that ϕ_T is $(\mu_i)_{i \in I}$-regular. For certain μ_i, strong μ_i-regularity will be implied by genericity, and the following Proposition allows us to deduce that the filtration splits in more cases.

Proposition 10.12. *Suppose that $(\mu_{1i})_{i \in I}, (\mu_{2i})_{i \in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+)^I$ are tuples of characters such that ϕ_T is $(\mu_{1i})_{i \in I}$ and $(\mu_{2i})_{i \in I}$-regular. Then if $V \simeq \boxtimes_{i \in I} T_{\mu_i^\Gamma} \in \mathrm{Tilt}({}^L G^I)$ occurs as a direct summand of the tensor product $\boxtimes_{i \in I} T_{\mu_{1i}^\Gamma} \otimes T_{\mu_{2i}^\Gamma} \in \mathrm{Tilt}({}^L G^I)$ then it follows that ϕ_T is $(\mu_i)_{i \in I}$-regular.*

Proof. It suffices to show that, given $(\mu_{1i})_{i\in I}, (\mu_{2i})_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+)^I$ with $V_1 := \boxtimes_{i\in I} T_{\mu_{1i}^\Gamma}$ and $V_2 := \boxtimes_{i\in I} T_{\mu_{2i}^\Gamma}$ such that we know the filtration on $T_{V_1}(\mathrm{nEis}(S_{\phi_T}))$ and $T_{V_2}(\mathrm{nEis}(S_{\phi_T}))$ splits, the same is true for the filtration on $T_V(\mathrm{nEis}(S_{\phi_T}))$, where $V \in \mathrm{Tilt}\ (^L G^I)$ is a direct summand of $V_1 \boxtimes V_2$. To do this, we can use the isomorphism

$$T_{V_1} T_{V_2}(\mathrm{nEis}(S_{\phi_T}))|_\Delta \cong T_{V_1 \boxtimes V_2}(\mathrm{nEis}(S_{\phi_T}))$$

coming from the fusion product, where $\Delta : \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{BW_{\overline{\mathbb{Q}}_p}^{I \sqcup I}} \to \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{BW_{\overline{\mathbb{Q}}_p}^I}$ is the natural map given by diagonal restriction. By assumption, the diagonal restriction of the filtration on the LHS splits. Moreover, by the compatibilities of the filtration, we know that the filtration on the LHS refines the filtration on the RHS. In particular, we deduce that the filtration on $T_{V_1 \boxtimes V_2}(\mathrm{nEis}(S_{\phi_T}))$ has a splitting, and since V is a direct summand of $V_1 \boxtimes V_2$, we know that the filtration also splits on $T_V(\mathrm{nEis}(S_{\phi_T}))$.

In particular, by considerations of highest weight, the tilting module $\boxtimes_{i\in I} T_{(\mu_{1i}+\mu_{2i})^\Gamma}$ is always a direct summand of $\boxtimes_{i\in I} T_{\mu_{1i}^\Gamma} \boxtimes T_{\mu_{2i}^\Gamma}$. Using this, we deduce the following Corollary.

Corollary 10.13. *If ϕ_T is $(\mu_{1i})_{i\in I}$ and $(\mu_{2i})_{i\in I}$-regular then it is also $(\mu_{1i} + \mu_{2i})_{i\in I}$-regular.*

This motivates the following definition.

Definition 10.14. We say a parameter ϕ_T is normalized regular if it is integral with weakly normalized regular mod ℓ reduction and if there exists a set of of elements μ_k for $k = 1, \ldots, n$ such that:
  (1) The μ_k generate $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ in the sense that any $\mu = \sum_{k=1}^n n_k \mu_k$ as elements in $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ for some $n_k \in \mathbb{Z}$.
  (2) ϕ_T is strongly μ_k-regular, for all $k = 1, \ldots, n$.

This allows us to state the following corollary.

Corollary 10.15. *Suppose that ϕ_T is normalized regular then it is $(\mu_i)_{i\in I}$-regular for all finite index sets I and $\mu_i \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$. In particular, the sheaf $\mathrm{nEis}(S_{\phi_T})$ is a (weak) tilting eigensheaf with eigenvalue ϕ given by the composite of ϕ_T with the natural map $^L T \to {}^L G$.*

Proof. We recall that $\mathrm{nEis}(S_{\phi_T})$ being a (weak) tilting eigensheaf means that, for all $V \in \mathrm{Tilt}\ (^L G^I)$, we know that we have an isomorphism

$$T_V(\mathrm{nEis}(S_{\phi_T})) \cong \mathrm{nEis}(S_{\phi_T}) \otimes r_V \circ \phi$$

but we do not know that these isomorphisms satisfy the desired compatibilities. To show this, it suffices to show the filtration on $T_V(\mathrm{nEis}(S_{\phi_T}))$ just splits (not necesarily uniquely) for all highest weight tilting modules $V = \boxtimes_{i\in I} T_{\mu_i^\Gamma}$ corresponding to the Γ-orbits of varying $(\mu_i)_{i\in I} \in (\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+)^I$. Since ϕ_T is strongly μ_k-regular for the $k = 1, \ldots, n$ appearing in the definition of normalized regularity, it follows, as in the proof of Theorem 10.10, that the filtration on $T_V(\mathrm{nEis}(S_{\phi_T}))$ splits for any tuple of the generating cocharacters $(\mu_i)_{i\in I}$ such that $\mu_i = \mu_k$ for all $i \in I$ and varying $k = 1, \ldots, n$ appearing in Definition 10.14. However, since every tuple of cocharacters μ_i, can be written as a linear combination of μ_k for $k = 1, \ldots, n$, this follows from Corollary 10.13.

By choosing μ_k for $k = 1, \ldots, n$ to be, for example a set of minuscule/quasi-minuscule generators of $\mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$, this allows to see that μ-regularity for all μ is actually an open condition on the variety of unramified twists of a fixed ϕ_T. This will be sufficient for most applications of the Eisenstein series to descirbing the cohomology of local shtukas and Shimura varieties in the Grothendieck group. However, for more refined applications of our results it becomes important to spell out the precise assumptions needed to deduce μ-regularity for all μ. Ideally, it should be implied by

genericity of ϕ_T for all μ, as is suggested by Conjecture 1.29. For $G = \mathrm{GL}_n$, we show how to verify this from the results proven above.

Corollary 10.16. *For $G = \mathrm{GL}_n$, if ϕ_T is an integral generic L-parameter (In particular, its mod ℓ-reduction is weakly normalized regular by Lemma 3.13) then the sheaf $\mathrm{nEis}(S_{\phi_T})$ is a (weak) tilting eigensheaf.*

Proof. We give the proof for $\Lambda = \overline{\mathbb{F}}_\ell$, with the other cases being strictly easier. It suffices to show for all tuples $(\mu_i)_{i \in I} \in (\mathbb{X}_\bullet(T_{\overline{\mathbb{Q}}_p})^+)^I$ that ϕ_T is $(\mu_i)_{i \in I}$ regular. By Lemma 3.15, we know that if ϕ_T is generic then it is strongly μ-regular for the cocharacter $\mu = (1, 0, \ldots, 0)$ corresponding to the standard representation V_{std} of $\hat{G} \simeq \mathrm{GL}_n$. By Proposition 10.13, it suffices to show that if ϕ_T is generic then μ-regularity holds for the fundamental coweights $\mu = \omega_i = (1^i, 0^{n-i})$ for $i = 1, \ldots, n$. We recall that the ω_i are minuscule in this case and therefore it follows by Lemma B.1 that the representation $\Lambda^i(V_{\mathrm{std}}) \simeq V_{\omega_i}$ is irreducible and thereby equal to the tilting module T_{ω_i}. However, now the result follows from Proposition 10.12 and the fact that $\Lambda^i(V_{\mathrm{std}})$ is a direct summand of $V_{\mathrm{std}}^{\otimes n}$. $\square$

With our main results in hand, we can move into applications.

11. Applications

Now we will deduce some applications to the cohomology of local shtuka spaces. In §9.1, we will use our eigensheaf to derive an analogue of an averaging formula of Shin for the cohomology of local shtuka spaces. In §9.2 and §9.3, we will discuss a refined version of this averaging formula, and use it to derive a very explicit formula for the isotypic parts of local shtuka spaces with respect to parabolic inductions of characters coming from normalized regular parameters. By combining this with a shtuka analogue of Boyer's trick, we will show that this gives rise to a geometric construction of intertwining operators, and recovers a result analogous to a result of Xiao and Zhu [XZ17] on the irreducible components of affine Deligne-Lusztig varieties, but on the generic fiber.

Let's first recall the key definitions. We say a local shtuka datum is a triple (G, b, μ) for μ a geometric dominant cocharacter of G and $b \in B(G, \mu)$ an element of the μ-admissible locus of the Kottwtiz set of G (Definition 2.5). We let E be the reflex field of μ. The triple (G, b, μ) defines a diamond

$$\mathrm{Sht}(G, b, \mu)_\infty \to \mathrm{Spd}(\breve{E})$$

parameterizing modifications $\mathcal{F}_b \dashrightarrow \mathcal{F}_G^0$ with meromorphy bounded by μ on X[7]. It carries an action of $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$ and a (non-effective) descent datum from $\breve{E}$ down to E. This allows us to consider the tower of quotients

$$\mathrm{Sht}(G, b, \mu)_\infty / \underline{K} =: \mathrm{Sht}(G, b, \mu)_K$$

for varying open compact subgroups $K \subset G(\mathbb{Q}_p)$. We write S_μ for the Λ-valued sheaf attached to the highest weight tilting module T_μ of $W_E \ltimes \hat{G}$ as in the previous section. This is given by pulling back the sheaf on $\mathrm{Hck}_{G,E}$ defined by T_μ and Theorem 4.2 along the natural map $\mathrm{Sht}(G, b, \mu) \to \mathrm{Hck}_{G,E}$. In particular, the sheaf S_μ is equivariant with respect to the actions of $G(\mathbb{Q}_p)$ and $J_b(\mathbb{Q}_p)$ by construction. Letting $\mathrm{Sht}(G, b, \mu)_{K, \mathbb{C}_p}$ denote the base-change of these spaces to $\mathbb{C}_p$, we can now define the complex

$$R\Gamma_c(G, b, \mu) := \mathrm{colim}_{K \to \{1\}} R\Gamma_c(\mathrm{Sht}(G, b, \mu)_{K, \mathbb{C}_p}, S_\mu)$$

---

[7]Note that this is the space denoted $\mathrm{Sht}(G, b, \mu^{-1})$ in [SW20]. We find that our convention simplifies certain formulae.

of $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E$-modules. We now want to disentangle the $G(\mathbb{Q}_p)$ and $J_b(\mathbb{Q}_p)$ action. To do this, for π (resp. ρ) a smooth irreducible representation of $G(\mathbb{Q}_p)$ (resp. $J_b(\mathbb{Q}_p)$) on Λ-modules, we define the π (resp. ρ)-isotypic part. I.e the complexes

$$R\Gamma_c(G, b, \mu)[\pi] := R\Gamma_c(G, b, \mu) \otimes_{H(G)} \pi$$

and

$$R\Gamma_c(G, b, \mu)[\rho] := R\Gamma_c(G, b, \mu) \otimes_{H(J_b)} \rho$$

where $H(G) := C_c^\infty(G(\mathbb{Q}_p), \Lambda)$ (resp. $H(J_b)$) is the usual smooth Hecke algebra of G (resp. J_b). We recall that $\mathrm{Sht}(G, b, \mu)_\infty$ has dimension equal to $\langle 2\hat{\rho}, \mu \rangle =: d$. The complexes $R\Gamma_c(G, b, \mu)[\pi]$ (resp. $R\Gamma_c(G, b, \mu)[\rho]$) are concentrated in degrees $-d \leq i \leq d$ and are valued in admissible $J_b(\mathbb{Q}_p)$ (resp. $G(\mathbb{Q}_p)$) representations, which are moreover of finite length with $\overline{\mathbb{Q}}_\ell$-coefficients [FS21, Page 317]. Similarly, we define the complexes

$$R\Gamma_c^\flat(G, b, \mu)[\pi] := R\mathrm{Hom}(R\Gamma_c(G, b, \mu), \pi)$$

and

$$R\Gamma_c^\flat(G, b, \mu)[\rho] := R\mathrm{Hom}(R\Gamma_c(G, b, \mu), \rho)$$

We will make regular use of the relationship between these complexes and Hecke operators. We write

$$T_\mu : \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \to \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{BW_E}$$

for the Hecke operator defined by the representation T_μ. We then have the following lemma.

Lemma 11.1. *[FS21, Section IX.3] Given a local shtuka datum (G, b, μ) as above and π (resp. ρ) a smooth representation of $G(\mathbb{Q}_p)$ (resp. $J_b(\mathbb{Q}_p)$) on Λ-modules, we can consider the associated sheaves $\rho \in D(J_b(\mathbb{Q}_p), \Lambda) \simeq \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G^b, \Lambda)$ and $\pi \in D(G(\mathbb{Q}_p), \Lambda) \simeq \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G^1, \Lambda)$ on the HN-strata $j_b : \mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$ and $j_1 : \mathrm{Bun}_G^1 \hookrightarrow \mathrm{Bun}_G$, respectively. There then exists an isomorphism*

$$R\Gamma_c(G, b, \mu)[\rho] \simeq j_1^* T_\mu j_{b!}(\rho)$$

of complexes of $G(\mathbb{Q}_p) \times W_E$-modules and an isomorphism

$$R\Gamma_c(G, b, \mu)[\pi] \simeq j_b^* T_{\mu^{-1}} j_{1!}(\pi)$$

of complexes of $J_b(\mathbb{Q}_p) \times W_E$-modules, where $\mu^{-1} := -w_0(\mu)$ is a dominant inverse of μ. Similarly, we have an isomorphism

$$R\Gamma_c^\flat(G, b, \mu)[\rho] \simeq j_1^* T_\mu j_{b*}(\rho)$$

of complexes of $G(\mathbb{Q}_p) \times W_E$-modules, and an isomorphism

$$R\Gamma_c^\flat(G, b, \mu)[\pi] \simeq j_b^* T_{\mu^{-1}} j_{1*}(\pi)$$

of complexes of $J_b(\mathbb{Q}_p) \times W_E$-modules.

We note that, since we used the tilting module T_μ in the definition of $R\Gamma_c(G, b, \mu)$ and the Hecke operator T_μ, the complex $R\Gamma_c(G, b, \mu)$ will in general be different from the usual complex defined with respect to the highest weight module V_μ. However, they will agree when we impose the following condition on μ with respect to our coefficient system Λ.

Definition 11.2. For $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$, we will say μ is tilting if the representation $V_\mu \in \mathrm{Rep}_\Lambda(\hat{G})$ lies in the full subcategory $\mathrm{Tilt}_\Lambda(\hat{G})$ of tilting modules or equivalently if it is irreducible with coefficients in Λ.

Remark 11.3. If $\Lambda = \overline{\mathbb{Q}}_\ell$ this condition always holds. Moreover, by Lemma B.1, this always holds if μ is minuscule. In Appendix B, we give more insight into this notion.

We now consider a toral parameter $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\Lambda)$ with associated smooth character $\chi : T(\mathbb{Q}_p) \to \Lambda$. Unless otherwise stated, we will assume that ϕ_T is integral with weakly normalized regular mod ℓ-reduction. Given such a ϕ_T, we set ϕ to be the L-parameter of G induced via the natural embedding ${}^L T(\Lambda) \to {}^L G(\Lambda)$ and consider the sheaf $\mathrm{nEis}(S_{\phi_T}) \in \mathrm{D}_{\mathsf{lis}}(\mathrm{Bun}_G, \Lambda)$ given by Theorem 10.10. We begin our analysis by relating our eigensheaves to an averaging formula of Shin.

11.1. **The Averaging Formula.** For μ a geometric dominant cocharacter with reflex field E, we write $r_\mu : W_E \ltimes \hat{G} \to \mathrm{GL}(T_\mu)$ for the map defined by T_μ. Since the Hecke operator T_{μ^Γ} attached to $T_{\mu^\Gamma} \in \mathrm{Rep}\ ({}^L G)$ factors through the Hecke operator T_μ attached to T_μ (cf. [FS21, Pages 313-315]). If ϕ_T is μ-regular then by definition we have an isomorphism

$$T_\mu(\mathrm{nEis}(S_{\phi_T})) \simeq r_\mu \circ \phi|_{W_E} \otimes \mathrm{nEis}(S_{\phi_T})$$

of objects in $\mathrm{D}_{\mathsf{lis}}(\mathrm{Bun}_G, \Lambda)^{BW_E}$. Now let's apply the restriction functor $j_{\mathbf{1}}(\ )$ to both sides of this isomorphism. By the description of the stalks, we know that the RHS is equal to $r_\mu \circ \phi|_{W_E} \otimes \pi$, where $\pi := i_B^G(\chi)$ is the normalized parabolic induction of the character χ attached to ϕ_T by class field theory. We can also simplify the RHS. In particular, first off note that, since any G-bundle $\mathcal{F}_G$ on X that occurs as a modification $\mathcal{F}_G \dashrightarrow \mathcal{F}_G^0$ of type μ lies in the set $B(G, \mu)$ by [Rap18, Proposition A.9], we have an isomorphism

$$j_{\mathbf{1}}(T_\mu(\mathrm{nEis}(S_{\phi_T}))) \simeq j_{\mathbf{1}} T_\mu(\mathrm{nEis}(S_{\phi_T})|_{B(G,\mu)})$$

where here we view $B(G, \mu)$ as the open subset of Bun_G defined by the identification $B(G) \simeq |\mathrm{Bun}_G|$ describing the underlying topological space $|\mathrm{Bun}_G|$ of Bun_G [Vie, Theorem 1.1], where $B(G)$ has the natural topology given by the partial ordering. We can further refine this by applying excision with respect to the locally closed stratification by the Harder-Narasimhan strata $\mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$ for $b \in B(G, \mu)$, using [FS21, Proposition VII.7.3]. The excision spectral sequence then tells us that the LHS has a filtration whose graded pieces are isomorphic to:

$$j_{\mathbf{1}}(T_\mu(j_{b!} j_b^*(\mathrm{nEis}(S_{\phi_T}))))$$

In particular, in $K_0(G(\mathbb{Q}_p) \times W_E, \Lambda)$, the Grothendieck group of Λ-valued smooth admissible $G(\mathbb{Q}_p)$-representations with a continuous action of W_E, this tells us that we have an equality:

$$(11) \qquad \sum_{b \in B(G,\mu)} [j_{\mathbf{1}}(T_\mu(j_{b!} j_b^*(\mathrm{nEis}(S_{\phi_T}))))] = [r_\mu \circ \phi|_{W_E} \otimes \pi]$$

Now, using the description of the stalks of $\mathrm{nEis}(S_{\phi_T})$ and Lemma 11.1, we can spell out the LHS more clearly. In particular, we define the following.

Definition 11.4. For ϕ_T an arbitrary toral parameter with induced parameter ϕ and $b \in B(G)$, we define the complex of smooth admissible $J_b(\mathbb{Q}_p)$-representations $\mathrm{Red}_{b,\phi}$ as follows. If $b \notin B(G)_{\mathsf{un}}$, we set $\mathrm{Red}_{b,\phi}$ to be equal to 0, and, if $b \in B(G)_{\mathsf{un}}$, we set $\mathrm{Red}_{b,\phi}$ to be equal to

$$\bigoplus_{w \in W_b} i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{1/2}[\langle 2\hat{\rho}, \nu_b \rangle] \in \mathrm{D}(J_b(\mathbb{Q}_p), \Lambda)$$

where $\delta_{P_b}^{1/2}$ is the modulus character of J_b defined by the standard parabolic $P_b \subset G$ with Levi factor $M_b \simeq J_b$.

This allows us to deduce the following from equation (11).

Theorem 11.5. *For $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\Lambda)$ an integral parameter with weakly normalized regular mod ℓ reduction, if $\pi := i_B^G(\chi)$ is the normalized parabolic induction of the smooth character χ attached to ϕ_T then, for any geometric dominant cocharacter μ such that ϕ_T is μ-regular, we have an equality*

$$\sum_{b \in B(G,\mu)} [R\Gamma_c(G, b, \mu)[\mathrm{Red}_{b,\phi}]] = [r_\mu \circ \phi|_{W_E} \otimes \pi]$$

in $K_0(G(\mathbb{Q}_p) \quad W_E, \Lambda)$. Similarly, one also has an equality[8]

$$\sum_{b \in B(G,\mu)} [R\Gamma_c^\flat(G, b, \mu)[\mathrm{Red}_{b,\phi}]] = [r_\mu \quad \phi j_{W_E} \quad \pi]$$

in $K_0(G(\mathbb{Q}_p) \quad W_E, \Lambda)$.

Remark 11.6. We note that, in the above analysis, we didn't necessarily have to restrict to the HN-strata Bun_G^1 or even a single cocharacter. In particular, by considering Hecke operators defined by representations in Tilt $({}^L G^I)$ for a finite index set I, we could have deduced an analogous formula for shtuka spaces with I legs for an arbitrary finite index set I. We could have also restricted to any HN-stratum; however, if the HN-stratum is not defined by a basic element, the answer is not as clean as above. In particular, given a G-bundle F_b on X corresponding to a general element $b \in B(G)$, it is to the best of our knowledge completely unknown exactly which G-bundles F_G occur as modifications $F_G \dashrightarrow F_b$ of type μ. It would be interesting to understand this question better. We leave it to the reader to work out the precise statements of these more general implications.

We can use this claim to deduce the averaging formula for an arbitrary toral parameter ϕ_T when $\Lambda = \overline{\mathbb{Q}}_\ell$, by viewing both sides as trace forms on $K_0(T(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ and using a continuity argument. We note that, in this case, μ is always tilting so we have that $R\Gamma_c(G, b, \mu)$ and $R\Gamma_c^\flat(G, b, \mu)$ are just the usual complexes. We recall that $f : K_0(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell) ! \overline{\mathbb{Q}}_\ell$ is a trace form if it can be written as $\mathrm{tr}(\delta j \ )$ for $\delta \in C_c^1 (G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$. We now fix a $\delta \in C_c^1 (G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ and $\gamma \in W_E$, we define the following functions attached to this datum

$$f_L^{\delta,\gamma} : K_0(T(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell) ! \overline{\mathbb{Q}}_\ell$$

$$\chi \ \mapsto \ \mathrm{tr}(\delta \quad \gamma j i_B^G(\chi) \quad r_\mu \quad \iota(\chi) j_{W_E})$$

$$f_R^{\delta,\gamma} : K_0(T(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell) ! \overline{\mathbb{Q}}_\ell$$

$$\chi \ \mapsto \ \sum_{b \in B(G,\mu)_{\mathrm{un}}} \sum_{w \in W_b} \mathrm{tr}(\delta \quad \gamma j R\Gamma_c(G, b, \mu)[i_{B_b}^{J_b}(\chi^w) \quad \delta_{P_b}^{1/2}])$$

where $\iota(\chi) \ ' \ \phi_T$ is the isomorphism given by local class field theory. We have the following lemma.

Lemma 11.7. *The functions $f_L^{\delta,\gamma}$ and $f_R^{\delta,\gamma}$ define trace forms on $K_0(T(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$.*

Proof. This follows from the fact that normalized parabolic induction takes trace forms to trace forms as can be checked from the characterization of trace forms in the trace Paley-Wiener theorem [BDK86], and the fact that $\mathrm{tr}(\delta \quad \gamma j R\Gamma_c(G, b, \mu)[\ ])$ defines a trace form on $K_0(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ by [HKW22, Theorem 6.5.4].

This gives the following.

Theorem 11.8. *For $\phi_T : W_{\mathbb{Q}_p} ! {}^L T(\overline{\mathbb{Q}}_\ell)$ an arbitrary toral parameter with associated character χ, and $i_B^G(\chi) =: \pi$, we have equalities*

$$\sum_{b \in B(G,\mu)} [R\Gamma_c(G, b, \mu)[\mathrm{Red}_{b,\phi}]] = [r_\mu \quad \phi j_{W_E} \quad \pi]$$

and

$$\sum_{b \in B(G,\mu)} [R\Gamma_c^\flat(G, b, \mu)[\mathrm{Red}_{b,\phi}]] = [r_\mu \quad \phi j_{W_E} \quad \pi]$$

in $K_0(G(\mathbb{Q}_p) \quad W_E, \overline{\mathbb{Q}}_\ell)$.

---

[8]To deduce this other claim, one can simply act by Verdier duality on the filtration giving rise to (11) and then relax the contragradients, or apply Corollary 11.13.

Proof. It suffices to show that the trace forms $f_L^{\delta,\gamma}(\chi)$ and $f_R^{\delta,\gamma}(\chi)$ agree for varying δ and γ and all $\chi \in K_0(T(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$. We define the difference $\Delta_{\delta,\gamma} := f_L^{\delta,\gamma}(\chi) - f_R^{\delta,\gamma}(\chi)$. We say that a subset $S \subseteq K_0(T(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ is dense if $\Delta_{\delta,\gamma}(x) = 0$ for all $x \in S$ implies that $\Delta_{\delta,\gamma} = 0$. Using Theorem 11.5 and Lemma 10.15, we can reduce to showing that the subset S of all characters χ which are normalized regular and admit a $\overline{\mathbb{Z}}_\ell$-lattice is dense. This is relatively easy to show. In particular, if we view $\Delta_{\delta,\gamma}$ as a regular function on the variety of unramified twists of a fixed character χ then the set of characters admitting a $\overline{\mathbb{Z}}_\ell$ lattice is Zariski-dense. Moreover, the locus where χ is normalized regular is also clearly Zariski dense, since it is implied by insisting that χ precomposed with sums of coroots is not the norm or trivial character for the sums of coroots appearing in differences of distinct weights of V_{μ_k}, for a finite list of generating cocharacters μ_k with $k = 1, \ldots, n$ appearing in the definition of μ-regularity, and it is also clear for the condition of weak normalized regularity. Therefore, the claim follows, using the previous Lemma.

This theorem is compatible with existing results. We recall that Shin [Shi12] and Bertoloni-Meli [Ber21], have described similar averaging formulas. In particular, given a refined endoscopic datum $\mathfrak{e} = (H, \mathcal{H}, s, \eta)$ (Definition C.1), Shin and Bertoloni-Meli define maps

$$\mathrm{Red}_b^{\mathfrak{e}}(-) : K_0^{st}(H(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell) \to K_0(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$$

where $K_0^{st}(H(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ denotes the Grothendieck group of stable virtual $\overline{\mathbb{Q}}_\ell$-representations of $H(\mathbb{Q}_p)$. If we are given an L-parameter ϕ which factors as $L_{\mathbb{Q}_p} \xrightarrow{\phi^H} \mathcal{H} \xrightarrow{{}^L\eta} {}^L G$ then, using the local Langlands correspondence for G, we are able to attach a stable distribution $S\Theta_\phi \in K_0^{st}(H(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ which should satisfy the endoscopic character identities as in [Kal16, Conjecture D]. The averaging formula (Conjecture C.2) is a conjectural formula for

$$\sum_{b \in B(G,\mu)} R\Gamma_c^\flat(G, b, \mu)[\mathrm{Red}_b^{\mathfrak{e}}(S\Theta_\phi)]$$

in $K_0(G(\mathbb{Q}_p) \times W_E, \overline{\mathbb{Q}}_\ell)$. Our averaging formula is related to the case when $\mathfrak{e}_{\mathsf{triv}} = (G, 1, {}^L G, \mathrm{id})$ is the trivial endoscopic datum. In particular, if ϕ_T is a generic toral parameter then, by Lemma 3.18, ϕ should define an actual L-parameter with trivial monodromy, and we can consider the L-packet $\Pi_\phi(G)$ under the local Langlands correspondence for G appearing in Assumption 7.5. By Assumption 7.5 (3), the members of the L-packet will be given by the irreducible constituents of $i_B^G(\chi)$. Therefore, we have that $S\Theta_\phi = [\pi]$ in $K_0^{st}(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$, and in the appendix we verify that the following is true.

Proposition 11.9. *Let $\chi : W_{\mathbb{Q}_p} \to \overline{\mathbb{Q}}_\ell$ be a smooth generic character, so that, using Lemma 3.18, we have an equality*

$$S\Theta_\phi = [\pi]$$

in $K_0(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)^{\mathsf{st}}$ under the local Langlands correspondence appearing in Assumption 7.5. Then we always have

$$[\mathrm{Red}_{b,\phi}] = \mathrm{Red}^{\mathfrak{e}_{\mathsf{triv}}}([\pi])$$

in the Grothendieck group $K_0(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$, and Conjecture C.2 holds true for the L-parameter ϕ attached to χ.

We would now like to refine our averaging formula further. In particular, using Theorem 8.3, we can upgrade this equality in the Grothendieck group to a genuine isomorphism of complexes.

11.2. **The Refined Averaging Formula and Intertwining Operators.** Consider an element $b \in B(G)_{\mathsf{un}}$ with HN-dominant reduction $b_T \in B(T)$. For $w \in W_b$, we set $\rho_{b,w} := i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{1/2}$ to be the twisted normalized parabolic induction, and consider $j_b : \mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$, the inclusion of the locally closed HN-strata corresponding to b. Temporarily, we will work with a more general integral

toral parameter. As seen in §10, the sheaf $\mathrm{nEis}(S_{\phi_T}) \in \mathrm{D}_{\mathsf{lis}}(\mathrm{Bun}_G, \Lambda)$ with its desired properties might not be well-defined for $\Lambda \in \{\overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell\}$ if ϕ_T isn't integral with weakly normalized regular mod ℓ-reduction; however, we note that, since b_T is HN-dominant, when $\Lambda = \overline{\mathbb{F}}_\ell$, we always have that $\mathrm{nEis}^{b_T}(S_{\phi_T})$ is only supported on Bun_G^b, since $\mathrm{Bun}_B^{b_T}$ will only parametrize split reductions. Moreover, $\mathrm{nEis}^{b_T}(S_{\phi_T})$ will be isomorphic to $j_{b!}(\rho_{b,1})$ by Proposition 9.4. Therefore, by the same procedure carried out in §10, we always get a well-defined sheaf $\mathrm{nEis}^{b_T}(S_{\phi_T}) \in \mathrm{D}_{\mathsf{lis}}(\mathrm{Bun}_G, \Lambda)$ for any integral toral parameter ϕ_T such that we have an isomorphism

(12)
$$j_{b!}(\rho_{b,1})[\langle 2\hat{\rho}, \nu_b \rangle] \simeq \mathrm{nEis}^{b_T}(S_{\phi_T})$$

and, if ϕ_T is an integral parameter with weakly generic mod ℓ-reduction, then, as in §10, Theorem 8.3 extends in a natural way to $\mathrm{nEis}^{b_T}(S_{\phi_T})$. We can act by $\mathbb{D}_{\mathrm{Bun}_G}(\ )$ on both sides of (12). By the commutation of Eisenstein series with Verdier duality, the RHS of (12) becomes

$$\mathrm{nEis}^{b_T}(S_{\phi_{\overline{T}}}) \simeq j_{b!}(\rho_{b,1}^\vee \otimes \delta_{P_b})[\langle 2\hat{\rho}, \nu_b \rangle]$$

where $\rho_{b,1}^\vee = (i_{B_b}^{J_b}(\chi) \otimes \delta_{P_b}^{1/2})^\vee = i_{B_b}^{J_b}(\chi^{-1}) \otimes \delta_{P_b}^{-1/2}$ denotes the contragradient. On the other hand, the LHS of (12) becomes

$$j_{b*}(\mathbb{D}_{\mathrm{Bun}_G^b}(\rho_{b,1}))[\langle 2\hat{\rho}, \nu_b \rangle]$$

We now need to be a bit careful. In particular, we recall that $\mathrm{Bun}_G^b \simeq [\ast / \widetilde{J}_b]$, where $\widetilde{J}_b$ is the group diamond parameterizing automorphisms of $\mathcal{F}_b$, and we are implicitly using the identification $\mathrm{D}_{\mathsf{lis}}(\mathrm{Bun}_G^b, \Lambda) \simeq \mathrm{D}(J_b(\mathbb{Q}_p), \Lambda)$ given by pullback along the natural map $p : [\ast/\widetilde{J}_b] \to [\ast/\underline{J_b(\mathbb{Q}_p)}]$, as in [FS21, Proposition V.2.2, VII.7.1]. Therefore, we need to account for the shifts and twists given by $p^!$. As in the proof of Proposition 6.16, we can use that the natural section s of p is an iterated fibration of positive Banach-Colmez space to show that $p^!(\ ) \simeq p^*(\ \otimes \delta_{P_b})[-2\langle 2\hat{\rho}, \nu_b \rangle]$, and therefore the LHS of (12) becomes $j_{b*}(\rho_{b,1}^\vee \otimes \delta_{P_b})[\langle 2\hat{\rho}, \nu_b \rangle - 2\langle 2\hat{\rho}, \nu_b \rangle] = j_{b*}(\rho_{b,1}^\vee \otimes \delta_{P_b})[-\langle 2\hat{\rho}, \nu_b \rangle]$. In conclusion, we have an isomorphism:

$$j_{b*}(\rho_{b,1}^\vee \otimes \delta_{P_b})[-\langle 2\hat{\rho}, \nu_b \rangle] \simeq j_{b!}(\rho_{b,1}^\vee \otimes \delta_{P_b})[\langle 2\hat{\rho}, \nu_b \rangle]$$

Relaxing the contragradients and cancelling the shifts and twists, we deduce an isomorphism:

$$j_{b*}(\rho_{b,1}) \simeq j_{b!}(\rho_{b,1})$$

Now given $w \in W_b$, we can replace S_{ϕ_T} by $S_{\phi_T^w}$ in the above argument, where ϕ_T^w is the conjugate of ϕ_T by w. This tells us that, if ϕ_T^w is integral with weakly generic mod ℓ reduction, we have an isomorphism:

$$j_{b*}(\rho_{b,w}) \simeq j_{b!}(\rho_{b,w})$$

In conclusion, we deduce the following.

Proposition 11.10. *For $b \in B(G)_{\mathsf{un}}$, $w \in W_b$, and $\rho_{b,w} = i_{B_b}^{J_b}(\chi^w)$ as defined above, where χ is the character attached to an integral toral parameter ϕ_T such that its conjugate ϕ_T^w has weakly generic mod ℓ reduction, we have an isomorphism*

$$j_{b!}(\rho_{b,w}) \simeq j_{b*}(\rho_{b,w})$$

of objects in $\mathrm{D}_{\mathsf{lis}}(\mathrm{Bun}_G, \Lambda)$.

Remark 11.11. Note that if we want this to hold for all $b \in B(G)_{\mathsf{un}}$ and $w \in W_b$ this is equivalent to assuming that the mod ℓ-reduction of ϕ_T is generic, since W_G acts transitively on the Γ-orbits of coroots.

Remark 11.12. If b is basic then this precisely says that the sheaf defined by $\rho_{b,w}$ is inert in the sense of [Han20, Definition 2.19]. In particular, this Proposition, in conjunction with [Han20, Theorem 2.22] and Lemma 3.18, seems to suggest that inert sheaves should correspond precisely to the representations whose semi-simplified L-parameter comes from the semi-simplification of a

parameter with non-trivial monodromy. For example, if one takes the constant sheaf on Bun_G^1 and considers $j_{\mathbf{1}!}(\Lambda)$ then we have that $\mathbb{D}_{\mathsf{Bun}_G}(j_{\mathbf{1}!}(\Lambda)) \simeq j_{\mathbf{1}*}(\Lambda) \simeq \Lambda$ which is not isomorphic to $j_!(\Lambda)$. Similarly, we see that the L-parameter attached to the trivial representation comes from the semi-simplification of a parameter with non-trivial monodromy (the Steinberg parameter).

Now consider μ a geometric dominant cocharacter of G with reflex field E and an element $b \in B(G, \mu)$. Applying $j_{\mathbf{1}*}T_\mu(-)$ to both sides of the previous isomorphism, we conclude, using Lemma 11.1, an isomorphism:

$$R\Gamma_c(G, b, \mu)[\rho_{b,w}] \simeq R\Gamma_c^\flat(G, b, \mu)[\rho_{b,w}]$$

We record this now.

Corollary 11.13. *Let (G, b, μ) be a local shtuka datum. For $b \in B(G)_{\mathsf{un}}$, $w \in W_b$, and ϕ_T an integral toral parameter such that ϕ_T^w has weakly generic mod ℓ reduction, there is an isomorphism*

$$R\Gamma_c(G, b, \mu)[\rho_{b,w}] \simeq R\Gamma_c^\flat(G, b, \mu)[\rho_{b,w}]$$

of complexes of $G(\mathbb{Q}_p) \times W_E$-modules. In particular, by Remark 11.11, if the mod ℓ-reduction of ϕ_T is generic, then this is true for all $w \in W_b$.

We now claim that the cohomology of $R\Gamma_c(G, b, \mu)[\rho_{b,w}]$ should be concentrated in degree $\langle 2\hat{\rho}, \nu_b \rangle$, for $\rho_{b,w}$ as above. To do this, let's put ourselves back in the position of an integral ϕ_T with weakly normalized regular mod ℓ reduction. We saw that in the previous section that the excision spectral sequence applied to $\mathrm{nEis}(S_{\phi_T})_{/B(G,\mu)}$ gives rise to a filtration whose graded pieces are isomorphic to $j_{b!}j_b^*(\mathrm{nEis}(S_{\phi_T}))$, but Lemma 11.10 implies that these graded pieces are also isomorphic to $j_{b*}j_b^*(\mathrm{Eis}(S_{\phi_T}))$. In particular, this allows us to deduce that the edge maps in the excision spectral sequence split, and therefore the sequence degenerates, giving an isomorphism:

$$\bigoplus_{b \in B(G,\mu)} j_{b!}j_b^*(\mathrm{nEis}^{b_T}(S_{\phi_T})) \simeq \mathrm{nEis}(S_{\phi_T})_{/B(G,\mu)}$$

We now would like to apply the eigensheaf property. So fix a geometric dominant cocharacter, and assume that ϕ_T is μ-regular. If $\pi = i_B^G(\chi)$ is the normalized parabolic induction of χ as above then, using our description of the stalks, we deduce the following "refined averaging formula".

Theorem 11.14. *For $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\Lambda)$ an integral toral parameter with weakly normalized regular mod ℓ-reduction and μ a geometric dominant cocharacter such that ϕ_T is μ-regular, we have an isomorphism*

$$\bigoplus_{b \in B(G,\mu)_{\mathsf{un}}} \bigoplus_{w \in W_b} R\Gamma_c(G, b, \mu)[\rho_{b,w}] = \bigoplus_{b \in B(G,\mu)} R\Gamma_c(G, b, \mu)[\mathrm{Red}_{b,\phi}] \simeq (i_B^G(\chi) \otimes r_\mu \circ \phi_{|W_E})[\langle 2\hat{\rho}, \nu_b \rangle]$$

of complexes of $G(\mathbb{Q}_p) \times W_E$-modules.

Unless otherwise stated, we will from now on assume that ϕ_T is integral with weakly normalized regular mod ℓ reduction. Using the definition of the functors $\mathrm{Red}_{b,\phi}$, we can give a very explicit description of the complexes $R\Gamma_c(G, b, \mu)[\mathrm{Red}_{b,\phi}]$, for $b \in B(G, \mu)_{\mathsf{un}}$.

Corollary 11.15. *For μ a geometric dominant cocharacter with reflex field E such that ϕ_T is μ-regular, fixed $b \in B(G, \mu)_{\mathsf{un}}$, and varying $w \in W_b$, the complex $R\Gamma_c(G, b, \mu)[\rho_{b,w}]$ is isomorphic to $\phi_{b,w}^\mu \otimes \pi[\langle 2\hat{\rho}, \nu_b \rangle]$, for $\phi_{b,w}^\mu$ a representation of W_E. Moreover, we have an isomorphism*

$$\bigoplus_{b \in B(G,\mu)_{\mathsf{un}}} \bigoplus_{w \in W_b} \phi_{b,w}^\mu \simeq r_\mu \circ \phi_{|W_E}$$

of W_E-representations.

This leads to a natural question. How can we describe the W_E-representations $\phi_{b,w}^{\mu}$ in terms of the weights appearing in $r_{\mu} \circ \phi|_{W_E}$. We recall, by Corollary 2.9, that the orbit of b_T under the Weyl group W_G can be described as $w(b_T)$ for $w \in W_b$ varying; moreover, using Corollary 2.9 and Remark 10.6, we see that we have a correspondence between $B(G, \mu)_{\mathsf{un}}$ and the set of Weyl orbits of weights which can occur in the representation $r_{\mu}|_{\hat{G}^{\Gamma}}$. In particular, given $\bar{\nu} \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})$, we consider the subspace

$$\bigoplus_{\substack{\nu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \\ \nu_{\Gamma} = \bar{\nu}}} \tilde{\nu} \circ \phi_T|_{W_E} \otimes r_{\mu}(\nu)$$

of $(r_{\mu} \circ \phi)|_{W_E}$, where we note that if we forget the Galois action then this identifies with the $\bar{\nu}$ weight space of $r_{\mu}|_{\hat{G}^{\Gamma}}$ by Lemma 2.8. Now the refined averaging formula suggests the following.

Conjecture 11.16. *For all geometric dominant cocharacters μ such that ϕ_T is μ-regular, an unramified element $b \in B(G, \mu)_{\mathsf{un}}$, and a Weyl group element $w \in W_b$, we have an isomorphism*

$$\bigoplus_{\substack{\widetilde{w(b_T)} \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \\ \widetilde{w(b_T)}_{\Gamma} = w(b_T)}} \widetilde{w(b_T)} \circ \phi_T|_{W_E} \otimes r_{\mu}(\widetilde{w(b_T)}) \simeq \phi_{b,w}^{\mu}$$

of W_E-representations, where b_T is a HN-dominant reduction of b[9].

For the rest of this section, let us look at some cases where this can be shown explicitly, using a shtuka analogue of Boyer's trick. To illustrate the idea, we begin with a particularly nice example, where Theorem 11.14 and Conjecture 11.16 can be checked by hand.

Example 11.17. Let $G = \mathrm{GL}_2$ and $\mu = (1, 0)$. Write $\phi_T = \phi_1 \oplus \phi_2$, and consider the set $B(G, \mu)$. It consists of two elements: the μ-ordinary element and the basic element. Only the μ-ordinary element lies in $B(G, \mu)_{\mathsf{un}}$; therefore, only this element contributes to the expression in Theorem 11.14. Namely, if b_{μ} denotes the μ-ordinary element, we note that $\langle 2\hat{\rho}, \nu_{b_{\mu}} \rangle = \langle 2\hat{\rho}, \mu \rangle = 1$. We conclude that Theorem 11.14 is an isomorphism

$$R\Gamma_c(G, b_{\mu}, \mu)[\chi \otimes \delta_B^{1/2}] \oplus R\Gamma_c(G, b_{\mu}, \mu)[\chi^{w_0} \otimes \delta_B^{1/2}] \simeq i_B^{GL_2}(\chi) \otimes \phi[-1]$$

of $G(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$-representations. This can be seen through direct computation. In particular, we have an isomorphism $J_{b_{\mu}} \simeq T$, and, since μ is minuscule, we have that $S_{\mu} \simeq \Lambda[1](\frac{1}{2})$. The space $\mathrm{Sht}(G, b_{\mu}, \mu)_{\mathit{1}, \mathbb{C}_p}$ is the moduli space parameterizing modifications $O(-1) \oplus O \dashrightarrow O^2$ of type $(1, 0)$. Every such modification is determined by an injection $O(-1) \hookrightarrow O$ of line bundles. Formally, this implies that the space $\mathrm{Sht}(G, b, \mu)_{\mathit{1}, \mathbb{C}_p}$ as a space with $\mathrm{GL}_2(\mathbb{Q}_p)$-action is parabolically induced from the space parameterizing such injections as a space with $T(\mathbb{Q}_p)$ action. Here $T(\mathbb{Q}_p)$ acts on the space of injections $O(-1) \hookrightarrow O$ via the scaling action precomposed with projection to the first factor of $T(\mathbb{Q}_p)$. This is a manifestation of the fact that $\mathrm{Sht}(G, b_{\mu}, \mu)_{\mathit{1}, \mathbb{C}_p}$ is a J_b-torsor over the flag variety $\underline{(G/B)(\mathbb{Q}_p)} \simeq \mathbb{P}^1(\mathbb{Q}_p) \to \mathbb{P}^1_{\mathbb{C}_p} \simeq \mathrm{Gr}_{G, \leq\mu, \mathbb{C}_p}$, where the last isomorphism is the Bialynicki-Birula map. In particular, note that the compactly supported cohomology of $\underline{G/B(\mathbb{Q}_p)}$ is precisely the space of compactly supported functions on $(G/B)(\mathbb{Q}_p)$. All in all, this allows us to conclude isomorphisms

$$R\Gamma_c(G, b_{\mu}, \mu)[\chi \otimes \delta_B^{1/2}] = \mathrm{Ind}_B^G(\chi \otimes \delta_B^{1/2}) \otimes \phi_1[-1] = i_B^G(\chi) \otimes \phi_1[-1]$$

$$R\Gamma_c(G, b_{\mu}, \mu)[\chi^{w_0} \otimes \delta_B^{1/2}] = \mathrm{Ind}_B^G(\chi^{w_0} \otimes \delta_B^{1/2}) \otimes \phi_2[-1] \simeq i_B^G(\chi^{w_0}) \otimes \phi_2[-1]$$

where there is a cancellation of the $\frac{1}{2}$ Tate twist in S_{μ} and the Tate twist coming from $\delta_B^{1/2}$, as in §3, with the Tate twists coming from the compactly supported cohomology of J_b. However, if χ is

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[9]We note that there is a minus sign appearing because we took the derived tensor product of $R\Gamma_c(G, b, \mu)$ with $\rho_{b,w}[-\langle 2\hat{\rho}, \nu_b \rangle]$

attached to a generic parameter ϕ_T, this implies that $i_B^G(\chi)$ is irreducible as in Example 3.9, and it follows that we have an isomorphism $i_B^G(\chi) \simeq i_B^G(\chi^{w_0})$, which allows us to conclude the result.

Now let's generalize this example. In particular, recall that $B(G, \mu)$ has a distinguished μ-ordinary element, denoted b_μ, which is the maximal element with respect to the partial ordering on $B(G, \mu)$, and has the property that $\tilde{\mu} = \nu_{b_\mu}$, where $\tilde{\mu}$ is the weighted average over the Galois orbit of μ as in §2.1. If we write μ_T for μ viewed as a geometric cocharacter of T in the negative Weyl chamber defined by the choice of Borel, we can see that b_μ admits a HN-dominant reduction to the unique element $b_{\mu_T} \in B(T, \mu_T)$. In other words, the element b_μ always lies in $B(G, \mu)_{\mathsf{un}} := B(G, \mu) \setminus B(G)_{\mathsf{un}}$. Conjecture 11.16 suggests to us that this should give rise to the contribution given by the highest weight of $T_\mu j_{\hat{G}^\Gamma}$, which will have multiplicity one. We now prove the following result using a shtuka analogue of Boyer's trick [Boy99] proven by Gaisin-Imai [GI16].

Proposition 11.18. *For μ any geometric dominant cocharacter with reflex field E, $b_\mu \in B(G, \mu)_{\mathsf{un}}$ the μ-ordinary element with HN-dominant reduction b_{μ_T}, $w \in W_b$ varying, and ϕ_T any toral parameter, we have an isomorphism*

$$R\Gamma_c(G, b_\mu, \mu)[\rho_{b_\mu,w}] \simeq w(\mu_T) \otimes \phi_T j_{W_E} \otimes i_B^G(\chi^w)[\langle 2\hat{\rho}, \nu_{b_\mu}\rangle]$$

of $W_E \times G(\mathbb{Q}_p)$-representations.

Proof. We note that the element $b_\mu \in B(G, \mu)$ is Hodge-Newton reducible in the sense of [RV14, Definition 4.5]. In particular, b_μ is induced from the unique element $b_{\mu_T} \in B(T, \mu_T)$ via the natural map $B(T) \to B(G)$. Consider a rank k vector bundle of the form $\bigoplus_{i=1}^k O(n_i)$ for $n_i \in \mathbb{Z}$ and suppose we have a modification:

$$\bigoplus_{i=1}^k O(n_i) \dashrightarrow O^n$$

Then it is easy to see that such a modification will be determined by a tuple of modifications

$$O(n_i) \dashrightarrow O$$

for all $i = 1, \ldots, k$. If we apply the Tannakian formalism (See [GI16, Lemma 4.1]), this tells us that the space $\mathrm{Sht}(G, b_\mu, \mu)_\infty{}_{,\mathbb{C}_p}$ parameterizing modifications of the form

$$\mathcal{F}_{b_\mu} \dashrightarrow \mathcal{F}_G^0$$

will be determined by the spaces $\mathrm{Sht}(T, w(b_{\mu_T}), w(\mu_T))_\infty{}_{,\mathbb{C}_p}$ parametrizing modifications of the form

$$\mathcal{F}_{w(b_{\mu_T})} \dashrightarrow \mathcal{F}_T^0$$

with meromorphy equal to $w(\mu_T)$ for varying $w \in W_{b_\mu}$. More specifically, $\mathrm{Sht}(G, b, \mu)_\infty{}_{,\mathbb{C}_p}$ will be parabolically induced from $\mathrm{Sht}(T, w(b_T), w(\mu_T))_\infty{}_{,\mathbb{C}_p}$ for any $w \in W_{b_\mu}$ (See [GI16, Corollary 4.14, Proposition 4.21] for details) as spaces with $T(\mathbb{Q}_p)$ action[10]. In particular, this tells us that the moduli space $\mathrm{Sht}(G, b_\mu, \mu)_\infty{}_{,\mathbb{C}_p}$ parameterizing modifications of meromorphy $\leq \mu$ is actually equal to the open subspace $\mathrm{Sht}(G, b, \mu)_\infty^\mu$ parameterizing modifications of meromorphy equal to μ. This is because any modification induced from a modification

$$\mathcal{F}_{b_{\mu_T}} \dashrightarrow \mathcal{F}_T^0$$

of type μ_T will be of type μ, which implies that we have an isomorphism $S_\mu \simeq \Lambda[d](\frac{d}{2})$, where $d = \langle 2\hat{\rho}, \nu_{b_\mu}\rangle = \langle 2\hat{\rho}, \mu\rangle$ using [FS21, Proposition VI.7.5]. Here we need to be a bit careful since S_μ is the pullback of the sheaf associated to the tilting module T_μ not V_μ as per usual. However, we note

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[10]More specifically, Imai-Gaisin show that $\mathrm{Sht}(G, b, \mu)_\infty$ is parabolically induced from $\mathrm{Sht}(M_b, b_{M_b}, \mu)_\infty$ as a space with $M_b(\mathbb{Q}_p)$-action, where b_{M_b} is the canonical basic reduction such that $J_{b_{M_b}} = J_b$. However, it follows from the analysis of Imai-Gaisin that, since b_{M_b} admits a further reduction to $B(T)$, the diagonally embedded $J_b = M_b$ acts trivially, and that this space is parabolically induced from T

that the above discussion tells us that the Newton strata in the Schubert cell/variety $\mathrm{Gr}_{G,\ \mu,\mathbb{C}_p}^{b_\mu} = \mathrm{Gr}_{G,\mu,\mathbb{C}_p}^{b_\mu}$ has only non-empty intersection with the semi-infinite cells $S_{G,w(\mu_T),\mathbb{C}_p}$ indexed by the Weyl group orbits of the highest weight, using the Remark proceeding 4.7, and since both T_μ and V_μ have highest weight with multiplicity one the discrepancy doesn't matter via Corollary 4.9. It remains to describe the complex $R\Gamma_c(G, b_\mu, \mu)$. Using our above observations and in particular [GI16, Theorem 4.23][11], it follows that we have an isomorphism

$$R\Gamma_c(G, b_\mu, \mu)[\rho_{b,w}] \simeq \mathrm{Ind}_B^G(R\Gamma_c(T, b_T, \mu_T)[\chi^w \quad \delta_B^{1/2}])[\ \hbar 2\hat{\rho}, \nu_b \hbar](\ \hat{\rho}, \mu \hbar)$$

of complexes of $G(\mathbb{Q}_p) \quad W_E$-representations. Moreover, by our analysis in §3, we know that

$$R\Gamma_c(T, b_T, \mu_T)[\chi^w \quad \delta_B^{1/2}] \simeq \chi^w \quad \delta_B^{1/2} \quad w(\mu_T) \quad \phi_T j_{W_E}(\hbar\hat{\rho}, \mu \hbar)$$

and in turn we have a chain of isomorphisms

$$R\Gamma_c(G, b_\mu, \mu)[\rho_{b,w}] \simeq \mathrm{Ind}_B^G(R\Gamma_c(T, b_T, \mu_T)[\chi^w \quad \delta_B^{1/2}])[\ \hbar 2\hat{\rho}, \nu_b \hbar](\ \hat{\rho}, \mu \hbar) \simeq$$
$$\mathrm{Ind}_B^G(\chi^w \quad \delta_B^{1/2}) \quad w(\mu_T) \quad \phi_T j_{W_E}(\hbar\hat{\rho}, \mu \hbar \quad \hbar\hat{\rho}, \mu \hbar)[\ \hbar 2\hat{\rho}, \nu_b \hbar] \simeq$$
$$i_B^G(\chi^w) \quad w(\mu_T) \quad \phi_T j_{W_E}[\ \hbar 2\hat{\rho}, \nu_b \hbar]$$

of complexes of $G(\mathbb{Q}_p) \quad W_E$-representations which gives the desired result.

Remark 11.19. We note that in the proof we did not use any of our results on geometric Eisenstein series. It would be interesting to generalize some of these computations to some non-principal situations. In particular, if one works with a general parabolic P with Levi factor M and a supercuspidal parameter $\phi_M : W_{\mathbb{Q}_p} \ ! \ ^L M$ and assumes that Fargues' conjecture holds on the Levi subgroup M, then the results of [GI16] guarantee that one has similar formulas relating the cohomology of the shtuka spaces of G to basic local shtuka spaces of M for Hodge-Newton reducible $b \ 2 \ B(G, \mu)$ admitting a reduction to M. The description of the eigensheaf in Fargues' conjecture, as described for odd unramified unitary groups in [BHN22] and partially for GSp_4 in [Ham21b], gives one a very explicit description of these basic local shtuka spaces, which would in turn give a computational approach to understanding and generalizing these formulas beyond the principal case. We fully anticipate that some of the methods we use in the principal case generalize to the non-principal case; however, the results seem much more technical, and these computations would give a nice foothold into the problem.

This explicit calculation has some interesting consequences. In particular, we already saw in Example 11.17 that relating Proposition 11.18 to Theorem 11.14 required using the existence of an isomorphism $i_B^G(\chi) \simeq i_B^G(\chi^w)$; i.e. intertwining operators. This phenomenon actually persists. In particular, we deduce the following.

Corollary 11.20. *Let* $\chi : T(\mathbb{Q}_p) \ ! \ \Lambda$ *be a character obtained from an integral toral parameter* ϕ_T *whose mod ℓ-reduction is weakly normalized regular. Suppose that there exists a geometric dominant cocharacter* μ *which is not fixed by any element of* W_G *and such that* ϕ_T *is* μ-regular. *Then, for all* $w \ 2 \ W_G$, *there is an isomorphism*

$$i_{\chi,w} : i_B^G(\chi) \ \overset{'}{!} \ i_B^G(\chi^w)$$

of smooth $G(\mathbb{Q}_p)$-*representations.*

Proof. Since μ is not fixed by any element of W_G, this implies that $M_{b_\mu} = J_{b_\mu} = T$ and $W_{b_\mu} = W_G$. The previous Proposition then tells us that we have an isomorphism

$$R\Gamma_c(G, b, \mu)[\chi^w] \simeq w(\mu_T) \quad \phi_T j_{W_E} \quad i_B^G(\chi^w)$$

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[11]Strictly speaking, Imai-Gaisin work with $\overline{\mathbb{Q}}_\ell$ coefficients, but it is easy to see the proof extends to other coefficient systems.

for all $w \in W_G$. On the other hand, since ϕ_T is μ-regular, Corollary 11.15 tells us that the LHS must be isomorphic as a $G(\mathbb{Q}_p)$ representation to $i_B^G(\chi)$. The claim follows.

In particular, the refined averaging formula, together with the direct computation of $R\Gamma_c(G, b, \mu)[\chi^w]$ provided above, gives rise to an isomorphism: $i_{\chi,w} : i_B^G(\chi) \cong i_B^G(\chi^w)$. If $\Lambda = \overline{\mathbb{Q}}_\ell$ this recovers the following special case of A.2.

Corollary 11.21. *Suppose that $\chi : T(\mathbb{Q}_p) \to \overline{\mathbb{Q}}_\ell$ is a normalized regular character admitting a $\overline{\mathbb{Z}}_\ell$-lattice then we have isomorphisms $i_B^G(\chi) \cong i_B^G(\chi^w)$ for all $w \in W_G$.*

This suggests an interesting relationship between the theory of geometric Eisenstein series over the Fargues-Fontaine curve and the classical theory of intertwining operators and the Langlands quotient, which we hope is explored more in the future. This prospect becomes even more exciting in the ℓ-modular situation. The theory of intertwining operators in this context has been partially explored by Dat [Dat05, Sections 6-8], and the Langlands quotient theorem does not naively hold in this context, as the following example illustrates.

Example 11.22. [12] Suppose that $\ell \neq 2$, $\ell \mid p - 1$, and $G = \mathrm{GL}_2$. We let $\chi = \delta_B^{1/2}$ be the modulus character. We note that, by our assumption that $\ell \mid p - 1$, we have an isomorphism $j \ j \cong 1_T$. It follows that we have that $\delta_B^{1/2} \cong \delta_B^{1/2}$. We consider the usual short exact sequence

$$0 \to 1_G \to i_B^G(\delta_B^{1/2}) \to \mathrm{St}_G \to 0$$

where St_G denotes the Steinberg representation. Acting by smooth duality actually gives a splitting of the short exact sequence and in turn a chain of isomorphisms

$$i_B^G(\delta_B^{1/2}) \cong \mathrm{St}_G \oplus 1_G \cong i_B^G(\delta_B^{1/2})$$

of smooth $G(\mathbb{Q}_p)$-representations. In particular, we see that $i_B^G(\delta_B^{1/2})$ does not have a unique irreducible quotient in this case, so that the Langlands quotient theorem cannot naively hold.

Let's now explore a case in which Proposition 11.18 can be used to verify Conjecture 11.16. Suppose that μ is a geometric dominant cocharacter such that the image $\mu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$ is quasi-minuscule or minuscule with respect to the pairing with $\mathbb{X}^*(\hat{T})$. In this case, we recall that the orbit of the highest weight space $T_\mu j_{\hat{G}^\Gamma}(b_\mu)$ forms a closed orbit under the relative Weyl group W_G, where $b_\mu \in B(G, \mu)_{\mathsf{un}}$ is the μ-ordinary element. It then follows that all the weight spaces of $T_\mu j_{\hat{G}^\Gamma}$ in this orbit will be given by the κ-invariants of $w(b_{\mu_T}) \in B(T) \cong \mathbb{X}_*(\hat{T})$, for $w \in W_G$ varying. However, by Proposition 11.18, we see that all the weight spaces of this form come from the contribution of the μ-ordinary element to the refined averaging formula. If μ is minuscule with respect to the above pairing this is the only weight space in $T_\mu j_{\hat{G}^\Gamma}$, and we see that Conjecture 11.16 is true. If μ is quasi-minuscule, the only other element in $B(G, \mu)_{\mathsf{un}}$ is the basic element, denoted $\mu^\flat$. By Corollary 2.9, the highest weight representation $T_\mu j_{\hat{G}^\Gamma}$ admits a central weight space $T_\mu j_{\hat{G}^\Gamma}(\mu_T^\flat)$ in this case, where $\mu_T^\flat \in B(T) \cong \mathbb{X}_*(\hat{T})$ is the (unique) reduction to T of $\mu^\flat \in B(G, \mu)_{\mathsf{un}}$. We deduce the following from the refined averaging formula.

Corollary 11.23. *Let μ be a geometric dominant cocharacter such that ϕ_T is strongly μ-regular with reflex field E. Assume that μ is quasi-minuscule with respect to the pairing with $\mathbb{X}^*(\hat{T})$. Let $\mu^\flat \in B(G, \mu)$ be the unique basic element. It follows by Corollary 2.9 that $\mu^\flat$ is unramified in this case. We let $\mu_T^\flat$ be its unique reduction to $B(T)$. There is an isomorphism*

$$R\Gamma_c(G, b, \mu^\flat)[i_{B_b}^{J_b}(\chi)] \cong \bigoplus_{\substack{\mu_T^\flat \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p}) \\ \mu_{T\Gamma}^\flat = \mu_T^\flat}} \tilde{\mu}_T^\flat \ \phi_T j_{W_E} \ T_\mu(\tilde{\mu}_T^\flat) \ i_B^G(\chi)$$

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[12]We thank Robert Kurinczuk for bringing this example to our attention.

of complexes of W_E $G(\mathbb{Q}_p)$-modules.

Proof. This follows from combining Theorem 11.14 and Proposition 11.18. Noting that, by the strong μ-regularity assumption on ϕ_T, the Weil group action of the contribution of the highest weight to the refined averaging formula must be distinct from the Weil group action on the contribution of the central weight space, by the vanishing of the H^0s for the differences of these weight spaces.

Remark 11.24. If G is split then the condition that the central weight space of $T_\mu j_{\hat{G}^\Gamma}$ is non-zero cannot occur if μ is minuscule. However, if G is not split then it can occur that the central weight space is non-trivial even if μ is minuscule. For example, if one considers an odd quasi-split unitary group U_n and the cocharacter $(1, 0, \ldots, 0, 0)$ then the σ-centralizer will be isomorphic to U_n, and therefore the basic element $b \in B(G, \mu)$ lies in $B(G)_{\mathsf{un}}$. A more in depth characterization of when this can occur is given in [XZ17, Remark 4.2.11]. This result is in some sense a generic fiber manifestation of some of the results in [XZ17]. Here, in the case that G is unramified, Xiao and Zhu relate the irreducible components of affine Deligne-Luztig varieties to the central weight spaces appearing above, and use these irreducible components to construct cohomological correspondences on the special fibers of certain Shimura varieties using uniformization. These affine Deligne-Luztig varieties are precisely the special fibers of a natural integral model of $\mathrm{Sht}(G, \mu^\flat, \mu)_{\mathfrak{1}} / \underline{K}$ for a choice of hyperspecial subgroup $K \quad G(\mathbb{Q}_p)$.

APPENDIX A. INTERTWINING OPERATORS AND THE IRREDUCIBILITY OF PRINCIPAL SERIES

We want to show the irreducibility of principal series representations obtained from the characters χ attached to parameters $\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\overline{\mathbb{Q}}_\ell)$ satisfying the Conditions in 3.8. We let $\chi : T(\mathbb{Q}_p) \to \mathbb{C}$ be the character attached to ϕ_T via local class field theory and a fixed isomorphism $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$ sending $p^{1/2}$ to the fixed choice of square root in $\overline{\mathbb{Q}}_\ell$. Our goal will be to show the following two facts.

Proposition A.1. *Suppose that χ is a regular character in the sense that it is not fixed under any $w \in W_G$ then if we have an isomorphism*

$$i_B^G(\chi) \simeq i_B^G(\chi^{w_0})$$

the representation $i_B^G(\chi)$ is irreducible.

Proposition A.2. *If $\chi : T(\mathbb{Q}_p) \to \mathbb{C}$ is a generic character then, for all $w \in W_G$, we have an isomorphism*

$$i_B^G(\chi) \simeq i_B^G(\chi^w)$$

of smooth $G(\mathbb{Q}_p)$-representations.

By combining these two Propositions, we deduce the following.

Corollary A.3. *For χ a generic regular character, the induction $i_B^G(\chi)$ is always irreducible.*

The idea behind proving such results is due to Speh and Vogan [SV80, Theorem 3.14] in the archimedean case. They study the reducibility of principal series using the Langlands classification [BW80, Section IV]. Strictly speaking, their analysis is for the archimedean place, but it is easy to see that it extends to the case of a p-adic group using the analogous Langlands classification there [BW80, Section XI.2]. They (roughly) break the problem of understanding the reducibility points of principal series representations into two parts:

(1) ([SV80, Theorem 3.14(a)]) Understanding the reducibility points of non-unitary principal series with respect to the parabolic inductions from T to M_i for $i \in J$, where M_i is the rank 1 Levi subgroup of G whose relative Dynkin diagram is given by $fig \quad J$.

(2) ([SV80, Theorem 3.14(b)]) Understanding the reducibility points of unitary principal series representations with respect to induction from T to a (not necessarily rank 1) proper Levi subgroup of G.

We will now explain this heuristic in our case. To see analogous analysis worked out more explicitly for specific p-adic reductive groups, we point the reader to [Tad94, Section 7], [Mui97, Section 3], [Mat10, Section 3] for a small sample. For (2), a very definitive answer to such questions can be found in the paper of Keys [Key82]. In particular, by [Key82, Corollary 1] the number of irreducible components of such a unitary parabolic induction can be computed in terms of the Knapp-Stein R-Group [KS72; Kna73a; Kna73b], which Keys determines for all split groups. While this is very interesting, we will not address this here. In particular, we have the following.

Corollary A.4. *If χ is a regular unitary character then the normalized parabolic induction $i_B^G(\chi)$ is irreducible.*

Proof. This follows from the Bruhat decomposition and the fact that $i_B^G(\chi)$ is unitary and therefore fully decomposable (See [Cas95, Theorem 6.6.1] or [Bru61]).

Now consider $\mathbb{X}(T_{\overline{\mathbb{Q}}_p}) \otimes \mathbb{R} \simeq \mathbb{X}(A) \otimes \mathbb{R} \simeq \mathbb{R}^d$ the set of unramified characters. For $s \in \mathbb{R}^d$, we write ν^s for the associated unramified character. We will say that ν^s is positive (resp. strictly positive) if, for all simple (reduced) positive coroots $\alpha_{i,A}$, the precomposition of ν^s with $\alpha_{i,A}$ is positive (resp. strictly positive), or in other words that s lies in the positive Weyl chamber of $\mathbb{X}(A) \otimes \mathbb{R}$ defined by the Borel. We write $\chi = \mu_\chi \nu^{s_\chi}$, where μ_χ is a unitary character and ν^{s_χ} is an unramified character of T, for some $s_\chi \in \mathbb{R}^d$. We now consider intertwining operators. Recall that, for $w \in W_G$ and $s \in \mathbb{R}^d$, we have the intertwining operator

$$I_w(\mu_\chi, s) : i_B^G(\mu_\chi \nu^s) \to i_B^G((\mu_\chi \nu^s)^w)$$

$$f(g) \mapsto \int_{U_w} f(w^{-1}ug)du$$

where $U_w := U^- \cap wU^- w^{-1}$ and U^- is the unipotent radical of the opposite Borel. This integral will converge if ν^s lies sufficiently deep in the dominant Weyl chamber, and admits a meromorphic continuation as a function of s (where one allows s to be complex and imposes this constraint on the real part). Away from these poles, it gives rise to an intertwining operator between $i_B^G(\mu_\chi \nu^s)$ and $i_B^G((\mu_\chi \nu^s)^w)$ in the usual representation theory sense. For our purposes, we will be interested in the intertwining operator $I_{w_0}(\mu_\chi, s)$ for the element of longest length w_0. In this case, one can see that the operator is convergent for all s which are strictly positive, and the image of the operator is the unique irreducible Langlands quotient of $i_B^G(\mu_\chi \nu^s)$ (See [BW80, Sections XI.2.6, XI.2.7]). This quotient has multiplcity one and therefore the intertwining space between $i_B^G(\chi)$ and $i_B^G(\chi^{w_0})$ is one-dimensional. It follows that, if ν^{s_χ} is strictly positive, it suffices to exhibit an isomorphism $i_B^G(\chi) \simeq i_B^G(\chi^{w_0})$ to show irreducibility. We will now use this kind of analysis to prove Proposition A.1.

Proof. (Proposition A.1) First off note that, for all χ, we have an equality

$$[i_B^G(\chi)] = [i_B^G(\chi^{w_0})]$$

in $K_0(G(\mathbb{Q}_p))$ (See for example [Dij72, Theorem 4]). This allows us to, without loss of generality, assume that ν^{s_χ} is positive. Now, consider the set of $i \in J$ such that the precomposition of ν^{s_χ} with $\alpha_{i,A}$ is equal to 0. This defines a parabolic P_χ of G which we decompose as $P_\chi = M_\chi A_\chi N_\chi$, where A_χ is the maximal split torus in the center of M_χ. If J_{M_χ} denotes the vertices of the relative Dynkin diagram of M_χ then $J_{M_\chi} \subset J$ corresponds to the set of simple positive coroots where this precomposition vanishes. Now, set $\nu_1 := \nu^{s_\chi}|_{A \cap M_\chi}$ and $\nu_2 := \nu^{s_\chi}|_{A_\chi}$. We consider the parabolic induction

$$i_{B \cap M_\chi}^{M_\chi}(\mu_\chi \otimes \nu_1)$$

where we now note that $\mu_\chi \otimes \nu_1$ is unitary by construction. Therefore, $i_{B \cap M_\chi}^{M_\chi}(\mu_\chi \otimes \nu_1)$ is unitary and thereby fully decomposable. It follows that, since $i_{B \cap M_\chi}^{M_\chi}(\mu_\chi \otimes \nu_1)$ and $i_{B \cap M_\chi}^{M_\chi}((\mu_\chi \otimes \nu_1)^{w_0^{M_\chi}})$ are

equal in the Grothendieck group, we have an isomorphism $i_{B\backslash M_\chi}^{M_\chi}(\mu_\chi \quad \nu_1) \simeq i_{B\backslash M_\chi}^{M_\chi}((\mu_\chi \quad \nu_1)^{w_0^{M_\chi}})$, where $w_0^{M_\chi}$ is the element of longest length of the Weyl group of M_χ. Now, we have an isomorphism:

$$i_B^G(\chi) \simeq i_{P_\chi}^G((i_{B\backslash M_\chi}^{M_\chi}(\mu_\chi \quad \nu_1)) \quad \nu_2)$$

Since χ is regular then, it follows by Lemma A.4, that the unitary induction $i_{B\backslash M_\chi}^{M_\chi}(\mu_\chi \quad \nu_1)$ is irreducible. Therefore, the RHS is the induction of an irreducible tempered representation times an unramified character ν_2 satisfying the property that $\hbar\alpha_{i,A}, \nu_2 i > 0$ for all $i \; 2 \; \mathcal{J} \; \cap \mathcal{J}_{M_\chi}$ by construction. Again applying the Langlands classification [BW80, Sections XI.2.6, XI.2.7], the intertwining operator attached to the parabolic P_χ and the element $w_0 w_0^{M_\chi}$ converges for $s = s_\chi$, and since it maps to a unique quotient of multiplicity one it suffices to exhibit an isomorphism between $i_{P_\chi}^G((i_{B\backslash M_\chi}^{M_\chi}(\mu_\chi \quad \nu_1)) \quad \nu_2)$ and the induction twisted by $w_0 w_0^{M_\chi}$. However, since we just saw that $i_{B\backslash M_\chi}^{M_\chi}((\mu_\chi \quad \nu_1)^{w_0^{M_\chi}}) \simeq i_{B\backslash M_\chi}^{M_\chi}(\mu_\chi \quad \nu_1)$, it suffices to show we have an isomorphism $i_B^G(\chi) \simeq i_B^G(\chi^{w_0})$. This establishes the claim.

Now we just need to show Proposition A.2.

Proof. We claim that this reduces to the analogous question for G a group of rank 1. In particular, let's consider for $i \; 2 \; \mathcal{J}$ the simple positive (reduced) coroot $\alpha := \alpha_{i,A}$ and the rank 1 parabolic $P_\alpha = M_\alpha N_\alpha A_\alpha$ attached to it. As before, we write $\nu_1^\alpha := \nu^{s_\chi} j_{A\backslash M_\alpha}$ and $\nu_2^\alpha := \nu^{s_\chi} j_{A_\alpha}$. For all simple positive coroots α, we have an isomorphism

$$i_B^G(\chi) \simeq i_{P_\alpha}^G((i_{B\backslash M_\alpha}^{M_\alpha}(\mu_\chi \quad \nu_1^\alpha)) \quad \nu_2^\alpha)$$

However, if w_α is the simple reflection corresponding to α, we have that

$$i_B^G(\chi^{w_\alpha}) \simeq i_{P_\alpha}^G((i_{B\backslash M_\alpha}^{M_\alpha}((\mu_\chi \quad \nu_1^\alpha)^{w_\alpha}) \quad \nu_2^\alpha)$$

Therefore, if we can show the existence of an isomorphism:

$$i_{B\backslash M_\alpha}^{M_\alpha}(\mu_\chi \quad \nu_1^\alpha) \simeq i_{B\backslash M_\alpha}^{M_\alpha}((\mu_\chi \quad \nu_1^\alpha)^{w_\alpha})$$

It will imply that we have an isomorphism:

$$i_B^G(\chi) \simeq i_B^G(\chi^{w_\alpha})$$

Now we can proceed by induction on the length of Weyl group elements. We just described the base case, but then, by replacing χ with χ^{w_α} we can proceed by considering another simple reflection attached to another simple positive coroot distinct from α. We note that, since χ being generic is a condition for all coroots (not just simple), at each step of the induction we are tasked with showing the following.

Proposition A.5. *Let G be a absolutely simple, simply connected, quasi-split connected reductive group of split rank 1. Let α be the unique simple (reduced) positive coroot of G and w_α the corresponding simple reflection. Then, for all χ a generic character of $T(\mathbb{Q}_p)$, we have an isomorphism*

$$i_B^G(\chi) \simeq i_B^G(\chi^{w_\alpha})$$

of smooth $G(\mathbb{Q}_p)$-representations.

It remains to justify the absolutely simple simply connected assumption. To do this, note that given a G not satisfying these conditions, we can find a central isogeny $f : \tilde{G} \! \rightarrow \! G$, where $\tilde{G}$ is a product of torii and absolutely simple simply connected groups. If we let $\tilde{B}$ be the preimage of the Borel B with maximal torus given by $\tilde{T}$ the preimage of T then, since $\mathrm{Ker}(f)$ is contained in the center, we have an isomorphism $\tilde{G}/\tilde{B} \simeq G/B$. This implies that we have an isomorphism:

$$i_B^G(\chi) j_{\tilde{G}(\mathbb{Q}_p)} \simeq i_B^G(\chi j_{\tilde{T}})$$

Moreover, since f will induce an isomorphism on the root spaces, it follows that if χ is generic with respect to G then $\chi j_{\tilde{T}}$ is generic with respect to $\tilde{G}$. This reduces us to exhibiting the desired isomorphism for $\tilde{G}$. Now, we prove the Proposition A.5 through brute force. In particular, we will use Tits' classification theorem [Tit79] (See also [Car01] for the classification in rank 1). We adopt the same notation as in [Tit79]. Since we are assuming the group to be quasi-split, there are two cases.

A.0.1. $^1A_{1,1}^1$. In this case, we have that $G = \mathrm{SL}_2$. We saw in Example 3.12 that genericity guaranteed irreducibility aside from the case where $\chi^2 \simeq \mathbf{1}$, but since this is a unitary character it still follows that we have the desired isomorphism.

A.0.2. $^2A_{2,1}^1$. In this case, the group cannot be split; in particular, we have that $G = \mathrm{SU}_3$ is a quasi-split special unitary group attached to a quadratic extension $E/\mathbb{Q}_p$. We saw in Example 3.17 that χ being generic guaranteed irreducibility and hence the desired isomorphism.

APPENDIX B. TILTING COCHARACTERS

We consider a general quasi-split connected reductive group $G/\mathbb{Q}_p$, and a geometric dominant cocharacter $\mu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}_p}})^+$. For $\Lambda \in \{\overline{\mathbb{Q}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{F}}_\ell\}$, we are interested in understanding the condition of μ being tilting (Definition 11.2). Recall that this means that the representation $V_\mu \in \mathrm{Rep}(\hat{G})$ attached to μ lies in the subcategory $\mathrm{Tilt}(\hat{G})$. If $\Lambda = \overline{\mathbb{Q}}_\ell$ this is always true, and so we fix $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell\}$ in what follows. Since this only involves the representation theory $\hat{G}$ we may, without loss of generality, assume G is split in what follows. This is simply the question of when the highest weight module V_μ of $\hat{G}$ is irreducible with Λ-coefficients. This question has been studied extensively (See for example [Jan03, Pages 283-286] for a comprehensive overview). In the first two sections, we discuss some general theory to determine when μ is tilting in this split case, and then provide a table summarizing when μ is tilting in the case that μ is a fundamental coweight.

B.1. **General Theory.** We assume G is a split connected reductive group throughout this section, with Langlands dual group $\hat{G}$. For μ a dominant cocharacter, the condition that μ is tilting is equivalent to showing that the highest weight G-module $V_\mu \in \mathrm{Rep}(\hat{G})$ is simple. We begin with the following lemma.

Lemma B.1. *If μ is minuscule then it is tilting.*

Proof. In this case, the weights of V_μ form a closed Weyl group orbit. It follows that V_μ is always irreducible and therefore tilting.

We would now like to provide a finer criterion for irreducibility. To do this, we will introduce some notation. Given a coroot ν and $r \in \mathbb{Z}$, we consider the affine reflection of $\mathbb{X}_*(T) \otimes \mathbb{R}$ given as

$$s_{\nu,r}(\mu) := s_\nu(\mu) + r\nu$$

where $s_\nu \in W_G$ is the reflection attached to ν. We set W_ℓ to be the subgroup generated by reflections $s_{\nu,n\ell}$, where ν is a coroot and $n \in \mathbb{Z}$, and write ρ for the sum of all coroots of G. Elements $w \in W_\ell$ act on $\mu \in \mathbb{X}_*(T) \otimes \mathbb{R}$, via the standard dot action $w \cdot \mu := w(\mu + \rho) - \rho$. In other words, we regard $s_{\nu,n\ell}$ as a reflection around the hyperplane:

$$\{\mu \in \mathbb{X}_*(T) \otimes \mathbb{R} \mid \langle \mu + \rho, \nu \rangle = n\ell\}$$

It follows that the standard alcove for this action is given by

$$C = \{\mu \in \mathbb{X}_*(T) \otimes \mathbb{R} \mid 0 < \langle \mu + \rho, \nu \rangle < \ell\}$$

and we denote the closure by $\overline{C}$. We now have the following slightly more general criterion for the irreducibility of V_μ.

Proposition B.2. *[Jan03, Corollary 5.6] Suppose that $\mu \in \overline{C} \setminus \mathbb{X}_*(T)^+$ then μ is tilting.*

For a given μ, this will give us a lower bound on the ℓ for which μ is tilting. However, it is only a sufficient condition and not necessary. In particular, note that we have the following.

Theorem B.3. *[Mat00, Theorem 2.6] If $\mu = (\ell - 1)(\rho)$ then μ is tilting.*

So $V_{(\ell-1)(\rho)}$ will always be simple, but $(\ell - 1)(\rho)$ will not usually lie in $\overline{C} \setminus \mathbb{X}_*(T)^+$. Moreover, if we define the Coxeter number $h = \max_\nu \{\langle \rho, \nu \rangle + 1\}$ ranging over all coroots ν then it is easy to see that $C \setminus \mathbb{X}_*(T) \neq \emptyset$ is equivalent to $\ell \geq h$, and so, for small ℓ, Proposition B.2 tells us nothing. To tackle these more general cases, we introduce a sum formula for the characters of the representations. Namely, for $V \in \text{Rep}(\hat{G})$, we write $\text{ch}(V) := \sum_{\nu \in \mathbb{X}_*(T)} \dim(V(\nu))e^\nu$ for the character of V. For μ a dominant cocharacter, we write $\chi(\mu) := \text{ch}(V_\mu)$ for the character of V_μ. Then we have the following.

Proposition B.4. *[Jan03, Section 8.19] For each $\mu \in \mathbb{X}_*(T)^+$, there is a filtration of $\hat{G}$-modules*

$$V_\mu^2 \subseteq V_\mu^1 \subseteq V_\mu^0 = V_\mu$$

such that

$$\sum_{i>0} \text{ch}(V_\mu^i) = \sum_\nu \sum_{0 < m\ell < \langle \mu + \rho, \nu \rangle - i} \nu_\ell(m\ell)\chi(s_{\nu,m\ell} \cdot \mu)$$

where $\nu_\ell(\ )$ is the ℓ-adic valuation and ν ranges over all coroots. Moreover, we have that V_μ/V_μ^1 is isomorphic to the irreducible socle of V_μ. In particular, we see that μ is tilting if and only if

$$\sum_\nu \sum_{0 < m\ell < \langle \mu + \rho, \nu \rangle - i} \nu_\ell(m\ell)\chi(s_{\nu,m\ell} \cdot \mu) = 0$$

This generalizes Proposition B.2 and gives a computational method for verifying when μ is tilting. See for example [Jan03, Section 8.20] for this worked out for $G = \text{SL}_4$ and $\ell > 3$, [GS88] for a table answering this question for G an exceptional group and certain μ, and [BW71] for an analogous table for certain exceptional groups and low rank classical groups. In general, a precise classification of when μ is tilting for all G seems to be quite complicated, and to the best of our knowledge is unknown. However, when G is of type A_{n-1}, there exists a complete classification.

Proposition B.5. *[Jan91, Page 113] For G of type A_{n-1}, μ is tilting if and only if for each coroot ν of G the following is satisfied. Write $\langle \mu + \rho, \nu \rangle - i = a\ell^s + b\ell^{s+1}$, with $a, b, s \in \mathbb{N}$ and $0 < a < \ell$. Then there have to be positive coroots $\beta_0, \beta_1, \ldots, \beta_b$ such that $\langle \mu + \rho, \beta_i \rangle - i = \ell^{s+1}$ for $1 \leq i \leq b$, $\langle \mu + \rho, \beta_0 \rangle - i = ap^s$, $\nu = \sum_{i=0}^b \beta_i$, and $\sum_{i=1}^b \beta_i$ is a coroot.*

For general types, we will content ourselves with describing the fundamental coweights, where we can give a full description of the tilting condition.

B.2. **The Tilting Condition for Fundamental Coweights.** We assume that G is a split adjoint group, and let $\hat{\alpha}_j$ denote the simple roots, where we use the enumeration as in [Bou68, Pages 250-275]. We choose fundamental coweights characterized by $\langle \varpi_i, \hat{\alpha}_j \rangle = \delta_{ij}$. We will be interested in the question of when the representation V_{ϖ_i} is irreducible. In this case, we have a complete classification [Jan03, Pages 286-287],[Jan91, Section 4.6]. We note that, if ϖ_i is minuscule, this is automatic by Lemma B.1, so in what follows we simply provide a list of ℓ for the non-minuscule ϖ_i (See [LR08, Page 221] for a classification). This namely implies the case of A_n is trivial, since all fundamental weights are minuscule. Additionally, we recall that our results will only apply if ℓ is very good in the sense of [FS21, Page 33], so we have also enumerated the condition that ℓ is very good for the different types[13].

---

[13]We warn the reader if G is non-split there can also be additional constraints on ℓ.

Type of G	Type of $\hat{G}$	μ	ℓ	ℓ very good
$B_n, n \geq 2$	C_n	$\varpi_i, 1 < i \leq n$	$\ell \mid j \binom{n+1-(i+j)/2}{(i-j)/2}, 0 \leq j < i, j \equiv i \mod 2$	$\ell \neq 2$
$C_n, n \geq 2$	B_n	$\varpi_i, 1 \leq i < n$	$\ell = 2$	$\ell \neq 2$
$D_n, n \geq 4$	D_n	$\varpi_i, 1 < i < n-1$	$\ell = 2$	$\ell \neq 2$
E_6	E_6	ϖ_1, ϖ_6	$\cdot$	$\ell \neq 2,3$
		ϖ_3, ϖ_5	$\ell = 2$	
		ϖ_2	$\ell = 3$	
		ϖ_4	$\ell = 2,3$	
E_7	E_7	ϖ_1	$\ell = 2$	$\ell \neq 2,3$
		ϖ_2	$\ell = 3$	
		ϖ_3, ϖ_5	$\ell = 2,3$	
		ϖ_4	$\ell = 2,3,13$	
		ϖ_6	$\ell = 2,7$	
		ϖ_7	$\cdot$	
E_8	E_8	ϖ_1	$\ell = 2$	$\ell \neq 2,3,5$
		ϖ_2	$\ell = 2,3,7$	
		ϖ_3	$\ell = 2,3,19$	
		ϖ_4	$\ell = 2,3,5,13,19$	
		ϖ_5	$\ell = 2,3,5$	
		ϖ_6	$\ell = 2,3,5,7$	
		ϖ_7	$\ell = 2,3,5$	
		ϖ_8	$\cdot$	
F_4	F_4	ϖ_1	$\ell = 2$	$\ell \neq 2,3$
		ϖ_2, ϖ_3	$\ell = 2,3$	
		ϖ_4	$\ell = 3$	
G_2	G_2	ϖ_1	$\ell = 3$	$\ell \neq 2,3$
		ϖ_2	$\ell = 2$	

By comparing the fourth and fifth columns, we deduce the following.

Proposition B.6. *For G a split adjoint connected reductive group over $\mathbb{Q}_p$, if ℓ is very good then ϖ_i for any $i \in J$ is tilting for all G of type A_n, C_n, D_n, E_6, F_4, and G_2.*

Appendix C. Relationship to the Classical Averaging Formula, by Alexander Bertoloni-Meli

In this appendix, we show that the averaging formula proven in Theorem 11.8 is compatible with existing formulas and conjectures in the literature.

C.1. **Averaging Formulas.** To begin, we recall the general statement of these averaging formulas. Such formulas first appeared in the book of Harris–Taylor ([HT01]) and are classically deduced by studying the geometry of the mod-p fibers of Shimura varieties. These fibers admit a Newton stratification in terms of the set $B(G, \mu)$ and the strata are uniformized by Rapoport–Zink spaces and Igusa varieties. The cohomological consequence of this is the formula of Mantovan ([Man05])

(13)
$$\sum_i \varinjlim_{K \subseteq G(\mathbb{A}_f)} (-1)^i H_c^i(\mathrm{Sh}(G,X)_K, \overline{\mathbb{Q}}_\ell) = \sum_{b \in B(G,\mu)} R\Gamma_c^\flat(G,b,\mu)[\sum_j (-1)^j \varinjlim_{K^p \subseteq G(\mathbb{A}_f^p)} H_c^j(\mathrm{Ig}_{K^p}^b, \overline{\mathbb{Q}}_\ell)],$$

in $K_0(G(\mathbb{A}_f) \times W_{E_\mu})$, where $\mathrm{Sh}(G,X)_K$ (resp. $\mathrm{Ig}^b_{K^p}$) is the Shimura variety (resp. Igusa variety) determined by the associated data. [14] Averaging formulas can then be deduced by studying isotypic pieces of the above formula.

In order to precisely state these averaging formulas, we first recall some facts about stable characters following [Hir04]. Let G be a connected reductive group with Levi subgroup M and parabolic P. Let $K_0(G(\mathbb{Q}_p), \mathbb{C})^{st} \subset K_0(G(\mathbb{Q}_p), \mathbb{C})$ denote the subgroup of virtual representations with stable character. Then the normalized Jacquet module and parabolic induction functors induce morphisms

$$i_P^G : K_0(M(\mathbb{Q}_p), \mathbb{C}) \to K_0(G(\mathbb{Q}_p), \mathbb{C}), \qquad r_P^G : K_0(G(\mathbb{Q}_p), \mathbb{C}) \to K_0(M(\mathbb{Q}_p), \mathbb{C}).$$

Moreover, one can show these operations preserve stability so that we get homomorphisms

$$i_P^G : K_0(M(\mathbb{Q}_p), \mathbb{C})^{st} \to K_0(G(\mathbb{Q}_p), \mathbb{C})^{st}, \qquad r_P^G : K_0(G(\mathbb{Q}_p), \mathbb{C})^{st} \to K_0(M(\mathbb{Q}_p), \mathbb{C})^{st}.$$

Now let G^* denote the unique quasi-split group that is an inner form of G. We assume that G arises as an extended pure inner twist of G^* and fix this extra structure (G, ϱ, z). One can work with more general G using Kaletha's theory of rigid inner twists, but this is not necessary to explore the connections to this paper where G is always quasi-split.

We also need to introduce endoscopy for G. For convenience, we recall:

Definition C.1. A *refined endoscopic datum* for a connected reductive group G over a local field F is a tuple $(H, \mathcal{H}, s, \eta)$ which consists of

> a quasi-split group H over F,
> an extension $\mathcal{H}$ of W_F by $\widehat{H}$ such that the map $W_F \to \mathrm{Out}(\widehat{H})$ coincides with the map $\rho_H : W_F \to \mathrm{Out}(\widehat{H})$ induced by the action of W_F on $\widehat{H} \subset {}^L H$,
> an element $s \in Z(\widehat{H})$,
> an L-homomorphism $\eta : \mathcal{H} \to {}^L G$,

satisfying the condition:

> we have $\eta(\widehat{H}) = Z_{\widehat{G}}(s)^\circ$.

Now suppose that $(H, \mathcal{H}, s, \eta)$ is a refined endoscopic datum for G. Then after fixing splittings of $G, H, \widehat{G}, \widehat{H}$, there is a canonical endoscopic transfer of distributions inducing a morphism

$$\mathrm{Trans}_H^G : K_0(H(\mathbb{Q}_p), \mathbb{C})^{st} \to K_0(G(\mathbb{Q}_p), \mathbb{C}).$$

Furthermore, suppose we have a refined endoscopic datum $(H_M, \mathcal{H}_M, s, \eta_M)$ of M such that H_M is a Levi subgroup of H and the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\eta} & {}^L G \\
\uparrow & & \uparrow \\
\mathcal{H}_M & \xrightarrow{\eta_M} & {}^L M.
\end{array}
$$

The datum $(H_M, \mathcal{H}_M, \mathcal{H}, M, s, \eta)$ along with these compatibilities is called an *embedded* endoscopic datum in [Ber21; BS22]. Our fixed choice of splittings of G, H and their duals determines from P a parabolic subgroup P_{H_M} of H with Levi subgroup H_M. We then have an equality

$$(14) \qquad \mathrm{Trans}_H^G \circ i_{P_{H_M}}^H = i_P^G \circ \mathrm{Trans}_{H_M}^M.$$

There is also a compatibility of Trans and r which we now recall. A refined endoscopic datum $\mathfrak{e} = (H, \mathcal{H}, s, \eta)$ of G and a Levi subgroup $M \subset G$ can be upgraded to the structure of an embedded endoscopic datum in potentially many non-equivalent ways and these are parametrized by a set $D(M, \mathfrak{e}) = W(\widehat{H}) \backslash W(M, H) / W(\widehat{M})$, where $W(M, H)$ is defined to be the subset of the Weyl group

---

[14]We note that the degree shift built into the Satake sheaf $\mathcal{S}_\mu$ causes a sign in the above formula that does not appear in much of the literature on this topic. This is explained in Remark 2.4.4 of [HKW22].

$W(\widehat{G})$ of $\widehat{G}$ such that $Z_{{}^L H}((w \rtimes \eta)^{-1}(Z(\widehat{M})))$ surjects onto W_F, and where $W(\widehat{H})$ is identified with a subgroup of $W(\widehat{G})$ via η. Then we have

$$r_P^G \circ \mathrm{Trans}_H^G = \sum_{D(M,\mathsf{e})} \mathrm{Trans}_{H_M}^M \circ r_{P_{H_M}}^H.$$

We now define the map $\mathrm{Red}_b^{\mathsf{e}}$ which plays a crucial role in the statement of the averaging formula. We define

(15) $$\mathrm{Red}_b^{\mathsf{e}} : K_0(H(\mathbb{Q}_p), \mathbb{C})^{st} \to K_0(J_b(\mathbb{Q}_p), \mathbb{C})$$

by

$$\mathrm{Red}_b^{\mathsf{e}} = \Big(\sum_{D(M,H)} \mathrm{Trans}_{J_b}^{H_M} \circ r_{P_H^{op}}^H \Big) \circ \overline{\delta}_P^{\frac{1}{2}},$$

where $\overline{\delta}_P$ is the transport of the modulus character for P to J_b.

Then the (still largely conjectural) averaging formula gives a relation satisfied by $R\Gamma_c^\flat(G,b,\mu)$ at Langlands (or Arthur) parameters ϕ for which there is an associated stable distribution $S\Theta_{\phi,G}$ on G satisfying endoscopic character identities as in [Kal16, Conjecture D]. In particular, this is the case for all tempered L-parameters. To describe the expected formula, fix such a parameter ϕ and suppose that $(H, \mathcal{H}, s, \eta)$ is an endoscopic datum such that ϕ factors as $L_F \xrightarrow{\phi^H} \mathcal{H} \to {}^L G$. Then we expect

Conjecture C.2 (Averaging Formula). *We expect the following equality in $K_0(G(\mathbb{Q}_p) \rtimes W_{E_\mu})$:*

$$\sum_{b \in B(G,\mu)} [R\Gamma_c^\flat(G,b,\mu)[\mathrm{Red}_b^{\mathsf{e}}(S\Theta_{\phi^H,H})]] = \mathrm{Trans}_H^G(S\Theta_{\phi^H,H}) \otimes \mathrm{tr}(r_\mu \circ \phi|_{W_{E_\mu}} \rtimes s).$$

In particular, when e is the trivial endoscopic datum given by $\mathsf{e}_{\mathsf{triv}} = (G, {}^L G, 1, \mathrm{id})$ then we expect

$$\sum_{b \in B(G,\mu)} [R\Gamma_c^\flat(G,b,\mu)[\mathrm{Red}_b^{\mathsf{e}_{\mathsf{triv}}}(S\Theta_{\phi,G})]] = S\Theta_{\phi,G} \otimes r_\mu \circ \phi|_{W_{E_\mu}}.$$

For GL_n, this is known in the trivial endoscopic case for all representations by [Shi12]. Because L-packets are singletons for GL_n, the trivial endoscopic case implies the endoscopic versions of the formula. In [BN21], the formula is proven for discrete parameters and elliptic endoscopy of unramified GU_n, for n odd. In [Ber21], a strategy is outlined to prove this formula in the elliptic endoscopic cases using the cohomology of Igusa and Shimura varieties. This strategy should (eventually) yield results comparable to [BN21] whenever adequate global results are known about the Langlands correspondence and the cohomology of Shimura and Igusa varieties.

The averaging formulas imply strong results about $R\Gamma_c^\flat(G,b,\mu)$. For instance, in [BN21, §6] it is shown that the averaging formula for each elliptic endoscopic group and for ϕ a supercuspidal parameter implies the Kottwitz conjecture as in [RV14, Conjecture 7.3].

C.2. **Proof of Proposition 11.9.** The averaging formula in §11.1 corresponds to the case of the trivial endoscopic triple $\mathsf{e}_{\mathsf{triv}} = (G, {}^L G, 1, \mathrm{id})$. Hence to check that Theorem 11.8 agrees with C.2, we just need to check that $\mathrm{Red}_b^{\mathsf{e}_{\mathsf{triv}}}(S\Theta_{\phi,G})$ coincides with $\mathrm{Red}_{b,\phi}$ for ϕ induced from ϕ_T generic. Since ϕ_T is generic, by Lemma 3.18 ϕ should give rise to a well-defined L-parameter with trivial monodromy. Therefore, under the LLC_G appearing in Assumption 7.5, we are assuming the parameter ϕ has an L-packet given by the irreducible constituents of $i_B^G(\chi)$, by Assumption 7.5 (3). Suppose first that $b \in B(G)_{\mathsf{un}}$. Then we have

$$\mathrm{Red}_{b,\phi} = \sum_{w \in W_G/W_{M_b}} i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{1/2}[\langle 2\hat{\rho}_G, \nu_b\rangle].$$

The set $D(M, \mathfrak{e})$ is a singleton and corresponds to the trivial embedded datum where $H_M = M$. Note that $r^{G}_{P^{op}_b}(i^G_B(\chi)) = r^G_{P_b}(i^G_B(\chi))$ and that the latter term can be simplified by the geometric lemma of [BZ77].

$$
\begin{aligned}
\mathrm{Red}^{\mathsf{etriv}}_b(S\Theta_{\phi,G}\ ) &= (\mathrm{Trans}^{J_b}_{M_b} \quad r^G_{P^{op}_b})(i^G_B(\chi)) \quad \overline{\delta}^{\frac{1}{2}}_P \\
&= \mathrm{Trans}^{J_b}_{M_b} \quad (\sum_{w \, \slashed{\in} \, W_G/W_{M_b}} i^{M_b}_{B \setminus M_b} \chi^w) \quad \overline{\delta}^{\frac{1}{2}}_P \\
&= (\sum_{W_G/W_{M_b}} i^{J_b}_{B_b}) \quad \mathrm{Trans}^{T_b}_T \chi^w \quad \overline{\delta}^{\frac{1}{2}}_P \\
&= \mathrm{Red}_{b,\phi}.
\end{aligned}
$$

where the third equality is (14).

Now consider the case where $b \, \slashed{\in} \, B(G)_{\mathsf{un}}$. We must show that $\mathrm{Red}^{\mathsf{etriv}}_b(i^G_B(\chi)) = 0$, for which it suffices to show that $\mathrm{Trans}^{J_b}_{M_b}(i^{M_b}_{B \setminus M_b}(\chi^w)) = 0$ for each $w \, \slashed{\in} \, W_G/W_{M_b}$. This follows from the fact that T does not transfer to J_b by assumption and the character of $i^{M_b}_{B \setminus M_b}(\chi^w)$ is supported on the conjugates of T as per [Dij72, Theorem 3].

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