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CHAPTER 3

Semi Log Canonical Pairs

We have seen in Section **??** that in order to compactify the moduli theory of higher dimensional varieties we need stable pairs (**??**). That is, pairs (X, Δ) with semi log canonical singularities and ample log canonical class $K_X + \Delta$. The aim of this Chapter is to study these stable pairs, especially their singularities.

In general X is neither normal nor irreducible. Such varieties can be studied either using semi log resolutions or by focusing on their normalization. Both of these approaches have difficulties.

A stable curve C has ordinary nodes, and we can encode C by giving a triple $(\bar{C}, \bar{D}, \tau)$ where $\bar{C}$ is the normalization of C, $\bar{D} \subset \bar{C}$ is the preimage of the nodes and $\tau : \bar{D} \to \bar{D}$ is an involution which tells us which point pairs of $\bar{D}$ are identified in C.

Correspondingly, a higher dimensional stable variety has ordinary self-intersection in codimension 1, and we will encode (X, Δ) by a quadruple

$$(\bar{X}, \bar{D}, \bar{\Delta}, \tau)$$

where $\pi : \bar{X} \to X$ is the normalization, $\bar{D}$ the preimage of the double normal crossing locus of X, $\bar{\Delta}$ the preimage of Δ and the involution τ tells us which point pairs in $\bar{D}$ are identified in X. (Since X can have rather more complicated self-intersections in higher codimensions, τ is an actual involution only on the normalization of $\bar{D}$.)

It is easy to see that $(\bar{X}, \bar{D}, \bar{\Delta}, \tau)$ uniquely determines (X, Δ). Our principal aim is to understand which quadruples come from an slc pair (X, Δ).

Section 1 gives the precise definitions and works out the complete theory for surfaces.

Section 2 studies the existence question for X. The main result (23) says that an easy to state finiteness condition is necessary and sufficient for the existence of X if $(\bar{X}, \bar{D} + \bar{\Delta})$ is lc. As a corolary of our methods, we also prove that if (X, Δ) is slc then X is Du Bois (44).

Even if (X, Δ) exists, $K_X + \Delta$ may not be $\mathbb{Q}$-Cartier. It is easy to see that if $K_X + \Delta$ is $\mathbb{Q}$-Cartier then the Poincaré residue map $\omega_{\bar{X}}(\bar{D}) \to \omega_{\bar{D}}$ has to be compatible with the involution τ (14). If $(\bar{X}, \bar{D} + \bar{\Delta})$ is lc, we show in Section 3 that the converse also holds (54). The key ingredient is a new definition of the Poincaré residue map at log canonical centers of codimension ≥ 2.

Then we turn to other ways of studying semi log canonical pairs. Section 4 contains various resolution theorems that are useful for non-normal schemes and Section 5 investigates finite ramified covers.

Section 6 gives an example of a surface with normal crossing singularities whose canonical ring is *not* finitely generated. Thus the minimal model program fails for semi log canonical varieties. This makes it very hard to use the subtler techniques of Chapter 2. For moduli theory it would be especially useful to establish the non-normal analog of the existence of dlt models (**??**?).

1. Semi-log-canonical singuarities

In this section we define semi-log-canonical pairs (X, Δ) and prove some of their basic properties. If X is normal, then semi-log-canonical is equivalent to log-canonical, hence we concentrate entirely on the case when X is not normal. We first study the normalization $(\bar{X}, \bar{\Delta})$ of (X, Δ). The key difficulty is then to reconstruct (X, Δ) from $(\bar{X}, \bar{\Delta})$ and to show that various good properties of $(\bar{X}, \bar{\Delta})$ descend to (X, Δ). Most of these will be accomplished only in subsequent sections.

Demi-normal schemes.

DEFINITION 1. Recall that, by Serre's criterion, a scheme X is normal iff it is S_2 and regular at all codimension 1 points. As a weakening of normality, it is natural to consider schemes that are S_2 and whose codimension 1 points are either regular or ordinary nodes (**??**). Such schemes will be called *demi-normal*. The initial "d" is supposed to remind us that they have double normal crossings in codimension 1. (I really would like to call these schemes "semi-normal," but that name is already taken.)

The *demi normalization* of a scheme is usually not defined. (What should the demi normalization of $(x^n = y^n) \subset \mathbb{A}^2$ be for $n \geq 3$?) However, if $j : U \hookrightarrow X$ is an open subscheme with only regular points and ordinary nodes such that $X \setminus U$ has codimension ≥ 2, then $\operatorname{Spec}_X j_* \mathcal{O}_U$ is the smallest demi normal scheme dominating X. It is called the demi normalization of X.

Roughly speaking, the concept of semi-log-canonical is obtained by replacing "normal" with "demi-normal" in the definition of log canonical (**??**), but some basic definitions and foundational results need to be in place first.

2 (Normalization of demi-normal schemes). Let X be a demi-normal scheme and $\pi : \bar{X} \to X$ its normalization. The *conductor ideal*

$$\operatorname{cond}_X := \operatorname{Hom}_X\big(\pi_* \mathcal{O}_{\bar{X}}, \mathcal{O}_X\big) \subset \mathcal{O}_X \tag{2.1}$$

is the largest ideal sheaf on X that is also an ideal sheaf on $\bar{X}$. We write it as $\operatorname{cond}_{\bar{X}}$ when we view cond_X as an ideal sheaf on $\bar{X}$. The *conductor subschemes* are defined as

$$D := \operatorname{Spec}_X\big(\mathcal{O}_X / \operatorname{cond}_X\big) \quad \text{and} \quad \bar{D} := \operatorname{Spec}_{\bar{X}}\big(\mathcal{O}_{\bar{X}} / \operatorname{cond}_{\bar{X}}\big). \tag{2.2}$$

Since X is S_2, $D \subset X$ and $\bar{D} \subset \bar{X}$ are both of pure codimension 1. Since X has only nodes at its codimension 1 points, D and $\bar{D}$ are generically reduced. Thus D and $\bar{D}$ are both reduced divisors.

Let $x_i \in D$ be a generic point. Then $\mathcal{O}_{X,x_i}$ is an ordinary node, thus, if $\operatorname{char} k(x_i) \neq 2$, $\pi : \bar{D} \to D$ is an étale double cover in a neighborhood of x_i.

In general, $\bar{D} \to D$ is not everywhere étale and not even flat, but the map between the normalizations $\pi^n : \bar{D}^n \to D^n$ has degree 2 over every irreducible component. Thus it defines a Galois involution $\tau : \bar{D}^n \to \bar{D}^n$.

Note that in general, τ does not define an involution of $\bar{D}$, not even set-theoretically. As a simple example, consider $X := (xyz = 0) \subset \mathbb{A}^3$. Here $D \subset \mathbb{A}^3$ is the 3 coordinate axes with a triple point at the origin. $\bar{X}$ is the disjoint union of 3 planes each containing a pair of intersecting lines $\bar{D}_i$ and $\bar{D}$ is their disjoint union.

The origin $0 \in D$ has 3 preimages in $\bar{D}$ and they would all have to be in the same τ-orbit.

Proposition 3. *Let X be demi-normal. The triple $(\bar{X}, \bar{D}^n, \tau)$ defined in (2) uniquely determines X.*

Proof. Note that $\pi : \bar{X} \to X$ is a finite surjection and $\pi \circ n : \bar{D}^n \to X$ is τ-ivariant. Assume that $\pi' : \bar{X} \to X'$ is another finite surjection such that $\pi' \circ n : \bar{D}^n \to X$ is τ-ivariant. We prove that there is a unique $g : X \to X'$ such that $\pi' = g \circ \pi$; giving a characterization of X.

Let $X^* \subset X \times X'$ be the image of (π, π'). Let $x \in X$ be a codimension 1 point. Then either X is smooth at x, hence $\bar{X} \to X^* \to X$ are isomorphisms near x, or X has a node at x with preimage $\bar{x} \in \bar{X}$. By assumption $\bar{x} \to X^*$ factors through $x \to X^*$, hence again $X^* \to X$ is an isomorphisms near x. Since X is S_2, this implies that the first projection $X^* \to X$ is an isomorphism. Thus the second projection $X^* \to X'$ gives the required g. □

Note that in the general framework of Section 2, the proof of (3) is equivalent to saying that the relation $(n, n \circ \tau) : \bar{D}^n \to \bar{X} \times \bar{X}$ generates a set-theoretic equivalence relation $R \rightrightarrows X$ and X is the geometric quotient of $\bar{X}$ by R.

4 (Main problems). In proving that the moduli problem of stable varieties satisfies the valuative criterion of properness, we need to construct degenerations. That is, given a flat family of stable varieties $\{X_c : c \in C^0\}$ over an open curve $C^0 \subset C$, we would like to extend the family across the points $p \in C \setminus C^0$ (at least after a finite base change). In Section **??**, our method is to first construct essentially $\left(\bar{X}_p, \bar{D}_p, \tau_p\right)$ and then recover from it X_p. This, however, turns out to be quite difficult, and we have to deal with 2 main problems.

Question 5. Let $\tilde{X}$ be a normal variety, $\tilde{D} \subset \tilde{X}$ a reduced divisor with normalization $\tilde{n} : \tilde{D}^n \to \tilde{D}$ and $\tilde{\tau} : \tilde{D}^n \to \tilde{D}^n$ an involution. Under what conditions does there exist a demi-normal variety X with normalization $\left(\bar{X}, \bar{D}, \tau\right)$ as in (2) such that $\left(\tilde{X}, \tilde{D}, \tilde{\tau}\right) = \left(\bar{X}, \bar{D}, \tau\right)$?

Question 6. Assume that X is demi-normal and $K_{\bar{X}} + \bar{D}$ is $\mathbb{Q}$-Cartier. Under what conditions is K_X also $\mathbb{Q}$-Cartier?

In order to answer (5), first we may aim to describe the closed fibers of the putative $\pi : \tilde{X} \to X$. Since $\tilde{D}^n \to X$ is $\tilde{\tau}$-invariant, we see that for any closed point $q \in \tilde{D}^n$, the points $\tilde{n}(q) \in \tilde{X}$ and $\tilde{n}(\tilde{\tau}(q)) \in \tilde{X}$ must be in the same fiber of π. The relation $\tilde{n}(q) \sim \tilde{n}(\tilde{\tau}(q))$ generates an equivalence relation on the closed points of $\tilde{X}$. A necessary condition for the existence of X is that this equivalence relation be finite, that is, it should have finite equivalence classes.

Even assuming finiteness, the first question seems rather intractable in general, as shown by the examples in [**Kol08**, Sec.2]. Thus we consider the case when $\left(\tilde{X}, \tilde{D}\right)$ is assumed lc. The main result of Section 2 gives a positive answer to (5) when $\tilde{\tau}$ is compatible with the lc structure in a weak sense (23).

The second question (6) may, at first, seem puzzling in view of the formula $\pi^* K_X \sim_{\mathbb{Q}} K_{\bar{X}} + \bar{D}$ (8.5). However, as the examples (15) show, in general K_X need not be $\mathbb{Q}$-Cartier, not even if $K_{\bar{X}} + \bar{D}$ is Cartier. We show in (14) that a necessary condition is that the different (13) be τ-invariant. We prove in Section 3 that the converse holds if $\left(\tilde{X}, \tilde{D}\right)$ is lc (54), but not in general (16). An explicit study of the surface case is in (17).

7 (Divisors and divisorial sheaves on demi-normal schemes). Let X be demi-normal. $\mathbb{Z}$-divisors whose support does not contain any irreducible component of the conductor $D_X \subset X$ (2.2) form a subgroup

$$\mathrm{Weil}^*(X) \subset \mathrm{Weil}(X). \tag{7.1}$$

A rank 1 reflexive sheaf which is locally free at the generic points of D_X is called a *divisorial sheaf* on X. Divisorial sheaves form a subgroup

$$\mathrm{Cl}^*(X) \subset \mathrm{Cl}(X). \tag{7.2}$$

As usual, the product of two divisorial sheaves L_1, L_2 is given by

$$L_1 \hat{\otimes} L_2 := \bigl(L_1 \otimes L_2\bigr)^{**},$$

the double dual or reflexive hull of the usual tensor product. For powers we use the notation $L^{[m]} := \bigl(L^{\otimes m}\bigr)^{**}$.

Let B be a $\mathbb{Z}$-divisor whose support does not contain any irreducible component of the conductor. Then there is a closed subset $Z \subset X$ of codimension ≥ 2 such that $X^0 := X \setminus Z$ has only smoth and double nc points and $B^0 := B|_{X^0}$ has smooth support. Thus B^0 is a Cartier divisor on X^0 and $\mathcal{O}_{X^0}(B^0)$ is an invertible sheaf. Let $j : X^0 \hookrightarrow X$ denote the natural injection and set

$$\mathcal{O}_X(B) := j_*\mathcal{O}_{X^0}(B^0). \tag{7.3}$$

This establishes a surjective homomorphism $\mathrm{Weil}^*(X) \to \mathrm{Cl}^*(X)$.

Similarly, K_{X^0} is a Cartier divisor on X^0 and $\omega_{X^0} \cong \mathcal{O}_{X^0}(K_{X^0})$ an invertible sheaves. For every $m \in \mathbb{Z}$, we get the rank 1 reflexive sheaves

$$\omega_X := j_*\omega_{X^0} \quad \text{and} \quad \omega_X^{[m]}(B) := j_*\bigl(\omega_{X^0}^m(B^0)\bigr). \tag{7.4}$$

Thus it makes sense to talk about K_X or B being Cartier or $\mathbb{Q}$-Cartier, even if B is a $\mathbb{Q}$-divisor. (Even on a nodal curve C one has to be rather careful about viewing a node $p \in C$ as a Weil divisor such that $2[p]$ is Cartier. Fortunately, in the slc case, we only need to deal with divisors in $\mathrm{Weil}^*(X)$.)

Let $\pi : \bar{X} \to X$ be the normalization. For any B in $\mathrm{Weil}^*(X)$, let $\bar{B}$ denote the divisorial part of $\pi^{-1}(B)$, as a divisor on $\bar{X}$. This establishes a one-to-one correspondence between $\mathbb{Z}$-divisors (resp. $\mathbb{Q}$-divisors) on X whose support does not contain any irreducible component of the conductor $D_X \subset X$ and $\mathbb{Z}$-divisors (resp. $\mathbb{Q}$-divisors) on $\bar{X}$ whose support does not contain any irreducible component of $\bar{D}_X \subset \bar{X}$.

8. Let Y be a scheme with only double nc points and $\pi : \bar{Y} \to Y$ its normalization. Then $\bar{Y}$ and the conductors $D \subset Y$ and $\bar{D} \subset \bar{Y}$ are smooth. The natural map $\pi : \bar{D} \to D$ is an étale double cover with Galois involution τ. From

$$\pi_*\omega_{\bar{Y}} = \mathrm{Hom}_Y\bigl(\pi_*\mathcal{O}_{\bar{Y}}, \omega_Y\bigr)$$

we conclude that $\pi_*\omega_{\bar{Y}} = \omega_Y(-D)$. (Note that D is not a Cartier divisor on X.) Since the conductor ideals $\mathcal{O}_Y(-D) = \mathcal{O}_{\bar{Y}}\bigl(-\bar{D}\bigr)$ agree, the latter is equivalent to $\omega_{\bar{Y}} = \pi^*\omega_Y\bigl(-\bar{D}\bigr)$. Since $\bar{D}$ is a Cartier divisor, we can take it to the other side to obtain the natural isomorphism

$$\pi^*\omega_Y = \omega_{\bar{Y}}\bigl(\bar{D}\bigr). \tag{8.1}$$

If X is an arbitrary demi-normal scheme, we can apply the above consideration to an open subset $X^0 \subset X$ such that $X \setminus X^0$ has codimension ≥ 2. By pushing forward from X^0 (resp. $\bar{X}^0$) to X (resp. $\bar{X}$) we obtain that

$$\pi_* \omega_{\bar{X}} = \omega_X(-D) \quad \text{and} \quad \left(\pi^* \omega_X\right)^{**} = \omega_{\bar{X}}\left(\bar{D}\right), \tag{8.2}$$

where the double dual is necessary in general since the pull back of an S_2 sheaf need not be S_2. Similarly, for any $\mathbb{Z}$-divisor B and integer m we obtain a natural isomorphism

$$\left(\pi^* \omega_X^{[m]}(B)\right)^{**} \cong \omega_{\bar{X}}^{[m]}\left(m\bar{D} + \bar{B}\right). \tag{8.3}$$

If Δ is a $\mathbb{Q}$-divisor, $m\Delta$ is integral and $m(K_X + \Delta)$ is Cartier, this simplifies to

$$\pi^*\left(\omega_X^{[m]}(m\Delta)\right) \cong \omega_{\bar{X}}^{[m]}\left(m\bar{D} + m\bar{\Delta}\right), \tag{8.4}$$

which we frequently abbreviate as

$$\pi^*\left(K_X + \Delta\right) \sim_{\mathbb{Q}} K_{\bar{X}} + \bar{D} + \bar{\Delta}. \tag{8.5}$$

It is a little more interesting to study which sections of $\omega_{\bar{X}}^{[m]}\left(m\bar{D} + m\bar{\Delta}\right)$ descend to a section of $\omega_X^{[m]}(m\Delta)$. The only question is at the generic points of D, hence we can work on X^0 and ignore Δ.

We give an answer in terms of the Poincaré residue map (**??**)

$$\mathcal{R} : \omega_{\bar{X}^0}(\bar{D}^0) \to \left(\omega_{\bar{X}^0}(\bar{D}^0)\right)|_{\bar{D}^0} = \omega_{\bar{D}^0}.$$

By taking tensor powers, we get

$$\mathcal{R}^{\otimes m} : \left(\omega_{\bar{X}^0}(\bar{D}^0)\right)^{\otimes m} \to \omega_{\bar{D}^0}^m.$$

As a local model, we can take $X := (xy = 0) \subset \mathbb{A}^2$. A generator of ω_X is given by $\sigma := \mathcal{R}\left((xy)^{-1} d(xy)\right)$. Note that

$$\mathcal{R}\Big(\frac{d(xy)}{xy}\Big)\Big|_{(y=0)} = \frac{dx}{x} \quad \text{and} \quad \mathcal{R}\Big(\frac{d(xy)}{xy}\Big)\Big|_{(x=0)} = -\frac{dy}{y}.$$

The two residues differ by a minus sign, thus we obtain the following:

Proposition 9. *A section ϕ of $\omega_{\bar{X}}^{[m]}\left(m\bar{D} + m\bar{\Delta}\right)$ descends to a section of $\omega_X^{[m]}(m\Delta)$ iff $\mathcal{R}^{\otimes m}(\phi)$ is τ-invariant if m is even and τ-anti-invariant if m is odd.* □

Remark 10. While it is not necessary, it is instructive to compute the dualizing sheaf of the curve singularity C_n given by the n coordinate axes in $\mathbb{A}_k^n$. Its normalization $\bar{C}_n$ is the disjoint union of n lines. Let $\bar{P} \subset \bar{C}_n$ be the preimage of the origin of C_n; it is n points, each with multiplicity 1. By taking a generic projection, we see that there is an exact sequence

$$0 \to \omega_{C_n} \to \omega_{\bar{C}_n}\left(\bar{P}\right) \xrightarrow{\sum \mathcal{R}} k \to 0$$

where the map $\sum \mathcal{R}$ sends a 1-form η to the sum of its residues at the points $\bar{P}$.

From this we see that, for $n \geq 3$, the sheaves ω_{C_n} are not locally free.

11 (Semi-resolutions). Let X be a demi-normal scheme over a field of characteristic 0 and $X^0 \subset X$ an open subset that has only smooth points ($x_1 = 0$), double nc points ($x_1^2 - ux_2^2 = 0$) and pinch points ($x_1^2 = x_2^2 x_3$) such that $X \setminus X^0$ has codimension ≥ 2. We show in (87) that there is a projective birational morphism $f : X' \to X$ such that

(1) X' has only smooth points, double nc points and pinch points,
(2) f is an isomorphism over X^0, and
(3) $\operatorname{Sing} X'$ maps birationally onto $\operatorname{Sing} X$.

We call any such $f : X' \to X$ a *semi-resolution* of X.

Moreover, given (X, Δ), we can choose $f : X' \to X$ such that

(4) the normalization of X' is a log resolution (**??**?) of $(\bar{X}, \bar{D} + \bar{\Delta})$.

We call any such $f : X' \to X$ a *log semi-resolution* of (X, Δ). See (90) for details.

Note that by (3), X' is smooth at the generic point of any f-exceptional divisor.

Semi-log-canonical.

Definition–Lemma 12. Let X be a demi-normal scheme over a field of characteristic 0 and Δ an effective $\mathbb{Q}$-divisor whose support does not contain any irreducible component of the conductor $D \subset X$ (2.2).

The pair (X, Δ) is called *semi-log-canonical* or *slc* if

(1) $K_X + \Delta$ is $\mathbb{Q}$-Cartier, and
(2) one of the following equivalent conditions holds
  (a) $\bigl(\bar{X}, \bar{D} + \bar{\Delta}\bigr)$ is lc where $\bar{D} \subset \bar{X}$ is the conductor (2.2) on $\bar{X}$ and $\bar{\Delta}$ is the divisorial part of $\pi^{-1}(\Delta)$, or
  (b) $a(E, X, \Delta) \geq -1$ for every exceptional divisor E for every semi-resolution of X (11).

Note that (2.b) is the exact analog of the definition of log canonical given in (**??**).

In order to see that the conditions (2.a) and (2.b) are equivalent, let $f : Y \to X$ be any semi-resolution and $\bar{Y} \to Y$ the normalization. Then we have a commutative diagram

$$\begin{array}{ccc} \bar{Y} & \stackrel{\pi_Y}{\longrightarrow} & Y \\ \bar{f} \downarrow & & \downarrow f \\ \bar{X} & \stackrel{\pi}{\longrightarrow} & X \end{array}$$

and $\pi^*\bigl(K_X + \Delta\bigr) \sim_{\mathbb{Q}} K_{\bar{X}} + \bar{D} + \bar{\Delta}$ by (8.5).

Since Y is smooth at the generic points of $\operatorname{Ex}(f)$, we see that π_Y is an isomorphism over the generic points of $\operatorname{Ex}(f)$. Thus

$$a(E, X, \Delta) = a(E, \bar{X}, \bar{D} + \bar{\Delta}) \tag{12.3}$$

for every exceptional divisor E. Thus (2.a) $\Rightarrow$ (2.b) and, using (**??**), the converse also follows from (11.4); see also (90). $\square$

The discrepancy $a(E, X, \Delta)$ is not defined if $K_X + \Delta$ is not $\mathbb{Q}$-Cartier, thus (12.2.b) does not make sense unless (12.1) holds. By contrast, (12.2.a) makes sense if $K_{\bar{X}} + \bar{D} + \bar{\Delta}$ is $\mathbb{Q}$-Cartier, even if $K_X + \Delta$ is not. The point of Question (6) is to understand the difference between these two. The answer is given in terms of the different (**??**), which we recall next.

13. Let $(Y, D + \Delta)$ be a pair where Y is normal, D a reduced divisor and Δ a $\mathbb{Q}$-divisor whose support does not contain any irreducible component of D. Let $\sigma : D^n \to D$ be the normalization. Assume that $m\Delta$ is an integral divisor and $m(K_Y + D + \Delta)$ is a Cartier divisor. By (**??**) there is a unique $\mathbb{Q}$-divisor $\operatorname{Diff}_{D^n} \Delta$ on D^n such that

(1) $m \cdot \operatorname{Diff}_{D^n} \Delta$ is integral and $m(K_{D^n} + \operatorname{Diff}_{D^n} \Delta)$ is Cartier, and

(2) the mth tensor power of the Poincaré residue map (**??**) extends to a natural isomorphism

$$\sigma^*\big(\omega_Y^{[m]}(mD+m\Delta)\big) \cong \omega_{D^n}^{[m]}(m\cdot \operatorname{Diff}_{D^n}\Delta).$$

Note that the Poincaré residue isomorphism is defined over the snc locus of $(Y, D+\Delta)$ and the different is then chosen as the unique $\mathbb{Q}$-divisor for which the extension is an isomorphism.

Let us now apply the above to $(\bar X, \bar D, \bar\Delta)$ obtained as the normalization of a pair (X, Δ). Using (8.5), for m sufficiently divisible, we have isomorphisms

$$\sigma^*\pi^*\big(\omega_X^{[m]}(m\Delta)\big) \cong \sigma^*\big(\omega_{\bar X}^{[m]}(m\bar D+m\bar\Delta)\big) \cong \omega_{\bar D^n}^{[m]}(m\cdot \operatorname{Diff}_{\bar D^n}\bar\Delta). \tag{13.3}$$

Note that the composite $\bar D^n \to \bar X \to X$ is τ-invariant. hence the composite isomorphism in (13.3) is also τ-invariant. As noted above, the isomorphism

$$\sigma^*\pi^*\big(\omega_X^{[m]}(m\Delta)\big) \cong \omega_{\bar D^n}^{[m]}(m\cdot \operatorname{Diff}_{\bar D^n}\bar\Delta).$$

uniquely determines the different $\operatorname{Diff}_{\bar D^n}\bar\Delta$. Thus we have proved the following:

PROPOSITION 14. *Let X be demi-normal and Δ a $\mathbb{Q}$-divisor whose support does not contain any irreducible component of the conductor $D\subset X$. Let $(\bar X, \bar D, \bar\Delta)$ and $\tau : \bar D \to \bar D$ be as in (2) and (7). If $K_X+\Delta$ is $\mathbb{Q}$-Cartier then* $\operatorname{Diff}_{\bar D^n}\bar\Delta$ *is τ-invariant.*

Note that the τ-invariance of $\operatorname{Diff}_{\bar D^n}\bar\Delta$ depends only on the codimension 2 points of X and we prove in (17) that the converse of (14) holds outside a codimension ≥ 3 subset of X. Thus the key question is whether there are futher conditions at higher codimension points or not. We settle this if $(\bar X, \bar D, \bar\Delta)$ is dlt (19), but the general case, treated in Section 3, seems more subtle. As the examples (16) show, there are further conditions if $(\bar X, \bar D, \bar\Delta)$ is not lc.

Semi-log-canonical surfaces.

Let us start with a series of examples of non-slc surfaces which seem quite close to being slc.

EXAMPLE 15. In $\mathbb{A}^4$ consider the surface S that consists of 3 planes, $P_{xy} := (z=t=0), P_{yz} := (x=t=0), P_{zt} := (x=y=0)$. Its normalization is the disjoint union $\bar S = P_{xy} \amalg P_{yz} \amalg P_{zt}$ and, correspondingly, the conductor D has 3 pieces $L_1 := (x=0) \subset P_{xy}$, $L_1'+L_2' := (yz=0) \subset P_{yz}$ and $L_2 := (t=0) \subset P_{zt}$. Its normalization $\bar D^n$ is the disjoint union of the 4 lines L_i, L_i'. Thus $(\bar S, \bar D)$ is dlt and both $K_{\bar S}$ and $\bar D$ are Cartier.

We see that the origin appears with coefficient 0 in the different on L_1 and L_2 but with coefficient 1 on L_1' and L_2'. The involution τ interchanges L_1 with L_1' and L_2 with L_2'. Thus $\operatorname{Diff}_{\bar D} 0$ is not τ-invariant, hence ω_S is not Cartier and not even $\mathbb{Q}$-Cartier.

Note that S is a cone over a curve $C\subset \mathbb{P}^3$ which is a chain of 3 lines, ω_C has degree -1 on the two ends and 0 on the middle line. Thus ω_C is not $\mathbb{Q}$-linearly equivalent to a rational multiple of the hyperplane class and (**??**) also implies that ω_S is not $\mathbb{Q}$-Cartier.

The next example shows that in the non-lc case there is no numerical condition that decides whether a demi-normal surface has $\mathbb{Q}$-Cartier canonical class or not.

EXAMPLE 16. We describe a flat family of demi-normal surfaces parametrized by $\mathbb{C}^* \times \mathbb{C}^*$ such that the canonical class of the fibers is $\mathbb{Q}$-Cartier for a Zariski dense set of pairs $(\lambda, \mu) \in \mathbb{C}^* \times \mathbb{C}^*$ and not $\mathbb{Q}$-Cartier for another Zariski dense set of pairs.

Start with a cone S over a hyperelliptic curve and two rulings $C_x, C_y \subset S$. Take two copies of S and glue them together by the isomorphisms $C_x^1 \to C_x^2$ and $C_y^1 \to C_y^2$ which are multiplication by $\lambda \in \mathbb{C}^*$ (resp. $\mu \in \mathbb{C}^*$) to get a non-normal surface $T(\lambda, \mu)$. We show that its canonical class is $\mathbb{Q}$-Cartier iff λ/μ is a root of unity.

To get concrete examples, fix an integer $a \geq 0$ and set

$$S := \bigl(z^2 = xy(x^{2a} + y^{2a})\bigr) \subset \mathbb{A}^3 \quad \text{and} \quad C := C_x + C_y$$

where $C_x = (y = z = 0)$ and $C_y = (x = z = 0)$. Note that C is not Cartier but $2C = (xy = 0)$ is. Furthermore, ω_S is locally free with generator $z^{-1}dx \wedge dy$ and so $\omega_S^2(2C)$ is locally free with generator

$$\frac{1}{xyz^2}\bigl(dx \wedge dy\bigr)^{\otimes 2} = \frac{1}{x^2y^2(x^{2a} + y^{2a})}\bigl(dx \wedge dy\bigr)^{\otimes 2}.$$

The restriction of $\omega_S^2(2C)$ to C_x is thus locally free with generator

$$\frac{1}{x^2(x^{2a} + y^{2a})}\Bigl(dx \wedge \frac{dy}{y}\Bigr)^{\otimes 2}\Big|_{C_x} = \frac{1}{x^{2+2a}}\bigl(dx\bigr)^{\otimes 2}.$$

Hence the different on C_x is the origin with coefficient $1+a$. Similarly, the restriction of $\omega_S^2(2C)$ to C_y is locally free with generator $y^{-2-2a}\bigl(dy\bigr)^{\otimes 2}$.

Take now 2 copies S_i with coordinates (x_i, y_i, z_i) for $i \in \{1, 2\}$. Let $\tau(\lambda, \mu) : C_1 \to C_2$ be an isomorphism such that $\tau(\lambda, \mu)^* x_2 = \lambda x_1$ and $\tau(\lambda, \mu)^* y_2 = \mu y_1$. Let $T(\lambda, \mu)$ be obtained by gluing $C_1 \subset S_1$ to $C_2 \subset S_2$ using $\tau(\lambda, \mu)$.

Assume that $\omega_{T(\lambda,\mu)}^{2m}$ is locally free with generator σ. Then the restriction of σ to S_i is of the form

$$\sigma|_{S_i} = \frac{1}{x_i^{2m} y_i^{2m} (x_i^{2a} + y_i^{2a})^m}\bigl(dx_i \wedge dy_i\bigr)^{\otimes 2m} \cdot f_i(x_i, y_i, z_i)$$

for some f_i such that $f_i(0, 0, 0) \neq 0$. Furthermore,

$$\tau^*\Bigl(\sigma|_{S_2}\Bigr)|_{C_2} = \Bigl(\sigma|_{S_1}\Bigr)|_{C_1}.$$

Further restricting to the x-axis, this gives

$$\frac{1}{(\lambda x_1)^{2m+2am}}\bigl(\lambda dx_1\bigr)^{\otimes 2m} f_2(\lambda x_1, 0, 0) = \frac{1}{x_1^{2m+2am}}\bigl(dx_1\bigr)^{\otimes 2m} f_1(x_1, 0, 0).$$

which implies that

$$f_2(0, 0, 0) = \lambda^{2am} f_1(0, 0, 0).$$

Similarly, computing on the y-axis we obtain that

$$f_2(0, 0, 0) = \mu^{2am} f_1(0, 0, 0).$$

If $\lambda^{2am} \neq \mu^{2am}$, these imply that $f_1(0, 0, 0) = f_2(0, 0, 0) = 0$, hence $\omega_{T(\lambda,\mu)}^{[2m]}$ is not locally free. If $\lambda^{2am} = \mu^{2am}$ then $f_1(x_1, y_1, z_1) \equiv 1$ and $f_2(x_2, y_2, z_2) \equiv \lambda^{2am}$ give a global generator of $\omega_{T(\lambda,\mu)}^{[2m]}$.

For $a \geq 1$, we have our required examples. As λ, μ vary in $\mathbb{C}^* \times \mathbb{C}^*$, we get a flat family of demi-normal surfaces $T(\lambda, \mu)$. The set of pairs (λ, μ) such that λ/μ

is a root of unity is a Zariski dense subset of $\mathbb{C}^* \times \mathbb{C}^*$ whose complement is also Zariski dense.

Note, however, that for $a = 0$, $\omega^{[2]}_{T(\lambda,\mu)}$ is locally free for every λ, μ. In this case, $S := \big(z^2 = xy\big) \subset \mathbb{A}^3$ is a quadric cone and $T(\lambda, \mu)$ is slc. (In fact $T(\lambda, \mu)$ is isomorphic to the reducible quartic cone $(x^2 + y^2 + z^2 + t^2 = xy = 0) \subset \mathbb{A}^4$ for every λ, μ.)

We are now ready to prove the converse of (14) for surfaces.

THEOREM 17. *Let X be demi-normal and Δ a $\mathbb{Q}$-divisor whose support does not contain any irreducible component of the conductor $D \subset X$. Let $(\bar{X}, \bar{D}, \bar{\Delta})$ and $\tau : \bar{D} \to \bar{D}$ be as in (2) and (7). The following are equivalent.*

(1) $\operatorname{Diff}_{\bar{D}^n} \bar{\Delta}$ *is τ-invariant.*
(2) *There is a codimension 3 set $W \subset X$ such that $(X \setminus W, \Delta|_{X\setminus W})$ is slc.*

Proof. We have already seen in (14) that (2) ⇒ (1).

The converse is étale local near codimension 2 points of X. We can thus localize at such a point $p \in X$ and assume from now on that X is an affine surface.

The conductor $D \subset X$ is thus a curve and by passing to a suitable étale neighborhood of p we may assume that the irreducible components of D are analytically irreducible at p. (This will make book-keeping easier.)

It is easiest to use case analysis, relying on some of the classification results in (**??**), but we use only (**??**).

(17.3) *Plt normalization case.* Assume that there is an irreducible component $\bar{X}_1 \subset \bar{X}$ such that $(\bar{X}_1, \bar{D}_1 + \bar{\Delta}_1)$ is plt. By (**??**), $\bar{D}_1$ is a smooth curve, $\lfloor \bar{\Delta}_1 \rfloor = 0$, and and $(\bar{D}_1, \operatorname{Diff}_{\bar{D}_1} \bar{\Delta}_1)$ is klt by adjunction (**??**).

There are 2 cases:

(i) If τ is an involution of $\bar{D}_1$ then $\bar{X} = \bar{X}_1$ is the only component and $X = X_1$ is not normal.

(ii) If τ maps $\bar{D}_1$ to another double curve $\bar{D}_2$, then, by the τ-invariance of the different (17.1), $\operatorname{Diff}_{\bar{D}_1} \bar{\Delta}_1 = \operatorname{Diff}_{\bar{D}_2} \bar{\Delta}_2$ and so $(\bar{X}_2, \bar{D}_2 + \bar{\Delta}_2)$ is also plt by inversion of adjunction (**??**). Thus $\bar{X} = \bar{X}_1 + \bar{X}_2$ has 2 irreducible components, both plt and the $\bar{X}_i$ are also irreducible components of X.

In the first case, chosoe $m \in \mathbb{N}$ such that $m \operatorname{Diff}_{\bar{D}_1} \bar{\Delta}_1$ is a $\mathbb{Z}$-divisor and let σ be a τ-invariant generator of $\omega^{2m}_{\bar{D}_1}(2m \cdot \operatorname{Diff}_{\bar{D}_1} \bar{\Delta}_1)$. Since

$$H^0\big(\bar{X}, \omega^{[2m]}_{\bar{X}}(2m\bar{D}_1 + 2m\bar{\Delta})\big) \twoheadrightarrow H^0\big(D_1, \omega^{2m}_{\bar{D}_1}(2m \cdot \operatorname{Diff}_{\bar{D}_1} \bar{\Delta}_1)\big)$$

is surjective, we can lift σ to a generator $\phi \in H^0\big(\bar{X}, \omega^{[2m]}_{\bar{X}}(2m\bar{D}_1 + 2m\bar{\Delta})\big)$, and by (9), ϕ descends to a nowhere zero section

$$\Phi_0 \in H^0\big(X \setminus p, \omega^{[2m]}_X(2m\Delta)|_{X\setminus p}\big)$$

which then extends to a local generator $\Phi \in H^0\big(X, \omega^{[2m]}_X(2m\Delta)\big)$.

In the second case, for $i = 1, 2$, pick local generators

$$\sigma_i \in H^0\big(\bar{D}_i, \omega^{2m}_{\bar{D}_i}(2m \operatorname{Diff}_{\bar{D}_i} \bar{\Delta})\big)$$

that are interchanged by τ and lift them back to sections

$$\phi_i \in H^0\big(\bar{X}_i, \omega^{[2m]}_{\bar{X}_i}(2m\bar{D}_i + 2m\bar{\Delta}_i)\big).$$

As before, the pair (ϕ_1, ϕ_2) descends to a section

$$\Phi_0 \in H^0\bigl(X \setminus p, \omega_X^{[2m]}(2m\Delta)|_{X\setminus p}\bigr)$$

which then extends to a section $\Phi \in H^0\bigl(X, \omega_X^{[2m]}(2m\Delta)\bigr)$. Then Φ is a local generator of $\omega_X^{[2m]}(2m\Delta)$, thus $2m(K_X+\Delta)$ is Cartier at P.

(17.4) *Non-plt normalization case.*

Let $(p_j \in \bar{D}_j)$ be the irreducible components of $\bar{D}^n$. Since $(\bar{X}, \bar{D}+\bar{\Delta})$ is lc but not plt at any of the preimages of p, we see that

$$\operatorname{Diff}_{\bar{D}_j}\bigl(\bar{D}-\bar{D}_j+\bar{\Delta}\bigr) = 1\cdot[p_j] \quad \text{for every } j.$$

Thus we can pick local generators

$$\sigma_j \in H^0\bigl(\bar{D}_j, \omega_{\bar{D}_j}^{2m}(2m[p_j])\bigr) = H^0\bigl(\bar{D}_j, \omega_{\bar{D}_j}^{2m}\bigl(2m \operatorname{Diff}_{\bar{D}_j}(\bar{D}-\bar{D}_j+\bar{\Delta})\bigr)\bigr)$$

that have residue 1 at p_j and such that together they give a τ-invariant section of $\omega_{\bar{D}^n}^{2m}(2m\operatorname{Diff}_{\bar{D}^n}\bar{\Delta})$. By (**??**), $\bar{D}$ is a curve with only nodes. Since the σ_j all have the same residue, by (9), they descend to a section

$$\sigma \in H^0\bigl(\bar{D}, \omega_{\bar{D}}^{2m}(2m\operatorname{Diff}_{\bar{D}}\bar{\Delta})\bigr).$$

As before, σ lifts back to $\phi \in H^0\bigl(\bar{X}, \omega_{\bar{X}}^{[2m]}(2m\bar{D}+2m\bar{\Delta})\bigr)$ and then descends to

$$\Phi \in H^0\bigl(X, \omega_X^{[2m]}(2m\Delta)\bigr). \quad \square$$

(17.5) Note the key point of the proof: on a smooth pointed curve $p \in C$, the fiber of $\omega_C(p)$ over p is not just a 1-dimensional vector space, but the residue gives a canonical isomorphism $\omega_C(p)|_p \cong \mathbb{C}$. A difficulty in higher dimensions is that there is no similar canonical isomorphism.

For instance, if X is a cone with vertex p over an Abelian variety A then there is a natural isomorphism

$$\omega_X^{[m]}|_x = \omega_X^m|_x \cong H^0(A, \omega_A)^{\otimes m}$$

and the latter does not have a canonical isomorphism with $\mathbb{C}$. (Indeed, as A moves in the moduli space of Abelian varieties, the $H^0(A, \omega_A)^{\otimes m}$ are fibers of an ample line bundle on the moduli space.)

We return to this in Section 3.

Divisorial semi-log-terminal.

Definition 18. An slc pair (X, Δ) is *divisorial semi-log-terminal* or *dslt* if $a(E, X, \Delta) > -1$ for every exceptional divisor E over X such that (X, Δ) is not semi-snc (**??**) at the generic point of $\operatorname{center}_X E$.

By (12.3), this implies that $\bigl(\bar{X}, \bar{D}+\bar{\Delta}\bigr)$ is dlt. The converse is not quite true, for instance $S := (xy = zt = 0) \subset \mathbb{A}^4$ is not dslt but its normalization is dlt. We see, however, in (19) that this difference appears only in codimension 2.

If $(Y, B+\Delta)$ is dlt and B is reduced, then $(B, \operatorname{Diff}_B \Delta)$ is dslt, and this is the main reason for our definition. However, if $(B, \operatorname{Diff}_B \Delta)$ is dslt then $(Y, B+\Delta)$ need not be dlt. For instance, take $Y := (x_1 \cdots x_n + x_{n+1}^m = 0) \subset \mathbb{A}^{n+1}$ and $B := (x_{n+1} = 0)$. Then B is simple normal crossing but (Y, B) is not, hence (Y, B) is not dlt.

It may be useful to develop a variant of dlt/dslt that is compatible both with adjunction and inversion of adjunction. There are, however, enough flavors of “log terminal” floating around, so we will not do this.

In the dlt case, we have the following positive answer to (6).

PROPOSITION 19. *Let* (X, Δ) *be a demi-normal pair. Assume that*

(1) *there is a codimension 3 set* $W \subset X$ *such that* $(X \setminus W, \Delta|_{X\setminus W})$ *is dslt and*
(2) *the normalization* $(\bar X, \bar D, \bar\Delta)$ *of* (X, Δ) *is dlt.*

Then

(3) *the irreducible components of* X *are normal,*
(4) $K_X + \Delta$ *is* $\mathbb{Q}$*-Cartier and*
(5) (X, Δ) *is dslt.*

Proof. We may assume that X is affine. At a codimension 2 point $p \in X$, the pair (X, Δ) is either snc, and hence the irreducible components of X are normal near p, or $(\bar X, \bar D, \bar\Delta)$ is plt above p and we are in case (17.3.ii). Thus again the irreducible components of X are normal.

Let $X_1, \dots, X_n$ be the irreducible components of X with normalization $\bar X_j \to X_j$. Let $\bar W \subset \bar X$ be the preimage of W.

Set $B_j := X_j \cap \bigl(X_1 \cup \cdots \cup X_{j-1}\bigr)$, as a divisor on X_j. By (**??**), $\mathcal{O}_{\bar X_j}(-\bar B_j)$ is a CM sheaf, hence $\operatorname{depth}_{\bar X_j \cap \bar W} \mathcal{O}_{\bar X_j}(-\bar B_j) \ge 3$. By (**??**?), this implies that

$$H^1\bigl(X_j \setminus W, \mathcal{O}_{X_j}(-B_j)|_{X_j\setminus W}\bigr) = H^1\bigl(\bar X_j \setminus \bar W, \mathcal{O}_{\bar X_j}(-\bar B_j)|_{\bar X_j\setminus \bar W}\bigr) = 0.$$

Hence, by (20.2), each X_j is S_2 and hence normal. Thus X and $\bar X$ have the same irreducible components.

There is an $m > 0$ such that $m(K_X + \Delta)|_{X_j}$ is locally free for every j. We can now apply (20.3) to $L := \omega_X^m(m\Delta)|_{X\setminus W}$ to conclude that $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

Let E be an exceptional divisor over X such that $a(E, X, \Delta) = -1$. We need to prove that (X, Δ) is snc at the generic point of $\operatorname{center}_X E$. By localizing, we may assume that $\operatorname{center}_X E =: p \in X$ is a closed point and (X, Δ) is dlt outside p. By assumption, we are done if $\operatorname{codim}_X p \le 2$.

Thus assume that $\dim X \ge 3$ and let $(X_i, D_i + \Delta_i)$ denote the irreducible components of $(\bar X, \bar D + \bar\Delta)$. By permuting the indices, we may assume that E is an exceptional divisor over X_1. Then $(X_1, D_1 + \Delta_1)$ is snc at p. If X_i is any other irreducible component such that $\dim_p(X_1 \cap X_i) \ge \dim X - 1$, then adjunction and inversion of adjunction shows that there is an exceptional divisor E_i over X_i with discrepancy -1 whose center is $p \in X_i$. Thus $(X_i, D_i + \Delta_i)$ is also snc at p. Since X is S_2, the complement of any codimension ≥ 2 subset is still connected [**Har62**], thus every $(X_j, D_j + \Delta_j)$ is snc at p.

We claim that $\dim T_pX = \dim X + 1$. Note that $\dim T_pX_1 = \dim X$ and for any $i \ne 1$, $\dim T_p(X_1 + X_i) = \dim X + 1$. Thus we are done if $T_p(X_1 + X_i) = T_p(X_1 + X_j)$ for every $i \ne 1 \ne j$. For this it is enough to find a tangent vector

$$v \in \bigl(T_p(X_1 + X_i) \cap T_p(X_1 + X_j)\bigr) \setminus T_pX_1.$$

Note that $(X_1 \cap X_i)$ and $(X_1 \cap X_j)$ are divisors in X_1, hence their intersection has dimension $\ge \dim X - 2$. Since $\dim X \ge 3$, we conclude that $X_i \cap X_j$ is strictly larger than p. Thus $\dim_p(X_i \cap X_j) \ge \dim X - 1$. In particular, $X_i \cap X_j$ has a tangent vector v which is not a tangent vector to $X_1 \cap X_i \cap X_j$.

Thus X has embedding dimension $\dim X + 1$ and so it is snc. □

PROPOSITION 20. *Let X be affine, pure dimensional and $X_1, \dots, X_m$ the irreducible components of X. Let $W \subset X$ be a closed subset of codimension ≥ 3. Let F be a coherent sheaf on X and set*

$$I_j := \ker\Big[F|_{X_1\cup\cdots\cup X_j} \to F|_{X_1\cup\cdots\cup X_{j-1}}\Big].$$

Assume that $H^1\big(X_j \setminus W, I_j|_{X_j\setminus W}\big) = 0$ for $j \geq 2$. Then

1. *The restriction maps*
$$H^0\big(X \setminus W, F|_{X\setminus W}\big) \to H^0\big(X_1 \cup \cdots \cup X_j \setminus W, F|_{X_1\cup\cdots\cup X_j\setminus W}\big)$$
are surjective.
2. *If $\operatorname{depth}_W F \geq 2$ then $\operatorname{depth}_W F|_{X_1\cup\cdots\cup X_j} \geq 2$ for every j.*
3. *If $F|_{X_j\setminus W} \cong \mathcal{O}_{X_j\setminus W}$ for every j then $F \cong \mathcal{O}_X$.*

Proof. The first claim follows from the cohomology sequence of

$$0 \to I_j \to F|_{X_1\cup\cdots\cup X_j} \to F|_{X_1\cup\cdots\cup X_{j-1}} \to 0$$

and induction on j. If $\operatorname{depth}_W F|_{X_1\cup\cdots\cup X_j} < 2$ then $F|_{X_1\cup\cdots\cup X_j\setminus W}$ has a section ϕ which does no extend to a section of $F|_{X_1\cup\cdots\cup X_j}$. By lifting ϕ back to a section of $F|_{X\setminus W}$, we would get a contradiction. This proves (2).

Finally, we prove by induction on j that, under the assumptions of (3), $F|_{X_1\cup\cdots\cup X_j\setminus W}$ has a nowhere zero section. For $j = 1$ we have assumed this. Next we lift the section, going from $j - 1$ to j.

Since $H^1\big(X_j \setminus W, I_j|_{X_j\setminus W}\big) = 0$, we have a surjection

$$H^0\big(X_1\cup\cdots\cup X_j\setminus W, F|_{X_1\cup\cdots\cup X_j\setminus W}\big) \twoheadrightarrow H^0\big(X_1\cup\cdots\cup X_{j-1}\setminus W, F|_{X_1\cup\cdots\cup X_{j-1}\setminus W}\big).$$

Thus $F|_{X_1\cup\cdots\cup X_j\setminus W}$ has a section σ_j which is nowhere zero on $X_1 \cup \cdots \cup X_{j-1} \setminus W$. Note that $\sigma_j|_{X_j\setminus W}$ is the section of a trivial line bundle. Thus, if it vanishes at all, then it vanishes along a Cartier divisor D_j on X_j.

Since $F|_{X_1\cup\cdots\cup X_j}$ is S_2, $X_j \cap \big(X_1 \cup \cdots \cup X_{j-1}\big)$ has pure codimension 1 in X_j. Thus, if $D_j \neq 0$ then $D_j \cap \big(X_1 \cup \cdots \cup X_{j-1}\big)$ is a nonempty codimension 2 set of $X_1 \cup \cdots \cup X_{j-1}$. On the other hand, W has codimension 3 and σ_j does not vanish on $X_1 \cup \cdots \cup X_{j-1} \setminus W$.

This implies that $D_j = 0$ and so σ_j is nowhere zero on $X_1 \cup \cdots \cup X_j \setminus W$. □

21 (Dslt models). In the study of lc pairs (X, Δ) it is very useful that there is a dlt model, that is, a projective, birational morphism $f : (X', \Delta') \to (X, \Delta)$ such that (X', Δ') is dlt, $K_{X'} + \Delta' \sim_{\mathbb{Q}} f^*(K_X + \Delta)$ and every f-exceptional divisor has discrepancy -1 (**??**?).

It would be convenient to have a similar result for slc pairs. An obvious obstruction is given by codimension 1 self-intersections of the irreducible components of X. Indeed, this is not allowed on a dslt pair but a semi resolution can not remove such self-intersections.

Every demi-normal scheme has a natural double cover that removes such codimension 1 self-intersections (22), thus it is of interest to ask for dslt models assuming that every irreducible component of X is normal in codimension 1.

22 (A natural double cover). Every demi-normal scheme has a natural double cover, constructed as follows.

Let X^0 be a scheme whose singularties are double nc points only.

First let us work over $\mathbb{C}$. Let $\gamma : S^1 \to X^0(\mathbb{C})$ be a path that intersects the singular locus only finitely many times. Let $c(\gamma) \in \mathbb{Z}/2\mathbb{Z}$ be the number of these intersection points where γ moves from one local component to another. It is easy to see that $c : \pi_1(X^0, p) \to \mathbb{Z}/2\mathbb{Z}$ is a well defined group homomorphism. Let $\pi^0 : \tilde{X}^0 \to X^0$ be the corresponding double cover.

For a general scheme, we can construct $\tilde{X}^0$ as follows. Let $\pi^0 : \bar{X}^0 \to X^0$ denote its normalization with conductors $D^0 \subset X^0$, $\bar{D}^0 \subset \bar{X}^0$ and Galois involution $\tau : \bar{D}^0 \to \bar{D}^0$. Take two copies $\bar{X}^0_1 \amalg \bar{X}^0_2$ and on $\bar{D}^0_1 \amalg \bar{D}^0_2$ consider the involution

$$\rho(p, q) = \big(\tau(q), \tau(p)\big).$$

Note that $\big(\bar{D}^0_1 \amalg \bar{D}^0_2\big)/\rho \cong \bar{D}^0$ but the isomorphism is non-canonical. Let $\tilde{X}^0$ be the universal pushout (46) of

$$\big(\bar{D}^0_1 \amalg \bar{D}^0_2\big)/\rho \leftarrow \big(\bar{D}^0_1 \amalg \bar{D}^0_2\big) \hookrightarrow \big(\bar{X}^0_1 \amalg \bar{X}^0_2\big).$$

Then $\pi^0 : \tilde{X}^0 \to X^0$ is an étale double cover and the irreducible components of $\tilde{X}^0$ are smooth. The normalization of $\tilde{X}^0$ is a disjoint union of two copies of the normalization of X^0.

Another way to construct $\pi^0 : \tilde{X}^0 \to X^0$ is the following. There is a natural quotient map $q : \pi_* \mathcal{O}_{\bar{X}^0} \to \pi_* \mathcal{O}_{\bar{D}^0}$ and τ decomposes the latter as the τ-invariant part $\mathcal{O}_D$ and the τ-anti-invariant part, call it L_D. Then $q^{-1}\mathcal{O}_D \subset \pi_* \mathcal{O}_{\bar{X}^0}$ is naturally $\mathcal{O}_X$ and $L_X := q^{-1}(L_D) \subset \pi_* \mathcal{O}_{\bar{X}^0}$ is also an invertible sheaf. Its tensor square is $\mathcal{O}_X$, since $L_D \cdot L_D = \mathcal{O}_D$ (multiplication as in $\pi_* \mathcal{O}_{\bar{D}^0}$). Thus L_X is 2-torsion in $\operatorname{Pic}(X^0)$ and

$$\tilde{X}^0 = \operatorname{Spec}_{X^0}\big(\mathcal{O}_X + L_X\big).$$

Let now X be a demi-normal scheme and $j : X^0 \hookrightarrow X$ an open subset with double nc points only and such that $X \setminus X^0$ has codimenson ≥ 2. Let $\pi^0 : \tilde{X}^0 \to X^0$ be as above. Then $j_* \pi^0_* \mathcal{O}_{\tilde{X}^0}$ is a coherent sheaf of algebras on X. Set

$$\tilde{X} := \operatorname{Spec}_X j_* \pi^0_* \mathcal{O}_{\tilde{X}^0}$$

with projection $\pi : \tilde{X} \to X$.

By construction, $\tilde{X}$ is S_2, π is étale in codimension 1 and the normalization of $\tilde{X}$ is a disjoint union of two copies of the normalization of X. Furthermore, the irreducible components of $\tilde{X}$ are smooth in codimension 1.

However, as shown by the examples (48), (49) and (50), in general the irreducible components of $\tilde{X}$ need not be normal.

2. Quotients by finite equivalence relations

In this Section we answer question (5) for slc pairs.

THEOREM 23. *Let $\tilde{X}$ be a normal variety, $\tilde{D} \subset \tilde{X}$ a reduced divisor, $\tilde{\Delta}$ a $\mathbb{Q}$-divisor on $\tilde{X}$ and $\tilde{\tau} : \tilde{D}^n \to \tilde{D}^n$ an involution on the normalization $\tilde{n} : \tilde{D}^n \to \tilde{D}$. Assume that*

(1) $\big(\tilde{X}, \tilde{D} + \tilde{\Delta}\big)$ *is lc,*
(2) $\tilde{\tau}$ *maps log canonical centers of* $\big(\tilde{D}^n, \operatorname{Diff}_{\tilde{D}^n} \tilde{\Delta}\big)$ *to log canonical centers, and*
(3) $(\tilde{n}, \tilde{n} \circ \tilde{\tau}) : \tilde{D}^n \to \tilde{X} \times \tilde{X}$ *generates a finite equivalence relation* $R(\tilde{\tau}) \rightrightarrows \tilde{X}$ *(26).*

Then there is a demi-normal pair $\big(X := \tilde{X}/R(\tilde{\tau}), \Delta\big)$ *whose normalization is* $\big(\tilde{X}, \tilde{D}+\tilde{\Delta}, \tilde{\tau}\big)$ *(cf. (2)).*

As noted after (6), the assumption (23.3) is obviously necessary. The theorem can fail without (23.1); in the examples of [**Hol63**, p.342] and [**Kol08**, 10], $\tilde{X}$ is a smooth 3-fold, $\tilde{D}$ has cusps along a curve, (23.2–3) both hold yet X does not exist. Here $\big(\tilde{X}, \tilde{D}\big)$ is not lc but it is not far from it; $\big(\tilde{X}, \frac{5}{6}\tilde{D}\big)$ is lc.

In order to prove (23), we develop a general theory of geometric quotients by finite set-theoretic equivalence relations. There are many cases when geometric quotients do not exist; see [**Kol08**, Sec.2] for a discussion of several such examples. On the positive side, we show in (43) that if a set-theoretic equivalence relation $R \rightrightarrows X$ satisfies a series of rather restrictive conditions, then the geometric quotient X/R exists.

The proof of (23) then boils down to showing that the relation $(n, n\circ\tau)$: $\bar{D}^n \to \bar{X}\times\bar{X}$ generates a set-theoretic equivalence relation $R \rightrightarrows X$ which satisfies the assumptions of (43).

Note that if $\big(\tilde{D}^n, \operatorname{Diff}_{\tilde{D}^n}\tilde{\Delta}\big)$ is τ-invariant, then assumption (23.2) holds.

Finite equivalence relations.

Definition 24. Let X and R be S-schemes. A pair of morphisms σ_1, σ_2 : $R \rightrightarrows X$, or equivalently a morphism $\sigma : R \to X\times_S X$ is called a *pre-relation.* A pre-relation is called *finite* if the σ_i are both finite and a *relation* if σ is a closed embedding.

To any finite pre-relation $\sigma : R \to X\times_S X$ one can associate a finite relation $i : \sigma(R) \hookrightarrow X\times_S X$. For the purposes in this section, there is no substantial difference between $\sigma : R \to X\times_S X$ and $i : \sigma(R) \hookrightarrow X\times_S X$. (By contrast, a key idea of Section **??**? is to exploit this difference using stacks.)

Definition 25 (Set theoretic equivalence relations). Let X and R be reduced S-schemes. We say that a morphism $\sigma : R \to X\times_S X$ is a *set theoretic equivalence relation* on X if, for every geometric point $\operatorname{Spec} K \to S$, we get an equivalence relation on K-points

$$\sigma(K) : \operatorname{Mor}_S(\operatorname{Spec} K, R) \hookrightarrow \operatorname{Mor}_S(\operatorname{Spec} K, X) \times \operatorname{Mor}_S(\operatorname{Spec} K, X).$$

Equivalently,

(1) σ is geometrically injective.
(2) (reflexive) R contains the diagonal Δ_X.
(3) (symmetric) There is an involution τ_R on R such that $\tau_{X\times X}\circ\sigma\circ\tau_R = \sigma$ where $\tau_{X\times X}$ denotes the involution which interchanges the two factors of $X\times X$.
(4) (transitive) For $1 \le i < j \le 3$ set $X_i := X$ and let $R_{ij} := R$ when it maps to $X_i \times_S X_j$. Then the coordinate projection of $\operatorname{red}\big(R_{12}\times_{X_2} R_{23}\big)$ to $X_1\times_S X_3$ factors through R_{13}:

$$\operatorname{red}\big(R_{12}\times_{X_2} R_{23}\big) \to R_{13} \xrightarrow{\pi_{13}} X_1\times_S X_3.$$

Note that the fiber product need not be reduced, and taking the reduced structure is essential.

26 (Equivalence closure). Let $R \hookrightarrow Y \times Y$ be a finite pre-relation, R reduced. There is a smallest set theoretic equivalence relation generated by R which is constructed as follows.

First we have to add the diagonal of $Y \times Y$ to R and make R symmetric with respect to the interchange of the two factors. Then we have $R^1 \hookrightarrow Y \times Y$ which is reflexive and symmetric.

Achieving transitivity may be an infinite process. Assume that we have already constructed $R^i \hookrightarrow Y \times Y$ with projections $\sigma_1^i, \sigma_2^i : R^i \to Y$. R^{i+1} is obtained by replacing R^i by the image

$$R^{i+1} := (\sigma_1^i \circ \tau_1^i, \sigma_2^i \circ \tau_2^i)\big(R^i \times_Y R^i\big) \subset Y \times Y, \tag{26.1}$$

where the maps are defined by the following diagram.

$$\begin{array}{ccccccc} & & & R^i \times_Y R^i & & & \\ & & \swarrow \tau_1^i & & \tau_2^i \searrow & & \\ & R^i & & & & R^i & \\ \sigma_1^i \swarrow & & \searrow \sigma_2^i & & \sigma_1^i \swarrow & & \searrow \sigma_2^i \\ Y & & & Y \ = \ Y & & & Y \end{array} \tag{26.2}$$

At the end we obtain a countable union of reduced subschemes

$$R \subset R^1 \subset R^2 \subset \cdots \subset Y \times Y$$

and finite projections $\sigma_1^j, \sigma_2^j : R^j \rightrightarrows Y$. In general, instead of an algebraic relation, we obtain a *pro-finite* set theoretic equivalence relations.

Definition 27 (Geometric quotients). Let $\sigma_1, \sigma_2 : R \rightrightarrows X$ be a set theoretic equivalence relation. We say that $q : X \to Y$ is a *categorical quotient* of X by R if

(1) $q \circ \sigma_1 = q \circ \sigma_2$, and
(2) $q : X \to Y$ is universal with this property. That is, given any $q' : X \to Y'$ such that $q' \circ \sigma_1 = q' \circ \sigma_2$, there is a unique $\pi : Y \to Y'$ such that $q' = \pi \circ q$.

If $\sigma_1, \sigma_2 : R \rightrightarrows X$ is finite, we say that $q : X \to Y$ is a *geometric quotient* of X by R if, in addition,

(3) $q : X \to Y$ is finite and
(4) for every geometric point $\operatorname{Spec} K \to S$, the fibers of $q_K : X_K(K) \to Y_K(K)$ are the $\sigma\big(R_K(K)\big)$-equivalence classes of $X_K(K)$.

Somewhat sloppily, we refer to the last property by saying that "the geometric fibers of q are the R-equivalence classes."

It is not hard to see [**Kol08**, 17] that the assumptions (1–3) imply (4), but in our applications we will check (4) directly.

The geometric quotient is denoted by X/R.

There are three cases when the construction of the geometric quotient is easy.

Lemma 28. *Let $R \rightrightarrows X$ be a finite, set theoretic equivalence relation and assume that there is a finite morphism $q' : X \to Y'$ such that $q' \circ \sigma_1 = q' \circ \sigma_2$. Set*

$$\mathcal{O}_Y := \ker\Big[q'_* \mathcal{O}_X \stackrel{\sigma_1^* - \sigma_2^*}{\longrightarrow} (q' \circ \sigma_i)_* \mathcal{O}_R\Big].$$

Then $Y = X/R$.

Proof. Y clearly satisfies the assumptions (27.1–3) and the geometric fibers of $X \to Y$ are finite unions of R-equivalence classes. As we noted above, by [**Kol08**, 17], Y also satisfies the assumption (27.4). $\square$

Lemma 29. *Let $R \rightrightarrows X$ be a finite, set theoretic equivalence relation with X, R reduced and over a field of characteristic 0. Let $\pi : X' \to X$ and $q' : X' \to Z$ be finite surjections. Assume that one of the following holds:*

(1) *X, Z are semi normal and the geometric fibers of q' are exactly the preimages of R-equivalence classes, or*
(2) *Z, X are normal, the $\sigma_i : R \to X$ are open and, over a dense open subset of Z, the geometric fibers of q' are exactly the preimages of R-equivalence classes*

Then $Z = X/R$.

Proof. Let $X^* \subset Z \times X$ be the image of X' under the diagonal map (q', π).

In the first case, every geometric fiber of π is contained in a geometric fiber of q', thus we see that the projection $X^* \to X$ is one-to-one on geometric points. Since X is semi normal, this implies that $X^* \cong X$. Thus we get a morphism $q : X \to Z$ whose geometric fibers are exactly the R-equivalence classes.

Therefore, $q \circ \sigma_1$ agrees with $q \circ \sigma_2$ on geometric points. Since R is reduced, this implies that $q \circ \sigma_1 = q \circ \sigma_2$. Define $p : Y \to Z$ as in (28). Since X is reduced, so is Y. The geometric fibers of $X \to Y$ are finite unions of R-equivalence classes. On the other hand, every geometric fiber of $X \to Y$ is contained in a geometric fiber of $X \to Z$ which is a single R-equivalence classe. Thus $X \to Z$ and $X \to Y$ have the same fibers, hence $Y \to Z$ is an isomorphism on geometric points. Since Z is semi normal, this implies that $Y \cong Z$.

In the second case, the same argument gives that $X^* \to X$ is birational. Since it is also finite, $X^* \cong X$ since the latter is normal. We know that $q \circ \sigma_1 = q \circ \sigma_2$ holds over a dense open subset of X, hence over a dense open subset of R. Thus $q \circ \sigma_1 = q \circ \sigma_2$ everywhere. Construct $p : Y \to Z$ as before. Here p is birational and finite, hence an isomorphism since X is normal. □

Lemma 30. *Let X be an excellent scheme over a field of characteristic 0 that is normal and of pure dimension d. Let $R \rightrightarrows X$ be a finite, set theoretic equivalence relation. Let $R^d \subset R$ denote the d-dimensional part of R. Then*

(1) *$R^d \rightrightarrows X$ is a finite, set theoretic equivalence relation* [**BB04**, 2.7]*,*
(2) *the geometric quotient X/R^d exists, and*
(3) *X/R^d is normal.*

Proof. Let us prove first that $R^d \rightrightarrows X$ is a set theoretic equivalence relation. The only question is transitivity. (Easy examples show that transitivity can fail if X is not normal [**Kol08**, 29].) Note that $\sigma_i : R^d \to X$ is finite with normal target. Hence, by (32), $R^d \times_X R^d \to R^d$ is open. In particular, $R^d \times_X R^d$ has pure dimension d. Thus the image of the finite morphism $R^d \times_X R^d \to R$ in (25.3) lies in R^d. Therefore $R^d \rightrightarrows X$ is a set theoretic equivalence relation. It is then necessarily finite.

Next assume that X/R^d exists and let $Y \to X/R^d$ be the normalization. Since X is normal, the quotient morphism $X \to X/R^d$ lifts to $\tau : X \to Y$. Thus $\tau \circ \sigma_1 = \tau \circ \sigma_2$ on a dense open set, hence equality holds everywhere. By the universal property of geometric quotients (27.2), $X/R^d = Y$ is normal.

It is sufficient to construct a geometric quotient one irreducible component at a time. Thus assume that X is irreducible and let $m = \deg \sigma_i$.

Consider the m-fold product $X \times \cdots \times X$ with coordinate projections π_i. Let R_{ij} (resp. Δ_{ij}) denote the preimage of R (resp. of the diagonal) under (π_i, π_j).

A geometric point of $\cap_{ij} R_{ij}$ is a sequence of geometric points $(x_1, \dots, x_m)$ such that any 2 are R-equivalent and a geometric point of $\cap_{ij} R_{ij} \setminus \cup_{ij} \Delta_{ij}$ is a sequence $(x_1, \dots, x_m)$ that constitutes a whole R-equivalence class. Let X' be the normalization of the closure of $\cap_{ij} R_{ij} \setminus \cup_{ij} \Delta_{ij}$. Note that every $\pi_\ell : \cap_{ij} R_{ij} \to X$ is finite, hence the projections $\pi'_\ell : X' \to X$ are finite.

The symmetric group S_m acts on $X \times \cdots \times X$ by permuting the factors and this lifts to an S_m-action on X'. Over a dense open subset of X, the S_m-orbits on the geometric points of X' are exactly the R-equivalence classes. Thus, by (29), $X'/S_m \cong X/R$. Hence the construction of X/R^d is reduced to the construction of X'/S_m. This is discussed in (31). □

31 (Quotients by finite group actions). Quotients by finite group actions are discussed at many places. The quasi projective case is quite elementary; see, for instance [**Sha94**, Sec.I.2.3] or the more advanced [**Mum70**, Sec.12]. For general schemes and algebraic spaces, the quotients are constructed in some unpublished notes of Deligne. See [**Knu71**, IV.1.8] for a detailed discussion of this method. In all cases, the geometric quotient X/G exists.

32 (Chevalley's criterion). (cf. [**Gro67**, IV.14.4.4]) Let X, Y be schemes of pure dimension d and Y normal. Then every quasi finite morphism $f : X \to Y$ is universally open. That is, for every $Z \to Y$, the induced morphism $X \times_Y Z \to Z$ is open.

Note also that if $f_i : X_i \to Y$ are open then so is $X_1 \times_Y X_2 \to Y$.

DEFINITION 33. Let $R \rightrightarrows X$ be a finite relation and $g : Y \to X$ a finite morphism. Then

$$g^* R := R \times_{(X \times X)} (Y \times Y) \rightrightarrows Y$$

defines a finite relation on Y. It is called the *pull-back* of $R \rightrightarrows X$. (Strictly speaking, it should be denoted by $(g \times g)^* R$.)

Note that if R is a set theoretic equivalence relation then so is $g^* R$ and the $g^* R$-equivalence classes on the geometric points of Y map injectively to the R-equivalence classes on the geometric points of X.

If X/R exists then, by (28), $Y/g^* R$ also exists and the natural morphism $Y/g^* R \to X/R$ is injective on geometric points. If, in addition, g is surjective then $Y/g^* R \to X/R$ is finite and an isomorphism on geometric points. Thus, if X is seminormal and the characteristic is 0, then $Y/g^* R \cong X/R$.

Let $h : X \to Z$ be a finite morphism and R a finite relation. Then the composite $R \rightrightarrows X \to Z$ defines a finite pre-relation. If, in addition, R is a set theoretic equivalence relation and the geometric fibers of h are subsets of R-equivalence classes, then $R \rightrightarrows X \to Z$ corresponds to a set theoretic equivalence relation

$$h_* R := (h \times h)(R) \subset Z \times Z,$$

called the *push forward* of $R \rightrightarrows X$. If $Z/h_* R$ exists, then, by (28), X/R also exists and the natural morphism $X/R \to Z/h_* R$ is finite and an isomorphism on geometric points.

Stratified equivalence relations.

We saw in (30) that pure dimensional equivalence relations behave well on normal schemes. In our intended applications, for instance in (23), we start with a normal scheme $\tilde{X}$ but the inductive nature of the proof leads to equivalence relations on schemes that are neither normal nor pure dimensional. Furthermore, in (23), the equivalence relation generated by

$$(\tilde{n}, \tilde{n} \circ \tilde{\tau}) : \tilde{D}^n \to \tilde{X} \times \tilde{X}$$

is not pure dimensional, since we always have to add the diagonal of $\tilde{X} \times \tilde{X}$ and $\dim \tilde{D}^n = \dim \tilde{X} - 1$.

Our aim is to show that $R \rightrightarrows X$ is still well behaved if X and R can be decomposed into normal and pure dimensional pieces and some strong semi-normality assumptions hold about the closures of the strata. To do these, we need the concept of a stratification.

DEFINITION 34. Let X be a scheme. A *stratification* of X is a decomposition of X into a finite disjoint union of reduced and locally closed subschemes. We will deal with stratifications where the strata are pure dimensional and indexed by the dimension. Then we write $X = \cup_i S_i X$ where $S_i X \subset X$ is the i-dimensional stratum. Such a stratified scheme is denoted by (X, S_*). We also assume that $\cup_{i \leq j} S_i X$ is closed for every j.

The *boundary* of (X, S_*) is the closed subscheme

$$BX := \cup_{i < \dim X} S_i X = X \setminus S_{\dim X} X.$$

Let (X, S_*) and (Y, S_*) be stratified schemes. We say that $f : X \to Y$ is a *stratified morphism* if $f\bigl(S_i X\bigr) \subset S_i Y$ for every i. Equivalently, if $S_i X = f^{-1}\bigl(S_i Y\bigr)$ for every i.

Let (Y, S_*) be a stratified scheme and $f : X \to Y$ a quasi-finite morphism such that $f^{-1}\bigl(S_i Y\bigr)$ has pure dimension i for every i. Then $S_i X := f^{-1}\bigl(S_i Y\bigr)$ defines a stratification of X, denoted by $\bigl(X, f^{-1} S_*\bigr)$. We say that $f : X \to (Y, S_*)$ is *stratifiable*.

Let (X, S_*) be a stratified scheme and $f : X \to Y$ a quasi-finite morphism such that $f^{-1}\bigl(f(S_i X)\bigr) = S_i X$ for every i. Then $S_i Y := f\bigl(S_i Y\bigr)$ defines a stratification of Y, denoted by $\bigl(Y, f_*\bigl(S_*\bigr)\bigr)$. We say that $f : (X, S_*) \to Y$ is *stratifiable*.

DEFINITION 35. Let (X, S_*) be stratified. A relation $\sigma_i : R \rightrightarrows (X, S_*)$ is called *stratified* if each σ_i is stratifiable and $\sigma_1^{-1} S_* = \sigma_2^{-1} S_*$. Equivalently, there is a stratification $(R, \sigma^{-1} S_*)$ such that $r \in \sigma^{-1} S_i R$ iff $\sigma_1(r) \in S_i X$ iff $\sigma_2(r) \in S_i X$.

Let $\sigma_i : R \rightrightarrows (X, S_*)$ be a stratified set theoretic equivalence relation and $f : (X, S_*) \to Y$ a stratifiable morphism. If the geometric fibers of f are subsets of R-equivalence classes then the push forward (33)

$$f_* R \rightrightarrows \bigl(Y, f_*\bigl(S_*\bigr)\bigr)$$

is also a stratified set theoretic equivalence relation.

By contrast, the pull-back of a stratified relation by a stratified morphism is not always stratified. (Sufficient conditions are given in (36).) As an example, let X be a nodal curve with $S_1 = X$, $R \rightrightarrows X$ the identity relation and $g : Z \to X$ the normalization. Then $(g^{-1} S)_1 = Z$ but $g^* R$ has 3 components. Besides the identity, it has 0 dimensional components showing that the 2 preimages of the node are equivalent.

LEMMA 36. *Assume that the strata of (Y, S_*) are all normal.*

(1) *Let $f_i : X_i \to (Y, S_*)$ be stratifiable quasi-finite morphisms. Then the induced maps $X_1 \times_Y X_2 \rightrightarrows X_i \rightrightarrows Y$ are all stratifiable.*

(2) *Let* $R \rightrightarrows (Y, S_*)$ *be a stratified relation. Then the pull-back* $g^*R \rightrightarrows (X, g^{-1}S_*)$ *by a stratified morphism* g *is also a stratified relation.*
(3) *Let* $R \rightrightarrows (Y, S_*)$ *be a stratified relation. Then its equivalence closure (26) is a stratified pro-finite relation.*

Proof. The conditions need to be checked one stratum at a time, hence we may assume that Y is normal and of pure dimension d.

By (32), the f_i are universally open and the X_i also have pure dimension d. Thus $X_i \times_Y X_2 \to Y$ is also open hence $X_i \times_Y X_2$ has pure dimenion d, proving (1).

Similarly, $g^*R := R \times_{(Y \times Y)} (X \times X) \to R$ is also open. Thus g^*R has pure dimension d and so $g^*R \rightrightarrows (X, g^{-1}S_*)$ is stratifiable.

To see (3), we need to show that all the pre-relations R^i constructed in (26) are stratified. By induction on i, assume that the maps $\sigma^i_j : R^i \rightrightarrows Y$ are stratified. By (1), the fiber products $\tau^i_j : R^i \times_Y R^i \rightrightarrows R^i$ are also stratified. Hence all arrows in the diagram (26.2) are stratified, and so the composites along the outer edges of the triangle are also stratified. Thus all the $\sigma^{i+1}_1 : R^{i+1} \rightrightarrows Y$ are also stratified. □

Definition 37. Let X be an excellent scheme. We consider 4 normality conditions on stratifications.

(N) We say that (X, S_*) has *normal strata*, or that it satisfies condition (N), if each S_iX is normal.

(SN) We say that (X, S_*) has *seminormal boundary*, or that it satisfies condition (SN), if X and the boundary $BX = \cup_{i<\dim X} S_iX$ are both seminormal.

(HN) We say that (X, S_*) has *hereditarily normal strata*, or that it satisfies condition (HN), if
  (a) X satisfies (N),
  (b) the normalization $\pi : X^n \to X$ is stratifiable, and
  (c) its boundary $B(X^n)$ satisfies (HN).

(HSN) We say that (X, S_*) has *hereditarily seminormal boundary*, or that it satisfies condition (HSN), if
  (a) X satisfies (SN),
  (b) the normalization $\pi : X^n \to X$ is stratifiable, and
  (c) its boundary $B(X^n)$ satisfies (HSN).

(In order to get a correct inductive definition, we should add that the empty scheme satisfies all these conditions.)

Note that if (X, S_*) satisfies (HN) or (HSN) then (X^n, π^*S_*) also satisfies (HN) or (HSN).

Remark 38. Condition (N) is quite reasonable and usually easy to satisfy but condition (HN) is more subtle. As an example, take

$$X = \left(x^2 = y^2(y + z^2)\right) \subset \mathbb{A}^3$$

with $S_1X = (x = y = 0)$. Then S_1X and S_2X are both smooth. The normalization of X is

$$X^n = \left(x_1^2 = y + z^2\right) \subset \mathbb{A}^3$$

where $x_1 = x/y$ and the preimage of S_1X is $(y = x_1^2 - z^2 = 0)$ which is not normal.

Actually, one of the trickiest parts of (HN) is to know when the normalization $\pi : X^n \to X$ is stratifiable. For example, let

$$X := (x = y = 0) \cup (z = t = 0) \subset \mathbb{A}^4$$

with $S_1X = (x = y = z = 0)$. As before, S_1X and S_2X are both smooth but $\pi : X^n \to X$ is not stratifiable since the preimage of S_1X has a 0-dimensional irreducible component.

Note also that while every scheme has a stratification satisfying (N) (and probably even (HN)), the conditions (SN) and (HSN) are usually impossible to satisfy since they pose restrictions on the *closures* of strata.

The conditions (SN) and (HSN) may seem less natural, and indeed they may not be the best conditions to consider. It would have been possible to require semi normality for the closure of every S_iX or even for the closure of any union of irreducible components of strata. The main objective in chosing (SN) and (HSN) was to find the weakest assumptions that make the proof of (43) work.

By contrast, the next conditions are chosen to yield the strongest conclusions in (44).

DEFINITION 39. Let (X, S_*) be a stratified scheme. Following (34) a subscheme $j : Z \hookrightarrow X$ is called *stratified* if j is a stratifiable morphisms. Equivalently, if $Z \cap S_iX$ is the union of some irreducible components of S_iX for every i. We say that (X, S_*) satisfies the *stratified closure property* if for every irreducible component $W \subset S_iX$, the injection of its closure $j : \bar{W} \to X$ is stratified.

(DB) We say that (X, S_*) is *Du Bois*, or that it satisfies condition (DB), if it has the stratified closure property and every stratified subscheme $Z \hookrightarrow X$ is Du Bois (**??**?).

(HDB) We say that (X, S_*) is *hereditarily Du Bois*, or that it satisfies condition (HDB), if
  (a) (X, S_*) satisfies (HN),
  (b) the normalization $\pi : X^n \to X$ is stratifiable, and
  (c) the boundary of the normalization $B(X^n)$ satisfies (HDB).

Note that by (**??**?) a Du Bois scheme is semi normal, thus (HDB) implies (HSN). Moreover, every stratified subscheme $j : Z \hookrightarrow X$ is semi normal.

The main excuse for all these definitions is that they are satisfied in one significant case:

EXAMPLE 40. Let (X, Δ) be lc. Let $S_i^*(X, \Delta) \subset X$ be the union of all $\leq i$-dimensional lc centers (**??**?) of (X, Δ) and

$$S_iX := S_i^*(X, \Delta) \setminus S_{i-1}^*(X, \Delta).$$

We call this the *log canonical stratification* or *lc stratification* of (X, Δ).

By (**??**?) [**KK09**, 1.4] the lc stratification (X, S_*) satisfies all of the conditions (N), (SN), (HN), (HSN), (DB), (HDB). Furthermore, if $D \subset \lfloor \Delta \rfloor$ is a divisor with normalization $\bar{D}$ then, by (**??**), $\bar{D} \to X$ is a stratified morphism from the lc stratification of $(\bar{D}, \operatorname{Diff}^*_{\bar{D}} \Delta)$ to the lc stratification of (X, Δ).

As a consequence of (43) and (44), we will obtain that the conditions (N), (SN), (HN), (HSN), (DB), (HDB) also hold if (X, Δ) is slc.

LEMMA 41. *Let (X, S_*) and (Y, S_*) be normal stratified spaces over a field of characteristic zero and $f : X \to Y$ a finite stratified morphism. If (X, S_*) satisfies one of the conditions (N), (SN), (HN), (HSN), (DB), (HDB) then so does (Y, S_*).*

Proof. The questions are local on Y. Let us check first (N).

Pick $y \in S_iY$. In order to check that S_iY is normal at y, we can replace Y by any affine neighborhood of y. Thus we may assume that X, Y are irreducible, affine, S_iY is closed in Y and S_iX is closed in X. Let ϕ be a regular function on the normalization $\bigl(S_iY\bigr)^n$. Since S_iX is normal, the pull back $f^*\phi$ is a regular function on S_iX. We can lift it to a regular function Φ_X on X. Since Y is normal,

$$\Phi_Y := \tfrac{1}{\deg X/Y} \operatorname{tr}_{X/Y} \Phi_X$$

is a regular function on Y whose restriction to S_iY is ϕ. Thus S_iY is normal.

A similar argument with BX instead of S_iX shows the (SN) case. Again we may assume that X, Y are irreducible and affine. Let ψ be a regular function on the semi normalization $\bigl(BY\bigr)^{sn}$. Since $BX = \operatorname{red} f^{-1}(BY)$ is semi normal, the pull back $f^*\psi$ is a regular function on BX, thus it lifts to a regular function Ψ_X on X. As before, $\Psi_Y := \frac{1}{\deg X/Y} \operatorname{tr}_{X/Y} \Psi_X$ is a regular function on Y whose restriction to BY is ψ. Thus BY is semi normal.

The hereditary cases follow by induction using the maps $(BX)^n \to (BY)^n$.

The Du Bois cases follow from (**??**). □

Example 42. Similar results do not hold in positive characteristic, not even if f is separable. For instance, let $f : \mathbb{A}^3_{xyz} \to \mathbb{A}^3_{uvw}$ be given by $f(x, y, z) = (x, y, z^2 + zy)$. Then f is finite of degree 2 and separable. The preimage of the cuspidal curve $(v = u^2 + w^3 = 0)$ is the curve $(y = x^2 + z^6 = 0)$, which has 2 branches tangent to each other if the characteristic is not 2. In characteristic 2, the (reduced) preimage is the smooth curve $(y = x + z^3 = 0)$.

The following is the main result of this section.

Theorem 43. *Let (X, S_*) be an excellent scheme or algebraic space over a field of characteristic 0 with a stratification as in (34). Assume that (X, S_*) satisfies the conditions (HN) and (HSN). Let $R \rightrightarrows X$ be a finite, set theoretic, stratified equivalence relation. Then*

(1) *the geometric quotient X/R exists,*
(2) *$\pi : X \to X/R$ is stratifiable and*
(3) *$\bigl(X/R, \pi_*S_*\bigr)$ also satisfies the conditions (HN) and (HSN).*

Complement 44. *Notation and assumptions as in (43). If (X, S_*) satisfies the condition (HDB) then $\bigl(X/R, \pi_*S_*\bigr)$ also satisfies (HDB).*

In particular, if (X, Δ) is slc then X is Du Bois.

Proof. The proof is by induction on $d := \dim X$. We follow the inductive plan in [**Kol08**, 30].

Let $(X^n, S^n_*) \to (X, S_*)$ be the normalization of X and $R^n \rightrightarrows X^n$ the pull-back of R. By (36), R^n is also a finite, set theoretic, stratified equivalence relation.

Let $X^{nd} \subset X^n$ (resp. $R^{nd} \subset R^n$) be the union of all d-dimensional irreducible components. By (30) R^{nd} is a finite, set theoretic, stratified equivalence relation on X^{nd}, the geometric quotient X^{nd}/R^{nd} is normal and the quotient map $X^{nd} \to X^{nd}/R^{nd}$ is stratifiable. By (41) the push forward of $R^n|_{X^{nd}}$ to X^{nd}/R^{nd} satisfies the conditions (HN) and (HSN).

Let X^{nl} be the union of all lower dimensional irreducible components of X^n. By a slight abuse of notation, we can view R^{nd} as an equivalence relation on X^n which is the identity on X^{nl}. Thus

$$X^n/R^{nd} = \bigl(X^{nd}/R^{nd}\bigr) \amalg X^{nl}.$$

Let $q : X^n \to X^n/R^{nd}$ denote the quotient map. Then $q_*S^n_*$ is a stratification which agrees with the push forward of $R^n|_{X^{nd}}$ on X^{nd}/R^{nd} and with $R^n|_{X^{nl}}$ on X^{nl}. Thus $\big(X^n/R^{nd}, q_*S^n_*\big)$ also satisfies the conditions (HN) and (HSN).

Furthermore, R^n descends to a stratified equivalence relation q_*R^n on X^n/R^{nd} which is the identity outside the boundary

$$B\big(X^n/R^{nd}\big) = B\big(X^{nd}/R^{nd}\big) \ \amalg \ X^{nl}.$$

By induction on the dimension, the geometric quotient of $B\big(X^n/R^{nd}\big)$ by the restriction of q_*R^n exists. Let us denote it by $B\big(X^n/R^{nd}\big)/q_*R^n$.

By (46) we get a universal push-out diagram

$$\begin{array}{ccccc} B\big(X^{nd}/R^{nd}\big) \ \amalg \ X^{nl} & = & B\big(X^n/R^{nd}\big) & \hookrightarrow & X^n/R^{nd} \\ & & \downarrow & & \downarrow \\ & & B\big(X^n/R^{nd}\big)/q_*R^n & \to & Y. \end{array}$$

We claim that $Y = X^n/R^n$. To see this note first that the geometric fibers of $X^n \to Y$ are exactly the R^n equivalence classes. On the boundary this holds by induction and on the open part this follows from (30). Second, X^n/R^{nd} is normal and $B\big(X^n/R^{nd}\big)/q_*R^n$ is semi normal. Thus $X^n/R^{nd} \to Y$ and $B\big(X^n/R^{nd}\big)/q_*R^n \to Y$ both lift to the semi-normalization of Y. By the universality of the push-out, this implies that Y is semi-normal. Thus $Y = X^n/R^n$ by (29). As we noted in (33), $X/R = X^n/R^n$.

The open stratum of X/R is also the open stratum of X^n/R^{nd} which is normal. The lower dimensional strata of X/R are also strata of $B\big(X^n/R^{nd}\big)/q_*R^n$. These are normal by induction. Thus $\big(X/R, \pi_*S_*\big)$ satisfies condition (N). We have seen that both $Y = X/R$ and its bundary $B\big(X^n/R^{nd}\big)/q_*R^n$ are semi normal. Thus $\big(X/R, \pi_*S_*\big)$ also satisfies condition (SN).

Note that X^n/R^{nd} is normal and $X^n/R^{nd} \to Y = X/R$ is an isomorphism at all d-dimensional generic poins. Hence the normalization of X/R is an open and closed subscheme of X^n/R^{nd}. We have seen during the proof that $\big(X^n/R^{nd}, q_*S^n_*\big)$ satisfies the conditions (HN) and (HSN), hence the same holds for the normalization of X/R. Together with the previous comments, these show that X/R satisfies the conditions (HN) and (HSN).

Assume finally that (X, S_*) is Du Bois. Then $\big(X^n/R^{nd}, q_*S^n_*\big)$ is Du Bois by (**??**) [**KK09**, 2.3] and then $\big(X/R, \pi_*S_*\big)$ is Du Bois by (**??**) [**KK09**, 1.5]. □

45 (Proof of (23)). Assume that $\big(\tilde X, \tilde D + \tilde\Delta\big)$ is lc. Let $\tilde S_*$ be the lc-stratification constructed in (40). We saw in (40) that $(\tilde X, \tilde S_*)$ satisfies all of the conditions (N), (SN), (HN), (HSN), (DB), (HDB).

As we noted in (40), $\tilde n : \tilde D^n \to \tilde X$ is stratified and $\tilde\tau$ is stratified by assumption (23.2). Thus $(\tilde n, \tilde n \circ \tilde\tau) : \tilde D^n \to \tilde X \times \tilde X$ is a stratified relation. Then by (36), its equivalence closure $\tilde R \rightrightarrows \tilde X$ is a stratified equivalence relation.

Thus the assumptions of (43) are satisfied and the geometric quotient $\tilde X/\tilde R$ exists. Set $X := \tilde X/\tilde R$ and let Δ be the image of $\tilde\Delta$. We claim that X is demi normal and $\big(\tilde X, \tilde D + \tilde\Delta\big)$ is its normalization.

Let $\tilde W \subset \tilde X$ be the union of lc centers of codimension ≥ 2 and $W \subset X$ its image in X. Then $\tilde\tau$ is an involution on $\tilde D \setminus \tilde W$, and the universal push out of

$$\big(\tilde D \setminus \tilde W\big)/\tilde\tau \leftarrow \big(\tilde D \setminus \tilde W\big) \hookrightarrow \big(\tilde X \setminus \tilde W\big)$$

is isomorphic to $X \setminus W$. Thus $X \setminus W$ has duble nc points only.

Let $X^d \to X$ be the demi normalization of X (1). Since $\tilde{X}$ is normal, the quotient map $\pi : \tilde{X} \to X$ lifts to $\pi^d : \tilde{X} \to X^d$. The involution $\tilde{\tau}$ is π^d-equivariant outside $\tilde{W}$, hence it is π^d-equivariant. By (28.1), these imply that $X^d = \tilde{X}/\tilde{R} = X$, thus X is demi normal. Moreover, $\big(\tilde{X}, \tilde{D}, \tilde{\tau}\big) = \big(\bar{X}, \bar{D}, \tau\big)$ holds over $X \setminus W$, hence everywhere. □

During the proof of (43) we have used the following theorem of [**Art70**, Thm.3.1]. For elementary proofs, see [**Fer03, Rao74**] or [**Kol08**, Sec.6].

THEOREM 46. *Let X be a Noetherian algebraic space over a Noetherian base scheme S. Let $Z \subset X$ be a closed subspace and $g : Z \to V$ a finite surjection. Then there is a universal push-out diagram of algebraic spaces*

$$\begin{array}{ccc} Z & \hookrightarrow & X \\ g\downarrow & & \downarrow \pi \\ V & \hookrightarrow & Y := X/(Z \to V) \end{array}$$

Furthermore, π is finite, $V \to Y$ is a closed embedding, $Z = \pi^{-1}(V)$ and the natural map $\ker\big[\mathcal{O}_Y \to \mathcal{O}_V\big] \to \ker\big[\mathcal{O}_X \to \mathcal{O}_Z\big]$ is an isomorphism.

Pro-finite equivalence relations.

In general it is quite hard to see when a finite pre-relation $R \rightrightarrows Y$ generates a finite set theoretic equivalence relations. Here are some examples which show that problems can occur in high codimension, even for dlt pairs.

EXAMPLE 47. Fix two points $a, b \in \mathbb{A}^1$ and consider two involutions $\tau_1 : x \mapsto a - x$ and $\tau_2 : x \mapsto b - x$. They correspond to a finite pre-relation

$$\sigma : \mathbb{A}^1 \amalg \mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{A}^1,$$

where $\sigma(x) = (x, a - x)$ and $\sigma(x) = (x, b - x)$. Note that the composite $\tau_1 \circ \tau_2$ is traslation by $b - a$. Thus it has infinite order if $a \neq b$ and the characteristic is 0.

EXAMPLE 48. Let $X := \mathbb{A}^3$ with coordinates (x, y, t) and $D_1 := (y = 0), D_2 := (x = 0)$ two hyperplanes. Let $L := (x = y = 0)$ be the t-axis. For $a, b \in \mathbb{C}$ define involutions on D_i by

$$\tau_1(x, 0, t) \mapsto (x, 0, a - t) \quad \text{and} \quad \tau_2(0, y, t) \mapsto (0, y, b - t).$$

Note that $D_1/(\tau_1) = \operatorname{Spec} \mathbb{C}[x, t(a - t)]$ and $D_2/(\tau_2) = \operatorname{Spec} \mathbb{C}[y, t(b - t)]$. Thus on $X \setminus L$ the τ_i generate a finite equivalence relation and we obtain a finite morphism

$$\pi^0 : \big(X \setminus L\big) \to \big(X \setminus L\big)/(\tau_1, \tau_2).$$

Both involutions act on L. Note that $\tau_2|_L \circ \tau_1|_L$ is translation by $b - a$, hence has infinite order if $a \neq b$ and the characteristic is 0.

This shows that π^0 can not be extended to a finite morphism on X.

EXAMPLE 49. Pick involutions $r_1, r_2, r_3 \in PGL(2, \mathbb{C})$ such that any 2 of them generate a finite subgroup but the 3 together generate an infinite subgroup.

Consider $X = \mathbb{A}^3 \times \mathbb{P}^1$. Let x_i be the coordinates on $\mathbb{A}^3$ and $D_i := (x_i = 0) \times \mathbb{P}^1$. On D_i consider the involution τ_i which is the identity on D_i and r_i on the $\mathbb{P}^1$-factor. Let $R \rightrightarrows X$ be the pro-finite set theoretic equivalence relation generated by the $\tau_i : i = 1, 2, 3$.

Note that

$$\pi_1 : \bigl(X \setminus D_1\bigr) \times \mathbb{P}^1 \to \bigl(X \setminus D_1\bigr) \times \bigl(\mathbb{P}^1/\langle r_2 r_3\rangle\bigr)$$

is finite, thus $R|_{X\setminus D_1}$ is a finite set theoretic equivalence relation. Similarly, $\bigl(X \setminus D_i\bigr)/\bigl(R|_{X\setminus D_1}\bigr)$ exists for $i = 2, 3$. Set $\mathbb{P}^1_0 := \{0\} \times \mathbb{P}^1$. Then the geometric quotient

$$\bigl(X \setminus \mathbb{P}^1_0\bigr)/\bigl(R|_{X\setminus \mathbb{P}^1_0}\bigr)$$

exists.

Note, however, that the restriction of R to $\mathbb{P}^1_0$ is not a finite equivalence relation since the subgroup generated by r_1, r_2, r_3 is infinite. Thus R is not a finite relation and there is no geometric quotient of X by R.

In order to find such r_1, r_2, r_3, its is easier to work with $SO(3,\mathbb{R}) \cong SU(2,\mathbb{C})$. Let $L_i \subset \mathbb{R}^3$ be 3 lines such that the angles between them are rational multiples of π. Let r_i denote the reflections determined by the lines L_i. By assumption, the angle between any 2 lines is a rational multiple of π, hence any 2 rotations generate a finite dihedral group.

The finite subgroups of $G \subset SO(3,\mathbb{R})$ are all known. If G is not cyclic or dihedral, then any rotation in G has order ≤ 6. Thus, as soon as the denominator of the angle between L_i, L_j is large enough, the subgroup generated by r_1, r_2, r_3 is infinite.

Example 50. In $\mathbb{R}^n$ consider the hyperplanes

$$H_0 := (x_1 = 1), H_n := (x_n = 0) \quad \text{and} \quad H_i := (x_i = x_{i+1}) \quad \text{for } i = 1, \dots, n-1.$$

Note that any n of these hyperplanes have a common point but the intersection of all $n+1$ of them is empty.

Let r_i denote the reflection on H_i. Each r_i is defined over $\mathbb{Z}$ and maps $\mathbb{Z}^n$ to itself. Thus any n of the r_i generate a reflection group which has a fixed point and preserves a lattice. These are thus finite groups. By contrast, all $n+1$ of them generate a reflection group with no fixed point. It is thus an infinite group.

As in (49) consider $X = \mathbb{A}^{n+1} \times \mathbb{A}^n$. Let $x_0, \dots, x_n$ be the coordinates on $\mathbb{A}^{n+1}$ and $D_i := (x_i = 0) \times \mathbb{A}^n$. On D_i consider the involution τ_i which is the identity on D_i and r_i on the $\mathbb{A}^n$-factor. Let $R \rightrightarrows X$ be the pro-finite set theoretic equivalence relation generated by the $\tau_i : i = 0, \dots, n$. We see that R is not a finite equivalence relation on X but it restricts to a finite equivalence relation on $\bigl(\mathbb{A}^{n+1} \setminus \{0\}\bigr) \times \mathbb{A}^n$.

The next examples have normal irreducible components but they are only lc.

Example 51. Let X be an affine variety, $p \in X$ a point and $D_1, D_2 \subset X$ divisors such that $D_1 \cap D_2 = p$. Take two copies $(X^i, D^i_1 + D^i_2)$ for $i = 1, 2$.

Choose an isomorphism $\phi(\lambda, \mu)$

$$\Bigl(\bigl(D^1_1 \setminus \{p^1\}\bigr) \times \mathbb{A}^1\Bigr) \amalg \Bigl(\bigl(D^1_2 \setminus \{p^1\}\bigr) \times \mathbb{A}^1\Bigr) \to \Bigl(\bigl(D^2_1 \setminus \{p^2\}\bigr) \times \mathbb{A}^1\Bigr) \amalg \Bigl(\bigl(D^2_2 \setminus \{p^2\}\bigr) \times \mathbb{A}^1\Bigr)$$

where $\phi(\lambda, \mu) = \bigl(1_{D_1} \times \lambda\bigr) \amalg \bigl(1_{D_2} \times \mu\bigr)$ is the identity on the D_i and multiplication by λ (resp. by μ) on the $\mathbb{A}^1$-factor of D_1 (resp. D_2).

The corresponding geometric quotient $Y^*(\lambda, \mu)$ is a non-normal variety whose irreducible components $\bigl(X^i \setminus \{p^i\}\bigr) \times \mathbb{A}^1$ intersect along $\bigl((D^i_1 + D^i_2) \setminus \{p^i\}\bigr) \times \mathbb{A}^1$.

When can we extend this to a non-normal variety $Y(\lambda, \mu) \supset Y^*(\lambda, \mu)$ whose irreducible components are $X^1 \times \mathbb{A}^1$ and $X^2 \times \mathbb{A}^1$?

Assume that $f^i = \sum f^i_j t^j$ is a function on $X^i \times \mathbb{A}^1$ such that f^1, f^2 glue together to a regular fuction on $Y^*(\lambda, \mu)$. The compatibility conditions are

$$f^1_j|_{D^1_1} = \lambda^j f^2_j|_{D^2_1} \quad \text{and} \quad f^1_j|_{D^1_2} = \mu^j f^2_j|_{D^2_2}.$$

In particular, we get that

$$f^1_j(p^1) = \lambda^j f^2_j(p^2) \quad \text{and} \quad f^1_j(p^1) = \mu^j f^2_j(p^2).$$

If λ/μ is not a root of unity, this implies that the f^i are constant on $\{p^i\} \times \mathbb{A}^1$. Thus there is no scheme or algebraic space $Y(\lambda, \mu) \supset Y^*(\lambda, \mu)$ whose irreducible components are $X^1 \times \mathbb{A}^1$ and $X^2 \times \mathbb{A}^1$.

Let us see now some log canonical examples satsfying the above assumptions.

(51.1) Set $X = \mathbb{A}^2, D_1 = (x = 0)$ and $D_2 = (y = 0)$. Then $(X, D_1 + D_2)$ is dlt. Thus we see that gluing in codimension 2 is not automatic for dlt pairs.

(51.2) Set $X = (xy - uv = 0), D_1 = (x = u = 0), D_2 = (y = v = 0)$ and $\Delta = (x = v = 0) + (y = u = 0)$. Here $(X, D_1 + D_2 + \Delta)$ is lc but not dlt. We can replace Δ by some other divisor whose coefficients are < 1, but $(X, D_1 + D_2 + \Delta)$ can never be dlt. Thus we see that gluing in codimension 3 is not automatic for lc pairs.

(51.3) Similar examples exists in any dimension. Let X be the cone over $\mathbb{P}^1 \times \mathbb{P}^n$, D_i the cone over $(i : 1) \times \mathbb{P}^n$ and Δ the cone over some $\mathbb{P}^1 \times B$ where $B \sim_{\mathbb{Q}} (n+1)H$ on $\mathbb{P}^n$. These examples can be lc but not dlt.

Thus we see that gluing in any codimension is not automatic for lc pairs.

52 (Polarization questions). The above examples were all local, but they can be easily compactified. However, none of them can be realized on lc pairs (X, D) such that $K_X + D$ is ample. That is, I do not know if there are $\big(\tilde X, \tilde D + \tilde\Delta, \tilde\tau\big)$ as in (23) such that $K_{\tilde X} + \tilde D + \tilde\Delta$ is ample yet $(n, n \circ \tau) : \bar D^n \to \bar X \times \bar X$ generates a non-finite set-theoretic equivalence relation.

Note, however, that a pre-relation that is compatible with an ample line bundle does not always generate a finite set-theoretic equivalence relation.

Indeed, [**BT09**] gives examples of étale pre-relations $R_C \rightrightarrows C$ on smooth curves C of genus ≥ 2 that generate non-finite equivalence relations. These are even compatible with the ample canonical line bundle.

The following obvious finiteness condition turns out to be quite useful. Note that its assumptions are satisfied if (X, Δ) is lc, S_* is the stratification by lc centers as in (40) and Z does not contain any lc center.

LEMMA 53. *Let (X, S_*) be a stratified space satisfying (N) and $Z \subset X$ a closed subspace which does not contain any of the irreducible components of the S_iX. Let $\sigma_i : R \rightrightarrows (X, S_*)$ be a pro-finite, stratified set theoretic equivalence relation. Assume that $R|_{X \setminus Z}$ is a finite set theoretic equivalence relation. Then R is also a finite set theoretic equivalence relation.*

Proof. Since R is a union of finite relations, it is enough to check that R has finitely many irreducible components. The latter can be checked one stratum at a time, hence we may assume that X is normal. Since every irreducible component of R dominates an irreducible component of X, finitness over the dense open set $X \setminus Z$ implies finiteness. □

3. Descending line bundles to geometric quotients

In this Section we answer question (6) for slc pairs.

THEOREM 54. *Let X be a demi-normal scheme and Δ a $\mathbb{Q}$-divisor on X. As in (2), let $(\bar{X}, \bar{D}, \bar{\Delta})$ be the normalization of (X, Δ) and $\tau : \bar{D}^n \to \bar{D}^n$ the corresponding involution. The following are equivalent:*

(1) *(X, Δ) is slc.*
(2) *$(\bar{X}, \bar{D} + \bar{\Delta})$ is lc and $\operatorname{Diff}_{\bar{D}^n} \bar{\Delta}$ is τ-invariant.*

55 (Plan of the proof). We have seen in (14) that (1) $\Rightarrow$ (2).

By (12), for the converse we only need to prove that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. The set of points where $K_X + \Delta$ is $\mathbb{Q}$-Cartier is open. After localizing at a generic point of the locus where $K_X + \Delta$ is not $\mathbb{Q}$-Cartier, we may assume that X is local with closed point $x \in X$, $k = k(x)$ is algebraically closed and $K_X + \Delta$ is $\mathbb{Q}$-Cartier on $X^0 := X \setminus \{x\}$.

Choose $m > 0$ such that $m\Delta$ is a $\mathbb{Z}$-divisor, $m(K_{\bar{X}} + \bar{D} + \bar{\Delta})$ is Cartier and $m(K_X + \Delta)$ is Cartier on X^0. Let $p : \bar{X}_L \to \bar{X}$ denote the total space of the line bundle $\mathcal{O}_{\bar{X}}\bigl(m(K_{\bar{X}} + \bar{D} + \bar{\Delta})\bigr)$; that is, $p_*\mathcal{O}_{\bar{X}_L} = \sum_{r \geq 0} \mathcal{O}_{\bar{X}}\bigl(-rm(K_{\bar{X}} + \bar{D} + \bar{\Delta})\bigr)$.

Set $\bar{D}_L := p^{-1}\bar{D}$ and $\bar{\Delta}_L := p^{-1}\bar{\Delta}$. Then $(\bar{X}_L, \bar{D}_L + \bar{\Delta}_L)$ is lc. The normalization $\bar{D}_L^n$ of $\bar{D}_L$ can be obtained either as the fiber product $\bar{D}^n \times_{\bar{D}} \bar{D}_L$ or as the total space of the line bundle $\mathcal{O}_{\bar{D}^n}\bigl(m(K_{\bar{D}^n} + \operatorname{Diff}_{\bar{D}^n} \bar{\Delta})\bigr)$. In particular, $\operatorname{Diff}_{\bar{D}_L^n} \bar{\Delta}_L = p^* \operatorname{Diff}_{\bar{D}^n} \bar{\Delta}$ and the lc centers of $\bigl(\bar{D}_L^n, \operatorname{Diff}_{\bar{D}_L^n} \bar{\Delta}_L\bigr)$ are the preimages of the lc centers of $\bigl(\bar{D}^n, \operatorname{Diff}_{\bar{D}^n} \bar{\Delta}\bigr)$.

The τ-invariance of $\operatorname{Diff}_{\bar{D}^n} \bar{\Delta}$ is equivalent to saying that τ lifts to an involution $\tau_L : \bar{D}_L^n \to \bar{D}_L^n$ and $\operatorname{Diff}_{\bar{D}_L^n} \bar{\Delta}_L$ is τ_L-invariant.

(Since we are working locally, $(\bar{X}_L, \bar{D}_L + \bar{\Delta}_L)$ is isomorphic to $(\bar{X}, \bar{D}, \bar{\Delta}) \times \mathbb{A}^1$, but τ_L may not be the product of τ with the identity on $\mathbb{A}^1$.)

Thus we have an lc pair $(\bar{X}_L, \bar{D}_L + \bar{\Delta}_L)$ and an involution $\tau_L : \bar{D}_L^n \to \bar{D}_L^n$ that maps log canonical centers of $\bigl(\bar{D}_L^n, \operatorname{Diff}_{\bar{D}_L^n} \bar{\Delta}_L\bigr)$ to log canonical centers. We are in a situation considered in (23). We have just established that (23.1–2) both hold.

(55.1) In order to apply (23), we need to check assumption (23.3). That is, we need to prove that $(n_L, \tau_L \circ n_L) : \bar{D}_L^n \to \bar{X}_L \times \bar{X}_L$ generates a *finite* set theoretic equivalence relation $\bar{R}_L \rightrightarrows \bar{X}_L$.

Note that finiteness holds over X^0. Indeed, we assumed that $m(K_X + \Delta)$ is Cartier on X^0; let $X_L^0 \to X^0$ denote the total space of $\mathcal{O}_X\bigl(m(K_X + \Delta)|_{X^0}\bigr)$. Set $\bar{X}_L^0 := p^{-1}(X^0) \subset \bar{X}_L$. There is a natural finite morphism $\bar{X}_L^0 \to X_L^0$ and in fact $X_L^0 = \bar{X}_L^0 / \bar{R}_L^0$ where $\bar{R}_L^0$ denotes the restriction of $\bar{R}_L$ to $\bar{X}_L^0$. Thus $\bar{R}_L^0 :\rightrightarrows \bar{X}_L^0$ is finite and therefore $\bar{R}_L :\rightrightarrows \bar{X}_L$ is finite iff it is finite over $\bar{X}_L \setminus \bar{X}_L^0$.

In order to study the latter, let $\bar{x}_1, \dots, \bar{x}_r \in \bar{X}$ be the preimages of x. If none of the $\bar{x}_i$ are lc centers of $(\bar{X}, \bar{D} + \bar{\Delta})$ then every lc center of $(\bar{X}_L, \bar{D}_L + \bar{\Delta}_L)$ intersects $\bar{X}_L^0$, hence $\bar{R}_L \rightrightarrows \bar{X}_L$ is finite by (53).

Thus we are left with the case when at least one of the $\bar{x}_i$ is an lc center. Then, by (54.2) and adjunction (**??**), all the $\bar{x}_i$ are lc centers of $(\bar{X}, \bar{D} + \bar{\Delta})$.

The fiber of $p : \bar{X}_L \to \bar{X}$ over $\bar{x}_i$ is the 1-dimensional k-vectorspace

$$V_i := \mathcal{O}_{\bar{X}}\bigl(m(K_{\bar{X}} + \bar{D} + \bar{\Delta})\bigr) \otimes_{\bar{X}} k(\bar{x}_i)$$

Thus $\bar{X}_L \setminus \bar{X}_L^0 = V_1 \cup \dots \cup V_r$ and τ_L gives a collection of isomorphisms $\tau_{ijk} : V_i \to V_j$. (A given $\bar{x}_i$ can have several preimages in $\bar{D}^n$ and each of these gives an isomorphism of V_i to some V_j. Thus there could be several isomorphisms from V_i to V_j for fixed i, j.) The τ_{ijk} generate a groupoid. All possible composites

$$\tau_{ij_2k_2} \circ \tau_{j_2j_3k_3} \circ \cdots \circ \tau_{j_nik_n} : V_i \to V_i$$

generate a subgroup of $\operatorname{Aut}(V_i)$, called the *stabilizer* $\operatorname{stab}(V_i)$. Note that $\bar{R}_L \rightrightarrows \bar{X}_L$ is a finite set theoretic equivalence relation iff $\operatorname{stab}(V_i) \subset \operatorname{Aut}(V_i) = k^*$ is a finite subgroup for every i.

In the surface case (17), the key step was to observe that, for m even, the Poincaré residue map gives a canonical isomorphism

$$\mathcal{R}^m : V_i \cong H^0\bigl(\bar{x}_i, \omega^m_{\bar{x}_i}\bigr)$$

and the right hand side is canonically isomorphic to $k(\bar{x}_i)$. With these choices, the τ_{ijk} become isomorphisms of *fields*

$$\tau_{ijk} : k(\bar{x}_i) \cong k(\bar{x}_j)$$

and $\operatorname{stab}(V_i)$ is the identity. (The last assertion does not quite follow from what we said before, but at least we see that $\operatorname{stab}(V_i)$ is a subgroup of the Galois group of $k(\bar{x}_i)/k$, hence finite.)

As noted in (17.5), in the higher dimensional case the V_i are not canonically isomorphic to $k(\bar{x}_i)$. Instead, in (67), we extract from a dlt model of $(\bar{X}_L, \bar{D}_L + \bar{\Delta}_L)$ (**??**) klt pairs (Z_i, Δ_{Z_i}) with $m(K_{Z_i} + \Delta_{Z_i}) \sim 0$ such that, for every i, j, k

(55.2) the Poincaré residue map gives a canonical isomorphism

$$\mathcal{R}^m : V_i \cong H^0\bigl(Z_i, \omega^{[m]}_{Z_i}(m\Delta_{Z_i})\bigr), \quad \text{and}$$

(55.3) $\tau_{ijk} : V_i \to V_j$ becomes the pull-back by a birational map $\phi_{jik} : Z_j \dashrightarrow Z_i$.

From these we conclude that

$$\operatorname{stab}(V_i) \subset \operatorname{im}\Bigl[\operatorname{Bir}\bigl(Z_i, \Delta_{Z_i}\bigr) \to \operatorname{Aut}_k H^0\bigl(Z_i, \omega^{[m]}_{Z_i}(m\Delta_{Z_i})\bigr)\Bigr]. \tag{55.4}$$

(There does not seem to be an obvious definition of birational self maps of an arbitrary pair (Z, Δ_Z); see (56) for our approach.) A variant (57) of a result of [**NU73**] allows us to conclude that the right hand side of (55.4) is finite, hence the stabilizers $\operatorname{stab}(V_i) \subset \operatorname{Aut}(V_i) = k^*$ are finite.

Thus $\bar{R}_L \rightrightarrows \bar{X}_L$ is a finite set theoretic equivalence relation and the geometric quotient $\bar{X}_L/\bar{R}_L$ exists by (23).

(55.5) For technical reasons it is more convenient to continue with the complement of the zero section $\bar{X}_S \subset \bar{X}_L$. Note that $p : \bar{X}_S \to \bar{X}$ is a G_m-torsor. We view $p : \bar{X}_S \to \bar{X}$ as a Seifert bundle over $\bar{X}$ (69).

Next we use that, by construction, $\bar{D}_S + \bar{\Delta}_S$ is G_m-invariant and τ_S is G_m-equivariant. It is easy to check that the G_m-action descends to a proper G_m-action on $\bar{X}_S/\bar{R}_S$ and the fibers of $\bar{X}_S/\bar{R}_S \to X$ are homogeneous under G_m. By (72) $\bar{X}_S/\bar{R}_S \to X$ is a Seifert G_m-bundle. The general theory of Seifert bundles (70) then shows that there are unique torsion free, rank 1, coherent sheaves L_i on X with multiplication maps $L_i \otimes L_j \to L_{i+j}$ such that L_M is locally free for some $M > 0$, and

$$\bar{X}_S/\bar{R}_S = \operatorname{Spec}_X \textstyle\sum_{i\in\mathbb{Z}} L_i.$$

As we noted earlier, the restriction of $\bar{X}_L/\bar{R}_L \to X$ to X^0 is the total space of the line bundle $\mathcal{O}_{X^0}\bigl(m(K_{X^0} + \Delta|_{X^0})\bigr)$. Since $x \in X$ has codimension ≥ 2, this implies that

$$L_{-r} = j_*\mathcal{O}_{X^0}\bigl(rm(K_{X^0} + \Delta|_{X^0})\bigr) = \mathcal{O}_X\bigl(rm(K_X + \Delta)\bigr),$$

where $j : X^0 \hookrightarrow X$ is the natural injection. Thus $Mm(K_X + \Delta)$ is Cartier. □

In the rest of the section we prove the auxiliary results that we used above.

Homotheties.

Definition 56. Let X be an integral k-variety and denote by $\mathcal{K}(X, \omega_X^{[m]})$ the $k(X)$-vectorspace of rational sections of $\omega_X^{[m]}$.

Given a nonzero rational section η of $\omega_X^{[m]}$, for some $m > 0$, let $k\eta \subset \mathcal{K}(X, \omega_X^{[m]})$ denote the 1-dimensional k-subspace generated by η.

Examples of such pairs $(X, k\eta)$ are obtained as follows. Let X be a proper, normal, geometrically integral k-variety and Δ a (not necessarily effective) $\mathbb{Q}$-divisor on X such that $K_X + \Delta \sim_{\mathbb{Q}} 0$. Then $m(K_X + \Delta) \sim 0$ for some $m > 0$ and

$$H^0\big(X, \omega_X^{[m]}(m\Delta)\big) \subset \mathcal{K}\big(X, \omega_X^{[m]}\big)$$

is a 1-dimensional k-subspace. This gives a pair $(X, k\eta)$ but there is no sensible way to pick an actual form $\eta \in k\eta$.

Given a birational map $\phi : X' \dashrightarrow X$, we get another pair $(X', k\eta') := \big(X', k\phi^*\eta\big)$. Any such $(X', k\eta')$ is called a *birational model* of $(X, k\eta)$.

A pair $(X, k\eta)$ is called *lc* (resp. *klt*) if for every birational model (X', η'), every pole of η' has order $\leq m$ (resp. $< m$). As usual (cf. (**??**)), it is sufficient to check this on one model where X' is smooth and the support of (η') is a snc divisor.

A birational map $\phi : X \dashrightarrow X$ is called a *homothety* of the pair $(X, k\eta)$ if $\phi^*\eta = \lambda\eta$ for some $\lambda = \lambda(\phi) \in k$, called the *scale*. The scale does not depend on the choice of $\eta \in k\eta$. All homotheties form a group $\operatorname{Homothety}(X, k\eta)$ and the scale gives a representation

$$\Lambda : \operatorname{Homothety}(X, k\eta) \to k^*.$$

The following theorem is the log version of the key finiteness result of [**NU73**]; see also [**Uen75**, Sec.14].

Theorem 57. *If $(X, k\eta)$ is klt then the scale representation* $\operatorname{Homothety}(X, k\eta) \to k^*$ *has finite image.*

Remark 58. The method of [$\mathbf{K^+92}$, Sec.12] shows that (57) should also hold if (X, η) is lc, but the proof may need MMP for lc pairs.

On the other hand, (57) fails if $(X, k\eta)$ is not lc. For instance, the image of the scale representation for $(\mathbb{A}^1_t, kdt)$ is k^* since $d(\lambda t) = \lambda dt$. Here dt has a double pole at infinity hence $(\mathbb{A}^1_t, kdt)$ is not lc.

The first step in the proof is reduction to the case when X is smooth, projective and η is a rational section of ω_X. Since the poles then are assumed to have order < 1, there are no poles and η is an actual section of ω_X. In the latter case, (63) proves that the image of the scale representation consists of roots of unity whose degree is bounded by the middle Betti number of X. Since there are only finitely many such roots of unity, this will complete the proof of (57).

59 (Reduction to $\eta \in H^0(X, \omega_X)$). Pick a birational model $(X, k\eta)$ such that X is smooth and (η) is a snc divisor. Since $(X, k\eta)$ is klt, we can write $(\eta) = mD - m\Delta$ where D is an effective $\mathbb{Z}$-divisor and $\lfloor\Delta\rfloor = 0$. Then $mK_X \sim -m\Delta + mD$ and we can view η as an isomorphism

$$\eta : \mathcal{O}_X(-m\Delta) \cong \big(\omega_X(-D)\big)^{\otimes m}.$$

Thus η defines an algebra structure and a cyclic cover

$$\tilde{X} := X\big[\sqrt[m]{\eta}\big] := \operatorname{Spec}_X \sum_{i=0}^{m-1} \omega_X(-D)^{\otimes i}\big(\lfloor i\Delta \rfloor\big)$$

with projection $p : \tilde{X} \to X$. Since p ramifies only along the snc divisor Δ, $\tilde{X}$ is klt (**??**?). Note that $\omega_{\tilde{X}}$ has a section $\tilde{\eta}$, given by the $i = 1$ summand in

$$1 \in H^0(X, \mathcal{O}_X(D)) \subset p_*\omega_{\tilde{X}} = \sum_{i=0}^{m-1} \operatorname{Hom}_X\Big(\omega_X(-D)^{\otimes i}\big(\lfloor i\Delta \rfloor\big), \omega_X\Big).$$

Assume now that $\phi : X \dashrightarrow X$ is a homothety with scale λ. Fix an mth root $\sqrt[m]{\lambda}$. Then ϕ lifts to a rational algebra map

$$\phi' : \phi^* \sum_{i=0}^{m-1} \omega_X(-D)^{\otimes i}\big(\lfloor i\Delta \rfloor\big) \dashrightarrow \sum_{i=0}^{m-1} \omega_X(-D)^{\otimes i}\big(\lfloor i\Delta \rfloor\big)$$

which is the natural isomorphis $\phi^*\omega_X \to \omega_X$ multiplied by $\big(\sqrt[m]{\lambda}\big)^i$ on the ith summand. Thus we get a homothety $\tilde{\phi}$ of $(\tilde{X}, \tilde{\eta})$ whith scale $\sqrt[m]{\lambda}$. Therefore, if the scale representation $\operatorname{Homothety}(\tilde{X}, k\tilde{\eta}) \to k^*$ has finite image then so does the scale representation $\operatorname{Homothety}(X, k\eta) \to k^*$.

Next we compare the pull-back of holomorphic forms with the pull-back map on integral cohomology. Note that one can pull back holomorphic forms by rational maps, but one has to be careful when pulling back integral cohomology classes by rational maps.

60. Let $f : M \to N$ be a map between oriented compact manifolds of dimension m. Then one can define a push forward map

$$f_* : H^i(M, \mathbb{Z})/(\text{torsion}) \to H^i(N, \mathbb{Z})/(\text{torsion})$$

as follows. Cup product with $\alpha \in H^i(M, \mathbb{Z})/(\text{torsion})$ gives

$$\alpha\cup : H^{m-i}(N, \mathbb{Z}) \to H^m(M, \mathbb{Z}) = \mathbb{Z} \quad \text{given by} \quad \beta \mapsto \alpha \cup f^*\beta.$$

Since the cup product $H^i(N, \mathbb{Z}) \times H^{m-i}(N, \mathbb{Z}) \to H^m(N, \mathbb{Z}) = \mathbb{Z}$ is unimodular, there is a unique class $\gamma \in H^i(N, \mathbb{Z})/(\text{torsion})$ such that $\gamma \cup \beta = \alpha \cup f^*\beta$ for every β. Set $f_*\alpha := \gamma$.

Note that if $\alpha = f^*\gamma$ for some $\gamma \in H^i(N, \mathbb{Z})$ then

$$f^*\gamma \cup f^*\beta = f^*(\gamma \cup \beta) = \deg f \cdot (\gamma \cup \beta)$$

shows that $f_*(f^*\gamma) = \deg f \cdot \gamma$. Thus $f_* \circ f^* : H^*(N, \mathbb{Z}) \to H^*(N, \mathbb{Z})$ is multiplication by $\deg f$. In particular, if $\deg f = 1$ then $f_* \circ f^*$ is the identity.

LEMMA 61. *Let X, X', Y be smooth proper varieties over $\mathbb{C}$ and $g : X \dashrightarrow Y$ a map. Let $f : X' \to X$ be a birational morphism such that $(g \circ f) : X' \to Y$ is a morphism. Then the following diagram is commutative*

$$\begin{array}{ccc} H^0(Y, \Omega^i_Y) & \hookrightarrow & H^i(Y(\mathbb{C}), \mathbb{C}) \\ g^* \downarrow & & \downarrow f_* \circ (g \circ f)^* \\ H^0(X, \Omega^i_X) & \hookrightarrow & H^i(X(\mathbb{C}), \mathbb{C}) \end{array}$$

Proof. For a holomorphic i form ϕ, let $[\phi]$ denotes its cohomology class in $H^i(\ ,\mathbb{C})$. Pull-back by a morphism commutes with taking cohomology class, thus $\big[(g\circ f)^*\phi\big] = (g\circ f)^*[\phi]$. On the other hand, $(g\circ f)^*\phi = f^*\big(g^*\phi\big)$. Thus $(g\circ f)^*[\phi] = f^*\big[g^*\phi\big]$. As noted in (60), $f_*\circ f^*\big[g^*\phi\big] = \big[g^*\phi\big]$. Thus $f_*\circ(g\circ f)^*[\phi] = f_*\circ f^*\big[g^*\phi\big] = \big[g^*\phi\big]$. □

Corollary 62. *Let X be a smooth proper variety over $\mathbb{C}$ and $g : X \dashrightarrow X$ a birational map. Then every eigenvalue of $g^* : H^0(X,\Omega^i_X) \to H^0(X,\Omega^i_X)$ is an algebraic integer of degree $\le \dim H^i(X(\mathbb{C}),\mathbb{C})$.*

Proof. Let $f : X' \to X$ be a birational morphism such that $(g\circ f) : X' \to Y$ is a morphism. By (61), every eigenvalue of $g^* : H^0(X,\Omega^i_X) \to H^0(X,\Omega^i_X)$ is also an eigenvalue of $f_*\circ(g\circ f)^* : H^i(X(\mathbb{C}),\mathbb{Z}) \to H^i(X(\mathbb{C}),\mathbb{Z})$. The latter is given by an integral matrix, hence its eigenvalues are algebraic integers. □

Warning 62.1. Although $f_*\circ(g\circ f)^* : H^i(X(\mathbb{C}),\mathbb{Z}) \to H^i(X(\mathbb{C}),\mathbb{Z})$ does not depend on the choice of f, the correspondence $g \mapsto f_*\circ(g\circ f)^*$ is not a group homomorphism. In fact, usually $f_*\circ(g\circ f)^*$ is not invertible; it need not even have maximal rank.

Corollary 63. *Let X be a smooth proper variety over $\mathbb{C}$ and $g : X \dashrightarrow X$ a birational map. Then every eigenvalue of $g^* : H^0(X,\omega_X) \to H^0(X,\omega_X)$ is a root of unity of degree $\le \dim H^{\dim X}(X(\mathbb{C}),\mathbb{C})$.*

Proof. Assume that η is an eigenform and $g^*\eta = \lambda\eta$. Since $\eta\wedge\bar\eta$ is a (singular) volume form,

$$\int_{X(\mathbb{C})}\eta\wedge\bar\eta = \int_{X(\mathbb{C})} g^*(\eta\wedge\bar\eta) = (\lambda\bar\lambda)\int_{X(\mathbb{C})}\eta\wedge\bar\eta.$$

Thus $|\lambda| = 1$ and, by (62), it is an algebraic integer.

Let $\sigma\in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ be any field automorphism. By conjugating everything by σ, we get $g^\sigma : X^\sigma \dashrightarrow X^\sigma$ such that $\big(g^\sigma\big)^*\eta^\sigma = \lambda^\sigma\eta^\sigma$. Thus λ^σ also has absolute value 1. We complete the proof by noting that an algebraic integer is a root of unity iff all of its conjugates have absolute value 1 [**?**]. □

Poincaré residue map.

64. Let (X,Δ) be dlt with Δ either effective or snc and $f : X \to Y$ a morphism with connected fibers such that $\omega_X^{[m]}(m\Delta)\sim_f 0$. Let $Z\subset X$ be a lc center of (X,Δ) such that $s := f(Z)\subset Y$ is a closed point. For $m>0$ and even, we have the general Poincaré residue map (**??**)

$$\mathcal{R}^m_{X\to Z} : \omega_X^{[m]}(m\Delta)|_Z \xrightarrow{\cong} \omega_Z^{[m]}(m\cdot \mathrm{Diff}^*_Z\Delta). \tag{64.1}$$

Note that $H^0(Z,\mathcal{O}_Z)$ is a finite field extension of $k(s)$ and by our assumptions there is a natural map

$$f_*\big(\omega_X^{[m]}(m\Delta)\big)\otimes_Y k(s) \to H^0\big(Z,\omega_X^{[m]}(m\Delta)|_Z\big).$$

Taking global sections in (64.1) gives a nonzero map

$$\mathcal{R}^m_{X\to Z} : f_*\big(\omega_X^{[m]}(m\Delta)\big)\otimes_Y k(s) \to \mathcal{K}(Z,\omega_Z^{[m]}). \tag{64.2}$$

and the resulting $(X,k\eta)$ is lc (cf. (**??**.1)). Furthermore, if Z is a minimal lc center then $(X,k\eta)$ is klt.

The following result shows, that, for minimal lc centers, (64.2) is essentially independent of the choice of Z.

PROPOSITION 65. *Let (X,Δ) be dlt (with Δ effective) over a field k and $f: X\to Y$ a proper morphism with connected fibers such that $\omega_X^{[m]}(m\Delta)\sim_f 0$ for some $m>0$ even. Let Z_1,Z_2 be minimal lc centers of (X,Δ) such that $f(Z_1)=f(Z_2)=s\in S$ is a closed point. Then there is a birational map $\phi: Z_2 \dashrightarrow Z_1$ such that the following diagram commutes*

$$\begin{array}{ccc} f_*\big(\omega_X^{[m]}(m\Delta)\big)\otimes_Y k(s) & = & f_*\big(\omega_X^{[m]}(m\Delta)\big)\otimes_Y k(s) \\ \mathcal{R}^m_{X\to Z_1}\downarrow & & \downarrow \mathcal{R}^m_{X\to Z_2} \\ \mathcal{K}(Z_1,\omega_{Z_1}^{[m]}) & \stackrel{\phi^*}{\longrightarrow} & \mathcal{K}(Z_2,\omega_{Z_2}^{[m]}) \end{array}$$

Proof. By (**??**) it is sufficient to prove this in case there is an lc center W of (X,Δ) that dominates s, contains Z_1,Z_2 as divisors and such that W is birational to a $\mathbb{P}^1$-bundle $\mathbb{P}^1\times U$ with Z_1,Z_2 as sections. Thus projection to U provides a birational isomorphism $\phi: Z_2\dashrightarrow Z_1$.

Since $\mathcal{R}_{X\to Z_i}=\mathcal{R}_{W\to Z_i}\circ\mathcal{R}_{X\to W}$ (**??**.2), it is sufficient to check the commutativity of the diagram

$$\begin{array}{ccc} f_*\big(\omega_W^{[m]}(m\,\mathrm{Diff}^*_W\Delta)\big)\otimes_Y k(s) & = & f_*\big(\omega_W^{[m]}(m\,\mathrm{Diff}^*_W\Delta)\big)\otimes_Y k(s) \\ \mathcal{R}^m_{W\to Z_1}\downarrow & & \downarrow \mathcal{R}^m_{W\to Z_2} \\ \mathcal{K}(Z_1,\omega_{Z_1}^{[m]}) & \stackrel{\phi^*}{\longrightarrow} & \mathcal{K}(Z_2,\omega_{Z_2}^{[m]}) \end{array} \tag{65.1}$$

Note that $L:=H^0(U,\mathbb{P}_U)$ is a finite field extension of $k(s)$ and (65.1) is equivalent to the commutativity of the following diagram of isomorphisms of 1-dimensional L-vector spaces.

$$\begin{array}{ccc} H^0\big(W,\omega_W^{[m]}(m\,\mathrm{Diff}^*_W\Delta)\big) & = & H^0\big(W,\omega_W^{[m]}(m\,\mathrm{Diff}^*_W\Delta)\big) \\ \mathcal{R}^m_{W\to Z_1}\downarrow & & \downarrow \mathcal{R}^m_{W\to Z_2} \\ H^0\big(Z_1,\omega_{Z_1}^{[m]}(m\,\mathrm{Diff}^*_{Z_1}\Delta)\big) & \stackrel{\phi^*}{\longrightarrow} & H^0\big(Z_2,\omega_{Z_2}^{[m]}(m\,\mathrm{Diff}^*_{Z_2}\Delta)\big) \end{array} \tag{65.2}$$

This in turn can be checked over the generic point of U. This reduces us to the case when $W=\mathbb{P}^1_L$ with coordinates $(x{:}y)$, $Z_1=(0{:}1)$ and $Z_2=(1{:}0)$. A generator of $H^0\big(\mathbb{P}^1,\omega_{\mathbb{P}^1}(Z_1+Z_2)\big)$ is dx/x which has residue 1 at Z_1 and -1 at Z_2. Thus (65.2) commutes for m even and anticommutes for m odd. □

DEFINITION 66. Let (X,Δ) be lc and $W\subset X$ an lc center. Let $f:(X',\Delta')\to(X,\Delta)$ be a log resolution and $Z\subset X'$ a minimal among the lc centers of (X',Δ') that dominate W. By (64), we obtain a Poincaré residue map

$$\mathcal{R}^m_{X\to Z}:\omega_X^{[m]}(m\Delta)\otimes k(W)\to\mathcal{K}\big(Z_{k(W)},\omega_Z^{[m]}\big),$$

defining a pair $\big(Z_{k(W)},k(W)\eta\big)$.

Note that this depends on the choice of f and Z.

Let $D\subset\lfloor\Delta\rfloor$ be a divisor with normalization $\pi:D^n\to D$. If $W\subset D$ then every irreducible component of $\pi^{-1}(W)$ is an lc center of $\big(D^n,\mathrm{Diff}^*_{D^n}\Delta\big)$ (**??**); let $W_D\subset\bar D$ be any one of them. Let Z_D denote a corresponding choice of Z as above.

THEOREM 67. *Notation and assumptions as in (66). Then*

(1) *$(Z_{k(W)},k(W)\eta)$ is independent of f and Z up to homothety.*

(2) *The construction of* $\big(Z_{k(W)}, k(W)\eta\big)$ *is local in the Zariski (and even in the strict étale) topology.*

(3) *If* $W \subset D$ *then there is a birational map* $\phi : Z_D \dashrightarrow Z$ *such that for* m *sufficiently divisible, the following diagram commutes*

$$\begin{array}{ccc} \omega_X^{[m]}(m\Delta) \otimes k(W) & \stackrel{\mathcal{R}_{X\to \bar D}}{\longrightarrow} & \omega_{\bar D}^{[m]}(m \operatorname{Diff}^*_{\bar D} \Delta) \otimes k(W_D) \\ \mathcal{R}^m_{X\to Z} \downarrow & & \downarrow \mathcal{R}^m_{\bar D \to Z_D} \\ \mathcal{K}(Z, \omega_Z^{[m]}) & \stackrel{\phi^*}{\longrightarrow} & \mathcal{K}(Z_D, \omega_{Z_D}^{[m]}). \end{array}$$

Proof. Let $(X^m, \Delta^m) \to (X, \Delta)$ be a dlt model of (X, Δ). Then (65) proves (1) if we choose Z on X^m and (**??**) takes care of every other choice. Part (2) is clear form the construction.

In order to see (3), let $D^m \subset X^m$ denote the birational tarnsform of D. Then $D^m \to \bar D$ is a dlt model. By (1) we may choose Z_D on D^m and take $Z = Z_D$. Then (3) is just the composition property (**??**.2) of the Poincaré residue map. □

Seifert bundles.

Seifert bundles were introduced to algebraic geometry in the works of [**OW75, Dol75, Pin77**]. The main emphasis has been on the case of smooth varieties and orbifolds. We need to study Seifert bundles over non-normal bases, so we go through the basic definitions.

Notation 68. G_m denotes the multiplicative group scheme $GL(1)$. As a scheme over $\operatorname{Spec} A$, it is $\operatorname{Spec}_A A[t, t^{-1}]$. For any natural number $r > 0$, the rth roots of unity form the subgroup scheme

$$\mu_r := \operatorname{Spec}_A A[t, t^{-1}]/(t^r - 1).$$

These are all the subgroup schemes of G_m. (Note that μ_r is nonreduced when the characteristic divides r.)

Every linear representation $\rho : G_m \to GL(W)$ is completely reducible, and the same holds for $\mu_r \subset G_m$ (see, for instance, [**SGA70**, I.4.7.3]). This implies that every quasi coherent sheaf with a G_m-action is a direct sum of eigensubsheaves. The set of vectors $\{v : \rho(\lambda)(v) = \lambda^i v\}$ is called the λ^i-*eigenspace*. We use this terminology also for μ_M-actions. In this case i is determined modulo M.

If a group G acts on a scheme X via $\rho : G \to \operatorname{Aut}(X)$, we get an action on rational functions on X given by $f \mapsto f \circ \rho(g^{-1})$. (The inverse is needed mostly for noncommutative groups only.)

Thus if G_m acts on itself by multiplication, we get an induced action on $A[t, t^{-1}]$ where $\lambda \in G_m(\bar k)$ acts as $t^i \mapsto \lambda^{-i} t^i$. Thus t^i spans the λ^{-i}-eigenspace.

A G_m-action on an A-algebra R is equivalent to a $\mathbb{Z}$ grading $R = \sum_{i\in\mathbb{Z}} R_i$ where R_i is the λ^{-i}-eigenspace.

The natural G_m-action on G_m/μ_M corresponds to the algebra $\sum_{i \in M\mathbb{Z}} A \cong A[t^M, t^{-M}]$.

Definition 69. Let X be a semi-normal scheme (or algebraic space) over S. A *Seifert bundle* (or a Seifert G_m-bundle) over X is a reduced scheme (or algebraic space) Y together with a morphism $f : Y \to X$ and a G_m-action on Y satisfying the following conditions.

(1) f is affine and G_m-equivariant (with the trivial G_m-action on X).

(2) The natural map $\mathcal{O}_X \to \big(f_*\mathcal{O}_Y\big)^{G_m}$ is an isomorphism,

(3) For every point $x \in X$, the G_m-action on the reduced fiber $\operatorname{red} Y_x$ is isomorphic to the natural G_m-action on $G_m/\mu_{m(x)}$ for some $m(x) \in \mathbb{N}$, called the *multiplicity* of the fiber over x.

(If X is normal, one usually assumes that $m(x) = 1$ at the generic point, but for X reducible this is not a natural condition.)

One can thus view the theory of Seifert bundles as a special chapter of the study of algebraic G_m-actions. The emphasis is, however, quite different.

THEOREM 70. *Let X be a pure dimensional semi-normal scheme (or algebraic space). There is a one-to-one correspondence between*

(1) *Seifert G_m-bundles $f : Y \to X$, and*
(2) *graded $\mathcal{O}_X$-algebras $\sum_{i\in\mathbb{Z}} L_i$ such that*
  (a) *each L_i is a torsion free, coherent sheaf on X whose rank is 1 or 0 at the generic points,*
  (b) *$L_i \otimes L_j \to L_{i+j}$ are isomorphisms at the generic points,*
  (c) *L_M is locally free for some $M > 0$, and*
  (d) *$L_i \otimes L_M \to L_{i+M}$ is an isomorphism for every i.*

Proof. Let $f : Y \to X$ be a Seifert bundle. Since $f : Y \to X$ is affine, $f_*\mathcal{O}_Y$ is a quasicoherent sheaf with a G_m-action. Thus it decomposes as a sum of quasicoherent G_m-eigensubsheaves

$$f_*\mathcal{O}_Y = \textstyle\sum_{j\in\mathbb{Z}} L_j, \tag{70.3}$$

where L_j is the λ^{-j} eigensubsheaf, with multiplication maps $m_{ij} : L_i \otimes L_j \to L_{i+j}$. Note that $L_0 = \mathcal{O}_X$ by (69.2).

Pick any point $x \in X$. By assumption $\operatorname{red} Y_x \cong G_m/\mu_{m(x)}$, thus $t^{-m(x)}$ on G_m descends to an invertible function h_x on Y_x which is a G_m-eigenfunction with eigencharacter $m(x)$. There is an affine neighborhood $x \in U \subset X$ such that h_x lifts to an invertible function h_U on $f^{-1}(U)$ which is a G_m-eigenfunction with eigencharacter $m(x)$. This h_U is a generator of $L_{m(x)}$ on U and the multiplication maps $L_i \otimes L_{m(x)} \to L_{i+m(x)}$ are isomorphisms over U for every i.

Setting $M = m(X) := \operatorname{lcm}\{m(x) : x \in X\}$, we see that L_M is locally free on X and the multiplication maps $L_i \otimes L_M \to L_{i+M}$ are isomorphisms for every i.

If x is a generic point, then $L_i \otimes k(x) \cong k(x)$ if $m(x)$ divides i and $L_i \otimes k(x) = 0$ otherwise.

We still need to prove that the L_i are torsion free and coherent. Any torsion section of L_i is killed by $L_i^{\otimes M} \to L_{Mi}$, hence it would give a nilpotent section of $\mathcal{O}_Y$, a contradiction. Thus every L_i is torison free. Coherence is a local question, thus assume that X is affine. For a generic point $x_g \in X$, $L_i \otimes k(x_g) \neq 0$ iff $m(x_g)|i$ iff $L_{-i} \otimes k(x_g) \neq 0$. Thus there is a section $s \in H^0(X, L_{-i})$ that is a generator at all generic points x_g such that $L_i \otimes k(x_g) \neq 0$. Then the composite

$$L_i \cong L_i \otimes \mathcal{O}_X \xrightarrow{(1,s)} L_i \otimes L_{-i} \to L_0 \cong \mathcal{O}_X$$

is an isomorphism at the generic points, hence an injection. Thus every L_i is a coherent sheaf on X.

Conversely, assume that $\sum_{i\in\mathbb{Z}} L_i$ satisfies the conditions of (70.2.a–d). Then $\sum_{i\in\mathbb{Z}} L_i$ is generated by the coherent submodule $\sum_{-m\leq i\leq m} L_i$. Thus $Y := \operatorname{Spec}_X \sum_{i\in\mathbb{Z}} L_i$ is affine over X. The grading gives a G_m-action.

Pick any $x \in X$, then the fiber Y_x over x is $\operatorname{Spec}_x\big(\sum_{i\in\mathbb{Z}} L_i \otimes k(x)\big)$. By (71), $L_i \otimes k(x)$ is nilpotent unless L_i is locally free at x and $L_i^{\otimes r} \to L_{ri}$ is an isomorphism near x for every r. Hence the reduced fiber is $\operatorname{Spec}_x \sum_{i\in m(x)\mathbb{Z}} k(x) \cong k(x)[t, t^{-1}]$ for some $m(x) \in \mathbb{N}$.

Lemma 71. *Let L, M be rank 1 torsion free sheaves and assume that there is a surjective map $h : L \otimes M \twoheadrightarrow \mathcal{O}_X$. Then L, M are both locally free.*

Proof. Pick $x \in X$. By assumption there is an affine nieghborhood $x \in U$ and sections $\alpha \in H^0(U, L), \beta \in H^0(U, M)$ such that $h(\alpha \otimes \beta)$ is invertible.

Let $\gamma \in H^0(U, L)$ be arbitrary. Then $h(\gamma \otimes \beta) = f \cdot h(\alpha \otimes \beta)$ for some $f \in \mathcal{O}_U$, thus $h((\gamma - f\alpha) \otimes \beta) = 0$. Thus $\gamma - f\alpha$ is zero on the open set where M is locally free, hence it is zero since L is torsion free. Thus α generates $L|_U$ and so L is locally free. □

Proposition 72. *Let $f : Y \to X$ be a Seifert G_m-bundle. Let $R_X \rightrightarrows X$ be a finite, set theoretic equivalence relation and $R_Y \rightrightarrows Y$ a G_m-equivariant finite, set theoretic equivalence relation. Assume that the geometric quotients X/R_X and Y/R_Y both exist. Then*

(1) *f descends to $f/R : Y/R_Y \to X/R_X$ iff $f(R_Y) \subset R_X$, and*
(2) *$f/R : Y/R_Y \to X/R_X$ is a Seifert G_m-bundle iff $R_Y \to R_X$ is surjective.*

Proof. The first part follows from the universal property of geometric quotients (27.2).

Next assume that $f/R : Y/R_Y \to X/R_X$ exists. Since $Y \to X$ is affine and $X \to X/R_X$ is finite, $Y \to X/R_X$ is also affine. Since $Y \to Y/R_Y$ is finite, $Y/R_Y \to X/R_X$ is affine by Chevalley's theorem. Since $R_Y \rightrightarrows Y$ is G_m-equivariant, the G_m action descends to Y/R_Y, again by the universal property of geometric quotients.

The only remaining question is about the fibers of $Y/R_Y \to X/R_X$. Pick a point $x \in X/R_X$, let $x_i \in X$ be its preimages and $Y_i \subset Y$ the reduced Seifert fiber over x_i. Then

$$\operatorname{red}(f/R)^{-1}(x) = \big(\amalg_i Y_i\big)/\big(R_Y|_{\amalg_i Y_i}\big).$$

Thus $\operatorname{red}(f/R)^{-1}(x)$ is a union of G_m-orbits and it is irreducible iff for every i, j, every point of Y_i is R_Y-equivalent to some point of Y_j. Using the G_m-action, this holds iff for every i, j, some point of Y_i is R_Y-equivalent to some point of Y_j. The latter holds iff $R_Y \to R_X$ is surjective. □

4. Semi log resolutions

The aim of this section is to discuss resolution theorems that are useful in the study of semi log canonical varieties.

Definition 73 (Simple normal crossing). Let k be a field, X a k-scheme and $D = \sum a_i D_i$ a Weil divisor on X with the D_i irreducible.

We say that (X, D) has *simple normal crossing* or *snc* at a point $p \in X$ if X is smooth at p and there are local coordinates $x_1, \dots, x_n$ such that $\operatorname{Supp} D \subset (x_1 \cdots x_n = 0)$ near p. Alternatively, if for each D_i there is a $c(i)$ such that $D_i = (x_{c(i)} = 0)$ near p.

We say that (X, D) has *normal crossing* or *nc* at a point $p \in X$ if $(\hat{X}_K, D|_{\hat{X}_K})$ is snc at p where $\hat{X}_K$ denotes the completion at p and K is an algebraic closure of $k(p)$.

Let $p \in D$ be a nc point of multiplicity 2. If the characteristic is different from 2, then, in suitable local coordinates, D can be given by an equation $x_1^2 - ux_2^2 = 0$ where $u \in \mathcal{O}_{p,X}$ is a unit. D is snc at p iff u is a square.

For example, $(y^2 = x^2 + x^3) \subset \mathbb{A}^2$ is nc but it is not snc at the origin. Similarly, $(x^2 + y^2 = 0) \subset \mathbb{A}^2$ is nc but it is snc only if $\sqrt{-1}$ is in the base field k.

We say that (X, D) is snc (resp. nc) if it is snc (resp. nc) for every $p \in X$.

Given (X, D), there is a largest open set $U \subset X$ such that $(U, D|_U)$ is snc (resp. nc). This open set is called that *snc* (resp. *nc*) *locus* of (X, D).

Definition 74 (Log resolution). Let k be a perfect field, X a reduced k-scheme and D a Weil divisor on X. A *log resolution* of (X, D) is a proper birational morphism $f : X' \to X$ such that $\bigl(X', D' := \operatorname{Supp}\bigl(f^{-1}(D) + \operatorname{Ex}(f)\bigr)\bigr)$ has snc. (In particular, all of the irreducible components of D' have codimension 1.) Here $\operatorname{Ex}(f)$ denotes the exceptional set of f, that is, the set of points where f is not a local isomorphism. We also say that $f : (X', D') \to (X, D)$ is a log resolution.

The basic existence result on resolutions was established by [**Hir64**]. We also need a strengthening of it, due to [**Sza94**], see also [**BM97**, Sec.12].

Theorem 75 (Existence of log resolutions). *Let X be an algebraic space of finite type over a field of characteristic 0 and D a Weil divisor on X.*

(1) [**Hir64**] *(X, D) has a log resolution.*
(2) [**Sza94, BM97**] *(X, D) has a log resolution $f : X' \to X$ such that f is an isomorphism over the snc locus of (X, D).*

Corollary 76 (Resolution in families). *Let C be a smooth curve over a field of characteristic 0, $f : X \to C$ a flat morphism and D a divisor on X. Then there is a log resolution $g : Y \to X$ such that $g_*^{-1}D + \operatorname{Ex}(g) + Y_c$ is a snc divisor for every $c \in C$ where Y_c denotes the fiber over a point c.*

Proof. Let $p : X' \to X$ be any log resolution of (X, D). There are only finitely many fibers $f \circ p$ of such that $p^{-1}D + \operatorname{Ex}(p) + X'_c$ is not a snc divisor. Let these be $\{X'_{c_i} : i \in I\}$. Let $p' : Y \to X'$ be a log resolution of $\bigl(X', p^{-1}D + \operatorname{Ex}(p) + \sum_i X'_{c_i}\bigr)$ that is an isomorphism over $X' \setminus \sum_i X'_{c_i}$ and set $g = p' \circ p$. Then $g_*^{-1}D + \operatorname{Ex}(g) + \sum_i Y_{c_i}$ is a snc divisor. Thus, if $c = c_i$ for some i then $g_*^{-1}D + \operatorname{Ex}(g) + Y_c$ is a snc divisor. For other $c \in C$, $g_*^{-1}D + \operatorname{Ex}(g) + Y_c$ is a snc divisor, except possibly near Y_c. By construction, the map p' is an isomorphism near Y_c, and $p^{-1}D + \operatorname{Ex}(p) + X'_c$ is snc divisor, hence so is $g_*^{-1}D + \operatorname{Ex}(g) + Y_c$. □

Next we show how (75.2) can be reduced to the Hironaka-type resolution theorems presented in [**Kol07**]. The complication is that the Hironaka method and its variants proceed by induction on the multiplicity. Thus, for instance, the method would normally blow up every triple point of D before dealing with the non-snc double points. In the present situation, however, we want to keep the snc triple points untouched.

We can start by resolving the singularities of X, thus it is no restriction to assume from the beginning that X is smooth. To facilitate induction, we work with a more general resolution problem.

Definition 77. Consider the object $(X, I_1, \dots, I_m, E)$ where X is a smooth variety, the I_j are ideal sheaves of Cartier divisors and E a snc divisor. We say that $(X, I_1, \dots, I_m, E)$ has *simple normal crossing* or *snc* at a point $p \in X$ if X is smooth at p and there are local coordinates $x_1, \dots, x_r, x_{r+1}, \dots, x_n$ and an injection $\sigma : \{1, \dots, r\} \to \{1, \dots, m\}$ such that

(1) $I_{\sigma(i)} = (x_i)$ near p for $1 \le i \le r$ and $p \notin \operatorname{cosupp} I_j$ for every other I_j;
(2) $\operatorname{Supp} E \subset (\prod_{i>r} x_i = 0)$ near p.

Thus $E + \sum_j \operatorname{cosupp} I_j$ has snc support near p, but we also assume that no two of $E, \operatorname{cosupp} I_1, \dots, \operatorname{cosupp} I_m$ have a common irreducible component near p. Furthermore, the I_j are assumed to vanish with multiplicity 1, but we do not care about the multiplicities in E. The definition is chosen mainly to satisfy the following restriction property:

(3) Assume that I_1 is the ideal sheaf of a smooth divisor $S \subset X$, $E + S$ is a snc divisor and that none of the irreducible components of S is contained in E or in $\operatorname{cosupp} I_j$ for $j > 1$. Then $(X, I_1, \dots, I_m, E)$ is snc near S iff $(S, I_2|_S, \dots, I_m|_S, E|_S)$ is snc. □

The set of all points where $(X, I_1, \dots, I_m, E)$ is snc is open. It is denoted by $\operatorname{snc}(X, I_1, \dots, I_m, E)$.

DEFINITION 78. Let $Z \subset X$ be a smooth, irreducible subvariety that has simple normal crossing with E (cf. [**Kol07**, 3.25]). Let $\pi : B_Z X \to X$ denote the blow-up with exceptional divisor $F \subset B_Z X$. Define the *birational transform* of $(X, I_1, \dots, I_m, E)$ as

$$(X' := B_Z X, I'_1, \dots, I'_m, E' := \pi^{-1}_{\mathrm{tot}} E) \tag{78.1}$$

where $I'_j = g^* I_j(-F)$ if $Z \subset \operatorname{cosupp} I_j$ and $I'_j = g^* I_j$ if $Z \not\subset \operatorname{cosupp} I_j$. Note that if Z has codimension 1, then $X' = X$ but $I'_j = I_j(-Z)$ whenever $Z \subset \operatorname{cosupp} I_j$.

By an elementary computation, the birational transform commutes with restriction to a smooth subvariety (cf. [**Kol07**, 3.62]). As in [**Kol07**, 3.29] we can define blow-up sequences.

The assertion (75.2) will be a special case of the following result.

PROPOSITION 79. *Let X be a smooth variety, E an snc divisor on X and I_j ideal sheaves of Cartier divisors. Then there is a smooth blow-up sequence*

$$\Pi : (X_r, I_1^{(r)}, \dots, I_m^{(r)}, E^{(r)}) \to \cdots \to (X_1, I_1^{(1)}, \dots, I_m^{(1)}, E^{(1)}) = (X, I_1, \dots, I_m, E)$$

such that

(1) $(X_r, I_1^{(r)}, \dots, I_m^{(r)}, E^{(r)})$ *has snc everywhere,*
(2) *for every j,* $\operatorname{cosupp} I_j^{(r)}$ *is the birational transform of (the closure of)* $\operatorname{cosupp} I_j \cap \operatorname{snc}(X, I_1, \dots, I_m, E)$, *and*
(3) Π *is an isomorphism over* $\operatorname{snc}(X, I_1, \dots, I_m, E)$.

Proof. The proof is by induction on $\dim X$ and on m.

Step 79.i. Reduction to the case where I_1 is the ideal sheaf of a smooth divisor.

Apply order reduction [**Kol07**, 3.107] to I_1. (Technically, to the marked ideal $(I_1, 2)$; see [**Kol07**, Sec.3.5].) In this process, we only blow up a center Z if the (birational transform of) I_1 has order ≥ 2 along Z. These are contained in the non-snc locus. A slight problem is that in [**Kol07**, 3.107] the transformation rule used is $I_1 \mapsto \pi^* I_1(-2F)$ instead of $I_1 \mapsto \pi^* I_1(-F)$ as in (78.1). Thus each blow-up for $(I_1, 2)$ corresponds to two blow ups in the sequence for Π: first we blow up $Z \subset X$ and then we blow up $F \subset B_Z X$.

At the end the maximal order of $I_1^{(r)}$ becomes 1. Since $I_1^{(r)}$ is the ideal sheaf of a Cartier divisor, $\operatorname{cosupp} I_1^{(r)}$ is a disjoint union of smooth divisors.

Step 79.ii. Reduction to the case when (X, I_1, E) is snc.

The first part is an easier version of Step (79.iii), and should be read after it. Let S be an irreducible component of E. Write $E = S + E'$ and consider the restriction $(S, I_1|_S, E'|_S)$. By induction on the dimension, there is a blow-up sequence $\Pi_S : S_r \to \cdots \to S_1 = S$ such that $\bigl(S_r, (I_1|_S)^{(r)}, (E'|_S)^{(r)}\bigr)$ is snc and Π_S is an isomorphism over $\operatorname{snc}(S, I_1|_S, E'|_S)$. The "same" blow-ups give a blow-up sequence $\Pi : X_r \to \cdots \to X_1 = X$ such that $\bigl(X_r, I_1^{(r)}, E^{(r)}\bigr)$ is snc near S_r and Π is an isomorphism over $\operatorname{snc}(X, I_1, E)$.

We can repeat the procedure for any other irreducible component of E. Note that as we blow up, the new exceptional divisors are added to E, thus $E^{(s)}$ has more and more irreducible components as s increases. However, we only add new irreducible components to E that are exceptional divisors obtained by blowing up a smooth center that is contained in (the birational transform of) $\operatorname{cosupp} I_1$. Thus these automatically have snc with I_1. Therefore the procedure needs to be repeated only for the original irreducible components of E.

After finitely many steps, (X, I_1, E) is snc near E and X and $\operatorname{cosupp} I_1$ are smooth. Thus (X, I_1, E) is snc everywhere.

(If we want to resolve just one (X, I_j, E), we can do these steps in any order, but for a functorial resolution one needs an ordering of the index set of E and proceed systematically.)

Step 79.iii. Reduction to the case when $(X, I_1, \dots, I_m, E)$ is snc near $\operatorname{cosupp} I_1$.

Assume that (X, I_1, E) is snc. Set $S := \operatorname{cosupp}(I_1)$. If an irreducible component $S_i \subset S$ is contained in $\operatorname{cosupp} I_j$ for some $j > 1$ then we blow up S_i. This reduces $\operatorname{mult}_{S_i} I_1$ and $\operatorname{mult}_{S_i} I_j$ by 1. Thus eventually none of the irreducible components of S are contained in $\operatorname{cosupp} I_j$ for $j > 1$. Thus we may assume that the $I_j|_S$ are ideal sheaves of Cartier divisors for $j > 1$ and consider the restriction $(S, I_2|_S, \dots, I_m|_S, E|_S)$.

By induction there is a blow-up sequence $\Pi_S : S_r \to \cdots \to S_1 = S$ such that

$$\bigl(S_r, (I_2|_S)^{(r)}, \dots, (I_m|_S)^{(r)}, (E|_S)^{(r)}\bigr) \quad \text{is snc}$$

and Π_S is an isomorphism over $\operatorname{snc}(S, I_2|_S, \dots, I_m|_S, E|_S)$. The "same" blow-ups give a blow-up sequence $\Pi : X_r \to \cdots \to X_1 = X$ such that the restriction

$$\bigl(S_r, I_2^{(r)}|_{S_r}, \dots, I_m^{(r)}|_{S_r}, E^{(r)}|_{S_r}\bigr) \quad \text{is snc}$$

and Π is an isomorphism over $\operatorname{snc}(X, I_1, \dots, I_m, E)$. (Since we use only order 1 blow-ups, this is obvious. For higher orders, one would need the Going-up theorem [**Kol07**, 3.84], which holds only for D-balanced ideals. Every ideal of order 1 is D-balanced [**Kol07**, 3.83], that is why we do not need to worry about subtleties here.)

As noted in (77.3), this implies that

$$\bigl(X_r, \mathcal{O}_{X_r}(-S_r), I_2^{(r)}, \dots, I_m^{(r)}, E^{(r)}\bigr) \quad \text{is snc near } S_r.$$

Note, furthermore, that $S_r = \operatorname{cosupp} I_1^{(r)}$, hence $\big(X_r, I_1^{(r)}, \dots, I_m^{(r)}, E^{(r)}\big)$ is snc near $\operatorname{cosupp} I_1^{(r)}$.

Step 79.iv. Induction on m.

By Step 3, we can assume that $(X, I_1, \dots, I_m, E)$ is snc near $\operatorname{cosupp} I_1$. Apply (79) to $(X, I_2, \dots, I_m, E)$. The resulting $\Pi : X_r \to X$ is an isomorphism over $\operatorname{snc}(X, I_2, \dots, I_m, E)$. Since $\operatorname{cosupp} I_1$ is contained in $\operatorname{snc}(X, I_2, \dots, I_m, E)$, all the blow up centers are disjoint from $\operatorname{cosupp} I_1$. Thus $\big(X_r, I_1^{(r)}, \dots, I_m^{(r)}, E^{(r)}\big)$ is also snc.

Finally, we may blow up any irreducible component of $\operatorname{cosupp} I_j^{(r)}$ that is not the birational transform of an irreducible component of $\operatorname{cosupp} I_j$ which intersects $\operatorname{snc}(X, I_1, \dots, I_m, E)$. □

80 (Proof of (75)). Let D_j be the irreducible components of D. Set $I_j := \mathcal{O}_X(-D_j)$ and $E := \emptyset$. Note that (X, D) is snc at $p \in X$ iff $(X, I_1, \dots, I_m, E)$ is snc at $p \in X$.

If X is a variety, we can apply (79) to $(X, I_1, \dots, I_m, E)$ to get $\Pi : X_r \to X$ and $(X_r, I_1^{(r)}, \dots, I_m^{(r)}, E^{(r)})$. Note that $E^{(r)}$ contains the whole exceptional set of Π, thus the support of $D' = \Pi_*^{-1} D + \operatorname{Ex}(\Pi)$ is contained in $E^{(r)} + \sum_j \operatorname{cosupp} I_j^{(r)}$. Thus D' is snc. By (79.3), Π is an isomorphism over the snc locus of (X, D).

The resolution constructed in (79) commutes with smooth morphisms and with change of fields [**Kol07**, 3.34.1–2], at least if in (78) we allow reducible blow-up centers.

As in [**Kol07**, 3.42–45], we conclude that (75) and (79) also hold for algebraic and analytic spaces over a field of characteristic 0.

Starting with (X, D), the above proof depends on an ordering of the irreducible components of D. This is an artificial device, but I don't know how to avoid it. This is very much connected with the difficulies of dealing with general nc divisors. □

81. It should be noted that (75.2) fails for nc instead of snc. The simplest example is given by the *pinch point* $D := (x^2 = y^2 z) \subset \mathbb{A}^3 =: X$. Here (X, D) has nc outside the origin. At a point along the z-axis, save at the origin, D has 2 local analytic branches. As we go around the origin, these 2 branches are interchanged. We can never get rid of the pinch point without blowing up the z-axis.

Note that (X, D) is not snc along the z-axis, thus in constructing a log resolution as in (75.2), we are allowed to blow up the z-axis.

This leads to the following general problem:

PROBLEM 82. For each n, describe the smallest class of singularities $\mathcal{S}_n$ such that for every (X, D) of dimension n there is a proper birational morphism $f : X' \to X$ such that

(1) (X', D') has only singularities in $\mathcal{S}_n$, and
(2) f is an isomorphism over the nc locus of (X, D).

In dimension 2 we can take, up to étale equivalence, $\mathcal{S}_2 = \{(xy = 0) \subset \mathbb{A}^2\}$ and in dimension 3 we can almost certainly take

$$\mathcal{S}_3 = \{(xy = 0), (xyz = 0), (x^2 = y^2 z) \subset \mathbb{A}^3\}.$$

(Bierstone and Milman informed me that this can be proved using their method of resolution [**BM97**].) In higher dimensions, there is not even a clear conjecture on what $\mathcal{S}_n$ should be.

It is natural to deal with the problem inductively. For this it is better to view $\mathcal{S}_n$ as a set of polynomials in n-variables, and allowing any number of dummy variables.

One could then hope that $\mathcal{S}_n$ consists of $\mathcal{S}_{n-1}$ plus a few other polynomials $f_i(x_1,\dots,x_n)$ such that, for every i the singularities of $(f_i=0)\setminus\{0\}$ are in $\mathcal{S}_{n-1}$. That is, for every $(c_1,\dots,c_n)\neq(0,\dots,0)$, $f_i(x_1-c_1,\dots,x_n-c_n)$ is in $\mathcal{S}_{n-1}$ (up to an analytic change of coordinates).

Unfortunately, already in dimension 4, the situation is more complicated, as illustrated by the following example.

Example 83. In $\mathbb{A}^4_{xyzu}$ consider the hypersurface

$$H := \bigl((x+uy+u^2z)(x+\epsilon uy+\epsilon^2u^2z)(x+\epsilon^2uy+\epsilon u^2z)=0\bigr)$$

where ϵ is a primite 3rd root of unity. Note that $H\cap(u=c)$ consists of 3 planes intersecting transversally if $c\neq 0$ while $H\cap(u=0)$ is a triple plane. The $\mathbb{Z}_3$-action $(x,y,z,u)\mapsto(x,y,z,\epsilon u)$ permutes the 3 irreducible components of H and $\mathbb{A}^4_{xyzu}/\mathbb{Z}_3=\mathbb{A}^4_{xyzt}$ where $t=u^3$. By explicit computation, the image of H in the quotient is the irreducible hypersurface

$$D=\bigl(x^3+ty^3+t^2z^3-3txyz=0\bigr)\subset\mathbb{A}^4_{xyzt}.$$

The singular locus of D can be parametrized as

$$\operatorname{Sing} D=\operatorname{im}\bigl[(u,z)\mapsto(\epsilon u^2z,-(1+\epsilon)uz,z,u^3)\bigr].$$

Since the quotient map $H\to D$ is étale away from $(t=0)$, we conclude that $\bigl(\mathbb{A}^4_{xyzt},D\bigr)$ is nc outside $(t=0)$. By explicit computation,

$$(t=0)\cap\operatorname{Sing} D=(x=y=t=0).$$

The singularity $D\subset\mathbb{A}^4$ can not be improved by futher smooth blow-ups whose centers are disjoint from the nc locus. Indeed, in our case the complement of the nc locus is the z-axis, hence there are only 2 choices for such a smooth blow-up center.

(1) Blow up $(x=y=z=t=0)$. In the chart $x=x_1t_1, y=y_1t_1, z=z_1t_1, t=t_1$ we get the birational transform

$$D'=\bigl(x_1^3+t_1y_1^3+t_1^2z_1^3-3t_1x_1y_1z_1=0\bigr).$$

(2) Blow up $(x=y=t=0)$. In the chart $x=x_1t_1, y=y_1t_1, z=z_1, t=t_1$ we get the birational transform

$$D''=\bigl(t_1x_1^3+t_1^2y_1^3+z_1^3-3t_1x_1y_1z_1=0\bigr).$$

In both cases we get a hypersurface which is, up to a coordinate change, isomorphic to D.

Note, however, that not every birational morphism between smooth 4-folds is a composite of smooth blow-ups, and I do not know if the singularity of D can be improved by some other birational morphism.

The transversal singularity type along the z-axis is a degenerate cusp of multiplicity 2. Indeed, in the coordinates $x_2=x/z-(y/z)^2, y_2=y/z, t_2=t-\frac{1}{2}(y^3-3xy)$ the equation of $D\cap(z=c)$ (where $c\neq 0$) becomes

$$t_2^2+x_2^2(x_2+\tfrac{3}{4}y_2^2)=0.$$

Thus $\mathcal{S}_4$ also contains the 3-variable polynomial $x^2+y^2(y+z^2)$, which is, however, not in $\mathcal{S}_3$.

Definition 84 (Semi snc). The ideal local model of an snc $\mathbb{Q}$-divisor is given by $D = \sum_{i=1}^{n} a_i(x_i = 0)$ on $X = \mathbb{A}^n$. We can also view this as sitting on $\mathbb{A}^{n+1}$, where $X = (x_{n+1} = 0)$ and D is defined using the other coordinates.

Following this example, we can define a non-normal version of snc where $X \subset \mathbb{A}^{n+1}$ is defined by the product of some of the coordinates and D is defined using the remaining coordinates.

For $n = 2$ we get three possible local models.

(1) $S = (z = 0) \subset \mathbb{A}^3$ and $D = a_x(x|_S = 0) + a_y(y|_S = 0)$. This is the usual normal case.

(2) $S = (yz = 0) \subset \mathbb{A}^3$ and $D = a_x(x|_S = 0)$. Note that as a Weil divisor, D has two irreducible components, namely $D_1 := (x = y = 0)$ and $D_2 := (x = z = 0)$. The support of the Weil $\mathbb{R}$-divisor $a_1D_1 + a_2D_2$ is always snc, but the pair $(S, a_1D_1 + a_2D_2)$ is semi-snc only if $a_1 = a_2$. It is easy to see that $a_1D_1 + a_2D_2$ is $\mathbb{R}$-Cartier only if $a_1 = a_2$.

(3) $S = (xyz = 0) \subset \mathbb{A}^3$ and $D = 0$.

Let Y be a smooth variety and $\sum_{i \in I} B_i$ a snc divisor. Let $I_X, I_D \subset I$ be disjoint subsets and $c : I_D \to \mathbb{R}$ a function. Then

$$X := \sum_{i \in I_X} B_i$$

is an snc divisor on Y, which we view now as a subscheme, and

$$D := \sum_{i \in I_D} c(i) B_i|_X$$

is a Weil (even Cartier) $\mathbb{R}$-divisor on X. We call any such (X, D) an *embedded semi-snc* pair.

Let X be a reduced variety and D a Weil $\mathbb{Q}$-divisor on X. We say that (X, D) is *semi-snc* if every point $x \in X$ has an open neighborhood $x \in U$ such that $(U, D|_U)$ is isomorphic to an embedded semi-snc pair.

Note that, by our definition, neither of the following examples are semi-snc:

$$\big((xy = 0), (x = y = 0)\big) \subset \mathbb{A}^3 \quad \text{or} \quad \big((xy = 0), (x = z = 0)\big) \subset \mathbb{A}^3.$$

As in (73), one can also define *semi-nc*.

85 (Semi log resolutions). What is the right notion of resolution or log resolution for non-normal varieties?

The simplest choice is to make no changes and work with resolutions. In particular, if $X = \cup_i X_i$ is a reducible scheme and $f : X' \to X$ is a resolution then $X' = \cup_i X'_i$ such that each $X'_i \to X_i$ is a resolution. Note that we have not completely forgotten the gluing data determining X since $f^{-1}(X_i \cap X_j)$ is part of the exceptional set, and so we keep track of it.

There are, however, several inconvenient aspects. For instance, $f_*\mathcal{O}_{X'} \neq \mathcal{O}_X$, and this makes it difficult to study the Picard group of X or the cohomology of line bundles on X using X'. Another problem is that although $\operatorname{Ex}(f)$ tells us which part of X_i intersects the other components, it does not tell us anything about what the actual isomorphism is between $(X_i \cap X_j) \subset X_i$ and $(X_i \cap X_j) \subset X_j$.

It is not clear how to remedy these problems for an arbitrary reducible scheme, but we are dealing with with schemes that have only double normal crossing in codimension 1.

We can thus look for $f : X' \to X$ such that X' has only double normal crossing singularities and f is an isomorphism over codimension 1 points of X.

As in (75), this works for simple nc but not in general. We need to allow at least pinch points.

DEFINITION 86 (Pinch points). Let X be a smooth variety over a field of characteristic $\neq 2$ and $D \subset X$ a divisor. We say that D has a *pinch point* at $p \in D$ if, in suitable local coordinates, D can be defined by the equation $x_1^2 - x_2^2 x_3 = 0$.

Note that this notion is invariant under field extensions and even completion. Indeed, if the singular set of D is a codimension 2 smooth subvariety, then D can be locally given by an equation $ax_1^2 + bx_1x_2 + cx_2^2 = 0$ where a, b, c are regular functions. If the quadratic part of the equation is a square times a unit, then, after a coordinate change, we can write the equation as $x_1^2 + cx_2^2 = 0$. This gives a pinch point after a field extension and completion iff the linear term of c is independent of x_1, x_2. Thus we can take $x_3 = -c$ to get the equation $x_1^2 - x_2^2 x_3 = 0$.

Let us blow up $Z := (x_1 = x_2 = 0)$. The normalization of D is contained in the affine charts with coordinates $x_1' := x_1/x_2, x_2, \dots, x_n$. If we introduce $x_3' := x_3 - {x_1'}^2$ then the normalization of D is given by $(x_3' = 0)$. The preimage of Z is the smooth divisor $x_2 = 0$ and the involution on it is $(x_1', 0, 0, x_4, \dots, x_n) \mapsto (-x_1', 0, 0, x_4, \dots, x_n)$.

A function f defines a τ-invariant divisor iff

$$f(x_1', x_2, x_4, \dots, x_n) = \begin{cases} g({x_1'}^2, x_4, \dots, x_n) + x_2 h(x_1', x_2, x_4, \dots, x_n), & \text{or} \\ x_1' g({x_1'}^2, x_4, \dots, x_n) + x_2 h(x_1', x_2, x_4, \dots, x_n). \end{cases}$$

In the first case f is τ-invariant and descends to a regular function on D. In the second case f is not τ-invariant, but f^2 descends to a regular function on D.

In particular, $(x_1 = x_3 = 0) \subset (x_1^2 = x_2^2 x_3)$ is not a Cartier divisor but it is $\mathbb{Q}$-Cartier since $2(x_1 = x_3 = 0) = (x_3 = 0)$ is Cartier.

Conversely, let Y be a smooth variety, $B \subset Y$ a smooth divisor and τ an involution on B whose fixed point set $F \subset B$ has pure codimension 1 in B. Let $Z := B/\tau$ and X the universal push-out of $Z \leftarrow B \hookrightarrow Y$, cf. [**Art70**, Thm.3.1]. Then X has only nc and pinch points.

To see this, pick a point $p \in F$ and local coordinates $y_1, \dots, y_n$ such that $B = (y_1 = 0)$, $\tau^* y_2|_B = -y_2|_B$ and $\tau^* y_i|_B = y_i|_B$ for $i > 2$. Then

$$x_1 := y_1 y_2, x_2 := y_1, x_3 := y_2^2 \quad \text{and} \quad x_i := y_{i-1} \quad \text{for } i > 3$$

give local coordinates on X with the obvious equation $x_1^2 - x_2^2 x_3 = 0$.

THEOREM 87. *Let X be a reduced scheme over a field of characteristic 0. Let $X^{ncp} \subset X$ be an open subset such that X^{ncp} has only smooth points $(x_1 = 0)$, double nc points $(x_1^2 - ux_2^2 = 0)$ and pinch points $(x_1^2 - x_2^2 x_3 = 0)$. Then there is a projective birational morphism $f : X' \to X$ such that*

(1) *X' has only smooth points, double nc points and pinch points,*
(2) *f is an isomorphism over X^{ncp},*
(3) $\operatorname{Sing} X'$ *maps birationally onto the closure of* $\operatorname{Sing} X^{ncp}$.

If X' has any pinch points then they are on an irreducible component of $B \subset \operatorname{Sing} X'$ along which X' is nc but not snc. Then, by (87.3), X is nc but not snc along $f(B)$. Thus we obtain the following simple nc version.

COROLLARY 88. *Let X be a reduced scheme over a field of characteristic 0. Let $X^{snc2} \subset X$ be an open subset which has only smooth points ($x_1 = 0$) and simple nc points of multiplicity ≤ 2 ($x_1x_2 = 0$). Then there is a projective birational morphism $f : X' \to X$ such that*

(1) *X' has only smooth points and simple nc points of multiplicity ≤ 2,*
(2) *f is an isomorphism over X^{snc2},*
(3) *$\operatorname{Sing} X'$ maps birationally onto the closure of $\operatorname{Sing} X^{snc2}$.* □

89 (Proof of (87)). The method of [**Hir64**] reduces the multiplicity of a scheme starting with the highest multiplicity locus. We can use it to find a proper birational morphism $g_1 : X_1 \to X$ such that every point of X_1 has multiplicity ≤ 2 and g_2 is an isomorphism over X^{ncp}. Thus by replacing X by X_1 we may assume to start with that every point of X has multiplicity ≤ 2.

The next steps of the Hironaka method would not distinguish the nc locus (that we want to keep intact) from the other multiplicity 2 points (that we want to eliminate). Thus we proceed somewhat differently.

Let $n : \bar{X} \to X$ be the normalization with reduced conductor $\bar{B} \subset \bar{X}$.

Near any point of X, in local analytic or étale coordinates we can write X as

$$X = \big(y^2 = g(\mathbf{x})h(\mathbf{x})^2\big) \subset \mathbb{A}^{n+1}$$

where $(\mathbf{x}) := (x_1, \dots, x_n)$ and g has no multiple factors. (We allow g and h to have common factors.) The normalization is then given by

$$\bar{X} = \big(z^2 = g(\mathbf{x})\big) \quad \text{where } z = y/h(\mathbf{x}).$$

Here $\bar{B} = (h(\mathbf{x}) = 0)$ and the involution $\tau : (z, \mathbf{x}) \mapsto (-z, \mathbf{x})$ is well defined on $\bar{B}$. (By contrast, the τ action on $\bar{X}$ depends on the choice of the local coordinate system.)

Thus we have a pair $(Y_2, B_2) := (\bar{X}, \bar{B})$ plus an involution $\tau_2 : B_2 \to B_2$ such that for every $b \in B_2$ there is an étale neighborhood U_b of $\{b, \tau_2(b)\}$ such that τ_2 extends (nonuniquely) to an involution τ_{2b} of $(U_b, B_2|_{U_b})$.

Let us apply an étale local resolution procedure (as in [**Wło05**] or [**Kol07**]) to (Y_2, B_2). Let the first blow up center be $Z_2 \subset Y_2$. Since the procedure is étale local, we see that $U_b \cap Z_2$ is τ_{2b}-invariant for every $b \in B_2$. Let $Y_3 \to Y_2$ be the blow up of Z_2 and let $B_3 \subset Y_3$ be the birational transform of B_2. Then τ_2 lifts to an involution τ_3 of B_3 and the τ_{2b} lift to extensions on suitable neighborhoods. Moreover, the exceptional divisor of $Y_3 \to Y_2$ intersected with B_3 is τ_3-invariant. In particular, there is an ample line bundle L_3 on Y_3 such that $L_3|_{B_3}$ is τ_3-invariant.

At the end we obtain $g : Y_r \to Y_2 = \bar{X}$ such that

(1) Y_r is smooth and $\operatorname{Ex}(g) + B_r$ is an snc divisor,
(2) B_r is smooth and τ lifts to an involution τ_r on B_r, and
(3) there is a g-ample line bundle L such that $L|_{B_r}$ is τ_r-invariant.

The fixed point set of τ_r is a disjoint union of smooth subvarieties of B_r. By blowing up those components whose dimension is $< \dim B_r - 1$, we also achieve (after replacing $r + 1$ by r) that

(4) the fixed point set of τ_r has pure codimension 1 in B_r.

Let $Z_r := B_r/\tau_r$ and X_r the universal push-out of $Z_r \leftarrow B_r \hookrightarrow Y_r$. As we noted in (86), X_r has only nc and pinch points.

Further, let D be a divisor on Y_r such that $D|_{B_r}$ is τ_r-invariant. As noted in (86), $2D$ is the pull back of a Cartier divisor on X_r. In particular, if D is ample then X_r is projective. □

We would like not just a semi resolution of X but a log resolution of the pair (X, D). Thus we need to take into account the singularities of D as well. As we noted in (81), this is not obvious even when X is a smooth 3-fold. The following weaker version, which gives the expected result only for the codimension 1 part of the singular set of (X, D), will be sufficient for us.

THEOREM 90. *Let X be a reduced scheme over a field of characteristic 0 and D a Weil divisor on X. Let $X^{nc2} \subset X$ be an open subset which has only nc points of multiplicity ≤ 2 and $D|_{X^{nc2}}$ is smooth and disjoint from $\operatorname{Sing} X^{nc2}$. Then there is a projective birational morphism $f : X' \to X$ such that*

(1) *the local models for $\big(X', D' := f_*^{-1}(D) + \operatorname{Ex}(f)\big)$ are*
   (a) *(Smooth) $X' = (x_1 = 0)$ and $D' = (\prod_{i\in I} x_i = 0)$ for some $I \subset \{2, \dots, n+1\}$,*
   (b) *(Double nc) $X' = (x_1^2 - ux_2^2 = 0)$ and $D' = (\prod_{i\in I} x_i = 0)$ for some $I \subset \{3, \dots, n+1\}$, or*
   (c) *(Pinched) $X' = (x_1^2 = x_2^2 x_3)$ and $D' = (\prod_{i\in I} x_i = 0) + D_2$ for some $I \subset \{4, \dots, n+1\}$ where either $D_2 = 0$ or $D_2 = (x_1 = x_3 = 0)$.*

(2) *f is an isomorphism over X^{nc2}.*
(3) *$\operatorname{Sing} X'$ maps birationally onto the closure of $\operatorname{Sing} X^{nc2}$.*
(4) *Let $\bar X \to X$ be the normalization, $\bar B \subset \bar X$ the closure of the conductor of $\bar X^{nc2} \to X^{nc2}$ and $\bar D \subset \bar X$ the preimage of D. Then f is a log resolution of $\big(\bar X, \bar B + \bar D\big)$.*

As before, (90) implies the simple nc version:

COROLLARY 91. *Let X be a reduced scheme over a field of characteristic 0 and D a Weil divisor on X. Let $X^{snc2} \subset X$ be an open subset which has only snc points of multiplicity ≤ 2 and $D|_{X^{snc2}}$ is smooth and disjoint from $\operatorname{Sing} X^{snc2}$. Then there is a projective birational morphism $f : X' \to X$ such that*

(1) *the local models for $\big(X', D' := f_*^{-1}(D) + \operatorname{Ex}(f)\big)$ are*
   (a) *(Smooth) $X' = (x_1 = 0)$ and $D' = (\prod_{i\in I} x_i = 0)$ for some $I \subset \{2, \dots, n+1\}$, or*
   (b) *(Double snc) $X' = (x_1x_2 = 0)$ and $D' = (\prod_{i\in I} x_i = 0)$ for some $I \subset \{3, \dots, n+1\}$.*

(2) *f is an isomorphism over X^{snc2}.*
(3) *$\operatorname{Sing} X'$ maps birationally onto the closure of $\operatorname{Sing} X^{snc2}$.*

92 (Proof of (90)). First we use (87) to reduce to the case when X has only double nc and pinch points. Let $\bar X \to X$ be the normalization and $\bar B \subset \bar X$ the conductor. Here $\bar X$ and $\bar B$ are both smooth.

Next we want to apply embedded resolution to $(\bar X, \bar B + \bar D)$. One has to be a little careful with D since the preimage $\bar D \subset \bar X$ need not be τ-invariant.

As a first step, we move the support of $\bar D$ away from $\bar B$. As in [**Kol07**, 3.102] this is equivalent to multiplicity reduction for a suitable ideal $I_D \subset \mathcal{O}_{\bar B}$. Let us now apply multiplicity reduction for the ideal $I_D + \tau^* I_D$. All the steps are now τ-invariant, so at the end we obtain $g : Y_r \to \bar X$ such that $B_r + D_r + \operatorname{Ex}(g)$ has only snc along B_r and τ lifts to an involution τ_r.

As in the proof of (87), we can also assume that the fixed locus of τ_r has pure codimension 1 in B_r and that there is a g-ample line bundle L such that $L|_{B_r}$ is τ_r-invariant.

As in the end of (89), let X_r be the universal push-out of $B_r/\tau_r \leftarrow B_r \hookrightarrow Y_r$. Then (X_r, D'_r) has the required normal form along $\operatorname{Sing} X_r$. The remaining singularities of D'_r can now be resolved as in (75). □

The following analog of (75) is still open:

Problem 93. Let X be a reduced scheme over a field of characteristic 0 and D a Weil divisor on X. Let $X^{snc} \subset X$ be the largest open subset such that $(X^{snc}, D|_{X^{snc}})$ is semi snc. Is there a projective birational morphism $f : X' \to X$ such that

(1) (X', D') is semi snc and
(2) f is an isomorphism over X^{snc}?

The following weaker version is sufficient for many applications. We do not guarantee that $f : X' \to X$ is an isomorphism over X^{snc}, only that f is an isomorphism over an open subset $X^0 \subset X^{snc}$ that intersects every semi log canonical center of $(X^{snc}, D|_{X^{snc}})$. (One can see easily that the latter are exactly the irreducible components of intersections of irreducible components of X^{snc} and of $D|_{X^{snc}}$.) This implies that we do not introduce any "unnecessary" f-exceptional divisors with discrepancy -1. The latter is usually the key property that one needs.

Unfortunately, the proof only works in the quasi projective case.

Proposition 94. *Let X be a reduced quasi projective scheme over a field of characteristic 0 and D a Weil divisor on X. Let $X^0 \subset X$ be an open subset such that $(X^0, D|_{X^0})$ is semi snc. There is a projective birational morphism $f : X' \to X$ such that*

(1) *(X', D') is an embedded semi snc pair and*
(2) *f is an isomorphism over the generic point of every semi log canonical center of $(X^0, D|_{X^0})$.*

Proof. In applications it frequently happens that $X + B$ is a divisor on a variety Y and $D = B|_X$. Applying (75.2) to $(Y, X + B)$ gives (94). In general, not every (X, D) can be obtained this way, but one can achieve something similar at the price of introducing other singularities.

Take an embedding $X \subset \mathbb{P}^N$. Pick a finite set $W \subset X$ such that each semi log canonical center of $(X^0, D|_{X^0})$ contains a point of W.

Choose $d \gg 1$ such that the scheme theoretic base locus of $\mathcal{O}_{\mathbb{P}^N}(d)(-X)$ is X near every point of W. Taking a complete intersection of $(N - \dim X - 1)$ general members in $|\mathcal{O}_{\mathbb{P}^N}(d)(-X)|$, we obtain $Y \supset X$ such that Y is smooth at every point of W. (Here we use that X has only hypersurface singularities near W.)

For every D_i choose $d_i \gg 1$ such that the scheme theoretic base locus of $\mathcal{O}_{\mathbb{P}^N}(d_i)(-D_i)$ is D_i near every point of W. For each i, let $D_i^Y \in |\mathcal{O}_{\mathbb{P}^N}(d_i)(-D_i)|$ be a general member.

We have thus constructed a pair $(Y, X + \sum D_i^Y)$ such that

(1) $(Y, X + \sum D_i^Y)$ is snc near W, and
(2) $(X, \sum D_i^Y|_X)$ is isomorphic to $(X, \sum D_i)$ in a suitable neighborhood of W.

By (75.2) there is a semi log resolution of

$$f : (Y', X' + \sum B_i) \to (Y, X + \sum D_i^Y)$$

such that f is an isomorphism over an open neighborhood of W. Then $f|_{X'} : X' \to X$ is the log resolution we want. $\square$

Definition 95 (Total transform). Let X be a smooth variety and $D \subset X$ a nc divisor. An irreducible subvariety $Z \subset D$ is called a *closed stratum* if, at a general point $z \in Z$, the intersection of the local analytic branches of D that pass through z is Z. If D is snc, then Z is an irreducible component of the intersection of some of the irreducible components of D.

In general, Z can be singular. For smooth Z, let $\pi : B_Z X \to X$ denote the blow-up of Z with exceptional divisor $E_Z \subset B_Z X$. Let $D' \subset B_Z X$ denote the birational transform of D.

Then $E_Z + D' \subset B_Z X$ is a nc divisor, called the *total transform* of D in $B_Z X$.

Let $J_Z \subset \mathcal{O}_X$ denote the ideal sheaf of $Z \subset X$ and $I_Z \subset \mathcal{O}_D$ denote the ideal sheaf of $Z \subset D$. Then

$$D' = \operatorname{Proj}_D \sum_{m \geq 0} I_Z^m \quad \text{and} \quad E_Z = \operatorname{Proj}_Z \sum_{m \geq 0} S^m\big(J_Z/J_Z^2\big).$$

Note that a priori the total transform also depends on X, that is, the embedding of D into a smooth variety. We claim, however, that any two total transforms are canonically isomorphic. In fact, we construct the total transfrom for non-embedded nc schemes as well.

Thus let D be a nc scheme and $Z \subset D$ an irreducible, smooth, closed stratum. In the trivial case, when Z is an irreducible component of D, the total transform is D itself. Thus assume from now on that Z has codimension at least 2. This implies that D is singular along Z.

Let $I_Z \subset \mathcal{O}_D$ denote the ideal sheaf of $Z \subset D$. As in the embedded case, the birational transform of D is given by the blow-up of D along Z:

$$D' = B_Z D = \operatorname{Proj}_D \sum_{m \geq 0} I_Z^m.$$

The preimage of Z in D' is thus

$$Z' := \operatorname{Proj}_D \sum_{m \geq 0} I_Z^m / I_Z^{m+1}.$$

Since D is singular along Z, we know that $J_Z/J_Z^2 = I_Z/I_Z^2$. Thus

$$E_Z = \operatorname{Proj}_Z \sum_{m \geq 0} S^m\big(I_Z/I_Z^2\big),$$

and there is a natural injection $Z' \hookrightarrow E_Z$ coming from the surjections $S^m\big(I_Z/I_Z^2\big) \twoheadrightarrow I_Z^m/I_Z^{m+1}$. Thus we can glue D' and E_Z along Z' to obtain the total transform $\pi : E_Z + D' \to D$.

The total transform commutes with étale maps $D^* \to D$.

5. Ramified covers

In this section we study finite ramified morphisms between demi normal schemes. We consider only morphisms that are unramified over the generic points of the conductor. This restriction is satisfied in our applications, but the general case is needed in some other contexts [**?**].

Definition 96 (Ramified covers).

A finite morphism of demi normal schemes $\pi : \tilde{X} \to X$ is called a *ramified cover* of degree m if there is a dense open subset $U \subset X$ which contains the generic points of the conductor D_X such that π is étale and has degree m over $\pi_U : \tilde{U} \to U$. In this case there is an open subscheme $j : X^0 \hookrightarrow X$ whose complement has codimension ≥ 2 such that π is finite and flat of degree m over X^0. Indeed, we can take $X^0 = U \cup X^{ns}$. Set $\tilde{X}^0 = \pi^{-1}(X^0)$ and $\pi^0 : \tilde{X}^0 \to X^0$ the induced map. Since $\tilde{X}$ is S_2,

$$j_*\big(\pi^0_*\mathcal{O}_{\tilde{X}^0}\big) = \pi_*\mathcal{O}_{\tilde{X}}.$$

In particular, $\pi : \tilde{X} \to X$ is uniquely determined by the finite, flat morphism $\pi^0 : \tilde{X}^0 \to X^0$.

If $\pi_U : \tilde{U} \to U$ is Galois with Galois group G then the G action on $\tilde{U}$ extends to a proper G-action on $\tilde{X}$. The action is free on $\tilde{U}$, hence it is free on an open set that contains the generic points of the conductor $D_{\tilde{X}}$.

Conversely, let $\tilde{X}$ be a demi normal scheme with a proper action of a finite group G. The geometric quotient $X := \tilde{X}/G$ exists by (31). Let $\tilde{U} \subset \tilde{X}$ be the largest open set on which the G-action is free. Then $\tilde{U} \to \tilde{U}/G$ is étale. (We could take this as the definition of a free action.) Thus if $\tilde{U}$ contains the generic points of the conductor $D_{\tilde{X}}$ then $\tilde{X} \to X = \tilde{X}/G$ is a ramified cover of X.

Let $g : Y \to X$ be a morphism with Y demi normal. The pull back $\tilde{X} \times_X Y \to Y$ defines a finite, flat, ramified cover over $g^{-1}(X^0)$. Thus, if $Y \setminus g^{-1}(X^0)$ has codimension ≥ 2, then there is a unique ramified cover $\tilde{Y} \to Y$ that agrees with $\tilde{X} \times_X Y \to Y$ over $g^{-1}(X^0)$. There is always a morphism $\tilde{Y} \to \tilde{X} \times_X Y$ which is an isomorphism iff $\tilde{X} \times_X Y$ is S_2.

97 (Pull-back and push-forward of divisors). The pull back of a Weil divisor by π can be defined as follows.

Take any Weil divisor B on X, restrict it to X^0 as in (96), pull it back and then extend uniquely to a Weil divisor $\tilde{B} =: \pi^*B$ on $\tilde{X}$.

If B is Cartier, then π^*B is Cartier on $\tilde{X}$ and agrees with the usual pull back. Conversely, if π^*B is Cartier then $m \cdot B$ is also Cartier. Thus the pull back and the norm (**??**) take $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors to $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors.

98 (Hurwitz formula). (cf. [**Har77**, Sec.IV.2]) Let $g : X' \to X$ be a ramified cover of an n-dimensional demi normal scheme defined over a field k.

The *ramification divisor* of g is defined as

$$R(g) = \sum_{F \subset X'} R(F)[F] := \sum_{F \subset X'} \operatorname{length}_{k(F)}\big(\Omega_{X'/X}\big)_F[F], \tag{98.1}$$

where the summation is over all prime divisors of X' and $\Omega_{X'/X}$ denotes the sheaf of relative differentials. If $r(F)$ denotes the ramification index of g along F then $R(F) \geq r(F) - 1$. The ramification is called *tame* along F if $R(F) = r(F) - 1$. This holds iff $\operatorname{char} k(F)$ does not divide $r(F)$.

The support of $g\bigl(R(g)\bigr)$ is called the *branch divisor* of g.

The Hurwitz formula says that $K_{X'} = g^*K_X + R$. More generally, if Δ is any $\mathbb{Q}$-divisor on X, then the $\mathbb{Q}$-divisor $g^*\Delta - R$ makes sense and then

$$K_{X'} + g^*\Delta - R = g^*(K_X + \Delta). \tag{98.2}$$

Thus $K_X + \Delta$ is $\mathbb{Q}$-Cartier iff $K_{X'} + g^*\Delta - R$ is.

In general $g^*\Delta - R$ need not be effective, but there are three important special cases when it is effective and the pull back formula is very simple.

The first is when $R = 0$, that is, when g is unramified in codimension 1. Then

$$K_{X'} + g^*\Delta = g^*(K_X + \Delta). \tag{98.3}$$

The second is when g is tamely ramified and $\Delta = B + \Delta_1$ where B is an integral divisor whose support contains the branch divisor of g. Then $g^*(B)$ contains the support of R and $g^*B = R + \operatorname{red} g^*B$. Thus we obtain the pull back formula

$$K_{X'} + \operatorname{red} g^*B + g^*\Delta_1 = g^*(K_X + B + \Delta_1). \tag{98.4}$$

More generally, assume that for every $D_i \subset X$, the coeffcient of D_i in Δ is at least $1 - \frac{1}{r_i}$ for some $r_i \geq \sup_j\{e_{ij}\}$ where D'_{ij} are the irreducible components of g^*D_i and e_{ij} denotes the ramification index along D'_{ij}. We can then write $\Delta = \sum_i\bigl(1 - \frac{1}{r_i}\bigr)D_i + \Delta'$ where $\Delta' \geq 0$. This gives as the formula

$$K_{X'} + \sum_{ij}\bigl(1 - \tfrac{e_{ij}}{r_i}\bigr)D'_{ij} + g^*\Delta' = g^*\Bigl(K_X + \sum_i\bigl(1 - \tfrac{1}{r_i}\bigr)D_i + \Delta'\Bigr). \tag{98.5}$$

In particular, if $\Delta' = 0$ and $e_{ij}|r_i$ for every i, j then every coefficient of the pull-back also has the form $1 - \frac{1}{r_{ij}}$ for some integer r_{ij}.

The following general principle compares discrepancies under finite morphisms. A result of this type first appeared in [**Rei80**].

Proposition 99. *Let $g : X' \to X$ be a finite, separable morphism between n-dimensional normal varieties defined over a field k. Let Δ be a $\mathbb{Q}$-divisor on X and Δ' a $\mathbb{Q}$-divisor on X' such that $K_{X'} + \Delta' = g^*(K_X + \Delta)$. Then*

(1) $\operatorname{discrep}(X', \Delta') \geq \operatorname{discrep}(X, \Delta)$;
(2) $(\deg g)(\operatorname{discrep}(X, \Delta) + 1) \geq (\operatorname{discrep}(X', \Delta') + 1)$ *if one of the following conditions holds*
   (a) $\operatorname{char} k = 0$,
   (b) $\deg X'/X < \operatorname{char} k$, *or*
   (c) X'/X *is Galois and* $\operatorname{char} k \nmid \deg X'/X$.

Proof: Consider the fiber product diagram with exceptional divisors given below:

$$\begin{array}{ccccc} F & \subset & Y' & \stackrel{f'}{\to} & X' \\ \downarrow & & \downarrow h & & \downarrow g \\ E & \subset & Y & \stackrel{f}{\to} & X. \end{array} \tag{99.4}$$

Near the generic point of F we compute that

$$\begin{aligned} K_{Y'} &= {f'}^*(K_{X'}+\Delta') + a(F,X',\Delta')F \\ &= {f'}^* g^*(K_X+\Delta) + a(F,X',\Delta')F \\ &= h^* f^*(K_X+\Delta) + a(F,X',\Delta')F, \quad \text{and} \\ K_{Y'} &= h^* K_Y + R(F)F \\ &= h^* f^*(K_X+\Delta) + a(E,X,\Delta)h^*E + R(F)F \\ &= h^* f^*(K_X+\Delta) + \bigl(r(F)a(E,X,\Delta) + R(F)\bigr)F. \end{aligned}$$

This shows that

$$a(F,X',\Delta')+1 = r(F)\bigl(a(E,X,\Delta)+1\bigr) + \bigl(R(F)+1-r(F)\bigr).$$

Since $R(F)+1 \geq r(F) \geq 1$ this implies (1) if $\operatorname{discrep}(X,\Delta) \geq -1$. Otherwise $\operatorname{discrep}(X,\Delta) = -\infty$ and there is nothing to prove.

Conversely, if one of the conditions of (99.2.a–c) hold then $R(F)+1 = r(F)$ and so

$$\bigl(a(E,X,\Delta)+1\bigr) = \frac{1}{r(F)}\bigl(a(F,X',\Delta')+1\bigr) \geq \frac{1}{\deg g}\bigl(a(F,X',\Delta')+1\bigr).$$

We are done if these considerations apply to all possible divisors E and F. Given any divisor E over X, we get a divisor F over X' from the diagram (99.4). The converse is proved in [**KM98**, 2.45]. □

COROLLARY 100. *Notation and assumptions as above. If (X,Δ) is klt (resp. lc) then so is (X',Δ'). Conversely, if (X',Δ') is klt (resp. lc) then so is (X,Δ) provided one of the conditions of (99.2.a–c) hold.* □

Next we consider ramified covers with cyclic Galois group. These are easy to construct and especially useful in the study of slc pairs.

101 (μ_m-covers). Let $\pi : \tilde X \to X$ be a ramified cover with Galois group μ_m. Since μ_m is reductive, its action decomposes $\pi_*\mathcal{O}_{\tilde X}$ into a sum of eigensheaves L_i, each of rank 1. Multiplication gives maps $L_1^{\otimes i} \to L_i$, hence there are divisors D_i such that

$$\pi_*\mathcal{O}_{\tilde X} = \sum_{i=0}^{m-1} L_1^{[i]}(D_i). \tag{101.1}$$

The $i \equiv 0 \mod m$ eigensubsheaf is isomorphic to $\mathcal{O}_X$, hence we get an isomorphism $\gamma : L_1^{[m]}(D_m) \cong \mathcal{O}_X$.

Since L_1 tends to be negative, we usually choose $L := L_1^{[-1]}$ as our basic sheaf and $D := D_m$ as the key divisor. Then γ corresponds to a section s of $L^{[m]}$ whose zero divisor is D.

Conversely, let X be a demi normal scheme, L a divisorial sheaf on X and s a section of $L^{[m]}$ for some $m > 0$ that does not vanish along any irreducible component of the conductor $D_X \subset X$. Set $D := (s)$ and $\Delta := \frac{1}{m}D$. The section s can be identified with an isomorphism

$$\gamma_s : L^{[-m]}\bigl(\lfloor m\Delta\rfloor\bigr) = L^{[-m]}\bigl(D\bigr) \cong \mathcal{O}_X. \tag{101.2}$$

This in turn defines an algebra structure on

$$\mathcal{O}_X + L^{[-1]} + \cdots + L^{[-(m-1)]},$$

where, for $i+j<m$ the multiplication $L^{[-i]}\times L^{[-j]}\to L^{[-(i+j)]}$ is the tensor product and for $i+j\geq m$ we compose the tensor product with the isomorphisms

$$L^{[-(i+j)]}=L^{[-(i+j-m)]}\otimes L^{[-m]}\stackrel{1\otimes\gamma_s}{\longrightarrow}L^{[-(i+j-m)]}\otimes\mathcal{O}_X=L^{[-(i+j-m)]}.$$

The spectrum of this algebra gives $\tilde{X}$ over $X\setminus D$, but it is usually quite singular over D since we have not yet found the correct divisors D_i. By the universal property of the normalization, D_i is the largest divisor such that $\bigl(L^{[-i]}(D_i)\bigr)^{[m]}\subset\mathcal{O}_X$. That is, $D_i=\lfloor\frac{i}{m}D\rfloor=\lfloor i\Delta\rfloor$ and so

$$\tilde{X}=\operatorname{Spec}_X\Bigl(\mathcal{O}_X+L^{[-1]}\bigl(\lfloor\Delta\rfloor\bigr)+\cdots+L^{[-(m-1)]}\bigl(\lfloor(m-1)\Delta\rfloor\bigr)\Bigr).\tag{101.3}$$

We frequently write $\tilde{X}=:X\bigl[L,\sqrt[m]{s}\bigr]$ to emphasize its dependence on L and s. Alternatively, let I_s be the ideal sheaf of $\sum_{i=0}^{\infty}L^{[-i]}\bigl(\lfloor i\Delta\rfloor\bigr)$ generated by $\phi-\gamma_s(\phi)$ where ϕ is any local section of $L^{[-m]}\bigl(\lfloor m\Delta\rfloor\bigr)$. Then

$$X\bigl[L,\sqrt[m]{s}\bigr]=\operatorname{Spec}_X\Bigl(\sum_{i=0}^{\infty}L^{[-i]}\bigl(\lfloor i\Delta\rfloor\bigr)\Bigr)/I_s.\tag{101.4}$$

Duality for finite morphisms now gives that

$$\pi_*\omega_{\tilde{X}}=\omega_X+\omega_X\hat{\otimes}L\bigl(-\lfloor\Delta\rfloor\bigr)+\cdots+\omega_X\hat{\otimes}L^{[m-1]}\bigl(-\lfloor(m-1)\Delta\rfloor\bigr).\tag{101.5}$$

Note also that

$$\pi_*\pi^{[*]}\bigl(L(-\lfloor\Delta\rfloor)\bigr)=L\bigl(-\lfloor\Delta\rfloor\bigr)+\mathcal{O}_X+L^{[-1]}\bigl(\lfloor\Delta\rfloor\bigr)+\cdots+L^{[-(m-2)]}\bigl(\lfloor(m-2)\Delta\rfloor\bigr),$$

which shows that

$$\pi^{[*]}\bigl(L(-\lfloor\Delta\rfloor)\bigr)\cong\mathcal{O}_{\tilde{X}}.\tag{101.6}$$

102 (Normal forms of μ_m-covers). There are several ways to change L and s without changing the corresponding μ_m-cover.

First of all, if $(i,m)=1$ then the same cover is constructed if we think of the ith summand as the basic divisorial sheaf. That is using $L^i\bigl(\lfloor -i\Delta\rfloor\bigr)$ and the isomorphism

$$\bigl(L^{[i]}\bigl(\lfloor -i\Delta\rfloor\bigr)\bigr)^{[m]}=L^{[mi]}\bigl(m\lfloor -i\Delta\rfloor\bigr)\cong\mathcal{O}_X\bigl(mi\Delta+m\lfloor -i\Delta\rfloor\bigr).\tag{102.1}$$

Second, if D_0 is any divisor then by replacing L by $L(-D_0)$ and Δ by $\Delta-D_0$ gives the same μ_m-cover. Thus we can always assume that $\lfloor\Delta\rfloor=0$.

Finally, there is the choice of the isomorphism $s:\mathcal{O}_X\cong L^{[m]}(-m\Delta)$. Given two such isomorphisms s_i, their quotient $u:=s_1/s_2$ is a unit in $\mathcal{O}_X$. If $u=v^m$ is an mth power, then acting by v on L shows that the two μ_m-covers are isomorphic. Thus we should think of s as an element

$$\bar{s}\in H^0(X,\mathcal{O}_X)^{\otimes m}\setminus\operatorname{Isom}_X\bigl(\mathcal{O}_X(m\Delta),L^{[m]}\bigr).\tag{102.2}$$

Different choices of $\bar{s}$ can result in quite different covers. For instance, if $C=\mathbb{A}^1\setminus\{0\}$ with coordinate x then $C\bigl[\mathcal{O}_C,\sqrt[m]{1}\bigr]$ is the reducible plane curve $y^m=1$ while $C\bigl[\mathcal{O}_C,\sqrt[m]{x}\bigr]$ is the irreducible plane curve $y^m=x$.

If the residue characteristics do not divide m, then $X\bigl[\mathcal{O}_X,\sqrt[m]{u}\bigr]\to X$ is étale and the two μ_m-covers $X\bigl[L,\sqrt[m]{s_1}\bigr]$ and $X\bigl[L,\sqrt[m]{s_2}\bigr]$ become isomorphic after pulling back to $X\bigl[\mathcal{O}_X,\sqrt[m]{u}\bigr]$. In particular, they have isomorphic étale covers.

However, they can be quite different in positive characteristic. For $m = p$ the above example gives $C\bigl[\mathcal{O}_C, \sqrt[p]{1}\bigr]$ which is the nonreduced plane curve $y^p = 1$ while $C\bigl[\mathcal{O}_C, \sqrt[p]{x}\bigr]$ is the smooth plane curve $y^p = x$.

We can summarize these discussions as follows.

COROLLARY 103. *Let X be a demi normal scheme over a field k and $U \subset X$ an open subset which contains every generic point of the conductor D_X. Assume that* $\operatorname{char} k$ *does not divide m. Then there is a natural one-to-one correspondance between the following 3 sets.*

(1) *Étale Galois covers $\tilde U \to U$ plus an isomorphism* $\operatorname{Gal}(\tilde U/U) \cong \mu_m$.
(2) *Ramified Galois covers $\tilde X \to X$ whose branch divisor is in $X \setminus U$ plus an isomorphism* $\operatorname{Gal}(\tilde X/X) \cong \mu_m$.
(3) *Triples $(L, \Delta, \bar s)$ where*
    (a) *L is a divisorial sheaf on X,*
    (b) *Δ is a $\mathbb{Q}$-divisor whose support is in $X \setminus U$ such that $\lfloor \Delta \rfloor = 0$ and $m\Delta$ is a $\mathbb{Z}$-divisor and*
    (c) $\bar s \in H^0(X, \mathcal{O}_X)^{\otimes m} \setminus \operatorname{Isom}_X\bigl(\mathcal{O}_X(m\Delta), L^{[m]}\bigr)$. □

104 (Local properties of μ_m-covers). Given X and $(L, \Delta, \bar s)$ as in (103.3), let $\pi : \tilde X \to X$ be the corresponding μ_m-cover. Write $\Delta = \sum (m_i/r_i) D_i$ where $(m_i, r_i) = 1$ and assume that $\operatorname{char} k$ does not divide m.

By (71), for $x \in X$ the evaluation of the product

$$L^{[-i]}\bigl(\lfloor i\Delta \rfloor\bigr) \times L^{[i-m]}\bigl(\lfloor (m-i)\Delta \rfloor\bigr) \to L^{[-m]}\bigl(\lfloor m\Delta \rfloor\bigr) \otimes k(x) \cong k(x)$$

is zero, unless $i\Delta$ is a $\mathbb{Z}$-divisor near x and $L^{[-i]}\bigl(\lfloor i\Delta \rfloor\bigr)$ is locally free.

This implies the following:

(1) The number of preimages of x equals the number of indices $0 \le j < m$ such that $j\Delta$ is a $\mathbb{Z}$-divisor near x and $L^{[-j]}\bigl(j\Delta\bigr)$ is locally free at x.
(2) π is étale at $x \in X$ iff L is locally free at x and $x \notin \operatorname{Supp} \Delta$.
(3) The ramification index of π over D_i is r_i.

DEFINITION 105 (Index 1 covers). Let (X, Δ) be a demi normal pair. Write $\Delta = B + \Delta'$ where B is a $\mathbb{Z}$-divisor and $\lfloor \Delta' \rfloor = 0$.

The *index* of (X, Δ) at a point $x \in X$, denoted by $\operatorname{index}_x(X, \Delta)$ is the smallest positive integer m such that $m\Delta$ is a $\mathbb{Z}$-divisor and $\omega_X^{[m]}(m\Delta)$ is locally free at x. (If there is no such m, set $\operatorname{index}_x(X, \Delta) = \infty$.)

For a subset $Z \subset X$, let $\operatorname{index}_Z(X, \Delta)$ be the least common multiple of $\operatorname{index}_x(X, \Delta)$ for all $x \in Z$. We write $\operatorname{index}(X, \Delta) := \operatorname{index}_X(X, \Delta)$

Thus $K_X + \Delta$ is $\mathbb{Q}$-Cartier iff $\operatorname{index}(X, \Delta) < \infty$.

Pick a point $x \in X$ and set $m = \operatorname{index}_x(X, \Delta)$. After replacing X with an open neighborhood of x, we may assume that there is an isomorphism $s : \mathcal{O}_X \cong \omega_X^{[m]}(m\Delta)$. Thus $L := \omega_X^{[-1]}(-B)$, Δ' and s determine a μ_m-cover $\pi : \tilde X \to X$. Since m is the smallest, (104.1) implies that $\pi^{-1}(x)$ consists of a single point $\tilde x$, hence we get a pointed scheme $(\tilde x \in \tilde X)$. Note that $(\tilde x \in \tilde X)$ depends on s if we work Zariski locally, but it does not depend on s if we work étale locally. Thus, in the latter case, we can talk about *the* index 1 cover of $(x \in X, \Delta)$.

$$\pi_* \mathcal{O}_{\tilde X} = \sum_{i=0}^{m-1} \omega_X^{[i]}\bigl(iB + \lfloor i\Delta' \rfloor\bigr) \quad \text{and} \quad \pi_* \omega_{\tilde X}\bigl(\tilde B\bigr) = \sum_{i=0}^{m-1} \omega_X^{[1-i]}\bigl((1-i)B - \lfloor i\Delta' \rfloor\bigr).$$

As in (101.6), the $i = 1$ summand shows that $\omega_{\tilde{X}}(\tilde{B}) \cong \mathcal{O}_{\tilde{X}}$. Furthermore, the μ_m-action on $\omega_{\tilde{X}}(\tilde{B}) \otimes k(\tilde{x})$ is the standard representation, hence it is faitful.

THEOREM 106. *In each of the following 4 cases, taking the index 1 cover gives a natural one-to-one correspondence between the sets described in (a) and (b). Local is understood in the étale topology and* $\operatorname{char} k(x) \not| m$ *is always assumed.*

(1) (a) *Local demi normal schemes* $(x \in X)$ *such that* $\operatorname{index}_x X = m$.
(b) *Local demi normal schemes* $(\tilde{x} \in \tilde{X})$ *such that* $\omega_{\tilde{X}}$ *is localy free with a proper* μ_m*-action that is free outside a codimension* ≥ 2 *subset and the induced action on* $\omega_{\tilde{X}} \otimes k(\tilde{x})$ *is faithful.*

(2) (a) *Local demi normal pairs* $(x \in X, B)$ *such that* $\operatorname{index}_x(X, B) = m$.
(b) *Local demi normal pairs* $(\tilde{x} \in \tilde{X}, \tilde{B})$ *such that* $\omega_{\tilde{X}}(\tilde{B})$ *is localy free with a proper* μ_m*-action that is free outside a codimension* ≥ 2 *subset and the induced action on* $\omega_{\tilde{X}}(\tilde{B}) \otimes k(\tilde{x})$ *is faithful.*

(3) (a) *Local demi normal pairs* $\big(x \in X, \Delta\big)$ *where* $\Delta = \sum_i (1 - \frac{1}{r_i}) D_i$ *with* $r_i \in \mathbb{N}$ *such that* $\operatorname{index}_x(X, \Delta) = m$.
(b) *Local demi normal schemes* $(\tilde{x} \in \tilde{X})$ *such that* $\omega_{\tilde{X}}$ *is localy free with a proper* μ_m*-action that is free on a dense open subset that contains all generic points of the conductor* $D_{\tilde{X}}$ *and the induced action on* $\omega_{\tilde{X}} \otimes k(\tilde{x})$ *is faithful.*

(4) (a) *Local demi normal pairs* $\big(x \in X, B + \Delta\big)$ *where* $\Delta = \sum_i (1 - \frac{1}{r_i}) D_i$ *with* $r_i \in \mathbb{N}$ *such that* $\operatorname{index}_x(X, B + \Delta) = m$.
(b) *Local demi normal pairs* $(\tilde{x} \in \tilde{X}, \tilde{B})$ *such that* $\omega_{\tilde{X}}(\tilde{B})$ *is localy free with a proper* μ_m*-action that is free on a dense open subset that contains all generic points of* $\tilde{B} + D_{\tilde{X}}$ *and the induced action on* $\omega_{\tilde{X}}(\tilde{B}) \otimes k(\tilde{x})$ *is faithful.*

Moreover, in all cases the pair $(X, 0)$ *(resp.* (X, B), (X, Δ), $(X, B + \Delta)$*) is klt (or lc or slc) iff the index 1 cover* $(\tilde{X}, 0)$ *(resp.* $(\tilde{X}, \tilde{B})$, $(\tilde{X}, 0)$, $(\tilde{X}, \tilde{B})$*) is klt (or lc or slc).*

Proof. Strating with $(X, B + \Delta)$, the conctruction of $(\tilde{X}, \tilde{B})$ was done in (105) and we also saw that $\omega_{\tilde{X}}(\tilde{B})$ is localy free and induced μ_m-action on $\omega_{\tilde{X}}(\tilde{B}) \otimes k(\tilde{x})$ is faithful.

The pull back of the canonical class is computed in (98) and (99) shows the last claim about the properties klt, lc or slc. □

The following two special cases are especially important.

COROLLARY 107. (1) *A singularity* $(x \in X)$ *is lt iff it is a quotient of an index 1 canonical singularity* $\big(\tilde{x} \in \tilde{X}\big)$ *by a proper* μ_m*-action that is free outside a codimension* ≥ 2 *subset.*

(2) *A singularity* $\big(x \in X, \Delta\big)$ *where* $\Delta = \sum_i (1 - \frac{1}{r_i}) D_i$ *with* $r_i \in \mathbb{N}$ *is klt iff it is a quotient of an index 1 canonical singularity* $(\tilde{x} \in \tilde{X})$ *by a proper* μ_m*-action.* □

6. Canonical rings of normal crossing surfaces

In this section we show, following [**?**], that the minimal model program does not work for varieties with semi log canonical singularities. Problems arise even for surfaces with normal crossing singularities.

Proposition 108. *There are irreducible, projective surfaces of general type with only normal crossing singularities whose canonical ring is not finitely generated.*

Proof. Let $S \to \mathbb{P}^1 \times \mathbb{P}^1$ be a double cover ramified along a curve B of bidegree $(6,6)$ and $\pi_i : S \to \mathbb{P}^1$ the coordinate projections. The canonical class of S is the pull-back of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$, hence ample.

Pick distinct points $p, q, r_1, \dots, r_6 \in \mathbb{P}^1$ and choose B such that

$$B \cap \{p\} \times \mathbb{P}^1 = \{p\} \times \{r_1, r_2, r_3, r_4, 2r_5\} \quad \text{and} \quad B \cap \{q\} \times \mathbb{P}^1 = \{q\} \times \{r_1, r_2, r_3, r_4, 2r_6\}.$$

Set $F_p := \pi_1^{-1}(p)$ and $F_q := \pi_1^{-1}(q)$ with normalizations of $\bar{F}_p, \bar{F}_q$. Note that F_p is singular at $p_0 = (p, r_5)$ and F_q is singular at $q_0 = (q, r_6)$. Furthermore, $\pi_2 : \bar{F}_p \to \mathbb{P}^1$ and $\pi_2 : \bar{F}_q \to \mathbb{P}^1$ both ramify over the points $\{r_1, r_2, r_3, r_4\}$, hence there are two isomorphisms $\tau_F, \tau'_F : \bar{F}_p \cong \bar{F}_q$ that commute with π_2. Let $p_1, p_2 \in F_p \subset S$ be the 2 preimages of (p, r_6) and $q_1, q_2 \in F_q \subset S$ be the 2 preimages of (p, r_5).

As in (23), the triple $(S, F_p + F_q, \tau_F)$ defines a demi-normal surface $T := S/R(\tau_F)$. Note that T has 2 triple points, P with preimages $p_0, q_1, q_2 \in S$ and Q with preimages $q_0, p_1, p_2 \in S$. Set $Z := \{p_0, q_1, q_2, q_0, p_1, p_2\}$. Then τ_F is an involution on $(F_p + F_q) \setminus Z$ and $T \setminus \{P, Q\}$ has only double nc points. Note that T is a double cover of $\big(\mathbb{P}^1/(p \sim q)\big) \times \mathbb{P}^1$, hence projective. By (9),

$$H^0\big(T, \omega_T^{[m]}\big) = \Big\{ s \in H^0\big(S \setminus Z, \omega_{S \setminus Z}^m(mF_p + mF_q)\big) : s|_{F_p + F_q} \text{ is } (-1)^m \tau_F\text{-invariant} \Big\}.$$

We show next that the canonical ring $\sum_{m \geq 0} H^0\big(T, \omega_T^{[m]}\big)$ is not finitely generated. This is, however, caused by the singularities of T; in fact, K_T is not even $\mathbb{Q}$-Cartier. So at the end we construct a surface T_1 with only normal crossing singularities whose canonical ring is isomorphic to the canonical ring of T.

Near the two triple points $P, Q \in T$, we are in the situation described in (108.1). In particular, we know that the $\mathcal{O}_T$-algebra $\sum_{m \geq 0} \omega_T^{[m]}$ is not finitely generated, not even locally near P or Q.

Local computation (108.1) Let $C_1 := (xy = 0) \subset \mathbb{C}^2_{x,y} =: S_1$. Let $C_{21} := (u_1 = 0) \subset \mathbb{C}^2_{u_1, v_1} =: S_{21}$ and $C_{22} := (v_2 = 0) \subset \mathbb{C}^2_{u_2, v_2} =: S_{22}$. Set $S_2 := S_{21} \amalg S_{22}$ and $C_2 := C_{21} \amalg C_{22}$

The gluing is defined by $\sigma : C_1 \setminus (0,0) \to C_2$ sending $(0, y) \mapsto (0, y) \in C_{21}$ and $(x, 0) \mapsto (x, 0) \in C_{22}$.

Note that $T := (S_1 \amalg S_2)/\sigma$ is not a nc surface. Rather, it has a triple point with embedding dimension 4. A local model is given by

$$(t_1 = t_2 = 0) \cup (t_2 = t_3 = 0) \cup (t_3 = t_4 = 0) \subset \mathbb{C}^4.$$

The isomorphism is given by $(x, y) \mapsto (0, x, y, 0)$, $(u_1, v_1) \mapsto (v_1, u_1, 0, 0)$ and $(u_2, v_2) \mapsto (0, 0, v_2, u_2)$.

A local generator of $\omega_{S_{21}}(C_{21})$ is $u_1^{-1} du_1 \wedge dv_1$, and the restriction $\omega_{S_{21}}(C_{21})|_{C_{21}} = \omega_{C_{21}}$ is given by the Poincaré residue map

$$\tfrac{df}{f} \wedge dg|_{(f=0)} \mapsto dg|_{(f=0)}.$$

Thus $\omega_{S_{21}}^m(mC_{21})|_{C_{21}} = (dv_1)^m \cdot \mathcal{O}_{C_{21}}$. The situation on C_{22} is similar.

On the other hand, a local generator of $\omega_{S_1}(C_1)$ is $(xy)^{-1} dx \wedge dy$. Its restriction to C_1 gives a local generator η of ω_{C_1}. Note that

$$\eta|_{(y=0)} = -\tfrac{dx}{x} \quad \text{and} \quad \eta|_{(x=0)} = \tfrac{dy}{y}. \tag{108.2}$$

Thus

$$\omega_{S_1}^m(mC_1)|_{C_1} = \eta^m \cdot \mathcal{O}_{C_1}.$$

The interesting feature appears when we compute that

$$\sigma^*(dv_1)^m = y^m \cdot \eta|_{(x=0)} \quad \text{and} \quad \sigma^*(du_2)^m = (-x)^m \cdot \eta|_{(y=0)}.$$

Thus the image of the restriction map

$$\omega_T^m \to \omega_{S_1}^m(mC_1) \quad \text{is} \quad (xy, x^m, y^m) \cdot \left(\tfrac{dx\wedge dy}{xy}\right)^m. \tag{108.3}$$

Local finite generation fails since the $\mathbb{C}[x,y]$-algebra

$$\sum_{m\geq 0}(xy, x^m, y^m)\cdot w^m \subset \mathbb{C}[x,y,w] \quad \text{is not finitely generated,}$$

where $w = (xy)^{-1}dx\wedge dy$ is a formal variable taking care of the grading. Indeed, for every m, the element $xy\cdot w^m$ needs to be added as a new generator.

To go from the local infinite generation to global infinite generation we consider the natural map

$$\rho : \sum_{m\geq 0} H^0(T, \omega_T^{[m]}) \to \sum_{m\geq 0}\omega_T^{[m]}.$$

Assume that for all $m \gg 1$ there are global sections $t_m \in H^0(T, \omega_T^{[m]})$ such that $\rho(t_m)$ is not contained in the subsheaf of $\omega_T^{[m]}$ generated by the $\omega_T^{[i]}$ for $i < m$. Then t_m is not contained in the subalgebra generated by the $H^0(T, \omega_T^{[i]})$ for $i < m$, hence $\sum_{m\geq 0} H^0(T, \omega_T^{[m]})$ is not finitely generated.

Since ω_S is ample and F_p, F_q are nef, we see that $\omega_S^m(mF_p + mF_q)(-F_p - F_q)$ is globally generated for $m \gg 1$. Sections of $\omega_S^m(mF_p + mF_q)(-F_p - F_q)$ vanish along $F_p + F_q$, hence they automatically glue and descend to sections of $\omega_T^{[m]}$.

Thus if $s_m \in H^0(S, \omega_S^m(mF_p + mF_q))$ vanishes along $F_p + F_q$ with multiplicity 1, then we obtain a corresponding $t_m \in H^0(T, \omega_T^{[m]})$ which, up to a unit, equals $xy\cdot\big((xy)^{-1}dx\wedge dy\big)^m$ in (108.3). Thus $\sum_{m\geq 0} H^0(T, \omega_T^m)$ is not finitely generated.

Finally we construct T_1. Let $S_1 \to S$ be obtained by blowing up p_1, p_2, q_1, q_2 with the corresponding exceptional curves are $E_{p_1}, E_{p_2}, E_{q_1}, E_{q_2} \subset S_1$. The normalization of our surface will be S_1 with conductor $D_1 := \bar F_p + \bar F_q + E_{p_1} + E_{p_2} + E_{q_1} + E_{q_2}$. Fix isomorpisms $\tau_p : E_{p_1} \cong E_{p_2}$ and $\tau_q : E_{q_1} \cong E_{q_2}$ that map $\bar F_p \cap E_{p_1}$ to $\bar F_p \cap E_{p_2}$ and $\bar F_q \cap E_{q_1}$ to $\bar F_q \cap E_{q_2}$. Let τ_1 be the involution on the normalization of D_1 which is τ_F on $\bar F_p + \bar F_q$, τ_p on $E_{p_1} + E_{p_2}$ and τ_q on $E_{q_1} + E_{q_2}$.

Set $T_1 := S_1/R(\tau_1)$ with normalization map $n : S_1 \to T_1$. T_1 has only normal crossing singularities, 2 of them triple points. By (8),

$$n^*\omega_{T_1} = \omega_{S_1}(D_1) = \omega_{S_1}(\bar F_p + \bar F_q + E_{p_1} + E_{p_2} + E_{q_1} + E_{q_2})$$

and this line bundle has negative degree along the 4 curves $E_{q_1}, E_{q_2}, E_{p_1}, E_{p_2}$. Therefore, every section of $\omega_{S_1}^m(mD_1)$ is the pull-back of a section of $\omega_S^m\big(mF_p + mF_q\big)$. Therefore

$$\sum_{m\geq 0} H^0\big(T_1, \omega_{T_1}^m\big) = \sum_{m\geq 0} H^0\big(T, \omega_T^{[m]}\big).$$

Note that T_1 is projective since T is projective that $\omega_{T_1}^{-1}$ is relatively ample on $T_1 \to T$. □

Note 109. The explicit computation in (**??**) is a special case of the following general result:

Let X be a reduced, S_2 surface and F a rank 1 sheaf on X. Then the $\mathcal{O}_X$-algebra $\sum_{m\geq 0} F^{[m]}$ is finitely generated iff $F^{[m]}$ is locally free for some $m>0$.

It seems that the minimal model of a typical nc surface has such singularities and its canonical ring is not finitely generated.

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