Math 416 HW

Unless stated otherwise, HW is from Shafarevich: Basic algebraic geometry.

Due Sept. 29. Sec I.1: 2,3,4 (+ give examples), 5,6,9
Sec I.2: 2,3,4,5,6,15,17

Due Oct. 6. Sec I.3: 1,4,5,6
Sec.I.4: 2,3,5,9,10
Sec.I.5: 1,5,6

Due Oct 13. Sec 1.6: 1, 2, 10
More problems:
a) Find a counter example to the Corollary on p.77 (in Sec I.6.3) and correct its statement.
b) Apply the method of Sec I.6.4 in the following situations:
i) Which quartic surfaces in $\mathbb{P}^3$ contain a line (or a plane conic or a plane cubic curve)?
ii) For which pairs (n, d) is it true that every hypersurface of degree d in $\mathbb{P}^n$ contains a line?

Solution to (b.ii). We follow the method of the cubic surfaces in Shafarevich.

It is enough to find ONE hypersurface and on it ONE line near which the dimension of all lines on that hypersurface is the expected one. I do the case $d = 2n - 3$.

The line will be $(x_0 = \cdots = x_{n-2} = 0)$. Nearby lines are of the form $x_i = s_i x_{n-1} + t_i x_n$ for $i = 0, \dots, n-2$. Hypersurafces containing this line have equation

$$\sum_{i=0}^{n-2} f_i x_i = 0 \quad \text{where } \deg f_i = 2n - 4.$$

Now look at the special hypersurface

$$\sum_{i=0}^{n-2} x_{n-1}^{2i} x_n^{2(n-2-i)} x_i = 0.$$

When we compute

$$\sum_{i=0}^{n-2} x_{n-1}^{2i} x_n^{2(n-2-i)} (s_i x_{n-1} + t_i x_n),$$

each monomial will have a t_i or s_i as a coefficient. Thus the line is on the hypersurface iff the last expression is identically 0 iff all the t_i, s_i are 0.

Note. This hypersurface actually contains a lot of lines since it contains the $(n-2)$-plane $(x_{n-1} = x_n = 0)$.

Due Oct 20. Sec II.1: 2, 4, 5, 6, 9, 10, 12, 13, 14, 15, 16

Extra problem:

i) Use Bézout's theorem (IV.2.1, example 1 on p.236) to show that a cubic hypersurface in $\mathbb{P}^n$ has at most 2^{n+1} singular points (or singular along a whole curve).

ii) Consider the cubic hypersurface, given by equations

$$\sum_1^m x_i^3 = \sum_1^m y_i^3 \quad \text{and} \quad \sum_1^m x_i = \sum_1^m y_i.$$

Find and count its singular points.

iii) Generalize the above to higher degree hypersurfaces.

iv) If possible, try to find examples that have more singular points than the one in ii). (I do not know what the best result should be, most likely it is not known at all, so this is an open ended question.)

Due Oct 27. Sec II.3: 6, 9–13,

Extra problems:

i) For which $n \geq 3$ is $\mathbb{C}[x_1, \ldots, x_n]/(x_1^2 + \cdots + x_n^2)$ a UFD?

ii) Resolve the following singularities and draw the corresponding dual graphs. (At least start these, you may finish them for next time if it is going slow.)

$x^2 + y^2 + z^{n+1} = 0$

$x^2 + y^2 z + z^{n-1} = 0$

$x^2 + y^3 + z^4 = 0$

$x^2 + y^3 + yz^3 = 0$

$x^2 + y^3 + z^5 = 0$

Due Nov 10. Sec II.5: 2, 4, 6

Sec II.6: 1–6

Extra problems:

1. Prove that the normalization of a graded ring is also graded.

2. For any $m, a \in \mathbb{N}$ find the normalization of the curve $S(m, a) := (z^m = x^a) \subset \mathbb{C}^2$.

3. For any $m, a, b \in \mathbb{N}$ find the normalization of the surface $S(m, a, b) := (z^m = x^a y^b) \subset \mathbb{C}^3$. (Hint. Use the previous results. Depending on how you do this, the answer may look complicated. Consider starting with low values of m. Partial answers should also be submitted.)

Due Nov 17. Sec III.1: 1,5,8–10, 16–18

Sec III.2: 1,3,6

Extra problems:

Consider the proposed intersection formula

$$X \cdot Y \overset{?}{=} \sum_{p \in X \cap Y} \mathcal{O}_p / \big(I(X), I(Y)\big) \qquad (*)$$

in the following cases.

1. In $\mathbb{P}^4$ let X be the union of two 2-planes meeting at a point p and let Y be another 2-plane. Compute the rhs of (*) when Y does not pass through p and also when Y does pass through p.

2. Let $C \subset \mathbb{P}^3$ be the image of $\mathbb{P}^1(s:t)$ by the map $(s:t) \mapsto (s^4 : s^3t : st^3 : t^4)$. Let $X \subset \mathbb{P}^4$ be the cone over C with vertex p. Let Y be a 2-plane. Compute the rhs of (*) when Y does not pass through p and also when Y does pass through p.

(In both cases, start with cases where the equations for both X and Y look simple. Then you may try the general cases later.)

3. In both of these cases, what can you say about the local rings $\mathcal{O}_{p,X}$?

Due Nov 24. 1. Let C be a smooth, projective curve. Let $\Gamma(\Omega_C)$ be the vector space of differential 1-forms without poles.

a) Show that $\Gamma(\Omega_C)$ is finite dimensional.

b) Explain how $\Gamma(\Omega_C)$ defines a map of C to a projective space of dimension $\dim \Gamma(\Omega_C) - 1$. Let $\bar{C} \subset \mathbb{P}^{g-1}$ denote the image.

c) Show that if $C_1 \cong C_2$ then there is a linear change of coordinates $\Phi : \mathbb{P}^{g-1} \to \mathbb{P}^{g-1}$ such that $\Phi(\bar{C}_1) = \bar{C}_2$.

2. Assume that C is hyperelliptic of genus ≥ 2.

a) Use the computation we did in class to show that $\bar{C} \cong \mathbb{P}^1$ and $C \to \mathbb{P}^1$ is the unique 2:1 map of C to $\mathbb{P}^1$.

b) Let $B(C) \subset \mathbb{P}^1$ denote the set of $2g+2$ branch points. Show that $C_1 \cong C_2$ if and only if there is a linear change of coordinates $\Phi : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\Phi(B(C_1)) = B(C_2)$.

3. Let $C_{m,f}$ denote the affine curve $y^m = f(x)$ where f has no multiple roots.

a) Show that $C_{m,f}$ is smooth.

b) Assume that m is rel.prime to $\deg f$. Let $\bar{C}_{m,f} \subset \mathbb{P}^2$ denote the corresponding projective curve. Show that it has one point at infinty and compute the normalization of its local ring.

c) As we did in class, use these to compute the genus of the normalization of $\bar{C}_{m,f}$.

d) Show that the genus agrees with the topological genus.

e) Compute the vector space of differential 1-forms without poles.