THE DUAL COMPLEX OF A G-VARIETY

LOUIS ESSER

ABSTRACT. We introduce a new invariant of G-varieties, the dual complex, which roughly measures how divisors in the complement of the free locus intersect. We show that the top homology group of this complex is an equivariant birational invariant of G-varieties. As an application, we demonstrate the non-linearizability of certain large abelian group actions on smooth hypersurfaces in projective space of any dimension and degree at least 3.

1. INTRODUCTION

Let G be a finite group which acts on a smooth projective variety X. A major problem in algebraic geometry is the classification of such G-varieties up to G-birational equivalence, where X and Y are *G-birationally equivalent* if there are invariant open sets $U \subset X$, $V \subset Y$ and an equivariant isomorphism $\varphi : U \xrightarrow{\cong} V$.

The geometry of G-varieties may be packaged into invariants that distinguish G-birational equivalence classes. This circle of ideas begins with the observation that the presence of an H-fixed point is a G-birational invariant of X for $H \subset G$ abelian, over an algebraically closed field of characteristic zero [15, App. A]. The recent introduction of *equivariant Burnside groups* [11] has led to many new advances in this area. The G-birational invariants which take values in these groups incorporate the birational types of fixed loci of all abelian subgroups $H \subset G$, actions of H on the normal spaces to these fixed loci, and residual actions on their orbits. They have found numerous applications in classifying G-varieties of low dimension, such as projective representations of finite groups [12, 18], algebraic tori [13], and actions on other rationally connected threefolds [5].

Another tool with significant applications in birational geometry is the *dual complex* of a simple normal crossings divisor. This is a simplicial (quasi-)complex which describes how the irreducible components of the divisor intersect. Such objects may be associated to (log resolutions of) singularities [6] and Calabi-Yau pairs [10] in such a way that the homotopy type of the complex is independent of the resolution chosen.

Existing G-birational invariants do not directly measure how components of the non-free locus of a G-variety intersect. Therefore, in this paper, we introduce a notion of dual complexes for G-varieties, which record intersection data for the part of the non-free locus that "looks most like the boundary of a toric variety". We show that the top homology group of this complex is a G-birational invariant (Theorem 2.3) and use this result to find new examples of non-birational actions. In particular, we prove that actions of rank n

2020 *Mathematics Subject Classification.* 14L30, 14E08 (primary); 14J70 (secondary).
Key words and phrases. equivariant birational geometry, hypersurfaces.

abelian subgroups $G \subset \mathrm{Aut}(\mathbb{P}^{n+1})$ on invariant smooth hypersurfaces $X \subset \mathbb{P}^{n+1}$ of degree $d \geq 3$ are *never* G-birational to a toric action on a smooth toric variety, except possibly in the case of cubic surfaces (Theorem 3.2).

1.1. **Notation.** G will always denote a finite group. We will only consider smooth and projective G-varieties X with faithful action. All varieties are over an algebraically closed field k of characteristic zero. For a subgroup $H \subset G$, X^H is the locus in X on which H acts trivially. We say that a G-variety has *abelian stabilizers* if for every closed point $x \in X$, $\mathrm{stab}_G(x) \coloneqq \{g \in G : g \cdot x = x\}$ is abelian. The *rank* of a finite abelian group H is the minimal number of generators of H.

Acknowledgements. Thank you to Nathan Chen, János Kollár, Burt Totaro, Ravi Vakil, and Ziquan Zhuang for useful conversations and comments. Thank you to Yuri Tschinkel for suggesting an alternative approach to Theorem 3.2.

2. Dual Complexes

Suppose X is a smooth projective G-variety with abelian stabilizers. By Luna's slice theorem [14, Lemme III.1], for any closed point $p \in X$, there is an invariant affine open neighborhood U of p and a $\mathrm{stab}_G(p)$-equivariant morphism $\varphi : U \to T_pX$ such that φ is étale at p and $\varphi(p) = 0$. In other words, the action is "étale-locally linear". Therefore, for any subgroup $H \subset \mathrm{stab}_G(p)$, $T_pX^H \cong (T_pX)^H$ and X^H is smooth at p (though globally X^H may be a union of connected components of different dimensions). It follows that the representation of $\mathrm{stab}_G(p)$ on T_pX is faithful. Thus, for any abelian group $H \subset G$, any connected component of X^H has codimension at least $\mathrm{rank}(H)$ in X.

The connected components Z of X^H for which $\mathrm{codim}(Z) = \mathrm{rank}(H)$ are particularly well-behaved, by the following lemma.

Lemma 2.1. *Let X be a G-variety with abelian stabilizers, $H \subset G$ an abelian subgroup, and Z a connected component of X^H such that $\mathrm{codim}(Z) = \mathrm{rank}(H) = k$. Then Z is a component of an intersection $D_1 \cap \cdots \cap D_k$ of divisors, where the general point $x \in D_i$ has nontrivial cyclic stabilizer H_i, and $H = \bigoplus_{i=1}^k H_i$. Moreover, these divisors have simple normal crossings in a neighborhood of Z.*

Proof. For any point $p \in Z$, we may diagonalize the action of H on T_pX to $T_pZ \oplus \chi_1 \oplus \cdots \oplus \chi_k$, where the χ_i are characters of H that do not depend on the choice of point $p \in Z$. The representation $\chi_1 \oplus \cdots \oplus \chi_k$ of H must be faithful. Since H has rank k, it follows that $H = \bigoplus_{i=1}^k H_i$, where

$$H_i \coloneqq \ker(\chi_1) \cap \cdots \cap \widehat{\ker(\chi_i)} \cap \cdots \cap \ker(\chi_k)$$

is a nontrivial cyclic group for each i. The subgroup H_i is the stabilizer of a hyperplane in T_pX, so this hyperplane is the tangent space to a divisor D_i in X^{H_i} passing through p. The divisors $D_1, \ldots, D_k$ have simple normal crossings at p since all of their intersections are smooth of the expected dimension near p [17, Lemma 0BIA]. $\square$

We refer to the components Z of X^H with $\operatorname{codim}(Z) = \operatorname{rank}(H)$ as the *maximal rank strata* of the G-variety X. Next, we construct a complex measuring the intersections of these special strata.

Definition 2.2. The *dual complex* $\mathcal{D}_G(X)$ of a smooth projective G-variety X of dimension n is a CW complex constructed as follows. The 0-skeleton of $\mathcal{D}_G(X)$ is a union of points indexed by divisorial components of X^H, as H ranges over all nontrivial cyclic abelian subgroups of G. At the kth step, attach a $(k-1)$-simplex for each connected component Z of X^H with $\operatorname{codim}(Z) = \operatorname{rank}(H)$, for $H \subset G$ abelian of rank k. By Lemma 2.1, Z is a connected component of the intersection $D_1 \cap \cdots \cap D_k$, where the divisors $D_1, \ldots, D_k$ each have nontrivial stabilizer and simple normal crossings along Z. The new simplex is attached along the simplices corresponding to $D_1, \ldots, D_k$ and the components of their intersections containing Z.

The result of this construction is a *simplicial quasicomplex* $\mathcal{D}_G(X)$ of dimension at most $n-1$. That is, it is a CW complex whose cells are simplices such that the collection of simplices is closed under the face relation and the intersection of any two is a union of some of their faces. We refer to [2, Section 7] for more about quasicomplexes and related terminology.

Theorem 2.3. *The homology group $H_{n-1}(\mathcal{D}_G(X))$ is a G-birational invariant of smooth projective varieties of dimension n with abelian stabilizers.*

Proof. By the G-equivariant weak factorization theorem [1], any birational map of smooth projective G-varieties may be decomposed into a sequence of blowups and blowdowns in smooth centers. If the varieties both have abelian stabilizers, we may guarantee that the intermediate varieties have abelian stabilizers as well (cf. [11, Proposition 3.6]). Therefore, it suffices to show that the homology group H_{n-1} of the dual complex is unchanged by a single blowup $X' := \mathrm{Bl}_W(X) \to X$ of a variety X of dimension n with abelian stabilizers in a G-stable center W, which is a disjoint union of smooth irreducible varieties. We may assume that G acts transitively on the connected components of W.

There are two cases to consider. First, suppose that the rank of the stabilizer of a general point in W equals $\operatorname{codim}(W)$. Then each component of W is a maximal rank stratum. We claim that in this case $\mathcal{D}_G(X')$ is the stellar subdivision of the complex $\mathcal{D}_G(X)$ in the simplices corresponding to the components of W. Note that distinct components of W are disjoint, so no simplex in $\mathcal{D}_G(X)$ contains multiple faces corresponding to components of W. It follows that we can perform all these stellar subdivisions simultaneously, and it's enough to consider the neighborhood of a single component.

To see that the effect of the blowup is a stellar subdivision, let Z be any connected component of W, and $D_1, \ldots, D_k$ the divisors from Lemma 2.1 containing Z. Each D_i is fixed by a nontrivial cyclic subgroup $H_i \subset G$ and the sum of these subgroups has rank k. Denote by E_Z the exceptional divisor over Z. Any maximal rank stratum contained in Z is a component of an intersection of the form $D_1 \cap \cdots \cap D_k \cap D'_1 \cap \cdots \cap D'_\ell$ for some additional divisors $D'_1, \ldots, D'_\ell$, where the stabilizer of a general point on each is a nontrivial cyclic group. Furthermore, these divisors have simple normal crossings in a neighborhood of the intersection. It's well known that the dual complex of the blow up of a stratum in an snc intersection is the stellar subdivision of the original complex (see, e.g., [2, Section 7]). It remains to see that all the strata of the blowup "count" for our notion of dual complex, namely that for each proper subset $I \subset \{1, \ldots, k\}$, $E_Z \cap D_I \cap D'_1 \cap \cdots \cap D'_\ell$ is a maximal rank stratum in X', where $D_I = \bigcap_{i \in I} D_i$.

Indeed, the stabilizer of a general point p in E_Z is the intersection of $H \subset \mathrm{GL}((N_{Z/X})_p)$ with the scalar transformations on this normal space, which must be a nontrivial cyclic group since $\mathrm{rank}(H) = \dim((N_{Z/X})_p)$. This group and any $|I| < k$ summands H_i of H generate a rank $|I|+1$ abelian subgroup. The intersection $E_Z \cap D_I \cap D'_1 \cap \cdots \cap D'_\ell$ is therefore fixed by a group of rank $|I|+1+\ell$, as desired. Since maximal rank strata are intersections of maximal rank divisors, no other new maximal rank strata are introduced by the blowup. Stellar subdivision induces a homeomorphism of complexes, so the homology of $\mathcal{D}_G(X')$ is the same as that of $\mathcal{D}_G(X)$ in this case.

The second case is that the stabilizer of a general point in W has rank smaller than $\mathrm{codim}(W)$. For this case, we require the following lemma:

Lemma 2.4. *Let $H = \mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_n$ be a rank n abelian group acting diagonally on $\mathbb{A}^n$. Then any H-stable smooth subvariety $Y \subset \mathbb{A}^n$ through $0 \in \mathbb{A}^n$ is a coordinate plane.*

Proof. Let $Y = V(I)$ for $I \subset k[x_1, \ldots, x_n]$ a prime ideal. Since I is H-invariant, we may choose a generating set $S := \{f_1, \ldots, f_t\}$ for I such that $g \cdot f_i = \chi_i(g) f_i$ for some character χ_i of H.

Since Y is smooth and passes through 0, its tangent cone is some linear subspace of $\mathbb{A}^n$. It is also H-stable, so we may assume the initial ideal of I (which cuts out the tangent cone) is $\mathrm{in}(I) = (x_1, \ldots, x_r)$, where $\dim(Y) = n - r$. Among the linear forms in $\mathrm{in}(I)$, only $x_1, \ldots, x_r$ are preserved up to scaling by H, so the generating set contains an element of the form $x_1 + h_1$, where h_1 only has terms of degree at least 2. Since x_1 and h_1 must have the same G-degree, it follows that x_1 divides h_1. Since I is prime, $x_1 \in I$. Similarly, $x_2, \ldots, x_r \in I$. For dimension reasons, $I = (x_1, \ldots, x_r)$, so Y is a coordinate plane. □

Returning to the proof of Theorem 2.3, we claim that the assumption on the stabilizers of points of W means that W does not contain a maximal rank stratum of dimension 0. Suppose by way of contradiction that W does contain such a point p with stabilizer H, with $\mathrm{rank}(H) = n$. There is a neighborhood U of p and an H-equivariant morphism $\varphi : U \to T_pX$ which is étale at p and sends p to 0, by Luna's slice theorem. Restricting to a smaller open $V \subset U$ containing p where φ is étale, we have that $\phi(V \cap W)$ is an H-stable closed subvariety of an open neighborhood of 0 in $T_pX \cong \mathbb{A}^n$ which passes through 0 and is smooth there. The group H is rank n and acts linearly on T_pX. After diagonalizing this action, we may apply Lemma 2.4 to conclude that $\varphi(V \cap W)$ coincides with a coordinate plane near 0 in T_pX. Every coordinate plane is the image of a maximal rank stratum, so W and some stratum Z coincide étale-locally near x. Hence W is a union of maximal rank strata, a contradiction.

Since W contains no point with stabilizer of rank n, it also does not contain any stratum containing such a point. Therefore, the subcomplex of $\mathcal{D}_G(X)$ consisting of the closures of all $(n-1)$-simplices remains the same in $\mathcal{D}_G(X')$, where each stratum is replaced by its strict transform, and a collection of strata intersects in X if and only if it does in X'. There could be new cells in $\mathcal{D}_G(X')$ corresponding to exceptional divisors over W and their intersections with existing strata. However, since every point of W has stabilizer of rank less than n, all new k-cells satisfy $k < n-1$ and don't appear in the boundary of an $(n-1)$-simplex. Therefore, the boundary map on cells of dimension $n-1$ remains unchanged, and the homology groups $H_{n-1}(\mathcal{D}_G(X))$ and $H_{n-1}(\mathcal{D}_G(X'))$ are isomorphic. □

The other homology groups of the dual complex are not G-birational invariants. As a simple example, suppose the action of $G = \mathbb{Z}/2$ on a smooth variety X of dimension at least 2 has an isolated fixed point p. Since the codimension of p is greater than 1, the point is not a maximal rank stratum and is not accounted for in the dual complex. However, blowing up p gives a new fixed divisor, which will increase the number of connected components of the dual complex by one.

3. Applications

In this section, we consider applications of the invariant $H_{n-1}(\mathcal{D}_G(X))$ defined in Section 2 to the classification of G-actions up to birational equivalence. First, note that this group always vanishes unless the dual complex contains $(n-1)$-cells, which can only occur if G contains an abelian group of rank at least $n = \dim(X)$. Toric varieties form one natural class with actions of this kind.

Example 3.1 (Smooth Toric Varieties)**.** Let X be a smooth projective toric variety of dimension n and $G \subset \mathbb{G}_{\mathrm{m}}^n$ a full rank finite subgroup of the torus. The non-free locus for the G-action is precisely the toric boundary, and each stratum of the boundary has maximal rank by the assumption that G has rank n. Since X is smooth and projective, its fan is the span of the faces of some simplicial polytope P in the lattice N of one-parameter subgroups [9, Theorem VII.3.11]. Therefore, the dual complex $\mathcal{D}_G(X)$ is simply P. In particular, $\mathcal{D}_G(X)$ is homeomorphic to the sphere S^{n-1}, so the top homology of $\mathcal{D}_G(X)$ is $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$.

A faithful G-action on a variety X is *linearizable* if there is a G-birational map $X \dashrightarrow \mathbb{P}(V)$, where the action on $\mathbb{P}(V)$ is via a faithful linear representation $G \to \mathrm{GL}(V)$. When G is abelian, we may diagonalize the G-action on V to show that G is a subgroup of the torus in $\mathbb{P}(V)$. If G also has rank $n = \dim(X)$, then Theorem 2.3 and Example 3.1 show that $H_{n-1}(\mathcal{D}_G(X)) \cong \mathbb{Z}$ whenever the action on X is linearizable.

Rank n abelian subgroups also act on many hypersurfaces in $\mathbb{P}^{n+1}$. We demonstrate that these actions are never linearizable when the degree of the hypersurface is at least 3, except for the case of cubic surfaces.

Theorem 3.2. *Let n and d be positive integers such that $n \geq 1$, $d \geq 3$, and $(n,d) \neq (2,3)$. Then for any rank n abelian subgroup $G \subset \mathrm{Aut}(\mathbb{P}^{n+1})$ and any G-invariant smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d, X is not G-birational to an action by a subgroup of the torus on a smooth projective toric variety. In particular, the action is not linearizable.*

Proof. Every rank n abelian group G contains an elementary abelian subgroup $(\mathbb{Z}/p)^{\oplus n}$, for some prime p. If X is not H-birational to a toric action for $H \subset G$, it certainly will not be G-birational to such an action, so we can and do assume $G \cong (\mathbb{Z}/p)^{\oplus n}$ from now on.

The group $G \subset \mathrm{PGL}(V)$ has some central extension $\tilde{G} \subset \mathrm{GL}(V)$, where $V \cong k^{n+2}$. We may assume that there is a G-fixed point p on X, or else $H_{n-1}(\mathcal{D}_G(X)) = 0$, while toric actions satisfy $H_{n-1} \cong \mathbb{Z}$. The line $\ell_p \subset k^{n+2}$ is a one-dimensional sub-$\tilde{G}$-representation of V. After twisting V by this representation (which doesn't change the image G), we can assume that $\tilde{G}$ acts trivially on ℓ_p. It follows that G lifts to $\mathrm{GL}_{n+2}(k)$, and we can diagonalize the action of G on V so that $V \cong \chi_{\mathrm{triv}} \oplus \chi_1 \oplus \cdots \oplus \chi_{n+1}$. In other words, $\mathbb{P}(V)$ is a compactification of the linear representation $\chi_1 \oplus \cdots \oplus \chi_{n+1}$ of G.

We claim that two of $\chi_{\mathrm{triv}}, \chi_1, \ldots, \chi_{n+1}$ must be the same, as long as $n \geq 3$. (The theorem automatically holds for curves of degree $d \geq 3$ and surfaces of degree $d \geq 4$ because these are all irrational, so we ignore these cases.) Indeed, if the characters are all distinct, the only G-fixed points on $\mathbb{P}^{n+1}$ are the coordinate points. We may assume X contains the coordinate point p of x_0 (which G acts on by the trivial character). Then $T_pX \subset T_p\mathbb{P}^{n+1}$ is a sub-G-representation. Since $T_p\mathbb{P}^{n+1}$ is a sum of distinct characters, T_pX coincides with the tangent space to a coordinate hyperplane, which we can assume is $\{x_1 = 0\}$ after relabeling. It also follows that $T_pX \cong \chi_2 \oplus \cdots \oplus \chi_{n+1}$ is faithful, so $\chi_2, \ldots, \chi_{n+1}$ form a basis for $G^* \cong \mathbb{F}_p^n$. By Lemma 2.1, p must be the intersection of n divisors on X which are each fixed by a nontrivial subgroup of G. These divisors are precisely $\{x_2 = 0\} \cap X, \ldots, \{x_{n+1} = 0\} \cap X$. Indeed, only coordinate strata in $\mathbb{P}^{n+1}$ are fixed by a nontrivial subgroup, and X cannot contain any codimension 2 coordinate plane of $\mathbb{P}^{n+1}$ when it is smooth of degree $d > 1$ and $n \geq 3$ [8, Corollary 6.26].

Since the hyperplanes $\{x_2 = 0\}, \ldots, \{x_{n+1} = 0\}$ all are fixed by a nontrivial subgroup, the collection $\chi_1, \ldots, \widehat{\chi_i}, \ldots, \chi_{n+1}$ does *not* form a basis of G^* for $i = 2, \ldots, n+1$. This implies χ_1 is trivial, contradicting the assumption that $\chi_{\mathrm{triv}}, \chi_1, \ldots, \chi_{n+1}$ are distinct.

We may therefore suppose that the G-action on $V \cong k^{n+2}$ has the following form: each summand of $(\mathbb{Z}/p)^{\oplus n}$ acts by multiplication by pth roots of unity on a distinct coordinate $x_2, \ldots, x_{n+1}$, and the action on the first two coordinates is trivial. Since X cannot contain a codimension 2 coordinate plane of $\mathbb{P}^{n+1}$, the only divisors fixed by nontrivial subgroups in X are the intersections $D_i := X \cap \{x_i = 0\}$, $i = 2, \ldots, n+1$, where D_i is fixed by the ith summand $\mathbb{Z}/p$ of G. Every intersection of $k < n$ of these divisors has a rank k stabilizer, so it is smooth and irreducible of dimension $n - k \geq 1$. Thus, the $(n-2)$-skeleton of $\mathcal{D}_G(X)$ is the same as the $(n-2)$-skeleton of the standard $(n-1)$-simplex. However, $D_2 \cap \cdots \cap D_{n+1}$ is a union of d distinct points (or else the divisors do not intersect transversely), so $\mathcal{D}_G(X)$ has d top-dimensional simplices with identical boundary. Therefore, $H_{n-1}(\mathcal{D}_G(X)) \cong \mathbb{Z}^{d-1}$. By Theorem 2.3 and Example 3.1, X is not G-birational to a smooth toric variety with action by a full rank finite subgroup of the torus whenever $d \geq 3$. This implies in particular that the G-action on X is not linearizable. □

Example 3.3 (Exceptional Cases)**.** Theorem 3.2 does not hold for the case of quadrics, where $d = 2$, and for cubic surfaces, where $(n, d) = (2, 3)$. For quadrics, any G-action with a G-fixed point p is linearizable, becuase the projection from p gives a G-equivariant birational map to projective space. Of course, an action by an abelian G without a fixed point is not linearizable, since linearizable actions have fixed points.

There is also a counterexample for smooth cubic surfaces, which are always rational. Let $X = \{x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0\} \subset \mathbb{P}^3$ be the Fermat cubic surface and $G \subset \mathrm{Aut}(X)$ the normal Klein four-subgroup of the S_4 permuting the four coordinates. Each involution in G fixes a line and three isolated points in X (these involutions are of type 2B in the classification of Dolgachev-Duncan [7]). No nontrivial element of G fixes a positive genus curve, so G is conjugate to a subgroup of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ or $\mathrm{Aut}(\mathbb{P}^2)$ in the plane Cremona group [3, Theorem 5]. Since G also fixes a point, the subgroup must be linearizable.

Remark 3.4. There is an alternate approach to the second part of the proof of Theorem 3.2 suggested by Yuri Tschinkel, using the formalism of *incompressible symbols* in equivariant Burnside groups. We sketch the argument below in the case of $G := (\mathbb{Z}/d)^{\oplus n}$ acting on the first n coordinates of a hypersurface X of dimension n and degree d.

The invariant in the equivariant Burnside group for this action contains a divisor symbol of the form $(H, G/H \circlearrowright k(Y), \beta)$. Here $Y := \{x_0 = 0\} \cap X$ is a smooth hypersurface of degree d in $\mathbb{P}^n$ with stabilizer $H := \mathbb{Z}/d$, the action $G/H = (\mathbb{Z}/d)^{\oplus(n-1)} \circlearrowright k(Y)$ is the residual action of G/H on the function field of Y, and β is a character of $\mathbb{Z}/d$ in the normal space to Y.

Such a divisor symbol is *incompressible* roughly if it cannot arise as the symbol for an exceptional divisor of a blowup of a lower dimensional subvariety (for a precise definition, see [12, Definition 3.3]). Assuming by induction the nonlinearizability of the residual action on $k(Y)$, one can prove the incompressibility of the corresponding divisor symbol. It follows that such a symbol cannot appear in the invariant associated to a linearizable action of G on $\mathbb{P}^n$.

For every pair n, d in the theorem, there are smooth hypersurfaces to which the result applies. For instance, if

$$X := \left\{ \sum_{i=0}^{n+1} a_i x_i^d = 0 \right\} \subset \mathbb{P}^{n+1},$$

then X is invariant under any subgroup of the action by multiplication by dth roots of unity on the coordinates. That group has rank $n+1$.

When the degree d is small compared to the dimension n, it is not known whether smooth hypersurfaces $X_d \subset \mathbb{P}^{n+1}$ are rational. Even a very general X is only known to be non-rational when the degree exceeds roughly $\log_2(n)$ [16]. In contrast, Theorem 3.2 shows that *no* action by a rank n abelian group on a hypersurface of dimension $n \geq 3$ and degree $d \geq 3$ is linearizable. This suggests the following question:

Question 3.5. For which finite groups $G \subset \mathrm{Aut}(\mathbb{P}^{n+1})$ and positive integers n, d is it true that the G-action on any G-invariant smooth hypersurface of degree d in $\mathbb{P}^{n+1}$ is not linearizable?

A positive answer for G trivial and some n, d is equivalent to the statement that no smooth hypersurface of degree d in $\mathbb{P}^{n+1}$ is rational.

Besides the groups in Theorem 3.2, some other interesting cases of Question 3.5 in low dimensions are known. For instance, if $G \cong \mathbb{Z}/3$ is the group acting by third roots of unity on the first coordinate x_0 in $\mathbb{P}^3$, then the induced action on any G-invariant smooth cubic surface X is not linearizable, even though X is rational. Indeed, a cyclic group of prime order in the Cremona group $\mathrm{Cr}(2)$ fixing a curve of positive genus is not even *stably* G-birational to a linear action on $\mathbb{P}^2$ [4, Corollary 1.2]. However, Question 3.5 appears to be open for low rank abelian groups acting on hypersurfaces of small degree and large dimension.

References

[1] D. Abramovich, K. Karu, K. Matsuki, J. Włodarczyk. Torification and factorization of birational maps. *J. Amer. Math. Soc.* **15** (2002), no. 3, 531–572. 3

[2] D. Arapura, P. Bakhtary, and J. Włodarczyk. Weights on cohomology, invariants of singularities, and dual complexes. *Math. Ann.* **357** (2013), no. 2, 513–550. 3, 4

[3] J. Blanc. Finite abelian subgroups of the Cremona group of the plane. PhD thesis, Université de Genève, 2006. Thèse no. 3777. `arXiv:0610368`. 7

[4] F. Bogomolov and Yu. Prokhorov. On stable conjugacy of finite subgroups of the plane Cremona group, I. *Cent. Eur. J. Math.*, **11** (2013), no. 12, 2099–2105. 8

[5] I. Cheltsov, Yu. Tschinkel, and Z. Zhang. Equivariant geometry of the Segre cubic and the Burkhardt quartic. `arxiv:2308.15271` (2023). 1

[6] T. de Fernex, J. Kollár, and C. Xu. The dual complex of singularities. In: *Higher dimensional algebraic geometry—in honour of Professor Yujiro Kawamata's sixtieth birthday*, vol. 74 of *Adv. Stud. Pure Math.*, 103–129. Math. Soc. Japan, Tokyo, 2017. 1

[7] I. Dolgachev and A. Duncan. Automorphisms of cubic surfaces in positive characteristic. *Izv. Ross. Akad. Nauk Ser. Mat.* **83** (2019), no. 3, 15–92. 7

[8] D. Eisenbud and J. Harris. *3264 and All That–A Second Course in Algebraic Geometry*. Cambridge University Press, Cambridge, 2016. 6

[9] G. Ewald. *Combinatorial Convexity and Algebraic Geometry*. Graduate Texts in Math. **168**, Springer, New York, 1996. 5

[10] J. Kollár and C. Xu. The dual complex of Calabi-Yau pairs. *Invent. Math.* **205** (2016), no. 3, 527–557. 1

[11] A. Kresch and Yu. Tschinkel. Equivariant birational types and Burnside volume. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **23** (2022), 1013–1052. 1, 3

[12] A. Kresch and Yu. Tschinkel. Equivariant Burnside groups and representation theory. *Selecta Math. (N.S.)* **28** (2022), Paper No. 81. 1, 7

[13] A. Kresch and Yu. Tschinkel. Equivariant Burnside groups and toric varieties. *Rend. Circ. Mat. Palermo (2)* **72** (2023), 3013–3039. 1

[14] D. Luna, Slices étales. *Bull. Soc. Math. France*, Paris, Mémoire **33** (1973), 81–105. 2

[15] Z. Reichstein and B. Youssin. Essential dimensions of algebraic groups and a resolution theorem for G-varieties. *Can. J. Math.* **52** (2000), no. 5, 1018–1056, with an appendix by J. Kollár and E. Szabó. 1

[16] S. Schreieder. Stably irrational hypersurfaces of small slopes. *J. Amer. Math.* Soc. **32** (4):1171–1199, 2019. 7

[17] The Stacks project authors. *The Stacks project*. `https://stacks.math.columbia.edu`, 2023. 2

[18] Yu. Tschinkel, K. Yang, and Z. Zhang. Equivariant birational geometry of linear actions. `arXiv:2302.02296` (2023). 1

Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544-1000, USA

Email address: `esserl@math.princeton.edu`