

NON-TORSION BRAUER GROUPS IN POSITIVE CHARACTERISTIC

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ABSTRACT. Unlike the classical Brauer group of a field, the Brauer-Grothendieck group of a singular scheme need not be torsion. We show that there exist integral normal projective surfaces over a large field of positive characteristic with non-torsion Brauer group. In contrast, we demonstrate that such examples cannot exist over the algebraic closure of a finite field.

1. INTRODUCTION

One way of extending the notion of the classical Brauer group of a field to any scheme X is by defining the Brauer-Grothendieck group $\mathrm{Br}(X) = H_{\text{ét}}^2(X, \mathbb{G}_m)$. Just as for fields, this group is torsion for any regular integral noetherian scheme [5, Corollaire 1.8]. However, this no longer holds for singular schemes. Chapter 8 of [3] lists several counterexamples. For instance, if R is the local ring of the vertex of a cone over a smooth curve of degree $d \geq 4$ in $\mathbb{P}_{\mathbb{C}}^2$, then $\mathrm{Br}(R)$ contains the additive group of \mathbb{C} . There are then affine Zariski open neighborhoods of the vertex with non-torsion Brauer group [3, Example 8.2.2]. However, the additive group of k is torsion when k has positive characteristic, so analogous constructions do not work there. There are also reducible varieties in arbitrary characteristic with non-torsion Brauer group [3, Section 8.1]. This leaves open the following question, communicated by Colliot-Thélène and Skorobogatov in an unpublished draft of [3]:

Question 1.1. If X is an integral normal quasi-projective variety over a field k of positive characteristic, is $\mathrm{Br}(X)$ a torsion group?

To analyze this question, we use a result concerning the Brauer group of a normal variety X with only isolated singularities p_1, \dots, p_n . Suppose X is defined over an algebraically closed field k of arbitrary characteristic. Let K be the function field of X , $R_i = \mathcal{O}_{X, p_i}$ be the local ring at each singularity, and R_i^{h} its henselization. Then, $\mathrm{Br}(X)$ is given by the exact sequence (see [3, Section 8.2], elaborating on [5, §1, Remarque 11 (b)])

$$(1) \quad 0 \rightarrow \mathrm{Pic}(X) \rightarrow \mathrm{Cl}(X) \rightarrow \bigoplus_{i=1}^n \mathrm{Cl}(R_i^{\mathrm{h}}) \rightarrow \mathrm{Br}(X) \rightarrow \mathrm{Br}(K).$$

This sequence indicates that one way for $\mathrm{Br}(X)$ to be large is for a singularity to have a large henselian local class group with divisors that do not extend globally to X . This

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idea is illustrated by a counterexample given by Burt Totaro, which shows that $\text{Br}(X)$ is non-torsion for X a hypersurface of degree $d \geq 3$ in \mathbb{P}_k^4 with a single node [3, Proposition 8.2.3]. Here k is any algebraically closed field with characteristic not 2.

We will show that counterexamples to Question 1.1 exist in dimension 2 if and only if k is not the algebraic closure of a finite field.

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2. A SURFACE COUNTEREXAMPLE

The following construction is taken from [7], Example 1.27. Take a smooth cubic curve D in the projective plane and a quartic curve Q that meets it transversally. Let $Y = \text{Bl}_{q_1, \dots, q_{12}} \mathbb{P}^2$, where q_1, \dots, q_{12} are the points of intersection. On Y , the proper transform C of D satisfies $C^2 = -3$. Unlike rational curves, not all negative self-intersection higher genus curves may be contracted to yield a projective surface. Rather, the contraction might only exist as an algebraic space. However, in this case, C is contractible.

Proposition 2.1. *There exists a normal projective surface X and a proper birational morphism $Y \rightarrow X$ whose exceptional locus is exactly C .*

Proof. The Picard group of Y is the free abelian group on H , the pullback of a general line in \mathbb{P}^2 , and the exceptional divisors E_1, \dots, E_{12} . Then, we claim that the line bundle $L := \mathcal{O}_Y(4H - \sum_i E_i) = \mathcal{O}_Y(H + C)$ defines a basepoint-free linear system on Y . Indeed, no point outside of the E_i can be a base point, and the proper transforms of both $D +$ a line and Q belong to the linear system. These don't intersect on the exceptional locus by the transversality assumption. Therefore, the system defines a morphism from Y to projective space with image X' . This morphism is birational because we have an injective map $H^0(Y, \mathcal{O}_Y(H)) \hookrightarrow H^0(Y, L)$ and $\mathcal{O}_Y(H)$ is the pullback of a very ample line bundle on \mathbb{P}^2 .

The exceptional locus of the morphism $Y' \rightarrow X'$ is precisely the union of the irreducible curves in Y on which L has degree zero. If C' an irreducible curve with $C' \cdot (H + C) = 0$, clearly C' is not supported on the E_i , or the intersection would be positive. Therefore, $H \cdot C' > 0$, meaning $C \cdot C' < 0$. This means $C' = C$, so the exceptional locus is the curve C , which is mapped to a point. Thus, X' is a surface birational to Y and $Y \rightarrow X'$ is an isomorphism away from C . Finally, passing to the normalization X of X' , we may assume X is a normal projective surface; the normalization will also be an isomorphism away from C , and the image of C will again be a point in X . \square

The resulting singularity p of X has minimal resolution with exceptional set exactly C , a smooth elliptic curve. Singularities satisfying this condition are *simple elliptic singularities*. Over \mathbb{C} , such singularities are completely classified. A simple elliptic singularity with $C^2 = -3$ is known as an \tilde{E}_6 singularity, and is complex analytically isomorphic to a cone over C [8]. Here, we present a computation of the henselian local class group $\text{Cl}(R^h)$ of the singularity that works in any characteristic.

Consider the pullback of the desingularization $Y \rightarrow X$ to a “henselian neighborhood”:

$$\begin{array}{ccc} Y^{\text{h}} & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \text{Spec}(R^{\text{h}}) & \longrightarrow & X. \end{array}$$

The scheme Y^{h} is regular and $Y^{\text{h}} \setminus C \cong \text{Spec}(R^{\text{h}}) \setminus \{\mathfrak{m}\}$, so we have an exact sequence $0 \rightarrow \mathbb{Z} \cdot [C] \rightarrow \text{Pic}(Y^{\text{h}}) \rightarrow \text{Cl}(R^{\text{h}}) \rightarrow 0$. Here, the first map is injective since $\mathcal{O}_{Y^{\text{h}}}(C)$ has nonzero degree on C . It suffices, therefore, to compute $\text{Pic}(Y^{\text{h}})$. To do so, we’ll first consider infinitesimal neighborhoods of C in Y .

The sequence of infinitesimal neighborhoods $C = C_1 \subset C_2 \subset \dots$ is defined by powers of the ideal sheaf \mathcal{I}_C . Notably, these C_n are the same regardless of whether we consider them inside Y or inside the henselian neighborhood Y^{h} . The normal bundle to C in Y gives obstructions to extending line bundles to successive neighborhoods, but we’ll show that all line bundles extend uniquely. The group $\varprojlim_n \text{Pic}(C_n)$ in the proposition below is also the Picard group of the formal neighborhood of C in Y .

Proposition 2.2. *The restriction map $\varprojlim_n \text{Pic}(C_n) \rightarrow \text{Pic}(C)$ is an isomorphism.*

Proof. It’s enough to show that the maps $\text{Pic}(C_{n+1}) \rightarrow \text{Pic}(C_n)$ are all isomorphisms for $n \geq 1$. Each extension $C_n \subset C_{n+1}$ is a first-order thickening, since C_n is defined in C_{n+1} by the square-zero ideal sheaf $\mathcal{I}_C^n/\mathcal{I}_C^{n+1}$. Associated to such a thickening is a long exact sequence in cohomology [9, 0C6Q]

$$\dots \rightarrow H^1(C, \mathcal{I}_C^n/\mathcal{I}_C^{n+1}) \rightarrow \text{Pic}(C_{n+1}) \rightarrow \text{Pic}(C_n) \rightarrow H^2(C, \mathcal{I}_C^n/\mathcal{I}_C^{n+1}) \rightarrow \dots$$

We may take the outer cohomology groups to be over C since the underlying topological spaces are the same. As sheaves of abelian groups on C , we have $\mathcal{I}_C^n/\mathcal{I}_C^{n+1} \cong \mathcal{O}_C(-nC)$, a multiple of the conormal bundle. But $C^2 = -3$ in Y so this last bundle has degree $3n > 0$. Since C is genus 1, the higher cohomology of $\mathcal{O}_C(-nC)$ vanishes and $\text{Pic}(C_{n+1}) \rightarrow \text{Pic}(C_n)$ is an isomorphism for all n . \square

Now, we need only compare $\varprojlim_n \text{Pic}(C_n)$ to $\text{Pic}(Y^{\text{h}})$. Using the Artin approximation theorem [1, Theorem 3.5], the map $\text{Pic}(Y^{\text{h}}) \rightarrow \varprojlim_n \text{Pic}(C_n)$ is injective with dense image. However, the topology of the latter group is discrete in this setting because each group of the inverse limit is $\text{Pic}(C)$. Therefore, the map is surjective also and $\text{Pic}(Y^{\text{h}}) \cong \text{Pic}(C)$.

Theorem 2.3. *Let k be an algebraically closed field that is not the algebraic closure of a finite field and X be the surface defined in the proof of Proposition 2.1. Then, $\text{Br}(X)$ is non-torsion.*

Proof. From the above, we have the identification $\text{Cl}(R^{\text{h}}) \cong \text{Pic}(Y^{\text{h}})/\mathbb{Z} \cdot \mathcal{O}_{Y^{\text{h}}}(C) \cong \text{Pic}(C)/\mathbb{Z} \cdot \mathcal{O}_C(C)$. Since $\deg_C \mathcal{O}_C(C) = 3$, the class group is then an extension of $\mathbb{Z}/3$ by $\text{Pic}^0(C) \cong C(k)$, where $C(k)$ is the group of k -rational points of the elliptic curve C with a chosen identity point. Since $k \neq \overline{\mathbb{F}}_p$, $C(k)$ has infinite rank [4, Theorem 10.1]. Note that

in contrast, $C(k)$ is torsion for an elliptic curve C over $\overline{\mathbb{F}}_p$, because every point is defined over \mathbb{F}_{p^m} for some m .

However, the global class group of X is quite small: $\text{Cl}(Y) = \text{Pic}(Y) \cong \mathbb{Z}^{13}$ since Y is the blow up of \mathbb{P}^2 in 12 points and $\text{Cl}(X) \cong \text{Cl}(Y)/\mathbb{Z} \cdot [C]$. Therefore, the cokernel of the restriction map $\text{Cl}(X) \rightarrow \text{Cl}(R^h)$ in (1) contains non-torsion elements, so $\text{Br}(X)$ does too. \square

To complement the above result, we also prove:

Theorem 2.4. *Suppose that X is an integral normal surface over the algebraic closure $k = \overline{\mathbb{F}}_p$ of a finite field. Then $\text{Br}(X)$ is torsion.*

Proof. The strategy is similar to the previous theorem. Here, the crucial fact is that all possible “building blocks” of the henselian local class group - abelian varieties over k , the additive group of k , and the multiplicative group of k - are torsion.

Since the singularities of a normal surface are isolated, we may apply the exact sequence (1). The group $\text{Br}(K)$ is always torsion, so if we can prove $\bigoplus_{i=1}^n \text{Cl}(R_i^h)$ is as well, the result follows. Let $Y \rightarrow X$ be a desingularization. We focus on the base change $\pi : Y^h \rightarrow \text{Spec}(R^h)$ to the henselian local ring at just one singular point p . Let $E = \pi^{-1}(p)$ be the scheme-theoretic fiber. We may choose Y such that E_{red} is a union of irreducible curves F_j which are smooth and meet pairwise transversely, with no three containing a common point.

The following argument is due to Artin [2]. Suppose G is the free abelian group of divisors supported on E , and consider the map $\alpha : \text{Pic}(Y^h) \rightarrow \text{Hom}(G, \mathbb{Z})$ given by $L \mapsto (D \mapsto D \cdot L)$. This restricts to a map $\alpha|_G : G \rightarrow \text{Hom}(G, \mathbb{Z})$ that is injective because the intersection matrix of the curves F_j is negative definite. Since G and $\text{Hom}(G, \mathbb{Z})$ are free abelian groups of equal rank, $G \rightarrow \text{Hom}(G, \mathbb{Z})$ also has finite cokernel. This allows us to find an effective Cartier divisor $Z = \sum_j r_j F_j$ with all $r_j > 0$ such that $\alpha(Z) = \alpha(-H)$ for an ample line bundle H on Y^h [2, p. 491]. If we restrict this Cartier divisor to the scheme associated to Z , the resulting line bundle $\mathcal{O}_Z(-Z)$ has positive degree on every irreducible component of Z , so it is ample. We’ll examine infinitesimal neighborhoods of the closed subscheme Z in Y^h .

As before, for every $n \geq 1$, there is an exact sequence

$$\cdots \rightarrow H^1(Z, \mathcal{O}_Z(-nZ)) \rightarrow \text{Pic}(Z_{n+1}) \rightarrow \text{Pic}(Z_n) \rightarrow H^2(Z, \mathcal{O}_Z(-nZ)) \rightarrow \cdots$$

Because $\dim(Z) = 1$, the last group is always zero. Since $\mathcal{O}_Z(-Z)$ is ample, the first group is zero for $n \gg 0$ by Serre vanishing, which holds on any projective scheme [6, Theorem II.5.2]. Therefore, the inverse limit $\varprojlim_n \text{Pic}(Z_n)$ is constructed as a finite series of extensions of $\text{Pic}(Z)$ by finite-dimensional k -vector spaces. Applying Artin approximation, we have that $\text{Pic}(Y^h) \rightarrow \varprojlim_n \text{Pic}(E_n)$ is injective with dense image. However, for large n , the scheme E_n is nested between two infinitesimal neighborhoods of Z , where all restrictions of Picard groups are surjective (use a similar exact sequence to the above,

e.g. [9, 09NY]). It follows that $\varprojlim_n \text{Pic}(E_n) \cong \varprojlim_n \text{Pic}(Z_n)$ and that both have the discrete topology, so $\text{Pic}(Y^h) \cong \varprojlim_n \text{Pic}(Z_n)$.

Next, let \bar{Z} be the disjoint union of the schemes $r_j F_j$, where $r_j F_j$ is the subscheme of Y^h cut out by the ideal sheaf of F_j to the power r_j . Then $f : \bar{Z} \rightarrow Z$ is a finite map that is an isomorphism away from the finite set of intersection points and such that $\mathcal{O}_Z \subset f_* \mathcal{O}_{\bar{Z}}$. It follows (see [9, 0C1M, 0C1N]) that $\text{Pic}(Z)$ is a finite sequence of extensions of $\text{Pic}(\bar{Z})$ by quotients of $(k, +)$ or $(k, *)$. Lastly, $\text{Pic}(\bar{Z}) = \bigoplus_j \text{Pic}(r_j F_j)$, where each summand is built from finite-dimensional k -vector spaces and $\text{Pic}(F_j) \cong \mathbb{Z} \oplus \text{Pic}^0(F_j)$. Since the $\text{Pic}^0(F_j)$ are groups of k -points of abelian varieties over k , they are all torsion.

Taken together, all of this implies that G and $\text{Pic}(Y^h)$ have equal rank. Since $G \rightarrow \text{Hom}(G, \mathbb{Z})$ is injective, the first map in the excision sequence of class groups $0 \rightarrow G \rightarrow \text{Pic}(Y^h) \rightarrow \text{Cl}(R^h) \rightarrow 0$ is injective also. Therefore, the quotient $\text{Cl}(R^h)$ is a torsion group, as desired. \square

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