Compatibility of the Fargues-Scholze and Gan-Takeda Local Langlands

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Notation

We let:

1. $\ell \neq p$ be distinct primes.
2. $W_{\mathbb{Q}_p}$ be the Weil group of $\mathbb{Q}_p$
3. $G = GL_n(\mathbb{Q}_p)$ the set of $n \times n$ invertible matrices.
4. We fix isomorphisms $i : \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$.

Notation

We set:

1. $\Pi(G)$ to be isomorphism classes of smooth irreducible representations of $G(\mathbb{Q}_p)$.
2. $\Phi(G)$ to be the set of conjugacy classes of admissible homomorphisms:

$$\phi : W_{\mathbb{Q}_p} \times SL(2, \overline{\mathbb{Q}}_\ell) \to GL_n(\overline{\mathbb{Q}}_\ell)$$

3. $\Phi^W(G)$ to be the set of conjugacy classes of continuous semisimple homomorphisms:

$$\phi : W_{\mathbb{Q}_p} \to GL_n(\overline{\mathbb{Q}}_\ell)$$

Theorem (Harris-Taylor/Henniart/Scholze)

For every $n \geq 1$, there exists a unique bijection:

$$\Pi(G) \xrightarrow{LLC_n} \Phi(G)$$

$$\pi \mapsto \phi_\pi$$

generalizing local class field theory and characterized by the preservation of character twists, L, ϵ, and γ factors in pairs of representations.

Supercuspidal Parameters and the Langlands' classification

1. For $n = 1$, we note that we have an isomorphism $W_{\mathbb{Q}_p}^{ab} \simeq \mathbb{Q}_p^*$. Therefore, if we are given a character $\chi : \mathbb{Q}_p^* \to \mathbb{C}^*$, we get a character of $W_{\mathbb{Q}_p}$.
2. Recall that $\Phi(G)$ parametrizes maps:

$$\phi : W_{\mathbb{Q}_p} \times SL(2, \mathbb{C}) \to GL_n(\mathbb{C})$$

   Why does this $SL(2, \mathbb{C})$ factor appear?
3. We can divide the set $\Phi^W(G)$ as follows. For $GL_{h_1} \times \cdots GL_{h_k} \simeq M \subset GL_n$ a Levi subgroup, we write $\Phi_M^W(G)$ for the set of parameters $\phi : W_{\mathbb{Q}_p} \to GL_n(\mathbb{C})$ which factor through $M(\mathbb{C})$, but no smaller Levi subgroup.
4. Equivalently, $\phi = \phi_1 \oplus \cdots \oplus \phi_k$, for ϕ_i an irreducible representation of dimension h_i.

Supercuspidal Parameters and the Langlands' classification

1. We say that $\Phi_M^W(G)$ is the set of supercuspidal parameters of M.
2. Analogously, $\pi \in \Pi(M)$ is said to be supercuspidal if, for all Levi subgroups $M' \subset M$ which are Levi factors of parabolics with unipotent radical U, the representation $\pi_U = \pi/\{\pi(u)v - v | u \in U\}$ of M' is trivial.
3. If M is the Levi factor of a parabolic P with unipotent radical N. Given $\pi_M \in \Pi(M)$, we define the parabolic induction $Ind_P^G(\pi_M)$:

$$\{f : G(\mathbb{Q}_p) \to V \text{ locally constant } | f(mng) = \pi_M(m)f(g)\}$$

for $m \in M$ and $n \in N$.

Theorem (Bernstein-Zelevinsky)

$\pi \in \Pi(G)$ is either supercuspidal or an irreducible constituent of a parabolic induction of a supercuspidal representation $\pi_M \in \Pi(M)$.

Supercuspidal Parameters and the Langlands' classification

1. Now supercuspidal parameters $\phi \in \Phi_M^W(G)$ of M should correspond bijectively to supercuspidal representations π_M of M.
2. Would then like to say that $Ind_P^G(\pi_M)$ corresponds to ϕ under LLC_n.
3. Problem: $Ind_P^G(\pi_M)$ is not always irreducible, so this construction would lead to all these irreducible constituents having the same attatched parameter $\Phi^W(G)$ (so the map LLC_n will not be a bijection!).

Supercuspidal Parameters and the Langlands' classification

Example

Let $\mathbf{1}_T$ be the trivial representation of the maximal torus $\mathbb{Q}_p^* \times \mathbb{Q}_p^* = T(\mathbb{Q}_p) \subset GL_2(\mathbb{Q}_p)$. Then $Ind_B^{GL_2}(\mathbf{1}_T)$ can be identified with locally constant functions on the flag variety $GL_2/B(\mathbb{Q}_p) = \mathbb{P}^1(\mathbb{Q}_p) \to \mathbb{C}$, with $G(\mathbb{Q}_p)$ acting via conjugation. The constant functions are acted on trivially. We get an exact sequence of $G(\mathbb{Q}_p)$-representations:

$$0 \to \mathbf{1}_G \to Ind_B^{GL_2}(\mathbf{1}_T) \to St \to 0$$

Supercuspidal Parameters and the Langlands' classification

In this case, we need a way to distinguish the two representations $\mathbf{1}_G$ and St. This is precisely why we need to include the $SL(2,\mathbb{C})$-factor:

$$\phi : W_{\mathbb{Q}_p} \times SL(2,\mathbb{C}) \to GL_2(\mathbb{C})$$

The fact that they are irreducible constituents of $Ind_B^{GL_2}(\mathbf{1}_T)$ implies that restriction to $W_{\mathbb{Q}_p}$ should be trivial. There are two possible extensions:

1. $1_{\mathbf{G}} \otimes \mathbf{1}_{SL(2,\mathbb{C})}$, where $\mathbf{1}_{SL(2,\mathbb{C})}$ is the trivial representation.
2. $1_{\mathbf{G}} \boxtimes \nu(2)$, where $\nu(2) : SL(2,\mathbb{C}) \hookrightarrow GL_2(\mathbb{C})$ is the standard embedding.

These correspond (up to twists) to the two irreducible constituents of $Ind_B^{GL_2}(\mathbf{1}_T)$ under LLC_2!

Supercuspidal Parameters and the Langlands' Classification

1. This construction works in general and reduces the construction of LLC_n to showing a bijection between supercuspidal representations $\pi \in \Pi(GL_n)$ and irreducible n-dimensional representations ϕ of $W_{\mathbb{Q}_p}$ for varying $n \geq 1$.
2. The key insight of Harris-Taylor was that one could realize the correspondence for supercuspidal representations in the cohomology of some (local) Shimura varieties.
3. To motivate this, first let's consider the case that $n = 1$. We recall that it boils down to an isomorphism $\mathbb{Q}_p^* \simeq W_{\mathbb{Q}_p}^{ab}$.

The Geometry of Formal Groups

1. We recall that under this isomorphism, we have
$$W_{\mathbb{Q}_p}^{ab} \simeq \mathbb{Z}_p^* \times p^{\mathbb{Z}}$$
.
2. The fixed field of $\mathbb{Z}_p^*$ is $\breve{\mathbb{Q}}_p$ the completion of the maximal unramified extension.
3. We let $\mathbb{Q}_p^{rm}$ be the field fixed by $\mathbb{Z}_p^*$, so we have that $\mathbb{Q}_p^{ab} = \mathbb{Q}_p^{rm}\breve{\mathbb{Q}}_p$. What is $\mathbb{Q}_p^{rm}$?
4. The local Kronecker-Weber theorem tells us that $\mathbb{Q}_p^{rm}$ is the completion of $\bigcup_{n\geq 1} \mathbb{Q}_p(\mu_{p^n})$.
5. Geometrically, we can think of this as:
$$\bigcup_{n\geq 1} (\mathrm{Ker}[p^n] : \mathbb{G}_m \to \mathbb{G}_m)$$

The Geometry of Formal Groups

1. We note that this is completely determined by local datum around the identity. In particular, we could have replaced $\mathbb{G}_m$ by its formal completion at the identity, denoted $\hat{\mathbb{G}}_m$.
2. A 1-dimensional formal group over $\breve{\mathbb{Z}}_p$ is a power series $\mathcal{G} \in \breve{\mathbb{Z}}_p[|X, Y|]$ such that:
   - $\mathcal{G}(X, Y) = X + Y +$ (h.o.t).
   - $\mathcal{G}(X, Y) = \mathcal{G}(Y, X)$
   - $\mathcal{G}(\mathcal{G}(X, Y), Z) = \mathcal{G}(X, \mathcal{G}(Y, Z))$.
   - There exists $i(X) \in \breve{\mathbb{Z}}_p[|X|]$ for which $\mathcal{G}(X, i(X)) = 0$.
3. (Ex) $\hat{\mathbb{G}}_m$ is the formal group given by $(1 + X)(1 + Y) - 1 = X + Y + XY$. Also, for all $a \in \mathbb{Z}_p$, we have an endomorphism

$$[a]_{\hat{\mathbb{G}}_m}(X) = (1 + X)^a - 1 = \sum_{n=1}^{\infty} \binom{a}{n} X^n$$

The Geometry of Formal Groups

1. Note that
$$[p]_{\hat{\mathbb{G}}_m}(X) \cong X^p \mod p$$
We say the height of $\hat{\mathbb{G}}_m$ is 1.
2. We can obtain more 1-dimensional formal groups by looking at elliptic curves. For example, super-singular elliptic curves gives rise to height 2 1-dimensional formal groups!
3. We let LT_n be the formal scheme over $Spf(\breve{\mathbb{Z}}_p)$ whose $\mathbb{Z}$-connected components parametrize isomorphism classes of 1-dimensional height n formal $\breve{\mathbb{Z}}_p$-modules.
4. We have an isomorphism:
$$LT_n \simeq Spf(\breve{\mathbb{Z}}_p[|X_1, \dots, X_{n-1}|])$$
this has an action of $D^*_{\frac{1}{n}}$ coming from automorphisms of the formal group.

The Geometry of Formal Groups

1. For varying $m \in \mathbb{N}_{\geq 0}$, we let $LT_{n,m} \to LT_n$ be the space parametrizing trivializations of the torsion $\mathcal{G}[p^m] \simeq (\mathbb{Z}/p^m\mathbb{Z})^n$. This tower has an action of $GL_n(\mathbb{Z}_p)$.
2. We pass to the rigid generic fiber and this gives us a tower

$$\lim_{m\to\infty} LT_{n,m,\breve{\mathbb{Q}}_p} = LT_{n,\infty,\breve{\mathbb{Q}}_p}$$

   of rigid spaces over $\breve{\mathbb{Q}}_p$. This is an example of a perfectoid space. Note that, if $n = 1$, this gives us disjoint copies of the "perfectoid" field $\bigcup_{n\geq 1} \breve{\mathbb{Q}}_p(\mu_{p^n})$ that realizes the maximal abelian extension!
3. Suggest that the cohomology of this space should realize local Langlands for GL_n!

The Cohomology of the Lubin-Tate Tower

Definition

1. Let $\pi \in \Pi(GL_n)$ and $\rho \in \Pi(D^*_{\frac{1}{n}})$.
2. Let $\mathcal{H}(G) := C_c^\infty(GL_n(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ and $\mathcal{H}(J_b) := C_c^\infty(D^*_{\frac{1}{n}}(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ denote the usual smooth Hecke algebras.
3. Set $R\Gamma_c(LT_{n,\infty}) := \mathrm{colim}_{m\to\infty} R\Gamma_c(LT_{n,m}, \overline{\mathbb{Q}}_\ell)$.
4. Set $R\Gamma_c(LT_{n,\infty})[\pi] := R\Gamma_c(LT_{n,\infty}) \otimes^{\mathbb{L}}_{\mathcal{H}(GL_n)} \pi$ and $R\Gamma_c(LT_{n,\infty})[\rho] := R\Gamma_c(LT_{n,\infty}) \otimes^{\mathbb{L}}_{\mathcal{H}(D^*_{\frac{1}{n}})} \rho$.

The Cohomology of the Lubin-Tate Tower

Theorem (Harris-Taylor/Dat)

1. Let π be a supercuspidal representation of $GL_n(\mathbb{Q}_p)$ and $\rho := JL^{-1}(\pi) \in \Pi(D^*_{\frac{1}{n}}{}^*)$.
2. Then $R\Gamma_c(LT_{n,\infty})[\pi]$ (resp. $R\Gamma_c(LT_{n,\infty})[\rho]$) is concentrated in middle degree $n-1$, and its cohomology decomposes as a $J_b(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$ (resp. $G(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$) representation as:

$$\rho \boxtimes \phi^\vee \otimes |\cdot|^{(1-n)/2}$$

and

$$\pi \boxtimes \phi \otimes |\cdot|^{(1-n)/2}$$

respectively, where ϕ is the Harris-Taylor parameter attached to π.

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2. $G/\mathbb{Q}_p$ be a connected reductive group.
3. $W_{\mathbb{Q}_p}$ be the Weil group of $\mathbb{Q}_p$
4. $\hat{G}$ the Langlands dual group of G viewed as a reductive group over $\overline{\mathbb{Q}}_\ell$
5. ${}^L G := W_{\mathbb{Q}_p} \ltimes \hat{G}$
6. We fix isomorphisms $i : \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$ and $j : \overline{\mathbb{Q}}_p \simeq \mathbb{C}$.

Notation

We set:

1. $\Pi(G)$ to be isomorphism classes of smooth irreducible representations of $G(\mathbb{Q}_p)$.
2. $\Phi(G)$ to be the set of conjugacy classes of admissible homomorphisms:

$$\phi : W_{\mathbb{Q}_p} \times SL(2, \overline{\mathbb{Q}}_\ell) \to {}^L G(\overline{\mathbb{Q}}_\ell)$$

3. $\Phi^W(G)$ to be the set of conjugacy classes of continuous semisimple homomorphisms:

$$\phi : W_{\mathbb{Q}_p} \to {}^L G(\overline{\mathbb{Q}}_\ell)$$

Theorem (Fargues-Scholze)

For any G, there exists a map:

$$\Pi(G) \xrightarrow{LLC_G^{FS}} \Phi^W(G)$$

$$\pi \mapsto \phi_\pi^{FS}$$

enjoying the following properties:

1. It is compatible with restriction of scalars, central characters, products, and isogenies.
2. It is is compatible with parabolic induction of representations.
3. It is compatible with the correspondence of Harris-Taylor/Henniart for $G = GL_n$ and its inner forms.

Key Question:

Can we show that the Fargues-Scholze Local Langlands correspondence is compatible with other instances of the correspondence? Namely, given a local Langlands correspondence:

$$\Pi(G) \xrightarrow{LLC_G} \Phi(G)$$

We expect a commutative diagram of the form:

$$\begin{array}{ccc} \Pi(G) & \xrightarrow{LLC_G} & \Phi(G) \\ & \searrow^{LLC_G^{FS}} & \downarrow \\ & & \Phi^W(G) \end{array}$$

where the right vertical arrow precomposes the map $\phi \in \Phi(G)$ with $g \in W_{\mathbb{Q}_p} \mapsto (g, \begin{pmatrix} |g|^{\frac{1}{2}} & 0 \\ 0 & |g|^{\frac{-1}{2}} \end{pmatrix}) \in W_{\mathbb{Q}_p} \times SL(2, \overline{\mathbb{Q}}_\ell)$

Theorem (Gan-Takeda/Gan-Tantono/Chan-Gan)

1. Let $L/\mathbb{Q}_p$ be a finite extension.
2. Let $G = Res_{L/\mathbb{Q}_p}GSp_4$ and $J = Res_{L/\mathbb{Q}_p}GU_2(D)$ be its unique non-split inner form, where D is the quaternion division algebra over L.
3. Then, for $H = G$ or J, up to the choice of the fixed isomorphism i, there exists a unique map:

$$LLC_H : \Pi(H) \to \Phi(H)$$

$$\pi \mapsto \{\phi_\pi : W_L \times SL(2, \overline{\mathbb{Q}}_\ell) \to \hat{H}(\overline{\mathbb{Q}}_\ell) = GSpin_5(\overline{\mathbb{Q}}_\ell) \simeq GSp_4(\overline{\mathbb{Q}}_\ell)\}$$

   characterized by preservation of character twists, L, ϵ, γ factors, and a condition on the Plancharel measure of a family of induced representations.
4. The L-packets satisfy the endoscopic character identities.

Remarks

1. The proof of compatibility will formally reduce to case of supercuspidal representations ($\implies$ discrete series). The L-parameters of such representations will be discrete (i.e ϕ does not factor through a Levi subgroup of GSp_4.)
2. Given such a parameter ϕ, the L-packets $\Pi_{\phi}(G) := LLC_G^{-1}(\phi)$ and $\Pi_{\phi}(J) := LLC_J^{-1}(\phi)$ have size 1 or 2, we say that ϕ is stable or endoscopic, respectively.
3. Let $std : GSp_4(\overline{\mathbb{Q}}_\ell) \rightarrow GL_4(\overline{\mathbb{Q}}_\ell)$ be the standard embedding.
   - (stable) $std \circ \phi$ is irreducible.
   - (endoscopic) $std \circ \phi \simeq \phi_1 \oplus \phi_2$, with ϕ_1 and ϕ_2 distinct irreducible 2-dimensional reps of $W_L \times SL(2, \overline{\mathbb{Q}}_\ell)$ such that $det(\phi_1) = det(\phi_2)$.

Remarks

We can further classify discrete parameters as follows:

- (supercuspidal) The SL_2-factor acts trivially. In this case, $\Pi_{\phi}(G)$ and $\Pi_{\phi}(J)$ consist only of supercuspidal representations.
- (mixed supercuspidal) The SL_2-factor acts non-trivially. In this case, $\Pi_{\phi}(G) = \{\pi_{disc}, \pi_{sc}\}$ and $\Pi_{\phi}(J)$ contain a mix of supercuspidal and (discrete series) non-supercuspidal representations. There are two cases:
  1. (Saito-Kurokawa Type) We have $std \circ \phi = \phi_0 \oplus \chi \boxtimes \nu(2)$ and $\Pi_{\phi}(J) = \{\rho_{disc}, \rho_{sc}\}$.
  2. (Howe-Piatetski–Shapiro Type) We have $std \circ \phi = \chi_1 \boxtimes \nu(2) \oplus \chi_2 \boxtimes \nu(2)$ and $\Pi_{\phi}(J) = \{\rho^1_{disc}, \rho^2_{disc}\}$.

The Main Theorem

Theorem (H)

The following is true.

1. For any $\pi \in \Pi(G)$ (resp. $\rho \in \Pi(J)$) such that the Gan-Takeda (resp. Gan-Tantono) parameter is not supercuspidal the Gan-Takeda (resp. Gan-Tantono) correspondence is compatible with the Fargues-Scholze correspondence.
2. If $L/\mathbb{Q}_p$ is unramified and $p > 2$, we have, for all $\pi \in \Pi(G)$ (resp. $\rho \in \Pi(J)$) such that the Gan-Takeda (resp. Gan-Tantono) parameter is supercuspidal the Gan-Takeda (resp. Gan-Tantono) correspondence is compatible with the Fargues-Scholze correspondence.

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Definition

1. Let $F = \lim_{x \mapsto x^p} \mathbb{C}_p$ be the tilt of the completed algebraic closure of $\mathbb{Q}_p$.
2. We set X to be the (schematic) Fargues-Fontaine curve of F.

Remark

X is a Dedekind scheme. Its closed points correspond to characteristic 0 untilts of F.

Theorem (Fargues)

- G-bundles on X correspond to elements of the Kottwitz set $B(G) := G(\breve{\mathbb{Q}}_p)/(b \sim gb\sigma(g)^{-1})$, where σ is the Frobenius on $\breve{\mathbb{Q}}_p$. In other words, we have an isomorphism:

$$B(G) \xrightarrow{\simeq} |Bun_G|$$

$$b \mapsto |Bun_G^b| = *$$

Remark

Elements of $B(G)$ parametrize G-isocrystals over k, which describe objects in the isogeny categories of p-divisible groups with extra structures.

Definition

1. Let $\mu \in X_*(G_{\overline{\mathbb{Q}}_p})^+$ be a minuscule dominant cocharacter with reflex field E.
2. Let $b \in B(G, \mu) \subset B(G)$ be the unique basic element. We call the triple (G, b, μ) a basic local Shimura datum.
3. $\breve{E} = E\breve{\mathbb{Q}}_p$
4. Let $\mathcal{E}_b$ be the bundle on X corresponding to $b \in B(G)$.
5. We define the diamond $Sht(G, b, \mu)_\infty \to Spd(\breve{E})$ to be the space parametrizing modifications

$$\mathcal{E}_b \dashrightarrow \mathcal{E}_0$$

with meromorphy bounded by μ. A local Shimura variety at infinite level.

Remark

The space $Sht(G, b, \mu)_\infty$ has commuting actions by $G(\mathbb{Q}_p) = Aut(\mathcal{E}_0)$ and $J_b(\mathbb{Q}_p) = Aut(\mathcal{E}_b)$, where J_b is the σ-centralizer of $b \in G(\breve{\mathbb{Q}}_p)$.

Example

Let $G = GL_n$ and $\mu = (1, 0 \ldots, 0)$, then $LT^\diamond_{n,\infty} \simeq Sht(GL_n, b, \mu)_\infty$. It parametrizes injections of the form:

$$\mathcal{O}(-\frac{1}{n}) \hookrightarrow \mathcal{O}_X^n$$

with cokernel of length 1 supported at a single closed point of X, where $\mathcal{O}(-\frac{1}{n})$ is the unique rank n bundle on X of slope $-\frac{1}{n}$. In this case, $J_b(\mathbb{Q}_p) = D^*_{1/n}$, where $D_{1/n}$ is the division algebra of invariant $1/n$.

The Cohomology of the Lubin-Tate Tower

Definition

1. Let (G, b, μ) be a local Shimura datum.
2. Let $\pi \in \Pi(G)$ and $\rho \in \Pi(J_b)$.
3. Let $\mathcal{H}(G) := C_c^{\infty}(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ and $\mathcal{H}(J_b) := C_c^{\infty}(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ denote the usual smooth Hecke algebras.
4. Set $R\Gamma_c(G, b, \mu) := \mathrm{colim}_{K \to \{1\}} R\Gamma_c(Sht(G, b, \mu)_\infty / \underline{K}, \overline{\mathbb{Q}}_\ell)$.
5. Set $R\Gamma_c(G, b, \mu)[\pi] := R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(G)} \pi$ and $R\Gamma_c(G, b, \mu)[\rho] := R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \rho$.

The Cohomology of the Lubin-Tate Tower

Theorem (Harris-Taylor/Dat)

1. Let $(G, b, \mu) = (GL_n, b, (1, 0, \dots, 0))$.
2. Let π be a supercuspidal representation of $G(\mathbb{Q}_p)$ and $\rho := JL^{-1}(\pi) \in \Pi(D^*_{\frac{1}{n}})$ with associated Weil parameter ϕ.
3. Then $R\Gamma_c(G, b, \mu)[\pi]$ (resp. $R\Gamma_c(G, b, \mu)[\rho]$) is concentrated in middle degree $n - 1$, and its cohomology decomposes as a $J_b(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$ (resp. $G(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$) representation as:

$$\rho \boxtimes \phi^\vee \otimes |\cdot|^{(1-n)/2}$$

and

$$\pi \boxtimes \phi \otimes |\cdot|^{(1-n)/2}$$

respectively.

Function-Sheaf Dictionary

To a locally spatial diamond or Artin v-stack X, Fargues-Scholze define a triangulated category $D(X) := D_{lis}(X, \overline{\mathbb{Q}}_\ell)$

Proposition (Fargues-Scholze)

1. For a connected reductive group $G/\mathbb{Q}_p$, define the v-stack $B\underline{G(\mathbb{Q}_p)} := [*/\underline{G(\mathbb{Q}_p)}]$.
2. We have an equivalence of categories:

$$D(B(\underline{G(\mathbb{Q}_p)})) \xrightarrow{\simeq} D(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$$

where the RHS is the unbounded derived category of smooth $\overline{\mathbb{Q}}_\ell$-representations of $G(\mathbb{Q}_p)$ and Verdier duality corresponds to smooth duality.

Function-Sheaf Dictionary

- We have an open immersion:

$$j : B\underline{G(\mathbb{Q}_p)} \simeq Bun_G^{\mathbf{1}} \hookrightarrow Bun_G$$

given by the inclusion of the HN-strata of Bun_G corresponding to the trivial bundle.

- Given $\pi \in \Pi(G)$, we can use the previous proposition to construct a sheaf:

$$j_!(\mathcal{F}_\pi) \in D(Bun_G)$$

- Following V.Lafforgue, Fargues-Scholze construct an excursion algebra acting on $D(Bun_G)$ via Hecke operators, which acts on this sheaf via eigenvalues determined by the parameter ϕ_π.

Hecke Operators

- We consider the diamond $Spd(\breve{\mathbb{Q}}_p)/Frob^{\mathbb{Z}}$, it parametrizes untilts $F^\sharp$ of F, which were precisely the closed points in X.
- We then have the Hecke-Stack:

$$\begin{array}{ccc} & Hck & \\ {\scriptstyle h^{\leftarrow}} \swarrow & & \searrow {\scriptstyle h^{\rightarrow} \times supp} \\ Bun_G & & Bun_G \times Spd(\breve{\mathbb{Q}}_p)/Frob^{\mathbb{Z}} \end{array}$$

parametrzing triples $(\mathcal{E}_0, \mathcal{E}_1, j, F^\sharp)$, where $j : \mathcal{E}_0 \to \mathcal{E}_1$ is a modification of two G-bundles on X away from the closed points defined by characteristic 0 untilt $F^\sharp$ of F.

Hecke Operators

- Given $V \in Rep_{\overline{\mathbb{Q}}_\ell}(^L G)$, Geometric Satake furnishes a sheaf $\mathcal{S}_V$ on Hck.
- We then define the Hecke operator:

$$T_V : D(Bun_G) \to D(Bun_G \times Spd(\breve{\mathbb{Q}}_p)/Frob^{\mathbb{Z}})$$

$$\mathcal{F} \mapsto R(h^{\rightarrow} \times supp)_{\natural}(h^{\leftarrow *}(\mathcal{F}) \otimes^{\mathbb{L}} \mathcal{S}_V)$$

Drinfeld's Lemma

Theorem (Fargues-Scholze)

The Hecke operator T_V induces a functor:

$$T_V : D(Bun_G) \to D(Bun_G)^{BW_{\mathbb{Q}_p}}$$

where $D(Bun_G)^{BW_{\mathbb{Q}_p}}$ is the derived category of objects in $D(Bun_G)$ with a continuous action of $W_{\mathbb{Q}_p}$.

- The values of the Fargues-Scholze parameter ϕ_π^{FS} are given by evaluating Hecke operators acting on $j_{1!}(\mathcal{F}_\pi) \in D(Bun_G)$ for varying elements of $W_{\mathbb{Q}_p}$, and the values of these Hecke operators are related to local Shimura varieties!

Hecke Operators and Local Shimura varieties

- We look at the Hecke Correspondence:

$$\begin{array}{ccccc} & & Hck_{\mu} & & \\ & \swarrow^{h^{\leftarrow}} & & \searrow^{h^{\rightarrow} \times supp} & \\ Bun_G^b \xrightarrow{j} Bun_G & & & & Bun_G \underset{j_1 \times id}{\longleftarrow} Bun_G^1 \end{array}$$

- We consider the sheaf:

$$(j_1)^* T_{\mu}(j_!(\mathcal{F}_{\rho})) \in D(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_{\ell})^{BW_{\mathbb{Q}_p}}$$

- This will be computed in terms of the space parametrizing modifications $\mathcal{E}_b \dashrightarrow \mathcal{E}_0$ of type μ. We get:

$$R\Gamma_c(G, b, \mu)[\rho][d](\frac{d}{2}) \simeq (j_1)^* T_{\mu}(j_{b!}(\mathcal{F}_{\rho}))$$

Compatibility for GL_n

- We combine this identification with the following:

Lemma (H,Koshikawa)

Let V be an irreducible representation of ${}^LG$, ϕ an irreducible representation of $W_{\mathbb{Q}_p}$, $b \in B(G)$, and $\rho \in \Pi(J_b)$.
If the cohomology sheaves of $T_V(j_{b!}(\mathcal{F}_\rho)) \in D(Bun_G)^{BW_{\mathbb{Q}_p}}$ admit a sub-quotient with action of ϕ then $r_V \circ \phi_\rho^{FS}$ also has such a sub-quotient.

Compatibility for GL_n

- If we let $(G, b, \mu) = (GL_n, b, (1, 0, \ldots, 0))$ and $\rho \in \Pi(D^*_{\frac{1}{n}})$ be a supercuspidal representation. Then, by Harris-Taylor, we know that:

$$R\Gamma_c(G, b, \mu)[\rho][n-1](\frac{n-1}{2}) \simeq j_1^* T_\mu j_{b!}(\mathcal{F}_\rho)$$

  admits a sub-quotient with $W_{\mathbb{Q}_p}$-action given by the semi-simplified parameter ϕ attatched to ρ
- Therefore, by the lemma $r_\mu \circ \phi_\rho^{FS} = \phi_\rho^{FS}$ also admits such a sub-quotient. Hence, we get that $\phi = \phi_\rho^{FS}$ and conclude compatibility for supercuspidal ρ.
- By compatibility of the Fargues-Scholze and Harris-Taylor correspondence with parabolic induction, we conclude compatibility in general.

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- Similar to the case of GL_n compatibility will be proven by describing the cohomology of certain local Shimura varieties attached to GSp_4/L for $L/\mathbb{Q}_p$ a finite extension.
- Consider $(G, b, \mu) := (Res_{L/\mathbb{Q}_p}(GSp_4), b, \mu)$, where $b \in B(G, \mu)$ is the unique basic element and μ is the Siegel cocharacter.
- Then $J_b = Res_{L/\mathbb{Q}_p} GU_2(D)$, where D is the quaternionic division algebra over L.
- Note that the rep of $\hat{G} \simeq GSp_4$ induced by μ is the standard embedding:

  $$std : GSp_4 \hookrightarrow GL_4$$

  and that $\langle 2\rho_G, \mu\rangle = 3$.
- We have a very general description of the cohomology assuming that the "refined LLC" is known for our group.

The Refined LLC

- Fix a discrete parameter $\phi : W_{\mathbb{Q}_p} \times SL(2, \mathbb{C}) \to GSp_4(\mathbb{C})$.
- We let $S_\phi := Cent(\phi, \hat{G})$.
- Note that we have $Z(\hat{G}) = GL_1$.
- The refined Local Langlands correspondence defines a bijection:

$$\Pi_\phi(GSp_4) \leftrightarrow \{\text{irred. reps } \tau \text{ of } S_\phi \text{ s.t } \tau|_{Z(\hat{G})} = \mathbf{1}\}$$

- And a bijection:

$$\Pi_\phi(GU_2(D)) \leftrightarrow \{\text{irred. reps } \tau \text{ of } S_\phi \text{ s.t } \tau|_{Z(\hat{G})} = id_{GL_1}\}$$

- These bijections are characterized by the endoscopic character identities proven by Chan-Gan, after fixing (B, ψ) a Whittaker datum.

The Refined LLC

- If ϕ is a stable discrete parameter then the L-packets are singletons and $S_\phi = Z(\hat{G}) = GL_1$.
- If ϕ is an endoscopic parameter $(std \circ \phi \simeq \phi_1 \oplus \phi_2)$. There is an identification

$$S_\phi = \{(a,b) \in GL_1 \times GL_1 | a^2 = b^2\} \subset GL_2 \times GL_2 \subset GL_4$$

  where $Z(\hat{G}) = GL_1$ embeds diagonally.
- We see that $\pi_0(S_\phi) \simeq \mathbb{Z}/2\mathbb{Z}$. The L-packet of $\Pi_\phi(GSp_4) = \{\pi^+, \pi^-\}$ is indexed by the reps τ_{π^+} and τ_{π^-} of S_ϕ defined by the trivial and non-trivial character of $\pi_0(S_\phi)$.
- The L-packet $\Pi_\phi(GU_2(D)) = \{\rho_1, \rho_2\}$ is indexed by the representations τ_{ρ_1} and τ_{ρ_2} corresponding to projection to the two GL_1 factors.

Hansen-Kaletha-Weinstein

Theorem (Hansen-Kaletha-Weinstein)

- Let $\rho \in \Pi_\phi(J_b)$.
- Let $K_0(G(\mathbb{Q}_p))$ (resp. $K_0(G(\mathbb{Q}_p))_{ell}$) be the Grothendieck group of (resp. elliptic) admissible $G(\mathbb{Q}_p)$-representations.
- If ϕ is a stable discrete (resp. supercuspidal) parameter, we have an equality

$$[R\Gamma_c(G, b, \mu)[\rho]] = -4\pi$$

  in $K_0(G(\mathbb{Q}_p))_{ell}$ (resp. $K_0(G(\mathbb{Q}_p))$).
- If ϕ is an endoscopic discrete parameter, we have an equality

$$[R\Gamma_c(G, b, \mu)[\rho]] = -2\pi^+ - 2\pi^-$$

Compatibility in the non-supercuspidal case

- We want to use the previous corollaries to begin making progress to compatibility. We first note that for any $\rho \in \Pi(J)$ (resp. $\pi \in \Pi(G)$) an irreducible constituent of a parabolic induction, compatibility follows from compatibility of Fargues-Scholze with parabolic induction and compatibility for GL_n and its inner forms.
- Therefore, we can assume ρ (resp. π) is supercuspidal, which implies that their L-parameter is discrete and we can use the previous corollaries.

Compatibility in the non-supercuspidal case

- It follows from the commutation of excursion and Hecke operators that the following is true.

Lemma

For any $\rho \in \Pi(J_b)$, if $\pi \in \Pi(G)$ occurs as a sub-quotient of $R\Gamma_c(G, b, \mu)[\rho]$ then $\phi_\rho^{FS} = \phi_\pi^{FS}$.

- Now, assume ϕ is of Howe-Piatetski–Schapiro type, write $\Pi_\phi(G) = \{\pi_{sc}, \pi_{disc}\}$ and $\Pi_\phi(J) = \{\rho_{sc}, \rho_{disc}\}$. The previous corollary gives us an equality

$$[R\Gamma_c(G, b, \mu)[\rho_{disc}]_{sc}] = -2\pi_{sc}$$

in $K_0(G(\mathbb{Q}_p))$.

- Thus, by the Lemma, we get $\phi_{\pi_{sc}}^{ss} = \phi_{\rho_{disc}}^{ss} = \phi_{\rho_{disc}}^{FS} = \phi_{\pi_{sc}}^{FS}$.

Compatibility in the supercuspidal case

- This allow us to reduce to checking compatibility for $\rho \in \Pi(J_b)$ with supercuspidal Gan-Tantono parameter ϕ.

Key Proposition

$$\bigoplus_{\rho' \in \Pi_\phi(J)} R\Gamma_c(G, b, \mu)[\rho']_{sc}$$

admits a non-zero W_L-stable sub-quotient with W_L-action given by $std \circ \phi \otimes |\cdot|^{-3/2}$.

- The previous Lemma and Corollaries tell us that the Fargues-Scholze parameter of all $\rho' \in \Pi_\phi(J_b)$ agree. Therefore, the same analysis as for GL_n allows us to conclude $std \circ \phi_\rho^{FS} = std \circ \phi$.

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We will now discuss the proof of the Key Proposition.

- As in Harris-Taylor, the key idea will be to relate the complex $\bigoplus_{\rho' \in \Pi_{\phi}(J)} R\Gamma_c(G, b, \mu)[\rho']_{sc}$ to the cohomology of a global Shimura variety.
- We let:
  - $F/\mathbb{Q}$ be a totally real field such that:
    - p is totally inert in F and $F_p \simeq L$.
    - q is an auxillary totally inert prime.
  - $\mathbf{G}$ a $\mathbb{Q}$-inner form of $Res_{F/\mathbb{Q}} GSp_4 =: \mathbf{G}^*$ such that:
    - $\mathbf{G}(\mathbb{R}) \simeq GSp_4(\mathbb{R}) \times GU_2(\mathbb{H})^{[F:\mathbb{Q}]-1}$
    - $\mathbf{G}_{\mathbb{Q}_p} \simeq Res_{L/\mathbb{Q}_p} GSp_4 = G$.
    - $\mathbf{G}_{F_v} \simeq GSp_4/F_v$ at all finite places v if $[F : \mathbb{Q}]$ is odd.
    - $\mathbf{G}_{F_v} \simeq GSp_4/F_v$ at all but the finite place q if $[F : \mathbb{Q}]$ is even.

Basic Uniformization

- We let $(\mathbf{G}, X)$ be the Shimura datum of abelian type, where $X_{\mathbb{C}}$ under the isomorphism $j : \overline{\mathbb{Q}}_p \simeq \mathbb{C}$ agrees with the conjugacy class of cocharacters defined by μ.
- Let $\mathbb{A}$ (resp. $\mathbb{A}_f$) denote the (resp. finite) adeles of $\mathbb{Q}$.
- For any compact open $K \subset \mathbf{G}(\mathbb{A}^f)$, we let $\mathcal{S}(\mathbf{G}, X)_K$ be the rigid analytic global Shimura variety over $\mathbb{C}_p$ of level K.
- We consider the infinite level Shimura variety:

$$\mathcal{S}(\mathbf{G}, X)_{K^p} = \lim_{K_p \to \{1\}} \mathcal{S}(\mathbf{G}, X)_{K^p K_p}$$

This is representable by a perfectoid space after completing the structure sheaf.

Basic Uniformization

- By results of Hansen, there exists a canonical $G(\mathbb{Q}_p)$-equivariant Hodge-Tate period map:

$$\pi_{HT} : \mathcal{S}(\mathbf{G}, X)_{K^p} \to \mathcal{F}\ell_{G,\mu^{-1}}$$

  where $\mathcal{F}\ell_{G,\mu^{-1}} := (G_{\mathbb{C}_p}/P_{\mu^{-1}})^{ad}$.
- For $b \in B(G, \mu)$, we define the Newton strata $\mathcal{S}(\mathbf{G}, X)^b_{K^p}$ by pulling back the Newton strata $\mathcal{F}\ell^b_{G,\mu^{-1}}$ along π_{HT}.

Basic Uniformization

- We let $\mathbf{G}'$ denote another $\mathbb{Q}$-inner form of $\mathbf{G}^*$ satisfying:
  - $\mathbf{G}'(\mathbb{R}) \simeq GU_2(\mathbb{H})^{[F:\mathbb{Q}]}$
  - $\mathbf{G}'_{\mathbb{Q}_p} \simeq J_b$.
  - $\mathbf{G}'(\mathbb{A}_f^p) \simeq \mathbf{G}(\mathbb{A}_f^p)$
- Under the assumptions on L, by results of Shen there exists a $\mathbf{G}(\mathbb{A}_f)$-equivariant isomorphism of diamonds over $\mathbb{C}_p$:

$$\lim_{K^p} \mathcal{S}(\mathbf{G}, X)^b_{K^p} \simeq \underline{\mathbf{G}'(\mathbb{Q})} \backslash \underline{\mathbf{G}'(\mathbb{A}^f)} \times_{Spd(\mathbb{C}_p)} Sht(G, b, \mu)_\infty / \underline{J_b(\mathbb{Q}_p)}$$

for the basic element $b \in B(G, \mu)$. Moreover,

$$\pi_{HT} : \lim_{K^p} \mathcal{S}(\mathbf{G}, X)^b_{K^p} \to \mathcal{F}\ell^b_{G,\mu^{-1}} \simeq Sht(G, b, \mu)_\infty / \underline{J_b(\mathbb{Q}_p)}$$

agrees with projection to the second factor.

Basic Uniformization

This allows us to deduce:

Corollary

There exists a natural $G(\mathbb{Q}_p) \times W_L$ and Hecke equivariant excision map:

$$\Theta : R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}^f)/K^p, \mathcal{L}_\xi) \to$$

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)$$

where (G, b, μ) is the local Shimura datum from before and $\mathcal{L}_\xi$ is the $\overline{\mathbb{Q}}_\ell$-local system defined by a representation of $\mathbf{G}$ of some highest weight ξ.

Boyer's Trick

Proposition

Let $R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)_{sc}$ denote the part of the cohomology where $G(\mathbb{Q}_p)$ acts via a supercuspidal representation. Then Θ induces an isomorphism:

$$\Theta : R\Gamma_c(G, b, \mu)_{sc} \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}^f)/K^p, \mathcal{L}_\xi) \simeq$$

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)_{sc}$$

of $G(\mathbb{Q}_p) \times W_L$-representations.

Globalization

Let $\rho \in \Pi(J)$ be a representation with supercuspidal Gan-Tantono parameter ϕ. Using the simple trace formula, we choose a globalization

$$\Pi' \in \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p, \mathcal{L}_\xi)$$

of ρ to a cusp form of $\mathbf{G}'(\mathbb{A}_F)$ of fixed level K^p satisfying the following:

- Π'_∞ is cohomological of regular weight ξ.
- Π' is unramified away from finite set of finite places S.
- Π' is an unramified twist of Steinberg at some finite set of finite places $\{q\} \subset S_{st}$.

Strong Multiplicity One

By combing the stable trace formula and the simple twisted trace formula of Kottwitz-Shelstad, we can deduce the following:

Proposition (H.)

Assume that $|S_{st}| \geq 3$. Let π be a cuspidal automorphic representation of $\mathbf{G}'$, $\mathbf{G}$, or $\mathbf{G}^* = GSp_4/F$ such that:

- $\pi^{S \cup \{\infty\}} \simeq \Pi'^{S \cup \{\infty\}}$
- π_v is an unramified twist of Steinberg at all $v \in S_{st}$
- π_∞ is cohomological of regular weight ξ

then we have an equality: $\phi_{\pi_p} = \phi$.

- If we let $\mathfrak{m} \subset \mathbb{T}^S$ be the maximal ideal in the commutative Hecke algebra defined by Π', we can localize the uniformization isomorphism at $\mathfrak{m}$

$$(R\Gamma_c(G,b,\mu)_{sc} \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p, \mathcal{L}_\xi))_{\mathfrak{m}} \simeq$$

$$R\Gamma_c(\mathcal{S}(\mathbf{G},X)_{K^p}, \mathcal{L}_\xi)_{\mathfrak{m},sc}$$

- We write $K^p = K^{S_{st}\cup\{p\}}K^p_{S_{st}}$ for $K^{S_{st}\cup\{p\}} \subset \mathbf{G}(\mathbb{A}_f^{S_{st}\cup\{p\}})$. Taking colimits on both sides as $K^p_{S_{st}} \to \{1\}$, we obtain an isomorphism

$$(R\Gamma_c(G,b,\mu)_{sc} \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^{S_{st}\cup\{p\}}, \mathcal{L}_\xi))^{st}_{\mathfrak{m}} \simeq$$

$$R\Gamma_c(\mathcal{S}(\mathbf{G},X)_{K^{S_{st}\cup\{p\}}}, \mathcal{L}_\xi)^{st}_{\mathfrak{m},sc}$$

on the summands where $\mathbf{G}$ and $\mathbf{G}'$ act via unramified twist of Steinberg at all $v \in S_{st}$.

Applying strong multiplicity one to both sides of the uniformization isomorphism, We have:

- All automorphic representations of $\mathbf{G}$ occurring in

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^{S_{st}\cup\{p\}}}, \mathcal{L}_\xi)^{st}_{\mathfrak{m},sc}$$

  have local constituent at $\mathbf{G}_{\mathbb{Q}_p} \simeq G$ with L-parameter equal to ϕ. Since ϕ is supercuspidal, we have:

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^{S_{st}\cup\{p\}}}, \mathcal{L}_\xi)^{st}_{\mathfrak{m},sc} = R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^{S_{st}\cup\{p\}}}, \mathcal{L}_\xi)^{st}_{\mathfrak{m}}$$

- Moreover, all automorphic representations of $\mathbf{G}'$ occurring in

$$(R\Gamma_c(G, b, \mu)_{sc} \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^{S_{st}\cup\{p\}}, \mathcal{L}_\xi))^{st}_{\mathfrak{m}}$$

  have local constiuent at $\mathbf{G}'_{\mathbb{Q}_p} \simeq J_b$ with L-parameter ϕ.

Galois Representations in the Cohomology of Shimura varieties

- This reduces us to checking that

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^{S_{st}\cup\{p\}}}, \mathcal{L}_\xi)^{st}_{\mathfrak{m}}$$

  has W_L-action given (up to multiplicity) by $std \circ \phi \otimes |\cdot|^{-3/2}$
- Kret and Shin show that the RHS is concentrated in middle degree (=3). Moreover, using recent work of Kisin-Shin-Zhu, they compute the trace of Frobenius on the Shimura variety over $\overline{\mathbb{Q}}$ in terms of the Satake parameters of a weak transfer τ of Π' to GSp_4.

Galois Representations in the Cohomology of Shimura varieties

Letting τ be such a (globally generic) weak transfer, we can identify the Galois action (up to multiplicity) with:

Theorem (Sorensen)

There exists, a unique (after fixing an isomorphism $i : \mathbb{C} \xrightarrow{\simeq} \overline{\mathbb{Q}}_\ell$) irreducible continuous representation $\rho_\tau : Gal(\overline{F}/F) \to GSp_4(\overline{\mathbb{Q}}_\ell)$ characterized by the property that, for each finite place $v \nmid \ell$ of F, we have:

$$WD(\rho_{\tau,i}|_{W_{F_v}})^{F-s.s} \simeq \phi_{\tau_v} \otimes |\cdot|^{-3/2}$$

where ϕ_{τ_v} is the semi-simplified Gan-Takeda parameter.

- Therefore $R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^{S_{st} \cup \{p\}}}, \mathcal{L}_\xi)^{st}_{\mathfrak{m},sc}$ has W_L-action by $std \circ \phi \otimes |\cdot|^{-3/2}$, and the Proposition follows.

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The Kottwitz Conjecture for GSp_4

- To show compatibility, it was enough to provide an indirect description of $R\Gamma_c(G, b, \mu)[\rho]$. Amazingly, by using compatibility, and the machinery of Fargues-Scholze, we can say much more about this complex. Namely, we can show:

Theorem (H)

Let $L/\mathbb{Q}_p$ be a finite unramified extension with $p > 2$ and ϕ a supercuspidal parameter. If ϕ is stable and $\Pi_\phi(G) = \{\pi\}$ and $\Pi_\phi(J) = \{\rho\}$. We have isomorphisms:

$$R\Gamma_c(G, b, \mu)[\pi] \simeq \rho \boxtimes (std \circ \phi)^\vee \otimes |\cdot|^{-3/2}[-3]$$

$$R\Gamma_c(G, b, \mu)[\rho] \simeq \pi \boxtimes (std \circ \phi) \otimes |\cdot|^{-3/2}[-3]$$

The Kottwitz Conjecture for GSp_4

Theorem (cont.)

If ϕ is endoscopic with $std \circ \phi \simeq \phi_1 \oplus \phi_2$ and $\Pi_\phi(G) = \{\pi^+, \pi^-\}$ and $\Pi_\phi(J) = \{\rho_1, \rho_2\}$. We have isomorphisms:

$$R\Gamma_c(G,b,\mu)[\pi] \simeq \begin{array}{c} \rho_1 \boxtimes \phi_1^\vee \otimes |\cdot|^{-3/2} \oplus \rho_2 \boxtimes \phi_2^\vee \otimes |\cdot|^{-3/2}[-3] \\ \rho_1 \boxtimes \phi_2^\vee \otimes |\cdot|^{-3/2} \oplus \rho_2 \boxtimes \phi_1^\vee \otimes |\cdot|^{-3/2}[-3] \end{array}$$

and

$$R\Gamma_c(G,b,\mu)[\rho] \simeq \begin{array}{c} \pi^+ \boxtimes \phi_1 \otimes |\cdot|^{-3/2} \oplus \pi^- \boxtimes \phi_2 \otimes |\cdot|^{-3/2}[-3] \\ \pi^+ \boxtimes \phi_2 \otimes |\cdot|^{-3/2} \oplus \pi^- \boxtimes \phi_1 \otimes |\cdot|^{-3/2}[-3] \end{array}$$

Both possibilities occur and knowing the precise form in one case determines the precise form of the cohomology in all other cases.

The Spectral Action

- We will explain the proof in the stable case for simplicity. This uses the spectral action.
- The cocharacter μ defines a vector bundle C_μ on the stack of Langlands parameters $[\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}]$, whose evaluation at a $\overline{\mathbb{Q}}_\ell$-point corresponding to $\phi : W_{\mathbb{Q}_p} \to \hat{G}(\overline{\mathbb{Q}}_\ell)$ is $r_\mu \circ \phi$.
- The spectral action $C_\mu * j_!(\mathcal{F}_\rho)$ is precisely $T_\mu j_!(\mathcal{F}_\rho)$.
- If ρ has supercuspidal Gan-Tantono ($=$ Fargues-Scholze) parameter then $\phi = \phi_\rho^{FS}$ defines a connected component of $[\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}]$ which (up to unramified twists) is given by:

$$[\overline{\mathbb{Q}}_\ell/S_\phi]$$

and the action of C_μ factors over restriction to this connected component.

The Spectral Action

- Vector bundles on $[\overline{\mathbb{Q}}_\ell/S_\phi]$ are the same thing as $W \in Rep_{\overline{\mathbb{Q}}_\ell}(S_\phi)$. Given such W, the spectral action gives us functors:

$$Act_W : \bigoplus_{b \in B(G)_{basic}} D(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell) \to \bigoplus_{b \in B(G)_{basic}} D(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$$

- In particular, if we consider $r_\mu \circ \phi|_{S_\phi} = id_{GL_1} \boxtimes \phi$ as a rep of $S_\phi \times W_L$. We get:

$$R\Gamma_c(G, b, \mu)[\rho] \simeq C_V * j_!(\mathcal{F}_\rho)[-3](-3/2) =$$

$$Act_{id_{GL_1}}(\rho) \boxtimes \phi \otimes |\cdot|^{-3/2}[-3]$$

The Spectral Action

- Recent work of Hansen tells us that, since ϕ_{ρ}^{FS} is supercuspidal, $R\Gamma_c(G, b, \mu)[\rho]$ is concentrated in middle degree 3.
- Therefore, by Hansen-Kaletha-Weinstein, we deduce that $Act_{id_{GL_1}}(\rho) \simeq \pi$ and the result follows.
- The spectral action is monoidal so in particular

$$Act_{id_{GL_1}^{-1}}(\pi) = Act_{id_{GL_1}^{-1}} \circ Act_{id_{GL_1}}(\rho) =$$

$$Act_{\mathbf{1}}(\rho) = \rho$$

so we get the same result for the π-isotypic part!

The non-minuscule Case

- Using the monoidal property, we can show that $Act_{id^{n}_{GL_1}}(\rho)$ (resp. $Act_{id^{n}_{GL_1}}(\pi)$) is equal to π (resp. ρ) if n is odd and is ρ (resp. π) if n is even.
- This gives us a complete description of $R\Gamma_c(G, b, \mu)[\rho]$ (resp. $R\Gamma_c(G, b, \mu)[\pi]$) for μ any cocharacter, which in turn implies Fargues' conjecture for GSp_4.
- Using compatibility of Fargues-Scholze with isogenies of groups, we can deduce compatibility for Sp_4 and $SU_2(D)$, arguing again using the spectral action, we can verify the Kottwitz's conjecture in the Grothendieck group.

Future Work

- Alexander Bertoloni-Meli and Kieh-Hieu Nguyen recently verified Kottwitz's conjecture in the Grothendieck group for odd unitary similitude groups. In joint work in progress, we verify compatibility for these groups and obtain more precise descriptions of the cohomology even in the non-minuscule case!
- Recently, Koshikawa reproved Caraini-Scholze's vanishing results, using their semi-perversity results and compatibility for GL_n and its inner forms. While the relevant semi-perversity statement is not known currently, our result allows the remaining part of the argument to go through for GSp_4!

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