

RATIONAL WEIGHTED PROJECTIVE HYPERSURFACES

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ABSTRACT. A very general hypersurface of dimension n and degree d in complex projective space is rational if $d \leq 2$, but is expected to be irrational for all $n, d \geq 3$. Hypersurfaces in weighted projective space with degree small relative to the weights are likewise rational. In this paper, we introduce rationality constructions for weighted hypersurfaces of higher degree that provide many new rational examples over any field. We answer in the affirmative a question of T. Okada about the existence of very general terminal Fano rational weighted hypersurfaces in all dimensions $n \geq 6$.

1. INTRODUCTION

An irreducible algebraic variety X over the field k is (k) -rational if there is a k -birational map $X \dashrightarrow \mathbb{P}_k^{\dim(X)}$. Determining whether a given variety is rational is often extremely difficult. One classical problem is to determine which hypersurfaces in \mathbb{P}_k^{n+1} are rational.

If $X \subset \mathbb{P}_k^{n+1}$ is a hypersurface of degree 1, it is a hyperplane, so it is isomorphic to \mathbb{P}_k^n , hence rational. When $d = 2$, X is a quadric, which is rational if and only if it contains a smooth k -point. For $d \geq 3$, the problem is much subtler. For instance, when $k = \mathbb{C}$, it's expected that the very general hypersurface of degree $d \geq 3$ in $\mathbb{P}_{\mathbb{C}}^{n+1}$ is not rational for any n , with the exception of cubic surfaces. However, this is unproven already for cubic hypersurfaces in $\mathbb{P}_{\mathbb{C}}^5$.

Recently, the more general question of rationality for hypersurfaces in weighted projective space $\mathbb{P}(a_0, \dots, a_{n+1})$ has also been considered [3, 4, 10, 11, 12]. Weighted projective hypersurfaces are a diverse class of varieties that include hypersurfaces in \mathbb{P}^{n+1} as well as natural geometric constructions such as cyclic covers of projective space. In this paper, we find many new examples of rational weighted projective hypersurfaces over any field k by introducing two main rationality constructions.

The first construction, Theorem 3.2, generalizes a result of J. Kollár showing that certain hypersurfaces in weighted projective space with “loop” equations are rational, among other remarkable properties [8, Section 5]. The generalization pertains to any hypersurface defined by a *Delsarte equation*, i.e., an equation with the same number of monomials as variables. The idea of the proof is to construct a birational map to \mathbb{P}^n directly, using the linear system generated by the monomials. This works under a certain gcd condition on the exponents in the equation.

The second construction, Theorem 3.3, shows that certain weighted projective hypersurfaces admit a birational quadric bundle structure with a section over a rational variety, and hence are rational.

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Just like in ordinary projective space, weighted hypersurfaces of “low degree” are easily shown to be rational. More precisely, a quasismooth weighted projective hypersurface $X_d \subset \mathbb{P}_{\mathbb{C}}(a_0, \dots, a_{n+1})$ (with $a_0 \geq a_1 \geq \dots \geq a_{n+1}$) is always rational whenever the following criterion holds (see Proposition 3.1 for a more general statement):

$$(I) \quad d < 2a_0 \text{ or } d = 2a_0 = 2a_1.$$

We use the constructions above to show that, contrary to the expectation in ordinary projective space, there *are* many examples with $d > 2 \max\{a_0, \dots, a_{n+1}\}$ where X is very general, quasismooth, and rational. In particular, we answer a question of T. Okada that arose from his study of rationality for terminal Fano threefold hypersurfaces [12]:

Question 1.1. [12, Question 1.3] Let $X_d \subset \mathbb{P}_{\mathbb{C}}(a_0, \dots, a_{n+1})$ be a very general quasismooth weighted projective hypersurface of dimension $n \geq 3$ with terminal singularities. Can X be rational without satisfying the degree criterion (I)?

We note that Question 1.1 is formulated for $n \geq 3$ because of the example of cubic surfaces in $\mathbb{P}_{\mathbb{C}}^3$, which are rational but fail the criterion (I). In addition, there are several other families of singular quasismooth weighted projective surfaces which are known to be rational, some of which even have ample canonical divisor (see [2]).

The answer to Question 1.1 is negative when $n = 3$, meaning that the degree criterion is necessary and sufficient for rationality in that case. Indeed, there are a total of 130 families of quasismooth terminal weighted projective Fano threefold hypersurfaces, among which 20 satisfy (I), and hence are rational. Of the remaining 110 families, 95 have Fano index 1. I. Cheltsov and J. Park proved that every quasismooth member of each of these 95 families is birationally rigid, and in particular not rational [3]. Okada showed that a very general member of the other 15 families is not rational [12, Theorem 1.1]. In fact, with the lone exception of cubic hypersurfaces in $\mathbb{P}_{\mathbb{C}}^4$, he proved that a very general hypersurface from any of the 110 families failing (I) is not stably rational.

We show that the answer to Question 1.1 is actually affirmative in all dimensions $n \geq 6$ (and for all $n \geq 3$ if we weaken “terminal” to “klt”).

Theorem 1.2. *For every integer $n \geq 3$, there exist positive integers d, a_0, \dots, a_{n+1} such that $d > 2 \max\{a_0, \dots, a_{n+1}\}$ and every quasismooth $X_d \subset \mathbb{P}_{\mathbb{C}}(a_0, \dots, a_{n+1})$ is a rational klt Fano variety (and there exist such quasismooth X). For $n \geq 6$, we can also make X terminal.*

The terminal cases for $n = 4, 5$ remain open. Though the question was originally formulated over the complex numbers, our methods yield quasismooth rational examples with $d > 2 \max\{a_0, \dots, a_{n+1}\}$ in every dimension over any field k . We also find examples with non-trivial moduli. We note that there is another approach to answering Question 1.1, at least for $n \geq 7$, using a different rationality construction due to M. Artebani and M. Chitayat (see Remark 4.4). It’s interesting to note that the various constructions all seem to produce terminal examples beginning only in dimension 6 or 7.

Section 2 defines the necessary terminology related to weighted projective varieties. Section 3 establishes rationality criteria for weighted projective hypersurfaces. In Section 4, we find some special choices for d, a_0, \dots, a_{n+1} that produce examples proving

Theorem 1.2. We also give a method of checking whether loop hypersurfaces are canonical or terminal using only the exponents. Finally, we show that there are families with non-trivial moduli satisfying the conditions of Theorem 1.2.

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2. PRELIMINARIES

Over a base field k , we define the *weighted projective space* with weights a_0, \dots, a_{n+1} to be the projective variety $\mathbb{P} := \mathbb{P}_k^{n+1}(a_0, \dots, a_{n+1}) := \text{Proj}(k[x_0, \dots, x_{n+1}])$. Here the a_i are positive integers and the variable x_i has weight a_i . We sometimes use the abbreviation $a^{(r)}$ to indicate that the weight a appears r times. The grading of the polynomial ring above corresponds to a $\mathbb{G}_{m,k}$ -action on affine space $\mathbb{A}_k^{n+2} = \text{Spec}(k[x_0, \dots, x_{n+1}])$ given by the ring homomorphism

$$\begin{aligned} k[x_0, \dots, x_{n+1}] &\longrightarrow k[x_0, \dots, x_{n+1}] \otimes_k k[t, t^{-1}], \\ x_i &\longmapsto x_i \otimes t^{a_i}. \end{aligned}$$

Then $\mathbb{P}_k(a_0, \dots, a_{n+1})$ is also identified with the universal quotient $(\mathbb{A}_k^{n+2} \setminus \{0\})/\mathbb{G}_{m,k}$. There is a third description of \mathbb{P} as a toric variety: it is the toric variety over k associated to the fan consisting of cones generated by proper subsets of the vectors $e_0, \dots, e_n, v_{n+1} \in \mathbb{R}^{n+1}$ in the lattice N generated by these vectors, where

$$v_{n+1} := -\frac{a_0}{a_{n+1}}e_0 - \dots - \frac{a_n}{a_{n+1}}e_n.$$

We call a set $\{c_1, \dots, c_r\}$ of integers *well-formed* if $\gcd(c_1, \dots, \widehat{c}_i, \dots, c_r) = 1$ for each $i = 1, \dots, r$. A weighted projective space is *well-formed* if its set of weights is so. Any weighted projective space is isomorphic, as a variety, to one which is well-formed.

Via the Proj construction, \mathbb{P} is equipped with a reflexive sheaf $\mathcal{O}(1)$, which is associated to a Weil divisor on X . In the well-formed case, the canonical class of \mathbb{P} is $K_{\mathbb{P}} = \mathcal{O}(-\sum_i a_i)$.

A subvariety $X \subset \mathbb{P}$ is *quasismooth* if its preimage $C_X^* \subset \mathbb{A}_k^{n+2} \setminus \{0\}$ under the quotient morphism is smooth over k . The affine variety C_X^* is called the (*punctured*) *affine cone* over X . The subvariety X is *well-formed* if \mathbb{P} is well-formed and the intersection of X with the singular locus of \mathbb{P} has codimension at least 2 in X . A *weighted projective hypersurface* is a subvariety of \mathbb{P} defined by a single polynomial $f(x_0, \dots, x_{n+1})$ with k -coefficients that is weighted homogeneous of degree d . A well-formed quasismooth hypersurface of degree d satisfies the adjunction formula $K_X = \mathcal{O}_X(d - \sum_i a_i)$.

A few special types of polynomials f appear in the paper. A polynomial f is *Delsarte* if it has the same number of monomials as variables. One example is a *loop polynomial*, which has the form:

$$x_0^{b_0} x_1 + x_1^{b_1} x_2 + \dots + x_{n+1}^{b_{n+1}} x_0.$$

We say that this is a loop polynomial of *type* $[b_0, \dots, b_{n+1}]$.

Unless all the weights equal 1, a well-formed weighted projective space is always singular, but its singularities are of a special class called cyclic quotient singularities. A *cyclic quotient singularity of type* $\frac{1}{r}(c_1, \dots, c_s)$ is a singular point étale-locally isomorphic to the point $0 \in \mathbb{A}^s / \mu_r$, where the group μ_r acts by $\zeta \cdot (t_1, \dots, t_s) = (\zeta^{c_1} t_1, \dots, \zeta^{c_s} t_s)$ for any $\zeta \in \mu_r$. We say this singularity is *well-formed* if $\gcd(r, c_1, \dots, \widehat{c_j}, \dots, c_s) = 1$ for all $j = 1, \dots, s$.

For the rest of this section take $k = \mathbb{C}$. Then a quasismooth weighted projective hypersurface also has only cyclic quotient singularities, whose types are determined by the weights and degree of the hypersurface (see [6, Lemma 2.5, Proposition 2.6]). There is a combinatorial condition called the *Reid-Tai criterion* that can be used to determine whether a cyclic quotient singularity of a particular type belongs to certain classes important to the Minimal Model Program. In the theorem below, we use $[x]$ to denote $x - \lfloor x \rfloor$, the fractional part of x .

Theorem 2.1. [13, Theorem 4.11] *Let $\frac{1}{r}(c_1, \dots, c_s)$ be a well-formed cyclic quotient singularity. This singularity is canonical (resp. terminal) if and only if*

$$\sum_{j=1}^s \left[\frac{ic_j}{r} \right] \geq 1$$

(resp. > 1) for all $i = 1, \dots, r - 1$.

We also note that all quotient singularities (over \mathbb{C}) are klt.

3. RATIONALITY CRITERIA FOR WEIGHTED PROJECTIVE HYPERSURFACES

Since every weighted projective space is a toric variety, it is rational (over any field). Hypersurfaces for which the degree is small compared to the weights are also rational. The proposition below is a generalization of the degree criterion (I) of T. Okada to an arbitrary field k .

Proposition 3.1. *Suppose that $X_d \subset \mathbb{P}_k(a_0, \dots, a_{n+1})$ is a well-formed quasismooth hypersurface over any field k . Assume one of the following two conditions holds:*

- (1) $d < 2a_0$;
- (2) $d = 2a_0 = \dots = 2a_r$, for some $r \geq 1$, and X contains a point in $\{x_{r+1} = \dots = x_{n+1} = 0\}$ defined over k .

Then X is rational over k .

In particular, a quasismooth $X_d \subset \mathbb{P}_k(a_0, \dots, a_{n+1})$ is always rational over an algebraically closed field k when $d \leq 2 \max\{a_0, \dots, a_{n+1}\}$, unless $d = 2a_0$ and only one weight equals half the degree. There are many non-rational examples in the latter case, such as double covers $X_{2a} \subset \mathbb{P}(a, 1^{(n+1)})$ of \mathbb{P}^n branched in a divisor of degree $2a$ with $a \geq n + 1$. (Indeed, X is smooth and $K_X = \mathcal{O}_X(c)$ has global sections in this case since $c := 2a - a - n - 1 \geq 0$.)

Proof. Let f be the weighted homogeneous polynomial defining the hypersurface X . First suppose that (1) holds. If there is any monomial of the form x_i in f , then we may write

the equation $f = 0$ as $x_i = g(x_0, \dots, \widehat{x}_i, \dots, x_{n+1})$. In this case, X is called a *linear cone*, and the change of coordinates $x_i \mapsto x_i - g$, defined over k , exhibits an isomorphism from X to $\{x_i = 0\} \cong \mathbb{P}(a_0, \dots, \widehat{a}_i, \dots, a_{n+1})$, which is rational.

If there is no monomial of the form x_i in f and $d < 2a_0$ holds, then there must be a monomial of the form x_0x_i in f , or else the affine cone over X would contain $(1, 0, \dots, 0) \in \mathbb{A}_k^{n+2}$ and be singular at this point, contradicting quasismoothness. After an appropriate coordinate change in x_i , defined over k , we can assume that x_0x_i is the only monomial involving x_0 , so that $f = 0$ can be written

$$x_0x_i = g(x_1, \dots, x_{n+1}).$$

There is then a k -birational map $X \dashrightarrow \mathbb{P}(a_1, \dots, a_{n+1})$ defined by forgetting the coordinate x_0 . Hence X is rational.

Now assume (2) holds. Then f is a sum of a quadratic form in x_0, \dots, x_r and terms involving other variables. The closed stratum where x_{r+1}, \dots, x_{n+1} vanish is isomorphic to \mathbb{P}^r , and by assumption its intersection with X contains a k -point. After a change of variables in x_0, \dots, x_r , we may assume this k -point has $x_0 \neq 0$ but $x_1 = \dots = x_r = 0$.

Since X contains this point, there is no monomial of the form x_0^2 in f . The quasismoothness condition at this point therefore requires that there is a monomial of the form x_0x_i with nonzero coefficient in f . From here, we can use the same argument from the proof of (1) to show that X is rational. \square

The remainder of this section presents two new rationality criteria for weighted projective hypersurfaces. These in particular produce many quasismooth rational examples with $d > 2 \max\{a_0, \dots, a_{n+1}\}$.

3.1. Delsarte hypersurfaces. A *Delsarte* polynomial is a weighted homogeneous polynomial with the same number of monomials as variables. Given a Delsarte polynomial f defining a weighted projective hypersurface in $\mathbb{P}(a_0, \dots, a_{n+1})$, we can associate to it an $(n+2) \times (n+2)$ matrix $B = (b_{ij})$, where the entries b_{ij} are determined from the equation f as follows:

$$f = \sum_{i=0}^{n+1} K_i \prod_{j=0}^{n+1} x_j^{b_{ij}}.$$

Here the K_i are nonzero constants in k . If the matrix B is invertible (over \mathbb{Q}), we say that f is an *invertible* Delsarte polynomial. In this case, we can find appropriate weights a_0, \dots, a_{n+1} which make f weighted homogeneous as follows. Let q_j be the sum of the entries of the j th row of B^{-1} . Then the equation $BB^{-1} = I_{n+2}$ means that for each $i = 0, \dots, n+1$, $\sum_{j=0}^{n+1} b_{ij}q_j = 1$. Define d to be the least common denominator of the q_j and $a_j := dq_j$. We observe that d always divides $\det(B)$. Then f is weighted homogeneous of degree d with weights a_0, \dots, a_{n+1} , and $\gcd(a_0, \dots, a_{n+1}) = 1$. This collection of weights (with the gcd condition) is uniquely determined by f . However, it is not always true that this set of weights is well-formed.

The theorem below shows that a Delsarte polynomial satisfying a certain gcd condition is rational. This generalizes a result of J. Kollár [8, Section 5] which proves rationality for certain hypersurfaces defined by a loop polynomial.

Theorem 3.2. *Suppose $X := \{f = 0\} \subset \mathbb{P}_k(a_0, \dots, a_{n+1})$ is an irreducible hypersurface defined by an invertible Delsarte polynomial in a well-formed weighted projective space over any field k . Let B be the matrix associated to f and d its weighted degree. If $d = |\det(B)|$, then X is rational over k .*

Proof. Let \mathcal{L} be the linear system generated by the monomials of f , where

$$f := \sum_{i=0}^{n+1} K_i \prod_{j=0}^{n+1} x_j^{b_{ij}}.$$

We claim that this linear system induces a birational map $|\mathcal{L}| : \mathbb{P}(a_0, \dots, a_{n+1}) \dashrightarrow \mathbb{P}^{n+1}$ if and only if $|\det(B)| = d$. (Note that while $\mathcal{O}(d)$ is not necessarily a line bundle on \mathbb{P} , it is a line bundle over an open set, so a space of global sections still induces a rational map as shown.)

To see this, view both $\mathbb{P}(a_0, \dots, a_{n+1})$ and \mathbb{P}^{n+1} as quotients of $\mathbb{A}^{n+2} \setminus \{0\}$, and let $M \cong \mathbb{Z}^{n+2}$ and $M' \cong \mathbb{Z}^{n+2}$ be the lattices of the respective tori inside these affine spaces. The map $|\mathcal{L}|$ is induced by the map of rings $\mathbb{C}[M'] \rightarrow \mathbb{C}[M]$ sending $y_i \mapsto \prod_{j=0}^{n+1} x_j^{m_{ij}}$. We observe that the corresponding map of lattices is $M' \xrightarrow{B^T} M$. The map on dual lattices is therefore $N \xrightarrow{B} N'$, and this descends to a map of quotient lattices:

$$N/(\mathbb{Z} \cdot (a_0, \dots, a_{n+1})) \xrightarrow{B} N'/(\mathbb{Z} \cdot (1, \dots, 1)).$$

This lattice map is dual to the corresponding map of tori inside the weighted projective spaces induced by $|\mathcal{L}|$. It is indeed well-defined because $(a_0, \dots, a_{n+1}) \mapsto (d, \dots, d)$ and it's easy to see that the map is injective (we've used here that B is invertible). It is also surjective if the vectors $Be_0, \dots, Be_{n+1}, v := (1, \dots, 1)$ generate $N' \cong \mathbb{Z}^{n+2}$, or equivalently if $e_0, \dots, e_{n+1}, B^{-1}v$ generate $B^{-1}\mathbb{Z}^{n+2}$. But this in turn is the same as $B^{-1}v$ being a generator of $B^{-1}\mathbb{Z}^{n+2}/\mathbb{Z}^{n+2}$, which is a finite abelian group of order $|\det(B)|$. But $B^{-1}v$ is the vector $(a_0/d, \dots, a_{n+1}/d)$, so its order in this group is the least common denominator d . Therefore, the map of lattices is an isomorphism if and only if $d = |\det(B)|$.

Now suppose $d = |\det(B)|$ holds. The transform of X under $|\mathcal{L}| : \mathbb{P}(a_0, \dots, a_{n+1}) \dashrightarrow \mathbb{P}^{n+1}$ is the hyperplane $\{K_0 y_0 + \dots + K_{n+1} y_{n+1} = 0\} \subset \mathbb{P}^{n+1}$. The restriction to X gives a birational map $X \dashrightarrow \mathbb{P}^n$, completing the proof. \square

This construction gives many new examples of rational weighted projective hypersurfaces. Quasismooth examples are more limited, though: a quasismooth Delsarte hypersurface over \mathbb{C} has an equation which is a sum of three types of atoms: Fermat, loop, and chain (see [9, Theorem 1] and [1, Section 2.2]). The theorem of Kollár covers the case where the defining polynomial is a single loop, but Theorem 3.2 gives quasismooth examples where it is a combination of multiple loops, for instance. This paper focuses on finding terminal examples, but the theorem can also be used to find new rational hypersurfaces X with K_X ample.

3.2. Rational quadric bundles. It's well-known that a smooth cubic hypersurface in \mathbb{P}^{n+1} of even dimension $n = 2m$ containing two disjoint m -planes is rational. One way to see this is to project away from one of the planes, which gives X the birational structure of a quadric bundle over \mathbb{P}^m with a section given by the second m -plane. The following criterion can be viewed as a generalization of this fact. Thanks to J. Kollár for suggesting a simpler formulation of the theorem.

Theorem 3.3. *Let $X := \{f = 0\} \subset \mathbb{P}_k(a_0, \dots, a_{n+1})$ be an irreducible weighted projective hypersurface over a field k and $1 \leq m \leq n - 1$ be an integer with the following properties:*

- (1) $\gcd\{a_0, \dots, a_m\} = 1$ and the set $\{a_{m+1}, \dots, a_{n+1}\}$ is well-formed;
- (2) Every monomial of f has degree 1 or 2 in the variables x_0, \dots, x_m and at least one has degree 1 in these variables.

Then X is rational over k .

This statement also holds, suitably interpreted, in the “degenerate” cases $m = 0, n, n+1$. These lead to easy rationality constructions along the lines of Proposition 3.1, so we will not consider them here.

Proof of Theorem 3.3. The idea of the proof is to look at the projection $X \dashrightarrow \mathbb{P}_k(a_{m+1}, \dots, a_{n+1})$. We can understand this map better using some explicit toric geometry. The weighted projective space $\mathbb{P}(a_0, \dots, a_{n+1})$ is the toric variety corresponding to the fan which is generated by proper subsets of the collection of vectors $e_0, \dots, e_n, v_{n+1} \in \mathbb{R}^{n+1}$ in the lattice N generated by these vectors, where

$$v_{n+1} := -\frac{a_0}{a_{n+1}}e_0 - \dots - \frac{a_n}{a_{n+1}}e_n.$$

There is a dominant toric rational map $p : \mathbb{P}(a_0, \dots, a_{n+1}) \dashrightarrow \mathbb{P}(a_{m+1}, \dots, a_{n+1})$ defined by forgetting the first $m + 1$ coordinates. In the toric picture, this is given by the quotient $N \rightarrow N'$ of lattices, where $N' = N / (N \cap \text{span}_{\mathbb{R}}\{e_0, \dots, e_m\})$. We can resolve the indeterminacy of p by a single toric blowup Y of $\mathbb{P}(a_0, \dots, a_{n+1})$ in the stratum $\{x_{m+1} = \dots = x_{n+1} = 0\}$, in particular the one obtained by performing the barycentric subdivision of the fan in the new ray spanned by $w := -a_0e_0 - \dots - a_me_m$. Under the assumption (1), w is a primitive lattice point, and it's not hard to show that the quotient of lattices now gives rise to a morphism $p : Y \rightarrow \mathbb{P}(a_{m+1}, \dots, a_{n+1})$.

The fibers of the toric morphism $p : Y \rightarrow \mathbb{P}(a_{m+1}, \dots, a_{n+1})$ can be read off from the fan of Y (for more details on how to do this in general, see, e.g., [7, Section 2]). In our case, we only need to identify the behavior over the open torus orbit $T \subset \mathbb{P}(a_{m+1}, \dots, a_{n+1})$. The homomorphism of lattices associated to p is surjective by condition (1) and the collection of cones of the fan of Y contained in the preimage of $0 \in N'_{\mathbb{R}}$ are precisely those generated by subsets of e_0, \dots, e_m, w . It follows that each fiber over the open torus orbit is the toric variety corresponding to the fan of these cones. Since $w = -a_0e_0 - \dots - a_me_m$, this is the weighted projective space $\mathbb{P}(a_0, \dots, a_m, 1)$, well-formed by (1). In fact, $p|_{p^{-1}(T)}$ can be identified with the second projection $\mathbb{P}(a_0, \dots, a_m, 1) \times T \xrightarrow{\pi_2} T$.

Now we analyze how X behaves under this transformation. If D_i is the toric divisor $\{x_i = 0\}$ in $\mathbb{P}(a_0, \dots, a_{n+1})$, then D_i pulls back to D_i in Y if $i = 0, \dots, m$ and to $D_i + a_i E$ if $i = m+1, \dots, n+1$, where E is the exceptional divisor of the blowup. Therefore, the total transform of X in Y is given by the equation

$$f(x_0, \dots, x_m, z^{a_{m+1}} x_{m+1}, \dots, z^{a_{n+1}} x_{n+1}) = 0.$$

Here z is section associated to E and the left-hand side above can be viewed as a section of the pullback of $\mathcal{O}(d)$ to Y .

Notice that this new equation is weighted homogeneous of degree d in the variables x_0, \dots, x_m, z for the weighted projective space $\mathbb{P}(a_0, \dots, a_m, 1)$, where z corresponds to the weight 1. The intersection of the total transform of X with each fiber over T gives the hypersurface in $\mathbb{P}(a_0, \dots, a_m, 1)$ defined by this equation. The affine chart $\{z \neq 0\} \subset \mathbb{P}(a_0, \dots, a_m, 1)$ is isomorphic to \mathbb{A}_k^{m+1} . Restricting p to this open set on each fiber over T gives $\mathbb{A}_k^{m+1} \times T \rightarrow T$.

The intersection \tilde{X} of X with $\mathbb{A}_k^{m+1} \times T$ is defined in the fiber of $\mathbb{A}_k^{m+1} \times T \rightarrow T$ over any point $t = [c_{m+1} : \dots : c_{2m+1}]$ of the scheme $T \subset \mathbb{P}(a_{m+1}, \dots, a_{2m+1})$ (for instance the generic point) by the equation

$$\{f(x_0, \dots, x_m, c_{m+1}, \dots, c_{2m+1}) = 0\} \subset \mathbb{A}_{k(t)}^{m+1}.$$

By condition (2), this equation is either linear or quadratic in the variables x_0, \dots, x_m . If it is linear, then the generic fiber of $\tilde{X} \rightarrow T$ is isomorphic to $\mathbb{A}_{k(T)}^m$, so X is rational. If it is quadratic, the generic fiber is an affine quadric over $k(T)$. But since every monomial of f has degree 1 or 2 in x_0, \dots, x_m , the point $0 \in \mathbb{A}_{k(T)}^m$ is contained in this quadric, and since at least one monomial has degree 1 in these variables, it is smooth there.

But an affine quadric over a field L containing a smooth L -point is L -rational, so we've shown that the generic fiber of the affine quadric bundle $\tilde{X} \rightarrow T$ is rational over $k(T)$. The field $k(T)$ is purely transcendental over k , so it follows that \tilde{X} is rational over k , and hence X is rational over k as well. \square

4. TERMINAL EXAMPLES

In this section, we'll use the rationality constructions from Section 3 to prove Theorem 1.2. The main sequence of examples will come from Theorem 3.2. Delsarte equations are of a special form, so it's not necessarily true that a Delsarte hypersurface is very general in its family. Further, these examples are rarely terminal. Nevertheless, we'll find a special choice of weights in each dimension $n \geq 7$ where these conditions hold.

The proposition below gives a sufficient condition for a hypersurface defined by a loop polynomial (over the complex numbers) to be terminal, based on the Reid-Tai criterion [13, Theorem 4.11]. Recall that $[x]$ denotes the fractional part $x - \lfloor x \rfloor$ of x .

Proposition 4.1. *Let $X \subset \mathbb{P}_{\mathbb{C}}(a_0, \dots, a_{n+1})$ be a well-formed hypersurface defined by a loop polynomial of type $[b_0, \dots, b_{n+1}]$ with $b_0, \dots, b_{n+1} \geq 2$. Suppose that*

$$(II) \quad \sum_{j=2}^{n+1} [(1 - b_{j-1} + b_{j-1}b_{j-2} - \dots + (-1)^{j-1}b_{j-1}b_{j-2} \cdots b_1)x] \geq 1 \text{ (resp., } > 1)$$

for every $x \in (0, 1)$. Then X is canonical (resp. terminal) in a neighborhood of the point $[1 : 0 : \dots : 0]$. In particular, if this condition holds for each cyclic permutation of the b_j , then X is canonical (resp. terminal).

The advantage of Proposition 4.1 is that one often needs only compute the sum of the first few terms in (II). Also, the condition does not directly involve the weights.

Proof. A loop polynomial with exponents b_j at least 2 defines a quasismooth hypersurface over \mathbb{C} . Since X is quasismooth at the coordinate point $[1 : 0 : \dots : 0]$ of the first variable, it has a cyclic quotient singularity there. The singularity is of type $\frac{1}{a_0}(a_2, \dots, a_{n+1})$ [6, Proposition 2.6].

We use the fact that f is homogeneous of degree d to give expressions for the a_j , modulo a_0 . Since $b_0a_0 + a_1 = d$, $d \equiv a_1 \pmod{a_0}$. Next, $b_1a_1 + a_2 = d$, so

$$a_2 = d - b_1a_1 \equiv (1 - b_1)a_1 \pmod{a_0}.$$

Proceeding inductively we get

$$a_j = (1 - b_{j-1} + b_{j-1}b_{j-2} - \dots + (-1)^{j-1}b_{j-1}b_{j-2} \cdots b_1)a_1 \pmod{a_0}, j = 2, \dots, n+1.$$

Note that $\gcd(a_0, a_1) = 1$, or else a_0, a_1 , and d share a common factor and following the loop gives that all weights share a common factor, a contradiction. It follows that a_1 is a unit in μ_{a_0} , and multiplication by a unit does not alter the singularity type. Hence the singularity at the coordinate point x_0 is of type $\frac{1}{a_0}(\beta_2, \dots, \beta_{n+1})$, where

$$\beta_j := 1 - b_{j-1} + b_{j-1}b_{j-2} - \dots + (-1)^{j-1}b_{j-1}b_{j-2} \cdots b_1.$$

Now, the Reid-Tai criterion [13, Theorem 4.11] states that this singularity is canonical (resp. terminal) iff

$$\sum_{j=2}^{n+1} \left\lceil \frac{i\beta_j}{a_0} \right\rceil \geq 1 \text{ (resp., } > 1),$$

for each $i = 1, \dots, a_0 - 1$. This sum is now the same as (II), but with x replaced by i/a_0 . Hence the inequality (II) for every $x \in (0, 1)$ is certainly enough to imply the Reid-Tai criterion. Finally, applying [6, Proposition 2.6] again, we note that every singularity at a point of X occurs in some stratum of the quotient singularities at the coordinate points, so if the corresponding inequality to (II) holds for each coordinate point, then X itself is canonical (resp. terminal). \square

We can now prove Theorem 1.2.

Proof of Theorem 1.2. For each dimension $n \geq 3$, we choose d, a_0, \dots, a_{n+1} in such a way that there is a loop polynomial

$$f := x_0^2 x_1 + x_1^2 x_2 + \cdots + x_n^2 x_{n+1} + x_{n+1}^3 x_0,$$

which is weighted homogeneous of degree d . Indeed, we can readily compute the required weights and degree using the matrix of the equation as in Section 3.1. We obtain:

$$a_i = 2^{n+1} + \sum_{j=1}^i (-1)^{n+2-j} 2^{j-1}, i = 0, \dots, n+1,$$

$$d = 3 \cdot 2^{n+1} + (-1)^{n+2}.$$

The first two weights differ by 1 and $d = \det(B)$, so the gcd condition of Theorem 3.2 holds.

Lemma 4.2. *The only monomials of weighted degree d in a_0, \dots, a_{n+1} are those in the loop polynomial f .*

Proof. We first note that all weights but the last, a_{n+1} , are strictly between $d/4$ and $d/2$. Indeed, the largest weight is $a_n < 2^{n+1} + 2^{n-1}$ so $2a_n = 2 \cdot 2^{n+1} + 2^n < d$. Conversely, the smallest weight besides a_{n+1} is a_{n-1} , which satisfies $a_{n-1} > 2^{n+1} - 2^{n-2}$, so $4a_{n-1} = 4 \cdot 2^{n+1} - 2^n > d$.

This shows that any monomial of degree d not involving the last variable is the product of exactly three terms. If $x_{k_1} x_{k_2} x_{k_3}$ has degree d (with $0 \leq k_1 \leq k_2 \leq k_3 \leq n$), then

$$\sum_{j=1}^{k_1} (-1)^{n+2-j} 2^{j-1} + \sum_{j=1}^{k_2} (-1)^{n+2-j} 2^{j-1} + \sum_{j=1}^{k_3} (-1)^{n+2-j} 2^{j-1} = (-1)^{n+2}.$$

We claim that $k_3 \leq k_2 + 1$. Suppose by way of contradiction that this is not the case. As an alternating sum of powers of 2, we can see that the k_3 sum has absolute value greater than 2^{k_3-2} and less than 2^{k_3-1} (if $k_3 \geq 2$); the same holds for the other sums. Thus if $k_3 > k_2 + 1$, the last sum has an absolute value of more than 2^{k_3-2} , which is at least twice the upper bound 2^{k_2-1} for the absolute value of the other two terms. It's therefore impossible for the sum to have absolute value 1 as long as $k_2 \geq 2$. (This leaves out the case $k_2 \leq 1$, but it's easy to check the same is still true in that setting.)

Thus we've shown $k_2 = k_3$ or $k_2 + 1 = k_3$. Either way, $x_{k_1} x_{k_2} x_{k_3}$ must now be one of the loop monomials since all the weights are distinct and the latter two terms belong to some unique loop monomial. Finally, we consider monomials involving the last variable x_{n+1} . We note that $2a_n + a_{n+1} = d$ and a_n is the largest weight, so any monomial of degree d involving x_{n+1} that is not $x_n^2 x_{n+1}$ must have four terms. We can check

$$2a_{n+1} + 2a_{n-1} \geq 2 \cdot 2^{n+1} - 2^n + 2 \cdot 2^{n+1} - 2^{n-1} = 4 \cdot 2^{n+1} - 2^n - 2^{n-1} > d,$$

where we note that a_{n-1} is the second-smallest weight. It follows that a monomial of degree d with four terms must involve at least three copies of x_{n+1} , and hence must be the last monomial $x_{n+1}^3 x_0$. \square

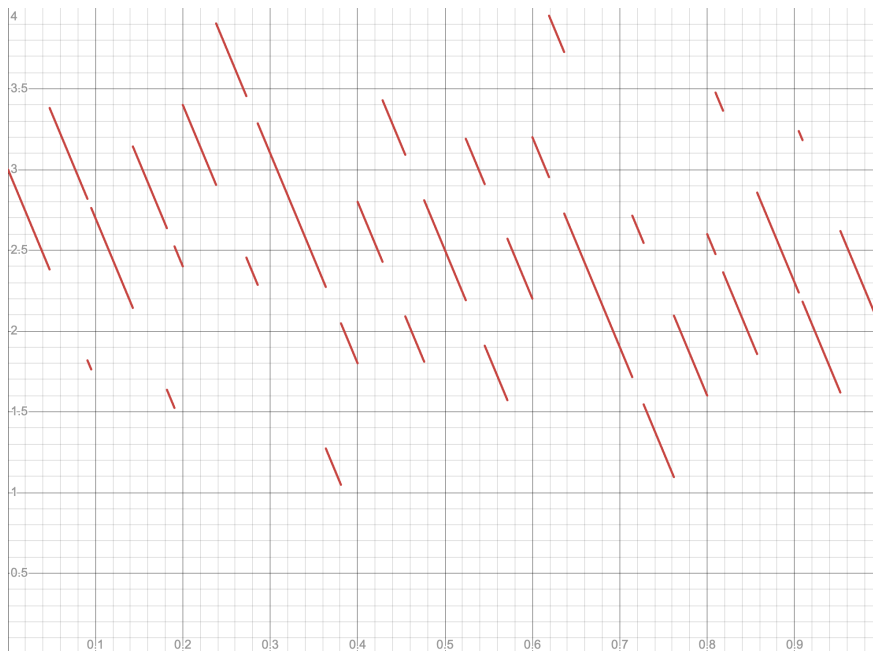


FIGURE 1. The sum of the first five terms in (II) for a sequence of exponents beginning $2, 2, 2, 2, 2, \dots$, namely the function $[-x] + [3x] + [-5x] + [11x] + [-21x]$. This function is greater than 1 on the interval $(0, 1)$, so the corresponding coordinate point is terminal.

Returning to the proof of Theorem 1.2, Lemma 4.2 shows that every hypersurface with degree d and weights a_0, \dots, a_{n+1} has the same equation f , up to varying the coefficients. The hypersurface is quasismooth if and only if all coefficients are nonzero. Theorem 3.2 shows that every quasismooth hypersurface in this family is rational. Every quasismooth member of the family is also a Fano variety by the adjunction formula, since the sum of the weights is greater than the degree.

We claim that this example also has terminal singularities whenever the dimension $n \geq 7$. This follows from Proposition 4.1 for $n \geq 8$. Indeed, it's actually true that the sum of the first eight terms from (II) is already greater than 1 on the interval $(0, 1)$ for every cyclic permutation of $[b_0, \dots, b_{n+1}] = [2, \dots, 2, 3]$, $n \geq 8$. Since the initial terms in the sum only depend on the first few of the values b_i , which are either all 2's or all 2's except a 3 in some position, we only need to graph a handful of different functions to verify this. (Or, more concretely, check a finite number of rational values in $(0, 1)$ for each of these functions.) As an example, the graph of the sum of the first five terms when b_1 through b_5 all equal 2 is shown in Figure 1. It is already greater than 1 at every point in the open unit interval, so we've actually shown that any coordinate point of a loop hypersurface where the "next five exponents" are 2 is terminal.

It turns out that the criterion (II) fails for our example when $n = 7$, but a computer check shows it is still terminal (using the usual version of Reid-Tai from Theorem 2.1). This gives an example where the criterion in Proposition 4.1 is not a necessary condition for X to be terminal.

This completes the proof of the theorem when $n \geq 7$. The loop example is not terminal when $n = 6$, but the theorem is proven for $n = 6$ in Example 4.3. \square

In the loop examples in the proof above, any two quasismooth members of the family are isomorphic to one another. In other words, the moduli space of quasismooth hypersurfaces is a single point.

However, we can use Theorem 3.3 to give new rational examples with non-trivial moduli answering Question 1.1. This construction also gives the only example I could find with $n = 6$:

Example 4.3. Consider the family of hypersurfaces $X_{23} \subset \mathbb{P}_{\mathbb{C}}(9^{(2)}, 8^{(2)}, 7^{(2)}, 5^{(2)})$ of dimension 6. In this example, the degree is more than twice the maximum of the weights, but we claim that every quasismooth member of the family is rational and terminal.

We first notice that every monomial of degree 23 with the given weights contains exactly one or two terms from the first four variables (the ones corresponding to weights 9 and 8), and there exist monomials of both of these kinds. In fact, in order for X to be quasismooth, it *must* contain both monomials that are linear and monomials that are quadratic in the first four variables. The general member of the family is indeed quasismooth. Hence the conditions of Theorem 3.3 are satisfied for any quasismooth X in the family (where we take $m = 3$). It follows that any such X is rational. A quick computation with the Reid-Tai criterion Theorem 2.1 also shows that this X is terminal. This completes the proof of Theorem 1.2 for $n = 6$.

This family has nontrivial moduli. Indeed, two quasismooth hypersurfaces in the family are isomorphic iff one is the image of the other under an automorphism of \mathbb{P} [5, Theorem 2.1]. There are 26 monomials of degree 23, so the projective space of weighted homogeneous polynomials of degree 23 is isomorphic to \mathbb{P}^{25} . On the other hand, $\text{Aut}(\mathbb{P})$ has dimension 15 (see, e.g., [5, Lemma 1.3]).

It's possible to find similar examples in all even dimensions $n \geq 6$.

Remark 4.4. Another recent rationality construction due to M. Artebani and M. Chitayat [2, Proposition 3.6] gives an alternative approach to Theorem 1.2 in the case $n \geq 7$. Indeed, they show that hypersurfaces of the form $X_{ac} \subset \mathbb{P}(c^{(k)}, a^{(\ell)})$, $\gcd(a, c) = 1, k, \ell \geq 1$ are always rational, provided that the equation f involves both variables of weight c and of weight a . (The very general X in particular has this property.) Geometrically, their rationality proof amounts to showing that the projection map $X \dashrightarrow \mathbb{P}^{k-1} \times \mathbb{P}^{\ell-1}$ is birational.

In order to find examples failing the degree criterion (I), we need $a, c > 2$, so the smallest possibilities are $c = 4, a = 3$. Using the Reid-Tai criterion, one can check that the example only has terminal singularities when $k > a$ and $\ell > c$. The smallest example that works for Theorem 1.2 is therefore $X_{12} \subset \mathbb{P}(4^{(4)}, 3^{(5)})$, which has dimension $n = 7$ (this is also the only example from this construction that works for $n = 7$). These examples have non-trivial moduli as well.

We were unable to find any terminal examples answering Question 1.1 affirmatively in dimensions 4 or 5, using any of the constructions mentioned above.

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