COMPATIBILITY OF THE FARGUES-SCHOLZE AND GAN-TAKEDA LOCAL LANGLANDS

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Abstract. Given a prime p, a finite extension $L/\mathbb{Q}_p$, a connected p-adic reductive group G/L, and a smooth irreducible representation π of $G(L)$, Fargues-Scholze [FS21] recently attached a semisimple Weil parameter to such π, giving a general candidate for the local Langlands correspondence. It is natural to ask whether this construction is compatible with known instances of the correspondence after semisimplification. For $G = GL_n$ and its inner forms, Fargues-Scholze and Hansen-Kaletha-Weinstein [HKW21] show that the correspondence is compatible with the correspondence of Harris-Taylor/Henniart [Hen00; HT01]. We verify a similar compatibility for $G = GSp_4$ and its unique non-split inner form $G = GU_2(D)$, where D is the quaternion division algebra over L, assuming that $L/\mathbb{Q}_p$ is unramified and $p > 2$. In this case, the local Langlands correspondence has been constructed by Gan-Takeda and Gan-Tantono [GT11; GT14]. Analogous to the case of GL_n and its inner forms, this compatibility is proven by describing the Weil group action on the cohomology of a local Shimura variety associated to GSp_4, using basic uniformization of abelian type Shimura varieties due to Shen [She17], combined with various global results of Kret-Shin [KS16] and Sorensen [Sor10] on Galois representations in the cohomology of global Shimura varieties associated to inner forms of GSp_4 over a totally real field. After showing the parameters are the same, we apply some ideas from the geometry of the Fargues-Scholze construction explored recently by Hansen [Han20], to give a more precise description of the cohomology of this local Shimura variety, verifying a strong form of the Kottwitz conjecture in the process.

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Acknowledgements

It is a pleasure to thank my advisor David Hansen for giving me this project and for numerous suggestions and ideas related to it, as well as Eric Chen, Arthur-César Le-Bras, Yoichi Mieda, Jack Sempliner, and Zhiyu Zhang for some nice conversations related to this work. Special thanks also go to Peter Scholze for sharing with me the draft of [FS21], pointing out some errors, and generously initiating me in these ideas during my masters thesis, Thomas Haines and Sug-Woo Shin for pointing out some errors in section 3 and section 5 respectively, and the MPIM Bonn for their hospitality during part of the completion of this project.

1. Introduction

1.1. **Background and Main Theorems.** Fix distinct primes $\ell \neq p$, let $\mathbb{Q}_p$ denote the p-adic numbers, and let $G/\mathbb{Q}_p$ be a connected reductive group. Set $\mathbb{C}_p := \hat{\overline{\mathbb{Q}}}_p$ to be the completion of the algebraic closure of $\mathbb{Q}_p$. We fix an isomorphism $i : \overline{\mathbb{Q}}_\ell \xrightarrow{\simeq} \mathbb{C}$. Let $W_{\mathbb{Q}_p}$ be the Weil group of $\mathbb{Q}_p$ and set $\hat{G}$ to be the reductive group over $\overline{\mathbb{Q}}_\ell$ with root datum dual to G. Let Q be the finite quotient through which $W_{\mathbb{Q}_p}$ acts on $\hat{G}$. We define the L-group ${}^LG := Q \ltimes \hat{G}$. We let $\Pi(G)$ denote the set of isomorphism classes of smooth irreducible representations of the p-adic group $G(\mathbb{Q}_p)$, and let $\Phi(G)$ denote the set of L-parameters, i.e the set of conjugacy classes of homomorphisms

$$\phi : W_{\mathbb{Q}_p} \times SL_2(\mathbb{C}) \to {}^LG(\mathbb{C})$$

where $SL_2(\mathbb{C})$ acts via an algebraic representation and $W_{\mathbb{Q}_p}$ acts via a continuous semisimple homomorphism in a way that commutes with the natural projection ${}^LG(\mathbb{C}) \to Q$, where ${}^LG(\mathbb{C})$ is endowed with the discrete topology. The local Langlands correspondence is a conjectural map

$$LLC_G : \Pi(G) \to \Phi(G)$$
$$\pi \mapsto \phi_\pi$$

that builds a bridge between L-parameters and the smooth irreducible representations of $G(\mathbb{Q}_p)$. Conjecturally (under some additional constraints on $\Phi(G)$ if G is not split), these maps should be surjective with finite fibers called L-packets and satisfy various properties such as compatibility with products, maps of L-groups, character twists, as well as L, ϵ, and γ-factors. Moreover, one expects that the correspondence is uniquely characterized by some such finite list of properties.

In general, the existence and uniqueness of such a correspondence is completely unknown. However, very recently, Fargues and Scholze [FS21], using the action of the excursion algebra on the moduli space of G-bundles on the Fargues-Fontaine curve, were able to construct a completely general candidate, analogous to the work of V. Lafforgue in the function field setting [Laf18]. Namely, they construct a map

$$LLC_G^{FS} : \Pi(G) \to \Phi^W(G)$$
$$\pi \mapsto \phi_\pi^{FS}$$

where $\Phi^W(G)$ denotes the set of conjugacy classes of continuous semisimple maps

$$\phi : W_{\mathbb{Q}_p} \to {}^LG(\overline{\mathbb{Q}}_\ell)$$

that commute with the projection ${}^LG(\overline{\mathbb{Q}}_\ell) \to Q$ as above. Fargues and Scholze show that their map has several good properties such as compatibility with parabolic induction; however, one would also like to check that this correspondence agrees with known instances of the local Langlands correspondence. Precisely, given a candidate for the local Langlands correspondence

$$LLC_G : \Pi(G) \to \Phi(G)$$
$$\pi \mapsto \phi_\pi$$

we expect a commutative diagram of the form

$$\begin{tikzcd} \Pi(G) \arrow[r, "LLC_G"] \arrow[dr, "LLC_G^{FS}"'] & \Phi(G) \arrow[d, "(-)^{ss}"] \\ & \Phi^W(G) \end{tikzcd}$$

where the semisimplification map $(-)^{ss}$ precomposes an L-parameter $\phi \in \Phi(G)$ with the map

$$g \in W_{\mathbb{Q}_p} \mapsto (g, \begin{pmatrix} |g|^{\frac{1}{2}} & 0 \\ 0 & |g|^{\frac{-1}{2}} \end{pmatrix}) \in W_{\mathbb{Q}_p} \times SL_2(\mathbb{C})$$

and then applies the fixed isomorphism $i^{-1} : \mathbb{C} \xrightarrow{\simeq} \overline{\mathbb{Q}}_\ell$, where $|\cdot| : W_{\mathbb{Q}_p} \to W_{\mathbb{Q}_p}^{ab} \simeq \mathbb{Q}_p^* \to \mathbb{C}^*$ is the norm character. We make the following definition.

Definition 1.1. For $\pi \in \Pi(G)$, we say that a local Langlands correspondence LLC_G is compatible with the Fargues-Scholze local Langlands correspondence if we have an equality: $\phi_\pi^{FS} = \phi_\pi^{ss}$, as conjugacy classes of Weil parameters.

For GL_n, the local Langlands correspondence was constructed by Harris-Taylor/Henniart [Hen00; HT01] and is uniquely characterized by the preservation of L, ϵ, and γ-factors. In this case, compatibility with the Fargues-Scholze local Langlands correspondence follows from the description of the cohomology of the Lubin-Tate and Drinfeld towers proven in [HT01] and was verified by Fargues and Scholze [FS21, Theorem I.9.6]. The main goal of this note is to extend compatibility of the correspondence to GSp_4 and its inner form. To this end, we now fix a finite extension $L/\mathbb{Q}_p$ and set G to be $Res_{L/\mathbb{Q}_p}GSp_4$ and J to be its unique non-split inner form $Res_{L/\mathbb{Q}_p}GU_2(D)$, where D/L is the quaternion division algebra. In this case, the local Langlands correspondence has been constructed by Gan-Takeda and Gan-Tantono, respectively [GT11; GT14]. It is constructed from the local Langlands correspondence for GL_n and theta lifting and admits a similar unique characterization in terms of the preservation of L, ϵ, and γ factors. We note that we can and do identify $\Phi(G)$ and $\Phi(J)$ with a subset of homomorphisms:

$$\phi : W_L \times SL_2(\mathbb{C}) \to \hat{G}(\mathbb{C}) = GSpin_5(\mathbb{C}) \simeq GSp_4(\mathbb{C})$$

This allows us to introduce a bit of terminology. Namely, we say that a parameter ϕ in $\Phi(G)$ or $\Phi(J)$ is supercuspidal if the $SL_2(\mathbb{C})$-factor acts trivially and ϕ does not factor through any proper Levi subgroup of GSp_4. This terminology is justified by the fact that this is precisely the case when the L-packets over ϕ contain only supercuspidal representations. In what follows, we will often abuse notation and drop the superscript $(-)^{ss}$ when speaking about such parameters, as in this case it merely corresponds to forgetting the trivially acting $SL_2(\mathbb{C})$-factor and applying the isomorphism i^{-1}. We now come to our main theorem.

Theorem 1.1. *The following is true.*

(1) *For any $\pi \in \Pi(G)$ (resp. $\rho \in \Pi(J)$) such that the Gan-Takeda (resp. Gan-Tantono) parameter is not supercuspidal, we have that the Gan-Takeda (resp. Gan-Tantono) correspondence is compatible with the Fargues-Scholze correspondence.*
(2) *If $L/\mathbb{Q}_p$ is unramified and $p > 2$, we have, for all $\pi \in \Pi(G)$ (resp. $\rho \in \Pi(J)$) such that the Gan-Takeda (resp. Gan-Tantono) parameter is supercuspidal, that the Gan-Takeda (resp. Gan-Tantono) correspondence is compatible with the Fargues-Scholze correspondence.*

Remark 1.1. As will be explained more below, the restrictions in the case where the parameter is supercuspidal are necessary to apply basic uniformization of the generic fiber of abelian type Shimura varieties due to Shen [She17]. If one were not to impose this assumption, the relevant Shimura varieties would have bad reduction at p, which, to the best of our knowledge, prevents the methods of Shen from working. In particular, if one could establish the expected description of basic locus in the sense of the isomorphism (2) of Definition 4.1, for Shimura varieties associated to the group $Res_{F/\mathbb{Q}}\mathbf{G}$, where $\mathbf{G}$ is an inner form of GSp_4 over a totally real field F with an inert prime p such that $F_p \simeq L$ for L any extension then our result would hold in complete generality.

As mentioned above, the proof of compatibility for GL_n uses the results of Harris-Taylor [HT01] on the cohomology of the Lubin-Tate/Drinfeld Towers. In particular, if one looks at the rigid generic fiber of the Lubin-Tate tower

$$\lim_{m\to\infty} LT_{n,m,\breve{\mathbb{Q}}_p}$$

a tower of $n-1$-dimensional rigid spaces over $\breve{\mathbb{Q}}_p$, for fixed $n \geq 1$ and varying $m \geq 1$, where $\breve{\mathbb{Q}}_p$ denotes the completion of the maximal unramified extension of $\mathbb{Q}_p$. The cohomology of this tower

$$R\Gamma_c(LT_{n,\infty}, \overline{\mathbb{Q}}_\ell) := \mathrm{colim}_{m\to\infty} R\Gamma_c(LT_{n,m,\mathbb{C}_p}, \overline{\mathbb{Q}}_\ell)$$

based changed to $\mathbb{C}_p$ carries commuting actions of $GL_n(\mathbb{Q}_p)$ and $D^*_{\frac{1}{n}}$, the units in the division algebra over $\mathbb{Q}_p$ of invariant $\frac{1}{n}$, as well as an action of the Weil group $W_{\mathbb{Q}_p}$. In particular, given $\pi \in \Pi(GL_n)$ (resp. $\rho \in \Pi(D^*_{\frac{1}{n}})$), we can consider the complexes

$$R\Gamma_c(LT_{n,\infty}, \overline{\mathbb{Q}}_\ell)[\pi] := R\Gamma_c(LT_{n,\infty}, \overline{\mathbb{Q}}_\ell) \otimes^{\mathbb{L}}_{\mathcal{H}(GL_n)} \pi$$

and

$$R\Gamma_c(LT_{n,\infty}, \overline{\mathbb{Q}}_\ell)[\rho] := R\Gamma_c(LT_{n,\infty}, \overline{\mathbb{Q}}_\ell) \otimes^{\mathbb{L}}_{\mathcal{H}(D^*_{\frac{1}{n}})} \rho$$

where $\mathcal{H}(GL_n) := C_c^\infty(GL_n(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ (resp. $\mathcal{H}(D^*_{\frac{1}{n}})$) is the usual smooth Hecke algebra of G (resp. $D^*_{\frac{1}{n}}$). Then the key result of Harris-Taylor [HT01] and later refined by Dat [Dat07] is as follows.

Theorem 1.2. *[HT01; Dat07] Fix $\pi \in \Pi(GL_n)$, a supercuspidal representation of $GL_n(\mathbb{Q}_p)$, let*

$$JL : \Pi(D^*_{\frac{1}{n}}) \to \Pi(GL_n(\mathbb{Q}_p))$$

be the map defined by the Jacquet-Langlands correspondence and $\rho := JL^{-1}(\pi) \in \Pi(D^*_{\frac{1}{n}})$ a Jacquet-Langlands lift of π. Then the complexes $R\Gamma_c(G, b, \mu)[\pi]$ and $R\Gamma_c(G, b, \mu)[\rho]$ are concentrated in middle degree $n-1$. The middle degree cohomology of $R\Gamma_c(G, b, \mu)[\pi]$ is isomorphic to

$$\rho \boxtimes \phi_\pi^\vee \otimes |\cdot|^{(1-n)/2}$$

as a $D^*_{\frac{1}{n}} \times W_{\mathbb{Q}_p}$ representation. Similarly, the middle degree cohomology of $R\Gamma_c(G, b, \mu)[\rho]$ is isomorphic to

$$\pi \boxtimes \phi_\pi \otimes |\cdot|^{(1-n)/2}$$

where $\phi_\pi \in \Phi^W(G)$ is the (semisimplified) Weil parameter associated to π by Harris-Taylor.

To see why this result is relevant for compatibility, we invoke the observation, due to Scholze-Weinstein [SW13; SW20], that, using Grothendieck-Messing theory, the Lubin-Tate tower at infinite level

$$LT_{n,\infty} := \lim_{m\to\infty} LT_{n,m,\breve{\mathbb{Q}}_p}$$

is representable by a space admitting a moduli interpretation as a space of shtukas over the Fargues-Fontaine curve. Namely, the space denoted $Sht(GL_n, b, \mu)_\infty$ in the notation of [SW20], where $b \in B(GL_n)$ is an element in the Kottwitz set of GL_n corresponding to a rank n isocrystal of slope

$\frac{1}{n}$ and $\mu = (1, 0, \ldots, 0, 0)$ is a dominant cocharacter of GL_n. If X denotes the Fargues-Fontaine curve, then this this parametrizes modifications

$$\mathcal{O}_X(-\frac{1}{n}) \to \mathcal{O}_X^n$$

of type $(1, 0, \ldots, 0, 0)$ (i.e this map is an embedding with cokernel a length 1 torsion sheaf on X), where $\mathcal{O}_X(-\frac{1}{n})$ is the unique rank n vector bundle on X of slope $-\frac{1}{n}$. This interpretation allows one to relate the complex $R\Gamma_c(LT_{n,\infty}, \overline{\mathbb{Q}}_\ell)[\pi]$ to the action of a Hecke operator $T_{\mu^{-1}}$ on Bun_G, acting on a sheaf $\mathcal{F}_\pi$ constructed from the supercuspidal representation π, where $\mu^{-1} = (0, 0, \ldots, 0, -1)$ is a dominant inverse of μ. The Fargues-Scholze parameter of π is built from the action of the excursion algebra on the sheaf $\mathcal{F}_\pi$, which in turn is built from Hecke operators equipped with a factorization structure coming from geometric Satake. It thus is reasonable to expect that the cohomology group $R\Gamma_c(LT_{n,\infty}, \overline{\mathbb{Q}}_\ell)[\pi]$ should have $W_{\mathbb{Q}_p}$-action given by the Fargues-Scholze parameter $\phi_\pi^{FS} : W_{\mathbb{Q}_p} \to {}^L GL_n(\overline{\mathbb{Q}}_\ell) \simeq GL_n(\overline{\mathbb{Q}}_\ell)$ composed with the highest weight representation of ${}^L GL_n(\overline{\mathbb{Q}}_\ell)$ corresponding to the dominant cocharacter μ^{-1}. However, this is just the dual representation of $GL_n(\overline{\mathbb{Q}}_\ell)$. Thus, using Theorem 1.2, we can see that

$$\phi_\pi^\vee = (\phi_\pi^{FS})^\vee$$

where the twist by the norm-character $|\cdot|^{(1-n)/2}$ is cancelled out by a perverse normalization (also related to the middle degree being the relevant one) in the definition of the Hecke operator $T_{\mu^{-1}}$. This implies compatibility for supercuspidal π, which, by using compatibility of the Fargues-Scholze correspondence with parabolic induction, is enough to conclude the general case. In a similar fashion, using the description of the ρ-isotypic part one can prove compatibility for the inner form $D^*_{\frac{1}{n}}$.

One may expect, given the above sketch of compatibility for GL_n, that, to prove Theorem 1.1, one must similarly provide a description of the cohomology of the $\pi \in \Pi(G)$ (resp. $\rho \in \Pi(J)$)-isotypic part of a local Shimura variety/shtuka space at infinite level associated to G. In the case where the associated L-parameters are supercuspidal, this is the content of the Kottwitz conjecture [RV14, Conjecture 7.3]. To this end, we consider the cohomology of a Shtuka space associated to the group $G = Res_{L/\mathbb{Q}_p} GSp_4$. Namely, the space denoted $Sht(G, b, \mu)_\infty$, where μ is the Siegel cocharacter and $b \in B(G)$ is a basic element in the Kottwitz set of G, corresponding to a rank 4 isocrystal with a polarization and automorphism group equal to $J = Res_{L/\mathbb{Q}_p}(GU_2(D))$. This space carries a commuting $G(\mathbb{Q}_p)$ and $J(\mathbb{Q}_p)$ action. The quotients

$$Sht(G, b, \mu)_K := Sht(G, b, \mu)_\infty / \underline{K}$$

for varying compact open $K \subset G(\mathbb{Q}_p)$ are, as before, representable by rigid analytic varieties over $Spd(\breve{L})$ of dimension 3, where $\breve{L} := L\breve{\mathbb{Q}}_p$. They define a tower of local Shimura varieties in the sense of Rapoport-Viehmann [RV14], which uniformize the basic locus of certain global Shimura varieties analogous to the Lubin-Tate case described above. Letting $Sht(G, b, \mu)_{K,\mathbb{C}_p}$ be the base-change of these spaces to $\mathbb{C}_p$, we can then consider the analog of the complexes described above

$$R\Gamma_c(G, b, \mu) := \mathrm{colim}_{K \to 1} R\Gamma_c(Sht(G, b, \mu)_{K,\mathbb{C}_p}, \overline{\mathbb{Q}}_\ell)$$

This complex is concentrated in degrees $0 \leq i \leq 6 = 2dim(Sht(G, b, \mu)_K)$ and admits an action of $G(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \times W_L$. This allows one to consider the ρ and π-isotypic parts, i.e we set

$$R\Gamma_c(G, b, \mu)[\rho] := R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(J)} \rho$$

and

$$R\Gamma_c(G, b, \mu)[\pi] := R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(G)} \pi$$

where $\mathcal{H}(G)$ (resp. $\mathcal{H}(J)$) are the usual smooth Hecke algebra of G (resp. J). To deduce compatibility, one needs to realize the (semi-simplified) Weil parameter ϕ_π (resp. ϕ_ρ) of Gan-Takeda (resp.

Gan-Tantono) in these two cohomology groups. We will sketch how to do this in the next section using uniformization and global methods. Interestingly, after knowing compatibility, one can use ideas from the geometry of the Fargues-Scholze construction to provide a more precise of the complexes $R\Gamma_c(G,b,\mu)[\rho]$ and $R\Gamma_c(G,b,\mu)[\pi]$. Namely, recent work of Hansen [Han20] allows us to deduce that if ϕ_ρ (resp. ϕ_π) is supercuspidal the complexes $R\Gamma_c(G,b,\mu)[\rho]$ (resp. $R\Gamma_c(G,b,\mu)[\pi]$) are concentrated in middle degree 3. It then follows from work of Hansen-Kaletha-Weinstein [HKW21] on a weakening of the Kottwitz conjecture and work of Fargues-Scholze [FS21, Section X.2] describing the Hecke action on objects with supercuspidal Fargues-Scholze parameter that one can actually deduce a strong form of the Kottwitz conjecture for these representations. To state this result, we first note that, if we are given a supercuspidal parameter $\phi : W_L \to GSp_4(\overline{\mathbb{Q}}_\ell)$, even though the parameter ϕ is irreducible, its composition with the standard embedding $std : GSp_4(\overline{\mathbb{Q}}_\ell) \hookrightarrow GL_4(\overline{\mathbb{Q}}_\ell)$ may not be. In particular, the size of the L-packets $\Pi_\phi(G) := LLC_G^{-1}(\phi)$ and $\Pi_\phi(J) := LLC_J^{-1}(\phi)$ over ϕ are governed by this.

(1) (stable) $std \circ \phi$ is irreducible. In this case, the L-packets each contain one supercuspidal member.
(2) (endoscopic) $std \circ \phi \simeq \phi_1 \oplus \phi_2$, where $\phi_i : W_L \to GL_2(\bar{\mathbb{Q}}_\ell)$ are distinct irreducible 2-dimensional representations with $det(\phi_1) = det(\phi_2)$. In this case, the L-packets over ϕ each contain two supercuspidal members.

This allows us to state our main consequence of Theorem 1.1, which (almost) verifies the strong form of the Kottwitz conjecture for GSp_4/L and $GU_2(D)/L$.

Theorem 1.3. *Let $L/\mathbb{Q}_p$ be an unramified extension with $p > 2$. Let π (resp. ρ) be members of the L-packet over a supercuspidal parameter $\phi : W_L \to GSp_4(\overline{\mathbb{Q}}_\ell)$ as above. Then the complexes*

$$R\Gamma_c(G,b,\mu)[\pi]$$

and

$$R\Gamma_c(G,b,\mu)[\rho]$$

are concentrated in middle degree 3.

(1) *If ϕ is stable supercuspidal, with singleton L-packets $\{\pi\} = \Pi_\phi(G)$ and $\{\rho\} = \Pi_\phi(J)$, then the cohomology of $R\Gamma_c(G,b,\mu)[\pi]$ in middle degree is isomorphic to*

$$\rho \boxtimes (std \circ \phi)^\vee \otimes |\cdot|^{-3/2}$$

as a $J(\mathbb{Q}_p) \times W_L$-module, and the cohomology of $R\Gamma_c(G,b,\mu)[\rho]$ in middle degree is isomorphic to

$$\pi \boxtimes std \circ \phi \otimes |\cdot|^{-3/2}$$

as a $G(\mathbb{Q}_p) \times W_L$-module.

(2) *If ϕ is an endoscopic parameter, with L-packets $\Pi_\phi(G) = \{\pi^+, \pi^-\}$ and $\Pi_\phi(J) = \{\rho_1, \rho_2\}$[1], the cohomology of $R\Gamma_c(G,b,\mu)[\pi]$ in middle degree is isomorphic to*

$$\rho_1 \boxtimes \phi_1^\vee \otimes |\cdot|^{-3/2} \oplus \rho_2 \boxtimes \phi_2^\vee \otimes |\cdot|^{-3/2}$$

or

$$\rho_1 \boxtimes \phi_2^\vee \otimes |\cdot|^{-3/2} \oplus \rho_2 \boxtimes \phi_1^\vee \otimes |\cdot|^{-3/2}$$

as a $J(\mathbb{Q}_p) \times W_L$-module. Similarly, the cohomology of $R\Gamma_c(G,b,\mu)[\rho]$ in middle degree is isomorphic to

$$\pi^+ \boxtimes \phi_1 \otimes |\cdot|^{-3/2} \oplus \pi^- \boxtimes \phi_2 \otimes |\cdot|^{-3/2}$$

or

$$\pi^+ \boxtimes \phi_2 \otimes |\cdot|^{-3/2} \oplus \pi^- \boxtimes \phi_1 \otimes |\cdot|^{-3/2}$$

[1]For an explanation of the notation, see the discussion at the end of section 2.2.

as a $G(\mathbb{Q}_p) \times W_L$-module. Here we write $std \circ \phi \simeq \phi_1 \oplus \phi_2$, with ϕ_i distinct irreducible 2-dimensional representations of W_L and $det(\phi_1) = det(\phi_2)$.

Moreover, both possibilities for the cohomology of $R\Gamma_c(G, b, \mu)[\rho]$ (resp. $R\Gamma_c(G, b, \mu)[\pi]$) in the endoscopic case occur for some choice of representation $\rho \in \Pi_\phi(J)$ (resp. $\pi \in \Pi_\phi(G)$). In particular, knowing the precise form of either $R\Gamma_c(G, b, \mu)[\rho]$ or $R\Gamma_c(G, b, \mu)[\pi]$ for some $\rho \in \Pi_\phi(J)$ or $\pi \in \Pi_\phi(G)$ determines the precise form of the cohomology in all other cases.

Remark 1.2. (1) Results of this form when $L = \mathbb{Q}_p$ have also been shown by Ito-Meida. [IM21]

(2) If one knew Arthur's multiplicity formula for inner forms of GSp_4 over totally real fields, one should be able to determine the cohomology in the endoscopic case more precisely, using basic uniformization and the more precise description of the cohomology of the global Shimura variety this multiplicity formula would provide. (See for example [Ngu19, Section 3.2] for this kind of analysis in the case of unitary groups.) However, to our knowledge the multiplicity formula is unknown in this case. In the case that $L = \mathbb{Q}_p$, one can apply what is known about the multiplicity formula for $GSp_4/\mathbb{Q}$ [Art04; GT19]. This is carried out by Ito-Mieda [IM21]. The correct answer, for the ρ-isotypic part, should be that, if $\rho = \rho_1$, we are in the first case, and if $\rho = \rho_2$, we are in the second case. Similarly, for the π-isotypic part, if $\pi = \pi^+$ is the unique generic member of the L-packet for a fixed choice of Whittaker datum, we should be in the first case and, if $\pi = \pi^-$, we should be in the second case. It might also be possible to show this using a weaker argument. Our analysis reduces us to checking that $R\Gamma_c(G, b, \mu)[\rho_1]$ admits a sub-quotient isomorphic to $\pi_+ \boxtimes \phi_1 \otimes |\cdot|^{-3/2}$, which may be possible to show through basic uniformization and a small global argument.

(3) We hope that the perspective we take on the Kottwitz conjecture in this paper will help provide further advancements in our knowledge of the cohomology of local Shimura varieties. In particular, by invoking the use of these very general geometric tools from the Fargues-Scholze construction, we require global input only to show compatibility, which, as we will see in the next section, requires substantially less than the input needed to determine the precise form of the cohomology such as a multiplicity formula for the automorphic spectrum.

We will now conclude the introduction by providing a sketch of the proof of Theorem 1.1.

1.2. **Proof Sketch of the Main Theorems.** As before, we set $G = Res_{L/\mathbb{Q}_p}GSp_4$ and $J = Res_{L/\mathbb{Q}_p}GU_2(D)$. Similar to the case of GL_n, the idea behind proving compatibility for G and J is to use the compatibility of the Fargues-Scholze local Langlands correspondence with parabolic induction to reduce to the case where π is a supercuspidal representation. However, this is a little bit more subtle then the case of GL_n. Unlike GL_n, the local Langlands correspondence for these groups is not a bijection. As seen before, the L-packets can be either of size 1 or 2. Given an L-parameter $\phi : W_L \times SL_2(\overline{\mathbb{Q}}_\ell) \to GSp_4(\overline{\mathbb{Q}}_\ell)$, there are three distinct possibilities.

(1) The L-packets $\Pi(G)_\phi$ and $\Pi(J)_\phi$ do not contain any supercuspidal representations.
(2) The L-packets $\Pi(G)_\phi$ and $\Pi(J)_\phi$ contain a mix of supercuspidal and non-supercuspidal representations.
(3) The L-packets $\Pi(G)_\phi$ and $\Pi(J)_\phi$ contain only supercuspidals.

Case (1) is straight forward. Since compatibility is known for GL_n and its inner forms and any proper Levi subgroup of G (resp. J) is a product of such groups, it follows from compatibility of the Fargues-Scholze correspondence with parabolic induction and products that the correspondences are compatible for any representation lying in such an L-packet.

Case (2) is a bit more subtle, here ϕ^{ss} factors through a Levi subgroup of $GSp_4(\overline{\mathbb{Q}}_\ell)$, but ϕ itself does not. In particular, the restriction to the SL_2 factor of ϕ is non-trivial. In this case, we can write $\Pi_\phi(G) = \{\pi_{disc}, \pi_{sc}\}$ (resp. $\Pi_\phi(J) := \{\rho_{disc}, \rho_{sc}\}$ or $\Pi_\phi(J) = \{\rho^1_{disc}, \rho^2_{disc}\}$, depending on whether the parameter is of Saito-Kurokawa or Howe-Piatetski–Schapiro type), where π_{disc} (resp. ρ^1_{disc}, ρ^2_{disc}, and ρ_{disc}) are non-supercuspidal (essentially) discrete series representation of G (resp. J), and π_{sc} (resp. ρ_{sc}) is a supercuspidal representation of G (resp. J). The key observation is that π_{disc} (resp. ρ_{disc}) is an irreducible sub-quotient of a parabolic induction, so, in this case, we can apply the same argument as in case (1) to deduce compatibility of the two correspondences. It remains to see that the same is true for π_{sc}. To do this, we use a description of the ρ-isotypic part of the Shtuka space $Sht(G, b, \mu)_\infty$ introduced in section 1.1. Namely, we consider the complex

$$R\Gamma_c(G, b, \mu)[\rho_{disc}] \simeq R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(G)} \rho_{disc}$$

of $G(\mathbb{Q}_p) \times W_L$-modules. Recent work of Hansen-Kaletha-Weinstein [HKW21] then tells us the form of this cohomology group (or rather a small variant thereof) as a $J(\mathbb{Q}_p)$-representation. In particular, if we let $R\Gamma_c(G, b, \mu)[\rho_{disc}]_{sc}$ denote the summand of $R\Gamma_c(G, b, \mu)[\rho_{disc}]$ where $J(\mathbb{Q}_p)$ acts via a supercuspidal representation, then in the Grothendieck group of admissible $J(\mathbb{Q}_p)$-representations of finite length $R\Gamma_c(G, b, \mu)[\rho_{disc}]_{sc}$ is equal to $-2\pi_{sc}$.

Similar to the case of $G = GL_n$, this complex describes the action of the Hecke operator T_μ acting on a sheaf $\mathcal{F}_{\rho_{disc}}$ constructed from ρ_{disc} on Bun_G. Moreover, the complex $R\Gamma_c(G, b, \mu)[\rho_{disc}]_{sc}$ can be interpreted as a complex of sheaves on the open Harder-Narasimhan(=HN)-strata $Bun_G^{\mathbf{1}} \subset Bun_G$ corresponding to the trivial G-bundle on the Fargues-Fontaine curve X. It follows from the above description in the Grothendieck group that the excursion algebra will act on this complex via eigenvalues valued in the parameter $\phi^{FS}_{\pi_{sc}}$. However, since the excursion algebra is built from Hecke operators, it will also commute with the action of Hecke operators on $\mathcal{F}_{\rho_{disc}}$. This allows us to conclude that it also must act via eigenvalues valued in $\phi^{FS}_{\rho_{disc}}$ giving a chain of equalities

$$\phi^{FS}_{\pi_{sc}} = \phi^{FS}_{\rho_{disc}} = \phi^{ss}_{\rho_{disc}} = \phi^{ss}_{\pi_{sc}}$$

where the first equality follows from the previous argument and the second equality follows from the above analysis of induced representations. Similarly, one deduces compatibility for ρ_{sc} by applying a similar argument to the π-isotypic part.

Case (3) is by far the most involved and takes up the majority of the paper. This is the case in which the L-parameter ϕ is supercuspidal. First, one can make a reduction to showing compatibility for just $\rho \in \Pi(J)$ with supercuspidal Gan-Tantono parameter ϕ, by using the commutation of Hecke operators and the excursion algebra similar to what was done in case (2). Now, the key point again is that the complex $R\Gamma_c(G, b, \mu)[\rho]$ describes the action of the Hecke operator T_μ on a sheaf $\mathcal{F}_\rho$ on Bun_G. To make further progress towards compatibility, we use that the Hecke operators can be in turn described using the spectral action of the derived category of perfect complexes on the stack of L-parameters, as constructed in [FS21, Chapter X]. In particular, a Hecke operator defines a vector bundle on the stack of L-parameters, whose action on the sheaf $\mathcal{F}_\rho$ via the spectral action is precisely T_μ. Using this, we argue using the support of the spectral action of certain averaging operators, considered by [AL21] in the case of GL_n, to show that, if $std \circ \phi \otimes |\cdot|^{-3/2}$ occurs as a W_L-stable sub-quotient of the complex

$$\bigoplus_{\rho' \in \Pi_\phi(J)} R\Gamma_c(G, b, \mu)[\rho']$$

we have an equality:

$$std \circ \phi^{FS}_\rho = std \circ \phi$$

for all $\rho \in \Pi_\phi(J)$. Now a GSp_4-valued parameter is in turn determined by its composition with std and its similitude character, which is precisely the central character of ρ. Therefore, since the Fargues-Scholze correspondence is compatible with central characters and isogenies, this is enough to conclude that $\phi = \phi_{\rho}^{FS}$. This reduces the question of showing compatibility for $\rho \in \Pi(GU_2(D))$ with supercuspidal Gan-Tantono parameter ϕ to the following.

Proposition 1.4. *Let ϕ be a supercuspidal parameter with associated L-packet $\Pi_{\phi}(J)$. Then the direct summand of*

$$\bigoplus_{\rho' \in \Pi_{\phi}(J)} R\Gamma_c(G, b, \mu)[\rho']$$

where $G(\mathbb{Q}_p)$ acts via a supercuspidal representation

$$\bigoplus_{\rho' \in \Pi_{\phi}(J)} R\Gamma_c(G, b, \mu)[\rho']_{sc}$$

is concentrated in middle degree 3 and admits a non-zero W_L-stable sub-quotient with W_L-action given by $std \circ \phi \otimes |\cdot|^{-3/2}$.

Just as one does in proving Theorem 1.2, the key idea is to directly relate the complex

$$\bigoplus_{\rho' \in \Pi_{\phi}(J)} R\Gamma_c(G, b, \mu)[\rho']_{sc}$$

to the cohomology of a global Shimura variety using basic uniformization of the generic fiber as proven by Shen [She17] and an analogue of Boyer's trick [Boy09]. This allows us to in turn prove Proposition 1.4 using global results on Galois representations in the cohomology of these Shimura varieties due to Kret-Shin [KS16] and Sorensen [Sor10]. More specifically, in this case the relevant Shimura datum is given by $(\mathbf{G}, X)$, where $\mathbf{G}$ is a $\mathbb{Q}$-inner form of $\mathbf{G}^* := Res_{F/\mathbb{Q}}(GSp_4)$ for $F/\mathbb{Q}$ a totally real extension with p inert and $F_p \simeq L$. The relevant uniformization result is then applicable if $L/\mathbb{Q}_p$ is an unramified extension and $p > 2$. To state the key consequence of this uniformization result, we introduce some notation. We let $\mathbb{A}$ and $\mathbb{A}_f$ denote the adeles and finite adeles of $\mathbb{Q}$, respectively. If $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$ denotes the level away from p and $K_p \subset G(\mathbb{Q}_p)$ denotes the level at p, we let $\mathcal{S}(\mathbf{G}, X)_{K_pK^p}$ be the rigid analytic Shimura variety over $\mathbb{C}_p$ of level K_pK^p. We set ξ be a regular weight of an algebraic representation $\mathcal{V}_{\xi}$ of $\mathbf{G}$ over $\mathbb{Q}$ and let $\mathcal{L}_{\xi}$ denote the associated $\overline{\mathbb{Q}}_{\ell}$ local system on $\mathcal{S}(\mathbf{G}, X)_{K_pK^p}$. We then define

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_{\xi}) := \operatorname{colim}_{K_p \to \{1\}} R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^pK_p}, \mathcal{L}_{\xi})$$

The basic uniformization result of Shen then furnishes a $\mathbb{Q}$-inner form $\mathbf{G}'$ of $\mathbf{G}$ satisfying that $\mathbf{G}'_{\mathbb{Q}_p} \simeq J$ and a $G(\mathbb{Q}_p) \times W_L$-invariant map

$$\Theta : R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f)/K^p, \mathcal{L}_{\xi}) \to R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_{\xi})$$

functorial in the level K^p. Here $\mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f)/K^p, \mathcal{L}_{\xi})$ denotes the space of algebraic automorphic forms of level K^p valued in the algebraic representation $\mathcal{V}_{\xi}$. We want to use this uniformization map to apply global results about the cohomology of $R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_{\xi})$ to study the action of W_L on $R\Gamma_c(G, b, \mu)$. To do this, we show an analogue of Boyer's trick, which says that the non-basic Newton strata of the adic flag variety $\mathcal{F}\ell_{G,\mu^{-1}} := (G/P_{\mu^{-1}})^{ad}$ are parabolically induced as spaces with $G(\mathbb{Q}_p)$-action. Using the Hodge-Tate period map from the Shimura variety $\mathcal{S}(\mathbf{G}, X)_{K^p}$ to $\mathcal{F}\ell_{G,\mu^{-1}}$, this implies that, if we pass to the part of the cohomology on both sides where $G(\mathbb{Q}_p)$ acts via a supercuspidal representation, we get a $W_L \times G(\mathbb{Q}_p)$-equivariant isomorphism:

$$\Theta_{sc} : R\Gamma_c(G, b, \mu)_{sc} \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f)/K^p, \mathcal{L}_{\xi}) \xrightarrow{\simeq} R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_{\xi})_{sc}$$

After showing this, we fix a ρ having supercuspidal Gan-Tantono parameter ϕ, and, via an argument using the simple trace formula, choose a globalization of ρ to a cuspidal automorphic representation Π' of $\mathbf{G}'$, which occurs as a $J(\mathbb{Q}_p)$-stable direct summand of $\mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f)/K^p, \mathcal{L}_{\xi})$ and is an unramified twist of Steinberg at some non-empty set of places S_{st}, for some sufficiently large regular weight ξ and sufficiently small level K^p. We set S to be a finite set of places outside of which Π'

is unramified. The Hecke eigenvalues of Π' then define a maximal ideal $\mathfrak{m} \subset \mathbb{T}^S$ in the abstract commutative Hecke algebra of $\mathbf{G}'$ away from the finite places S. Regarding both sides of Θ as $\mathbb{T}^S$-modules, we can localize at $\mathfrak{m}$ to get a map

$$\Theta_{\mathfrak{m}} : (R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p, \mathcal{L}_\xi))_{\mathfrak{m}} \to R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)_{\mathfrak{m}}$$

We write $K^p = K^{S_{st}\cup\{p\}}K^p_{S_{st}}$ for $K^{S_{st}\cup\{p\}} \subset \mathbf{G}(\mathbb{A}_f^{S_{st}\cup\{p\}})$. Taking colimits on both sides as $K^p_{S_{st}} \to \{1\}$, we see that $\Theta_{\mathfrak{m}}$ induces a map:

$$(R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^{S_{st}\cup\{p\}}, \mathcal{L}_\xi))_{\mathfrak{m}} \to R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^{S_{st}\cup\{p\}}}, \mathcal{L}_\xi)_{\mathfrak{m}}$$

We consider the projection on the LHS (resp. RHS) sides to the summand, where $\mathbf{G}'$ (resp. $\mathbf{G}$) acts via an unramified twist of the Steinberg representation at all places in S_{st}. This gives us a map:

$$\Theta^{st}_{\mathfrak{m}} : (R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^{S_{st}\cup\{p\}}, \mathcal{L}_\xi))^{st}_{\mathfrak{m}} \to R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^{S_{st}\cup\{p\}}}, \mathcal{L}_\xi)^{st}_{\mathfrak{m}}$$

The key point is now, by analyzing the simple twisted trace formula of Kottwitz-Shelstad [KS99] and stable trace formulas of Arthur [Art02], we can prove a strong multiplicity one type result (Proposition 5.4), for cuspidal automorphic representations that are unramified twists of Steinberg at some sufficiently large non-empty set of places and regular of weight ξ at infinity. This implies that the automorphic representations of $\mathbf{G}$ (resp. $\mathbf{G}'$) occurring on the RHS (resp. LHS) of $\Theta^{st}_{\mathfrak{m}}$ must have local constituent at p with Langlands parameter ϕ, since we localized at the Hecke eigensystem defined by Π' at the unramified places. Since all representations in the L-packet $\Pi_\phi(G)$ are supercuspidal this will imply that the projection of $R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^{S_{st}\cup\{p\}}}, \mathcal{L}_\xi)^{st}_{\mathfrak{m}}$ to the summand where $G(\mathbb{Q}_p)$ acts via a supercuspidal representation is an isomorphism. Therefore, since we know Θ_{sc} is an isomorphism, this implies that we have an isomorphism

$$\Theta^{st}_{\mathfrak{m},sc} : (R\Gamma_c(G, b, \mu)_{sc} \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^{S_{st}\cup\{p\}}, \mathcal{L}_\xi))^{st}_{\mathfrak{m}} \xrightarrow{\simeq} R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^{S_{st}\cup\{p\}}}, \mathcal{L}_\xi)^{st}_{\mathfrak{m}}$$

Moreover, since the local constituents of the automorphic representations of $\mathbf{G}'$ at p occurring in the LHS are all in the L-packet $\Pi_\phi(J)$, we can reduce Proposition 1.4 to showing that $R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)^{st}_{\mathfrak{m}}$ is concentrated in degree 3 and has W_L-action given (up to multiplicity) by $std \circ \phi \otimes |\cdot|^{-3/2}$. This follows from the analysis carried out in Kret-Shin [KS16]. In particular, it follows from their results that this complex will be concentrated in degree 3 and that the traces of Frobenius in $\Gamma_F := Gal(\overline{F}/F)$ on the étale cohomology of the associated global Shimura variety over $\overline{F}$ are given by $std \circ \phi_{\tau_v}$, where τ_v are the local constituents of some weak transfer τ of Π' to an automorphic representation of $Res_{F/\mathbb{Q}}GSp_4 =: \mathbf{G}^*$ and ϕ_{τ_v} is the associated Gan-Takeda parameter. This allows one, up to multiplicities, to describe the Galois action on the global Shimura variety in terms of the composition $std \circ \rho_\tau$, where ρ_τ is a global $GSp_4(\overline{\mathbb{Q}}_\ell)$-valued representation of the absolute Galois group of F constructed by Sorensen [Sor10] from τ characterized by the property that $iWD(std \circ \rho_\tau)|^{F-s.s.}_{W_{F_v}} \simeq \phi_{\tau_v} \otimes |\cdot|^{-3/2}$ for all but finitely many places v of F. This would give one precisely the desired description of the W_L-action on $R\Gamma_c(G, b, \mu)[\rho]_{sc}$ if one knew that $\phi_{\tau_p} = \phi_\rho$. Since Π' is globalization of ρ, one needs to choose τ to be a strong transfer of Π' at the prime p. This latter goal is accomplished using analysis of the simple trace formula as done in Kret-Shin [KS16, Section 6] combined with the character identities proven by Chan-Gan [CG15]. These results on strong transfers also aid us in deducing the strong multiplicity one type result mentioned above.

In section 2, we give an overview of the Gan-Takeda and Gan-Tantono local Langlands correspondence, putting it in the framework of the refined local Langlands correspondence of Kaletha in preparation for applications to the Kottwitz conjecture. In section 3, we describe the Fargues-Scholze local Langlands correspondence and related ideas, giving the proof of compatibility in cases (1) and (2) and reducing case (3) to Proposition 1.4, via some properties of the spectral action discussed in section 3.2. In section 4, we discuss basic uniformization of the relevant Shimura varieties and prove the aforementioned analogue of Boyer's trick, showing that the uniformization map Θ_{sc} is an isomorphism. In section 5, we analyze the simple trace formula with fixed central character in a fashion similar to Kret-Shin [KS16] to deduce the existence of the required strong transfers, as well as combine this with analysis of the simple twisted trace formula to deduce the required strong multiplicity one result. In section 6, we apply the results of section 5 combined with results of Kret-Shin [KS16] and Sorensen [Sor10] to compute the relevant Galois action on the global Shimura variety. Finally, in section 7, we put the results of the previous sections together to prove Proposition 1.4. We then conclude with the application to the proofs of Theorem 1.1 and 1.3, as well as formally deduce compatibility for the local Langlands correspondence for Sp_4 and its non quasi-split inner form $SU_2(D)$, as constructed by Gan-Takeda [GT10] and Choiy [Cho17], respectively. We finish the section with a brief discussion of an application to the cohomology of the related (non-minuscule) local Shtuka spaces.

Conventions and Notations

For a diamond or v-stack, we freely use the formalism in [Sch18], [FS21] of ℓ-adic cohomology of diamonds and v-stacks. We will fix isomorphisms $i : \overline{\mathbb{Q}}_\ell \xrightarrow{\simeq} \mathbb{C}$ and $j : \overline{\mathbb{Q}}_p \xrightarrow{\simeq} \mathbb{C}$ and use the (geometric) normalization of local class field theory that sends the Frobenius to the inverse of the uniformizer. For a supercuspidal L-parameter, we will often abuse notation and use ϕ to denote both the L-parameter and the semisimplified parameter ϕ^{ss}, as in this case this merely corresponds to forgetting the trivially acting $SL_2(\mathbb{C})$-factor and applying the isomorphism i. Normally, in the literature the space $Sht(G,b,\mu)_\infty$ parametrizes modifications $\mathcal{E}_0 \dashrightarrow \mathcal{E}_b$ with meromorphy μ. For us, they will denote the space parametrizing modifications of type μ^{-1}. This convention limits the appearances of duals (cf. Remark 3.8).

2. Local Langlands for GSp_4 and $GU_2(D)$

2.1. **Local Langlands for GSp_4.** In this section, we will describe the local Langlands correspndence of Gan-Takeda for the group $G := GSp_4/L$, where $L/\mathbb{Q}_p$ is a finite extension. We fix a choice of Whittaker datum $\mathfrak{m} := (B, \psi)$ throughout section 2, where B is the Borel and ψ is a generic character of L.

As before, we consider the set $\Phi(G)$ of admissible homomorphisms

$$\phi : W_L \times SL_2(\mathbb{C}) \to \hat{G}(\mathbb{C}) = GSpin_5(\mathbb{C}) \simeq GSp_4(\mathbb{C})$$

taken up to $\hat{G}$-conjugacy, where W_L acts via a continuous semisimple homomorphism with respect to the discrete topology and $SL_2(\mathbb{C})$ acts via an algebraic representation. Similarly, let $\Pi(G)$ denote the isomorphism classes of smooth irreducible representations of the group $G(L)$. We can now state the main theorem of Gan-Takeda.

Theorem 2.1. *[GT11] There is a surjective finite to one map*

$$LLC_G : \Pi(GSp_4) \to \Phi(GSp_4)$$

$$\pi \mapsto \phi_\pi$$

with the following properties:

(1) *π is an (essentially) discrete series representation of $GSp_4(L)$ if and only if its L-parameter does not factor through any proper Levi subgroup of $GSp_4(\mathbb{C})$.*
(2) *Given an L-parameter ϕ, we set $S_\phi := Z_{\hat{G}}(Im(\phi))$ to be the centralizer of ϕ. The fiber $\Pi_\phi(G)$ can be naturally parametrized by the set of irreducible characters of the component group*

$$A_\phi := \pi_0(S_\phi) \simeq \pi_0(S_\phi/Z(GSp_4))$$

which is either trivial or equal to $\mathbb{Z}/2\mathbb{Z}$. When $A_\phi = \mathbb{Z}/2\mathbb{Z}$, exactly one of the two representations in $\Pi_\phi(G)$ is generic for the fixed choice of Whittaker datum, and is indexed by the trivial character of A_ϕ.

(3) *The similitude character $sim(\phi_\pi)$ of $GSp_4(L)$ is equal to the central character ω_π via the isomorphism given by local class field theory.*
(4) *Given a character χ of L^* and letting $\lambda : GSp_4 \to L^*$ be the similitude character of $GSp_4(L)$, we have, via local class field theory, that the L-parameter of $\pi \otimes (\chi \circ \lambda)$ is equal to $\phi_\pi \otimes \chi$.*
(5) *If $\pi \in \Pi(GSp_4)$ is a representation, for any smooth irreducible representation σ of $GL_r(L)$, we have that*

$$\gamma(s, \pi \times \sigma, \psi) = \gamma(s, \phi_\pi \otimes \phi_\sigma, \psi)$$

$$L(s, \pi \times \sigma, \psi) = L(s, \phi_\pi \otimes \phi_\sigma, \psi)$$

$$\epsilon(s, \pi \times \sigma, \psi) = \epsilon(s, \phi_\pi \otimes \phi_\sigma, \psi)$$

where the RHS are the Artin local factors associated to the representations of $W_L \times SL_2(\mathbb{C})$ and the LHS are the local factors of Shahidi [Sha90] with respect to the morphisms of L-groups defined in [GT11, Section 4] in the case that π is a generic supercuspidal or non-supercuspidal and are the local factors defined by Townsend [Tow13] if π is a non-generic supercuspidal representation.

The map LLC_G is uniquely determined by the properties (1),(3), and (5), where one can take $r \leq 2$ in (5).

Remark 2.1. When the paper of Gan-Takeda was released there was no good theory of L,ϵ, and γ factors for nongeneric supercuspidal representations satisfying the usual properties. (See the 10-Commandents in [LR05]) Instead, to uniquely characterize the correspondence for these representations, they use an equality between the Plancharel measure on the family of inductions from $GSpin_5(L) \times GL_r(L) \simeq GSp_4(L) \times GL_r(L)$ to $GSpin_{2r+5}(L)$. However, this theory of L,ϵ, and γ factors was later constructed by Nelson Townsend in his PhD thesis [Tow13].

We now make the following definition.

Definition 2.1. Write $std : GSp_4 \hookrightarrow GL_4$ for the standard embedding. We say an L-parameter is stable if the L-packet $\Pi_\phi(G)$ has size 1 and is endoscopic if it has size 2. Equivalently, by Theorem 2.1 (2), this is equivalent to saying that the character group $A_\phi^\vee$ of the component group A_ϕ has cardinality 1 or 2, respectively. By [GT11, Lemma 6.2], this can be characterized as follows.

- (stable) $std \circ \phi$ is an irreducible representation of $W_L \times SL_2(\mathbb{C})$. In this case, $S_\phi = Z(\hat{G}) = \mathbb{G}_m$, so A_ϕ is trivial.
- (endoscopic) $std \circ \phi \simeq \phi_1 \oplus \phi_2$, where the $\phi_i : W_L \to GL_2(\overline{\mathbb{Q}}_\ell)$ for $i = 1, 2$ are distinct irreducible 2-dimensional representations of $W_L \times SL_2(\mathbb{C})$ with $det(\phi_1) = det(\phi_2)$. In this case, $A_\phi \simeq \mathbb{Z}/2\mathbb{Z}$. More specifically, one can compute that, under the identification $GSp_4(\mathbb{C}) \simeq (GL_2(\mathbb{C}) \times GL_2(\mathbb{C}))^0 = \{(g_1, g_2) \in GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) | det(g_1) = det(g_2)\}$, one has an identification

$$S_\phi \simeq \{(a, b) \in \mathbb{C}^* \times \mathbb{C}^* | a^2 = b^2\} \subset (GL_2(\mathbb{C}) \times GL_2(\mathbb{C}))^0$$

  where the center $Z(GSp_4)(\mathbb{C}) \simeq \mathbb{C}^*$ embeds diagonally.

For an L-parameter ϕ, we see that the size of the L-packet $\Pi_\phi(G)$ is at most 2, this allows us to subdivide into three cases:

(1) The L-packet $\Pi_\phi(G)$ does not contain any supercuspidal representations.
(2) The L-packet $\Pi_\phi(G)$ contains one supercuspidal and one non-supercuspidal.
(3) The L-packet contains only supercuspidals.

Case (2) is where the parameter ϕ does not factor through a Levi-subgroup, but its semisimplification ϕ^{ss} as defined in section 1 does. Case (2) does not occur when the parameter is stable, by definition. The relevant case is when the parameter is endoscopic. To understand this, we let $\nu(n)$ denote the unique n-dimension irreducible representation of $SL_2(\mathbb{C})$ then there are two cases:

(1) (Saito-Kurokawa Type) We have $std \circ \phi = \phi_0 \oplus \chi \boxtimes \nu(2)$, where ϕ_0 is a 2-dimensional irreducible representation of W_L and χ is a character, with $\chi^2 = det(\phi_0)$. Therefore, the semisimplification ϕ^{ss} satisfies: $std \circ \phi^{ss} = \phi_0 \oplus \chi \otimes |\cdot|^{\frac{1}{2}} \oplus \chi \otimes |\cdot|^{-\frac{1}{2}}$.
(2) (Howe-Piatetski–Shapiro Type) We have $std \circ \phi = \chi_1 \boxtimes \nu(2) \oplus \chi_2 \boxtimes \nu(2)$, where χ_1 and χ_2 are distinct characters of W_L satisfying $\chi_1^2 = \chi_2^2$. Therefore, the semisimplification ϕ^{ss} satisfies: $std \circ \phi^{ss} = \chi_1 \otimes |\cdot|^{\frac{1}{2}} \oplus \chi_1 \otimes |\cdot|^{-\frac{1}{2}} \oplus \chi_2 \otimes |\cdot|^{\frac{1}{2}} \oplus \chi_2 \otimes |\cdot|^{-\frac{1}{2}}$.

Remark 2.2. (1) The terminology here is explained by Arthur's classification [Art04] of the global automorphic representations of GSp_4 appearing in the papers [Kur78] and [HP79], respectively.
(2) We will mention in the next section how to distinguish these two cases via the number of supercuspidals in the L-packet $\Pi_{\phi}(GU_2(D))$ defined by the Gan-Tantono local Langlands correspondence.

Case (3) is the situation where the parameter ϕ is supercuspidal as defined in the introduction. In particular, in the supercuspidal case the restriction of the parameter ϕ to the $SL_2(\mathbb{C})$ factor is trivial, so the irreducible representations occurring in the decomposition of $std \circ \phi$ are just representations of W_L.

For the purposes of applying the weak form of the Kottwitz Conjecture proven in Hansen-Kaletha-Weinstein [HKW21], we formulate this correspondence in terms of the refined local Langlands of Kaletha [Kal16] with respect to the fixed choice of Whittaker datum $\mathfrak{m}$. Now, given a parameter ϕ that is either mixed supercuspidal or supercuspidal, we have by Theorem 2.1 (2) a correspondence between the L-packet $\Pi_{\phi}(G)$ and the set of irreducible characters $A_{\phi}^{\vee}$. This in turn gives rise to an irreducible character of the group S_{ϕ} via the composition:

$$S_{\phi} \to \pi_0(S_{\phi}) = A_{\phi}$$

This allows us to make the following definition.

Definition 2.2. For ϕ a supercuspidal or mixed-supercuspidal parameter ϕ as above and $\pi \in \Pi_{\phi}(G)$, we denote the character of S_{ϕ} given by the previous composition by τ_{π}.

2.2. **Local Langlands for $GU_2(D)$.** In this section, we describe the local Langlands correspondence for the unique non-split inner form $J = GU_2(D)$, the group of similitudes of the unique 2-dimensional Hermitian vector space over the quaternion division algebra D/L. As in the previous section, we let $\Pi(J)$ denote the set of irreducible admissible representations of J, and $\Phi(J)$ be the set of L-parameters of J. This is a subset of the previous set $\Phi(GSp_4)$ as we will now explain. J has a unique up to conjugacy minimal parabolic whose Levi factor is

$$D^* \times GL_1$$

This defines a form of the Siegel parabolic of GSp_4 and it determines a dual parabolic subgroup $P^{\vee}(\mathbb{C})$ in the dual group $GSp_4(\mathbb{C})$ of $GU_2(D)$. This is the Heisenberg parabolic subgroup of $GSp_4(\mathbb{C})$, its conjugacy class is said to be relevant for J while all other conjugacy classes of proper parabolics are said to be irrelevant. We say $\phi \in \Phi(GSp_4)$ is relevant if it does not factor through any irrelevant parabolic subgroups of $GSp_4(\mathbb{C})$. We define $\Phi(J)$ to be the subset of relevant ϕ in $\Phi(GSp_4)$. We set $B_{\phi} := \pi_0(Z_{Sp_4}(Im(\phi)))$. One has an exact sequence:

$$B_{\phi} \to A_{\phi} \to \langle \pm 1 \rangle \to 0$$

Implying that one has an injection on the group of irreducible characters $\hat{A}_\phi \hookrightarrow \hat{B}_\phi$, which identifies $\hat{A}_\phi$ as the subgroup of (index at most 2) of characters trivial on the image of the center $Z(Sp_4)(\mathbb{C})$. One can check that $\hat{B}_\phi \neq \hat{A}_\phi$ if and only if ϕ is relevant for $GU_2(D)$. Now we can state the main theorem of Gan and Tantono.

Theorem 2.2. *[GT14] There is a natural surjective finite-to-one map*

$$LLC_J : \Pi(GU_2(D)) \to \Phi(GU_2(D))$$
$$\rho \mapsto \phi_\rho$$

with the following properties:

1. *ρ is an (essentially) discrete series representation of $GU_2(D)$ if and only if its parameter ϕ_ρ does not factor through any proper Levi subgroup of $GSp_4(\mathbb{C})$.*
2. *For an L-parameter ϕ, the fiber $\Pi_\phi(J)$ can be naturally parametrized by the set $\hat{B}_\phi \setminus \hat{A}_\phi$. This set has size either 1 or 2.*
3. *The similitude character $sim(\phi_\rho)$ of ϕ_ρ is equal to the central character ω_ρ of ρ, via the isomorphism given by local class field theory.*
4. *Given a character χ of L^* and letting $\lambda : GU_2(D) \to L^*$ be the similitude character of $GU_2(D)$, we have, via local class field theory, that the L-parameter of $\rho \otimes (\chi \circ \lambda)$ is equal to $\phi_\rho \otimes \chi$.*
5. *If $\rho \in \Pi(GU_2(D))$ is a non-supercuspidal representation then, for any smooth irreducible representation σ of $GL_r(L)$, we have that*
$$\gamma(s, \rho \times \sigma, \psi) = \gamma(s, \phi_\rho \otimes \phi_\sigma, \psi)$$
$$L(s, \rho \times \sigma, \psi) = L(s, \phi_\rho \otimes \phi_\sigma, \psi)$$
$$\epsilon(s, \pi \times \sigma, \psi) = \epsilon(s, \phi_\rho \otimes \phi_\sigma, \psi)$$
where the RHS are the Artin local factors associated to the representations of $W_L \times SL_2(\mathbb{C})$ and the LHS are the local factors of Shahidi, as defined in [GT14, Section 8].
6. *Suppose that ρ is a supercuspidal representation. For any irreducible supercuspidal representation σ of $GL_r(L)$ with L-parameter ϕ_σ, if $\mu(s, \rho \boxtimes \sigma, \psi)$ denotes the Plancharel measure associated to the family of induced representations $I_P(\pi \boxtimes \sigma, s)$ on $GSpin_{r+4,r+1}$, where we have regarded $\rho \boxtimes \sigma$ as a representation of the Levi subgroup $GSpin_{4,1} \times GL_r \simeq GU_2(D) \times GL_r$, then $\mu(s, \rho \boxtimes \sigma)$ is equal to*

$$\gamma(s, \phi_\rho^\vee \otimes \phi_\sigma, \psi) \cdot \gamma(-s, \phi_\rho \otimes \phi_\sigma^\vee, \overline{\psi}) \cdot \gamma(2s, Sym^2\phi_\sigma \otimes sim(\phi_\rho)^{-1}, \psi)) \cdot \gamma(-2s, Sym^2\phi_\sigma^\vee \otimes sim(\phi_\rho), \overline{\psi})$$

The map LLC_J is uniquely determined by the properties (1), (3), (5), and (6), with $r \leq 4$ in (5) and (6).

We now further elaborate on the structure of the L-packets $\Pi_\phi(J) := LLC_J^{-1}(\phi)$ in the case where the parameter ϕ is mixed supercuspidal. If the parameter ϕ is of this form, then, it follows from [GT14, Proposition 5.4] and the description of LLC_J provided in [GT14, Section 7], that the L-packet $\Pi_\phi(J)$ has following structure, as alluded to in Remark 2.2 (2).

1. (Saito-Kurokawa Type) The L-packet $\Pi_\phi(J) = \{\rho_{disc}, \rho_{sc}\}$ contains one supercuspidal representation ρ_{sc} and one non-supercuspidal representation ρ_{disc}.
2. (Howe-Piatetski–Shapiro Type) The L-packet $\Pi_\phi(J) = \{\rho^1_{disc}, \rho^2_{disc}\}$ contains no supercuspidal representations.

We now would also like to briefly comment on the structure of the set $\hat{B}_\phi \setminus \hat{A}_\phi$, confirming the expectation that the size of the L-packets $\Pi_\phi(G)$ and $\Pi_\phi(J)$ is always the same.

- (stable) In the case that the L-parameter ϕ is stable, we have that $B_\phi = \mathbb{Z}/2\mathbb{Z}$ and, as noted in section 2.1, $A_\phi = 1$. This means the set $\hat{B}_\phi \setminus \hat{A}_\phi$ consists of one element corresponding to the non-trivial character.

- (endoscopic) In the case that the parameter ϕ is endoscopic, we have that the decomposition $std \circ \phi \simeq \phi_1 \oplus \phi_2$ induces an exact sequence

$$A_\phi = \mathbb{Z}/2\mathbb{Z} \xrightarrow{\Delta} B_\phi = Z(SL_2) \times Z(SL_2) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

and so the set $\hat{B}_\phi \setminus \hat{A}_\phi$ has size 2 and is indexed by two characters η_{+-} and η_{-+} each non-trivial on one of the two $\mathbb{C}^*$-factors under the isomorphism

$$S_\phi \simeq \{(a,b) \in \mathbb{C}^* \times \mathbb{C}^* | a^2 = b^2\} \subset (GL_2(\mathbb{C}) \times GL_2(\mathbb{C}))^0$$

from section 2.1.

We now wish to put this local Langlands correspondence for the inner form in the context of the refined local Langlands correspondence of Kaletha [Kal16]. We consider the Kottwitz set $B(G)$ [Kot85; RR96] and let $b \in B(G)$ be the basic element whose associated σ-centralizer $J_b = J$. This is the basic element whose slope homorphism is the dominant rational cocharacter of G given by $(1/4, 1/4, 1/4, 1/4)$. Let $Z(GSp_4) \simeq \mathbb{G}_m$ be the center. We recall that we have an isomorphism $\pi_1(G) \simeq X_*(Z(\hat{G})) \simeq \mathbb{Z}$ and that the κ-invariant of b is sent to the element $1 \in \mathbb{Z}$ under this isomorphism. This indexes the identity representation of $\mathbb{G}_m$, denoted $id_{\mathbb{G}_m}$. Thus, given a discrete parameter (= supercuspidal or mixed supercuspidal) $\phi : W_L \times SL_2(\mathbb{C}) \to GSp_4(\mathbb{C})$, the refined local Langlands correspondence asserts bijections

$$\Pi_\phi(G) \longleftrightarrow \{\text{irreducible algebraic representations } \tau \text{ of } S_\phi \text{ s.t } \tau|_{Z(\hat{G})} = \mathbf{1}\}$$

$$\pi \mapsto \tau_\pi$$

$$\Pi_\phi(J) \longleftrightarrow \{\text{irreducible algebraic representations } \tau \text{ of } S_\phi \text{ s.t } \tau|_{Z(\hat{G})} = id_{\mathbb{G}_m}\}$$

$$\rho \mapsto \tau_\rho$$

where $\mathbf{1}$ is the trivial representation. In section 2.1, we saw how for $\pi \in \Pi_\phi(G)$ to construct the desired τ_π. In the case of the inner form, the situation is a bit more tricky. Consider $\rho \in \Pi(J)$ with associated L-parameter ϕ_ρ. If ϕ_ρ is stable then $S_\phi = \mathbb{G}_m$ and τ_ρ is simply $id_{\mathbb{C}^*}$. If ϕ is endoscopic, then, as noted in section 2.1, we have an inclusion:

$$Z(GSp_4)(\mathbb{C}) = \mathbb{C}^* \xrightarrow{\Delta} S_\phi \simeq \{(a,b) \in \mathbb{C}^* \times \mathbb{C}^* | a^2 = b^2\}$$

We consider the characters $\tau_i : S_\phi \to \mathbb{C}^*$ for $i = 1, 2$ given by projecting to the first and second coordinate. After passing to the component groups, these give rise to the non-trivial and trivial character of A_ϕ and also satisfy the property that $\tau_i|_{\mathbb{C}^*} = id_{\mathbb{C}^*}$ on the diagonally embedded center as desired. The exact matching between ρ_{+-} and ρ_{-+} with τ_1 and τ_2 is pinned down by the endoscopy character relations as proven by Chan-Gan [CG15] with respect to explicit transfer factors determined by the fixed Whittaker datum $\mathfrak{m}$; however, for our purposes the choice ends up being irrelevant, so we simply make some choice and denote the representations in the L-packet $\Pi_\phi(J)$ corresponding to the projections τ_1 and τ_2 by ρ_1 and ρ_2, respectively. Similarly, for the representations obtained by pre-composing a character with the composition

$$S_\phi \to A_\phi$$

we denote the elements of the L-packet $\Pi_\phi(G)$ corresponding to the trivial (non-trivial) character of A_ϕ by π^+ (resp. π^-). We note that, by Theorem 2.1 (2), π^+ can be characterized by the unique $\mathfrak{m}$-generic representation of this L-packet.

Definition 2.3. Given a supercuspidal or mixed supercuspidal L-parameter ϕ as above and $\rho \in \Pi_\phi(J)$, we let τ_ρ be the irreducible representation of S_ϕ associated to it via the matching described above. Given $\pi \in \Pi_\phi(G)$ and $\rho \in \Pi_\phi(J)$, we set

$$\delta_{\pi,\rho} := \tau_\pi^\vee \otimes \tau_\rho$$

where $\tau_\pi^\vee$ denotes the contragredient.

Remark 2.3. Changing the choice of Whittaker datum scales the representations by a 1-dimensional character of S_ϕ that is trivial when restricted to the center, so in particular this pairing is independent of the choice of Whittaker datum.

3. The Fargues-Scholze Local Langlands Correspondence

We will now discuss the Fargues-Scholze local Langlands correspondence and deduce compatibility in the cases where the Gan-Takeda/Gan-Tantono parameter is not supercuspidal. We will then conclude by reducing the question of compatibility in the supercuspidal case to Proposition 1.4.

3.1. **Overview of the Fargues-Scholze Local Langlands Correspondence.** For now, let G be any connected reductive group over $\mathbb{Q}_p$. Since we are going to be using geometric Satake, we fix a choice of the square root of p in $\overline{\mathbb{Q}}_\ell$, so that half Tate-twists are well-defined. Fargues-Scholze [FS21] consider the moduli space of G-bundles on the Fargues-Fontaine curve X, denoted Bun_G. This moduli space is an Artin v-stack (in the sense of [FS21, Section IV.I]) and has the structure that the underlying points of its topological space $|Bun_G|$ are in natural bijection with elements of the Kottwitz set $B(G)$, where the slopes of the G-isocrystal associated to $b \in B(G)$ are the negatives of the slopes of the associated vector bundle $\mathcal{E}_b$ and the specializations between points of $|Bun_G|$ is dictated by the partial ordering on $B(G)$ induced by the kappa invariant and the slope homomorphism [Vie21]. In particular, the connected components of Bun_G are in bijection with $B(G)_{basic} \xrightarrow{\simeq} \pi_1(G)_\Gamma$. Specifically, for any $b \in B(G)_{basic}$, there is a unique open Harder-Narasimhan strata $Bun_G^b \subset Bun_G$ dense inside the associated connected component. We recall that the elements of $B(G)_{basic}$ parametrize extended pure inner forms of G, via sending an element $b \in B(G)_{basic}$ to its σ-centralizer $J_b/\mathbb{Q}_p$. For such a basic b, we have an identification $Bun_G^b \simeq [*/\underline{J_b(\mathbb{Q}_p)}] =: B\underline{J_b(\mathbb{Q}_p)}$ of the HN-strata defined by b and the classifying stack of $J_b(\mathbb{Q}_p)$. For any Artin v-stack X, Fargues-Scholze define a triangulated category $D_{\blacksquare}(X, \overline{\mathbb{Q}}_\ell)$ of solid $\overline{\mathbb{Q}}_\ell$-sheaves [FS21, Section VII.1] and isolate a nice full subcategory $D_{lis}(X, \overline{\mathbb{Q}}_\ell) \subset D_{\blacksquare}(X, \overline{\mathbb{Q}}_\ell)$ of étale-lisse $\overline{\mathbb{Q}}_\ell$-sheaves [FS21, Section VII.6.], which may be roughly thought of as the unbounded derived category of étale $\overline{\mathbb{Q}}_\ell$ sheaves on X, where one has made an enlargement to capture information about the topology of p-adic groups. In any case, the key point for us is that we have the following basic result.

Lemma 3.1. *[FS21, Proposition VII.7.1] There is an equivalence of categories*

$$D_{lis}(B\underline{J_b(\mathbb{Q}_p)}), \overline{\mathbb{Q}}_\ell) \simeq D(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$$

where the RHS denotes unbounded derived category of smooth $J_b(\mathbb{Q}_p)$-representations with coefficients in $\overline{\mathbb{Q}}_\ell$. Under this equivalence, Verdier duality corresponds to smooth duality.

Remark 3.1. The main reason for constructing this category D_{lis} is that, if one were to take the usual definition for the category of étale $\overline{\mathbb{Q}}_\ell$-sheaves on $B\underline{J_b(\mathbb{Q}_p)}$, this equivalence would no longer be true. In particular, one would obtain the bounded derived category of representations of $J_b(\mathbb{Q}_p)$ admitting a $J_b(\mathbb{Q}_p)$-stable $\overline{\mathbb{Z}}_\ell$-lattice, where the representation is continuous with respect to the ℓ-adic topology on the target. This would limit the scope of the Fargues-Scholze LLC as, in general, one wants to consider smooth $\overline{\mathbb{Q}}_\ell$-representations of $J_b(\mathbb{Q}_p)$, and hence the need for the enlargement of the derived category to D_{lis}.

Lemma 3.1 tells us that, given an irreducible smooth representation π of $G(\mathbb{Q}_p)$, we can consider the associated sheaf, denoted $\mathcal{F}_\pi$, on $Bun_G^{\mathbf{1}}$ the open HN-strata corresponding to the trivial element $\mathbf{1} \in B(G)$, and take the extension by zero along the open inclusion $j_!(\mathcal{F}_\pi)$. This realizes the representation π in terms of a sheaf on the moduli space Bun_G in an analogous way to how the function-sheaf dictionary realizes cuspidal automorphic forms as functions associated to sheaves in the context of curves over finite fields. Following V. Lafforgue [Laf18], Fargues and Scholze construct a semisimple Weil parameter associated to this sheaf by looking at the action of the excursion algebra on this category $D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)$. This relies on a form of the geometric Satake correspondence for the B_{dR}^+-affine Grassmannians. For any finite set I, let X^I be the product of I-copies of the diamond $X = Spd(\breve{\mathbb{Q}}_p)/Frob^{\mathbb{Z}}$. We then have the Hecke stack

$$\begin{tikzcd} & Hck \arrow[dl, "h^{\leftarrow}"'] \arrow[dr, "h^{\rightarrow}\times supp"] & \\ Bun_G & & Bun_G \times X^I \end{tikzcd}$$

defined as the functor that parametrizes, for S a perfectoid space in characteristic p together with a map $S \to X^I$ defining a tuple of Cartier divisors in the relative Fargues-Fontaine X_S over S, corresponding to characteristic 0 untilts $S_i^\sharp$ for $i \in I$ of S, a pair of G-torsors $\mathcal{E}_1$, $\mathcal{E}_2$ together with an isomorphism

$$\beta : \mathcal{E}_1|_{X_S \setminus \bigcup_{i\in I} S_i^\sharp} \xrightarrow{\simeq} \mathcal{E}_2|_{X_S \setminus \bigcup_{i\in I} S_i^\sharp}$$

where $h^{\leftarrow}((\mathcal{E}_1, \mathcal{E}_2, i, (S_i^\sharp)_{i\in I})) = \mathcal{E}_1$ and $h^{\rightarrow} \times supp((\mathcal{E}_1, \mathcal{E}_2, \beta, (S_i^\sharp)_{i\in I})) = (\mathcal{E}_2, (S_i^\sharp)_{i\in I})$. We set ${}^LG^I$ to be I-copies of the Langlands dual group of G, i.e ${}^LG = Q \ltimes \hat{G}(\overline{\mathbb{Q}}_\ell)$, where $\hat{G}$ is the reductive group having dual root datum to G and is viewed as a reductive group over $\overline{\mathbb{Q}}_\ell$. The Weil group acts on $\hat{G}$ via the induced action on root datum through some finite quotient Q, which we now fix. Let $Rep_{\overline{\mathbb{Q}}_\ell}({}^LG^I)$ denote the category of algebraic $\overline{\mathbb{Q}}_\ell$-representations of I-copies of ${}^LG$. For each element $W \in Rep_{\overline{\mathbb{Q}}_\ell}({}^LG^I)$, the geometric Satake correspondence of Fargues-Scholze [FS21, Chapter VI] furnishes a solid $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{S}_W$ on Hck. This allows us to define Hecke operators.

Definition 3.1. For each $W \in Rep_{\overline{\mathbb{Q}}_\ell}({}^LG^I)$, we define the Hecke operator

$$T_W : D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell) \to D_{\blacksquare}(Bun_G \times X^I)$$
$$A \mapsto R(h^{\rightarrow} \times supp)_\natural(h^{\leftarrow *}(A) \otimes^{\mathbb{L}} \mathcal{S}_W)$$

$\mathcal{S}_W$ is a solid $\overline{\mathbb{Q}}_\ell$-sheaf and the functor $R(h^{\rightarrow} \times supp)_\natural$ is the natural push-forward. I.e the left adjoint to the restriction functor in the category of solid $\overline{\mathbb{Q}}_\ell$-sheaves [FS21, Proposition VII.3.1].

Remark 3.2. These satisfy various compatibilities with respect to composition and restriction to the diagonal. In particular, given two representations $V, W \in Rep_{\overline{\mathbb{Q}}_\ell}({}^LG)$, we have that

$$(T_V \times id)(T_W)(\cdot)|_\Delta \simeq T_{V\otimes W}(\cdot)$$

where $\Delta : X \to X^2$ is the diagonal map.

We then consider $D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)^{BW^I_{\mathbb{Q}_p}}$, the category of objects in $D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)$ with continuous action by $W^I_{\mathbb{Q}_p}$. Examples of objects in this category are objects of $D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)$ tensored by a continuous representation of $W^I_{\mathbb{Q}_p}$, for a more precise description see [FS21, Section IX.1]. With this in hand, we then have the following theorem of Fargues-Scholze.

Theorem 3.2. *[FS21, Theorem I.7.2, Proposition IX.2.1, Corollary IX.2.3] The Hecke operator T_W for $W \in Rep_{\overline{\mathbb{Q}}_\ell}({}^LG^I)$*

$$T_W : D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell) \to D_{\blacksquare}(Bun_G \times X^I)$$

induces a functor

$$D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell) \to D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)^{BW^I_{\mathbb{Q}_p}}$$

and the induced endofunctors of $D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)$ given by forgetting the Weil group action preserve compact and ULA objects.

Remark 3.3. This should be thought of as a manifestation of Drinfeld's Lemma, where (roughly) the étale fundamental group of $Spd(\breve{\mathbb{Q}}_p)/Frob^{\mathbb{Z}} = X$ should be the same as $W_{\mathbb{Q}_p}$.

From now on, when talking about Hecke operators we shall always refer to this induced functor, which we will also abusively denote by T_W. Theorem 3.2 has direct implications for the cohomology of local Shimura varieties. To study this, consider a minuscule cocharacter μ with field of definition E and let $b \in B(G, \mu)$ be the unique basic element in the μ-admissible locus (See [RV14, Definition 2.3]). We say that the triple (G, b, μ) defines a local Shimura datum in the sense of Rapoport-Viehmann [RV14]. Attached to such a data, Scholze-Weinstein [SW20] construct a tower of spatial diamonds

$$p_K : (Sht(G, b, \mu)_K)_{K \subset G(\mathbb{Q}_p)} \to Spd(\breve{E})$$

for varying open compact $K \subset G(\mathbb{Q}_p)$. This is obtained by considering the space $Sht(G, b, \mu)_\infty$ which parametrizes modifications $\mathcal{E}_b \to \mathcal{E}_0$ with meromorphy bounded by μ, where $\mathcal{E}_b$ (resp. $\mathcal{E}_0$) is the bundle corresponding to $b \in B(G)$ (resp. the trivial bundle) on the Fargues-Fontaine curve. It has commuting actions by $G(\mathbb{Q}_p)$ and $J_b(\mathbb{Q}_p)$ given by acting via automorphisms on $\mathcal{E}_0$ and $\mathcal{E}_b$, respectively. The tower is then given by considering the quotients of this space for varying open compact $K \subset G(\mathbb{Q}_p)$ under the action of $G(\mathbb{Q}_p)$.

Definition 3.2. Let $Sht(G, b, \mu)_{K,\mathbb{C}_p}$ be the base-change of the above tower to $\mathbb{C}_p$. We define the complex

$$R\Gamma_c(G, b, \mu) := \mathrm{colim}_{K \to 1} R\Gamma_c(Sht(G, b, \mu)_{K,\mathbb{C}_p}, \overline{\mathbb{Q}}_\ell)$$

it carries commuting actions of $W_E \times G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$, where W_E is the Weil group of E. A priori it only has an action by the inertia group, but this space admits a non-effective Frobenius descent datum. We then define, for ρ a smooth admissible $J_b(\mathbb{Q}_p)$-representation, the complex

$$R\Gamma_c(G, b, \mu)[\rho] := R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \rho$$

where $\mathcal{H}(J_b)$ is the usual smooth Hecke algebra. We also define

$$R\Gamma^\flat_c(G, b, \mu)[\rho] := R\mathcal{H}om_{J_b(\mathbb{Q}_p)}(R\Gamma_c(G, b, \mu), \rho)[-2d](-d)$$

Similarly, for π a smooth admissible $G(\mathbb{Q}_p)$-representation, we define $R\Gamma_c(G, b, \mu)[\pi]$ and $R\Gamma^\flat_c(G, b, \mu)[\pi]$.

Remark 3.4. We note that, by Hom-Tensor duality, $R\mathcal{H}om_{\overline{\mathbb{Q}}_\ell}(R\Gamma_c(G, b, \mu)[\rho], \overline{\mathbb{Q}}_\ell)[-2d](-d)$ is isomorphic to $R\Gamma^\flat_c(G, b, \mu)[\rho^*]$, where ρ^* is the contragredient. We will end up using both of these cohomology groups throughout this manuscript. The former is more natural from the point of view of basic uniformization, while the latter is disposable to the results of Hansen-Kaletha-Weinstein [HKW21] on the Kottwitz conjecture.

To study these complexes, we specialize the above discussion of Hecke operators to the case where $W = V_{\mu^{-1}}$ is specified by the highest weight representation of highest weight μ^{-1} a dominant inverse of μ and $I = \{*\}$ is a singleton. The sheaf $\mathcal{S}_W$ will then be supported on the closed subspace $Hck_{\leq \mu^{-1}} = Hck_{\mu^{-1}}$ of Hck, parametrizing modifications with meromorphy bounded by or equal to μ^{-1}, where the equality follows by the minuscule assumption. The space $Hck_{\mu^{-1}}$ is cohomologically smooth of dimension $d := \langle 2\rho_G, \mu \rangle$, and the sheaf $\mathcal{S}_W$, as in the geometric Satake correspondence of [MV07], behaves like the intersection cohomology of this space, so we have $\mathcal{S}_W \simeq \overline{\mathbb{Q}}_\ell[d](\frac{d}{2})$. This implies that, to study the action of the Hecke operator T_W on Bun_G, we can look at the restriction of the diagram defining the Hecke correspondence to this subspace

$$\begin{tikzcd} & Hck_{\mu^{-1}} \arrow[dl, "h^{\leftarrow}_{\mu^{-1}}"'] \arrow[dr, "h^{\rightarrow}_{\mu^{-1}} \times supp"] & \\ Bun_G & & Bun_G \times Spd(\hat{E})/Frob^{\mathbb{Z}} \end{tikzcd}$$

In particular, we have an isomorphism:

$$T_{\mu^{-1}}(A) := T_W(A) \simeq R(h^{\rightarrow}_{\mu^{-1}} \times supp)_{\natural}(h^{\leftarrow *}_{\mu^{-1}}(A))[d](\frac{d}{2})$$

Now consider a smooth admissible representation π of $G(\mathbb{Q}_p)$ and apply the Hecke operator to the sheaf:

$$j_!(\mathcal{F}_\pi)$$

Then the fiber of $Hck_{\mu^{-1}}$ of $h^{\leftarrow}_{\mu^{-1}}$ over $Bun_G^{\mathbf{1}}$ is identified with

$$[Gr_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}]$$

the Schubert cell/variety associated to μ^{-1} in the B_{dR}^{+}-affine Grassmannian, quotiented out by $G(\mathbb{Q}_p)$ acting on the trivial bundle via automorphisms. The sheaf

$$T_{\mu^{-1}}j_!(\mathcal{F}_\pi)$$

is then supported on the HN-strata given by the Kottwitz elements in $B(G,\mu)$ since, by [Rap18, Proposition A.9], any G-bundle occurring as a modification of type μ^{-1} of the trivial bundle has associated Kottwitz element lying in this set. We then consider the restriction

$$j_b^{*}T_{\mu^{-1}}j_!(\mathcal{F}_\pi) \in D(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)^{BW_E}$$

where $j_b : Bun_G^b \hookrightarrow Bun_G$ is the inclusion of the open HN-strata defined by b. This will be supported on the Newton strata

$$[Gr^b_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}]$$

parametrizing modifications of type μ^{-1} of the trivial bundle such that the resulting bundle has associated Kottwitz element of type b after pulling back to each geometric point, modulo automorphisms of the trivial bundle. The space $Sht(G,b,\mu)_\infty$ defined above is a pro-étale $\underline{J_b(\mathbb{Q}_p)}$-torsor with respect to the $J_b(\mathbb{Q}_p)$-action by automorphisms of $\mathcal{E}_b$

$$Sht(G,b,\mu)_\infty \to Gr^b_{G,\mu^{-1}}$$

over this Newton strata. Using this description of the infinite level Shimura variety, it then follows from base change and the fact that the sheaves $\mathcal{S}_W$ are ULA over X (See [FS21, Chapter IX.3] for details) that we have an isomorphism

$$R\Gamma_c(G,b,\mu)[\pi][d](\frac{d}{2}) \simeq j_b^{*}T_{\mu^{-1}}j_!(\mathcal{F}_\pi) \in D(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)^{BW_E}$$

of $J_b(\mathbb{Q}_p)\times W_E$-modules. For our purposes, it will be also useful to have a description of the ρ-isotypic part of this cohomology in terms of Hecke operators, for ρ a smooth irreducible representation of J_b. In particular, analysis similar to the above gives us an isomorphism

$$R\Gamma_c(G,b,\mu)[\rho][d](\frac{d}{2}) \simeq j_{\mathbf{1}}^{*}T_\mu j_{b!}(\mathcal{F}_\rho)$$

as $G(\mathbb{Q}_p) \times W_E$-modules. We record these two isomorphisms as a corollary of the above discussion.

Corollary 3.3. *Given a local Shimura datum (G,b,μ) as above and π (resp. ρ) a smooth irreducible representation of $G(\mathbb{Q}_p)$ (resp. $J_b(\mathbb{Q}_p)$). There exists an isomorphism*

$$R\Gamma_c(G,b,\mu)[\rho][d](\frac{d}{2}) \simeq j_{\mathbf{1}}^{*}T_\mu j_{b!}(\mathcal{F}_\rho)$$

of complexes of $G(\mathbb{Q}_p) \times W_E$-modules and an isomorphism

$$R\Gamma_c(G,b,\mu)[\pi](d)(\frac{d}{2}) \simeq j_b^{*}T_{\mu^{-1}}j_{\mathbf{1}!}(\mathcal{F}_\pi)$$

of complexes of $J_b(\mathbb{Q}_p) \times W_E$-modules.

We have the following basic structural result which, in more generality, follows from the analysis in Fargues-Scholze, but, in the case of a local Shimura datum, also follows from standard finiteness results for rigid spaces. In particular, one can show the following.

Theorem 3.4. *[FS21, Corollary I.7.3] For a local Shimura datum (G, b, μ) as above, the cohomology groups of $R\Gamma_c^{\flat}(G, b, \mu)[\rho]$ and $R\Gamma_c(G, b, \mu)[\rho]$ are valued in admissible $G(\mathbb{Q}_p)$-representations of finite length admitting a smooth action of W_E. Moreover, they are concentrated in degrees $0 \leq i \leq 2d$.*

Remark 3.5. A sheaf $\mathcal{F} \in D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)$ being ULA is equivalent to its stalks at different HN-strata being valued in complexes of smooth admissible representations [FS21, Theorem V.7.1, Proposition VII.7.9], so indeed the admissibility of the above complex is a consequence of Theorem 3.2 and Corollary 3.3.

Fargues-Scholze use the endofunctors defined by the Hecke algebra on $D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)$ to define the excursion algebra.

Definition 3.3. For a finite set I, a representation $W \in Rep_{\overline{\mathbb{Q}}_\ell}({}^L G^I)$, maps $\alpha : \overline{\mathbb{Q}}_\ell \to \Delta^* W$ and $\beta : \Delta^* W \to \overline{\mathbb{Q}}_\ell$, and elements $(\gamma_i)_{i \in I} \in W_{\mathbb{Q}_p}$ for $i \in I$, one defines the excursion operator on $D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)$ to be the composition:

$$id = T_{\overline{\mathbb{Q}}_\ell} \xrightarrow{\alpha} T_{\Delta^* W} = T_W \xrightarrow{(\gamma_i)_{i \in I}} T_W = T_{\Delta^* W} \xrightarrow{\beta} T_{\overline{\mathbb{Q}}_\ell} = id$$

This defines a natural endomorphism of the identity functor on $D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)$. If one looks at the induced endofunctor given by the inclusion $D(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell) \subset D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)$ induced by the open immersion $j : Bun_G^{\mathbf{1}} \hookrightarrow Bun_G$ then one obtains a natural endomorphism of the identity functor on $D(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$. In other words, we get a family of compatible endomorphisms for all complexes of smooth representations of $G(\mathbb{Q}_p)$; namely, an element of the Bernstein center. One can verify that this excursion algebra satisfies similar properties to that considered by V. Lafforgue, so, using Lafforgue's reconstruction theorem [Laf18, Proposition 11.7], one can show the following.

Theorem 3.5. *To an irreducible smooth $\overline{\mathbb{Q}}_\ell$-representation π of $G(\mathbb{Q}_p)$ (or more generally $A \in D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)$ any Schur-irreducible object (i.e $End(A) = \overline{\mathbb{Q}}_\ell$)), there is a unique continuous semisimple map*

$$\phi_\pi^{FS} : W_{\mathbb{Q}_p} \to {}^L G(\overline{\mathbb{Q}}_\ell)$$

characterized by the property that for all I, W, α, β and $\gamma_i \in W_{\mathbb{Q}_p}$ the corresponding endomorphism of π defined above is given by multiplication by the scalar that results from the composite

$$\overline{\mathbb{Q}}_\ell \xrightarrow{\alpha} \Delta^* W = W \xrightarrow{(\phi_\pi(\gamma_i))_{i \in I}} W = \Delta^* W \xrightarrow{\beta} \overline{\mathbb{Q}}_\ell$$

By further studying the geometry of Bun_G and the Hecke stacks, one can deduce various good properties of this correspondence.

Theorem 3.6. *[FS21, Theorem I.9.6] The mapping defined above*

$$\pi \mapsto \phi_\pi^{FS}$$

enjoys the following properties:

1. *(Compatibility with Local Class Field Theory) If $G = T$ is a torus, then $\pi \mapsto \phi_\pi$ is the usual local Langlands correspondence*
2. *The correspondence is compatible with character twists, passage to contragredients, and central characters.*
3. *(Compatibility with products) Given two irreducible representations π_1 and π_2 of two connected reductive groups G_1 and G_2 over $\mathbb{Q}_p$, respectively. We have*

$$\pi_1 \boxtimes \pi_2 \mapsto \phi_{\pi_1}^{FS} \times \phi_{\pi_2}^{FS}$$

under the Fargues-Scholze local Langlands correspondence for $G_1 \times G_2$.

(4) *(Compatibility with parabolic induction) Given a parabolic subgroup $P \subset G$ with Levi factor M and a representation π_M of M, then the Weil parameter corresponding to any sub-quotient of $ind_P^G(\pi_M)$ the (normalized) parabolic induction is the composition*

$$W_{\mathbb{Q}_p} \xrightarrow{\phi^{FS}_{\pi_M}} {}^L M(\overline{\mathbb{Q}}_\ell) \to^L G(\overline{\mathbb{Q}}_\ell)$$

where the map ${}^L M(\overline{\mathbb{Q}}_\ell) \to {}^L G(\overline{\mathbb{Q}}_\ell)$ is the natural embedding.

(5) *(Compatibility with Harris-Taylor/Henniart LLC) For $G = GL_n$ or an inner form of G the Weil parameter associated to π is the (semi-simplified) parameter ϕ_π associated to π by Harris-Taylor/Henniart.*

(6) *(Compatibility with Restriction of Scalars) The above story works the same for G' a connected reductive group over any finite extension $E'/\mathbb{Q}_p$, where one then gets a Weil parameter valued on $W_{E'}$. If $G = Res_{E'/\mathbb{Q}_p}G'$ is the Weil restriction of some G'/E' then L-parameters for $G/\mathbb{Q}_p$ agree with L-parameters for G'/E' in the usual sense.*

(7) *(Compatibility with Isogenies) If $G' \to G$ is a map of reductive groups inducing an isomorphism of adjoint groups, π is an irreducible smooth representation of $G(E)$ and π' is an irreducible constituent of $\pi|_{G'(E)}$ then $\phi_{\pi'}$ is the image of ϕ_π under the induced map $\hat{G} \to \hat{G}'$.*

Remark 3.6. (1) By the restriction of scalars property, we can and will implicitly interpret the Fargues-Scholze parameter attached to a smooth irreducible representation of $Res_{L/\mathbb{Q}_p}(GSp_4)$ or $Res_{L/\mathbb{Q}_p}GU_2(D)$ as a continuous semisimple map:

$$\phi : W_L \to GSp_4(\overline{\mathbb{Q}}_\ell)$$

(2) In (5), the compatibility of the Fargues-Scholze local Langlands correspondence with the Harris-Taylor/Henniart local Langlands correpsondence for an arbitrary inner form of GL_n is not included in the paper of Fargues-Scholze [FS21]. However, it follows from the work of Hansen-Kaletha-Weinstein on the Kottwitz conjecture [HKW21, Theorem 1.0.3].

3.2. **The Spectral Action.** With these basic structural properties out of the way, we turn our attention to the "spectral action" on $D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)$, which will be very important to proving compatibility of the two correspondences in the case where the parameter is supercuspidal, as well as deducing applications to the Kottwitz conjecture. We recall that an L-parameter over $\overline{\mathbb{Q}}_\ell$ can be thought of as a continuous (not necessarily semisimple) homomorphism

$$\phi : W_{\mathbb{Q}_p} \to {}^L G(\overline{\mathbb{Q}}_\ell)$$

commuting with the natural projection to Q. One can use the classical construction of Grothendieck-Deligne to see that this coincides with the definition given in section 2 for GSp_4 after applying the isomorphism i, where the monodromy operation is recovered through the exponential of the action of $W_{\mathbb{Q}_p}$ on the ℓ-power roots of unity. Such a continuous map can be thought of as a continuous 1-cocycle $W_{\mathbb{Q}_p} \to \hat{G}(\overline{\mathbb{Q}}_\ell)$, with respect to the action of $W_{\mathbb{Q}_p}$ on $\hat{G}(\overline{\mathbb{Q}}_\ell)$. If we let $A/\mathbb{Z}_\ell$ be any $\mathbb{Z}_\ell$-algebra endowed with a topology given by writing $A = \mathrm{colim}_{A' \subset A} A'$, where A' is a finitely generated $\mathbb{Z}_\ell$-module with its ℓ-adic topology, then we can defined a moduli space, denoted $\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})$, over $\mathbb{Z}_\ell$, whose A-points are the continuous 1-cocyles $W_{\mathbb{Q}_p} \to \hat{G}(A)$ with respect to the natural action of $W_{\mathbb{Q}_p}$ on $\hat{G}(A)$. This defines a scheme considered in [Dat+20] and [Zhu20] which, by [FS21, Theorem I.9.1], can be written as a union of open and closed affine subschemes $\mathcal{Z}^1(W_{\mathbb{Q}_p}/P, \hat{G})$ as P runs through subgroups of wild inertia of W_E, where each $\mathcal{Z}^1(W_{\mathbb{Q}_p}/P, \hat{G})$ is a flat local complete intersection over $\mathbb{Z}_\ell$ of dimension $dim(G)$. This allows us to consider the Artin stack quotient $[\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})/\hat{G}]$, where $\hat{G}$ acts via conjugation. We then consider the base change to $\overline{\mathbb{Q}}_\ell$, denoted $[\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}]$ and referred to as the stack of Langlands parameters, as well as the category $Perf([\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}])$ of perfect complexes of coherent sheaves on this space.

We let $D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)^\omega$ denote the triangulated sub-category of compact objects in $D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)$ (which are precisely the objects with quasi-compact support on Bun_G and which restrict to compact objects in $D(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ for all $b \in B(G)$ by [FS21, Theorem V.4.1, Proposition VII.7.4]). We then have the key theorem of Fargues-Scholze.

Theorem 3.7. *[FS21, Corollary X.I.3] There exists a natural compactly supported $\overline{\mathbb{Q}}_\ell$-linear action of $Perf(\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G})$ on $D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)^\omega$ satisfying the property that the restriction along the map*

$$Rep_{\overline{\mathbb{Q}}_\ell}({}^LG) \to Perf([\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G})])^{BW_{\mathbb{Q}_p}}$$

induces the action of Hecke operators

$$Rep_{\overline{\mathbb{Q}}_\ell}({}^LG^I) \to End(D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)^\omega)^{BW_{\mathbb{Q}_p}^I}$$

for a varying finite index set I.

Remark 3.7. (1) Here the map

$$Rep_{\overline{\mathbb{Q}}_\ell}({}^LG) \to Perf([\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G})])^{BW_{\mathbb{Q}_p}}$$

associates to a representation V, with associated map $r_V : {}^LG(\overline{\mathbb{Q}}_\ell) \to GL(V)(\overline{\mathbb{Q}}_\ell)$ a vector bundle on $[\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G})]$ of rank equal to $dim(V)$ with $W_{\mathbb{Q}_p}$-action. This bundle, denoted C_V, has the property that its evaluation at a $\overline{\mathbb{Q}}_\ell$-point corresponding to a parameter $\phi : W_{\mathbb{Q}_p} \to {}^LG(\overline{\mathbb{Q}}_\ell)$ is precisely $r_V \circ \phi$.

(2) The compactly supported condition means that, for all $\mathcal{F} \in D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)^\omega$, the functor $Perf([\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G})]) \to D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)^\omega$ induced by acting on $\mathcal{F}$ factors through an action of $Perf([\mathcal{Z}^1(W_{\mathbb{Q}_p}/P, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G})])$, where P is a subgroup of wild inertia. Fargues and Scholze state this action in terms of a $(\infty, 1)$-category acting on an $(\infty, 1)$-category, we suppress this technicality for simplicity. However, we note that this enhanced action has the concrete implication that if one has a morphism, cone, or homotopy limit/colimit in the derived category $Perf([\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G})])$ that acting on an object $A \in D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)^\omega$ will produce a corresponding morphism, cone, or homotopy limit/colimit.

(3) In fact, Fargues-Scholze show that giving such a compactly supported $\overline{\mathbb{Q}}_\ell$-linear action is equivalent (when properly formulated) to giving a $Rep_{\overline{\mathbb{Q}}_\ell}(Q^I)$-linear monoidal functor

$$Rep_{\overline{\mathbb{Q}}_\ell}({}^LG^I) \to End_{\overline{\mathbb{Q}}_\ell}(D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)^\omega)^{BW_{\mathbb{Q}_p}^I}$$

In the case that $I = \{*\}$, the fact that the Hecke action satisfies this monoidal property is precisely Remark 3.2.

To study this spectral action, we consider, as in [FS21, Section VIII.3.], the coarse quotient in the category of schemes

$$\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}//\hat{G}$$

of $\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}$ by the action of $\hat{G}$ via conjugation. Given an L-parameter $\phi : W_{\mathbb{Q}_p} \to {}^LG(\overline{\mathbb{Q}}_\ell)$, it follows by [Dat+20, Proposition 4.13] or [FS21, Proposition VIII.3.2], that the $\hat{G}$-orbit of ϕ defines a closed $\overline{\mathbb{Q}}_\ell$-point of $\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}//\hat{G}$ if and only if ϕ is a semisimple parameter. Moreover, the natural map

$$\pi : [\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}] \to \mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}//\hat{G}$$

evaluated on a $\overline{\mathbb{Q}}_\ell$-point in the stack quotient defined by an L-parameter ϕ defines a closed $\overline{\mathbb{Q}}_\ell$-point in the coarse moduli space given by its semisimplification ϕ^{ss}. We can fit excursion operators into this picture as follows. We let $\mathcal{Z}^{spec}(G, \overline{\mathbb{Q}}_\ell) := \mathcal{O}(\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell})^{\hat{G}}$ be the ring of functions on the stack of L-parameters/the coarse moduli space, which we refer to as the spectral Bernstein center.

As noted above, the excursion operators define a family of commuting endomorphisms of the identity functor on $D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)$. We let $\mathcal{Z}^{geom}(G, \overline{\mathbb{Q}}_\ell)$ be the ring of such endomorphisms as in [FS21, Definition IX.0.2], which we refer to as the geometric Bernstein center. In [FS21, Corollary IX.0.3], Fargues and Scholze construct a canonical map of rings

$$\mathcal{Z}^{spec}(G, \overline{\mathbb{Q}}_\ell) \to \mathcal{Z}^{geom}(G, \overline{\mathbb{Q}}_\ell)$$

which is given by excursion operators in the following sense. By [FS21, Theorem VIII.3.6], there is an identification between $\mathcal{Z}^{spec}(G, \overline{\mathbb{Q}}_\ell)$ and the algebra of excursion operators. In particular, an excursion operator, as in Definition 3.3, associated to the datum I, W, α, β, and $\gamma_i \in W_{\mathbb{Q}_p}$ for $i \in I$ defines a function $f_{I,W,\alpha,\beta,(\gamma_i)_{i\in I}} \in \mathcal{O}_{[Z^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}]} = \mathcal{Z}^{spec}(G, \overline{\mathbb{Q}}_\ell)$ on $\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}//\hat{G}$, whose evaluation on the closed point of the coarse moduli space associated to a semisimple parameter $\phi : W_{\mathbb{Q}_p} \to {}^LG(\overline{\mathbb{Q}}_\ell)$ is precisely the scalar that results from the endomorphism:

$$\overline{\mathbb{Q}}_\ell \xrightarrow{\alpha} \Delta^* W = W \xrightarrow{(\phi(\gamma_i))_{i\in I}} W = \Delta^* W \xrightarrow{\beta} \overline{\mathbb{Q}}_\ell$$

We note that multiplication by $f_{I,W,\alpha,\beta,(\gamma_i)_{i\in I}}$ defines an endomorphism

$$\mathcal{O}_{[Z^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}]} \to \mathcal{O}_{[Z^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}]}$$

of the structure sheaf on the Artin stack $[Z^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}]$. If we act on a Schur-irreducible object $A \in D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)^\omega$ then we obtain an endomorphism

$$\{\mathcal{O}_{[Z^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}]} \star A = A \to \mathcal{O}_{[Z^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}]} \star A = A\} \in End(A) = \overline{\mathbb{Q}}_\ell$$

which will be precisely the scalar given by evaluating ϕ_A^{FS} on the excursion datum. In this way, we see that the action of excursion operators can be obtained through the spectral action. We will leverage this interpretation of excursion operators to prove the following key lemma.

Lemma 3.8. *Let $A \in D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)^\omega$ be any Schur-irreducible object with Fargues-Scholze parameter ϕ_A^{FS}. Set x to be the closed point defined by the parameter ϕ_A^{FS} in the coarse moduli space $\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}//\hat{G}$ and let $\pi^{-1}(x)$ denote the closed subset defined by the preimage in $[\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G})]$. Suppose we have $C \in Perf([\mathcal{Z}^1(W_E, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G})])$ with support disjoint from $\pi^{-1}(x)$ then C acts by zero on A via the spectral action.*

Proof. If we look at the action on $A \in D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)$ via the map

$$\mathcal{Z}^{spec}(G, \overline{\mathbb{Q}}_\ell) \to \mathcal{Z}^{geom}(G, \overline{\mathbb{Q}}_\ell)$$

given by excursion operators this factors through the maximal ideal $\mathfrak{m}_A \subset \mathcal{Z}^{spec}(G, \overline{\mathbb{Q}}_\ell) = \mathcal{O}(Z^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell})^{\hat{G}}$ defined by the closed point ϕ_A^{FS} in the coarse moduli space. By the conditions on the support of C, this implies that we can write the identity element as $1 = 1_C + 1_A \in \mathcal{Z}^{spec}(G, \overline{\mathbb{Q}}_\ell)$, where 1_C is a function that annihilates C and 1_A is in the annihilator of $\mathcal{Z}^{spec}(G, \overline{\mathbb{Q}}_\ell)/\mathfrak{m}_A$. We consider the spectral action of C on A

$$C \star A \in D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)$$

and look at the endomorphism induced by multiplication by 1 on C

$$C \star A \to C \star A$$

which is just the identity. However, since 1_C annihilates C, this is the same as the action of 1_A on $C \star A$, but, it follows by the above discussion that acting via multiplication by 1_A is the same as acting via the map

$$\mathcal{Z}^{spec}(G, \overline{\mathbb{Q}}_\ell) \to \mathcal{Z}^{geom}(G, \overline{\mathbb{Q}}_\ell)$$

given by excursion operators, and the action of 1_A after applying this map is zero. This would lead to a contradiction unless $C \star A$ is also zero. $\square$

To take advantage of this lemma, we now introduce the following endofunctors of $D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)^\omega$, which are analogues of the averaging operators considered by [AL21] in the Fargues-Scholze geometric Langlands correspondence for GL_n and by [FGV02; Gai04] in the classical geometric Langlands correspondence over function fields.

Definition 3.4. Let ϕ be a representation of $W_{\mathbb{Q}_p}$ and V a representation of ${}^LG$ with T_V the associated Hecke operator. We consider the endofunctor of $D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)$

$$A \mapsto R\Gamma(W_{\mathbb{Q}_p}, T_V(A) \otimes \phi^\vee)$$

where $R\Gamma(W_{\mathbb{Q}_p}, -) : D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)^{BW_{\mathbb{Q}_p}} \to D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)$ is the derived functor given by continuous group cohomology with respect to $W_{\mathbb{Q}_p}$. We denote this endofunctor by $Av_{V,\phi} : D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell) \to D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)$.

We now would like to realize the functor $Av_{V,\phi}$ as the spectral action of an object in $Perf([\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G})])$ similar to [AL21, Section 5.5]. An obvious guess would be that one should take the vector bundle C_V corresponding to the Hecke operator T_V, as in Remark 3.7 (1), and then twist this by the constant sheaf defined by $\phi^\vee$, which we denote by

$$C_V \otimes \phi^\vee \in Perf([\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G})])^{BW_{\mathbb{Q}_p}}$$

More precisely, this is the vector bundle with $W_{\mathbb{Q}_p}$-action whose evaluation at a $\overline{\mathbb{Q}}_\ell$-point corresponding to a L-parameter $\tilde{\phi} : W_{\mathbb{Q}_p} \to {}^LG(\overline{\mathbb{Q}}_\ell)$ is the vector space with $W_{\mathbb{Q}_p}$-action given by tensoring the representation

$$\tilde{\phi} : W_{\mathbb{Q}_p} \to {}^LG(\overline{\mathbb{Q}}_\ell) \xrightarrow{r_V} GL(V)$$

with $\phi^\vee$. To obtain the desired perfect complex, it is natural to apply $R\Gamma(W_{\mathbb{Q}_p}, -)$ to the vector bundle to $C_V \otimes \phi^\vee$, which we denote by $\mathcal{A}v_{V,\phi}$. We note that $\mathcal{A}v_{V,\phi}$ is a perfect complex. Indeed, as p is invertible in $\overline{\mathbb{Q}}_\ell$, the wild inertia $P \subset W_{\mathbb{Q}_p}$ will always act through a finite quotient on $C_V \otimes \phi^\vee$ and has no higher cohomology which implies that the invariants

$$(C_V \otimes \phi^\vee)^P$$

are a direct summand of the vector bundle $C_V \otimes \phi^\vee$. If we choose a generator $\tau \in I/P$ in the tame quotient of the inertia subgroup $I \subset W_{\mathbb{Q}_p}$ together with a Frobenius lift $\sigma \in W/P$ then the complex $\mathcal{A}v_{V,\phi}$ is computed as the homotopy limit of the diagram:

$$\begin{tikzcd} (C_V \otimes \phi^\vee)^P \arrow[r, "\tau-1"] \arrow[d, "\sigma-1"] & (C_V \otimes \phi^\vee)^P \arrow[d, "\sigma(1+\tau+\ldots+\tau^{p-1})-1"] \\ (C_V \otimes \phi^\vee)^P \arrow[r, "\tau-1"] & (C_V \otimes \phi^\vee)^P \end{tikzcd} \tag{1}$$

This gives a presentation of $\mathcal{A}v_{\phi,V}$ as a perfect complex. Then we have the following Lemma which is a verbatim generalization of [AL21, Lemma 5.7].

Lemma 3.9. *There exists a canonical identification*

$$Av_{V,\phi}(-) \simeq \mathcal{A}v_{V,\phi} \star (-)$$

of endofunctors of $D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)^\omega$

Proof. Equation (1) gives a diagram of perfect complexes on $[\mathcal{Z}^1(W_{\mathbb{Q}_p},\hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}]$. Acting via the spectral action on an object $\mathcal{F} \in D_{lis}(Bun_G,\overline{\mathbb{Q}}_\ell)$ then gives a diagram

$$\begin{array}{ccc} (T_V(\mathcal{F})\otimes\phi^\vee)^P & \xrightarrow{\tau-1} & (T_V(\mathcal{F})\otimes\phi^\vee)^P \\ \downarrow{\scriptstyle \sigma-1} & & \downarrow{\scriptstyle \sigma(1+\tau+\ldots+\tau^{p-1})-1} \\ (T_V(\mathcal{F})\otimes\phi^\vee)^P & \xrightarrow{\tau-1} & (T_V\otimes\phi^\vee)^P \end{array}$$

However, if we take the homotopy limit of the diagram in (1), the claim follows from the fact that the spectral action commutes with homotopy limits, as noted in Remark 3.7 (3). ☐

With this identification in hand, we can apply Lemma 3.8 to prove the following key consequence.

Lemma 3.10. *Let $A \in D_{lis}(Bun_G,\overline{\mathbb{Q}}_\ell)^\omega$ be a Schur-irreducible object with Fargues-Scholze parameter ϕ_A^{FS}, V a representation of ${}^LG$, and ϕ an irreducible representation of $W_{\mathbb{Q}_p}$. If the cohomology sheaves of $T_V(A) \in D_{lis}(Bun_G,\overline{\mathbb{Q}}_\ell)^{BW_{\mathbb{Q}_p}}$ with respect to the standard t-structure on $D_{lis}(Bun_G,\overline{\mathbb{Q}}_\ell)^{BW_{\mathbb{Q}_p}}$ have a non-zero sub-quotient as $W_{\mathbb{Q}_p}$-modules with $W_{\mathbb{Q}_p}$-action given by ϕ or $\phi(1)$ then $r_V\circ\phi_A^{FS}$ also has such a sub-quotient.*

Remark 3.8. During the creation of this manuscript, a similar result was obtained by Koshikawa through a similar but simpler proof [Kos21, Theorem 1.3]. As we will similarly conclude in Section 3.3, he shows that if the cohomology $R\Gamma_c(G,b,\mu)[\rho](\frac{d}{2})$ admits a sub-quotient with W_E-action given by an irreducible representation ϕ then the Fargues-Scholze parameter ϕ_ρ^{FS} admits a sub-quotient given by $\phi^\vee$. However, in his paper, the Shtuka space $Sht(G,b,\mu)_\infty$ parametrizes modifications of the form $\mathcal{E}_0 \dashrightarrow \mathcal{E}_b$ of type μ, whereas for us it parametrizes modifications of type μ^{-1}. This explains why, under our conventions, there is no dual appearing.

Proof. We first show that the assumption on the cohomology sheaves of $T_V(A)$ implies that averaging operator

$$Av_{\phi,V}(A) = R\Gamma(W_{\mathbb{Q}_p}, T_V(A)\otimes\phi^\vee)$$

is non-trivial. To do this, we consider the spectral sequence

$$E_2^{p,q} = H^p(W_{\mathbb{Q}_p}, H^q(T_V(A)\otimes\phi^\vee)) \Longrightarrow H^{p+q}(R\Gamma(W_{\mathbb{Q}_p}, T_V(A)\otimes\phi^\vee))$$

where cohomology is being taken with respect to the standard t-structure on $D_{lis}(Bun_G,\overline{\mathbb{Q}}_\ell)$. Now recall that the cohomological dimension of $W_{\mathbb{Q}_p}$ acting on finite dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces is 2 (where finite dimensionality follows from Theorem 3.2). Therefore, this sequence degenerates at the E_3 page. Moreover, the only non-zero degeneracy maps are given by

$$E_2^{0,q+1} = H^0(W_{\mathbb{Q}_p}, H^{q+1}(T_V(A)\otimes\phi^\vee)) \to E_2^{2,q} = H^2(W_{\mathbb{Q}_p}, H^q(T_V(A)\otimes\phi^\vee))$$

However, using local Tate-duality on the RHS, we can rewrite this differential as a map:

$$(H^{q+1}(T_V(A))\otimes\phi^\vee)^{W_{\mathbb{Q}_p}} \to H^0(W_{\mathbb{Q}_p}, H^q(T_V(A)\otimes\phi^\vee)^\vee(1))^\vee \simeq ((H^q(T_V(A))^\vee\otimes\phi(1))^{W_{\mathbb{Q}_p}})^\vee$$

Now the term on the RHS (resp. LHS) will only be non-zero if $\phi(1)$ (resp. ϕ) occurs as a sub-quotient of $H^q(T_V(A))$ (resp. $H^{q+1}(T_V(A))$). By assumption, this will be true for some value of q, but now, since the Euler-Poincaré characteristic of $W_{\mathbb{Q}_p}$ acting on $\overline{\mathbb{Q}}_\ell$-vector spaces is 0, one of these values being non-zero implies the $H^1(W_{\mathbb{Q}_p},-)$ of some cohomology sheaf of T_V must also be non-zero. This will then give rise to a non-zero contribution to the cohomology of $R\Gamma(W_{\mathbb{Q}_p}, T_V(A)\otimes\phi^\vee)$ so the averaging operator $Av_{V,\phi}(A)$ applied to A is non-zero. Lemma 3.9 therefore tells us that the spectral action of the perfect complex $\mathcal{A}v_{V,\phi}$ on A is non-trivial. If x denotes the closed $\overline{\mathbb{Q}}_\ell$-point in the coarse moduli space of Langlands parameters defined by ϕ_A^{FS} with preimage $\pi^{-1}(x)$ in the stack of Langlands parameters, Lemma 3.8 tells us that $\mathcal{A}v_{V,\phi}$ must have non-zero support on

$\pi^{-1}(x)$. The $\overline{\mathbb{Q}}_{\ell}$-points of $\pi^{-1}(x)$ correspond to the set of Langlands parameters whose semisimplification is precisely ϕ_{A}^{FS}. The previous analysis tells us that the evaluation of the perfect complex $R\Gamma(W_{\mathbb{Q}_p}, C_V \otimes \phi^{\vee})$ at some such point, corresponding to an L-parameter $\tilde{\phi} : W_{\mathbb{Q}_p} \to {}^{L}G(\overline{\mathbb{Q}}_{\ell})$, must be non-zero. This evaluation is precisely the complex

$$R\Gamma(W_{\mathbb{Q}_p}, r_V \circ \tilde{\phi} \otimes \phi^{\vee})$$

However, by again applying local Tate-duality, this can only be the case if $r_V \circ \tilde{\phi}$ has a sub-quotient isomorphic to ϕ or $\phi(1)$. Since ϕ is irreducible, this can only happen if $r_V \circ \tilde{\phi}^{ss} = r_V \circ \phi_{A}^{FS}$ has this property. □

We conclude this section by reviewing how the spectral action behaves on objects with supercuspidal Fargues-Scholze parameter along the lines of [FS21, Section X.2]. These results will be used to deduce a strong form of the Kottwitz conjecture from compatibility. We recall that a supercuspidal parameter, viewed as a continuous 1-cocyle with respect to the $W_{\mathbb{Q}_p}$-action on $\hat{G}(\overline{\mathbb{Q}}_{\ell})$, denoted $\phi : W_{\mathbb{Q}_p} \to \hat{G}(\overline{\mathbb{Q}}_{\ell})$, satisfies the property that it doesn't factor through any $\hat{P}(\overline{\mathbb{Q}}_{\ell})$ for P a parabolic subgroup of G. Equivalently, this is the same as insisting that, if $S_{\phi} := Z_{\hat{G}}(Im(\phi))$ as before, then the quotient $S_{\phi}/Z(\hat{G})^{\Gamma}$ is finite, where $\Gamma := Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. From this, it follows by deformation theory (See the proof of [Dat+20, Theorem 1.6] for more details) that the unramified twists of the parameter ϕ define a connected component

$$C_{\phi} \hookrightarrow [\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_{\ell}}/\hat{G}]$$

giving rise to a direct summand

$$Perf(C_{\phi}) \hookrightarrow Perf([\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_{\ell}}/\hat{G}])$$

Therefore, the spectral action gives rise to a corresponding direct summand

$$D_{lis}^{C_{\phi}}(Bun_G, \overline{\mathbb{Q}}_{\ell})^{\omega} \subset D_{lis}(Bun_G, \overline{\mathbb{Q}}_{\ell})^{\omega}$$

such that the Schur-irreducible objects in this subcategory all have Fargues-Scholze parameter given by an unramified twist of ϕ. Now, it follows by Proposition 3.14 and Theorem 3.6 (4), that, since ϕ is supercuspidal, the restriction of any object in $D_{lis}^{C_{\phi}}(Bun_G, \overline{\mathbb{Q}}_{\ell})^{\omega}$ to any non-basic HN-strata of Bun_G must be zero. Therefore, one obtains a decomposition

$$D_{lis}^{C_{\phi}}(Bun_G, \overline{\mathbb{Q}}_{\ell})^{\omega} \simeq \bigoplus_{b \in B(G)_{basic}} D^{C_{\phi}}(J_b(E), \overline{\mathbb{Q}}_{\ell})^{\omega}$$

where J_b is the σ-centralizer of b and $D^{C_{\phi}}(J_b(E), \overline{\mathbb{Q}}_{\ell})^{\omega} \subset D(J_b(E), \overline{\mathbb{Q}}_{\ell})^{\omega}$ is a full subcategory of the derived category of compact objects in smooth representations of $J_b(E)$. It also follows again by Theorem 3.6 (4) that the Schur-irreducible objects of any $D^{C_{\phi}}(J_b(E), \overline{\mathbb{Q}}_{\ell})^{\omega}$ must lie only in the supercuspidal components of the Bernstein center. Now to further analyze this we fix a character χ of $Z(G)$ and consider the subcategory

$$D_{lis,\chi}^{C_{\phi}}(Bun_G, \overline{\mathbb{Q}}_{\ell})^{\omega} \simeq \bigoplus_{b \in B(G)_{basic}} D_{\chi}^{C_{\phi}}(J_b(E), \overline{\mathbb{Q}}_{\ell})^{\omega}$$

where $D_{\chi}^{C_{\phi}}(J_b(E), \overline{\mathbb{Q}}_{\ell})^{\omega}$ is the derived subcategory of $D^{C_{\phi}}(J_b(E), \overline{\mathbb{Q}}_{\ell})^{\omega}$ generated by compact objects with fixed central character χ, via the natural isomorphism $Z(J_b)(\mathbb{Q}_p) \simeq Z(G)(\mathbb{Q}_p)$ (where we recall that J_b is an extended pure inner form of G). One can see that the spectral action of $Perf(C_{\phi})$ preserves this subcategory. Indeed, it follows by [FS21, Theorem I.8.2] that $Perf([\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_{\ell}}/\hat{G}])$ and in turn $Perf(C_{\phi})$ is generated under cones and retracts by the image of $Rep_{\overline{\mathbb{Q}}_{\ell}}({}^{L}G)$ in $Perf([\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_{\ell}}/\hat{G}])$. This reduces us to checking that Hecke operators preserve this subcategory, which, in turn reduces to the observation that, if one looks at the simultaneous action of $J_b(\mathbb{Q}_p) \times J_{b'}(\mathbb{Q}_p)$ on the space parametrizing modifications $\mathcal{E}_b \to \mathcal{E}_{b'}$, for b

and b' in $B(G)_{basic}$ that, under the canonical identification $Z(J_{b'})(\mathbb{Q}_p) \simeq Z(J_b)(\mathbb{Q}_p)$, the diagonally embedded center acts trivially. This follows since an element in the center of $J_b(\mathbb{Q}_p)$ acts on the modification by the inverse of an element in the corresponding center of $J_{b'}(\mathbb{Q}_p)$, where the inverse appears from the fact that $J_b(\mathbb{Q}_p)$ is acting on the left and $J_{b'}(\mathbb{Q}_p)$ is acting on the right.

Now, via local class field theory, we take χ to be the central character determined by ϕ and local class field theory (as in [Bor79, Section 10.1]). Since all Schur-irreducible objects in $D_{\chi}^{C_\phi}(J_b(E), \overline{\mathbb{Q}}_\ell)$ lie in the supercuspidal component of the Bernstein-center and have fixed central character χ and, by [AR04], supercuspidal representations are injective/projective in the category of smooth representations with fixed central character, we can write $D_{\chi}^{C_\phi}(J_b(E), \overline{\mathbb{Q}}_\ell)^\omega = \bigoplus_\pi Perf(\overline{\mathbb{Q}}_\ell) \otimes \pi$, where π runs over all supercuspidal representations of $J_b(E)$ with central character χ which, a priori, have Fargues-Scholze parameter given by an unramified twist of ϕ, but, by Theorem 3.6 (2), must indeed be equal to ϕ.

Now, the closed point of C_ϕ determined by the parameter ϕ gives rise to a closed embedding

$$[\overline{\mathbb{Q}}_\ell/S_\phi] \hookrightarrow C_\phi$$

and in turn a fully faithful embedding

$$Perf([\overline{\mathbb{Q}}_\ell/S_\phi]) \hookrightarrow Perf(C_\phi)$$

The above discussion and Lemma 3.8 imply that the action of $Perf(C_\phi)$ factors over this subcategory in the sense that everything not in the image of this must act trivially. All in all, we conclude that we have a decomposition

$$D_{lis,\chi}^{C_\phi}(Bun_G, \overline{\mathbb{Q}}_\ell)^\omega = \bigoplus_{b \in B(G)_{basic}} \bigoplus_{\pi_b} Perf(\overline{\mathbb{Q}}_\ell) \otimes \pi_b$$

where the π_b runs over all supercuspidal representations of $J_b(E)$ with Fargues-Scholze parameter $\phi_{\pi_b}^{FS} = \phi$. Moreover, the RHS carries an action of $Rep_{\overline{\mathbb{Q}}_\ell}(S_\phi)$, the category of finite-dimensional $\overline{\mathbb{Q}}_\ell$-representations of S_ϕ. Therefore, given $W \in Rep_{\overline{\mathbb{Q}}_\ell}(S_\phi)$, we get an object:

$$Act_W(\pi_b) \in \bigoplus_{b \in B(G)_{basic}} \bigoplus_{\pi_b} Perf(\overline{\mathbb{Q}}_\ell) \otimes \pi_b$$

Assume that $W|_{Z(\hat{G})^\Gamma}$ is isotypic, given by some character $\eta : Z(\hat{G})^\Gamma \to \overline{\mathbb{Q}}_\ell^*$. As $Z(\hat{G})^\Gamma$ is diagonalizable with characters given by $B(G)_{basic} \xrightarrow{\simeq} \pi_1(G)_\Gamma$ via the κ map, we obtain an element $b_\eta \in B(G)_{basic}$. Then $Act_W(\pi_b)$ is concentrated on the basic HN-strata given by $b' = b + b_\eta$. Therefore, we get an isomorphism

$$Act_W(\pi_b) \simeq \bigoplus_{\pi_{b'}} V_{\pi_{b'}} \otimes \pi_{b'}$$

where $V_{\pi_{b'}} \in Perf(\overline{\mathbb{Q}}_\ell)$ and $\pi_{b'}$ runs over all supercuspidals of $J_{b'}$ with Fargues-Scholze parameter $\phi_{\pi_{b'}}^{FS} = \phi$. With this in hand, we can elucidate the $W_{\mathbb{Q}_p}$-action on the Hecke operator applied to a smooth irreducible object with supercuspidal Fargues-Scholze parameter, similar to what was done in Lemma 3.10 in the general case. Namely, given $V \in Rep_{\overline{\mathbb{Q}}_\ell}({}^LG)$, we obtain a vector bundle on $[\overline{\mathbb{Q}}_\ell/S_\phi]$ with $W_{\mathbb{Q}_p}$-action given by ϕ. In other words, we have a functor:

$$Rep_{\overline{\mathbb{Q}}_\ell}({}^LG) \to Rep_{\overline{\mathbb{Q}}_\ell}(S_\phi)^{BW_{\mathbb{Q}_p}}$$

Theorem 3.7 and the above discussion imply that the action of the image of V in $Rep_{\overline{\mathbb{Q}}_\ell}(S_\phi)^{BW_{\mathbb{Q}_p}}$ acting via the spectral action on $D_{lis,\chi}^{C_\phi}(Bun_G, \overline{\mathbb{Q}}_\ell)$ is precisely the Hecke operator T_V. This tells us that, if we decompose $r_V \circ \phi$ viewed as a representation of S_ϕ as a direct sum $\bigoplus_{i \in I} W_i \otimes \sigma_i$ where $W_i \in Rep(S_\phi)$ is irreducible and σ_i is a continuous finite-dimensional representation of $W_{\mathbb{Q}_p}$, then we have an isomorphism

$$T_V(\pi) \simeq \bigoplus_{i \in I} Act_{W_i}(\pi) \otimes \sigma_i$$

as $J_{b'}(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$-modules. We now summarize the above discussion as a corollary for future use.

Corollary 3.11. *Let ϕ be a supercuspidal parameter of G, $b \in B(G)_{basic}$ a basic elment, $V \in Rep_{\overline{\mathbb{Q}}_\ell}({}^LG)$ an irreducible representation of some highest weight μ with dominant inverse μ^{-1}, and π_b a representation of $J_b(E)$ with Fargues-Scholze parameter equal to ϕ. We set $b_\mu \in B(G, \mu^{-1})$ to be the unique basic element and $b' = b + b_\mu$. If we decompose $r_V \circ \phi$ viewed as representation of S_ϕ as a direct sum $\bigoplus_{i \in I} W_i \otimes \sigma_i$, where $W_i \in Rep(S_\phi)$ is irreducible and σ_i is a continuous finite-dimensional representation of $W_{\mathbb{Q}_p}$, then there exists an isomorphism of $W_{\mathbb{Q}_p} \times J_{b'}(\mathbb{Q}_p)$-modules*

$$T_\mu(\pi_b) \simeq \bigoplus_{i \in I} Act_{W_i}(\pi) \otimes \sigma_i$$

where $Act_{W_i}(\pi) \simeq \bigoplus_{\pi_{b'}} V_{\pi_{b'}} \otimes \pi_{b'}$ with $V_{\pi_{b'}} \in Perf(\overline{\mathbb{Q}}_\ell)$ and $\pi_{b'}$ ranging over supercuspidal representation of $J_{b'}(\mathbb{Q}_p)$ with Fargues-Scholze parameter equal to ϕ.

Remark 3.9. As we will start to see in the next section, the work of Hansen [Han20], Hansen-Kaletha-Weinstein [HKW21], and compatibility of the Fargues-Scholze and Gan-Takeda/Gan-Tantono local Langlands correspondence will allow us to use this Corollary to prove Theorem 1.3. This is suggested already by Corollary 3.3, which shows us that $T_\mu(\pi_b)$ can be computed explicitly using the cohomology of local Shimura varieties.

3.3. **Compatibility with the Local Langlands for GSp_4 and $GU_2(D)$.** With the results of the previous section in place, we can now start making progress towards our goal of proving compatibility. So we again let $G = Res_{L/\mathbb{Q}_p} GSp_4$ and $J = Res_{L/\mathbb{Q}_p} GU_2(D)$, for $L/\mathbb{Q}_p$ a finite extension. As mentioned in section 1, the case where a representation $\pi \in \Pi(G)$ (resp. $\rho \in \Pi(J)$), is a sub-quotient of a parabolic induction easily follows from Theorem 3.6 (3), (4), (5), and compatibility of the (semi-simplified) Gan-Takeda (resp. Gan-Tantono) parameter with parabolic induction. We record this as a corollary now.

Corollary 3.12. *Let $\pi \in \Pi(G)$ (resp. $\rho \in \Pi(J)$) be representations occurring as a sub-quotient of a parabolic induction. For such π (resp. ρ), the Fargues-Scholze and Gan-Takeda (resp. Gan-Tantono) local Langlands correspondences are compatible.*

To tackle the remaining cases where the L-parameter ϕ is mixed supercuspidal or supercuspidal, we note that these are the cases where the L-parameter is discrete. (i.e the L-parameter does not factor through a Levi subgroup) This case is disposable to the results of Hansen-Kaletha-Weinstein [HKW21]. We state their main result now specialized to the case of GSp_4 and $GU_2(D)$.

Theorem 3.13. *[HKW21, Theorem 1.0.2] Let ϕ be a mixed supercuspidal or supercuspidal parameter and $S_\phi := Z_{\hat{G}}(Im(\phi))$ as before. Let $\Pi_\phi(G)$ and $\Pi_\phi(J)$ denote the L-packets over ϕ. Set $\pi \in \Pi_\phi(G)$ (resp. $\rho \in \Pi_\phi(J)$) to be smooth irreducible representations of G (resp. J). If ϕ is supercuspidal, we have the following equality in $K_0(G(\mathbb{Q}_p))$ the Grothendieck group of admissible $G(\mathbb{Q}_p)$-modules of finite length*

$$[R\Gamma_c^\flat(G, b, \mu)[\rho]] = -\sum_{\pi \in \Pi_\phi(G)} Hom_{S_\phi}(\delta_{\pi,\rho}, std \circ \phi)\pi$$

and the following equality in the Grothendieck group of admissible $J(\mathbb{Q}_p)$-representations of finite length

$$[R\Gamma_c^\flat(G,b,\mu)[\pi]] = -\sum_{\rho\in\Pi_\phi(J)} Hom_{S_\phi}(\delta_{\pi,\rho}^\vee,(std\circ\phi)^\vee)\rho$$

where $\delta_{\pi,\rho}$ is the algebraic representation of S_ϕ in Defition 2.3. Similarly, if ϕ is mixed supercuspidal, we have the same formulas for $[R\Gamma_c^\flat(G,b,\mu)[\rho]]$ (resp. $[R\Gamma_c^\flat(G,b,\mu)[\pi]]$) in the Grothendieck group $K_0(G(\mathbb{Q}_p))^{ell}$ (resp. $K_0(J(\mathbb{Q}_p))^{ell}$) of elliptic representations of $G(\mathbb{Q}_p)$ (resp. $J(\mathbb{Q}_p)$).

Remark 3.10.
1. To deduce the result for the π-isotypic part, we have implicitly used the two towers isomorphism $Sht(G,b,\mu)_\infty \simeq Sht(J,\hat{b},\mu^{-1})_\infty$, where μ^{-1} is a dominant inverse to μ and $\hat{b}=b^{-1}\in B(G,\mu^{-1})$ is the unique basic element [SW20, Corollary 23.3.2.]. This inverse explains the appearance of duals in the formula for the π-isotypic part.
2. We see that, via Corollary 3.3, this, in the case that ϕ is supercuspidal, should provide us insight into the multiplicity spaces $V_{\pi_{b'}}$ appearing in Corollary 3.11, assuming compatibility of the Fargues-Scholze and Gan-Tantono/Gan-Takeda local Langlands correspondences. Namely, we will see later (Theorem 3.17 and 3.18) that $R\Gamma_c(G,b,\mu)[\rho]\simeq R\Gamma_c^\flat(G,b,\mu)[\rho]$ and is concentrated in middle degree 3 if ϕ_ρ^{FS} is supercuspidal. Assuming compatibility, Corollary 3.11 will therefore tell us that $R\Gamma_c(G,b,\mu)[\rho]$ will be a direct sum over representations $\pi\in\Pi_\phi(G)$ with W_L-action given by $std\circ\phi_\rho^{FS} = std\circ\phi_\rho$ decomposed as a representation of S_ϕ. The summands in the decomposed S_ϕ-representation correspond to the weight spaces appearing in the above description in the Grothendieck group.

We now wish to write out the precise formula for the ρ and π-isotypic parts, using the refined local Langlands discussed in section 2.

1. (ϕ stable) In this case, the L-packet $\Pi_\phi(G)=\{\pi\}$ is a singleton so the RHS of the above formula for the ρ-isotypic part has one term
$$-\pi Hom_{S_\phi}(\delta_{\pi,\rho}, std\circ\phi_\rho)$$
In this case, $S_\phi=\mathbb{G}_m$ and $\delta_{\pi,\rho}$ is simply the identity representation. Thus, this Hom space gets identified with the characters of GL_4, so the formula reduces to
$$-4\pi$$
2. (ϕ endoscopic) In this case, the L-packet has size 2 and, as seen in section 2.1, $\Pi_\phi(G) = \{\pi^+,\pi^-\}$, where π^+ (resp. π^-) corresponds to the trivial (resp. non-trivial) character of the component group. ρ can be either of the two representations corresponding to the irreducible representation of S_ϕ given by τ_i for $i=1,2$ the projection to the two coordinates of S_ϕ. However, the RHS remains the same regardless of which one it corresponds to. So, without loss of generality, we assume that $\rho=\rho_1$. Then the RHS of the above formula for the ρ-isotypic part has two terms
$$\pi^+ Hom_{S_\phi}(\tau_1, std\circ\phi_\rho)$$
and
$$\pi^- Hom_{S_\phi}(\tau_1\otimes\tau_{\pi^-}\simeq\tau_2, std\circ\phi_\rho)$$
However, writing $std\circ\phi_\rho\simeq\phi_1\oplus\phi_2$, these get identified with
$$-\pi^+ Hom_{\overline{\mathbb{Q}}_\ell^*}(\overline{\mathbb{Q}}_\ell^*,\phi_1)$$
and
$$-\pi^- Hom_{\overline{\mathbb{Q}}_\ell^*}(\overline{\mathbb{Q}}_\ell^*,\phi_2)$$
which will both be identified with characters of GL_2. Thus, the RHS is equal to
$$-2\pi^+ - 2\pi^-$$

Similarly, for the π-isotypic part, we get that the RHS of the above formula is given by

$$-4\rho$$

in the stable case and

$$-2\rho_1 - 2\rho_2$$

in the endoscopic case.

As mentioned in the section 1.2, we will now use the previous result to perform a bootstrap to the supercuspidal representations occurring in the L-packets $\Pi_\phi(G)$ (resp. $\Pi_\phi(J)$), for ϕ a mixed supercuspidal parameter. For this, we will mention one last result from the Fargues-Scholze local Langlands correspondence.

Proposition 3.14. *[FS21, Section IX.7.1] For G any connected reductive group over $\mathbb{Q}_p$, the action of the excursion algebra on $D_{lis}(Bun_G, \overline{\mathbb{Q}}_\ell)$ commutes with Hecke operators and is compatible with restriction to the HN-strata Bun_G^b for $b \in B(G)$.*

Remark 3.11. The commutation of Hecke operators and the excursion algebra follows from the interpretation of the excursion algebra in terms of endomorphisms coming from multiplication by the ring of global functions of the stack of L-parameters, as discussed in Section 3.2.

From this, we can deduce the following useful corollary.

Corollary 3.15. *For G any connected reductive group with (G, b, μ) a local Shimura datum and $\pi \in \Pi(G)$ and $\rho \in \Pi(J_b)$ smooth irreducible representations. All smooth irreducible representations occurring in the cohomology of $R\Gamma_c(G, b, \mu)[\pi]$ have Fargues-Scholze parameter equal to ϕ_π^{FS}. Similarly, all smooth irreducible representations occurring in the cohomology of $R\Gamma_c(G, b, \mu)[\rho]$ have Fargues-Scholze parameter equal to ϕ_ρ^{FS}. The same is also true for $R\Gamma_c^\flat(G, b, \mu)[\rho]$ and $R\Gamma_c^\flat(G, b, \mu)[\pi]$.*

Proof. The first part follows immediately from Proposition 3.14 and Corollary 3.3. It remains to see the same is true for the complexes $R\Gamma_c^\flat(G, b, \mu)[\rho]$ and $R\Gamma_c^\flat(G, b, \mu)[\pi]$. We do this for the ρ-isotypic part, with the proof for the π-isotypic part being analogous. Consider the contragredient ρ^* then, by Hom-Tensor duality, we have an isomorphism

$$R\mathcal{H}om(R\Gamma_c(G, b, \mu)[\rho^*], \overline{\mathbb{Q}}_\ell)[-2d](-d) \simeq R\Gamma_c^\flat(G, b, \mu)[\rho]$$

Using this isomorphism, one can see if π' is an irreducible smooth representation of $G(\mathbb{Q}_p)$ occurring as a sub-quotient of the cohomology of $R\Gamma_c^\flat(G, b, \mu)[\rho]$ then the contragredient π'^* occurs as a sub-quotient of the cohomology of $R\Gamma_c(G, b, \mu)[\rho^*]$. By first part of the Corollary, this implies that

$$(\phi_{\pi'}^{FS})^\vee = (\phi_{\pi'^*})^{FS} = \phi_{\rho^*}^{FS} = (\phi_\rho^{FS})^\vee$$

where we have used Theorem 3.6 (2). Taking duals, we obtain the desired equality of parameters. $\square$

We now exploit this corollary to deduce compatibility in the mixed supercuspidal case.

Corollary 3.16. *Let ϕ be an L-parameter of Howe-Piatetski–Schapiro or Saito-Kurokawa type. Then, for any $\pi \in \Pi_\phi(G)$ (resp. $\rho \in \Pi_\phi(J)$), the Fargues-Scholze and Gan-Takeda (resp. Gan-Tantono) local Langlands correspondences are compatible.*

Proof. We give the proof for the Gan-Takeda local Langlands correspondence with the proof for the Gan-Tantono correspondence being completely analogous. If ϕ is of Saito-Kurokawa type then, as seen in section 2, we can write $\Pi_\phi(G) = \{\pi_{sc}, \pi_{disc}\}$ and $\Pi_\phi(J) = \{\rho_{sc}, \rho_{disc}\}$, where π_{sc} (resp. ρ_{sc}) is a supercuspidal representation of G (resp. J) and π_{disc} (resp. ρ_{disc}) is a non-supercuspidal representaiton. We note that the Gan-Takeda (resp. Gan-Tantono) correspondences are compatible with the Fargues-Scholze correspondence for π_{disc} (resp. ρ_{disc}), by Corollary 3.12.

If we let μ be the Siegel cocharacter and $b \in B(G,\mu)$ be the unique basic element. Then the σ-centralizer J_b is isomorphic to J and we can consider the complex

$$R\Gamma^{\flat}_{c}(G,b,\mu)[\rho_{disc}]$$

of $J(\mathbb{Q}_p) \times W_L$-representations. We then let $R\Gamma^{\flat}_{c}(G,b,\mu)[\rho_{disc}]_{sc}$ denote the direct summand of $R\Gamma^{\flat}_{c}(G,b,\mu)[\rho_{disc}]$, where $J(\mathbb{Q}_p)$ acts via a supercuspidal representation. Theorem 3.13 tells us that we can describe this complex in the Grothendieck group of admissible $G(\mathbb{Q}_p)$-representations of finite length as

$$[R\Gamma^{\flat}_{c}(G,b,\mu)[\rho_{disc}]_{sc}] = -2\pi_{sc}$$

which tells us that π_{sc}, occurs as a non-zero sub-quotient of the complex $R\Gamma^{\flat}_{c}(G,b,\mu)[\rho_{disc}]$. By Corollary 3.15, we know that we have an equality:

$$\phi^{FS}_{\rho_{disc}} = \phi^{FS}_{\pi_{sc}}$$

However, Corollary 3.12 tells us that $\phi^{FS}_{\rho_{disc}} = \phi^{ss}_{\rho_{disc}}$, which is equal to $\phi^{ss}_{\pi_{sc}}$, so we get the desired equality. The analysis in the Howe-Piatetski–Schapiro case is the same, where one can look at the ρ-isotypic part for any of the two non-supercuspidals in $\Pi_{\phi}(J)$. □

In the remaining part of this section, we address proving compatibility in the case where the parameter ϕ is supercuspidal. Before tackling the question of compatibility, we address some geometric properties of the sheaves $\mathcal{F}_{\rho}$, for ρ with supercuspidal Fargues-Scholze parameter. This will be leveraged in proving the strong form of the Kottwitz Conjecture for the ρ and π-isotypic parts in section 8, as mentioned in Remark 3.10 (2).

Now, considering again G a general connected reductive group, $b \in B(G)_{basic}$ a basic element, and a smooth irreducible representation ρ of the σ-centralizer $J_b(\mathbb{Q}_p)$. We will now address some further consequences of the Fargues-Scholze parameter ϕ^{FS}_{ρ} being supercuspidal. It turns out that the sheaves defined by representations with these parameters have interesting geometric properties, which were leveraged in [Han20] to prove various general results on the cohomology groups. In particular, Hansen shows the following:

Theorem 3.17. *[Han20, Theorem 1.1.] Let (G,b,μ) be a basic local Shimura datum with E the reflex field of μ as before, and let ρ be a smooth irreducible representation of $J_b(\mathbb{Q}_p)$. Suppose the following conditions hold:*

1. *The spaces $(Sht(G,b,\mu)_K)_{K\subset G(\mathbb{Q}_p)}$ occur in the basic uniformization at p of a global Shimura variety in the sense of Definition 4.1.*
2. *The Fargues-Scholze parameter $\phi^{FS}_{\rho} : W_{\mathbb{Q}_p} \rightarrow^{L} G(\overline{\mathbb{Q}}_{\ell})$ is supercuspidal.*

Then the complex $R\Gamma_c(G,b,\mu)[\rho]$ defined above is concentrated in middle degree $d = dim(Sht(G,b,\mu)_{\infty}) = \langle 2\rho_G, \mu\rangle$.

One of the key ideas in the argument is to exploit the behavior of the sheaf $j_{b!}(\mathcal{F}_{\rho})$ under Verdier duality, where $j_b : Bun^{b}_{G} \hookrightarrow Bun_G$ is the inclusion of the open HN-strata corresponding to $b \in B(G)_{basic}$. In particular, by Proposition 3.14 and Theorem 3.6 (4), one can see that the natural map $j_{b!}(\mathcal{F}_{\rho}) \rightarrow Rj_{b*}(\mathcal{F}_{\rho})$ is an isomorphism. Namely, Proposition 3.14 implies that a non-zero restriction of $Rj_{b*}(\mathcal{F}_{\rho})$ to any non-basic HN-strata must be valued in representations having Fargues-Scholze parameter ϕ^{FS}_{ρ} under the relevant twisted embedding, which is impossible since the σ-centralizers of non-basic elements are extended pure inner forms of proper Levi subgroups of G and, by assumption, the parameter ϕ^{FS}_{ρ} is supercuspidal. This implies that, if we apply Verdier duality to both sides of the isomorphism

$$j^{*}_{\mathbf{1}}T_{\mu}j_{b!}(\mathcal{F}_{\rho}) \simeq R\Gamma_c(G,b,\mu)[\rho][d](\frac{d}{2})$$

supplied by Corollary 3.3, we see that the LHS is isomorphic to

$$j_{\mathbf{1}}^{*}T_{\mu}j_{b!}(\mathcal{F}_{\rho^*}) \simeq R\Gamma_c(G,b,\mu)[\rho^*][d](\frac{d}{2})$$

On the other hand, on the RHS we act through Verdier duality on the tower $(Sht(G,b,\mu)_K)_{K\subset G(\mathbb{Q}_p)}$, which are smooth rigid spaces of dimension d. So, in particular, the dualizing object is isomorphic to $\overline{\mathbb{Q}}_{\ell}[2d](d)$. This allows one to deduce the following consequence for the cohomology groups $R\Gamma_c(G,b,\mu)[\rho]$.

Theorem 3.18. *[Han20, Theorem 1.3, Theorem 2.23] Fix a basic local Shimura datum (G,b,μ) and let ρ be representation of $J_b(\mathbb{Q}_p)$ with supercuspidal Fargues-Scholze parameter. Then there is a natural isomorphism*

$$R\mathcal{H}om(R\Gamma_c(G,b,\mu)[\rho],\overline{\mathbb{Q}}_{\ell}) \simeq R\Gamma_c(G,b,\mu)[\rho^*][2d](d)$$

as W_E-equivariant objects of $D(J_b(\mathbb{Q}_p),\overline{\mathbb{Q}}_{\ell})$, where ρ^* is the contragredient of ρ. In particular, we have a natural W_E-equivariant isomorphism of admissible $G(\mathbb{Q}_p)$-representations for all $0 \leq i \leq 2d$

$$H^i(R\Gamma_c(G,b,\mu)[\rho])^* \simeq H^{2d-i}(R\Gamma_c(G,b,\mu)[\rho^*])(d)$$

Remark 3.12. As noted in Remark 3.4, the LHS of the above formula is isomorphic to $R\Gamma_c^{\flat}(G,b,\mu)[\rho^*][2d](d)$, so it follows by cancelling the shifts and Tate twists and relaxing contragradients that one has an isomorphism

$$R\Gamma_c^{\flat}(G,b,\mu)[\rho] \simeq R\Gamma_c(G,b,\mu)[\rho]$$

as $J_b(\mathbb{Q}_p)\times W_E$-representations for all such ρ.

Now we turn our attention to the question of showing compatibility for supercuspidal parameters assuming Proposition 1.4. So again let $L/\mathbb{Q}_p$ be a finite extension and let $G := Res_{L/\mathbb{Q}_p}(GSp_4)$ be the restriction of scalars of GSp_4 and $J := Res_{L/\mathbb{Q}_p}GU_2(D)$ the unique non-split inner form as before. As we will see in section 8, it essentially follow from Theorem 3.13 and Corollary 3.15 that showing compatibility for $\rho \in \Pi(J)$ with supercuspidal Gan-Tantono parameter implies the corresponding statement for $\pi \in \Pi(G)$ with supercuspidal Gan-Takeda parameter. So we fix such a ρ and assume that the Gan-Tantono parameter ϕ_{ρ} is endoscopic supercuspidal with the stable case being strictly easier. We will write $std\circ\phi_{\rho} \simeq \phi_1 \oplus \phi_2$ for ϕ_i distinct irreducible 2-dimensional representations of W_L and let μ be the Siegel cocharacter. The Shtuka space $Sht(G,b,\mu)_{\infty}$ in this case will have dimension $d := \langle 2\rho_G,\mu\rangle = 3$. We will assume for the rest of this section that Proposition 1.4 is true.

Proposition 3.19. *Let ϕ be a supercuspidal parameter with associated L-packet $\Pi_{\phi}(J)$. Then the direct summand of*

$$\bigoplus_{\rho'\in\Pi_{\phi}(J)} R\Gamma_c(G,b,\mu)[\rho']$$

where $G(\mathbb{Q}_p)$ acts via a supercuspidal representation

$$\bigoplus_{\rho'\in\Pi_{\phi}(J)} R\Gamma_c(G,b,\mu)[\rho']_{sc}$$

is concentrated in middle degree 3 *and admits a non-zero W_L-stable sub-quotient with W_L-action given by $std\circ\phi\otimes|\cdot|^{-3/2}$.*

First, we combine this with the following lemma.

Lemma 3.20. *Let ϕ be a supercuspidal parameter then all representations in the L-packet $\Pi_{\phi}(J)$ have the same Fargues-Scholze parameter.*

Proof. We choose a $\pi \in \Pi_{\phi}(G)$ and then apply Corollary 3.15 to deduce that all representations occurring in the cohomology of $R\Gamma_c^{\flat}(G, b, \mu)[\pi]$ have Fargues-Scholze parameter equal to ϕ_{π}^{FS}. However, by Theorem 3.13, we have that all representations in $\rho \in \Pi_{\phi}(J)$ occur in the cohomology of $R\Gamma_c^{\flat}(G, b, \mu)[\pi]$, so their Fargues-Scholze parameters are the same as desired. □

With this in hand, we are ready to prove the key consequence of Proposition 1.4 using the results on the spectral action obtained in section 3.2.

Corollary 3.21. *Assume that $L/\mathbb{Q}_p$ is an unramified extension and that $p > 2$ and that Proposition 1.4 is true. Then, for ρ a representation of J with supercuspidal Gan-Tantono parameter ϕ, the Gan-Tantono and Fargues-Scholze correspondences coincide.*

Proof. As mentioned in the introduction, the key will be the isomorphism

$$\bigoplus_{\rho' \in \Pi_{\phi}(J)} j_{\mathbf{1}}^{*} T_{\mu} j_{b!}(\mathcal{F}_{\rho'}) \simeq \bigoplus_{\rho' \in \Pi_{\phi}(J)} R\Gamma_c(G, b, \mu)[\rho'][3](\frac{3}{2})$$

supplied by Corollary 3.3. Now Proposition 1.4 tells us that one of the summands on the RHS admits a sub-quotient with W_L-action given by ϕ_1 and one of them admits a sub-quotient with W_L-action given by ϕ_2. Applying Lemma 3.20 and 3.10 therefore tells us that $std \circ \phi_{\rho}^{FS}$ admits a sub-quotient isomorphic to ϕ_1 or $\phi_1(1)$ and a sub-quotient isomorphic to ϕ_2 or $\phi_2(1)$. This gives four possibilities for what the parameter $std \circ \phi_{\rho}^{FS}$ is. Since ϕ_{ρ}^{FS} is a GSp_4-valued parameter only two of these are possible; namely, $\phi_1 \oplus \phi_2$ or $(\phi_1 \oplus \phi_2)(1)$. However, the second possibility can be ruled out since the similitude character of $std \circ \phi_{\rho}^{FS}$ must coincide with the central character of ρ by Theorem 3.6 (3) and (7), which agrees with the similitude character of $std \circ \phi_{\rho} = \phi_1 \oplus \phi_2$. Therefore, we conclude that $std \circ \phi = std \circ \phi_{\rho}^{FS}$ and, by the aforementioned equality of similitude characters of these two parameters and [GT11, Lemma 6.1], this is enough to conclude that $\phi_{\rho} = \phi_{\rho}^{FS}$, as conjugacy classes of GSp_4-valued parameters. □

4. Basic Uniformization

In this section, we will briefly review what basic uniformization of the generic fiber of a global Shimura variety means, following [Han20, Section 3.1]. Then we will apply it to our particular case and derive an analogue of Boyer's trick, providing useful consequences for the proof of Proposition 1.4.

4.1. A Review of Basic Uniformization.

We now recall briefly what basic uniformization means. Let $\mathbf{G}/\mathbb{Q}$ be a connected reductive group over $\mathbb{Q}$ and let $(\mathbf{G}, X)$ be a Shimura datum, with a conjugacy class of Hodge cocharacters $\mu : \mathbb{G}_{m,\mathbb{C}} \to \mathbf{G}_{\mathbb{C}}$. Fix a prime p, and set $G := \mathbf{G}_{\mathbb{Q}_p}$. Using our fixed isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_p$, we can and do regard μ as a conjugacy class of cocharacters $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$. This allows us to consider the μ-admissible locus $B(G, \mu)$ in the Kottwitz set of G. Let $\mathbb{A}_f$ denote the finite adeles of $\mathbb{Q}$ and $\mathbb{A}_f^p$ denote the finite adeles away from p. For any compact open subgroup $K \subset \mathbf{G}(\mathbb{A}_f)$, let $\mathcal{S}(\mathbf{G}, X)_K$ be the associated rigid analytic Shimura variety over $\mathbb{C}_p$ of level K. We let $K = K^p K_p$, where $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$ and $K_p \subset G(\mathbb{Q}_p)$ are open compact subgroups. We set

$$\mathcal{S}(\mathbf{G}, X)_{K^p} = \lim_{K_p \to \{1\}} \mathcal{S}(\mathbf{G}, X)_{K^p K_p}$$

If $(\mathbf{G}, X)$ is of pre-abelian type, this is (up to completing the structure sheaf) representable by a perfectoid space and in general it is a diamond. By the results of [Han16], there exists a canonical $G(\mathbb{Q}_p)$-equivariant Hodge-Tate period map

$$\pi_{HT} : \mathcal{S}(\mathbf{G}, X)_{K^p} \to \mathcal{F}\ell_{G,\mu^{-1}}$$

where $\mathcal{F}\ell_{G,\mu^{-1}} := (G_{\mathbb{C}_p}/P_{\mu^{-1}})^{ad}$ is the adic space associated to the flag variety defined by the parabolic $P_{\mu^{-1}} \subset G_{\mathbb{C}_p}$ given by μ^{-1} via the dynamical method. By the $G(\mathbb{Q}_p)$-equivariance, π_{HT} descends to a map:

$$\pi_{HT,K_p} : \mathcal{S}(\mathbf{G}, X)_{K_pK_p} \to [\mathcal{F}\ell_{G,\mu^{-1}}/\underline{K_p}]$$

We let $b \in B(G, \mu)$ be the unique basic element, and let $\mathcal{F}\ell^b_{G,\mu^{-1}}$ be the basic Newton stratum. This parametrizes, for S a perfectoid space in characterstic p, modifications $\mathcal{E}_0 \dashrightarrow \mathcal{E}$ of type μ^{-1} between the trivial G-bundle $\mathcal{E}_0$ on the relative Fargues-Fontaine curve X_S and $\mathcal{E}$ a bundle isomorphic to the G-bundle $\mathcal{E}_b$ corresponding to $b \in B(G)$ after pulling back to a geometric point of S. (G, b, μ) defines a local Shimura datum, as in section 3.1, so we may consider the infinite level Shimura variety/Shtuka space $Sht(G, b, \mu)_\infty$. By pulling back along π_{HT}, we get an open subspace $\mathcal{S}(\mathbf{G}, X)^b_{K^p} \subset \mathcal{S}(\mathbf{G}, X)_{K^p}$, which descends to an open subspace $\mathcal{S}(\mathbf{G}, X)^b_K$, for $K \subset \mathbf{G}(\mathbb{A}_f)$ an open compact. We now have the key definition.

Definition 4.1. We say a global Shimura datum $(\mathbf{G}, X)$ satisfies basic uniformization at p if there exists a unique up to isomorphism $\mathbb{Q}$-inner form $\mathbf{G}'$ of $\mathbf{G}$ satisfying

- $\mathbf{G}'_{\mathbb{A}^p_f} \simeq \mathbf{G}_{\mathbb{A}^p_f}$ as algebraic groups over $\mathbb{A}^p_f$,
- $\mathbf{G}'_{\mathbb{Q}_p} \simeq J_b$, where J_b is the inner form of G given by the σ-centralizer of the basic element $b \in B(G, \mu)$,
- $\mathbf{G}'(\mathbb{R})$ is compact modulo center,

and a $\mathbf{G}(\mathbb{A}_f)$-equivariant isomorphism of diamonds over $\mathbb{C}_p$

$$\lim_{K^p \to \{1\}} \mathcal{S}(\mathbf{G}, X)^b_{K^p} \simeq \underline{\mathbf{G}'(\mathbb{Q})}\backslash\underline{\mathbf{G}'(\mathbb{A}_f)} \times_{Spd(\mathbb{C}_p)} Sht(G, b, \mu)_\infty/\underline{J_b(\mathbb{Q}_p)} \tag{2}$$

such that under the identification $\mathcal{F}\ell^b_{G,\mu^{-1}} \simeq Sht(G, b, \mu)_{\infty,\mathbb{C}_p}/\underline{J_b(\mathbb{Q}_p)}$ the morphism

$$\pi_{HT} : \lim_{K^p \to \{1\}} \mathcal{S}(\mathbf{G}, X)^b_{K^p} \to \mathcal{F}\ell^b_{G,\mu^{-1}}$$

identifies with the projection

$$\underline{\mathbf{G}'(\mathbb{Q})}\backslash\underline{\mathbf{G}'(\mathbb{A}_f)} \times_{Spd(\mathbb{C}_p)} Sht(G, b, \mu)_{\infty,\mathbb{C}_p}/\underline{J_b(\mathbb{Q}_p)} \to Sht(G, b, \mu)_{\infty,\mathbb{C}_p}/\underline{J_b(\mathbb{Q}_p)}$$

where $\mathbf{G}(\mathbb{A}_f) \simeq \mathbf{G}'(\mathbb{A}^p_f) \times G(\mathbb{Q}_p)$ acts on the RHS via the natural action of $\mathbf{G}'(\mathbb{A}^p_f)$ on $\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)$ and $G(\mathbb{Q}_p)$ on $Sht(G, b, \mu)_\infty$. Moreover, if the reflex field of the cocharacter $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$ is $E/\mathbb{Q}_p$ then this isomorphism descends to an isomorphism of diamonds over $\breve{E} := E\breve{\mathbb{Q}}_p$.

We now mention some consequences of uniformization which will be key to us in what follows. Let $\mathcal{H}(J_b) := C^\infty_c(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ be the usual smooth Hecke algebra. We fix an algebraic representation of $\mathbf{G}/\mathbb{Q}$, denoted $\mathcal{V}_\xi$, of some regular highest weight ξ. The isomorphism $i : \overline{\mathbb{Q}}_\ell \xrightarrow{\simeq} \mathbb{C}$ then determines a $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{L}_\xi$ on the Shimura variety $\mathcal{S}(G, X)_{K^p}$. We now consider the space of algebraic automorphic forms valued in $\mathcal{V}_\xi$:

$$\mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p, \mathcal{L}_\xi) := \mathrm{colim}_{K_p \to \{1\}} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^pK_p, \mathcal{L}_\xi)$$

The isomorphism (2) then allows us to deduce an isomorphism

$$R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p, \mathcal{L}_\xi) \xrightarrow{\simeq} R\Gamma_c(\mathcal{S}(\mathbf{G}, X)^b_{K^p}, \mathcal{L}_\xi)$$

of $G(\mathbb{Q}_p) \times W_E$-modules, which when composed with the morphism

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)^b_{K^p}, \mathcal{L}_\xi) \to R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)$$

coming from excision with respect to the open strata $\mathcal{S}(\mathbf{G}, X)^b_{K^p} \hookrightarrow \mathcal{S}(\mathbf{G}, X)_{K^p}$, gives rise to the uniformization map mentioned in the introduction.

Corollary 4.1. *Assume that $(\mathbf{G}, X)$ satisfies basic uniformization at p, then there exists a $G(\mathbb{Q}_p) \times W_E$-equivariant map*

$$\Theta : R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p, \mathcal{L}_\xi) \to R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)$$

functorial in the level K^p.

4.2. **Boyer's Trick.** We will now be interested in applying uniformization to the situation we are interested in, proving an analogue of Boyer's trick [Boy09] and deducing some relevant consequences. The relevant results are due to Shen.

Theorem 4.2. *[She17] If $(\mathbf{G}, X)$ is a Shimura datum of abelian type and $p > 2$ is a prime where G is unramified then $(\mathbf{G}, X)$ satisfies basic uniformization at p.*

Now, consider a Shimura datum $(\mathbf{G}, X)$, where $\mathbf{G}$ is a $\mathbb{Q}$-inner form of $\mathbf{G}^* := Res_{F/\mathbb{Q}}GSp_4$, with $F/\mathbb{Q}$ a totally real field such that p is totally inert and $F_p \simeq L$ a fixed unramified extension of $\mathbb{Q}_p$ and assume that $\mathbf{G}_{\mathbb{Q}_p} \simeq Res_{L/\mathbb{Q}_p}GSp_4 = G$. We fix a level $K = K_pK^p \subset \mathbf{G}(\mathbb{A}_f)$ as before, and see that, under the assumptions, μ is the Siegel cocharacter and therefore the unique basic $b \in B(G, \mu)$ will have σ-centralizer given by $Res_{L/\mathbb{Q}_p}GU_2(D)$, with D/L the quaternionic division algebra. Since $L/\mathbb{Q}_p$ is unramified and $p > 2$, we can apply Theorem 4.2 to deduce basic uniformization at p. Let $\mathbf{G}'$ be the $\mathbb{Q}$-inner form defined above. Now we prove the following result which plays a similar role to Boyer's trick [Boy09] in the study of the cohomology of the Lubin-Tate/Drinfeld towers.

Lemma 4.3. *For $b \in B(G, \mu)$ non-basic, the adic Newton strata $\mathcal{F}\ell^b_{G,\mu^{-1}}$ is parabolically induced as a space with $G(\mathbb{Q}_p)$-action.*

Proof. There are two non-basic elements $b \in B(G, \mu)$. The μ-ordinary locus, defined by the maximal element $b^{max} \in B(G, \mu)$ with respect to the partial ordering on $B(G, \mu)$, and the intermediate strata corresponding to the element lying between the basic element and b^{max} with respect to the partial ordering on $B(G)$. For the μ-ordinary locus, the corresponding element $b \in B(G, \mu)$ admits a basic reduction to an element $b_T \in B(T)_{basic}$, where $T \simeq GL_1 \times GL_1$ is the maximal torus inside GSp_4/L. We have that $\mathcal{F}\ell^b_{G,\mu^{-1}}$ is the parabolic induction of the Newton strata $\mathcal{F}^{b_T}_{T,\mu^{-1}}$, which is in particular just a point. This gives us that $G/P_{\mu^{-1}}(L) \simeq \mathcal{F}\ell^b_{G,\mu^{-1}}$. Similarly, the intermediate strata corresponding to $b \in B(G, \mu)$ admits a basic reduction to $b_{GL_2 \times GL_1} \in B(GL_2 \times GL_1)_{basic}$. Then we can consider the element $b_{GL_2} \in B(GL_2)_{basic}$ induced by this and the analogous space $\mathcal{F}\ell^{b_{GL_2}}_{GL_2,\mu^{-1}_{GL_2}}$, where $\mu^{-1}_{GL_2}$ is the cocharacter induced by μ^{-1} via the natural projection $GL_2 \times GL_1 \to GL_2$. If we let $\mathcal{O}(\lambda)$ denote the vector bundle on the Fargues-Fontaine curve X of slope $\lambda \in \mathbb{Q}$ then $\mathcal{F}\ell^b_{G,\mu^{-1}}$ parametrizes modifications

$$\mathcal{O}(-\frac{1}{2}) \oplus \mathcal{O}(-1) \oplus \mathcal{O} \to \mathcal{O}^4$$

of type $\mu = (1, 1, 0, 0)$ of GSp_4-bundles at ∞. Now the flag variety $\mathcal{F}\ell^{b_{GL_2}}_{GL_2,\mu^{-1}_{GL_2}}$ parametrizes modifications

$$\mathcal{O}(-\frac{1}{2}) \to \mathcal{O}^2$$

of type $(1, 0)$, which in particular can be identified with the open subset $\Omega \subset \mathbb{P}^1_{\mathbb{C}_p}$ given by the Drinfeld upper half plane, via looking at the coordinates defining the modification. It is clear that every such GSp_4-modification uniquely determines such a modification of GL_2-bundles. Namely, such a modification is specified by modifications of the form $\mathcal{O}(-1) \to \mathcal{O}$ and $\mathcal{O}(-\frac{1}{2}) \to \mathcal{O}^2$ of type 1 and $(1, 0)$, respectively, using the symplectic similitude forms on the bundles. One now notes that, since there is a unique modification of the form $\mathcal{O}(-1) \to \mathcal{O}$ at a given Cartier divisor, the GSp_4-modification is uniquely determined by the induced GL_2-modification, so $\mathcal{F}\ell^b_{G,\mu^{-1}} =$

$\Omega \times^{Q(L)} GSp_4(L)$, where $Q(L)$ is the Klingen parabolic acting on Ω via the natural projection $Q(L) \to GL_2(L)$. $\square$

We now consider some irreducible representation ξ as above and look at the uniformization map

$$\Theta : R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p, \mathcal{L}_\xi) \to R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)$$

furnished by Corollary 4.1 and Theorem 4.2. We let $R\Gamma_c(G, b, \mu)_{sc}$ and $R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)_{sc}$ be the direct summands where $G(\mathbb{Q}_p)$ acts via a supercuspidal representation. Then we have the following key consequence of the previous lemma, which justifies why we are referring to this as Boyer's trick.

Proposition 4.4. *The uniformization map Θ induces an isomorphism*

$$\Theta_{sc} : R\Gamma_c(G, b, \mu)_{sc} \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p, \mathcal{L}_\xi) \xrightarrow{\simeq} R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)_{sc}$$

on the summand where $G(\mathbb{Q}_p)$ acts via a supercuspidal representation.

Proof. It follows from the above construction of the uniformization map that the cone of Θ is identified with

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)^{nbas}_{K^p}, \mathcal{L}_\xi)$$

where $\mathcal{S}(\mathbf{G}, X)^{nbas}_{K^p}$ denotes the non-basic locus (i.e the preimage of $\mathcal{F}\ell^{nbas}_{G,\mu^{-1}}$ under π_{HT}, the closed complement of the open Newton strata $\mathcal{F}\ell^{b}_{G,\mu^{-1}}$, where $b \in B(G, \mu)$ is the unique basic element). We want to show that the cohomology is parabolically induced as a $G(\mathbb{Q}_p)$ representation. For this, we consider the Hodge-Tate period morphism:

$$\pi_{HT} : \mathcal{S}(\mathbf{G}, X)^{nbas}_{K_p} \to \mathcal{F}\ell^{nbas}_{G,\mu^{-1}}$$

The Cartan-Leray spectral sequence then gives us:

$$E_2^{p,q} = H^p_c(\mathcal{F}\ell^{nbas}_{G,\mu^{-1}}, R^q\pi_{HT,*}(\mathcal{L}_\xi)) \Longrightarrow H^{p+q}_c(\mathcal{S}(\mathbf{G}, X)^{nbas}, \mathcal{L}_\xi)$$

Since the Hodge-Tate period map is $G(\mathbb{Q}_p)$-equivariant by construction this tells us that the maps defining this spectral sequence are also equivariant. Therefore, we are reduced to showing the following.

Lemma 4.5. *For all integers $p, q \geq 0$ the cohomology of $H^p_c(\mathcal{F}\ell^{nbas}_{G,\mu^{-1}}, R^q\pi_{HT,*}(\mathcal{L}_\xi))$ is parabolically induced as a $G(\mathbb{Q}_p)$-representation.*

Proof. We apply excision with respect to the locally closed stratification given by $\mathcal{F}\ell^{b}_{G,\mu^{-1}}$ for $b \in B(G, \mu)$ which is not basic. Since these Newton strata are stable under the $G(\mathbb{Q}_p)$ action this reduces us to showing that the cohomology of

$$H^p_c(\mathcal{F}\ell^{b}_{G,\mu^{-1}}, R^q\pi_{HT,*}(\mathcal{L}_\xi)|_{\mathcal{F}\ell^{b}_{G,\mu^{-1}}})$$

is parabolically induced as a $G(\mathbb{Q}_p)$-representation. However, this follows from Lemma 4.3. $\square$

$\square$

This result will be the key tool in allowing us to describe the W_L-action on the local Shimura variety by global methods. We start this global analysis by constructing strong transfers between GSp_4 and its inner forms over a number field F and proving a strong multiplicity one result.

5. Existence of Strong Transfers and a Strong Multiplicity One Result

In this section, we will show the existence of strong transfers of certain automorphic representations of an inner form of GSp_4 over a number field F, using the analysis of the trace formula similar to that of [KS16, Section 6]. We will then combine this with analysis of the simple twisted trace formula of Kottwitz-Shelstad [KS99], to deduce a kind of strong multiplicity one result for inner forms of GSp_4.

5.1. **The Simple Trace Formula and Existence of Strong Transfers.** In order to describe the Galois action on the global Shimura variety, we will need to construct strong transfers for inner forms of GSp_4/F over a totally real field F, this will allow us to compute the traces of Frobenius on the global Shimura variety in terms of the Langlands parameters of the strong transfer. The construction of strong transfers will be accomplished by applying the elliptic part of the stable trace formula with respect to the Lefschetz functions constructed by Kret-Shin [KS16] at the Steinberg/Infinite places and pseudo-coefficients at some finite number places where the representation has supercuspidal L-parameter, applying the character identities of Chan-Gan [CG15] to conclude equality of the orbital integrals at these latter places. First, we recall the key results of Kret-Shin on the trace formula with fixed central character. For now, we will work generally. Let $\mathbf{G}$ denote a connected reductive group over a number field F with center Z, write A_Z for the maximal $\mathbb{Q}$-split torus of $Res_{F/\mathbb{Q}}Z$, and set $A_{Z,\infty} = A_Z(\mathbb{R})^0$ to be the connected component of the identity. Let $\mathbb{A}_F$ be the adeles of F and write $\mathbf{G}(\mathbb{A}_F)^1$ for a choice of subgroup so that $\mathbf{G}(\mathbb{A}_F) = \mathbf{G}(\mathbb{A}_F)^1 \times A_{Z,\infty}$, as in [Art81, Page 11]. We consider a closed subgroup $\mathfrak{X} \subset Z(\mathbb{A}_F)$ which contains $A_{Z,\infty}$ such that $Z(F)\mathfrak{X}$ is closed inside $Z(\mathbb{A}_F)$ and a continuous character $\chi : (\mathfrak{X} \cap Z(F))\backslash\mathfrak{X} \to \mathbb{C}^*$.

Let v be a place of F and let $\chi_v : \mathfrak{X}_v \to \mathbb{C}^*$ be a smooth character. Write $\mathcal{H}(\mathbf{G}(F_v), \chi_v^{-1})$ for the space of smooth compactly supported functions on $\mathbf{G}(F_v)$ which transform under $\mathfrak{X}(F_v)$ via χ_v^{-1}. We also require the functions to be K_v-finite for some maximal compact subgroup K_v of $\mathbf{G}(F_v)$ if v is archimedean. Given a semisimple element $\gamma_v \in \mathbf{G}(F_v)$ and an admissible representation π_v of $\mathbf{G}(F_v)$ with central character χ_v on $\mathfrak{X}(F_v)$, we define the orbital integral and trace character for $f_v \in \mathcal{H}(\mathbf{G}(F_v), \chi_v^{-1})$ as follows. Let I_{γ_v} denote the connected centralizer of γ_v in $\mathbf{G}$. We have

$$O_{\gamma_v}(f_v) := \int_{I_{\gamma_v}(F_v)\backslash\mathbf{G}(F_v)} f_v(x^{-1}\gamma_v x)dx$$

and

$$tr(f_v|\pi_v) = tr(\pi_v(f_v)) := tr(\int_{\mathbf{G}(F_v)/Z(F_v)} f_v(g)\pi_v(g)dg)$$

where we have fixed compatible choices of Haar measure on $\mathbf{G}$ and Z throughout. We note that this operator is well defined since the above operator is of finite rank if v is finite and is of trace class if v is infinite, by the K_v-finiteness assumption.

We define the adelic Hecke algebra $\mathcal{H}(\mathbf{G}(\mathbb{A}_F), \chi^{-1})$, as well as the global orbital integrals by taking a restricted tensor product over the local Hecke algebras defined above and products of the local integrals. Write $\Gamma_{ell}(\mathbf{G})$ to be the set of F-elliptic conjugacy classes in $\mathbf{G}(F)$. Let $\mathbb{A}_{f,F}$ denote the finite adeles of F. We make the following assumptions in what follows

1. $\mathfrak{X} = \mathfrak{X}^\infty\mathfrak{X}_\infty$ for an open compact subgroup $\mathfrak{X}^\infty \subset Z(\mathbb{A}_{f,F})$ and $\mathfrak{X}_\infty = Z(F_\infty)$,
2. $\chi = \prod_v \chi_v$ with $\chi_v = 1$ at every finite place v.

We let $L^2_{disc,\chi}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F))$ denote the space of functions on $\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F)$ transforming under $\mathfrak{X}$ by χ and square-integrable on $\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F)^1/\mathfrak{X}(\mathbb{A}_F) \cap G(\mathbb{A}_F)^1$. Write $\mathcal{A}_{cusp,\chi}(\mathbf{G})$ for the set of isomorphism classes of cuspidal automorphic representations of $\mathbf{G}(\mathbb{A}_F)$ whose central characters restricted to $\mathfrak{X}$ are χ. For $f \in \mathcal{H}(\mathbf{G}(\mathbb{A}_F), \chi^{-1})$, define the invariant distributions $T^{\mathbf{G}}_{ell,\chi}$ and $T^{\mathbf{G}}_{disc,\chi}$ by

$$T^{\mathbf{G}}_{ell,\chi}(f) := \sum_{\gamma \in \Gamma_{ell}(\mathbf{G})} i(\gamma)^{-1} vol(I_\gamma(F)\backslash I_\gamma(\mathbb{A}_F)/\mathfrak{X}(\mathbb{A}_F))O_\gamma(f)$$

$$T^{\mathbf{G}}_{disc,\chi}(f) := tr(f|L^2_{disc,\chi}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F)))$$

where $i(\gamma)$ is the number of connected components in the centralizer of γ. Analogously, we define $T^{\mathbf{G}}_{cusp,\chi}(f)$ by taking the trace on the space of square-integrable cusp forms whose central character restricted to $\mathfrak{X}$ is χ. Let $\mathbf{G}^*$ denote the quasi-split inner form of $\mathbf{G}$ over F, with a fixed inner twist $\mathbf{G}^* \simeq \mathbf{G}$ over $\overline{F}$. Since Z is canonically identified with the center of $\mathbf{G}^*$, we may view the character χ as a central character datum for $\mathbf{G}^*$. We then let f^* denote a Langlands-Shelstad transfer of f to $\mathbf{G}^*$. One can construct such a transfer by lifting f along the surjection $\mathcal{H}(\mathbf{G}(\mathbb{A}_F)) \to \mathcal{H}(\mathbf{G}(\mathbb{A}_F), \chi^{-1})$ applying the transfer with trivial central character due to Waldspurger and then taking the image along the analogous surjection for $\mathbf{G}^*(\mathbb{A}_F)$. We let $\Sigma_{ell}(\mathbf{G}^*)$ denote the set of stable F-elliptic conjugacy classes in $\mathbf{G}^*(F)$. We define the stable elliptic distribution

$$ST^{\mathbf{G}^*}_{ell,\chi}(f^*) := \tau(\mathbf{G}^*) \sum_{\gamma \in \Sigma_{ell,\chi}(\mathbf{G}^*)} \tilde{i}(\gamma)^{-1} SO^{\mathbf{G}^*}_{\gamma,\chi}(f)$$

where $SO^{\mathbf{G}^*}_{\gamma,\chi}(f^*)$ denotes the stable orbital integral of f^* at γ, $\tau(\mathbf{G}^*)$ is the Tamagawa number of $\mathbf{G}^*$, and $\tilde{i}(\gamma)$ is the number of Galois fixed connected components of the centralizer of γ in $\mathbf{G}^*$. Let ξ be an irreducible representation of $\mathbf{G}_{F_\infty}$. Denote by $\chi_\xi : Z(F_\infty) \to \mathbb{C}^*$ the restriction of ξ to $Z(F_\infty)$. Write $f^{\mathbf{G}}_\xi \in \mathcal{H}(G(F_\infty), \chi_\xi^{-1})$ for a Lefschetz function associated with ξ. In other words, a function such that $tr(\pi_\infty(f^{\mathbf{G}}_\xi))$ computes the Euler-Poincaré characteristic for the relative Lie algebra cohomology of $\pi_\infty \otimes \xi$ for every irreducible admissible representation π_∞ of $\mathbf{G}(F_\infty)$ with central character χ_ξ. It follows by the Vogan-Zuckerman classification [VZ84] that, if ξ is regular, this will be non-zero if and only if π_∞ is discrete-series representation cohomological of regular weight ξ. For a finite place v_{st} of F, we let $f_{v_{st}} := f^{\mathbf{G}}_{Lef,v_{st}}$ denote a Lefschetz function at v_{st}. Morally, this should be characterized by the property that $tr(\pi_v(f_{v_{st}}))$ computes the Euler-Poincaré characteristic of the continuous group cohomology of $\mathbf{G}(F_{v_{st}})$ valued in $\pi_{v_{st}}$. In the case that the center is anisotropic, it follows from the computations in [BW00, Theorem XI.3.9] that this means that the trace of $f^{\mathbf{G}}_{Lef,v_{st}}$ will only be non-zero if π_v is 1-dimensional or an unramified twist of the Steinberg representation, for all irreducible admissible unitary π_v. In this case, they were originally constructed by Kottwitz; however, these results do not apply for the desired application, since the center of GSp_4 is not compact. For the construction of these functions in this case and the proof of the property that their traces detect when a representation is trivial or an unramified twist of the Steinberg, see [KS16, Appendix A]. Now we have the key lemma, which tells us that with respect to these choices of test functions we obtain a "simple" trace formula.

Lemma 5.1. *[KS16, Lemma 6.1, 6.2] Fix a central character datum $(Z(F_\infty), \chi)$ with $\chi_\xi = \chi$ as above. For an element $f \in \mathcal{H}(\mathbf{G}(\mathbb{A}_F), \chi_\xi^{-1})$ in the global Hecke algebra as above, assume that $f_\infty = f^{\mathbf{G}}_\xi \in \mathcal{H}(\mathbf{G}(F_\infty), \chi^{-1})$ is a Lefschetz function at ∞ and assume that $f_{v_{st}} = f^{\mathbf{G}}_{Lef,v_{st}}$ is a Lefschetz function at v_{st}. Then we have an equality:*

$$ST^{\mathbf{G}^*}_{ell,\chi}(f^*) = T^{\mathbf{G}}_{ell,\chi}(f) = T^{\mathbf{G}}_{disc,\chi}(f) = T^{\mathbf{G}}_{cusp,\chi}(f)$$

Let S_{st} and S_{sc} be disjoint finite sets of finite places, and let S_0 be a finite set of places contained in $S_{st} \cup S_{sc}$. Let S_∞ denote the infinite places of F. Set S to be a finite set of places containing $S_{st} \cup S_{sc} \cup S_\infty$. We assume that the inner twist $\mathbf{G}^*$ of $\mathbf{G}$ is trivialized away from S_0 and S_∞. In particular, $\mathbf{G}$ is unramified outside S and we can fix a reductive model for $\mathbf{G}$ and $\mathbf{G}^*$ over $\mathcal{O}_F[1/S]$. By abuse of notation, we write $\mathbf{G}$ and $\mathbf{G}^*$ for the integral models of these groups. The inner twist gives an isomorphism $\mathbf{G}^*_{F_v} \simeq \mathbf{G}_{F_v}$ and isomorphisms $\mathbf{G}^*_{\mathcal{O}_{F_v}} \simeq \mathbf{G}_{\mathcal{O}_{F_v}}$ of the hyperspecial subgroups determined by this model for the finite places $v \notin S$. The notion of unramified local representation on either side will be defined with respect to this fixed choice of hyperspecial level.

For the rest of the section, we will assume that $\mathbf{G}^* = GSp_4$. We assume throughout that π is a global cuspidal automorphic representation of the group $\mathbf{G}(\mathbb{A}_F)$ satisfying the following properties:

(1) π is cohomological of some regular weight ξ at infinity.

(2) π_v is unramified at all finite places $v \notin S$.
(3) π_v has supercuspidal L-parameter for $v \in S_{sc}$.
(4) π_v is an unramified twist of Steinberg at all finite places $v \in S_{st}$.

Note, as in Definition 2.1, we can further partition S_{sc} into S_{ssc}, where $std \circ \phi_{\pi_v}$ is irreducible (ssc = stable supercuspidal) and S_{esc}, where $std \circ \phi_{\pi_v}$ is reducible (esc = endoscopic supercuspidal).

The main result of this section shows the existence of strong transfers from $\mathbf{G}$ to $\mathbf{G}^*$ at the places in $S_{sc} \cup S_{st} \cup S_{\infty}$ for a certain class of automorphic representations of $\mathbf{G}$. It is essentially a more refined version of [KS16, Proposition 6.3] in the particular case that $\mathbf{G}^* = GSp_4$.

Theorem 5.2. *Suppose that S_{st} is non-empty. Given a π as above, there exists a cuspidal automorphic representation τ of $\mathbf{G}^*(\mathbf{A}_F)$ satisfying the following:*

- *$\tau^S \simeq \pi^S$.*
- *At all $v \in S_{sc} \cup S_{st} \cup S_{\infty}$, τ_v has the same Langlands parameter as π_v.*

Moreover, we can choose τ to be globally generic. The same is true with the roles of τ and π reversed.

Proof. First off note that, since π is Steinberg at some finite place, τ, if it exists, is automatically (essentially) tempered at all places. (cf. Remark 5.1) It follows, by [GT19, Remark 7.4.7], that the global L-packet of τ therefore contains a globally generic representation. So, if we can show the existence of some τ with the desired properties, that means we can find τ globally generic with the same properties. For the former, we now apply the trace formula.

Let $\mathfrak{X} = Z(F_{\infty})$ and $\chi = \chi_{\xi}$ as above. We set $f = \bigotimes_{v \in F} f_v$ to be a test function on $\mathbf{G}(\mathbb{A}_F)$ satisfying the following:

(1) $f_{\infty} = f_{\xi}^{\mathbf{G}}$ is a Lefschetz/Euler-Poincaré function for the representation ξ of $\mathbf{G}(F_{\infty})$.
(2) At $v \in S_{st}$, $f_v = f_{Lef,v}^{\mathbf{G}}$ is a Lefschetz function for $\mathbf{G}(F_v)$.
(3) At $v \in S_{ssc}$, $f_v = f_{\pi_v}$ is the pseudo-coefficient of π_v.
(4) At $v \in S_{esc}$, $f_v = f_{\pi_v^+} + f_{\pi_v^-}$, where $\{\pi_v^+, \pi_v^-\}$ is the L-packet over ϕ_{π_v} and $f_{\pi_v^{\pm}}$ is the pseudo-coefficient of $\pi_v^{\pm}$.
(5) At the finite places $v \notin S$, f_v is an arbitrary element of the unramified Hecke algebra.
(6) For $v \in S \setminus S_{st} \cup S_{\infty} \cup S_{sc}$, choose f_v to be an arbitrary function.

Given such a f, we choose a test function $f^* = \bigotimes_{v \in F} f_v^*$ on $\mathbf{G}^*(\mathbb{A}_F)$ satisfying the following:

(1) $f_{\infty}^* = f_{\xi}^{\mathbf{G}^*}$ is a Lefschetz/Euler-Poincaré for the representation ξ of $\mathbf{G}^*(F_{\infty})$.
(2) At $v \in S_{st}$, $f_v^* = f_{Lef,v}^{\mathbf{G}^*}$ is a Lefschetz function for $\mathbf{G}^*(F_v)$.
(3) At $v \in S_{ssc}$, $f_v^* = f_{\tau_v}$ is a pseudo-coefficient of τ_v, where τ_v is the unique supercuspidal representation of $\mathbf{G}^*(F_v)$ with Langlands parameter ϕ_{π_v}.
(4) At $v \in S_{esc}$, $f_v^* = f_{\tau_v^+} + f_{\tau_v^-}$, where $\{\tau_v^+, \tau_v^-\}$ is the L-packet over ϕ_{π_v} of $\mathbf{G}^*(F_v)$ and $f_{\tau_v^{\pm}}$ is the pseudo-coefficient of $\tau_v^{\pm}$.
(5) At the finite places $v \notin S$, $f_v^* = f_v$ is the same element of the unramified Hecke algebra.
(6) For $v \in S \setminus S_{st} \cup S_{\infty} \cup S_{sc}$, choose $f_v^* = f_v$.

Now we wish to check that f and f^* are matching up to a non-zero constant c, in the sense of [KS16, Section 5.5]. We can check this place by place. For the finite places $v \notin S_{st} \cup S_{sc} \cup S_{\infty}$, this is tautological. For all $v \in S_{\infty} \cup S_{st}$, this follows from [KS16, Lemma A.4, A.11]. For $v \in S_{sc}$, this follows from the character identities of Chan-Gan [CG15, Proposition 11.1]. Since the test functions are matching and $S_{st} \neq \emptyset$, we can apply Lemma 5.1 to conclude:

$$T_{cusp,\chi}^{\mathbf{G}^*}(f^*) = ST_{ell,\chi}^{\mathbf{G}^*}(f^*) = cT_{cusp,\chi}^{\mathbf{G}}(f)$$

Then, by linear independence of characters, we have a relationship

$$\sum_{\substack{\Pi' \in \mathcal{A}_{cusp,\chi}(\mathbf{G}^*) \\ \Pi'^S \simeq \pi^S}} m(\Pi') tr(f^*_{S_{st}\cup S_\infty \cup S_{sc}} | \Pi'_{S_{st}\cup S_\infty \cup S_{sc}}) = c \cdot \sum_{\substack{\Pi \in \mathcal{A}_{cusp,\chi}(\mathbf{G}) \\ \Pi^S \simeq \pi^S}} m(\Pi) tr(f_{S_{st}\cup S_\infty \cup S_{sc}} | \Pi_{S_{st}\cup S_\infty \cup S_{sc}})$$

where $m(\Pi)$ (resp. $m(\Pi')$) denotes the multiplicity of Π (resp. Π') in the cuspidal sepctrum of $\mathbf{G}$ (resp. $\mathbf{G}^*$). Now at the infinite places, as soon as $tr(f_v|\Pi_v) \neq 0$ at $v \mid \infty$ the regularity condition on ξ implies that Π_v is an (essentially) discrete series representation cohomological of regular weight ξ and that $tr(f_v|\Pi_v) = (-1)^{q(\mathbf{G}_v)}$ by the Vogan-Zuckerman classification of unitary cohomological representations, where $q(\mathbf{G})$ is the F-rank of the derived group of $\mathbf{G}$. At $v_{st} \in S_{st}$ it follows by [KS16, Lemma A.13] that Π_v is either an unramified twist of the Steinberg representation or is the trivial representation. If $\Pi_{v_{st}}$ were one-dimensional then the global representation would also be one-dimensional by a strong-approximation argument [KST20, Lemma 6.2], implying that Π_∞ cannot be tempered, which would contradict the fact Π_∞ is an (essentially) discrete series representation. Therefore, $\Pi_{v_{st}}$ is always an unramified twist of the Steinberg representation. In this case, the trace is also given by $(-1)^{q(\mathbf{G})}$. At the remaining $v \in S_{ssc}$ (resp. $v \in S_{esc}$), it follows from the definition of pseudo-coefficients that, if $tr(f_v|\Pi_v) \neq 0$, we have $\Pi_v \simeq \pi_v$ (resp. $\Pi_v \in \{\pi_v^+, \pi_v^-\}$) and that $tr(f_v|\Pi_v) > 0$. Similar considerations apply for $\Pi' \in \mathcal{A}_{cusp,\chi}(\mathbf{G}^*)$ occurring non-trivially in the LHS. In summary, by the above analysis, we can deduce that the RHS of the previous equation is non-zero for the term corresponding to π and that all the non-trivial terms on the RHS have the same sign. Therefore, the LHS is also non-zero, and we see, by choosing any non-zero term, that we obtain the desired τ. The converse direction works similarly, where the role of $\mathbf{G}$ and $\mathbf{G}^*$, are swapped. □

Remark 5.1. We note that we crucially used at the places $v \in S_{sc}$ that ϕ_{π_v} was supercuspidal. Otherwise, $tr(f_v|\Pi_v) \neq 0$ wouldn't necessarily imply that Π_v lies in the L-packet over ϕ_{π_v} without assuming that Π_v is tempered. However, by [KS16, Corollary 2.8] any representation of GSp_4 that is Steinberg at some non-empty finite set of places is tempered at all places. Therefore, we can relax this assumption at least for the forward direction of Theorem 5.2 to just assuming that τ_v is a discrete series representation at all $v \in S_{sc}$.

5.2. **The Stable and σ-twisted Simple Trace Formula.** For the proof of strong multiplicity one, we will need some more refined analysis of trace formulae. Namely, we will be interested in the discrete part of the stable trace formula in the particular case of $\mathbf{G}^* = GSp_4/F$, as discussed in [CG15, Section 7.1]. To this end, fix a central character datum $(\mathfrak{X}, \chi)$ as before. We recall that the unique elliptic proper endoscopic group of GSp_4/F is $C = GSO(2,2) \simeq (GL_2 \times GL_2)/\{(t, t^{-1}) | t \in GL_1\}$. Then, for a test function f^* on $\mathbf{G}^*(\mathbb{A}_F)$ as above, the discrete part of the stable trace formula is an equality:

$$I^{\mathbf{G}^*}_{disc,\chi}(f^*) = ST^{\mathbf{G}^*}_{disc,\chi}(f^*) + \frac{1}{4} ST^C_{disc,\chi}(f^C)$$

for f^C a matching test function on C. Here

$$I^{\mathbf{G}^*}_{disc,\chi}(f^*) = \sum_M |W(G,M)|^{-1} \cdot \sum_{s \in W(M,G)_{reg}} |det(s-1)_{\mathfrak{a}_M/\mathfrak{a}_G}|^{-1} \cdot tr(M_P(s,0) \cdot I^P_{disc,\chi}(0,f))$$

is a sum indexed over classes of standard Levi subgroups of $\mathbf{G}^*$. The precise definition of the terms will not be important for our purposes, but the interested reader can look at [Art02, Section 3]. We simply note that the term corresponding to $M = \mathbf{G}^*$ is precisely equal to $T^{\mathbf{G}^*}_{disc,\chi}(f^*)$, as defined in section 5.1. $ST^{\mathbf{G}^*}_{disc,\chi}$ is a stable distribution on $\mathbf{G}^*$, similar to $ST^{\mathbf{G}^*}_{ell,\chi}$, and $ST^C_{disc,\chi}$ is the analogous stable distribution on $C(\mathbb{A}_F)$. However, since C has no proper elliptic endoscopic group, we have

$$ST^C_{disc,\chi}(f^C) = I^C_{disc,\chi}(f^C) = T^C_{disc,\chi}(f^C) + (\text{other terms})$$

with the other terms indexed by proper standard Levi subgroups of C as above. We will be interested in combining this with the elliptic part of the twisted trace formula as described by Kottwitz-Shelstad [KS99] for the particular group $\tilde{\mathbf{G}} := GL_4 \times GL_1/F$ with respect to involution

$$\sigma : (g, e) \mapsto (J^t g^{-1} J^{-1}, edet(g))$$

where

$$J := \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}$$

We can enumerate the elliptic σ-twisted endoscopic groups as follows.

(1) $\mathbf{G}^* = GSp_4$
(2) $C_E = Res_{E/F}GL_2' := \{(g_1, g_2) \in Res_{E/F}GL_2 | det(g_1) = det(g_2)\}$
(3) $C_+^E = (GL_2 \times Res_{E/F}GL_1)/GL_1$

where E is an étale quadratic F-algebra and E is not split in case (3). The simple stable twisted trace formula says that if $\tilde{f}$ is a test function on $\tilde{\mathbf{G}}$ whose twisted orbital integral is supported on the regular elliptic set at at least 3 finite places, then we have an identity

$$I^{\tilde{\mathbf{G}},\sigma}_{disc,\chi}(\tilde{f}) = \frac{1}{2} ST^{\mathbf{G}^*}_{disc,\chi}(f^*) + \frac{1}{4}\sum_E ST^{C_E}_{disc,\chi}(f^{C_E}) + \frac{1}{8}\sum_{E \neq F^{\oplus 2}} ST^{C_+^E}_{disc,\chi}(f^{C_+^E})$$

where

- $\sum_E$ is a sum over étale quadratic F-algebras E,
- $\tilde{f}$, f^*, f^{C_E}, and $f^{C_+^E}$ are matching test functions,
- $ST^{\mathbf{G}^*}_{disc,\chi}$, $ST^{C_E}_{disc,\chi}$, and $ST^{C_+^E}_{disc,\chi}$ are the stable distributions appearing in the discrete part of the stable trace formula, as described above,
- $I^{\tilde{\mathbf{G}},\sigma}_{disc,\chi}$ is the invariant distribution which is the twisted analogue of $I^{\mathbf{G}^*}_{disc,\chi}$. It is given by [LW13, Theorem 14.3.1 and Proposition 14.3.2] and has the form

$$I^{\tilde{\mathbf{G}},\sigma}_{disc,\chi}(\tilde{f}) = \sum_M |W(G,M)|^{-1} \cdot \sum_{s \in W(M,G)_{reg}} |det(s-1)^{s\cdot\sigma}_{\mathfrak{a}_M/\mathfrak{a}_G}|^{-1} \cdot tr(M_P(s,0) \cdot I^P_{disc,\chi}(0,f) I_{P,disc}(\sigma))$$

where the sum runs over standard Levi subgroups M of G.

Now we want to apply these trace formulae with appropriately chosen test functions. We will assume that S_{st} is a finite set of places such that $|S_{st}| \geq 3$. Then, for all $v \in S_{st}$, we let f_v^* be a pseudo-coefficient for the unramified twist of Steinberg by χ_v. It follows by [CG15, Corollary 10.8], that we can choose the local constituent of the matching function $\tilde{f}$ at v to be the σ-twisted pseudo-coefficient of the Steinberg twisted by χ_v, as defined in [MW18]. These functions are supported on the regular elliptic set and therefore we can apply the simple twisted trace formula. Moreover, the twisted orbital integral of $\tilde{f}_v$ is a stable function, and hence the κ-orbital integral of $\tilde{f}_v$ is zero for all $\kappa \neq 1$. Therefore, it follows that the transfers of f_v to all elliptic twisted endoscopic groups of $(\tilde{\mathbf{G}}_{F_v}, \sigma)$ vanish, except possibly for $\mathbf{G}^*_{F_v}$. Thus, the simple twisted trace formula simplifies giving an equality:

$$I^{\tilde{\mathbf{G}},\sigma}_{disc,\chi}(\tilde{f}) = \frac{1}{2} ST^{\mathbf{G}^*}_{disc,\chi}(f^*)$$

Now we apply the discrete part of the stable trace formula for $\mathbf{G}^*$ to the RHS this gives us an equality:

$$I^{\mathbf{G}^*}_{disc,\chi}(f^*) - \frac{1}{4} ST^{C}_{disc,\chi}(f^{C}) = ST^{\mathbf{G}^*}_{disc,\chi}(f^*)$$

However, for any $v \in S_{st}$, the orbital integral f_v^* is again stable, so we see that its transfer to the endoscopic group C is zero. Hence, the second term vanishes. All in all, we obtain the following lemma.

Lemma 5.3. *For S_{st} a finite set of finite places with $|S_{st}| \geq 3$, f^* and $\tilde{f}$ matching test functions on $\mathbf{G}^*$ and $\tilde{\mathbf{G}}$, respectively, such that f_v^* is a pseudo-coefficient for the Steinberg representation twisted by χ_v and $\tilde{f}_v$ is the σ-twisted pseudo-coefficient for the Steinberg representation of $\tilde{\mathbf{G}}_{F_v}$ twisted by χ_v, we have an equality:*

$$\frac{1}{2} I^{\mathbf{G}^*}_{disc,\chi}(f^*) = I^{\tilde{\mathbf{G}},\sigma}_{disc,\chi}(\tilde{f})$$

relating spectral information on $\mathbf{G}^*$ to $\tilde{\mathbf{G}}$.

5.3. **Strong Multiplicity One.** We now would like to combine the analysis of sections 5.1 and 5.2 to deduce a strong multiplicity one result for $\mathbf{G}^* = GSp_4/F$ and certain inner forms. Our analysis is very similar to [CG15, Sections 10.5 and 10.6] and benefited from reading the proofs of [RW21, Proposition 10.1 and Theorem 11.4] in a paper of Rosner and Weissauer, where they prove a similar multiplicity one result using Weselmann's topological twisted trace formula [Wes12] instead of the simple twisted trace formula of Kottwitz-Shelstad. Let S_{st} and S_{sc} be disjoint finite sets of finite places. Let S_∞ denote the infinite set of places. Set $S_0 \subset S_{st} \cup S_{sc}$ and $S_{sc} \cup S_{st} \cup S_\infty \subset S$ to be finite sets of places as before. We let $\mathbf{G}$ be an inner form over F, as in Theorem 5.2, trivialized outside of $S_0 \cup S_\infty$. We have the following.

Proposition 5.4. *Assume that $|S_{st}| \geq 3$. Let π be a cuspidal automorphic representation of $\mathbf{G}^* = GSp_4/F$ or the above inner form $\mathbf{G}$ satisfying the following:*

1. *π is cohomological of regular weight ξ at infinity,*
2. *π is unramified outside of S,*
3. *π is an unramified twist of Steinberg at all places in S_{st},*
4. *π has supercuspidal L-parameter at all places in S_{sc}.*

If π' is a cuspidal automorphic representation of $\mathbf{G}^*$ satisfying (1), (3), and $\pi'^S \simeq \pi^S$ then its Langlands parameter at all places in S agrees with π. If π' is a cuspidal automorphic representation of $\mathbf{G}$ satisfying conditions (1), (3), and $\pi'^S \simeq \pi^S$ then its Langlands parameter at all places in $S_{st} \cup S_{sc} \cup S_\infty$ agrees with π.

Proof. Set χ to be the central character of π. We apply the above trace formulae with central character datum $(Z(\mathbb{A}_F), \chi)$. If π is a representation of $\mathbf{G}^* = GSp_4/F$, we take τ to be a globally generic member of the global L-packet of π, as in the proof of Theorem 5.2. If π is a representation of the inner form $\mathbf{G}$ then, using Theorem 5.2, we take τ to be a globally generic strong transfer τ of π to a cuspidal automorphic representation of $\mathbf{G}^*$, with Langlands parameter equal to ϕ_{π_v} at all places in $v \in S_{sc} \cup S_{st} \cup S_\infty$. Now we apply [Sor10, Theorem 1] to τ to deduce the existence of a strong transfer to a generic automorphic representation of $GL_4(\mathbb{A}_F)$, denoted $\tilde{\tau}$. It satisfies the following:

1. $\tilde{\tau}$ is a global theta lift of τ.
2. $\tilde{\tau}^\vee \otimes \chi \simeq \tilde{\tau}$.
3. For all places v, we have that $\phi_{\tilde{\tau}_v} = std \circ \phi_{\tau_v}$ as parameters. In particular, by the assumption that S_{st} is non-empty $\tilde{\tau}$ is automatically cuspidal.

We choose matching test functions $\tilde{f}$ and f^* on $\tilde{\mathbf{G}}$ and $\mathbf{G}^*$, respectively, such that, for $v \in S_{st}$, they are pseudo-coefficients for Steinberg twisted by χ_v, as in Lemma 5.3. We let f^*_∞ be a Lefschetz function for the discrete series L-packet given by ξ as before. Lemma 5.3 then gives us an equality:

$$\frac{1}{2} I^{\mathbf{G}^*}_{disc,\chi}(f^*) = I^{\tilde{\mathbf{G}},\sigma}_{disc,\chi}(\tilde{f})$$

We consider the part of the RHS corresponding to the cuspidal representation $\tilde{\tau}$ constructed above. By using linear independence of the unramified characters and the strong multiplicity one property for $\tilde{\mathbf{G}}$, the above identity implies an equality

$$c_1 \cdot \sum_{\Pi' \simeq \pi^S} m(\Pi') tr(f_S^* | \Pi'_S) = tr_\sigma(\tilde{\tau}_S | \tilde{f}_S) \tag{3}$$

where c_1 is a non-zero constant. Here $tr_\sigma(\tau_S|\tilde{f}_S)$ is the σ-twisted trace, as defined in [CG15, Section 5.16]. The LHS runs over automorphic representations satisfying the following:

(1) Π' has non-zero contribution to the discrete part of the trace formula $I^{\mathbf{G}^*}_{disc,\chi}$.
(2) The coefficient $m(\Pi')$ is the coefficient associated with the trace of Π' in $I^{\mathbf{G}^*}_{disc,\chi}$.

We can further simplify the LHS of (3) by noting that non-discrete spectrum representations which intervene in $I^{\mathbf{G}^*}_{disc,\chi}$ are parabolically induced from the discrete spectrum of proper Levi subgroups of $\mathbf{G}^*$. By [CG15, Section 5.8], we know that parabolically induced representations of $\mathbf{G}^*$ lift to parabolically induced representations of $\tilde{\mathbf{G}}$. Therefore, since τ is cuspidal, all terms occurring in in the LHS must all come from the discrete spectrum $T^{\mathbf{G}^*}_{disc,\chi}(f^*)$, by strong multiplicity one for $\tilde{\mathbf{G}}$. However, as in Lemma 5.1, we have an equality:

$$T^{\mathbf{G}^*}_{disc,\chi}(f^*) = T^{\mathbf{G}^*}_{cusp,\chi}(f^*)$$

In other words, we may assume that the sum on the LHS of (3) ranges over $\Pi' \in \mathcal{A}_{cusp,\chi}(\mathbf{G}^*)$, and that $m(\Pi')$ denotes the multiplicity in the cuspidal automorphic spectrum. Then we can rewrite the LHS as

$$\sum_{\substack{\Pi' \in \mathcal{A}_{cusp,\chi}(\mathbf{G}^*) \\ \Pi'^S \simeq \pi^S}} m(\Pi') tr(f_S^* | \Pi'_S)$$

Now, for the RHS, we apply the local character identities of Chan-Gan [CG15, Proposition 9.1], this tells us that we have an equality:

$$tr_\sigma(\tilde{\tau}_S | \tilde{f}_S) = c_2 \prod_{v \in S} \sum_{\pi'_v \in \Pi_{\phi_{\tau_v}}(\mathbf{G}^*_{F_v})} tr(f_v | \pi'_v)$$

for some non-zero constant c_2, where we have used property (3) of the representation $\tilde{\tau}$. In summary, we have concluded

$$\sum_{\substack{\Pi' \in \mathcal{A}_{cusp,\chi}(\mathbf{G}^*) \\ \Pi'^S \simeq \pi^S}} m(\Pi') tr(f_S^* | \Pi'_S) = c \cdot \prod_{v \in S} \sum_{\pi'_v \in \Pi_{\phi_{\tau_v}}(\mathbf{G}^*_{F_v})} tr(f_v | \pi'_v)$$

for some non-zero constant c. If π was a representation of $\mathbf{G}^*$, we know by our choice of τ that $\phi_{\tau_v} = \phi_{\pi_v}$ for all $v \in S$, so, by linear independence of characters at the places $v \in S \setminus S_\infty \cup S_{st}$, this tells us that the local constituents of some Π' occurring in the LHS with non-zero trace at $S_\infty \cup S_{st}$ are described by members of the L-packet over ϕ_{π_v} occurring with some multiplicity. Since the representation π' is by assumption cohomological of regular weight ξ and an unramified twist of Steinberg at all places in S_{st}, by arguing as in proof of Theorem 5.2, we have that $tr(f_{S_{st} \cup S_\infty} | \pi'_{S_{st} \cup S_\infty}) \neq 0$, so this gives us the desired claim for $\mathbf{G}^* = GSp_4$. Now, if π' is a representation of the inner form, we apply the character identities of Chan-Gan [CG15, Proposition 11.1]. This tells us that the RHS of the previous equation is equal to

$$c_3 \prod_{v \in S} \sum_{\rho_v \in \Pi_{\phi_{\tau_v}}(\mathbf{G}_{F_v})} tr(f_v^* | \rho_v)$$

for some non-zero constant c_3. Now, to rewrite the LHS, we apply the trace formula as in Theorem 5.2, with $\mathfrak{X} = Z(F_\infty)$. However, at all the places $v \in S_{sc}$, we choose f_v and f_v^* to be arbitrary matching functions instead of pseudo-coefficients. By linear independence of characters, we obtain a relationship

$$\sum_{\substack{\Pi' \in \mathcal{A}_{cusp,\chi_\xi}(\mathbf{G}^*) \\ \Pi'^S \simeq \pi^S}} m(\Pi') tr(f_S^* | \Pi'_S) = c_4 \cdot \sum_{\substack{\Pi \in \mathcal{A}_{cusp,\chi_\xi}(\mathbf{G}) \\ \Pi^S \simeq \pi^S}} m(\Pi) tr(f_S | \Pi_S)$$

for some non-zero constant c_4 and χ_ξ the central character of ξ. We note that all representations in the above sum must necessarily have central character equal that of π, by strong multiplicity one applied to the central characters, so, we can rewrite the above sum as being taken over $\mathcal{A}_{cusp,\chi}$. All in all, we obtain that

$$\cdot \sum_{\substack{\Pi \in \mathcal{A}_{cusp,\chi}(\mathbf{G}) \\ \Pi^S \simeq \pi^S}} m(\Pi) tr(f_S | \Pi_S) = c' \prod_{v \in S} \sum_{\rho_v \in \Pi_{\phi_{\tau_v}}(\mathbf{G}_{F_v})} tr(f_v^* | \rho_v)$$

for some non-zero constant c'. We know by our choice of τ that $\phi_{\tau_v} = \phi_{\pi_v}$ for all $v \in S_{sc} \cup S_{st} \cup S_\infty$, so, arguing as above, this implies the desired claim for the inner form. $\square$

Remark 5.2. Strong multiplicity one for globally generic automorphic representations of GSp_4 has been proven by Jiang-Soudry [JS07]. So, in the particular case that π and π' are representations of GSp_4, we could have just assumed that S_{st} is non-empty and then applied their results to a globally generic member in the global L-packets of π and π' to deduce the desired claim.

6. Galois Representations in the Cohomology of Shimura varieties

We now would like to combine the results of the previous section with results of Sorensen [Sor10] on the Galois representations associated to automorphic representations of $\mathbf{G}^* = GSp_4/F$ to say something about the Galois action of the global Shimura varieties occurring in basic uniformization. Let $F/\mathbb{Q}$ be a totally real field and $\mathbb{A}_{f,F}$ the finite adeles of F. Throughout, we will assume that τ is a cuspidal automorphic representation of $\mathbf{G}^*$ satisfying the same properties as in the previous section.

(1) τ_∞ is cohomological of some regular weight ξ of $\mathbf{G}^*(F_\infty)$.
(2) τ_v is unramified at all finite places outside of S.
(3) τ_v is an unramified twist of Steinberg at some finite set of finite places S_{st}.

We have the following key result of Sorensen.

Theorem 6.1. *[Sor10, Theorem A] Fix a globally generic τ as above such that S_{st} is non-empty. Then there exists, a unique (after fixing the isomorphism $i : \overline{\mathbb{Q}}_\ell \xrightarrow{\simeq} \mathbb{C}$) irreducible continuous representation $\rho_\tau : Gal(\overline{F}/F) \to GSp_4(\overline{\mathbb{Q}}_\ell)$ characterized by the property that, for each finite place $v \nmid \ell$ of F, we have*

$$iWD(\rho_\tau|_{W_{F_v}})^{F-s.s} \simeq \phi_{\tau_v} \otimes |\cdot|^{-3/2}$$

where $(-)^{F-s.s}$ denotes the Frobenius semisimplification and ϕ_{τ_v} is the Gan-Takeda parameter of τ_v.

Now let us fix τ with associated ρ_τ as above and assume that τ is a strong transfer of some cuspidal automorphic representation π of $\mathbf{G}$, as in Theorem 5.2. We assume tht S_{st} contains q an odd inert prime in the number field F and choose the inner form $\mathbf{G}$ to be of the following form, as in Kret-Shin [KS16, Section 8],

- $\mathbf{G}(\mathbb{R}) \simeq GSp_4(\mathbb{R}) \times GU_2(\mathbb{H})^{[F:\mathbb{Q}]-1}$,
- $\mathbf{G}_{F_v} \simeq GSp_4/F_v$ at all finite places v if $[F:\mathbb{Q}]$ is odd,
- $\mathbf{G}_{F_v} \simeq GSp_4/F_v$ at all but the finite place q if $[F:\mathbb{Q}]$ is even,

where $\mathbb{H}$ is the Hamilton quaternions. Let $A(\pi)$ be the set of isomorphism classes of cuspidal automorphic representations Π of $\mathbf{G}$ such that, for all $v \in S_{st}$, Π_v is an unramified twist of Steinberg, Π_∞ is ξ cohomological, and, for all $v \notin S_\infty \cup S_{st}$, $\Pi_v \simeq \pi_v$. Our main task now is to show that ρ_τ is realized in the π^∞ isotypic component of the Shimura variety associated to a Shimura datum $(\mathbf{G}, X)$. Let $Sh(\mathbf{G}, X)_{K,\overline{F}}$ be the associated Shimura variety over $\overline{F}$ which we recall is 3-dimensional. We set $\mathcal{L}_\xi$ to be the $\overline{\mathbb{Q}}_\ell$ local system associated to a irreducible representation of $\mathbf{G}$ over F of highest weight ξ on it as before, and let $H^i_c(Sh(\mathbf{G}, X)_K, \mathcal{L}_\xi)_{ss}$ denote the semisimplfication as a Hecke module of the compactly supported etale cohomology valued in $\mathcal{L}_\xi$. Choose $K \subset \mathbf{G}(\mathbb{A}_{f,F})$ a sufficiently small compact open subgroup such that π^∞ has a non-zero K-invariant vector. Let S_{bad} denote the set of prime numbers p for which either $p = 2$, the group $\mathbf{G}$ is ramified, or $K_p = \prod_{v|p} K_v$ is not hyperspecial. Then we define the virtual Galois representation

$$\rho^\pi_{shim} := (-1)^3 \sum_{\Pi \in A(\pi)} \sum_{i=0}^{6} (-1)^i [Hom_{\mathbf{G}(\mathbb{A}_{f,F})}(\Pi^\infty, H^i_c(Sh(\mathbf{G}, X)_{K,\overline{F}}, \mathcal{L}_\xi)_{ss})] \in K_0(\overline{\mathbb{Q}}_\ell(\Gamma_F)) \tag{4}$$

where $K_0(\overline{\mathbb{Q}}_\ell(\Gamma_F))$ denotes the Grothendieck group of continuous $\Gamma_F := Gal(\overline{F}/F)$-representations with coefficients in $\overline{\mathbb{Q}}_\ell$. We now define the rational number

$$a(\pi) := (-1)^{\frac{3}{2}} N_\infty^{-1} \sum_{\Pi \in A(\pi)} m(\Pi) \cdot ep(\Pi_\infty \otimes \xi)$$

where

1. $m(\Pi)$ is the multiplicity of Π in the automorphic spectrum of $\mathbf{G}$,
2. $N_\infty = |\Pi_\xi^{G(F_\infty)}| \cdot |\pi_0(G(F_\infty)/Z(F_\infty)| = 4$, where $\Pi_\xi^{G(F_\infty)}$ denotes the discrete series L-packet of representations of $G(F_\infty)$ cohomological of weight ξ,
3. $ep(\Pi_\infty \otimes \xi) := \sum_{i=0}^\infty dim(H^i(Lie(G(F_\infty)), K_\infty; \Pi_\infty \otimes \xi)$.

Then we have the following proposition of Kret-Shin.

Proposition 6.2. *[KS16, Proposition 8.2] With notation as above, for almost all finite F-places v not dividing a prime number in S_{bad} and all sufficiently large integers j, we have*

$$Tr(\rho^\pi_{shim}(Frob^j_v)) = a(\pi) q_v^{j\frac{3}{2}} Tr(std \circ \phi_{\pi_v})(Frob^j_v)$$

Moreover, the virtual representation ρ^π_{shim} is a true representation. In particular, the only non-zero term appearing in the above alternating sum occurs in middle degree (= 3).

Remark 6.1. The claim about it occurring in middle degree is part of the proof of Proposition not the statement. (See the discussion after equation (8.13) in [KS16])

We use this to deduce the following corollary.

Corollary 6.3. *The π^∞-isotypic component of $R\Gamma_c(Sh(\mathbf{G}, X)_{K^p,\overline{F}}, \mathcal{L}_\xi)$ is concentrated in degree 3 and has Γ_F-action given (up to multiplicity) by $std \circ \rho_\tau$.*

Proof. The first part follows immediately from the previous Proposition, and the second part follows from the identification of the traces. In particular, by the Brauer-Nesbitt Theorem, Cheboratev density theorem, and the condition characterizing ρ_τ, we can identify the Galois representation ρ^π_{shim} with the Galois representation $std \circ \rho_\tau$ occurring with multiplicity $a(\pi)$ and this proves the claim. □

7. Proof of the Key Proposition

We will now combine the results of the previous three sections to deduce some key consequences that will be used to derive Proposition 1.4. For this, using Krasner's lemma, we now fix a totally real number field F with two odd totally inert primes p and q such that $F_p \simeq L$ the fixed unramified extension of $\mathbb{Q}_p$. We fix the $\mathbb{Q}$-inner form $\mathbf{G}$ of $\mathbf{G}^* = Res_{F/\mathbb{Q}}GSp_4$ defined in section 6, and let $\mathbf{G}'$ be the inner form of $\mathbf{G}$ seen in Definition 4.1. Set ξ to be a regular weight of an algebraic representation of $\mathbf{G}$ over $\mathbb{Q}$. Let $K^p \subset \mathbf{G}(\mathbb{A}^{p\infty})$ be an open compact subgroup. We set $S_{sc} = \{p\}$ and S_{st} to be a finite set of finite places of $\mathbb{Q}$ containing q. We consider the uniformization map

$$\Theta : R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p, \mathcal{L}_\xi) \to R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi) \tag{5}$$

supplied by Theorem 4.2 and Corollary 4.1. Now fix a smooth irreducible supercuspidal representation ρ of $J = Res_{L/\mathbb{Q}_p}(GU_2(D))$. We have the following lemma.

Lemma 7.1. *Suppose ρ is a supercuspidal representation of J, then for sufficiently regular ξ and sufficiently small K^p, we can find a lift Π' to a cuspidal automorphic representation of $\mathbf{G}'$, which occurs as a $J(\mathbb{Q}_p)$-stable direct summand of $\mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f))/K^p, \mathcal{L}_\xi)$. Moreover, for all places in S_{st}, we can assume that the local constituents at $v \in S_{st}$ are unramified twists of the Steinberg representation.*

Proof. This follows from an argument using the simple trace formula. See for example [Han20, Proposition 2.9] or [Shi12]. We note in particular that cuspidality is vacuous, since $\mathbf{G}'(\mathbb{R})$ is compact modulo center by construction. □

So let Π' be a globalization of a fixed supercuspidal ρ to a cuspidal automorphic representation of $\mathbf{G}'$ for some sufficiently regular ξ and sufficiently small K^p. We can and do regard $\Pi'^{p\infty}$ as a representation of $\mathbf{G}(\mathbb{A}^p_f) \simeq \mathbf{G}'(\mathbb{A}^p_f)$. We set $K^p = K^p_S K^S$, where $K^S \subset \mathbf{G}(\mathbb{A}^S_f)$ is an open compact in the finite adeles away from S. We assume that S is sufficiently large such that outside of S the automorphic representation Π' is unramified, so in particular $K^S \subset \mathbf{G}(\mathbb{A}^S_f)$ is a product of hyperspecial subgroups away from S. We consider the abstract commutative Hecke algebra

$$\mathbb{T}^S := Z[\mathbf{G}(\mathbb{A}^S_f)//K^S)]$$

of bi-invariant compactly supported smooth functions on $\mathbf{G}(\mathbb{A}^S_f)$. We regard both sides of (5) as $\mathbb{T}^S$-modules and consider the maximal ideal $\mathfrak{m}$ defined by the Hecke eigenvalues of Π'^S. We then localize both sides of (5) at $\mathfrak{m}$ to obtain a map:

$$\Theta_{\mathfrak{m}} : (R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p, \mathcal{L}_\xi))_{\mathfrak{m}} \to R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)_{\mathfrak{m}}$$

We would like to apply Propositions 4.4 and 5.4 to conclude that this map is an isomorphism. However, to apply these results we need to make some more modifications. In particular, the automorphic representations of $\mathbf{G}'$ (resp. $\mathbf{G}$) occurring in the LHS (resp. RHS) of $\Theta_{\mathfrak{m}}$ are not necessarily unramified twists of Steinberg at all places in S_{st}. To remedy this, we set $K^p = K^p_{S_{st}} K^{\{p\}\cup S_{st}}$, where $K^{\{p\}\cup S_{st}} \subset \mathbf{G}(\mathbb{A}^{\{p\}\cup S_{st}}_f) \simeq \mathbf{G}'(\mathbb{A}^{\{p\}\cup S_{st}}_f)$ is an open compact subgroup. Then we consider the colimits

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^{\{p\}\cup S_{st}}}, \mathcal{L}_\xi) := \mathrm{colim}_{K^p_{S_{st}} \to \{1\}} R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p_{S_{st}} K^{\{p\}\cup S_{st}}}, \mathcal{L}_\xi)$$

and

$$\mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^{\{p\}\cup S_{st}}, \mathcal{L}_\xi) := \mathrm{colim}_{K^p_{S_{st}} \to \{1\}} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p_{S_{st}} K^{\{p\}\cup S_{st}}, \mathcal{L}_\xi)$$

Since $S_{st} \subset S$, the map $\theta_{\mathfrak{m}}$ gives rise to a map

$$(R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^{S_{st}\cup\{p\}}, \mathcal{L}_\xi))_{\mathfrak{m}} \to R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^{S_{st}\cup\{p\}}}, \mathcal{L}_\xi)_{\mathfrak{m}}$$

Now, we can consider the summand on the LHS (resp. RHS) where $\mathbf{G}'$ (resp. $\mathbf{G}$) acts via an unramified twist of the Steinberg representation at all places in S_{st}. This gives a map:

$$\Theta^{st}_{\mathfrak{m}} : (R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^{S_{st}\cup\{p\}}, \mathcal{L}_\xi))^{st}_{\mathfrak{m}} \to R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^{S_{st}\cup\{p\}}}, \mathcal{L}_\xi)^{st}_{\mathfrak{m}}$$

Moreover, passing to the sub-quotient where $G(\mathbb{Q}_p)$ acts via a supercuspidal representation, by Proposition 4.4, we obtain an isomorphism:

$$\Theta^{st}_{\mathfrak{m},sc} : (R\Gamma_c(G,b,\mu)_{sc} \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^{S_{st}\cup\{p\}}, \mathcal{L}_\xi))^{st}_{\mathfrak{m}} \xrightarrow{\simeq} R\Gamma_c(\mathcal{S}(\mathbf{G},X)_{K^{S_{st}\cup\{p\}}}, \mathcal{L}_\xi)^{st}_{\mathfrak{m},sc}$$

We now apply Proposition 5.4 to obtain the following.

Proposition 7.2. *Let $\rho \in \Pi(J)$ be a representation with supercuspidal Gan-Tantono parameter ϕ. Assume that $|S_{st}| \geq 3$. Then, for Π' a choice of globalization of ρ as in Lemma 7.1, unramified outside S with associated maximal ideal $\mathfrak{m} \subset \mathbb{T}^S$ in the Hecke algebra defined by the Hecke eigenvalues of Π'^S, the RHS (resp. LHS) of the map*

$$\Theta^{st}_{\mathfrak{m}} : (R\Gamma_c(G,b,\mu) \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^{S_{st}\cup\{p\}}, \mathcal{L}_\xi))^{st}_{\mathfrak{m}} \to R\Gamma_c(\mathcal{S}(\mathbf{G},X)_{K^{S_{st}\cup\{p\}}}, \mathcal{L}_\xi)^{st}_{\mathfrak{m}}$$

takes values in cuspidal automorphic representations of $\mathbf{G}$ (resp. $\mathbf{G}'$) whose local constituent at p has associated L-parameter ϕ. In particular, since all representations in the L-packet above ϕ are supercuspidal, this tells us that we have an equality: $R\Gamma_c(\mathcal{S}(\mathbf{G},X)_{K^{S_{st}\cup\{p\}}}, \mathcal{L}_\xi)^{st}_{\mathfrak{m},sc} = R\Gamma_c(\mathcal{S}(\mathbf{G},X)_{K^{S_{st}\cup\{p\}}}, \mathcal{L}_\xi)^{st}_{\mathfrak{m}}$. Therefore, we have an isomorphism

$$\Theta^{st}_{\mathfrak{m},sc} : (R\Gamma_c(G,b,\mu)_{sc} \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^{S_{st}\cup\{p\}}, \mathcal{L}_\xi))^{st}_{\mathfrak{m}} \xrightarrow{\simeq} R\Gamma_c(\mathcal{S}(\mathbf{G},X)_{K^{S_{st}\cup\{p\}}}, \mathcal{L}_\xi)^{st}_{\mathfrak{m}}$$

of complexes of $G(\mathbb{Q}_p) \times W_L$-modules.

Proof. We localized at $\mathfrak{m}$ corresponding to Π'^S and we are considering algebraic automorphic representations valued in the algebraic representation defined by ξ, where $\mathbf{G}'$ acts via an unramified twist of Steinberg at all places in S_{st}. Moreover, all these automorphic representations are automatically cuspidal since $\mathbf{G}'(\mathbb{R})$ is compact modulo center. Therefore, since $|S_{st}| \geq 3$ by construction, Proposition 5.4 applied to the inner form $\mathbf{G}'$ of $\mathbf{G}^*$ and the cuspidal automorphic representation representation Π' tells us that they all have Langlands parameter at $\{p\} = S_{sc}$ given by ϕ. This gives us the desired statement for the LHS.

For the RHS, by applying Theorem 5.2 with respect to the inner form $\mathbf{G}'$ of $\mathbf{G}^*$ and the cuspidal automorphic representation Π' we produce a strong transfer to a cuspidal automorphic representation τ of $\mathbf{G}^*$. Proposition 5.4 applied to the inner form $\mathbf{G}$ of $\mathbf{G}^*$ and the automorphic representation τ of $\mathbf{G}^*$ then tells us that the cuspidal automorphic representations of $\mathbf{G}$ occurring in $R\Gamma_c(\mathcal{S}(\mathbf{G},X)_{K^{S_{st}\cup\{p\}}}, \mathcal{L}_\xi)^{st}_{\mathfrak{m}}$ will all have Langlands parameter at p given by ϕ. □

Remark 7.1. Upon finishing the proof of compatibility (Theorem 8.1), we can use Corollary 3.15 to see that, for every representation ρ with supercuspidal Gan-Tantono parameter ϕ, all representations occurring in $R\Gamma_c(G,b,\mu)[\rho]$ must have semisimplified Gan-Takeda parameter equal to ϕ, which implies they must be supercuspidal, since ϕ is a supercuspidal parameter. Therefore, we have an equality

$$R\Gamma_c(G,b,\mu)[\rho]_{sc} = R\Gamma_c(G,b,\mu)[\rho]$$

for all such ρ. However, since all representations on the LHS of $\Theta^{st}_{\mathfrak{m}}$ have local constituent at p with supercuspidal Gan-Tantono parameter ϕ, this implies that the LHS of $\Theta^{st}_{\mathfrak{m}}$ equals its projection to the summand where $G(\mathbb{Q}_p)$ acts via a supercuspidal representation. So, a fortiori, the map $\Theta^{st}_{\mathfrak{m}}$ is already an isomorphism before projecting to the supercuspidal quotient.

We now combine this with Corollary 6.3 to deduce the following.

Corollary 7.3. *With notation as above, the map*

$$\Theta^{st}_{\mathfrak{m},sc} : (R\Gamma_c(G,b,\mu)_{sc} \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^{S_{st}\cup\{p\}}, \mathcal{L}_\xi))^{st}_{\mathfrak{m}} \xrightarrow{\simeq} R\Gamma_c(\mathcal{S}(\mathbf{G},X)_{K^{S_{st}\cup\{p\}}}, \mathcal{L}_\xi)^{st}_{\mathfrak{m}}$$

is an isomorphism of complexes of $G(\mathbb{Q}_p) \times W_L$-modules concentrated in degree 3 with W_L-action given up to multiplicity by $std \circ \phi \otimes |\cdot|^{-3/2}$.

Proof. Proposition 7.2 tells us that the LHS of $\Theta^{st}_{\mathfrak{m},sc}$ breaks up as a direct sum of $G(\mathbb{Q}_p) \times W_L$-modules of the form

$$R\Gamma_c(G, b, \mu)_{sc} \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \Pi'^{\{p,\infty\}\cup S_{st}}$$

for Π' a cuspidal automorphic representation of $\mathbf{G}'$ that has L-parameter ϕ at p, which is also cohomological of regular weight ξ at infinity and an unramified twist of Steinberg at all places in S_{st}. It suffices to prove the claim for each one of these summands. This summand will map to the $\Pi'^{\{p,\infty\}\cup S_{st}}$-isotypic part of the RHS by construction. Letting τ denote a strong transfer of Π' to a cuspidal automorphic representation of $\mathbf{G}^*$ given by Theorem 5.2, with associated Galois representation ρ_τ given by Theorem 6.1, we note, by Corollary 6.3, that the $\Pi'^{\{p,\infty\}\cup S_{st}}$-isotypic part will be concentrated in degree 3 and will have W_L-action given (up to multiplicity) by $std \circ \rho_\tau|_{W_L} \simeq \phi \otimes |\cdot|^{-3/2}$, by the property characterizing ρ_τ. This gives us the claim. □

With this in hand, we are finally ready conclude our key Proposition.

Proposition 7.4. *Let ϕ be a supercuspidal parameter with associated L-packet $\Pi_\phi(J)$. Then the direct summand of*

$$\bigoplus_{\rho' \in \Pi_\phi(J)} R\Gamma_c(G, b, \mu)[\rho']$$

where $G(\mathbb{Q}_p)$ acts via a supercuspidal representation

$$\bigoplus_{\rho' \in \Pi_\phi(J)} R\Gamma_c(G, b, \mu)[\rho']_{sc}$$

is concentrated in middle degree 3 and admits a non-zero W_L-stable sub-quotient with W_L-action given by $std \circ \phi \otimes |\cdot|^{-3/2}$.

Proof. This is an immediate consequence of Corollaries 7.2 and 7.3. □

In particular, using Corollary 3.21, we can deduce the following.

Corollary 7.5. *If $p > 2$ and $L/\mathbb{Q}_p$ is an unramified extension, then, for all $\rho \in \Pi(J)$ with supercuspidal Gan-Tantono parameter ϕ_ρ, the Fargues-Scholze and Gan-Tantono correspondences are compatible.*

8. Applications

We will now apply Corollary 7.5 to deduce some applications to the strong form of the Kottwitz conjecture and conclude the proof of Theorem 1.1. We begin with the latter.

Theorem 8.1. *The following is true.*

1. *For any $\pi \in \Pi(G)$ (resp. $\rho \in \Pi(J)$) such that the Gan-Takeda (resp. Gan-Tantono) parameter is not supercuspidal, we have that the Gan-Takeda (resp. Gan-Tantono) correspondence is compatible with the Fargues-Scholze correspondence.*
2. *If $L/\mathbb{Q}_p$ is unramified and $p > 2$, we have, for all $\pi \in \Pi(G)$ (resp. $\rho \in \Pi(J)$) such that the Gan-Takeda (resp. Gan-Tantono) parameter is supercuspidal, that the Gan-Takeda (resp. Gan-Tantono) correspondence is compatible with the Fargues-Scholze correspondence.*

Proof. Part (1) follows by Corollary 3.12 and Corollary 3.16. Part (2) for the Gan-Tantono local Langlands is precisely Corollary 7.5. It remains to show that for $L/\mathbb{Q}_p$ unramified and $p > 2$, π a smooth irreducible representation of GSp_4/L with supercuspidal Gan-Takeda ϕ_π parameter that the two correspondences are compatible. To show this, we consider the complex

$$R\Gamma^\flat_c(G, b, \mu)[\pi]$$

of $J(\mathbb{Q}_p) \times W_L$-representations. We know, by Theorem 3.13, that this admits sub-quotients as a $J(\mathbb{Q}_p)$-module given by ρ, for all ρ whose Gan-Tantono parameter ϕ_ρ is equal to the Gan-Takeda parameter ϕ_{π} of π. However, by Corollary 3.15, we know that these representations must have Fargues-Scholze parameter equal to ϕ_{π}^{FS}. Therefore, we get a chain of equalities

$$\phi_{\pi}^{FS} = \phi_{\rho}^{FS} = \phi_{\rho} = \phi_{\pi}$$

where we have used compatibility of the Gan-Tantono and the Fargues-Scholze correspondence for the middle equality. □

Now, with this out of the way, we turn our attention to proving some strong forms of the Kottwitz conjecture for these representations, verifying Theorem 1.3.

Theorem 8.2. *Let $L/\mathbb{Q}_p$ be an unramified extension with $p > 2$. Let π (resp. ρ) be members of the L-packet over a supercuspidal parameter $\phi : W_L \to GSp_4(\overline{\mathbb{Q}}_{\ell})$. Then the complexes*

$$R\Gamma_c(G, b, \mu)[\pi]$$

and

$$R\Gamma_c(G, b, \mu)[\rho]$$

are concentrated in middle degree 3.

(1) *If ϕ is stable supercuspidal, with singleton L-packets $\{\pi\} = \Pi_{\phi}(G)$ and $\{\rho\} = \Pi_{\phi}(J)$, then the cohomology of $R\Gamma_c(G, b, \mu)[\pi]$ in middle degree is isomorphic to*

$$\rho \boxtimes (std \circ \phi)^{\vee} \otimes |\cdot|^{-3/2}$$

as a $J(\mathbb{Q}_p) \times W_L$-module, and the cohomology of $R\Gamma_c(G, b, \mu)[\rho]$ in middle degree is isomorphic to

$$\pi \boxtimes std \circ \phi \otimes |\cdot|^{-3/2}$$

as a $G(\mathbb{Q}_p) \times W_L$-module.

(2) *If ϕ is an endoscopic parameter, with L-packets $\Pi_{\phi}(G) = \{\pi^+, \pi^-\}$ and $\Pi_{\phi}(J) = \{\rho_1, \rho_2\}$, the cohomology of $R\Gamma_c(G, b, \mu)[\pi]$ in middle degree is isomorphic to*

$$\rho_1 \boxtimes \phi_1^{\vee} \otimes |\cdot|^{-3/2} \oplus \rho_2 \boxtimes \phi_2^{\vee} \otimes |\cdot|^{-3/2}$$

or

$$\rho_1 \boxtimes \phi_2^{\vee} \otimes |\cdot|^{-3/2} \oplus \rho_2 \boxtimes \phi_1^{\vee} \otimes |\cdot|^{-3/2}$$

as a $J(\mathbb{Q}_p) \times W_L$-module. Similarly, the cohomology of $R\Gamma_c(G, b, \mu)[\rho]$ in middle degree is isomorphic to

$$\pi^+ \boxtimes \phi_1 \otimes |\cdot|^{-3/2} \oplus \pi^- \boxtimes \phi_2 \otimes |\cdot|^{-3/2}$$

or

$$\pi^+ \boxtimes \phi_2 \otimes |\cdot|^{-3/2} \oplus \pi^- \boxtimes \phi_1 \otimes |\cdot|^{-3/2}$$

as a $G(\mathbb{Q}_p) \times W_L$-module. Here we write $std \circ \phi_{\rho} \simeq \phi_1 \oplus \phi_2$, with ϕ_i distinct irreducible 2-dimensional representations of W_L and $det(\phi_1) = det(\phi_2)$.

Moreover, both possibilities for the cohomology of $R\Gamma_c(G, b, \mu)[\rho]$ (resp. $R\Gamma_c(G, b, \mu)[\pi]$) in the endoscopic case occur for some choice of representation $\rho \in \Pi_{\phi}(J)$ (resp. $\pi \in \Pi_{\phi}(G)$). In particular, knowing the precise form of either $R\Gamma_c(G, b, \mu)[\rho]$ or $R\Gamma_c(G, b, \mu)[\pi]$ for some $\rho \in \Pi_{\phi}(J)$ or $\pi \in \Pi_{\phi}(G)$ determines the precise form of the cohomology in all other cases.

Proof. We show the proof in the endoscopic case, with the stable case being strictly easier. We first note that, since $\phi = \phi_{\rho}^{FS}$ by Theorem 8.1, it follows by assumption that the Fargues-Scholze parameter ϕ_{ρ}^{FS} of ρ is supercuspidal. Therefore, by Remark 3.12, we have an isomorphism

$$R\Gamma_c(G, b, \mu)[\rho] \simeq R\Gamma_c^{\flat}(G, b, \mu)[\rho]$$

of $G(\mathbb{Q}_p) \times W_L$-modules. Moreover, by Theorem 3.17, we see that both are concentrated in middle degree 3. Applying Theorem 3.13, we get the following chain of equalities in the Grothendieck group of admissible $G(\mathbb{Q}_p)$-representations of finite length

$$-[H^3(R\Gamma_c(G,b,\mu)[\rho])] = [R\Gamma_c(G,b,\mu)[\rho]] = [R\Gamma_c(G,b,\mu)[\rho]^{KW}] = -\sum_{\pi\in\Pi_\phi(G)} Hom_{S_\phi}(\delta_{\pi,\rho}, std\circ\phi_\rho)\pi$$

Now we saw in the discussion proceeding Theorem 3.13 that the RHS takes the form:

$$-2\pi^+ - 2\pi^-$$

We set p_1 and p_2 to be the two representations of $S_\phi \simeq \{(a,b) \in \overline{\mathbb{Q}}_\ell^* \times \overline{\mathbb{Q}}_\ell^* | a^2 = b^2\} \subset (GL_2(\overline{\mathbb{Q}}_\ell) \times GL_2(\overline{\mathbb{Q}}_\ell))^0$ given by projecting to the first and second $\overline{\mathbb{Q}}_\ell^*$-factor, respectively. By Corollary 3.3, Corollary 3.11, and Theorem 8.1, we have an isomorphism of $G(\mathbb{Q}_p) \times W_L$-modules

$$R\Gamma_c(G,b,\mu)[\rho] \simeq Act_{p_1}(\rho)[-3] \boxtimes \phi_1 \otimes |\cdot|^{-3/2} \oplus Act_{p_2}(\rho)[-3] \boxtimes \phi_2 \otimes |\cdot|^{-3/2}$$

where $Act_{p_1}(\rho)$ and $Act_{p_2}(\rho)$ are a priori direct sum of shifts of supercuspidal representations of $G(\mathbb{Q}_p)$ with Fargues-Scholze ($=$ Gan-Takeda) parameter equal to ϕ. However, since we know that the LHS is a complex concentrated in middle degree 3, this implies, by the above description in the Grothendieck group, that one of the $Act_{p_1}(\rho)$ and $Act_{p_2}(\rho)$ is isomorphic to π^+ and the other is isomorphic to π^-. Without loss of generality, assume that

$$Act_{p_1}(\rho_1) \simeq \pi^+$$

and

$$Act_{p_2}(\rho_1) \simeq \pi^-$$

We let p_+ and p_- be the representation of S_ϕ determined by the trivial and non-trivial characters of the component group, respectively. Now, given two representations of S_ϕ, denoted W and W', it follows from Remark 3.7 (3) that we have an isomorphism:

$$Act_W \circ Act_{W'}(\cdot) \simeq Act_{W\otimes W'}(\cdot)$$

In turn, we get

$$Act_{p_1^\vee}(\pi^+) \simeq Act_{p_1^\vee} \circ Act_{p_1}(\rho_1) \simeq Act_{p_+}(\rho_1) \simeq \rho_1$$

where the last isomorphism follows since p_+ is the trivial representation. Similarly, depending on the values of $Act_{p_1}(\rho_2)$ and $Act_{p_2}(\rho_2)$ we can deduce that $Act_{p_2^\vee}(\pi^+)$ is isomorphic to ρ_1 or ρ_2. Now by Corollary 3.3, Corollary 3.11, and Theorem 8.1, we have an isomorphism

$$R\Gamma_c(G,b,\mu)[\pi^+] \simeq Act_{p_1^\vee}(\pi^+)[-3] \boxtimes \phi_1^\vee \otimes |\cdot|^{-3/2} \oplus Act_{p_2^\vee}(\pi^+)[-3] \boxtimes \phi_2^\vee \otimes |\cdot|^{-3/2}$$

Since $Act_{p_1^\vee}(\pi^+) \simeq \rho_1$ it therefore follows, by Theorem 3.13 and Remark 3.11, that $Act_{p_2^\vee}(\pi^+)$ must be isomorphic to ρ_2. Moreover, we know that $Act_{p_-} \circ Act_{p_1}(\rho_1) \simeq Act_{p_2}(\rho_1)$ and $Act_{p_-} \circ Act_{p_2}(\rho_1) \simeq Act_{p_1}(\rho_1)$. Therefore, we obtain that $Act_{p_-}(\pi^+) \simeq \pi^-$ and $Act_{p_-}(\pi^-) \simeq \pi^+$. This allows us to determine that

$$Act_{p_1^\vee}(\pi^-) \simeq Act_{p_2^\vee} \circ Act_{p_-}(\pi^-) \simeq Act_{p_2^\vee}(\pi^+) \simeq \rho_2$$

and

$$Act_{p_2^\vee}(\pi^-) \simeq Act_{p_1^\vee} \circ Act_{p_-}(\pi^-) \simeq Act_{p_1^\vee}(\pi^+) \simeq \rho_1$$

which will determine the cohomology of $R\Gamma_c(G,b,\mu)[\pi^-]$. It only remains to show that the value of $R\Gamma_c(G,b,\mu)[\rho_2]$ is determined. However, this follows since

$$Act_{p_2}(\rho_2) \simeq Act_{p_2} \circ Act_{p_2^\vee}(\pi^+) \simeq \pi^+$$

and

$$Act_{p_1}(\rho_2) \simeq Act_{p_1} \circ Act_{p_1^\vee}(\pi^-) \simeq \pi^-$$

$\square$

To conclude this section, we use Theorem 8.1 to deduce compatibility with the local Langlands correspondence for Sp_4 and its unique non quasi-split inner form $SU_2(D)$. These correspondences are described in the papers [GT10] and [Cho17] by Gan-Takeda and Choiy, respectively. For Sp_4, this is described as the unique correspondence which sits in the commutative diagram:

$$\begin{tikzcd} \Pi(GSp_4) \arrow[r, "LLC_{GSp_4}"] \arrow[d] & \Phi(GSp_4) \arrow[d, "\alpha"] \\ \Pi(Sp_4) \arrow[r, "LLC_{Sp_4}"] & \Phi(Sp_4) \end{tikzcd}$$

Here, the left vertical arrow is not a map at all, it is a correspondence defined by the subset of $\Pi(GSp_4) \times \Pi(Sp_4)$ consisting of pairs (π, ω) such that ω is a constituent of the restriction of π to Sp_4, and the right vertical arrow is the map on L-parameters induced by the natural map $GSpin_5(\mathbb{C}) \to SO_5(\mathbb{C})$. One has a similar characterization of the local Langlands correspondence for $SU_2(D)$. With this description of the correspondence, compatibility for Sp_4 and $SU_2(D)$ follows from Theorem 8.1 and Theorem 3.6 (7).

Corollary 8.3. *For π (resp. ρ) a smoooth irreducible representation of Sp_4/L (resp. $SU_2(D)$), with associated Gan-Takeda (resp. Choiy) parameter $\phi_\pi : W_L \times SL_2(\mathbb{C}) \to SO_5(\mathbb{C})$ (resp. ϕ_ρ) we have that:*

1. *The Fargues-Scholze and Gan-Takeda (resp. Choiy) local Langlands correspondences are compatible for any representation π (resp. ρ) such that ϕ_π (resp. ϕ_ρ) is not supercuspidal.*
2. *If $L/\mathbb{Q}_p$ is unramified and $p > 2$ then the Fargues-Scholze and Gan-Takeda (resp. Choiy) local Langlands correspondences are compatible for any representation π (resp. ρ) such that ϕ_π (resp. ϕ_ρ) is supercuspidal.*

Remark 8.1. We note that Corollary 3.11, Theorem 3.18, Remark 3.11, and [HKW21, Theorem 1.0.2] apply to a triple (G, b, μ), where μ is any cocharacter and $b \in B(G, \mu)$ is the unique basic element. Therefore, by applying the same kind of analysis as in the proof of Theorem 8.2, we can prove the analogue of Theorem 8.2 in the Grothendieck group of finite length admissible representations with a smooth action of W_E (cf. [HKW21, Conjecture 1.0.1]) for the cohomology of the local Shtuka spaces defined by the triple (G, b, μ). By Corollary 8.3, this works even in the case when $G = Sp_4$ and there are no Shimura varieties that these spaces uniformize. Moreover, in the case that $G = GSp_4$, one can also deduce from Theorem 8.3 that it is concentrated in middle degree $\langle 2\rho_G, \mu\rangle$, using the monoidal property of the *Act*-functors, for μ any cocharacter. This in particular will imply some form of Fargues' Conjecture for these groups.

References

[AL21] Johannes Anschütz and Arthur-César Le-Bras. "Averaging Functors in Fargues' Program for GL_n". In: *Preprint* (2021). arXiv:2104.04701.

[AR04] Jeffrey D. Adler and Alan Roche. "Injectivity, projectivity and supercuspidal representations". In: *J. London Math. Soc. (2)* 70.2 (2004), pp. 356–368. ISSN: 0024-6107. DOI: 10.1112/S0024610704005617. URL: https://doi.org/10.1112/S0024610704005617.

[Art02] James Arthur. "A stable trace formula. I. General expansions". In: *J. Inst. Math. Jussieu* 1.2 (2002), pp. 175–277. ISSN: 1474-7480. DOI: 10.1017/S1474-748002000051. URL: https://doi.org/10.1017/S1474-748002000051.

[Art04] James Arthur. "Automorphic representations of GSp(4)". In: *Contributions to automorphic forms, geometry, and number theory.* Johns Hopkins Univ. Press, Baltimore, MD, 2004, pp. 65–81.

[Art81] James Arthur. "The trace formula in invariant form". In: *Ann. of Math. (2)* 114.1 (1981), pp. 1–74. ISSN: 0003-486X. DOI: `10.2307/1971376`. URL: `https://doi.org/10.2307/1971376`.

[Bor79] A. Borel. "Automorphic L-functions". In: *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*. Proc. Sympos. Pure Math., XXXIII. Amer. Math. Soc., Providence, R.I., 1979, pp. 27–61.

[Boy09] Pascal Boyer. "Monodromie du faisceau pervers des cycles évanescents de quelques variétés de Shimura simples". In: *Invent. Math.* 177.2 (2009), pp. 239–280. ISSN: 0020-9910. DOI: `10.1007/s00222-009-0183-9`. URL: `https://doi.org/10.1007/s00222-009-0183-9`.

[BW00] A. Borel and N. Wallach. *Continuous cohomology, discrete subgroups, and representations of reductive groups*. Second. Vol. 67. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2000, pp. xviii+260. ISBN: 0-8218-0851-6. DOI: `10.1090/surv/067`. URL: `https://doi.org/10.1090/surv/067`.

[CG15] Ping-Shun Chan and Wee Teck Gan. "The local Langlands conjecture for GSp(4) III: Stability and twisted endoscopy". In: *J. Number Theory* 146 (2015), pp. 69–133. ISSN: 0022-314X. DOI: `10.1016/j.jnt.2013.07.009`. URL: `https://doi.org/10.1016/j.jnt.2013.07.009`.

[Cho17] Kwangho Choiy. "The local Langlands conjecture for the p-adic inner form of Sp_4". In: *Int. Math. Res. Not. IMRN* 6 (2017), pp. 1830–1889. ISSN: 1073-7928. DOI: `10.1093/imrn/rnw043`. URL: `https://doi.org/10.1093/imrn/rnw043`.

[Dat+20] Jean-François Dat et al. "Moduli of Langlands Parameters". In: *Preprint* (2020). arXiv:2009.06708.

[Dat07] J.-F. Dat. "Théorie de Lubin-Tate non-abélienne et représentations elliptiques". In: *Invent. Math.* 169.1 (2007), pp. 75–152. ISSN: 0020-9910. DOI: `10.1007/s00222-007-0044-3`. URL: `https://doi.org/10.1007/s00222-007-0044-3`.

[FGV02] E. Frenkel, D. Gaitsgory, and K. Vilonen. "On the geometric Langlands conjecture". In: *J. Amer. Math. Soc.* 15.2 (2002), pp. 367–417. ISSN: 0894-0347. DOI: `10.1090/S0894-0347-01-00388-5`. URL: `https://doi.org/10.1090/S0894-0347-01-00388-5`.

[FS21] L. Fargues and P. Scholze. "Geometrization of the local Langlands Correspondence". In: *Preprint* (2021). arXiv:2102.13459.

[Gai04] D. Gaitsgory. "On a vanishing conjecture appearing in the geometric Langlands correspondence". In: *Ann. of Math. (2)* 160.2 (2004), pp. 617–682. ISSN: 0003-486X. DOI: `10.4007/annals.2004.160.617`. URL: `https://doi.org/10.4007/annals.2004.160.617`.

[GT10] Wee Teck Gan and Shuichiro Takeda. "The local Langlands conjecture for Sp(4)". In: *Int. Math. Res. Not. IMRN* 15 (2010), pp. 2987–3038. ISSN: 1073-7928. DOI: `10.1093/imrn/rnp203`. URL: `https://doi.org/10.1093/imrn/rnp203`.

[GT11] Wee Teck Gan and Shuichiro Takeda. "The local Langlands conjecture for GSp(4)". In: *Ann. of Math. (2)* 173.3 (2011), pp. 1841–1882. ISSN: 0003-486X. DOI: `10.4007/annals.2011.173.3.12`. URL: `https://doi.org/10.4007/annals.2011.173.3.12`.

[GT14] Wee Teck Gan and Welly Tantono. "The local Langlands conjecture for GSp(4), II: The case of inner forms". In: *Amer. J. Math.* 136.3 (2014), pp. 761–805. ISSN: 0002-9327. DOI: `10.1353/ajm.2014.0016`. URL: `https://doi.org/10.1353/ajm.2014.0016`.

[GT19] Toby Gee and Olivier Taïbi. "Arthur's multiplicity formula for $\mathbf{GSp}_4$ and restriction to $\mathbf{Sp}_4$". In: *J. Éc. polytech. Math.* 6 (2019), pp. 469–535. ISSN: 2429-7100. DOI: `10.5802/jep.99`. URL: `https://doi.org/10.5802/jep.99`.

[Han16] D. Hansen. "Period Morphisms and variations of p-adic Hodge structures". In: *Draft* (2016). Available at home page of author.

[Han20] D. Hansen. "On the supercuspidal cohomology of basic local Shimura varieties". In: (2020). Available at home page of first named author.

[Hen00] Guy Henniart. "Une preuve simple des conjectures de Langlands pour $\mathrm{GL}(n)$ sur un corps p-adique". In: *Invent. Math.* 139.2 (2000), pp. 439–455. ISSN: 0020-9910. DOI: 10.1007/s002220050012. URL: https://doi.org/10.1007/s002220050012.

[HKW21] David Hansen, Tasho Kaletha, and Jared Weinstein. "On the Kottwitz conjecture for local Shimura varieties". In: *Preprint* (2021). arXiv:1709.06651.

[HP79] R. Howe and I. I. Piatetski-Shapiro. "A counterexample to the "generalized Ramanujan conjecture" for (quasi-) split groups". In: *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1.* Proc. Sympos. Pure Math., XXXIII. Amer. Math. Soc., Providence, R.I., 1979, pp. 315–322.

[HT01] Michael Harris and Richard Taylor. *The geometry and cohomology of some simple Shimura varieties.* Vol. 151. Annals of Mathematics Studies. With an appendix by Vladimir G. Berkovich. Princeton University Press, Princeton, NJ, 2001, pp. viii+276. ISBN: 0-691-09090-4.

[IM21] Tetsushi Ito and Yoichi Mieda. "Local Saito-Kurokawa A-packets and ℓ-adic cohomology of Rapoport-Zink tower for GSp_4". In: *Preparation* (2021).

[JS07] Dihua Jiang and David Soudry. "The multiplicity-one theorem for generic automorphic forms of GSp(4)". In: *Pacific J. Math.* 229.2 (2007), pp. 381–388. ISSN: 0030-8730. DOI: 10.2140/pjm.2007.229.381. URL: https://doi.org/10.2140/pjm.2007.229.381.

[Kal16] Tasho Kaletha. "The local Langlands conjectures for non-quasi-split groups". In: *Families of automorphic forms and the trace formula.* Simons Symp. Springer, [Cham], 2016, pp. 217–257.

[Kos21] T. Koshikawa. "On Eichler-Shimura relations for local Shimura varieties". In: *Preprint* (2021). arXiv:2106.10603.

[Kot85] Robert E. Kottwitz. "Isocrystals with additional structure". In: *Compositio Math.* 56.2 (1985), pp. 201–220. ISSN: 0010-437X. URL: http://www.numdam.org/item?id=CM_1985__56_2_201_0.

[KS16] Arno Kret and Sug-Woo Shin. "Galois representations for general symplectic groups". In: *Preprint* (2016). arXiv:1609.04223.

[KS99] Robert E. Kottwitz and Diana Shelstad. "Foundations of twisted endoscopy". In: *Astérisque* 255 (1999), pp. vi+190. ISSN: 0303-1179.

[KST20] Ju-Lee Kim, Sug Woo Shin, and Nicolas Templier. "Asymptotic behavior of supercuspidal representations and Sato-Tate equidistribution for families". In: *Adv. Math.* 362 (2020), pp. 106955, 57. ISSN: 0001-8708. DOI: 10.1016/j.aim.2019.106955. URL: https://doi.org/10.1016/j.aim.2019.106955.

[Kur78] Nobushige Kurokawa. "Examples of eigenvalues of Hecke operators on Siegel cusp forms of degree two". In: *Invent. Math.* 49.2 (1978), pp. 149–165. ISSN: 0020-9910. DOI: 10.1007/BF01403084. URL: https://doi.org/10.1007/BF01403084.

[Laf18] Vincent Lafforgue. "Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale". In: *J. Amer. Math. Soc.* 31.3 (2018), pp. 719–891. ISSN: 0894-0347. DOI: 10.1090/jams/897. URL: https://doi.org/10.1090/jams/897.

[LR05] Erez M. Lapid and Stephen Rallis. "On the local factors of representations of classical groups". In: *Automorphic representations, L-functions and applications: progress and prospects.* Vol. 11. Ohio State Univ. Math. Res. Inst. Publ. de Gruyter, Berlin, 2005, pp. 309–359. DOI: 10.1515/9783110892703.309. URL: https://doi.org/10.1515/9783110892703.309.

[LW13] Jean-Pierre Labesse and Jean-Loup Waldspurger. *La formule des traces tordue d'après le Friday Morning Seminar.* Vol. 31. CRM Monograph Series. With a foreword by Robert Langlands [dual English/French text]. American Mathematical Society, Providence, RI, 2013, pp. xxvi+234. ISBN: 978-0-8218-9441-5. DOI: 10.1090/crmm/031. URL: https://doi.org/10.1090/crmm/031.

[MV07] I. Mirković and K. Vilonen. "Geometric Langlands duality and representations of algebraic groups over commutative rings". In: *Ann. of Math. (2)* 166.1 (2007), pp. 95–143. ISSN: 0003-486X. DOI: 10.4007/annals.2007.166.95. URL: https://doi.org/10.4007/annals.2007.166.95.

[MW18] Colette Mœglin and J.-L. Waldspurger. "La formule des traces locale tordue". In: *Mem. Amer. Math. Soc.* 251.1198 (2018), pp. v+183. ISSN: 0065-9266. DOI: 10.1090/memo/1198. URL: https://doi.org/10.1090/memo/1198.

[Ngu19] Kieu Hieu Nguyen. "Un cas PEL de la conjecture de Kottwitz". In: *Preprint* (2019). arXiv:1903.11505.

[Rap18] Michael Rapoport. "Appendix to On the p-adic cohomology of the Lubin-Tate tower". In: *Ann. Sci. Éc. Norm. Supér. (4)* 51.4 (2018), pp. 811–863. ISSN: 0012-9593. DOI: 10.24033/asens.2367. URL: https://doi.org/10.24033/asens.2367.

[RR96] M. Rapoport and M. Richartz. "On the classification and specialization of F-isocrystals with additional structure". In: *Compositio Math.* 103.2 (1996), pp. 153–181. ISSN: 0010-437X. URL: http://www.numdam.org/item?id=CM_1996__103_2_153_0.

[RV14] Michael Rapoport and Eva Viehmann. "Towards a theory of local Shimura varieties". In: *Münster J. Math.* 7.1 (2014), pp. 273–326. ISSN: 1867-5778.

[RW21] Mirko Rosner and Rainer Weissauer. "Global liftings between inner forms GSp_4". In: *Preprint* (2021). arXiv:2103.14715.

[Sch18] P. Scholze. "Étale Cohomology of Diamonds". In: *Preprint* (2018). arXiv:1709.07343.

[Sha90] Freydoon Shahidi. "A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups". In: *Ann. of Math. (2)* 132.2 (1990), pp. 273–330. ISSN: 0003-486X. DOI: 10.2307/1971524. URL: https://doi.org/10.2307/1971524.

[She17] Xu Shen. "Perfectoid Shimura varieties of abelian type". In: *Int. Math. Res. Not. IMRN* 21 (2017), pp. 6599–6653. ISSN: 1073-7928. DOI: 10.1093/imrn/rnw202. URL: https://doi.org/10.1093/imrn/rnw202.

[Shi12] Sug Woo Shin. "Automorphic Plancherel density theorem". In: *Israel J. Math.* 192.1 (2012), pp. 83–120. ISSN: 0021-2172. DOI: 10.1007/s11856-012-0018-z. URL: https://doi.org/10.1007/s11856-012-0018-z.

[Sor10] Claus M. Sorensen. "Galois representations attached to Hilbert-Siegel modular forms". In: *Doc. Math.* 15 (2010), pp. 623–670. ISSN: 1431-0635. DOI: 10.1007/s00031-010-9092-7. URL: https://doi.org/10.1007/s00031-010-9092-7.

[SW13] Peter Scholze and Jared Weinstein. "Moduli of p-divisible groups". In: *Camb. J. Math.* 1.2 (2013), pp. 145–237. ISSN: 2168-0930. DOI: 10.4310/CJM.2013.v1.n2.a1. URL: https://doi.org/10.4310/CJM.2013.v1.n2.a1.

[SW20] P. Scholze and J. Weinstein. *Berkeley lectures on p-adic Geometry*. Vol. 389. Annals of Mathematics Studies. Princeton University Press, 2020.

[Tow13] Nelson J. Townsend. *Properties of Gamma factors for $GSp(4) \times GL(r)$ with r = 1, 2.* Thesis (Ph.D.)–University of California, San Diego. ProQuest LLC, Ann Arbor, MI, 2013, p. 69. ISBN: 978-1303-62050-8. URL: http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqm&rft_dat=xri:pqdiss:3605550.

[Vie21] Eva Viehmann. "On Newton strata in the B_{dR}^+-Grassmannian". In: *Preprint* (2021). arXiv:2101.07510.

[VZ84] David A. Vogan Jr. and Gregg J. Zuckerman. "Unitary representations with nonzero cohomology". In: *Compositio Math.* 53.1 (1984), pp. 51–90. ISSN: 0010-437X. URL: http://www.numdam.org/item?id=CM_1984__53_1_51_0.

[Wes12] Uwe Weselmann. “A twisted topological trace formula for Hecke operators and liftings from symplectic to general linear groups”. In: *Compos. Math.* 148.1 (2012), pp. 65–120. ISSN: 0010-437X. DOI: 10.1112/S0010437X11005641. URL: https://doi.org/10.1112/S0010437X11005641.

[Zhu20] X. Zhu. “Coherent Sheaves on the stack of Langlands Parameters”. In: *Preprint* (2020). arXiv:2008.02998.