SINGULARITIES OF PAIRS

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Contents

1. Introduction

Higher dimensional algebraic geometry has been one of the most rapidly developing research areas in the past twenty years. The first decade of its development centered around the formulation of the minimal model program and finding techniques to carry this program through. The proof of the existence of flips, given in [Mori88], completed the program in dimension three. These results, especially the progress leading up to [Mori88], are reviewed in several surveys. A very general overview is given in [Kollár87b]; many of the methods are explained in the series of lectures [CKM88]; a technically complete review for experts is found in [KaMaMa87].

The methods of [Mori88] allow us to understand three dimensional flips, but the question of how to proceed to higher dimensions remains a baffling one. Therefore the focus of the field moved in one of two major directions.

Internal Developments.

There has been a considerable internal development, as we have understood the implications of the minimal model program to the structure of threefolds.

Two major achievements in this direction are the study of log flips by [Shokurov92], and its subsequent application to the proof of the abundance theorem for threefolds in [Kawamata92]. Both of these results have been simplified and explained in detail in [Kollár et al.92]. Unfortunately, many of the necessary methods are rather technical and require considerable preparatory work. A further significant advance along these lines is the proof of the log abundance theorem by [KeMaM^{c}95].

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A study of rational curves on varieties was undertaken in [KoMiMo92,a,b,c] and in [Keel-M^{c}Kernan95]. Many of these results are described in [Kollár95b].

[Alexeev93,94] studies some questions about surfaces which were inspired by 3-dimensional problems. These results lead to a geometrically meaningful compactification of the moduli of surfaces of general type.

[Corti94,96] has been developing a method to use the minimal model program in order to understand birational transformations between varieties which are close to being rational.

A short overview of the above four directions is given in [Corti95].

Applications of higher dimensional geometry.

Another major theme of the last decade has been the spreading of the ideas of the minimal model program to other fields of algebraic geometry and beyond.

One of the most dramatic changes is that people have been discovering flips in many places. An early example is [Thaddeus94]. Later [Dolgachev-Hu94; Thaddeus96] showed that geometric invariant theoretic quotients are frequently related to each other by series of flips. A similar phenomenon was discovered in [Kapovich-Millson95]. The cone of curves appeared in the study of symplectic manifolds [Ruan93]. These articles use relatively few of the results of higher dimensional geometry. One of the reasons is that in the development of the minimal model program, the study of singular varieties attracted the greatest attention. For the minimal model program this is an indispensable part, but in the above applications most varieties are smooth.

Another collection of concepts developed in higher dimensional geometry is a new way of looking at singularities of pairs (X, D) where X is a variety and D a $\mathbb{Q}$-linear combination of divisors. Traditional approaches studied either the singularities of a variety X, or the singularities of a divisor D in a smooth variety, but did not concentrate on problems that occur when both X and D are singular.

The class of all pairs (X, D) is usually referred to as the "log category". (Everybody is rather vague about what the morphisms should be.) The terminology seems to derive from the observation that differential forms on a variety X with logarithmic poles along a divisor D should be thought of as analogs of holomorphic differential forms. Frequently, the adjective "log" indicates the analog of a notion or theorem in the log category. Unfortunately, the notion "logarithmic pole" is not the log analog of the notion "pole".

At the beginning, the log category was viewed by many as a purely technical construct, but during the last decade the importance of this concept gradually became indisputable. A large part of the evidence is provided by the numerous applications of these ideas and results in diverse questions of algebraic geometry.

The aim of these notes.

My intention is to explain the basic concepts and results of the log category, with a strong emphasis on applications. I am convinced that in the coming years these techniques will become an essential tool for algebraic geometers.

These notes are written for two very different kinds of reader. First, they are intended to serve as a first introduction for algebraic geometers not familiar with higher dimensional methods. Second, they also contain new results and simpler proofs of old results of interest to experts in higher dimensional geometry. Each section starts with the introductory parts, which in themselves constitute a coherent treatment and can be read without reference to the more advanced parts intended for experts. The technical parts in each section are separated from the introductory ones by the symbol $*****$.

Description of the sections.

Section 2 is a survey of the various generalizations of the Kodaira vanishing theorem which have been developed in connection with log varieties. Paragraphs (2.8–14) explain the simplest known proof of the basic versions. At the end, (2.16–17) provide a summary, without proofs, of the most general forms of the vanishing results. The general results can be reduced to the basic versions by some technical arguments which I do not find too illuminating. The interested reader should consult [KaMaMa87] or [Kollár95a]. In many instances the general variants are easier to apply, so at least the statements should be widely known.

Section 3 gives the basic definitions concerning the log category. The most important notion is the discrepancy (3.3). This provides a measure of how singular a pair (X, D) is. The most significant classes are defined in (3.5). (3.6–14) give examples and various methods of computing discrepancies. Finally (3.18–20) relates our notion to singular metrics on line bundles.

The first major application is in section 4. Inspired by [Xu94], we study Bertini type theorems for linear systems with base points. One of the nicest applications is (4.5). Its statement has nothing to do with the log category, but its proof uses log techniques in an essential way. (4.8) gives a slew of more technical Bertini–type theorems in the log category.

The best developed application of log techniques is presented in section 5. This concerns the study of linear systems of the form $K_X + L$, where L is ample. Numerous people have contributed to this direction [Fujita87; Demailly93; Ein-Lazarsfeld93; Kollár93a,b; Fujita94; Demailly94; Angehrn-Siu95; Helmke96; Kawamata96; Smith96]. The lectures of [Lazarsfeld96] provide a very readable introduction. My aim is to explain a version which works in all dimensions, but other versions give better results in low dimensions.

Section 6 contains the hardest part of the proof of the results in the previous section. The question is quite interesting in itself: Let L be an ample divisor on a variety X and $x \in X$ a point. We would like to construct a divisor B, such that B is numerically equivalent to L and B is rather singular at x but not too singular near x. It turns out that in order to get a reasonable answer we need to allow B to be a $\mathbb{Q}$-divisor. Also, the traditional measures of singularities, like the multiplicity, are not suitable for this problem. The precise result is given in (6.4–5). For the analogous theorems concerning singular metrics which blow up at a single point, see [Angehrn-Siu95].

In section 7 we compare the singularities of a pair (X, D) with the pair $(H, D|H)$ where $H \subset X$ is a hypersurface. This problem is closely related to the L^2 extension theorem (7.2) of [Ohsawa-Takegoshi87]. The precise conjecture (7.3), called "inversion of adjunction", was proposed by [Shokurov92]. This trick frequently allows us to reduce an n-dimensional problem to an $(n-1)$-dimensional question. For the applications the most important variant is (7.5). This implies that the notions *klt* and *lc* behave well under deformations (7.6–8). A significant application of inversion of adjunction is in the study of log flips [Shokurov92; Kollát et al.92, 17–18]; these results are not discussed here.

The notion of log canonical threshold is introduced in section 8. This concept provides a new way of measuring the singularities of pairs (X, D) which do not fit in the previous framework. The most striking aspect of this approach is a conjecture of [Shokurov92] (8.8). The section is devoted to some computations that tend to support the conjecture.

Sections 9–10 compare the log canonical threshold to previously known invariants of a hypersurface singularity, namely the complex singular index, the quasiadjunction constants of [Libgober83] and the Bernstein–Sato polynomial. This raises the possibility that conjecture (8.8) can be approached through the theory of variations of Hodge structures or through the study of D-modules. My hope is that experts of these fields will get interested in such questions.

Finally, section 11 contains a simplified proof of an old result of [Elkik81; Flenner81], asserting that canonical singularities are rational. The proof is simpler in that it does not use Grothendieck's general duality theory, but it is still not very short.

Terminology.

The terminology follows [Hartshorne77] for algebraic geometry. Some other notions, which are in general use in higher dimensional algebraic geometry, are defined below.

1.1.1. I use the words *line bundle* and *invertible sheaf* interchangeably. If D is a *Cartier divisor* on a variety X then $\mathcal{O}_X(D)$ denotes the corresponding line bundle. *Linear equivalence* of line bundles (resp. Cartier divisors) is denoted by $\cong$ (resp. $\sim$). *Numerical equivalence* is denoted by $\equiv$.

1.1.2. Let X be a normal variety. A $\mathbb{Q}$-*divisor* is a $\mathbb{Q}$-linear combination of Weil divisors $\sum a_i D_i$. A $\mathbb{Q}$-divisor is called $\mathbb{Q}$-*Cartier* if it is a $\mathbb{Q}$-linear combination of Cartier divisors $\sum e_i E_i$. Thus, a $\mathbb{Q}$-Cartier Weil divisor is a Weil divisor which is $\mathbb{Q}$-Cartier. (If X is smooth then any Weil divisor is also Cartier, but not in general.) The notion of $\mathbb{R}$-divisor etc. can be defined analogously.

1.1.3. Let X be a scheme and $D \subset X$ a divisor. A *log resolution* of (X, D) is a proper and birational morphism $f : Y \to X$ such that $f^{-1}(D) \cup$(exceptional set of f) is a divisor with global normal crossings. Log resolutions exist if X is an excellent scheme over a field of characteristic zero.

1.1.4. The *canonical line bundle* of a smooth variety X is denoted by K_X. By definition, $K_X = (\det T_X)^{-1}$ where T_X is the holomorphic tangent bundle. Thus $c_1(K_X) = -c_1(X)$.

If X is a normal variety, there is a unique divisor class K_X on X such that

$$\mathcal{O}_{X-\mathrm{Sing}(X)}(K_X|X - \mathrm{Sing}(X)) \cong K_{X-\mathrm{Sing}(X)}.$$

K_X is called the *canonical class of* X. The switching between the divisor and line bundle versions should not cause any problems.

1.1.5. A *morphism* between algebraic varieties is assumed to be everywhere defined. It is denoted by a solid arrow $\to$. A *map* is defined only on a dense open set. It is sometimes called a rational or meromorphic map to emphasize this fact. It is denoted by a broken arrow $\dashrightarrow$.

1.1.6. Let $g : U \dashrightarrow V$ be a map which is a morphism over the open set $U^0 \subset U$. Let $Z \subset U$ be a subscheme such that every generic point of Z is in U^0. The closure of $g(U^0 \cap Z)$ is called the *birational transform* of Z. It is denoted by $g_*(Z)$. (This notion is frequently called the proper or strict transform.)

If $f : V \dashrightarrow U$ is a birational map and $g = f^{-1}$ then we get the slightly strange looking notation $f_*^{-1}(Z)$.

1.1.7. As usual, $\llcorner x \lrcorner$ (round down) denotes the integral part of a real number x and $\{x\} := x - \llcorner x \lrcorner$ the fractional part. We also use the notation $\ulcorner x \urcorner := -\llcorner -x \lrcorner$ (round up).

If $D = \sum a_i D_i$ is a $\mathbb{Q}$-divisor, then $\llcorner D \lrcorner := \sum \llcorner a_i \lrcorner D_i$, similarly for $\{D\}$ and $\ulcorner D \urcorner$. When using this notation we always assume that the D_i are prime divisors and $D_i \neq D_j$ for $i \neq j$, since otherwise these operations are not well defined.

1.1.8. MMP stands for minimal model program, and log MMP for the log minimal model program. These notions are used only occasionally and familiarity with them is not necessary. See [Kollár87b; CKM88; KaMaMa87, Kollár et al.92] for details.

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2. Vanishing Theorems

Some of the most useful results in higher dimensional geometry are the various generalizations of Kodaira's vanishing theorem. These results are not new, and they have been surveyed ten years ago at Bowdoin [Kollár87a]. Since then we have understood their proofs much better, and a whole new range of applications was also discovered. The aim of this section is to explain the main ideas behind the proof and to present a typical application (2.5–7). Several other applications are contained in subsequent sections.

Throughout this section, the characteristic is zero.

For other treatments of these and related vanishing theorems see [KaMaMa87; Esnault-Viehweg92; Kollár95a].

Let us first recall Kodaira's vanishing theorem:

2.1 Theorem. [Kodaira53] *Let X be a smooth projective variety and L an ample line bundle on X. Then*

$$H^i(X, K_X \otimes L) = 0 \quad \textit{for } i > 0. \qquad \square$$

This result will be generalized in two directions.

(2.1.1) The first step is to weaken the condition "L ample" while keeping all the vanishing. The guiding principle is that if L is sufficiently close to being ample, vanishing should still hold.

(2.1.2) In order to understand the second step, we need to look at a typical application of a vanishing result. Let

$$A \xrightarrow{\partial_A} B \to C \xrightarrow{\partial_C} D$$

be an exact sequence (in applications this is part of a long exact sequence of cohomology groups). If $C = 0$, then ∂_A is surjective. More generally, if ∂_C is injective, then ∂_A is still surjective. Thus the injectivity of a map between cohomology groups can be viewed as a generalization of a vanishing theorem.

The formulation of the first step requires some definitions.

2.2 Definition. Let X be a proper variety and L a line bundle or a Cartier divisor on X.

(2.2.1) L is *nef* iff $\deg_C(L|C) \geq 0$ for every irreducible curve $C \subset X$.

(2.2.2) L is *big* iff $H^0(X, L^m)$ gives a birational map $X \dashrightarrow X' \subset \mathbb{P}$ to some projective space for $m \gg 1$. Equivalently, L is big iff $h^0(X, L^m) > const \cdot m^{\dim X}$ for $m \gg 1$.

(2.2.3) Both of these notions extend by linearity to $\mathbb{Q}$-Cartier divisors.

We are ready to formulate the simplest form of the general vanishing theorem about perturbations of ample line bundles:

2.3 Theorem. [Kawamata82; Viehweg82] *Let X be a smooth projective variety and L a line bundle on X. Assume that we can write $L \equiv M + \sum d_i D_i$ where M is a nef and big $\mathbb{Q}$-divisor, $\sum D_i$ is a normal crossing divisor and $0 \leq d_i < 1$ for every i. Then*

$$H^i(X, K_X \otimes L) = 0 \quad \text{for } i > 0.$$

2.4 Questions. The two main questions raised by this result are the following:

(2.4.1.1) Where do we find line bundles L which can be decomposed as $L \equiv M + \sum d_i D_i$?

(2.4.1.2) How to use a result like this?

There are two basic situations where line bundles with a decomposition $L \equiv M + \sum d_i D_i$ naturally arise.

(2.4.2.1) Let S be a normal surface and $f : S' \to S$ a resolution of singularities. Assume for simplicity that S has rational singularities. Then $f^* \operatorname{Pic}(S)$ and the exceptional curves D_i of f generate a finite index subgroup of $\operatorname{Pic}(S')$. Thus any line bundle L on S' can be decomposed as $L \equiv f^*M + \sum d_i D_i$ where M is a $\mathbb{Q}$-Cartier divisor on S and the D_i are the f-exceptional curves. Rational coefficients are usually inevitable.

(2.4.2.2) Let L be a line bundle on a variety X such that L^n has a section with zero set $D = \sum a_i D_i$. Then $L \equiv \sum (a_i/n) D_i$. In general $\sum D_i$ is not a normal crossing divisor. By Hironaka, there is a proper birational morphism $f : X' \to X$ such that $f^*L \equiv \sum b_i B_i$, where $\sum B_i$ is a normal crossing divisor.

This suggests that the best hope of using (2.3) is in questions which are birational in nature. The following example shows a rather typical application of this kind.

2.5 Example. Let X be a smooth proper variety of general type. Our aim is to express $H^0(X, sK_X)$ as an Euler characteristic, at least for $s \geq 2$.

(2.5.1) First approach.

Assume that X is a surface and let X' be its minimal model. Then $K_{X'}$ is nef and big, so

$$H^0(X, sK_X) = H^0(X', sK_{X'}) = \chi(X', sK_{X'}) \quad \text{for } s \geq 2.$$

In higher dimensions this still works if X has a minimal model X'. Unfortunately, the existence of a minimal model is unknown in general.

(2.5.2) Second approach.

We try to follow (2.4.2.2) with some modifications. Choose an auxiliary number $m \gg 1$ and $f : X' \to X$ such that:

X' is smooth, and

$|mK_{X'}| = |M| + F$, where M is free and big, and $F = \sum a_i F_i$ is a normal crossing divisor.

This is always possible by Hironaka. Further conditions on m will be imposed later. Thus

$$sK_{X'} = K_{X'} + (s-1)K_{X'} \equiv K_{X'} + \frac{s-1}{m}M + \frac{s-1}{m}F.$$

This is not quite what we want, since $\frac{s-1}{m}F$ can have coefficients that are bigger than one. To remedy this problem we just get rid of the excess coefficients in $\frac{s-1}{m}F$. We want to get a Cartier divisor, so we can subtract only integral multiples of Cartier divisors, and we also want to end up with coefficients between 0 and 1. These two conditions uniquely determine the choice of $\sum c_i F_i := \sum \llcorner (s-1)a_i/m \lrcorner F_i$. Set

$$\begin{aligned} L :=& (s-1)K_{X'} - \sum c_i F_i \\ \equiv& \frac{s-1}{m}M + \sum \left\{ \frac{(s-1)a_i}{m} \right\} F_i. \end{aligned} \tag{2.5.3}$$

The choice of L in (2.5.3) is set up so that vanishing applies to $K_{X'} + L$. We still need to check that we have not lost any sections of $\mathcal{O}(sK_{X'})$ by subtracting $\sum c_i F_i$.

Let $D = D' + \sum b_i F_i$ be any divisor in the linear system $|sK_{X'}|$. Assume that $m = rs$. Then $rD' + \sum rb_i F_i \in |mK_{X'}|$, thus $rb_i \geq a_i$ for every i. Therefore

$$b_i \geq a_i/r = sa_i/m \geq \llcorner (s-1)a_i/m \lrcorner = c_i. \tag{2.5.4}$$

Thus $D - \sum c_i F_i = D' + \sum (b_i - c_i) F_i$ is effective. Therefore

$$H^0(X, sK_X) = H^0(X', sK_{X'}) = H^0(X', K_{X'} + L).$$

Thus by (2.3)

$$H^0(X, sK_X) = \chi(X', K_{X'} + L).$$

Note. The choice of m has been left rather free. Different choices do lead to different models X'. Also, the estimate (2.5.4) is far from being sharp. In delicate situations it is worthwhile to check how much room it gives us.

The following theorems use this construction to compare plurigenera of étale covers:

2.6 Theorem. [Kollár95a, 15.4] *Let $p : Y \to X$ be a finite étale morphism between smooth and proper varieties of general type. Then*

$$h^0(X, sK_X) = \frac{1}{\deg p} h^0(Y, sK_Y) \quad \textit{for } s \geq 2.$$

Proof. Fix s and choose $f : X' \to X$ as in (2.5.2). Set $Y' = Y \times_X X'$. Then $p' : Y' \to X'$ is étale. As in (2.5.2) we construct L such that,

$$h^0(X, sK_X) = \chi(X', K_{X'} + L), \quad \text{and} \quad h^0(Y, sK_Y) = \chi(Y', K_{Y'} + p'^*L).$$

(This requires a little extra care; see [Kollár95a, 15.5] for details.) An Euler characteristic is multiplicative in étale covers, thus

$$\chi(X', K_{X'} + L) = \frac{1}{\deg p} \chi(Y', K_{Y'} + p'^*L).$$

The two formulas together imply (2.6). $\square$

This is just the baby version of the following result which compares plurigenera in possibly infinite covers. See [Kollár95a, 15.5] for the necessary definitions and the proof.

2.7 Theorem. *Let X be a smooth, proper variety of general type and $p : Y \to X$ a (possibly infinite) étale Galois cover with Galois group Γ. Then*

$$h^0(X, K_X^m) = \dim_\Gamma H^0_{(2)}(Y, K_Y^m) \quad \textit{for } m \geq 2.$$

(Here $H^0_{(2)}$ is the Hilbert space of holomorphic L^2 sections with respect to a metric pulled back from X and $\dim_\Gamma$ is the usual dimension in the theory of von Neumann algebras.)

2.8 Idea of the proof of the vanishing theorems.

The two main steps of the proofs, as outlined in [Kollár86b,Sec.5], are the following:

(2.8.1) Step 1.

Find several examples where the coherent cohomology of a sheaf comes from topological cohomology. The simplest example of this situation is given by Hodge theory. For the proof see any of the standard textbooks on Kähler geometry (e.g. [Wells73, V.4.1; Griffiths-Harris78, p.116]).

2.8.1.1 Theorem. *Let X be a smooth proper variety (or compact Kähler manifold) with structure sheaf $\mathcal{O}_X$. Let $\mathbb{C}_X \subset \mathcal{O}_X$ denote the constant sheaf. Then the natural map*

$$H^i(X, \mathbb{C}_X) \to H^i(X, \mathcal{O}_X) \quad \textit{is surjective for every } i. \qquad \square$$

We need that this also holds if X has quotient singularities. This is due to [Steenbrink77; Danilov78]. The more up-to-date "orbifold approach" is to notice that the usual proof for manifolds works with essentially no changes. We should still view X as being patched together from smooth coordinate charts, but instead of allowing patching data between different charts only, we admit patching data between a chart and itself, corresponding to the local group action. Once the conceptual difficulties are behind, the proof is really the same.

2.8.1.2 Remarks.

The analog of (2.8.1.1) also holds if X has rational singularities, but I do not know any simple proof, cf. [Kollár95a, Chap.12].

More generally, any variation of Hodge structures over X gives rise to a similar situation, see [Kollár86b,Sec.5; Saito91].

(2.8.2) Step 2.

By an auxiliary construction, which in this case is the study of cyclic covers, we find many related situations of a topological sheaf $\mathbf{F}$ and a coherent sheaf $\mathcal{F}$ together with natural surjections

$$H^i(X, \mathbf{F}) \to H^i(X, \mathcal{F}).$$

(By "topological sheaf" I mean a sheaf of abelian groups which is defined in terms of the classical topology of X.) Moreover, we try to achieve that $\mathbf{F}$ is supported over an open set $X^0 \subset X$. If X^0 is sufficiently small, for instance affine, then many cohomologies vanish over X^0, giving the vanishing of certain coherent cohomology groups.

(In the treatment of [Esnault-Viehweg92], a De Rham complex takes the place of the topological sheaf $\mathbf{F}$.)

We start the proof of (2.3) by constructing cyclic covers and studying their basic properties. This is the third time that I give a somewhat different treatment of cyclic covers (cf. [Kollár95a,Chap.9; Kollár95b,II.6]) but I am still unable to find one which I consider optimal.

2.9 Local construction of cyclic covers.

Let U be a smooth variety, $f \in \mathcal{O}_U$ a regular function and $n \geq 1$ a natural number. Let $D = (f = 0)$ be the zero divisor of f. We want to construct the cyclic cover corresponding to $\sqrt[n]{f}$. It is denoted by $U[\sqrt[n]{f}]$.

Let y be a new variable. $U[\sqrt[n]{f}] \subset U \times \mathbb{A}^1$ is defined by the equation $y^n = f$. Thus $U[\sqrt[n]{f}]$ is smooth at a point $(u, *)$ iff either $f(u) \neq 0$ or u is a smooth point of the divisor D.

Let $p : U[\sqrt[n]{f}] \to U$ be the coordinate projection. p is finite over U and étale over $U - D$. The cyclic group $\mathbb{Z}_n$ acts on $U[\sqrt[n]{f}]$ and the $\mathbb{Z}_n$-action gives an eigensheaf decomposition

$$p_* \mathcal{O}_{U[\sqrt[n]{f}]} = \mathcal{O}_U + y\mathcal{O}_U + \cdots + y^{n-1}\mathcal{O}_U.$$

Let $\bar{U}[\sqrt[n]{f}] \to U[\sqrt[n]{f}]$ be the normalization and $\bar{p} : \bar{U}[\sqrt[n]{f}] \to U$ the projection. The $\mathbb{Z}_n$-action lifts to a $\mathbb{Z}_n$-action on $\bar{U}[\sqrt[n]{f}]$, thus we get an eigensheaf decomposition

$$\bar{p}_* \mathcal{O}_{\bar{U}[\sqrt[n]{f}]} = \sum_{k=0}^{n-1} F_k, \quad \text{where} \quad F_k \supset y^k \mathcal{O}_U.$$

Each F_i is a rank one reflexive sheaf, hence invertible since U is smooth. Thus there are divisors D^k such that $F_k = y^k \mathcal{O}_U(D^k)$ and $\operatorname{Supp} D^k \subset \operatorname{Supp} D$.

In order to figure out the coefficients in D^k, we may localize at a smooth point $u \in \operatorname{Supp} D$. Thus we can assume that $f = vx_1^d$ where v is a unit at u and x_1 is a local coordinate at u.

A rational section $y^k x_1^{-j}$ of $y^k \mathcal{O}_U$ is integral over $\mathcal{O}_U$ iff

$$(y^k x_1^{-j})^n = y^{nk} x_1^{-nj} = f^k x_1^{-nj} = v^k x_1^{kd-nj}$$

is a regular function. Thus $j \le dk/n$. This shows that

$$\bar{p}_* \mathcal{O}_{\bar{U}[\sqrt[n]{f}]} = \sum_{k=0}^{n-1} y^k \mathcal{O}_U(\llcorner kD/n \lrcorner).$$

2.10 Local structure of $\bar{U}[\sqrt[n]{f}]$.

Assume that D is a normal crossing divisor. Pick $u \in U$ and let $D_1, \dots, D_s$ be the irreducible components of D passing through u. Choose local coordinates x_i at u such that $D_i = (x_i = 0)$. Let $B \subset U$ be a polydisc around u defined by $|x_i| < 1$ for every i.

$\pi_1(B - D) \cong \mathbb{Z}^s$ is generated by the loops around the divisors D_i.

Let $\bar{B} \subset \bar{p}^{-1}(B)$ be an irreducible component. $\bar{B} \to B$ is étale over $B - D$, thus it corresponds to a quotient $\mathbb{Z}^s \to \mathbb{Z}_n$. By Galois theory, $\bar{B}$ is a quotient of the cover corresponding to the subgroup $(n\mathbb{Z})^s \subset \mathbb{Z}^s$.

Let $\Delta \subset \mathbb{C}$ be the unit disc. The cover corresponding to $(n\mathbb{Z})^s \subset \mathbb{Z}^s$ is

$$\Delta^m \to B, \quad \text{given by} \quad (z_1, \dots, z_m) \mapsto (z_1^n, \dots, z_s^n, z_{s+1}, \dots, z_m).$$

This cover is smooth, hence $\bar{U}[\sqrt[n]{f}]$ has only quotient singularities.

2.11 Global construction of cyclic covers.

Let X be a smooth variety, L a line bundle on X and $s \in H^0(X, L^n)$ a section.

Let $U_i \subset X$ be an affine cover such that $L|U_i$ has a nowhere zero global section h_i. L is given by transition functions $h_i = \phi_{ij} h_j$. Let $s = f_i h_i^n$. s is a well defined section, thus $f_i = \phi_{ij}^{-n} f_j$.

The local cyclic covers are given by equations $y_i^n = f_i$. These are compatible if we set the transformation rules $y_i = \phi_{ij}^{-1} y_j$. This gives the global cyclic cover

$$X[\sqrt[n]{f}] = \cup_i U_i[\sqrt[n]{f_i}].$$

The invertible sheaves $y_i^k \mathcal{O}_{U_i}$ patch together to the line bundle L^{-k}, and so

$$p_* \mathcal{O}_{X[\sqrt[n]{f}]} = \mathcal{O}_X + L^{-1} + \cdots + L^{-(n-1)}.$$

Let $\bar{X}[\sqrt[n]{f}] \to X[\sqrt[n]{f}]$ be the normalization and $\bar{p} : \bar{X}[\sqrt[n]{f}] \to X$ the projection. The $\mathbb{Z}_n$-action gives the eigensheaf decomposition

$$\bar{p}_* \mathcal{O}_{\bar{X}[\sqrt[n]{f}]} = \sum_{k=0}^{n-1} L^{-k}(\llcorner kD/n \lrcorner).$$

2.12 Decomposing $\bar{p}_*\mathbb{C}$.

Until now everything worked in the Zariski as well as in the classical topology. From now on we have to use the classical topology.

In order to simplify notation set $Z = \bar{X}[\sqrt[n]{f}]$ and let $\mathbb{C}_Z \subset \mathcal{O}_Z$ denote the sheaf of locally constant functions. We have an eigensheaf decomposition

$$\bar{p}_*\mathbb{C}_Z = \sum G_k, \quad \text{such that} \quad G_k \subset L^{-k}(\llcorner kD/n \lrcorner).$$

It is not hard to write down the sheaves G_k explicitly (cf. [Kollár95a,9.16]), but the arguments are clearer and simpler if we do not attempt to do this. Their basic cohomological properties are easy to establish:

2.12.1 Proposition. *Notation as above. Write $D = \sum d_i D_i$.*

(2.12.1.1) $G_0 \cong \mathbb{C}_X$.

(2.12.1.2) For every $x \in X$ there is an open neighborhood $x \in U_x \subset X$ such that $H^i(U_x, G_j|U_x) = 0$ for $i > 0$.

(2.12.1.3) If $U \subset X$ is connected, $U \cap D \neq \emptyset$ and n does not divide d_i for every i, then $H^0(U, G_1|U) = 0$.

Proof. G_0 is the invariant part of $\bar{p}_*\mathbb{C}_Z$, which is $\mathbb{C}_X$.

Choose $x \in U_x \subset X$ such that $V_x = (\bar{p})^{-1}U_x \subset Z$ retracts to $(\bar{p})^{-1}(x)$. Then $H^i(V_x, \mathbb{C}_{V_x}) = 0$ for $i > 0$. Since $\bar{p}$ is finite,

$$H^i(W, \bar{p}_*\mathbb{C}_Z|W) = H^i((\bar{p})^{-1}W, \mathbb{C}_Z|(\bar{p})^{-1}W),$$

for every $W \subset X$. In particular for $W = U_x$ we obtain that $H^i(U_x, \bar{p}_*\mathbb{C}_Z|U_x) = 0$. This implies (2.12.1.2) since $H^i(U_x, G_j|U_x)$ is a direct summand of $H^i(U_x, \bar{p}_*\mathbb{C}_Z|U_x)$.

If U intersects D then we can find a point $u \in U \cap D$ which has a neighborhood where D is defined by a function $f = vx_1^d$ where v is a unit and x_1 a local coordinate.

In the local description of (2.9), any rational section of $y\mathcal{O}_U$ can be written as yg where g is a rational function. yg is a locally constant section iff $(yg)^n = fg^n$ is a locally constant section of $\mathcal{O}_U$. That is, when

$$g = cf^{-1/n} = cv^{-1/n}x^{-d/n} \quad \text{for some } c \in \mathbb{C}.$$

Since n does not divide d, this gives a rational function only for $c = 0$. □

The following result is a very general theorem about the injectivity of certain maps between cohomology groups. In (2.14) we see that it implies (2.3), at least when M is ample. The general case of (2.3) requires a little more work.

In some applications the injectivity part is important [Kollár86a,b; Esnault-Viehweg87], though so far the vanishing theorem has found many more uses.

The theorem is a culmination of the work of several authors [Tankeev71; Ramanujam72; Miyaoka80; Kawamata82; Viehweg82; Kollár86a,b,87a; Esnault-Viehweg86,87].

2.13 Theorem. *Let X be a smooth proper variety and L a line bundle on X. Let $L^n \cong \mathcal{O}_X(D)$ where $D = \sum d_i D_i$ is an effective divisor. Assume that $\sum D_i$ is a normal crossing divisor and $0 < d_i < n$ for every i. Let Z be the normalization of $X[\sqrt[n]{s}]$. Then:*

(2.13.1) $H^j(Z, \mathbb{C}_Z) \to H^j(Z, \mathcal{O}_Z)$ *is surjective for every* j.
(2.13.2) $H^j(X, G_1) \to H^j(X, L^{-1})$ *is surjective for every* j.
(2.13.3) For every j *and* $b_i \geq 0$ *the natural map*

$$H^j(X, L^{-1}(-\sum b_i D_i)) \to H^j(X, L^{-1}) \quad \textit{is surjective.}$$

(2.13.4) For every j *and* $b_i \geq 0$ *the natural map*

$$H^j(X, K_X \otimes L) \to H^j(X, K_X \otimes L(\sum b_i D_i)) \quad \textit{is injective.}$$

Proof. By (2.10) Z has only quotient singularities, thus (2.13.1) follows from (2.8.1.1).

The assumption $0 < d_i < n$ implies that $\llcorner D/n \lrcorner = \emptyset$, thus $F_1 = L^{-1}$. Therefore (2.13.2) is just (2.13.1) restricted to one $\mathbb{Z}_n$-eigenspace.

(2.13.3) and (2.13.4) are equivalent by Serre duality, thus it is sufficient to prove (2.13.3).

The main step is the following:

2.13.5 Claim. G_1 *is a subsheaf of* $L^{-1}(-\sum b_i D_i)$.

Proof. Both of these are subsheaves of L^{-1}, so this is a local question. We need to show that if $U \subset X$ connected, then

$$H^0(U, G_1|U) \subset H^0(U, L^{-1}(-\sum b_i D_i)|U). \tag{2.13.6}$$

$L^{-1}(-\sum b_i D_i)$ and L^{-1} are equal over $X - D$, thus (2.13.6) holds if $U \subset X - D$. If $U \subset X$ is connected and it intersects D then by (2.12.1.3) $H^0(U, G_1|U) = \{0\}$, thus

$$H^0(U, G_1|U) = \{0\} \subset H^0(U, L^{-1}(-\sum b_i D_i)|U) \quad \text{trivially.} \qquad \square$$

This gives a factorization

$$H^j(X, G_1) \to H^j(X, L^{-1}(-\sum b_i D_i)) \to H^j(X, L^{-1}).$$

The composition is surjective by (2.13.2) hence the second arrow is also surjective. $\square$

As a corollary, let us prove (2.3) in a special case:

2.14 Proof of (2.3) for M *ample.*

We have $L \equiv M + \sum_{i \geq 1} d_i D_i$. Choose n such that nd_i is an integer for every i and

$$nM \equiv nL - \sum_{i \geq 1} (nd_i) D_i \sim D_0,$$

where D_0 is a smooth divisor which intersects $\sum D_i$ transversally. (This is possible since M is ample.) Let $d_0 = 1/n$. Then $nL \sim \sum_{i \geq 0} (nd_i) D_i$, thus by (2.13.4),

$$H^i(X, K_X + L) \to H^i(X, K_X + L + b_0 D_0) \quad \text{is injective.}$$

By Serre vanishing, the right hand side is zero for $b_0 \gg 1$ and $i > 0$.

Thus $H^i(X, K_X + L) = 0$ for $i > 0$. $\square$

As an exercise in using (2.3), derive the following relative version of it.

2.15 Exercise. Let X be a smooth projective variety and L a line bundle on X. Assume that we can write $L \equiv M + \sum d_i D_i$ where M is a nef and big $\mathbb{Q}$-divisor, $\sum D_i$ is a normal crossing divisor and $0 \le d_i < 1$ for every i.

Let $f : X \to Y$ be a proper and birational morphism. Then

$$R^j f_*(K_X \otimes L) = 0 \quad \text{for } j > 0.$$

For the applications it is frequently useful that we have a vanishing even if $\sum d_i D_i$ is not a normal crossing divisor. This approach was first used extensively by [Nadel90] in the analytic setting which we discuss in (3.18–20). (See (3.5) for the definition of klt.)

2.16 Theorem. *Let X be a normal and proper variety and N a line bundle on X. Assume that $N \equiv K_X + \Delta + M$ where M is a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X and Δ is an effective $\mathbb{Q}$-divisor. Then there is an ideal sheaf $J \subset \mathcal{O}_X$ such that*

(2.16.1) $\operatorname{Supp}(\mathcal{O}_X/J) = \{x \in X | (X, \Delta)$ *is not klt at* $x\}$.

(2.16.2) $H^j(X, N \otimes J) = 0$ *for* $j > 0$.

Proof. Let $f : Y \to X$ be a log resolution of (X, Δ). Write

$$K_Y \equiv f^*(K_X + \Delta) + \sum a_i E_i.$$

This can be rewritten as

$$f^*N + \sum a_i E_i \equiv K_Y + f^*M, \quad \text{or as}$$

$$f^*N + \sum \ulcorner a_i \urcorner E_i \equiv K_Y + f^*M + \sum (\ulcorner a_i \urcorner - a_i) E_i.$$

By (2.3) and (2.15) we know that

$$H^j(Y, f^*N(\sum \ulcorner a_i \urcorner E_i)) = 0 \quad \text{and} \quad R^j f_*(f^*N(\sum \ulcorner a_i \urcorner E_i)) = 0 \quad \text{for } j > 0.$$

Thus by the Leray spectral sequence,

$$H^j(f_*(f^*N(\sum \ulcorner a_i \urcorner E_i))) = 0 \quad \text{for } j > 0.$$

By the projection formula,

$$f_*(f^*N(\sum \ulcorner a_i \urcorner E_i)) = N \otimes f_*(\mathcal{O}_Y(\sum \ulcorner a_i \urcorner E_i)).$$

If E_i is not an exceptional divisor, then $a_i \le 0$. Thus in $\sum \ulcorner a_i \urcorner E_i$ only f-exceptional divisors appear with positive coefficient. Therefore

$$f_*(\mathcal{O}_Y(\sum \ulcorner a_i \urcorner E_i)) = f_*(\mathcal{O}_Y(\sum_{a_i \le -1} \ulcorner a_i \urcorner E_i)).$$

The latter is an ideal sheaf in $\mathcal{O}_X$ whose cosupport is exactly the set of points over which there is a divisor with $a_i \le -1$. $\square$

The following is a summary of the most general versions of vanishing theorems. Proofs can be found in [KaMaMa87; Esnault-Viehweg92; Kollár95a]. For the latest results in the holomorphic category, see [Takegoshi95].

2.17 Theorem. *(General Kodaira Vanishing)*

Let X be a normal and proper variety and N a rank one, reflexive sheaf on X. Assume that $N \equiv K_X + \Delta + M$ where M is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X and (X, Δ) is klt.

(2.17.1) (Global vanishing)

Assume that M is nef and big. Then

$$H^i(X, N) = 0 \quad \text{for } i > 0.$$

(2.17.2) (Injectivity theorem)

Assume that M is nef and let D, E be effective Weil divisors on X such that $D + E \equiv mM$ for some $m > 0$. Then

$$H^j(X, N) \to H^j(X, N(D)) \quad \text{is injective for } j \geq 0.$$

(2.17.3) (Relative vanishing)

Let $f : X \to Y$ be a surjective morphism with generic fiber X_{gen}. Assume that M is f-nef and $M|X_{gen}$ is big. Then

$$R^j f_* N = 0 \quad \text{for } j > 0.$$

(Note that if f is generically finite then $M|X_{gen}$ is always big.)

(2.17.4) (Torsion freeness)

Let $f : X \to Y$ be a surjective morphism. Assume that $M \equiv f^* M_Y$ where M_Y is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y. Then

$R^j f_* N$ is torsion free for $j \geq 0$.

2.17.5 Remark. One has to be careful with the localization of (2.17.4). Namely, (2.17.4) is not true if f is a projective morphism but Y is not proper. An example is given by the Poincaré sheaves on Abelian varieties. The local version holds if the numerical equivalence is everywhere replaced by $\mathbb{Q}$-linear equivalence.

2.17.6 Remark. The Grauert-Riemenschneider vanishing theorem is the special case of (2.17.3) when X is smooth and $N = K_X$.

3. Singularities of Pairs

There are many ways to measure how singular a variety is. In higher dimensional geometry a new notion, called the discrepancy, emerged. This concept was developed to deal with the following two situations:

(3.1.1) Let X be a singularity and $f : Y \to X$ a resolution. We would like to measure how singular X is by comparing K_Y with $f^* K_X$, as Y runs through all resolutions.

(3.1.2) Let $D \subset \mathbb{C}^{n+1}$ be a hypersurface with defining equation $h = 0$. Let $f : Y \to \mathbb{C}^{n+1}$ be a birational morphism, Y smooth. We would like to measure how singular D is by comparing the order of vanishing of the Jacobian of f with the order of vanishing of $f^* h$ along exceptional divisors, as Y runs through all birational morphisms.

In order to illustrate the final definitions, it is very useful to look at the following examples. For simplicity we consider the first of the above cases.

3.2 Example. Let X be normal and assume that mK_X is Cartier. Let $f : Y \to X$ be a birational morphism, Y smooth. We can write

$$mK_Y = f^*(mK_X) + \sum (ma_i)E_i,$$

where the E_i are exceptional divisors and the a_i are rational. We frequently write this in the form

$$K_Y \equiv f^*K_X + \sum a_i E_i.$$

Our aim is to get a single invariant out of the numbers a_i, preferably one that is independent of the resolution. The straightforward candidates are $\min\{a_i\}$ and $\max\{a_i\}$. One can easily see that the latter depends on $f : Y \to X$, and its supremum as Y changes is always $+\infty$.

$\min\{a_i\}$ is somewhat better behaved, but it still depends on the choice of $f : Y \to X$ in most cases. To make things better, assume that $\sum E_i$ is a normal crossing divisor.

Let $Z \subset Y$ be a smooth subvariety and $p : B_Z Y \to Y$ the blow up with exceptional divisor $F \subset B_Z Y$. Set $g := f \circ p : B_Z Y \to X$. Then

$$K_{B_Z Y} \equiv g^*K_X + cF + \sum a_i E_i',$$

where E_i' is the birational transform of E_i on $B_Z Y$.

If Z is not contained in $\sum E_i$, then $c \geq 1$. Otherwise it is not hard to check that

$$c \geq \min\{a_i\} \quad \text{if} \quad \min\{a_i\} \geq -1,$$

but not in general. In fact, if $\min\{a_i\} < -1$ then there is a sequence of resolutions such that $\min\{a_i\} \to -\infty$ (3.4.1.4). In this case we say that X is not log canonical. For these singularities our invariant does not give anything. From the point of view of general singularity theory this is rather unfortunate, since most singularities are not log canonical. (In section 8 we introduce another invariant which is meaningful in the non log canonical case.)

Our point of view is, however, quite different. Our main interest is in smooth varieties, and we want to deal with singularities only to the extent they inevitably appear in the course of the minimal model program. In many situations it is precisely $\min\{a_i\}$ which tells us which varieties need to be considered.

If $\min\{a_i\} \geq -1$, then by (3.13) the minimum is independent of the choice of $f : Y \to X$ (assuming that Y is smooth and $\sum E_i$ is a normal crossing divisor).

More generally, one can put the two aspects mentioned in (3.1.1–2) together, and consider pairs (X, D) where X is a normal scheme and D a formal linear combination of Weil divisors $D = \sum d_i D_i$, $d_i \in \mathbb{R}$. It took people about 10 years to understand that this is not simply a technical generalization but a very fruitful — even basic — concept. For now just believe that this makes sense.

Since one cannot pull back arbitrary Weil divisors, we always have to assume that $K_X + D$ is $\mathbb{R}$-Cartier, that is, it is an $\mathbb{R}$-linear combination of Cartier divisors.

In the applications we almost always use $\mathbb{Q}$-coefficients, but for the basic definitions the coefficients do not matter.

The resulting notion can be related to more traditional ways of measuring singularities (for instance, multiplicity or arithmetic genus), but it is a truly novel way of approaching the study of singularities.

3.3 Definition. Let X be a normal, integral scheme and $D = \sum d_i D_i$ an $\mathbb{R}$-divisor (not necessarily effective) such that $K_X + D$ is $\mathbb{R}$-Cartier. Let $f : Y \to X$ be a birational morphism, Y normal. We can write

$$K_Y \equiv f^*(K_X + D) + \sum a(E, X, D)E, \tag{3.3.1}$$

where $E \subset Y$ are distinct prime divisors and $a(E, X, D) \in \mathbb{R}$. The right hand side is not unique because we allow nonexceptional divisors in the summation. In order to make it unique we adopt the following:

3.3.2 Convention. A nonexceptional divisor E appears in the sum (3.3.1) iff $E = f_*^{-1} D_i$ for some i, and then with the coefficient $a(E, X, D) = -d_i$. (Note the negative sign!)

Similarly, if we write $K_Y + D' \equiv f^*(K_X + D)$, then $D' = -\sum a(E, X, D)E$.

$a(E, X, D)$ is called the *discrepancy* of E with respect to (X, D). We frequently write $a(E, D)$ or $a(E)$ if no confusion is likely.

If $f' : Y' \to X$ is another birational morphism and $E' \subset Y'$ is the birational transform of E on Y' then $a(E, X, D) = a(E', X, D)$. In this sense $a(E, X, D)$ depends only on the divisor E but not on Y. This is the reason why Y is suppressed in the notation.

A more invariant description is obtained by considering a rank one discrete valuation ν of the function field $K(X)$. ν corresponds to a divisor $E \subset Y$ for some $f : Y \to X$. The closure of $f(E)$ in X is called the *center* of ν (or of E) on X. It is denoted by $\operatorname{Center}_X(\nu)$ or $\operatorname{Center}_X(E)$.

Thus we obtain a function

$$a(\ , X, D) : \{\text{divisors of } K(X) \text{ with nonempty center on } X\} \to \mathbb{R}.$$

(If X is proper over a field k then every divisor of $K(X)$ over k has a nonempty center.)

3.4 Definition. In order to get a global measure of the singularities of the pair (X, D) we define

$$\operatorname{discrep}(X, D) := \inf_E \{a(E, X, D) | E \text{ is exceptional with nonempty center on } X\},$$

$$\operatorname{totaldiscrep}(X, D) := \inf_E \{a(E, X, D) | E \text{ has nonempty center on } X\}.$$

3.4.1 Examples. (3.4.1.1) Let $E \subset X$ be a divisor different from any of the D_i. Then $a(E, X, D) = 0$, thus $\operatorname{totaldiscrep}(X, D) \le 0$.

(3.4.1.2) Let E be a divisor obtained by blowing up a codimension 2 smooth point $x \in X$ which is not contained in any of the D_i. Then $a(E, X, D) = 1$, thus $\operatorname{discrep}(X, D) \le 1$.

(3.4.1.3) If X is smooth then $K_Y = f^*K_X + E$ where E is effective and its support is the whole exceptional divisor. Thus $\operatorname{discrep}(X, 0) = 1$.

(3.4.1.4) (cf. [CKM88, 6.3]) Show that

$$\begin{array}{lrcl} \text{either} & \operatorname{discrep}(X, D) = -\infty, & \text{or} & -1 \le \operatorname{discrep}(X, D) \le 1, \quad \text{and} \\ \text{either} & \operatorname{totaldiscrep}(X, D) = -\infty, & \text{or} & -1 \le \operatorname{totaldiscrep}(X, D) \le 0. \end{array}$$

(3.4.1.5) Assume that the D_i are $\mathbb{R}$-Cartier. Let $D = \sum d_i D_i$ and $D' = \sum d'_i D_i$. If $d'_i \ge d_i$ for every i, then $\operatorname{discrep}(X, D') \le \operatorname{discrep}(X, D)$. □

Every restriction on $\operatorname{discrep}(X, D)$ defines a class of pairs (X, D). The following cases emerged as the most important ones:

3.5 Definition. Let X be a normal scheme and $D = \sum d_i D_i$ a (not necessarily effective) $\mathbb{R}$-divisor such that $K_X + D$ is $\mathbb{R}$-Cartier. We say that (X, D) or $K_X + D$ is

$$\left\{\begin{array}{l} \text{terminal} \\ \text{canonical} \\ \text{klt (or Kawamata log terminal)} \\ \text{plt (or purely log terminal)} \\ \text{lc (or log canonical)} \end{array}\right. \quad \text{iff} \quad \operatorname{discrep}(X) \left\{\begin{array}{l} > 0, \\ \ge 0, \\ > -1 \quad \text{and } \llcorner D \lrcorner \le 0, \\ > -1, \\ \ge -1. \end{array}\right.$$

Equivalently, one can define klt by the condition $\operatorname{totaldiscrep}(X, D) > -1$.

In order to get a feeling for these concepts, let us give some examples. In dimension two these notions correspond to well-known classes of singularities. The proof of the first two parts is an easy exercise using the minimal resolution. The last two cases are trickier. See, for instance, [Kollár et al.92,3].

3.6 Theorem. *Let $0 \in X$ be a (germ of a) normal surface singularity over $\mathbb{C}$. Then X is*

$$\begin{array}{rcl} \textit{terminal} & \Leftrightarrow & \textit{smooth;} \\ \textit{canonical} & \Leftrightarrow & \mathbb{C}^2/(\textit{finite subgroup of } SL(2,\mathbb{C}))\textit{;} \\ \textit{klt} & \Leftrightarrow & \mathbb{C}^2/(\textit{finite subgroup of } GL(2,\mathbb{C}))\textit{;} \\ \textit{lc} & \Leftrightarrow & \textit{simple elliptic, cusp, smooth, or a} \\ & & \textit{quotient of these by a finite group.} \quad \square \end{array}$$

Log canonical pairs appear naturaly in many contexts:

3.7 Proposition. *Let X be a normal toric variety with open orbit $T \subset X$ and set $D = X - T$. Then (X, D) is lc. If K_X is $\mathbb{Q}$-Cartier, then $(X, 0)$ is klt.*

Proof. Let $f : Y \to X$ be a toric resolution and set $E = Y - f^{-1}(T)$. By [Fulton93,4.3], $K_X \sim -D$ and $K_Y \sim -E$. Thus $K_Y + E \sim f^*(K_X + D)$, and so (X, D) is lc. The rest is easy (cf. (3.4.1.5)). □

As was observed by [Alexeev96, Sec.3], this easily implies that Baily–Borel compactifications are also log canonical:

3.7.1 Corollary. *Let D be a bounded symmetric domain and Γ an arithmetic subgroup of* $\operatorname{Aut}(D)$. *Let $(D/\Gamma)^*$ denote the Baily–Borel compactification of D/Γ. There is a natural choice for a $\mathbb{Q}$-divisor Δ, supported on the boundary, such that $((D/\Gamma)^*, \Delta)$ is lc.* $\square$

3.8 Example: Cones.

Let Y be a smooth variety and $E \subset Y$ a smooth divisor with normal bundle L^{-1}. If L is ample then E is contractible to a point, at least as an analytic space. Let $f : Y \to X$ be this contraction.

(3.8.1) K_X is $\mathbb{Q}$-Cartier iff K_E and $L|E$ are linearly dependent in $\operatorname{Pic} E$. Assume that this is the case and write

$$K_E \equiv -(1+a)(L|E), \quad \text{thus} \quad K_Y \equiv f^*K_X + aE.$$

(3.8.2) If K_E is ample then $a < -1$; X is not log canonical.

(3.8.3) If $K_E = 0$ then $a = -1$ and X is log canonical.

(3.8.4) If $-K_E$ is ample, that is E is Fano, we have 3 cases.

(3.8.4.1) $0 < L < -K_E$. Then $a > 0$ and X is terminal. Notice that there are very few Fano varieties for which this can happen, so this is a very rare case.

(3.8.4.2) $L = -K_E$. Then $a = 0$ and X is canonical. For every Fano variety we get one example, still only a few cases.

(3.8.4.3) $L > -K_E$. Then $-1 < a < 0$ and X is klt. For every Fano variety we get infinitely many cases.

3.9 Example. Let X be a smooth variety and $D \subset X$ a divisor. The discrepancies are all integers.

(3.9.1) Show that (X, D) is terminal iff $D = \emptyset$.

(3.9.2) We see in (7.9) that (X, D) is canonical iff D is reduced, normal and has rational singularities only.

(3.9.3) The case when (X, D) is log canonical does not seem to have a traditional name. Show that if $\dim X = 2$, then (X, D) is log canonical iff D has normal crossings only. In dimension three the list is much longer. D can have pinch points, rational double points, simple elliptic points (like $(x^2 + y^3 + z^6 = 0)$) and cusps (like $(xyz + z^p + y^q + z^r = 0)$).

(3.9.4) Let X be a smooth variety of dimension n and B a $\mathbb{Q}$-divisor. Assume that (X, B) is terminal (resp. canonical, plt, lc). Study the blow up of $x \in X$ to show that $\operatorname{mult}_x(B) < n - 1$ (resp. $\le n - 1$, $< n$, $\le n$). (The converse is not true, see (3.14).)

The definition (3.5) requires some understanding of all exceptional divisors of all birational modifications of X. The following lemmas reduce the computation of $\operatorname{discrep}(X, D)$ to a finite computation in principle.

3.10 Lemma. *Notation as above. Let $f : X' \to X$ be a proper, birational morphism and write $K_{X'} + D' \equiv f^*(K_X + D)$ (using (3.3.2)).*

(3.10.1) $a(E, X, D) = a(E, X', D')$ for every divisor E of $K(X)$.

(3.10.2) (X, D) is klt (resp. lc) iff (X', D') is klt (resp. lc).

(3.10.3) (X, D) is plt iff (X', D') is plt and $a(E, X, D) > -1$ for every exceptional divisor $E \subset X'$ of f.

(3.10.4) (X, D) *is terminal (resp. canonical) iff* (X', D') *is terminal (resp. canonical) and* $a(E, X, D) > 0$ *(resp.* $a(E, X, D) \geq 0$*) for every exceptional divisor* $E \subset X'$ *of* f. $\square$

3.11 Lemma. *Let* X *be a smooth scheme and* $\sum D_i$ *a normal crossing divisor. Set* $D = \sum d_i D_i$ *and assume that* $d_i \leq 1$ *for every* i. *Let* $x \in X$ *be a (not necessarily closed) point and* E *a divisor of* $K(X)$ *such that* $\operatorname{Center}_X(E) = x$. *Then*

(3.11.1) $a(E, X, D) \geq \operatorname{codim}(x, X) - 1 - \sum_{j: x \in D_j} d_j$.

(3.11.2) $\operatorname{totaldiscrep}(X, D) = \min\{0, -d_i\}$, *and*

(3.11.3) $\operatorname{discrep}(X, D) = \min\{1, 1 - d_i, 1 - d_i - d_j \colon D_i \cap D_j \neq \emptyset\}$.

Proof. Pick a birational morphism $f : Y \to X$ such that $E \subset Y$ is an exceptional divisor with general point $y \in E$. By localizing at $x = f(y)$ we may assume that x is a closed point. Pick a local coordinate system $\{y_j\}$ such that $E = (y_1 = 0)$. Possibly after reindexing, let $D_1, \dots, D_k$ be those divisors which pass through $f(y)$. Let x_i be a local coordinate system at x such that $D_i = (x_i = 0)$ for $i = 1, \dots, k$. Set $c_i = d_i$ for $i \leq k$ and $c_i = 0$ for $i > k$. We can write $f^* x_i = y_1^{a_i} u_i$ where $a_i > 0$ for $i \leq k$ and u_i is a unit at y. Then

$$f^* \frac{dx_i}{x_i^{c_i}} = a_i y_1^{(1-c_i)a_i - 1} u_i^{1-c_i} dy_1 + y_1^{(1-c_i)a_i} \omega_i, \quad \text{where } \omega_i \text{ is regular at } y.$$

Therefore in

$$f^* \frac{dx_1 \wedge \cdots \wedge dx_n}{x_1^{c_1} \cdots x_n^{c_n}}$$

the only terms which could have a pole at y are of the form

$$\begin{aligned} & y_1^{A_i} dy_1 \wedge \omega_1 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega_n \quad \text{where} \\ & A_i = -1 + \sum_{j=1}^{n} (1 - c_j) a_j = -1 + \sum_{j=1}^{k} (1 - d_j) a_j \geq k - 1 - \sum_{j=1}^{k} d_j. \end{aligned}$$

The rest is a simple computation. $\square$

3.12 Corollary. *Let* X *be a smooth scheme and* $\sum D_i$ *a normal crossing divisor. Then* $(X, \sum d_i D_i)$ *is*

$$\left\{ \begin{array}{l} \textit{terminal} \\ \textit{canonical} \\ \textit{klt} \\ \textit{plt} \\ \textit{lc} \end{array} \right. \qquad \textit{iff} \qquad \left\{ \begin{array}{ll} d_i < 1 & \textit{and } d_i + d_j < 1 \textit{ if } D_i \cap D_j \neq \emptyset, \\ d_i \leq 1 & \textit{and } d_i + d_j \leq 1 \textit{ if } D_i \cap D_j \neq \emptyset, \\ d_i < 1, & \\ d_i \leq 1 & \textit{and } d_i + d_j < 2 \textit{ if } D_i \cap D_j \neq \emptyset, \\ d_i \leq 1. & \end{array} \right. \qquad \square$$

3.13 Corollary. *Let* $(X, D = \sum d_i D_i)$ *be a pair and* $f : Y \to X$ *a log resolution of singularities. Let* $E_j \subset Y$ *be the exceptional divisors of* f. *Then* (X, D) *is lc iff* $\min\{a(E_j, X, D), -d_i\} \geq -1$.

If (X, D) *is lc then the following hold:*

(3.13.1) $\operatorname{totaldiscrep}(X, D) = \min\{a(E_j, X, D), -d_i\}$.
(3.13.2) $\operatorname{discrep}(X, \emptyset) = \min\{a(E_j, X, D), 1\}$.
(3.13.3) If $f_*^{-1}(\operatorname{Supp} D)$ *is smooth, then* $\operatorname{discrep}(X, D) = \min\{a(E_j, X, D), 1 - d_i\}$. □

In the canonical case, the conditions (3.12) imposed by codimension one and two points of X impose stronger restrictions than those imposed by higher codimension points. One might expect that this is also the case for arbitrary divisors. The following exercise shows that this is not the case:

3.14 Exercise. Let X be a smooth variety and D an effective $\mathbb{Q}$-divisor. Show that
(3.14.1) If $\operatorname{mult}_x D \leq 1$ for every $x \in X$ then (X, D) is canonical.
(3.14.2) If $\dim X = 2$ then the converse also holds.
(3.14.3) Give an example of a pair $(\mathbb{C}^2, D)$ such that $\operatorname{mult}_x D \leq 1$ outside the origin, $\operatorname{mult}_0 D = 1 + \epsilon$ and $(\mathbb{C}^2, D)$ is not even log canonical.
(3.14.4) If $\dim X \geq 3$ then the converse of (3.14.1) does not hold.
(3.14.5) For every $\epsilon > 0$ give an example of a pair $(\mathbb{C}^n, D)$ such that $\operatorname{mult}_x D \leq 1$ for x outside the origin, $\operatorname{mult}_0 D = 1 + \epsilon$ and $(\mathbb{C}^n, D)$ is not even log canonical.

Unfortunately, (3.4.1.3) does not characterize smooth points, except in dimension 2. The problem is that the discrepancy 1 is caused by codimenson 2 effects, and it gives very little information about X in higher codimension. The following question corrects this:

3.15 Conjecture. [Shokurov88] *Let X be a normal scheme, D an effective $\mathbb{R}$-divisor and $x \in X$ a closed point. Assume that $K_X + D$ is $\mathbb{R}$-Cartier. Then*

$$\inf_E \{a(E, X, D) \mid \operatorname{Center}_X(E) = x\} \leq \dim X - 1,$$

and equality holds iff X is smooth and $0 \notin \operatorname{Supp} D$. (This is proved for $\dim X \leq 3$ by [Markushevich96]*.)*

It is also possible to compare the discrepancies for pull-backs by finite morphisms, though the relationship is not as close as in (3.10).

3.16 Proposition. *Let $p : X \to Y$ be a finite and dominant morphism between normal varieties. Let D_Y be a $\mathbb{Q}$-divisor on Y and define D_X by the formula*

$$K_X + D_X = p^*(K_Y + D_Y), \quad \text{that is,} \quad D_X = p^* D_Y - K_{X/Y}.$$

Then
(3.16.1) $1+\operatorname{discrep}(Y, D_Y) \leq 1+\operatorname{discrep}(X, D_X) \leq \deg(X/Y)(1+\operatorname{discrep}(Y, D_Y))$.
(3.16.2) (X, D_X) is lc (resp. klt) iff (Y, D_Y) is lc (resp. klt).

Proof. (3.16.2) is a special case of (3.16.1).
In order to prove (3.16.1), let $f_Y : Y' \to Y$ be a proper birational morphism and $X' \to Y' \times_Y X$ the normalization of the (dominant component of the) fiber product with projection maps $f_X : X' \to X$ and $q : X' \to Y'$. Write

$$K_{X'} + D_{X'} = f_X^*(K_X + D_X) \quad \text{and} \quad K_{Y'} + D_{Y'} = f_Y^*(K_Y + D_Y).$$

$D_{X'}$ and $D_{Y'}$ are related by the formula

$$K_{X'} + D_{X'} = q^*(K_{Y'} + D_{Y'}).$$

In order to compare the coefficients in $D_{X'}$ and in $D_{Y'}$, we may localize at the generic point of a component $E_Y \subset \operatorname{Supp} D_{Y'}$. Let $E_X \subset \operatorname{Supp} D_{X'}$ be a component which dominates E_Y. Thus we are reduced to the case when $y \in Y'$ and $x \in X'$ are smooth pointed curves, $D'_Y = d_y[y]$, $D'_X = d_x[x]$ and $q : X' \to Y'$ has ramification index r at x. Here $d_x = -a(X, D_X, E_X)$, $d_y = -a(Y, D_Y, E_Y)$ and $r \le \deg(X/Y)$. Then $d_x = rd_y - (r-1)$, or equivalently,

$$a(X, D_X, E_X) + 1 = r(a(Y, D_Y, E_Y) + 1).$$

As $Y' \to Y$ runs through all proper birational morphisms, the corresponding morphisms $X' \to X$ do not give all possible proper birational morphisms. Nonetheless, every algebraic valuation of $K(X)$ with center on X appears on some $X' \to X$ by (3.17). This shows (3.16.1). $\square$

The proof used the next result which is essentially due to [Zariski39]. See also [Artin86, 5.1; Kollár95b,VI.1.3]. In many instances it can be used instead of the resolution of singularities.

3.17 Theorem. *Let X, Y be integral schemes of finite type over a field or over $\mathbb{Z}$, and $f : Y \to X$ a dominant morphism. Let $D \subset Y$ be an irreducible divisor and $y \in D$ the generic point. Assume that Y is normal at y. We define a sequence of schemes and maps as follows:*

$X_0 = X, f_0 = f$.

If $f_i : Y \dashrightarrow X_i$ is already defined, then let $Z_i \subset X_i$ be the closure of $f_i(y)$. Let $X_{i+1} = B_{Z_i} X_i$ and $f_{i+1} : Y \dashrightarrow X_{i+1}$ the induced map.

Then $\dim Z_n \ge \dim X - 1$ and X_n is regular at the generic point of Z_n for some $n \ge 0$. $\square$

The notion of a klt pair (X, Δ) also emerged naturally in the theory of singular metrics on line bundles. I just give the basic definition and prove its equivalence with the algebro-geometric one. For a more detailed exposition of the theory, see, for instance, [Demailly92,94].

3.18 Singular Metrics on Line Bundles. Let L be a line bundle on a complex manifold M. A *singular Hermitian metric* $|| \ ||$ on L is a Hermitian metric on $L|M - Z$ (where Z is a measure zero set) such that if $U \subset M$ is any open subset, $u : L|U \cong U \times \mathbb{C}$ a local trivialization and f a local generating section over U then

$$||f|| = |u(f)| \cdot e^{-\phi}$$

where $| \ |$ is the usual absolute value on $\mathbb{C}$ and $\phi \in L^1_{loc}(U)$. (The latter assumption assures that $\partial\bar{\partial}\phi$ exists as a current on M. We do not use it.)

We say that the metric is L^p on M if $e^{-\phi}$ is locally L^p for every point. (This is clearly independent of the local trivializations.)

3.19 Examples. (3.19.1) Let D be a divisor and $L = \mathcal{O}_X(D)$. L has a natural section f coming from the constant section 1 of $\mathcal{O}_X$. A natural choice of the metric on L is to set $||f|| = 1$ everywhere. This metric is singular along D. If h is a local equation of D at a point $x \in D$ then $h^{-1}f$ is a local generating section of L at x and

$$||h^{-1}f|| = e^{-\log |h|}.$$

(3.19.2) Let L be a line bundle on X. Assume that $L^n \cong M(D)$ for some line bundle M and effective divisor D. Let $||\ ||_M$ be a continuous Hermitian metric on M. We construct a singular metric $||\ ||_L$ on L as follows.

Let f be a local section of L at a point $x \in D$ and h a local equation for D at x. hf^n is a local section of M, thus we can set

$$||f||_L := (||hf^n||_M)^{1/n} e^{-log|h|/n}.$$

The first factor on the right is continuous and positive if f is a local generator. Thus $||\ ||_L$ is L^p iff the exponential factor

$$e^{-log|h|/n} = |h|^{-1/n} \quad \text{is } L^p.$$

(3.19.3) Assume that $D = n\sum d_i\Delta_i$ where $\Delta_i = (x_i = 0)$ for a local coordinate system. Then

$$||\ ||_L \text{ is } L^2 \text{ at } x \quad \Leftrightarrow \quad \prod |x_i|^{-d_i} \text{ is } L^2 \text{ near } x \quad \Leftrightarrow \quad d_i < 1 \text{ for every } i.$$

More generally we have:

3.20 Proposition. *Let X be a smooth manifold and D a divisor on X. Let L be a line bundle on X and assume that $L^n = M(D)$ for some line bundle M. Set $\Delta = D/n$. Let $||\ ||_L$ be the singular metric constructed on L as in (3.19.2). Then*

$$||\ ||_L \quad \textit{is } L^2 \quad \Leftrightarrow \quad (X, \Delta) \quad \textit{is klt.}$$

Proof. Let $f : Y \to X$ be a log resolution of (X, D) and $E \subset Y$ the exceptional divisor. Both properties are local in X, so pick a point $x \in D$ and fix a local coordinate system $\{x_i\}$. Let h be a local equation for D. Set $\omega_x = dx_1 \wedge \cdots \wedge dx_k$. $||\ ||_L$ is L^2 iff

$$\int |h|^{-2/n}\omega_x \wedge \bar{\omega}_x < \infty. \tag{3.20.1}$$

This is equivalent to saying that $|h|^{-1/n}\omega_x$ is L^2. The advantage of putting ω_x in is that in this form the condition is invariant under pull backs. Thus (3.20.1) is equivalent to $f^*(|h|^{-1/n}\omega_x)$ being L^2 on Y.

This is a local condition on Y, so pick a point and a local coordinate system $\{y_i\}$ such that every irreducible component $F_i \subset E + f_*^{-1}(D)$ is defined locally as $F_i = (y_i = 0)$. Set $\omega_y = dy_1 \wedge \cdots \wedge dy_k$. We can write

$$f^*(|h|^{-1/n}\omega_x) = \omega_y \cdot u \cdot \prod |y_i|^{a_i},$$

where u is invertible and $a_i = a(F_i, X, \Delta)$. Thus $f^*(|h|^{-1/n}\omega_x)$ is L^2 iff $a(F_i, X, \Delta) > -1$ for every i. This happens precisely when (X, Δ) is klt. $\square$

4. Bertini Theorems

The classical Bertini theorem says that on a smooth variety a general member of a base point free linear system is again smooth. Actually, the original version of the Bertini theorem applies in the case of linear systems with base points and it says the following:

4.1 Theorem. *(Bertini) Let X be a smooth variety over a field of characteristic zero and $|B_1, \dots, B_k|$ the linear system spanned by the (effective) divisors B_i. Let $B \in |B_1, \dots, B_k|$ be a general member. Then*

$$\operatorname{mult}_x B \le 1 + \inf_i \{\operatorname{mult}_x B_i\} \quad \textit{for every } x \in X. \qquad \square$$

If $x \notin B_i$ then $\operatorname{mult}_x B_i = 0$, and so this theorem includes the usual form as a special case.

Recently [Xu94] studied the case of linear systems with base points. [Xu94] proved a variant of (4.1) which also implies similar results about infinitely near singularities of B, though it is not clear to me whether his results can be interpreted in terms of multiplicities alone. The results of this section grew out of trying to understand his results in terms of discrepancies.

In order to get a better idea of what is possible, let us consider the simplest case of linear systems with base points: at each point there is a smooth member. The general member will not, in general, be smooth:

4.2 Examples. (4.2.1) Let $X = \mathbb{C}^n$ and $f \in \mathbb{C}[x_3, \dots, x_n]$ such that $(f = 0)$ has an isolated singularity at the origin. Consider the linear system $|B| = (\lambda x_1 + \mu x_1 x_2 + \nu f = 0)$. At each point there is a smooth member and the general member is singular at $(0, -\lambda/\mu, 0, \dots, 0)$. All general members are isomorphic to $(x_1 x_2 + f = 0)$. This way we can get any isolated cA-type singularity (4.3) for suitable choice of f.

(4.2.2) As above, let $n = 3$ and $f = x_3^{m+1}$, then general members have an A_m singularity, which is canonical but not terminal.

(4.2.3) Consider the linear system $\lambda(x^2 + zy^2) + \mu y^2$. At any point $x \in \mathbb{C}^3$ its general member has a cA-type singularity, but the general member has a moving pinch point.

4.3 Definition. Let $0 \in H \subset X$ (where X is smooth at 0) be a hypersurface singularity. In local coordinates $H = (g = 0)$; let g_2 denote the quadratic part of g. We say that H has type cA if either H is smooth or g_2 has rank at least 2 (as a quadratic form). Equivalently, there are suitable local analytic (or formal) coordinates $x_1, \dots, x_n$ such that $H = (x_1 x_2 + f(x_3, \dots, x_n) = 0)$ [AGV85,I.11.1].

Let $0 \in Y \subset X$ be a smooth hypersurface. If $H \cap Y$ has a cA-type singularity at 0 then so does H. By following the Hironaka resolution process, it is easy to see that a normal cA-type singularity is canonical.

In the smooth case one can give a very precise description of the possible singularities occurring on general members of linear systems.

4.4 Theorem. *Let X be a smooth variety over a field of characteristic zero and $|B|$ a linear system of Cartier divisors. Assume that for every $x \in X$ there is a $B(x) \in |B|$ such that $B(x)$ is smooth at x (or $x \notin B(x)$).*

Then a general member $B^g \in |B|$ has only type cA singularities.

Proof. The result is clear if $\dim X = 1$, thus assume that it holds for smaller dimensional schemes. By Noetherian induction it is sufficient to prove the following:

(4.4.1) For every irreducible subvariety $Z \subset X$ there is an open subset $Z^0 \subset Z$ such that a general member $B^g \in |B|$ has only type cA singularities at points of Z^0.

If $Z \not\subset \operatorname{Bs}|B|$ then let $Z^0 = Z - \operatorname{Bs}|B|$. The base point free Bertini then implies (4.4.1).

Next assume that $Z \subset \operatorname{Bs}|B|$ and $\operatorname{codim}(Z, X) = 1$. The assumptions imply that Z is smooth and $|B| - Z$ induces a base point free linear system on Z. Again by Bertini, the general member $B^g - Z$ intersects Z transversally. Thus at every point of Z the divisor B^g is either smooth or its local equation is $x_1x_2 = 0$.

Finally assume that $Z \subset \operatorname{Bs}|B|$ and $\operatorname{codim}(Z, X) > 1$. Pick a smooth point $z \in Z$. Let Y be a hypersurface in X such that:

(4.4.2.1) $Z \subset Y$ and Y is smooth at z, and

(4.4.2.2) Y is transversal to $B(z)$ at z.

Let $Y^0 \subset Y$ be an open set containing z such that Y^0 and $B(z) \cap Y^0$ are smooth. Let $|B_{Y^0}|$ be the restriction of the linear system $|B|$ to Y^0. By induction $B^g \cap Y^0$ has only type cA singularities, hence by (4.3) B^g has only type cA singularities at points of $Z \cap Y^0$. $\square$

The above results say that if $|B|$ is a linear system and for each $x \in X$ there is a $B(x) \in |B|$ which is not too singular at x, then the general $B^g \in |B|$ has only somewhat worse singularities. This raises the question: is there a class of singularities for which the general member does not get any worse?

Assume that **S** is such a class. By (4.2.1) **S** contains all cA-type singularities. Thus by (4.2.1) it also has to contain pinch points and maybe many more singularities. It is not at all clear that this process ever terminates with a reasonably small class **S**. I do not know what is the smallest class **S** (it is clear that it exists).

The following result provides one such example for **S**, under a mild assumption on the linear series. More examples are contained in (4.8).

4.5 Theorem. *Let X be a smooth variety over a field of characteristic zero and $|L|$ a linear system of Cartier divisors such that* $\operatorname{Bs}|L|$ *has codimension at least two. Assume that for every $x \in X$ there is a $B_x \in |L|$ such that B_x has a rational singularity at x.*

Then a general member $B \in |L|$ has only rational singularities.

The above result has nothing to do with discrepancies or with canonical pairs. Still, I have no idea how to prove it without the machinery of adjunction and canonical pairs. After translating the problem to our language, it becomes easy.

Proof. B is a Cartier divisor on a smooth variety, hence ω_B is locally free. By (11.1.1) B has rational singularities iff it has canonical singularities. By (7.9) the latter holds iff the pair (X, B) is canonical. Thus (4.5) is equivalent to the following version:

4.5.1 Theorem. *Let X be a smooth variety over a field of characteristic zero and $|L|$ a linear system of Cartier divisors such that* $\operatorname{Bs}|L|$ *has codimension at least two.*

Assume that for every $x \in X$ there is a $B_x \in |L|$ such that (X, B_x) is canonical at x.

Let $B \in |L|$ be a general member. Then (X, B) is canonical.

Proof. Let $f : Y \to X$ be a proper birational morphism such that Y is smooth, $f^*|L| = F + |M|$ where $|M|$ is base point free and $F +$ (f-exceptional divisors) has only normal crossings. Let $K_Y = f^*K_X + E$. For a given divisor $B \in |L|$, let $B^Y := f^*(B) - F \in |M|$ denote the corresponding member. We can write $B^Y = f_*^{-1}B + N$ where N is effective and empty for general $B \in |L|$. (If $\mathrm{Bs}\,|L|$ contains a divisor, then N is not effective and the rest of the proof does not work.) Then

$$K_Y + f_*^{-1}B_x = K_Y + B_x^Y - N_x = f^*(K_X + B_x) + (E - F - N_x).$$

By assumption (X, B_x) is canonical at x, thus $E-F$ is effective over a neighborhood of x. This holds for any x, thus $E - F$ is effective.

Choose $B \in |L|$ such that the corresponding B^Y is irreducible and intersects $E - F$ transversally. These are both nonempty and open conditions. Then

$$K_Y + f_*^{-1}B = K_Y + B^Y = f^*(K_X + B) + (E - F).$$

This shows that (X, B) is canonical. $\square$

* * * * *

In the rest of the section we study Bertini-type theorems that compare the properties klt, lc and so on for the general members and for generators of linear systems. There are two ways of approaching this problem. One can look at singularities of divisors B or singularities of pairs (X, B). The second variant is better suited for the present purposes. In some cases the two versions are equivalent (4.9).

4.6 Definition. Let X be a normal, integral scheme, $D = \sum d_i D_i$ a $\mathbb{Q}$-divisor (not necessarily effective) and $|B_j|$ (not necessarily complete) linear systems of Weil divisors. Let $0 \le b_j \le 1$ be rational numbers such that $K_X + D + \sum b_j B_j$ is $\mathbb{Q}$-Cartier. Let E be a divisor of the function field $\mathbb{C}(X)$ and define

$$a(E, X, D + \sum b_j|B_j|) := \sup\{a(E, X, D + \sum b_j B_j')|B_j' \in |B_j|\}.$$

In the above formula it is sufficient to let B_j' run through a finite set of divisors spanning $|B_j|$. In particular the supremum is a maximum and if the b_j are rational then so is $a(E, X, D + \sum b_j|B_j|)$. We define as in (3.4)

$$\begin{aligned}&\mathrm{discrep}(X, D + \sum b_j|B_j|) =\\ &\quad = \inf_E\{a(E, X, D + \sum b_j|B_j|)|E \text{ is exceptional with nonempty center on } X\}.\end{aligned}$$

As in (3.5), we say that $(X, D + \sum b_j|B_j|)$ or $K_X + D + \sum b_j|B_j|$ is *terminal, canonical, klt, plt* resp. *lc* if $\mathrm{discrep}(X, D + \sum b_j|B_j|) > 0, \ge 0, > -1$ and $d_i, b_j < 1\ \forall j, > -1$ resp. ≥ -1.

The following properties are straightforward from the definitions.

4.7 Lemma. *Notation as above.*

(4.7.1) If $|B_1|$ is base point free then

$$a(E,X,D+\sum b_j|B_j|)=a(E,X,D+\sum_{j\geq 2} b_j|B_j|).$$

(4.7.2) If $F_j\subset \operatorname{Bs}|B_j|$ is a divisor, then

$$a(E,X,D+\sum b_j|B_j|)=a(E,X,(D+\sum b_jF_j)+\sum b_j|B_j-F_j|).$$

(4.7.3) Assume that the B_j are $\mathbb{Q}$-Cartier. Let $f:X'\to X$ be a proper, birational morphism and write $K_{X'}+D'=f^*(K_X+D)$. Then

$$a(E,X,D+\sum b_j|B_j|)=a(E,X',D'+\sum b_jf^*|B_j|).\quad\square$$

The following is a summary of the Bertini-type theorems for linear systems. It should be clear from the proof that there are several other variants involving different values of the discrepancy. Also, one can be more precise concerning the interplay of the allowable coefficients d_i and b_j.

4.8 Theorem. *Let X be a normal, integral, excellent scheme over a field of characteristic zero, $D=\sum d_iD_i$ a $\mathbb{Q}$-divisor (not necessarily effective) and $\sum b_j|B_j|$ a formal sum of (not necessarily complete) linear systems of Weil divisors, $0\leq b_j\leq 1$. Assume that K_X+D and the B_j are $\mathbb{Q}$-Cartier. Let B_j^g be a general member of $|B_j|$. Then:*

(4.8.1) $(X,D+\sum b_j|B_j|)$ is lc $\Leftrightarrow$ $(X,D+\sum b_jB_j^g)$ is lc.

(4.8.2) $(X,D+\sum b_j|B_j|)$ is klt $\Leftrightarrow$ $(X,D+\sum b_jB_j^g)$ is klt, for $0\leq b_j<1$.

Assume that general members of $|B_1|$ are irreducible and (X,D) is klt. Then:

(4.8.2') $(X,D+b_1|B_1|)$ is plt $\Leftrightarrow$ $(X,D+b_1B_1^g)$ is plt.

Assume that $d_i\leq 1/2$ for every i. Then:

(4.8.3) $(X,D+\sum b_j|B_j|)$ is canonical $\Leftrightarrow$ $(X,D+\sum b_jB_j^g)$ is canonical, for $0\leq b_j\leq 1/2$.

(4.8.4) $(X,D+\sum b_j|B_j|)$ is terminal $\Leftrightarrow$ $(X,D+\sum b_jB_j^g)$ is terminal, for $0\leq b_j<1/2$.

Assume that general members of $|B_1|$ are irreducible and $D=\emptyset$. Then:

(4.8.3') $(X,b_1|B_1|)$ is canonical $\Leftrightarrow$ $(X,b_1B_1^g)$ is canonical.

(4.8.4') $(X,b_1|B_1|)$ is terminal $\Leftrightarrow$ $(X,b_1B_1^g)$ is terminal, for $0\leq b_1<1$.

Proof. Let $f:X'\to X$ be a proper birational morphism such that X' is smooth, $f^*|B_j|=F_j+|M_j|$ where $|M_j|$ is base point free for every j and $f^*D+\sum F_j+$ (f-exceptional divisors) has only normal crossings. By (4.7.1–3) we see that $(X,D+\sum b_j|B_j|)$ is lc (resp. klt) iff $(X',D'+\sum b_jF_j)$ is lc (resp. klt). Let $D'+\sum b_jF_j=\sum e_kE_k$. By (3.12), $(X',\sum e_kE_k)$ is lc (resp. klt) iff $e_k\leq 1$ (resp. $e_k<1$) for every k.

Let $M_j^g:=f^*B_j^g-F_j\in|M_j|$ be a general member. Then

$$K_{X'}+\sum e_kE_k+\sum b_jM_j^g=f^*(K_X+D+\sum b_jB_j^g),$$

and $\sum E_k + \sum M_j^g$ is still a normal crossing divisor by the usual Bertini theorem. Therefore

$$\begin{aligned} &K_X + D + \sum b_j B_j^g \quad \text{is lc (resp. klt)} \Leftrightarrow \\ &K_{X'} + \sum e_k E_k + \sum b_j M_j^g \quad \text{is lc (resp. klt)}, \Leftrightarrow \\ &e_k, b_j \le 1 \text{ (resp. } e_k, b_j < 1) \text{ for every } j, k. \end{aligned}$$

This shows (4.8.1–2).

In order to obtain (4.8.2') we need the additional remark that if $\sum E_k + M_1^g$ is a normal crossing divisor, M_1^g is irreducible and $e_k < 1$ for every k then $(X', \sum e_k E_k + b_1 M_1^g)$ is plt for every $b_1 \le 1$.

The proofs of the remaining assertions are similar. $(X, D + \sum b_j |B_j|)$ is canonical (resp. terminal) iff $e_k \le 0$ (resp. $e_k < 0$) for every k such that E_k is f-exceptional. Therefore $(X', \sum e_k E_k + \sum b_j M_j^g)$ is canonical (resp. terminal) iff $(X', f_*^{-1} D + \sum b_j M_j^g)$ is canonical (resp. terminal).

In (4.8.3–4) the coefficient 1/2 comes in because two divisors M_i^g and M_j^g may intersect. In (4.8.3'–4') we know that M_1^g is irreducible, thus we only have to make sure that its coefficient is at most 1 (resp. less than 1). □

It is frequently more convenient to have Bertini-type theorems which give information about the singularities of the general member of a linear system directly. This is rather straightforward for base point free linear systems (7.7). For linear systems with base points the situation is more difficult to analyze. The known cases of the inversion of adjunction conjecture (7.3) can be used to transform the $b_j = 1$ cases of (4.8) to Bertini-type results that concern only the singularities of divisors. I formulate it only for Cartier divisors. Using the notion of different (cf. [Kollár et al.92,16.6]) it can be stated in case (X, Δ) is klt and $|B|$ is a linear system of $\mathbb{Q}$-Cartier Weil divisors.

4.9 Corollary. *Let X be a scheme over a field of characteristic zero and $|B|$ a linear system of Cartier divisors such that the general member of $|B|$ is irreducible.*

(4.9.1) Assume that X is klt and for every $x \in X$ there is a $B(x) \in |B|$ such that $B(x)$ is klt at x (or $x \notin B(x)$).

Then B^g is klt for general $B^g \in |B|$.

(4.9.2) Assume that X is canonical and of index 1 (for instance smooth). Assume also that for every $x \in X$ there is a $B(x) \in |B|$ such that $B(x)$ is canonical at x (or $x \notin B(x)$).

Then B^g is canonical for general $B^g \in |B|$.

Proof. Let $B' \in |B|$ be any divisor. By (7.5) we see that $K_X + B'$ is plt at a point $x \in X$ iff B' is klt at x. Thus (4.9.1) follows from (4.8.2').

If X has index 1 and $|B|$ is a linear system of Cartier divisors, then every member of $|B|$ has index 1. Hence klt is the same as canonical. Thus (4.9.1) implies (4.9.2). □

4.9.3 Remark. It is expected that (4.9.2) remains true even if X has higher index canonical singularities.

5. Effective Base Point Freeness

In its simplest form the problem is the following:

5.1 Problem. Let X be a projective variety and L an ample line bundle on X. Try to construct a very ample (or maybe just globally generated) line bundle, using as little information about X and L as possible.

The first major result of this type is "Matsusaka's big theorem" which asserts the following:

5.2 Theorem. [Matsusaka72] *There is a function $\Phi(x, y, z)$ with the following property:*

If X is an n-dimensional smooth projective variety over a field of characteristic zero and L an ample line bundle on X, then

$$L^m \quad \text{is very ample for} \quad m \geq \Phi(n, (L^n), (K_X \cdot L^{n-1})).$$

[Matsusaka86] generalized this to the case when X has at worst rational singularities. The methods of [Matsusaka72,86] do not give any information about ϕ, beyond its existence.

Mukai pointed out that a reasonable bound can be expected if one tries to find a very ample line bundle of the form $K_X \otimes L^m$. The precise conjecture was formulated by [Fujita87]:

5.3 Conjecture. *Let Y be a smooth projective variety, and L an ample line bundle on Y. Then:*

(5.3.1) $K_Y \otimes L^m$ is globally generated for $m \geq \dim Y + 1$, and

(5.3.2) $K_Y \otimes L^m$ is very ample for $m \geq \dim Y + 2$.

Both of the bounds are sharp for $Y = \mathbb{P}^n$ and $L = \mathcal{O}(1)$. The following example gives many more such cases:

5.3.3 Example. (Lazarsfeld) Let $X \subset \mathbb{P}^{n+1}$ be a hypersurface of degree d and $L \subset \mathbb{P}^{n+1}$ a line intersecting X in distinct points $P_1, \dots, P_d$. Let $p : Y \to X$ be the blow up of $P_1, \dots, P_{d-1}$ with exceptional divisors E_i. Set $L = p^*\mathcal{O}_X(1)(-\sum_{i=1}^{d-1} E_i)$.

Show that L is nef and big, and in fact it is generated by global sections outside P_d. L is not always ample (for instance, if X contains a line through P_1) but L is ample for general X and L for d sufficiently large.

$K_Y \otimes L^n \cong p^*\mathcal{O}_X(d-2)(-\sum_{i=1}^{d-1} E_i)$, and this has P_d as its base point.

Another series of examples is in [Kawachi96].

5.3.4 Remarks. This conjecture is true in low dimensions. The case $\dim Y = 1$ is very easy. The surface case follows from [Reider88]. (5.3.1) is quite hard in dimension three [Ein-Lazarsfeld93]. A very readable introduction is provided by the lectures [Lazarsfeld96]. The first step in all dimensions was proved by [Demailly93] who showed that under the above assumptions $K_Y^2 \otimes L^m$ is very ample for $m \geq 12n^n$ where $n = \dim Y$.

These results seem to furnish rather strong evidence in favor of (5.3), but looking at the proofs gives a less optimistic picture. The method of [Ein-Lazarsfeld93] gives a base point freeness result assuming that (L^3) is large. In some vague sense only finitely many types of cases remain to be analyzed. The study of these cases requires considerable care and several ad hoc arguments. This is especially the case for the proof of variants of (5.4) in dimension 3 given by [Ein-Lazarsfeld93] and improved by [Fujita94].

Recently [Kawamata96] proved (5.3) in dimension 4, and [Smith95] showed that (5.3) holds in positive characteristic for L very ample. [Helmke96] considerably improved on the earlier methods. With his approach the low dimensional cases are now quite satisfactory, but for large dimensions (5.3) is still out of reach.

The above conjecture was given a more local form in [Ein-Lazarsfeld93]. As before, the actual bounds are inspired by the worst known example $(\mathbb{P}^n, \mathcal{O}(n))$.

5.4 Conjecture. *Let Y be a smooth projective variety, $y \in Y$ a closed point and L a nef and big line bundle on Y. Assume that if $y \in Z \subset Y$ is an irreducible (positive dimensional) subvariety then*

$$(L^{\dim Z} \cdot Z) \geq (\dim Y)^{\dim Z}, \quad \textit{and} \quad (L^{\dim Y}) > (\dim Y)^{\dim Y}.$$

Then $K_Y \otimes L$ has a section which is nonzero at y.

The higher dimensional situation was recently greatly clarified by [Demailly94; Angehrn-Siu95; Tsuji95]. Their results are weaker than (5.4), but the proofs are very natural and the bounds quite good.

5.5 Theorem. *Let Y be a smooth projective variety and L an ample line bundle on Y. Then*

(5.5.1) $K_Y \otimes L^m$ is generated by global sections for $m > \binom{\dim Y+1}{2}$.

(5.5.2) Global sections of $K_Y \otimes L^m$ separate points for $m \geq \binom{\dim Y+2}{2}$.

5.6 Idea of the proof. Assume for simplicity that the linear system $|L|$ is base point free. Let $D \in |L|$ be a general smooth member. For $m > 0$ we have an exact sequence

$$H^0(X, K_X \otimes L^{m+1}) \to H^0(D, K_D \otimes L^m) \to H^1(X, K_X \otimes L^m) = 0.$$

Thus an induction on the dimension produces sections, giving even the original conjecture (5.3).

In general we cannot assume that $|L|$ is nonempty, let alone that it has a smooth member. Assume instead that

$$L \equiv M + D + \Delta, \tag{5.6.1}$$

where M is a nef and big $\mathbb{Q}$-divisor, D a smooth (integral) divisor and Δ a $\mathbb{Q}$-divisor such that $\llcorner \Delta \lrcorner = \emptyset$ and everything is in normal crossing. As before we get an exact sequence

$$H^0(X, K_X \otimes L^{m+1}) \to H^0(D, K_X \otimes L^{m+1}|D) \to H^1(X, K_X \otimes L^{m+1}(-D)).$$

Observe that

$$\begin{aligned} K_X \otimes L^{m+1}(-D) &\equiv K_X + \Delta + mL + M, \quad \text{and} \\ K_X \otimes L^{m+1}|D &\equiv K_D + \Delta|D + (mL + M)|D. \end{aligned}$$

Thus by (2.3) $H^1(X, K_X \otimes L^{m+1}(-D)) = 0$, and we can run induction on the dimension as before, assuming that we can make the whole process work in the log version.

The assumption (5.6.1) seems strong, but it is easy to achieve. Pick a point $x \in X$ and assume that $(L^n) > 1$. By (6.1) we can find a $\mathbb{Q}$-divisor $L \equiv B$ such that $c := \operatorname{mult}_x B > 1$. Let $\pi : X' \to X$ be the blow up of x and $E \subset Y$ the exceptional divisor. Then $\pi^* B \equiv cE + B'$ and $c > 1$. After further blowing ups the normal crossing assumption can be satisfied, and we obtain a proper birational morphism $p : Y \to X$ such that $p^* B = \sum c_i E_i$ and $\max\{c_i\} > 1$. For suitable indexing the maximum is achieved for c_0. Assume for simplicity that $c_i < c_0$ for $i > 0$. (Paragraph (6.3.5) shows what to do otherwise.) Then

$$p^* L \equiv (1 - c_0^{-1}) p^* L + E_0 + \sum_{i>0} \frac{c_i}{c_0} E_i,$$

exactly as required for (5.6.1).

The problem is that the pull back of L is no longer ample, only nef. In the worst case $p^* L|D \equiv 0$ and $p^*(K_X \otimes L^{m+1})|D$ may have negative degree. The induction breaks down completely. This happens already for surfaces.

One way to get around this problem is to find $L \equiv B$ such that $c = \operatorname{mult}_x B \geq n$. At the level of the first blow up $p : Y \to X$ we get that

$$p^*(K_X \otimes L^{m+1}) \equiv K_Y + B' + (c - (n-1))E + p^* L^m. \tag{5.6.2}$$

The advantage of this situation is that the divisor in (5.6.2) is a pull back, so it has sections over the fibers of p.

The inductive assumptions become rather messy and there are further technical problems. Still, this idea can be made to work to get some results, see [Kollár95a, Ch.14].

The idea of [Angehrn-Siu95; Tsuji95] is to try to get a section right away. This is possible if we can prove that $H^1(X, K_X \otimes L^m \otimes m_x) = 0$ where m_x is the ideal sheaf of $x \in X$. This seems very hard to do. Fortunately, it is sufficient to produce an ideal sheaf $J \subset \mathcal{O}_X$ such that

(5.6.3.1) x is an isolated point of $\operatorname{Spec}(\mathcal{O}_X/J)$, and

(5.6.3.2) $H^1(X, K_X \otimes L^m \otimes J) = 0$.

The variant (2.16) of the vanishing theorem suggests such an approach:

Try to find a $\mathbb{Q}$-divisor B such that

(5.6.4.1) $B \equiv (m - \epsilon)L$, and

(5.6.4.2) B is not klt at x but is klt in a neighborhood of x.

The construction of such a divisor is not easy but turns out to be feasible once a few technical points are settled. Thus the essential part of the proof is postponed until section 6. □

Properties as in (5.6.4.2) will appear frequently, so we introduce a notation for it.

5.7 Definition. Let (X, D) be a pair. The set of points where (X, D) is klt is open, it is called the *klt locus* of (X, D). The complement of the klt locus is denoted by $\operatorname{Nklt}(X, D)$; it is called the *non-klt locus.*

Some authors call this the "locus of log canonical singularities". In my view this may be misleading.

Here I give an algebraic version of the proof of [Angehrn-Siu95]. I state a more general form which also applies to singular varieties.

5.8 Theorem. *Let (X, Δ) be a proper klt pair and M a line bundle. Assume that $M \equiv K_X + \Delta + N$, where N is a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Let $x \in X$ be a closed point and assume that there are positive numbers $c(k)$ with the following properties:*

(5.8.1) If $x \in Z \subset X$ is an irreducible (positive dimensional) subvariety then

$$(N^{\dim Z} \cdot Z) > c(\dim Z)^{\dim Z}.$$

(5.8.2) The numbers $c(k)$ satisfy the inequality

$$\sum_{k=1}^{\dim X} \frac{k}{c(k)} \leq 1.$$

Then M has a global section not vanishing at x.

Analogous results hold for separating points:

5.9 Theorem. *Let (X, Δ) be a proper klt pair and M a line bundle. Assume that $M \equiv K_X + \Delta + N$, where N is a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Let $x_1, x_2 \in X$ be closed points and assume that there are positive numbers $c(k)$ with the following properties:*

(5.9.1) If $Z \subset X$ is an irreducible subvariety which contains x_1 or x_2 then

$$(N^{\dim Z} \cdot Z) > c(\dim Z)^{\dim Z}.$$

(5.9.2) The numbers $c(k)$ satisfy the inequality

$$\sum_{k=1}^{\dim X} \sqrt[k]{2} \frac{k}{c(k)} \leq 1.$$

Then global sections of M separate x_1 and x_2.

Proof. Let us prove first (5.8).

First I claim that the inequalities (5.8.1) are satisfied if we replace N by $(1-\epsilon)N$ for $0 < \epsilon \ll 1$. This is a minor technical step which could easily have been built into the assumptions. A proof is given in (6.6.2).

Thus, by (6.4), there is a $\mathbb{Q}$-divisor $B \equiv (1-\epsilon)N$ such that x is an isolated point of $\operatorname{Nklt}(X, \Delta + B)$. We can write

$$M \equiv K_X + \Delta + B + \epsilon N.$$

By (2.16) there is an ideal sheaf $J \subset \mathcal{O}_X$ such that $\operatorname{Supp}(\mathcal{O}_X/J) = \operatorname{Nklt}(X, \Delta + B)$ and $H^i(X, M \otimes J) = 0$ for $i > 0$. In particular, the $i = 1$ case implies that

$$H^0(X, M) \twoheadrightarrow H^0(X, M \otimes (\mathcal{O}_X/J)) \twoheadrightarrow H^0(X, M \otimes (\mathcal{O}_X/m_x)) \quad \text{is surjective.}$$

Thus M has a section which does not vanish at x.

The proof of (5.9) is similar. We already know that M has sections which do not vanish at x_1 and at x_2. Thus global sections separate x_1 and x_2 iff there is an $i \in \{1,2\}$ and a global section $s \in H^0(X,M)$ such that $s(x_i) \neq 0$ and $s(x_{3-i}) = 0$. By (6.5) there is an $i \in \{1,2\}$ and a $\mathbb{Q}$-divisor $B \equiv (1-\epsilon)N$ such that x_i is an isolated point of $\operatorname{Nklt}(X, \Delta+B)$ and $x_{3-i} \in \operatorname{Nklt}(X, \Delta+B)$. As before, this implies the existence of the required section s. □

5.10 Proof of (5.5). Apply (5.8) and (5.9) with $X = Y$, $\Delta = \emptyset$ and $N = L^m$. Set $n = \dim Y$. In the first case set $c(k) = \binom{n+1}{2}$. This gives (5.5.1).

(5.9) implies (5.5.2) by setting $c(k) = \binom{n+2}{2}$ and using the inequality

$$\sum_{k=1}^{n} \sqrt[k]{2}k < \sum_{k=1}^{n} (1+\frac{1}{k})k = \binom{n+2}{2} - 1. \quad \square$$

* * * * *

The following is an application of (5.9) to varieties with generically large algebraic fundamental group. See [Kollár95a] for the relevant definitions and results.

5.11 Theorem. *Let X be a smooth proper variety with generically large algebraic fundamental group. Then:*

(5.11.1) If M is a big Cartier divisor on X the $K_X + M$ is also big.

(5.11.2) K_X is the limit of effective $\mathbb{Q}$-divisors.

Proof. One can choose a suitable birational model $p : Y \to X$ such that

$$p^*M \equiv N + \Delta + R$$

where N is an ample $\mathbb{Q}$-divsior, Δ is a fractional normal crossing divisor and R is effective. It is sufficient to prove that

$$K_Y + p^*M - R \equiv K_Y + N + \Delta$$

is big. If N has sufficiently large intersection number with any subvariety through a given point $y \in Y$, then by (5.9) global sections of $K_Y + p^*M - R$ separate points, thus it is big.

X has generically large algebraic fundamental group, thus there is a finite étale cover $q : Y' \to Y$ such that q^*N has large intersection number with any subvariety through a given point $y' \in q^{-1}(y)$. (In fact these two properties are equivalent.) Thus

$$K_{Y'} + q^*N + q^*\Delta \equiv q^*(K_Y + N + \Delta)$$

is big, hence so is $K_Y + N + \Delta$.

By induction, $mK_Y + M$ is big for any $m \geq 1$, thus

$$K_Y \equiv \lim_{m\to\infty} \frac{1}{m}(mK_Y + M)$$

is the limit of big $\mathbb{Q}$-divisors. A big $\mathbb{Q}$-divisor is also effective, proving (5.11.2). □

6. Construction of Singular Divisors

The aim of this section is to construct divisors which are "very singular" at a given point and not too singular elsewhere. The precise measure of what we mean by "very singular" is given by the notion of discrepancy. Actually, the construction is even weaker in the sense that we are able to guarantee only that the resulting divisor is not too singular in a neighborhood of our point.

The first step is to construct divisors which are as singular at a point as possible. The optimal result is achieved by an easy dimension count:

6.1 Lemma. *Let X be a proper and irreducible variety of dimension m, M a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X and $x \in X$ a smooth point. For every $\epsilon > 0$ there is an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor divisor $D = D(x, \epsilon)$ such that $M \equiv D$ and*

$$\operatorname{mult}_x D \geq \sqrt[m]{(M^m)} - \epsilon.$$

Proof. Fix $s, t > 0$ such that tM is Cartier and let $m \subset \mathcal{O}_X$ be the ideal sheaf of x. From the sequence

$$0 \to m^s \otimes \mathcal{O}_X(tM) \to \mathcal{O}_X(tM) \to (\mathcal{O}_X/m^s) \otimes \mathcal{O}_X(tM) \cong \mathcal{O}_X/m^s \to 0$$

we see that

$$h^0(X, m^s \otimes \mathcal{O}_X(tM)) > 0 \quad \text{if} \quad H^0(X, \mathcal{O}_X(tM)) > H^0(X, \mathcal{O}_X/m^s).$$

Since $x \in X$ is a smooth point,

$$H^0(X, \mathcal{O}_X/m^s) = \dim_k k[x_1, \ldots, x_m]/(x_1, \ldots, x_m)^s = \binom{m+s-1}{m} = \frac{s^m}{m!} + O(s^{m-1}).$$

By Riemann-Roch,

$$H^0(X, \mathcal{O}_X(tM)) = \frac{(M^m)}{m!} t^m + O(t^{m-1}).$$

Choose $t \gg 1$ and s such that $\sqrt[m]{(M^m)} > s/t > \sqrt[m]{(M^m)} - \epsilon$. Let $D(s, t, x)$ be the zero set of a nonzero section of $m^s \otimes \mathcal{O}_X(tM)$ and $D(x, \epsilon) = D(s, t, x)/t$. By construction $\operatorname{mult}_x D(x, \epsilon) \geq s/t$, as required. $\square$

The above divisor $D(x, \epsilon)$ has high multiplicity at x, but we cannot guarantee that it has low multiplicity elsewhere. The following example shows that, even for surfaces, forcing high multiplicity at one point can cause high multiplicities to appear at other points.

6.2 Exercises. (6.2.1) Let $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $M = \mathcal{O}(1, m)$. Then $(M^2) = 2m$, so by (6.1) for any point $x = (a, b) \in S$ and $d < \sqrt{2m}$ there is a $D_x \equiv M$ such that $\operatorname{mult}_x D_x > d$. Show that D_x contains the curve $\mathbb{P}^1 \times \{b\}$ with multiplicity at least $d - 1$.

(6.2.2) For any $m, n, d > 0$ construct a smooth surface S and an irreducible curve $C \subset S$ such that $(C^2) = d$ and there are two points $p, q \in C$ such that $\operatorname{mult}_p C = m$ and $\operatorname{mult}_q C = n$.

Assume that $m^2 > d$. Let $D = aC + C'$ ($C \not\subset \operatorname{Supp} C'$) be a $\mathbb{Q}$-divisor such that $D \equiv C$ and $\operatorname{mult}_p D \geq r$. $D = C$ is such a divisor, but we want to understand all such divisors as well. Show that $a \geq (mr - d)/(m^2 - d)$. Thus if $mr - d > 0$ then any such divisor has C as an irreducible component.

Assume now that $m^2 > d$ and $n > 2(m^2 - d)/(m - d)$. Then any $\mathbb{Q}$-divisor D which has multiplicity at least 1 at p has multiplicity at least 2 at q. Thus there is no way to make D not klt at p without making it much worse at q.

In order to illustrate the techniques involved, I first prove the surface version of (6.4). The proof is set up to emphasize the general methods, and it does not give the optimal result for surfaces.

6.3 Example-Theorem. *Let S be a normal projective surface, L an ample $\mathbb{Q}$-divisor on S and $x \in S$ a smooth point. Set $a^2 = (L^2)$ and $b = \min_{x \in C}(L \cdot C)$. Assume that*

$$1 > \frac{2}{a} + \frac{1}{b}.$$

Then there is an effective $\mathbb{Q}$-divisor $D \equiv L$ such that

(6.3.1) D is not log canonical at x, and

(6.3.2) D is klt in a punctured neighborhood of x.

Proof. The proof is in three steps, corresponding to the 3 main steps (6.7–9) of the higher dimensional argument.

(6.3.3) Step 1. Construction of a divisor which is singular at x.

Choose $c_1 > 2/a$. Then $(c_1 L)^2 > 4$, thus by (6.1) there is a divisor $D_1 \equiv c_1 L$ such that $\operatorname{mult}_x D_1 > 2$. Thus D_1 is not log canonical at x (3.9.4). If D_1 is klt (or even lc) in a punctured neighborhood of x, then go to step 3.

We are left with the case when D_1 is not lc in any punctured neighborhood of x. Write $D_1 = \sum e_i E_i$ where the E_i are irreducible curves and set $e = \max\{e_i | x \in E_i\}$. By assumption $e > 1$. Let

$$D_2 = \frac{1}{e} D_1 = \sum \frac{e_i}{e} E_i \quad \text{and} \quad C = \sum_{i:\ e_i = e, x \in E_i} E_i.$$

Then $D_2 \equiv (c_1/e)L$ and $D_2 - C$ is an effective $\mathbb{Q}$-divisor where each irreducible component containing x has coefficient less than 1.

If x is a singular point of C, then D_2 is not klt at x, again go to step 3. Thus we are left with the case when x is a smooth point of C, in particular, C is irreducible.

(6.3.4) Step 2. Induction on the dimension.

Let $c_2 > 1/b$ and choose $n \gg 1$ such that $nc_2 L$ is Cartier. Then

$$\deg_C(nc_2 L) = nc_2 \deg_C(L) \geq nc_2 b > n + 2g(C) - 1 \quad \text{for } n \gg 1,$$

thus $\mathcal{O}_C(nc_2 L|C)$ has a section s_C which vanishes at x to order n. That is, $(1/n)(s_C = 0)$ is a divisor on C which is numerically equivalent to $c_2 L|C$ and which is not klt at x.

We may assume that $H^1(S, \mathcal{O}_S(nc_2L)(-C)) = 0$, thus s_C can be lifted to a section s_S of $\mathcal{O}_S(nc_2L)$. By generic choice of s_S we may assume that s_S does not vanish along any irreducible component of D_1. Let $D_1' = (1/n)(s_S = 0)$. Then

$$D_1' + D_2 \equiv ((c_1/e) + c_2)L, \quad \text{and it is not klt at } x \text{ by (7.3.2).}$$

We can choose c_1, c_2 such that $(c_1/e) + c_2 < 1$.

(6.3.5) Step 3. Tie breaking.

The previous steps frequently yield a divisor $D_1 \equiv cL$ for some $c < 1$ such that D_1 is not klt at x and it is lc in a punctured neighborhood of x. We show that a small perturbation of D_1 gives the required D.

Choose $n \gg 1$ such that $n(1-c)L$ is Cartier and very ample. Let $D_1' \in |n(1-c)L|$ be a general divisor passing through x and set $D_2 = D_1 + (1/n)D_1'$. Then D_2 is not lc at x but lc in a punctured neighborhood of x. Choose $m \gg 1$ such that mL is Cartier and very ample. Let $D_2' \in |mL|$ be a general divisor. Then for $0 < \delta \ll 1$,

$$D = (1-\delta)D_2 + \frac{\delta}{m}D_2'$$

satisfies the requirements of (6.3). □

6.3.7 Remarks.

(6.3.7.1) Observe that the multiplicity of the divisor D at x does not necessarily predict that it is not lc at x. In step 2, the best lower bound for the multiplicity is $\operatorname{mult}_x D \geq 1 + (1/m) > 1$.

(6.3.7.2) The proof of (6.4) proceeds along the same lines. First we find a very singular divisor, and then we try to correct it, improving things one dimension at a time. There are some technical problems. In the surface case, if a divisor is klt at a point, it is smooth. In higher dimensions this is not true, and the main technical innovation of [Angehrn-Siu95] is to figure out how to deal with the resulting singularities.

The main result of this section is the following:

6.4 Theorem. *Let (X, Δ) be a projective klt pair and N a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Let $x \in X$ be a closed point and $c(k)$ positive numbers such that if $x \in Z \subset X$ is an irreducible (positive dimensional) subvariety then*

$$(N^{\dim Z} \cdot Z) > c(\dim Z)^{\dim Z}.$$

Assume that

$$\sum_{k=1}^{n} \frac{k}{c(k)} \leq 1.$$

Then there is an effective $\mathbb{Q}$-divisor $D \equiv N$ and an open neighborhood $x \in X^0 \subset X$ such that

(6.4.1) $(X^0, \Delta + D)$ is lc;

(6.4.2) $(X^0, \Delta + D)$ is klt on $X^0 - x$;

(6.4.3) $(X, \Delta + D)$ is not klt at x.

In order to separate points by global sections we need a version of the above result with two points. One might try to find a divisor which is lc at two given points and klt in a neighborhood of them. This is impossible in general (6.2.2). The following proof gives a weaker result which, however, is sufficient for our purposes.

6.5 Theorem. *Notation as above. Let $x, x' \in X$ be closed points such that if $Z \subset X$ is an irreducible (positive dimensional) subvariety such that $x \in Z$ or $x' \in Z$ then*

$$(N^{\dim Z} \cdot Z) > c(\dim Z)^{\dim Z}.$$

Assume also that

$$\sum_{k=1}^{n} \sqrt[k]{2}\frac{k}{c(k)} \leq 1.$$

Then, possibly after switching x and x', one can choose D as above such that in addition to (6.4.1–3) it also satisfies:

(6.5.1) $(X, \Delta + D)$ is not lc at x'.

* * * * *

Proof. The proof has several steps. Many of the intermediate results are of interest in their own right.

Proof.

6.6 Step 0. Reduction to N ample.

This step relies on two lemmas:

6.6.1 Lemma. *Let X be a proper scheme, N a nef and big divisor on X. Let $x \in X$ be a point and assume that there are numbers $c(k) > 0$ such that if $x \in Z \subset X$ is an irreducible subvariety then $(N^{\dim Z} \cdot Z) > c(\dim Z)$.*

Then we can write $N \equiv M + F$ where M is ample, F is effective and very small and if $x \in Z \subset X$ is an irreducible subvariety then $(M^{\dim Z} \cdot Z) > c(\dim Z)$.

Proof. Write $N = A + E$ where A is ample and E effective. Set $A_\epsilon = (1-\epsilon)N + \epsilon A$. Then A_ϵ is ample and $N = A_\epsilon + \epsilon E$. Furthermore, if $Z \subset X$ is a k-dimensional irreducible subvariety then

$$(A_\epsilon^k \cdot Z) = (1-\epsilon)^k (N^k \cdot Z) + \sum_{i=0}^{k-1} \epsilon (1-\epsilon)^i (A \cdot N^i \cdot A_\epsilon^{k-1-i} \cdot Z) \geq (1-\epsilon)^k (N^k \cdot Z).$$

This says that a nef and big divisor can be approximated by ample ones with uniform control over intersection numbers. We are done if we can exclude the possibility that $\inf_Z (N^k \cdot Z) = c(k)$. This is implied by (6.6.2). □

6.6.2 Lemma. *Let X be a proper scheme, N a nef and big divisor on X. Let $x \in X$ be a point and assume that if $x \in Z \subset X$ is an irreducible subvariety then $(N^{\dim Z} \cdot Z) > 0$.*

For every constant $C > 0$ there are only finitely many families of irreducible subvarieties $x \in Z \subset X$ such that $(N^{\dim Z} \cdot Z) < C$.

Proof. Write $N = A + E$ where A is ample and E effective. If $Z \not\subset E$ and $\dim Z = k$ then

$$(N^k \cdot Z) = (A^k \cdot Z) + \sum_{i=0}^{k-1} (E \cdot A^i \cdot (A+E)^{k-1-i} \cdot Z) \geq (A^k \cdot Z),$$

and there are only finitely many families of k-dimensional irreducible subvarieties of X such that $(A^k \cdot Z) < C$. By induction on the dimension, there are only finitely many families of k-dimensional irreducible subvarieties of E containing x such that $(N^k \cdot Z) < C$. □

By (6.6.1) we can write $N = M + F$ where M is ample, satisfies the assumptions of (6.4) and we can choose F small enough such that $(X, \Delta + F)$ is still klt. Find $D' \equiv M$ as required and then set $D = D' + F$. □

6.7 Step 1. Finding a singular divisor at x.

6.7.1. Theorem. *Let (X, Δ) be klt, projective of dimension n and $x \in X$ a closed point. Let H be an ample $\mathbb{Q}$-divisor on X such that $(H^n) > n^n$. Then there is an effective $\mathbb{Q}$-divisor $B_x \equiv H$ such that $(X, \Delta + B_x)$ is not lc at x.*

Proof. If x is smooth, this follows directly from (6.1). Moreover, from the proof we see that there is an $m > 0$ (depending on (X, Δ) and H but not on x) such that we can choose $B_x = (1/m)D_x$ where $D_x \in |mH|$.

In general, let $0 \in C$ be a smooth affine curve and $g : C \to X$ a morphism such that $x = g(0)$ and $g(c) \in X$ is smooth for $0 \neq c \in C$. For $c \neq 0$ pick a $\mathbb{Q}$-divisor $B_{g(c)}$ as above such that $B_{g(c)}$ is not lc at $g(c)$. It is natural to take $B_x := \lim_{c \to 0} B_{g(c)}$. Limits of $\mathbb{Q}$-divisors do not make too much sense in general (except as currents), but in our case one can attach a clear and precise meaning. By our construction, $B_{g(c)} = (1/m)D_{g(c)}$ where the $D_{g(c)}$ are Cartier divisors from the same linear system $|mH|$. After passing to a finite cover of C, we may assume that g lifts to a morphism $\tilde{g} : C \setminus \{0\} \to |mH|$ such that $D_{g(c)} = \tilde{g}(c)$. Thus we can take $B_x := (1/m)\tilde{g}(0)$.

By (7.8), B_x is not lc at x. □

6.8 Step 2. Inductive step.

The main part of the proof is the following:

6.8.1. Theorem. *Let (X, Δ) be klt, projective and $x \in X$ a closed point. Let D be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that $(X, \Delta + D)$ is lc in a neighborhood of x. Assume that* $\operatorname{Nklt}(X, \Delta + D) = Z \cup Z'$ *where Z is irreducible, $x \in Z$ and $x \notin Z'$. Set $k = \dim Z$. Let H be an ample $\mathbb{Q}$-divisor on X such that $(H^k \cdot Z) > k^k$. Then there is an effective $\mathbb{Q}$-divisor $B \equiv H$ and rational numbers $1 \gg \delta > 0$ and $0 < c < 1$ such that*

(6.8.1.1) $(X, \Delta + (1 - \delta)D + cB)$ is lc in a neighborhood of x, and

(6.8.1.2) $\operatorname{Nklt}(X, \Delta + (1 - \delta)D + cB) = Z_1 \cup Z_1'$, *$x \in Z_1, x \notin Z_1'$ and* $\dim Z_1 < \dim Z$.

6.8.1.3 Complement. *Assume in addition that $(X, \Delta + D)$ is not lc at $x' \in X$. Then we can choose B, δ and c such that $(X, \Delta + (1 - \delta)D + cB)$ is not lc at x'.*

Proof. By assumption there is a proper birational morphism $f : Y \to X$ and a divisor $E \subset Y$ such that $a(X, \Delta + D, E) = -1$ and $f(E) = Z$. Write $K_Y \equiv f^*(K_X + \Delta + D) + \sum e_i E_i$ where $E = E_1$ and so $e_1 = -1$. Let $Z^0 \subset Z$ be an open subset such that

(6.8.2.1) $f|E : E \to Z$ is smooth over Z^0, and

(6.8.2.2) if $z \in Z^0$ then $(f|E)^{-1}(z) \not\subset E_i$ for $i \neq 1$.

The following claim essentially proves (6.8.1) in case $x \in Z^0$.

6.8.3 Claim. *Notation as above. Choose $m \gg 1$ such that mH is Cartier. Then for every $z \in Z^0$ the following assertions hold:*

(6.8.3.1) There is a divisor $F_z \sim mH|Z$ such that $\operatorname{mult}_z F_z > mk$.

(6.8.3.2) $\mathcal{O}_X(mH) \otimes I_Z$ is generated by global sections and $H^1(X, \mathcal{O}_X(mH) \otimes I_Z) = 0$. In particular $H^0(X, \mathcal{O}_X(mH)) \to H^0(Z, \mathcal{O}_Z(mH|Z))$ is surjective.

(6.8.3.3) For any $F \sim mH|Z$ there is $F^X \sim mH$ such that $F^X|Z = F$ and $(X, \Delta + D + (1/m)F^X)$ is klt on $X - (Z \cup Z')$.

(6.8.3.4) Let $F_z^X \sim mH$ be such that $F_z^X|Z = F_z$. Then $(X, \Delta + D + (1/m)F_z^X)$ is not lc at z.

Proof. (6.8.3.1) is the usual multiplicity estimate (6.1), and (6.8.3.2) is satisfied once $m \gg 1$.

Let $|B| \subset |mH|$ be the linear subsystem consisting of those divisors B' such that either $Z \subset B'$ or $B'|Z = F$. By (6.8.3.2) $|B|$ is base point free on $X - Z$. Thus (4.8.2) implies (6.8.3.3).

Finally consider (6.8.3.4). Let y be the generic point of $(f|E)^{-1}(z)$. Write

$$K_Y \equiv f^*(K_X + \Delta + D) + \sum e_i E_i, \quad \text{where } E = E_1, \text{ and}$$
$$f^* F_z^X = F_z^Y + \sum m f_i E_i, \quad \text{where } F_z^Y = f_*^{-1} F_z^X. \text{ Thus}$$
$$K_Y + (1/m)F_z^Y + \sum (f_i - e_i)E_i \equiv f^*(K_X + \Delta + D + (1/m)F_z^X).$$

$(X, \Delta + D + (1/m)F_z^X)$ is not lc at z if $(Y, (1/m)F_z^Y + \sum(f_i - e_i)E_i)$ is not lc at y. $Z \not\subset F_z^X$, and therefore $f_1 = 0$. Thus $\sum(f_i - e_i)E_i = E + \sum_{i \neq 1}(f_i - e_i)E_i$ and by assumption none of the E_i for $i \neq 1$ contains y. Thus $(Y, (1/m)F_z^Y + \sum(f_i - e_i)E_i)$ is not lc at y iff $(Y, (1/m)f^*F_z^X + E)$ is not lc at y. By (7.5.2) the latter holds iff $(E, (1/m)f^*F_z^X|E = (1/m)(f|E)^*(F_z))$ is not lc at y. E is smooth at y, y has codimension k in E and $(1/m)(f|E)^*(F_z)$ has multiplicity $> k$. Thus $(E, (1/m)(f|E)^*(F_z))$ is not lc at y. $\square$

Next we intend to show that by continuity, there are divisors F_z^X as in (6.8.3) even if $z \in Z - Z^0$.

Pick $z_0 \in Z$ arbitrary. Let $0 \in C$ be a smooth affine curve and $g : C \to Z$ a morphism such that $z_0 = g(0)$ and $g(c) \in Z^0$ for general $c \in C$. For general $c \in C$ pick $F_c := F_{g(c)}$ as in (6.8.3.1). Let $F_0 = \lim_{c \to 0} F_c$. (The limit is defined as at the end of Step 1.)

6.8.4 Claim. *Notation as above. There is a divisor $F_0^X \in |mH|$ such that*

(6.8.4.1) $F_0^X|Z = F_0$,

(6.8.4.2) $(X, \Delta + D + (1/m)F_0^X)$ is klt on $X - (Z \cup Z')$,

(6.8.4.3) $(X, \Delta + D + (1/m)F_0^X)$ is lc at the generic point of Z,

(6.8.4.3) $(X, \Delta + D + (1/m)F_0^X)$ is not lc at z_0.

Proof. By (6.8.3.3) we can find F_0^X such that (6.8.4.1–2) are satisfied. F_0^X does not contain Z, thus $(X, \Delta + D + (1/m)F_0^X)$ is lc at the generic point of Z.

We can lift F_0^X to a family $F_c^X : c \in C$ such that $F_c^X|Z = F_c$. If $(X, \Delta + D + (1/m)F_0^X)$ is lc at z_0 then by (7.8) $(X, \Delta + (D + (1/m)F_c^X)$ is lc in a neighborhood of z_0 for general $c \in C$. This, however, contradicts (6.8.3.4). $\square$

To finish the proof of (6.8.1) set $B=(1/m)F_0^X$. $(X,\Delta+(1-\delta)D)$ is klt at the generic point of Z for every $\delta>0$. Choose $1\gg\delta>0$ such that $(X,\Delta+(1-\delta)D+B)$ is not lc at z_0 and then $0<c<1$ such that $(X,\Delta+(1-\delta)D+cB)$ is lc but not klt at z_0.

If $(X,\Delta+D)$ is not lc at x' then the same holds for $(X,\Delta+(1-\delta)D)$ for $1\gg\delta>0$ and any choice of c preserves this property. □

(6.8.1) is nearly enough to prove (6.4) by induction. The only problem is that in (6.8.1) we may end up with $(X,\Delta+(1-\delta)D+cB)$ such that $\operatorname{Nklt}(X,\Delta+(1-\delta)D+cB)$ has several irreducible components passing through z_0. This is taken care of by the next step.

6.9 Step 3. Tie breaking.

6.9.1 Lemma. *Let (X,Δ) be klt, projective and $x\in X$ a point. Let D be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that $(X,\Delta+D)$ is lc in a neighborhood of x. Let $\operatorname{Nklt}(X,\Delta+D)=\cup Z_i$ be the irreducible components; $x\in Z_1$. Let H be an ample $\mathbb{Q}$-divisor on X. Then for every $1\gg\delta>0$ there is an effective $\mathbb{Q}$-divisor $B\equiv H$ and $0<c<1$ such that*

(6.9.1.1) $(X,\Delta+(1-\delta)D+cB)$ is lc in a neighborhood of x, and

(6.9.1.2) $\operatorname{Nklt}(X,\Delta+(1-\delta)D+cB)=W\cup W'$ where $x\in W, x\notin W'$ and $W\subset Z_1$.

Proof. Choose $m\gg1$ such that mH is Cartier and so that $\mathcal{O}_X(mH)\otimes I_{Z_1}$ is generated by global sections. Let $B'\in|\mathcal{O}_X(mH)\otimes I_{Z_1}|$ be a general member. By (4.8.2) $(X,\Delta+(1-\delta)D+bB')$ is klt outside Z_1 in a neighborhood of x for $b<1$. It is definitely not lc along Z_1 for $1>b\gg\delta>0$. First choose $b=1/m$ and $1/m\gg\delta>0$. Then choose $0<c<1$ such that $(X,\Delta+(1-\delta)D+(c/m)B')$ is lc but not klt at x. Set $B=B'/m$. □

6.10 Step 4. Proof of (6.4).

We prove by induction the following theorem. The case $j=\dim X$ gives (6.4):

6.10.1 Theorem. *Notation and assumptions as in (6.4). Assume in addition that N is ample. Let $j\in\{1,\dots,n\}$. Then for every $b\geq\sum_{k=n-j}^{n}kc(k)^{-1}$ there is an effective $\mathbb{Q}$-divisor $D_j\equiv bN$ and an open neighborhood $x\in X^0\subset X$ such that*

(6.10.1.1) $(X^0,\Delta+D_j)$ is lc;

(6.10.1.2) $\operatorname{codim}(\operatorname{Nklt}(X^0,\Delta+D_j),X^0)\geq j$;

(6.10.1.3) $(X,\Delta+D_j)$ is not klt at x.

Proof. Set $D_0=\emptyset$. Assume that we already found D_j and we would like to get D_{j+1}.

If $j=0$, then apply (6.7.1). If $j>0$ then by (6.9) for every $\epsilon>0$ there is a divisor $D_j'\equiv(1-\delta)D_j+\epsilon N$ such that D_j' satisfies (6.10.1.1–3) and in addition either $Z:=\operatorname{Nklt}(X^0,\Delta+D_j)$ is irreducible of codimension at least j or it has codimension at least $j+1$. In the latter case we can take $D_{j+1}=D_j'+\alpha M$ where M is a general member of $|mN|$ for $m\gg1$ and α is suitable. (We may assume that $\epsilon<(j+1)c(j+1)^{-1}$.)

In the first case set $H=((j+1)c(j+1)^{-1}-\epsilon)N$. For $0<\epsilon\ll1$ we have that $(H^j\cdot Z)>j^j$. Thus we can apply (6.8.1) to obtain D_{j+1}. □

6.11 Step 5. Proof of (6.5).

The proof is very much similar to the proof of (6.4), I just outline the necessary changes. As before, we assume that N is ample.

As a first step of the induction we take $H = (\sqrt[n]{2}nc(n)^{-1} - \epsilon)N$. Then $(H^n) > 2n^n$ and as in (6.1) we can find a divisor $D_1 \equiv H$ such that $\operatorname{mult}_x D_1 > n$ and $\operatorname{mult}_{x'} D_1 > n$. This shows that $(X, \Delta + D_1)$ is not lc at the points x, x'. Choose $0 < c < 1$ such that $(X, \Delta + cD_1)$ is not klt at the points x, x' and is lc at one of them, say at x. There are two separate cases to consider.

(6.11.1) Case 1. If $(X, \Delta + cD_1)$ is not lc at x' then (6.8.1) can be used to continue exactly as in Step 4 to complete the proof.

(6.11.2) Case 2. What if $(X, \Delta + cD_1)$ is lc at x'? Then first we apply the tie-breaking method, to reduce to the case when $\operatorname{Nklt}(X, \Delta + cD_1)$ is irreducible in an open set containing x and x'. The tie-breaking may put us in the first case. Otherwise we are in the situation when $Z = \operatorname{Nklt}(X, \Delta + cD_1)$ is irreducible near x and contains x'.

We then proceed as in (6.8.1) but instead of trying to force high multiplicity at one point only, we do it at two points. Only the notation has to be changed. Proceeding inductively as in (6.10) we obtain (6.11).

If we always end up in case 2, then we may get a divisor D_n such that both x and x' are isolated points of $\operatorname{Nklt}(X, \Delta + D_n)$ and $(X, \Delta + D_n)$ is lc at both points. For the purposes of (5.9) this is not a problem at all (and may even be an advantage in general). In this case we can do a last tie breaking to achieve exactly (6.5). □

7. The L^2 Extension Theorem and Inversion of Adjunction

Let X be a variety and $S \subset X$ a Cartier divisor. If we know something about the singularities of S then we can usually assert that the singularities of X near S are not worse. For instance, if S is smooth, or rational, or CM then the same holds for X. In some cases the converse implication also holds. This fails for smooth or rational, but works for CM.

Our aim is to investigate the analogous problem for discrepancies. It can be formulated in two variants. This first one has been considered in complex analysis. The second version is the natural one from the algebraic geometry point of view: When talking about (X, B), we always compute with $K_X + B$, and $K_X + S + B|S = K_S + B$ by adjunction.

7.1 Questions. *Let $0 \in S \subset X$ be a Cartier divisor. Let $B = \sum b_i B_i$ be a $\mathbb{Q}$-divisor such that S is not among the B_i.*

(7.1.1) If $(S, B|S)$ is lc, is then (X, B) also lc?

(7.1.2) If $(S, B|S)$ is lc, is then $(X, S + B)$ also lc?

It is a priori clear that the second form is stronger, but it turns out that there is no real difference between them:

7.1.3 Lemma. [Manivel93] *The above two questions are equivalent.*

Proof. We need to prove that the first version implies the second one.

Let $S = (f = 0)$ and let $X_n \subset X \times \mathbb{A}^1$ be given by the equation $y^n - f = 0$, where y is the coordinate on $\mathbb{A}^1$. Let $p_n : X_n \to X$ be the projection and set $B_n := p_n^{-1}B$.

$S \cong S_n := (y=0)$ appears as a Cartier divisor on X_n and $B_n|S_n = B|S$ under this isomorphism.

By (7.1.1), (X_n, B_n) is lc. Observe that

$$K_{X_n}+B_n+(n-1)S_n = p_n^*(K_X+B), \quad \text{thus} \quad K_{X_n}+B_n \equiv p_n^*(K_X+B+(1-1/n)S).$$

From (3.16.2) we conclude that $(X, B+(1-1/n)S)$ is lc for every n, thus $(X, B+S)$ is lc. □

The first significant result toward answering (7.1) is the L^2 extension theorem of [Ohsawa-Takegoshi87], though the connection was first realized only later. I state a form of the theorem which is natural from the point of view of complex analysis. Instead of defining the notions "pseudoconvex" and "plurisubharmonic", it is sufficient to keep two special cases in mind:

— every convex subset $\Omega \subset \mathbb{C}^n$ is pseudoconvex,

— if g is holomorphic then $c\log|g|$ is plurisubharmonic for $c>0$.

7.2 Theorem. [Ohsawa-Takegoshi87] *Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain, and $H \subset \mathbb{C}^n$ a hyperplane intersecting Ω. Fix Lebesgue measures dm_n on $\mathbb{C}^n$ and dm_{n-1} on H. Then there is a constant C_Ω with the following property.*

Let ϕ be plurisubharmonic on Ω and f holomorphic on $\Omega \cap H$. Then f can be extended to a holomorphic function F on Ω such that

$$\int_\Omega |F|^2 e^{-\phi} dm_n \leq C_\Omega \int_{\Omega\cap H} |f|^2 e^{-\phi} dm_{n-1}. \quad \square$$

The following consequence relates this to (7.1):

7.2.1 Corollary. *Let $0 \in H \subset \mathbb{C}^n$ be a hyperplane.*

(7.2.1.1) Let g be a holomorphic function near 0 and let g_H denote the restriction of g to H. If $|g_H|^{-c}$ is L^2 near 0 then $|g|^{-c}$ is L^2 near 0.

(7.2.1.2) Let $B=\sum b_iB_i$ be a $\mathbb{Q}$-divisor such that H is not among the B_i. If $(H, B|H)$ is klt (resp. lc) then $(\mathbb{C}^n, B)$ is klt (resp. lc).

Proof. For Ω choose a small ball around 0. Pick $\phi = 2c\log|g|$ and $f\equiv 1$. We do not know what F is, but $|F| \geq 1/2$ in a neighborhood $0 \in \Omega' \subset \Omega$. Thus

$$\frac{1}{2}\int_{\Omega'} |g|^{-2c} dm_n \leq \int_\Omega |F|^2|g|^{-2c} dm_n \leq C_\Omega \int_{\Omega\cap H} |g|^{-2c} dm_{n-1}.$$

This shows the first part. In order to see the second part, choose m such that mb_i are all integers and let g be a function with $(g=0) = \sum (mb_i)B_i$. Applying (7.2.1.1) with $c=1/m$ gives the klt part of (7.2.1.2) by (3.20).

Finally $(\mathbb{C}^n, B)$ is lc iff $(\mathbb{C}^n, (1-\epsilon)B)$ is klt for every $\epsilon > 0$, thus the klt case implies the lc version. □

7.2.2 Remark. The application of the L^2 extension theorem for the constant function seems quite silly. After all, we need only that if $|g_H|^{-c}$ is L^2 then so is $|g|^{-c}$. A simple manipulation of integrals may give this result, but such a proof is not yet known.

Unaware of (7.2), [Shokurov92,3.3] proposed a conjecture along the lines of (7.1.1) for algebraic varieties. The conjecture was subsequently generalized in [Kollár et al.92, 17.3].

The conjecture (or similar results and conjectures) is frequently referred to as *adjunction* (if we assume something about X and obtain conclusions about S) or *inversion of adjunction* (if we assume something about S and obtain conclusions about X).

7.3 Conjecture. *Let X be a normal variety, S a normal Cartier divisor and $B=\sum b_iB_i$ a $\mathbb{Q}$-divisor. Assume that K_X+S+B is $\mathbb{Q}$-Cartier. Then*

$$\operatorname{totaldiscrep}(S, B|S) = \operatorname{discrep}\left(\operatorname{Center}\cap S\neq\emptyset, X, S+B\right),$$

where the notation on the right means that we compute the discrepancy using only those divisors whose center on X intersects S.

7.3.1 Remarks.

(7.3.1.1) The conjecture can also be formulated if $S\subset X$ is only a Weil divisor. For some applications this is crucial, but for us it is not necessary. Also, it leads to additional difficulties. The point is that if S is not Cartier, the usual adjunction formula $K_S=(K_X+S)|S$ fails. A suitable correction term needs to be worked out. Once this is settled, the proofs require little change. The interested reader should consult [Kollár et al.92], especially Chapters 16–17.

(7.3.1.2) [Kollár et al.92,17.12] shows that (7.3) is implied by the logMMP. Thus it is true if $\dim X\le 3$. Various special cases of this conjecture are very useful in the proof of the logMMP and in many other contexts, see, for instance, [Kollár et al.92,Ch.18] or [Corti94]. Therefore it is rather desirable to find a proof independent of the logMMP.

One inequality is easy to prove:

7.3.2 Proposition. [Kollár et al.92, 17.2] *Let X be a normal variety, S a Cartier divisor and $B=\sum b_iB_i$ a $\mathbb{Q}$-divisor. Assume that K_X+S+B is $\mathbb{Q}$-Cartier. Then*

$$\operatorname{totaldiscrep}(S, B|S)\ge \operatorname{discrep}(X, S+B).$$

Proof. (Strictly speaking, the left hand side is only defined if S itself is normal. The following proof furnishes a definition of the left hand side, which is the correct one for schemes S which are Cartier divisors on a normal scheme.)

Let $f:Y\to X$ be a log resolution of $(X, S+B)$ and set $S':=f_*^{-1}S$. We may also assume that S' is disjoint from $f_*^{-1}B$. Write $K_Y+S'\equiv f^*(K_X+S+B)+\sum e_iE_i$. By the usual adjunction formula,

$$K_{S'}=K_Y+S'|S',\quad \text{and}\quad K_X+S+B|S=K_S+B|S.$$

This gives that

$$K_{S'}\equiv f^*(K_S+B|S)+\sum e_i(E_i\cap S').$$

By assumption S' is disjoint from $f_*^{-1}B$, thus if $E_i\cap S'\neq\emptyset$ then E_i is f-exceptional. This shows that every discrepancy which occurs in $S'\to S$ also occurs among the exceptional divisors of $Y \to X$. It may of course happen that E_i is f-exceptional but $E_i \cap S'$ is not $f|S'$-exceptional. This is why we have totaldiscrep on the left hand side of the inequality. $\square$

7.3.2.1 Remark. In general there are many exceptional divisors E_j of $f : Y \to X$ which do not intersect S', and there is no obvious connection between the discrepancies of such divisors and the discrepancies occurring in $S' \to S$. This makes the reverse inequality nonobvious.

For most of the applications of (7.3) the crucial case is when one of the two sides is klt or lc. The case when S is smooth follows from (7.2.1). The singular cases are settled in [Kollár et al.92, Chapter 17]. The proof relies on the following connectedness result which is of interest itself.

7.4 Theorem. [Kollár et al.92, 17.4] *Let X be a normal variety (or analytic space) and $D = \sum d_i D_i$ an effective $\mathbb{Q}$-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $g : Y \to X$ be a log resolution of (X, D). Write*

$$K_Y = g^*(K_X + D) + \sum e_i E_i, \quad \textit{and set}$$
$$A = \sum_{i:e_i > -1} e_i E_i, \quad \textit{and} \quad F = - \sum_{i:e_i \le -1} e_i E_i.$$

Then $\operatorname{Supp} F = \operatorname{Supp} \llcorner F \lrcorner$ *is connected in a neighborhood of any fiber of g.*

Proof. By definition

$$\ulcorner A \urcorner - \llcorner F \lrcorner = K_Y + (-g^*(K_X + D)) + \{-A\} + \{F\},$$

and therefore by (2.17.3)

$$R^1 f_* \mathcal{O}_Y(\ulcorner A \urcorner - \llcorner F \lrcorner) = 0.$$

Applying g_* to the exact sequence

$$0 \to \mathcal{O}_Y(\ulcorner A \urcorner - \llcorner F \lrcorner) \to \mathcal{O}_Y(\ulcorner A \urcorner) \to \mathcal{O}_{\llcorner F \lrcorner}(\ulcorner A \urcorner) \to 0$$

we obtain that

$$g_* \mathcal{O}_Y(\ulcorner A \urcorner) \to g_* \mathcal{O}_{\llcorner F \lrcorner}(\ulcorner A \urcorner) \quad \text{is surjective.} \tag{7.4.1}$$

Let E_i be an irreducible component of A. Then either E_i is g-exceptional or E_i is the birational transform of some D_j whose coefficient in D is less than 1. Thus $\ulcorner A \urcorner$ is g-exceptional and $g_* \mathcal{O}_Y(\ulcorner A \urcorner) = \mathcal{O}_X$. Assume that $\llcorner F \lrcorner$ has at least two connected components $\llcorner F \lrcorner = F_1 \cup F_2$ in a neighborhood of $g^{-1}(x)$ for some $x \in X$. Then

$$g_* \mathcal{O}_{\llcorner F \lrcorner}(\ulcorner A \urcorner)_{(x)} \cong g_* \mathcal{O}_{F_1}(\ulcorner A \urcorner)_{(x)} + g_* \mathcal{O}_{F_2}(\ulcorner A \urcorner)_{(x)},$$

and neither of these summands is zero. Thus $g_* \mathcal{O}_{\llcorner F \lrcorner}(\ulcorner A \urcorner)_{(x)}$ cannot be the quotient of the cyclic module $\mathcal{O}_{x,X} \cong g_* \mathcal{O}_Y(\ulcorner A \urcorner)_{(x)}$. $\square$

As a corollary we obtain the following results which were proved by [Shokurov92] in dimension 3 and by [Kollár et al.92, 17.6–7] in general.

7.5 Theorem. *Let X be normal and $S \subset X$ an irreducible Cartier divisor. Let B be an effective $\mathbb{Q}$-divisor and assume that $K_X + S + B$ is $\mathbb{Q}$-Cartier. Then*

(7.5.1)

$$(X, S+B) \quad \textit{is plt near } S \quad \Leftrightarrow \quad (S, B|S) \quad \textit{is klt.}$$

(7.5.2) Assume in addition that B is $\mathbb{Q}$-Cartier and S is klt. Then

$$(X, S+B) \quad \textit{is lc near } S \quad \Leftrightarrow \quad (S, B|S) \quad \textit{is lc.}$$

Proof. In both cases the implication $\Rightarrow$ follows from (7.3.2).

In order to see (7.5.1), let $g : Y \to X$ be a resolution of singularities and as in (7.4) let

$$K_Y \equiv g^*(K_X + S + B) + A - F.$$

Let $S' = g_*^{-1} S$ and $F = S' + F'$. By adjunction

$$K_{S'} = g^*(K_S + B|S) + (A - F')|S'.$$

$(X, S+B)$ is plt iff $F' = \emptyset$ and $(S, B|S)$ is plt iff $F' \cap S' = \emptyset$. By (7.4) $S' \cup F'$ is connected, hence $F' = \emptyset$ iff $F' \cap S' = \emptyset$. This shows (7.5.1).

From the definition we see that

$$\begin{aligned} &(X, S+B) \text{ is lc} \quad \text{iff} \quad (X, S+cB) \quad \text{is plt for } c<1, \text{ and} \\ &(S, B|S) \text{ is lc} \qquad \text{iff} \quad (S, cB|S) \quad \text{is klt for } c<1. \end{aligned}$$

Thus (7.5.1) implies (7.5.2). $\square$

7.5.3 Exercise. Notation and assumptions as above. If $(X, S+B)$ is plt then $\llcorner B \lrcorner = \emptyset$ in a neighborhood of S.

As a corollary we obtain that klt and lc are open conditions in flat families:

7.6 Corollary. *Let (X, B) be a pair such that $K_X + B$ is $\mathbb{Q}$-Cartier and $g : X \to C$ a flat morphism to a smooth pointed curve $0 \in C$. Let $x \in X_0 = g^{-1}(0)$ be a closed point.*

(7.6.1) Assume that $(X_0, B|X_0)$ is klt at x, respectively

(7.6.2) assume that X_0 is klt at x and $(X_0, B|X_0)$ is lc at x.

Then there is an open neighborhood $x \in U \subset X$ such that $(U_c, B|U_c)$ is klt (resp. lc) for every $c \in C$.

Proof. Let $S = X_0$. By (7.5) $(X, S+B)$ is plt (resp. lc) in a neighborhood W of x. In the first case, $\llcorner B \lrcorner = \emptyset$ by (7.5.3). Thus $(W - S, (S+B)|W - S) = (W - S, B|W - S)$ is klt (resp. lc). By (7.7) there is an open subset $C^0 \subset C$ such that $(W_c, B|W_c)$ is klt (resp. lc) for $c \in C^0$. Set $U = W \cap g^{-1}(\{0\} \cup C^0)$ to conclude. $\square$

7.7 Proposition. [Reid80, 1.13] *Let X be a scheme over a field of characteristic zero, D a $\mathbb{Q}$-divisor and $|B|$ a base point free linear system of Cartier divisors. Assume that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $B' \in |B|$ be a general member. Then*

$$\operatorname{discrep}(B', D|B') \geq \operatorname{discrep}(X, D).$$

In particular, if (X, D) is lc (resp. klt, canonical, terminal) then $(B', D|B')$ is also lc (resp. klt, canonical, terminal).

Proof. Let $f : Y \to X$ be a log resolution of (X, D) and $C' := f_*^{-1}B' = f^*B'$. Then $g := f|C' : C' \to B'$ is a log resolution of $(B', D|B')$. Write $K_Y \equiv f^*(K_X + D) + \sum e_i E_i$. Then

$$\begin{aligned} K_{C'} = K_Y + C'|C' &\equiv (f^*(K_X + D + B') + \sum e_i E_i)|C' \\ &= g^*(K_{B'} + D|B') + \sum e_i(E_i|C'). \quad \square \end{aligned}$$

7.8 Corollary. *Let Y be a klt variety over $\mathbb{C}$ and $B_c : c \in C$ an algebraic family of $\mathbb{Q}$-divisors on Y parametrized by a smooth pointed curve $0 \in C$. Assume that (Y, B_0) is klt (resp. lc) at $y \in Y$.*

Then there is a Euclidean open neighborhood $y \in W \subset Y$ such that (Y, B_c) is klt (resp. lc) on W for $c \in C$ near 0.

Proof. B_c defines a $\mathbb{Q}$-divisor B on $X := Y \times C$. Let $g : X \to C$ be the projection. Apply (7.6) to obtain $(y, 0) \in U \subset X$. There are Euclidean neighborhoods $y \in W \subset Y$ and $0 \in D \subset C$ such that $W \times D \subset U$. $\square$

In general W cannot be chosen to be Zariski open.

The following result, predating (7.3), settles another special case.

7.9 Theorem. [Stevens88] *Let X be a normal variety such that ω_X is locally free. Let $S \subset X$ be a Cartier divisor. Then*

$$S \quad \textit{is canonical} \quad \Leftrightarrow \quad (X, S) \quad \textit{is canonical near } S.$$

Proof. If (X, S) is canonical then S is canonical by (7.3.2). In order to see the converse, let $f : Y \to X$ be a log resolution of (X, S) and set $T := f_*^{-1}S$.

We have exact sequences:

$$\begin{array}{l} 0 \longrightarrow \omega_Y \longrightarrow \omega_Y(T) \longrightarrow \omega_T \longrightarrow 0, \quad \text{and} \\ 0 \longrightarrow \omega_X \longrightarrow \omega_X(S) \xrightarrow{\ r\ } \omega_S. \end{array}$$

(If X is CM, then r is surjective, but we do not need this.) Pushing the first sequence forward to X, there are natural maps:

$$\begin{tikzcd} 0 \arrow[r] & f_*\omega_Y \arrow[r] \arrow[d] & f_*\omega_Y(T) \arrow[r] \arrow[d, "b"'] & f_*\omega_T \arrow[r] \arrow[d, "c"'] & R^1f_*\omega_Y = 0 \\ 0 \arrow[r] & \omega_X \arrow[r] & \omega_X(S) \arrow[r, "r"] & \omega_S. \end{tikzcd}$$

b is an injection and c is an isomorphism since S is canonical. Therefore $r \circ b$ is surjective, thus by the Nakayama lemma, b itself is surjective near S. Thus b is also an isomorphism near S. By shrinking X, we may assume that b is surjective. Since $\omega_X(S)$ is locally free, the pull back of b to Y gives a map

$$f^*(\omega_X(S)) \to \omega_Y(T).$$

That is, $f^*(\omega_X(S)) = \omega_Y(T)(-E)$ for some effective divisor E. Thus

$$K_Y + T = f^*(K_X + S) + E,$$

and (X, S) is canonical near S. $\square$

7.9.1 Remark. Looking at the proof we see that instead of assuming that ω_X is locally free and S is Cartier, it is sufficient to assume that $\omega_X(S)$ is locally free. This is important in some applications [Stevens88; Kollár-Mori92,3.1].

A study of the above commutative diagram also gives the following:

7.9.2 Corollary. [Ein-Lazarsfeld96, 3.1] *Let X be a smooth variety (or a variety with index one canonical singularities) and $S \subset X$ a Cartier divisor with desingularization $f : \bar{S} \to S$. Then there exists an ideal sheaf $J \subset \mathcal{O}_X$ such that*

$$0 \to \omega_X \to \omega_X(S) \otimes J \to f_*\omega_{\bar{S}} \to 0 \quad \textit{is exact,}$$

and $\operatorname{Supp}(\mathcal{O}_X/J)$ *is precisely the set of points where S is not canonical.* $\square$

8. The Log Canonical Threshold

Let (X, D) be a pair. In section 3 we used the notion of discrepancy to attach various invariants to (X, D) which provide a way of measuring how singular X and D are. If (X, D) is not log canonical, the values of these invariants are $-\infty$, hence they give very little information. The aim of this section is to develop another invariant, which becomes nontrivial precisely when the discrepancy is $-\infty$.

8.1 Definition. Let (X, Δ) be an lc pair, $Z \subset X$ a closed subscheme and D an effective $\mathbb{Q}$-Cartier divisor on X. The *log canonical threshold* (or *lc-threshold*) of D along Z with respect to (X, Δ) is the number

$$c_Z(X, \Delta, D) := \sup\{c \mid (X, \Delta + cD) \text{ is lc in an open neighborhood of } Z\}.$$

If $\Delta = 0$ then we use $c_Z(X, D)$ instead of $c_Z(X, 0, D)$. We frequently write $c_Z(D)$ instead of $c_Z(X, \Delta, D)$ if no confusion is likely. If $D = (f = 0)$ then we also use the notation $c_Z(X, \Delta, f)$ and $c_Z(f)$.

(3.20) shows that the lc-threshold has an equivalent analytic definition:

8.2 Proposition. *Let Y be a smooth variety over $\mathbb{C}$, $Z \subset Y$ a closed subscheme and f a nonzero regular function on Y. Then*

$$c_Z(Y, 0, f) = \sup\{c \colon |f|^{-c} \textit{ is locally } L^2 \textit{ near } Z\}. \quad \square$$

The following properties are clear from the definition:

8.3 Lemma. *Notation as above. Then:*

(8.3.1) $c_Z(X,\Delta,D) \geq 0$ *and* $c_Z(X,\Delta,D) = +\infty$ *iff* $D = 0$.

(8.3.2) $c_Z(X,\Delta,D) = \inf_{p\in Z} c_p(X,\Delta,D)$.

(8.3.3) If D is a Weil divisor, then $c_Z(X,\Delta,D) \leq 1$. □

8.4 Remark. There is a slightly more general situation where the above definition also makes sense. Instead of assuming that (X,Δ) is lc, it is sufficient to assume that (X,Δ) is lc on $X - \operatorname{Supp} D$. In this case $c_Z(X,\Delta,D)$ is negative if (X,Δ) is not lc along Z.

Next we turn to various techniques of computing and estimating the lc-threshold. We can rewrite (3.13) to give an effective computational method:

8.5 Proposition. *Let (X,Δ) be an lc pair, $Z \subset X$ a closed subscheme and D an effective $\mathbb{Q}$-Cartier divisor on X. Let $p : Y \to X$ be a proper birational morphism. Using the convention (3.3.2), write*

$$K_Y \equiv p^*(K_X+\Delta) + \sum a_i E_i, \quad \textit{and} \quad p^*D = \sum b_i E_i.$$

Then:

(8.5.1)

$$c_Z(X,\Delta,D) \leq \min_{i:\ p(E_i)\cap Z\neq\emptyset} \left\{\frac{a_i+1}{b_i}\right\}.$$

(8.5.2) Equality holds if $\sum E_i$ is a divisor with normal crossings only. In particular, $c_Z(X,\Delta,D) \in \mathbb{Q}$. □

(7.6) translates to an upper semicontinuity statement for the lc-threshold:

8.6 Lemma. *Let (X,Δ) be a klt pair, $x \in X$ a closed point and $D_t : t \in C$ an algebraic family of effective $\mathbb{Q}$-Cartier divisors. Pick a point $t_0 \in C$. Then*

$$c_0(X,\Delta,D_{t_0}) \leq c_0(X,\Delta,D_t), \quad \textit{for } t \textit{ near } t_0. \qquad \square$$

The log canonical threshold has been investigated earlier in different contexts. Some of these are explained in sections 9–10. Recent interest arose following [Shokurov92] who used various properties of lc-thresholds in order to establish the existence of log flips in dimension three. He proposed a rather striking conjecture (8.8), and proved it for surfaces. Later this was proved for threefolds in [Alexeev93]. The general case is still unknown. Before formulating the conjecture, we need a definition.

8.7 Definition. For every $n \in \mathbb{N}$ define a subset of $\mathbb{R}$ by

$$\mathcal{T}_n := \{c_x(X,D) | x \in X \text{ is klt, } \dim X = n \text{ and } D \text{ is an effective Weil divisor}\}.$$

By (8.3.3) and (8.5.2) we see that $\mathcal{T}_n \subset (0,1] \cap \mathbb{Q}$.

8.8 Conjecture. [Shokurov92] *For every n, the set $\mathcal{T}_n$ satisfies the ascending chain condition.*

8.8.1 Remark. This conjecture is only one example of a series of conjectures that assert the ascending or descending chain condition for various naturally defined invariants coming from algebraic geometry. See [Shokurov92; Kollár et al.92, Ch.18; Kollár94; Alexeev94; Ganter95] for further examples and for applications.

The rest of the section is devoted to various methods of computing the lc-threshold in several examples, and to study (8.8) in those cases. Most of the computations that we do are for X smooth. Thus working analytically we consider only the case $X=\mathbb{C}^n$. Even this special case of (8.8) is mysterious.

8.9 Example. Let $f\in\mathbb{C}[[x,y]]$ be an irreducible power series. By [Igusa77],

$$c_0(\mathbb{C}^2,f)=\frac{1}{m}+\frac{1}{n}\quad\text{where } m=\operatorname{mult}_0 f \text{ and } n/m \text{ is the first Puiseux exponent of } f.$$

See [Loeser87] for some higher dimensional generalizations.

The following result furnishes the basic estimates for the lc-threshold:

8.10 Lemma. *Let f be a holomorphic function near $0\in\mathbb{C}^n$ and $D=(f=0)$. Set $d=\operatorname{mult}_0 f$ and let f_d denote the degree d homogeneous part of the Taylor series of f. Let $T_0D:=(f_d=0)\subset\mathbb{C}^n$ be the tangent cone of D and $\mathbb{P}(T_0D):=(f_d=0)\subset\mathbb{P}^{n-1}$ the projectivized tangent cone of D. Then*

(8.10.1) $\frac{1}{d}\le c_0(D)\le\frac{n}{d}$.

(8.10.2) $c_0(D)=\frac{n}{d}$ *iff* $(\mathbb{P}^{n-1},\frac{n}{d}\mathbb{P}(T_0D))$ *is lc.*

(8.10.3) If $\mathbb{P}(T_0D)$ is smooth (or even lc) then $c_0(D)=\min\{1,\frac{n}{d}\}$.

(8.10.4) $c_0(T_0D)\le c_0(D)$.

Proof. Let $p:Y\to\mathbb{C}^n$ the blow up of $0\in\mathbb{C}^n$ with exceptional divisor $E\subset Y$. Then

$$K_Y=p^*K_{\mathbb{C}^n}+(n-1)E,\quad\text{and}\quad p^*D=p_*^{-1}D+dE.$$

Thus by (8.5.1), $c_0(D)\le(n-1+1)/d$. In particular, $c_0(D)=\frac{n}{d}$ iff $(\mathbb{C}^n,\frac{n}{d}D)$ is lc.

$$K_Y+E+\frac{n}{d}p_*^{-1}D=p^*(K_{\mathbb{C}^n}+\frac{n}{d}D),$$

and by (3.10.2) we see that

$$c_0(D)=\frac{n}{d}\quad\Leftrightarrow\quad(Y,E+\frac{n}{d}p_*^{-1}D)\quad\text{is lc.}$$

Observe that $E\cap p_*^{-1}D=\mathbb{P}(T_0D)$. We can apply inversion of adjunction (7.5) to see that

$$(Y,E+\frac{n}{d}p_*^{-1}D)\quad\text{is lc}\quad\Leftrightarrow\quad(\mathbb{P}^{n-1},\frac{n}{d}\mathbb{P}(T_0D))\quad\text{is lc.}$$

This shows (8.10.2) which implies (8.10.3).

In order to see the lower bound in (8.10.1), choose local coordinates such that the Taylor series of f has the form $x_1^d+\dots$. Consider the deformation $f_t=$

$t^{-d}f(tx_1, t^2x_2, \dots, t^2x_n)$. For $t \neq 0$ the singularity of $(f_t = 0)$ is isomorphic to $(f = 0)$, and for $t = 0$ we get $(x_1^d = 0)$. By (8.6) we see that

$$\frac{1}{d} = c_0(x_1^d) \leq c_0(D).$$

To see (8.10.4) we use the deformation $f_t = t^{-d}f(tx_1, tx_2, \dots, tx_n)$. For $t \neq 0$ the singularity of $(f_d = 0)$ is isomorphic to $(f = 0)$, and for $t = 0$ we get $(f_k = 0)$. By (8.6) we are done. □

8.11 Remarks. (8.11.1) It is not true in general that truncation of f yields a smaller value for c_0. For instance, let $f = x^2 + 2xy^2 + y^4$. Then $c_0(f) = 1/2$, but $c_0(x^2 + 2xy^2) = 3/4$.

(8.11.2) (8.10.1) shows that $c_0(D)^{-1}$ behaves roughly as $\operatorname{mult}_0(D)$. For this reason the number $c_0(D)^{-1}$ is sometimes called the Arnold multiplicity of D or of f.

(8.11.3) The estimate $c_0(D) \leq n/d$ holds for any $\mathbb{Q}$-divisor D, even if D contains components with negative coefficients.

Looking at the homogeneous leading term does not give a true indication of the subtle behaviour of the lc-threshold. In order to get better examples, we need to look at the weighted homogeneous case. The best way to study it is by using weighted blow ups, see [Reid80,87]. In many cases weighted blow ups can be reduced to an ordinary blow up using the following lemma, which is a direct consequence of (3.16).

8.12 Lemma. *Let $p : X \to Y$ be a finite and dominant morphism between normal varieties. Let Δ_Y be a $\mathbb{Q}$-divisor on Y and define Δ_X by the formula*

$$K_X + \Delta_X = p^*(K_Y + \Delta_Y), \quad \text{that is,} \quad \Delta_X = p^*\Delta_Y - K_{X/Y}.$$

Let D_Y be an effective $\mathbb{Q}$-divisor on Y and $Z \subset Y$ a closed subscheme. Then

$$c_Z(Y, \Delta_Y, D_Y) = c_{p^{-1}(Z)}(X, \Delta_X, p^*D_Y). \quad \square$$

8.12.1 Remark. If $\dim X = 2$ and X is klt, then X has quotient singularities. Thus, by (8.12), $\mathcal{T}_2$ can be determined by computing $c_0(Y, D)$ where Y is smooth. In general, determining $\mathcal{T}_n$ can be reduced to the case when Y is canonical, and, assuming MMP, to the case when Y is terminal (cf. [Kollár94, pp.267-268]). A reduction to the smooth case is not known.

We can now prove the analog of (8.10) in the weighted case:

8.13 Proposition. *Let f be a holomorphic function near $0 \in \mathbb{C}^n$. Assign rational weights $w(x_i)$ to the variables and let $w(f)$ be the weighted multiplicity of f (= the lowest weight of the monomials occurring in f). Then*

$$c_0(f) \leq \frac{\sum w(x_i)}{w(f)}.$$

Proof. We may assume that the weights $w(x_i)$ are natural numbers. Set $X \cong \mathbb{C}^n$ with coordinates z_i, $H_i = (z_i = 0)$ and let $p : X \to \mathbb{C}^n$ be given by $z_i \mapsto z_i^{w(x_i)} = x_i$. Let $F := f(z_1^{w(x_1)}, \dots, z_n^{w(x_n)})$ and note that $\operatorname{mult}_0 F = w(f)$. Then

$$K_X + \sum (1 - w(x_i))H_i + c(F = 0) = p^*(K_{\mathbb{C}^n} + c(f = 0)).$$

By (8.11.3) we obtain that if $(X, \sum(1 - w(x_i))H_i + c(F = 0))$ is lc then

$$\sum (1 - w(x_i)) + cw(f) \leq n, \quad \text{or equivalently,} \quad c \leq \frac{\sum w(x_i)}{w(f)}. \quad \square$$

When do we have equality? Let $b = \sum w(x_i)/w(f)$. As in the proof of (8.10), let $p : Y \to X$ denote the blow up of the origin with exceptional divisor E. We obtain that

$$\begin{aligned} &(X, \sum (1 - w(x_i))H_i + b(F = 0)) \quad \text{is lc, iff} \\ &(Y, E + \sum (1 - w(x_i))p_*^{-1}(H_i) + b \cdot p_*^{-1}(F = 0)) \quad \text{is lc.} \end{aligned}$$

A slight problem is that the coefficients of the H_i are negative, and inversion of adjunction fails with negative coefficients. Thus we can only assert that if the leading term of F defines a smooth (or just lc) hypersurface $(F_{w(f)} = 0) \subset \mathbb{P}^{n-1}$, then $(Y, E + b \cdot p_*^{-1}(F = 0))$ is lc. Subtracting the divisors $p_*^{-1}(H_i)$ only helps, thus we obtain the following:

8.14 Proposition. *Let f be a holomorphic function near $0 \in \mathbb{C}^n$ and $D = (f = 0)$. Assign integral weights $w(x_i)$ to the variables and let $w(f)$ be the weighted multiplicity of f. Let f_w denote the weighted homogeneous leading term of the Taylor series of f. Assume that*

$$(f_w(z_1^{w(x_1)}, \dots, z_n^{w(x_n)}) = 0) \subset \mathbb{P}^{n-1} \quad \textit{is smooth (or lc).}$$

Then

$$c_0(D) = \frac{\sum w(x_i)}{w(f)}. \quad \square$$

8.14.1 Remark. Using a weighted blow up, it is not hard to see that (8.14) also holds if f is semiquasihomogeneous, that is, if f_w has an isolated critical point at the origin ([AGV85,I.12.1]).

The following examples give some explicit formulas.

8.15 Example. (8.14) shows that

$$c_0(\sum x_i^{b_i}) = \min\{1, \sum \frac{1}{b_i}\}.$$

Define sets of numbers by

$$\mathcal{F}_n := \left\{ \sum_{i=1}^{n} \frac{1}{b_i} | b_i \in \mathbb{N} \right\} \cap (0, 1].$$

(8.14) shows that $\mathcal{F}_n \subset \mathcal{T}_n$. $\mathcal{F}_n$ satisfies the ascending chain condition for every n. The set of accumulation points of $\mathcal{F}_n$ is precisely $\mathcal{F}_{n-1}$. This shows that the sets $\mathcal{T}_n$ have plenty of accumulation points in the interval $[0,1]$.

8.16 Example. An equivalent formulation of the ascending chain condition is that each subset of the set has a maximal element. Thus, for instance, $\mathcal{T}_n \cap [0,1)$ has a maximal element. Let us denote it by $1-\delta'(n)$ (cf. [Kollár94, 5.3.3]). It is known that $\delta'(1)=1/2$, $\delta'(2)=1/6$ and $\delta'(3)=1/42$ [Kollár94, 5.4]. More generally, define a sequence a_i by the recursive formula

$$a_1=2, a_{k+1}=a_1a_2\cdots a_k+1.$$

From (8.14) we obtain that

$$c_0(x_1^{a_1}+\cdots+x_n^{a_n})=1-\frac{1}{a_{n+1}-1}.$$

It is possible that $\delta'(n)=1/(a_{n+1}-1)$ for every n. It is known that the maximal element of $\mathcal{F}_n\cap[0,1)$ is $1-1/(a_{n+1}-1)$ [Soundararajan95].

8.17 Example. An analysis of the proof of (8.14) shows that

$$c_0((\prod x_i^{a_i})(\sum x_i^{b_i}))=\min\left\{\frac{\sum_i 1/b_i}{1+\sum_i a_i/b_i},\frac{1}{a_1},\dots,\frac{1}{a_n}\right\}.$$

Define sets of numbers by

$$\mathcal{G}_n:=\left\{\frac{\sum_{i=1}^n 1/b_i}{1+\sum_{i=1}^n a_i/b_i}|a_i,b_i\in\mathbb{N}\right\}.$$

It is quite remarkable that $\mathcal{G}_n$ does not satisfy the ascending chain condition. For example, fix the numbers a_i and $b_1,\dots,b_{n-1}$ and let $b_n\to\infty$. Then

$$\lim_{b_n\to\infty}\frac{\sum_i^n 1/b_i}{1+\sum_i^n a_i/b_i}=\frac{\sum_i^{n-1} 1/b_i}{1+\sum_i^{n-1} a_i/b_i},$$

and the sequence is increasing iff

$$\frac{\sum_i^{n-1} 1/b_i}{1+\sum_i^{n-1} a_i/b_i}>\frac{1}{a_n}.$$

It is precisely in this case that the lc-threshold is computed by $\min\{1/a_i\}$ and not by the main part $(\sum_i 1/b_i)/(1+\sum_i a_i/b_i)$.

It is not hard to see that $\mathcal{G}_n\cap\mathcal{T}_n$ satisfies the ascending chain condition.

The last question that we consider in this section is the following. Let $f\in\mathbb{C}[[x_1,\dots,x_n]]$ be a power series. How well can one approximate $c_0(f)$ by computing c_0 of some polynomials? More specifically, let $f_{\le d}$ be the degree $\le d$ part of f. What can one say about the difference $c_0(f)-c_0(f_{\le d})$?

8.18 Example. (8.18.1) Assume that f defines an isolated singularity. Then f and $f_{\le d}$ differ only by a coordinate change for $d \gg 1$, thus $c_0(f) = c_0(f_{\le d})$.

(8.18.2) Let $f = (y+x^2+x^3+\dots)^2 \in \mathbb{C}[[x,y]]$. Then $c_0(f) = 1/2$. Furthermore,

$$f_{\le d} = (y + x^2 + x^3 + \cdots + x^{d-1})^2 - \llcorner (d-1)/2 \lrcorner x^{d+1} - \dots.$$

Change variables to $z = y + x^2 + x^3 + \cdots + x^{d-1}$. We get that

$$f_{\le d} = z^2 - \llcorner (d-1)/2 \lrcorner x^{d+1} - \dots.$$

Setting $w(z) = 1/2$ and $w(x) = 1/(d+1)$ shows that $c_0(f_{\le d}) = 1/2 + 1/(d+1)$.

Thus the best we can hope is that $c_0(f_{\le d})$ converges to $c_0(f)$ in a uniform way. Theorem (8.20) is a much more precise result. Together with (8.10.1) it implies the following estimate:

8.19 Proposition. *Let $f \in \mathbb{C}[[x_1, \dots, x_n]]$ be a power series and let $f_{\le d}$ denote the degree $\le d$ part of f. Then*

$$|c_0(f) - c_0(f_{\le d})| \le \frac{n}{d+1}. \quad \square$$

8.20 Theorem. [Demailly-Kollár96] *Let f, g be polynomials (or power series) in n variables. Then*

$$c_0(f+g) \le c_0(f) + c_0(g) \quad \textit{and} \quad c_0(fg) \le \min\{c_0(f), c_0(g)\}.$$

* * * * *

Proof. The estimate for $c_0(fg)$ is clear. The only surprising part is that it cannot be sharpened; for instance $c_0(x^a y^b) = \min\{1/a, 1/b\}$.

The proof of the additive part has two steps. The first is the computation of the lc-threshold for direct sums of functions. The formula is proved in [AGV84,vol.II. sec.13.3.5] for isolated singularities. Since the proof does not seem to generalize to the nonisolated case, we give an alternative argument in a more general setting (8.21).

The second step uses inversion of adjunction to go from $f(x_1, \dots, x_n) + g(y_1, \dots, y_n)$ to $f(x_1, \dots, x_n) + g(x_1, \dots, x_n)$. Let $B_X \subset \mathbb{C}^n$ be a small ball with coordinates x_i and B_Y a small ball with coordinates y_i. Let

$$\begin{aligned} G &= (f(x_1, \dots, x_n) + g(x_1, \dots, x_n) = 0) \subset B_X, \quad \text{and} \\ D &= (f(x_1, \dots, x_n) + g(y_1, \dots, y_n) = 0) \subset B_X \times B_Y. \end{aligned}$$

Set $F_i = (x_i = y_i)$. Applying (7.5) n-times to $cD + F_1 + \cdots + F_n$ we obtain that

$$(B_X, cG) \quad \text{is lc} \quad \Leftrightarrow \quad (B_X \times B_Y, cD + F_1 + \cdots + F_n) \quad \text{is lc.}$$

The latter clearly implies that $(B_X \times B_Y, cD)$ is lc. This shows that $c_0(B_X, G) \le c_0(B_X \times B_Y, D)$. By (8.21), $c_0(B_X \times B_Y, D) \le c_0(B_X, f) + c_0(B_X, g)$, which completes the proof. $\square$

8.21 Proposition. *Let (X_1, Δ_1) and (X_2, Δ_2) be lc pairs with marked points $x_i \in X_i$. Let*

$$X = X_1 \times X_2, \quad \text{and} \quad \Delta = \Delta_1 \times X_2 + X_1 \times \Delta_2$$

be their product; $x = (x_1, x_2) \in X$. Let f_i be a regular function on X_i, $D_i = (f_i = 0)$ and $D = (f_1 + f_2 = 0)$. Then

$$c_x(X, \Delta, D) = \min\{1, c_{x_1}(X_1, \Delta_1, D_1) + c_{x_2}(X_2, \Delta_2, D_2)\}.$$

Proof. In order to simplify notation, we pretend that X_i is local with closed point x_i. Let $p_i : Y_i \to X_i$ be log resolutions and write

$$K_{Y_i} = p_i^*(K_{X_i} + \Delta_i) + \sum_j a_{ij} E_{ij}, \quad \text{and} \quad p_i^* D_i = \sum_j b_{ij} E_{ij}.$$

Set $Y = Y_1 \times Y_2$ and let $p : Y \to X$ be the product morphism. p is a resolution of X but it is rarely a log resolution. Set $E'_{1j} = E_{1j} \times Y_2$ and $E'_{2j} = Y_1 \times E_{2j}$. Then

$$K_Y = p^*(K_X + \Delta) + \sum_{ij} a_{ij} E'_{ij}.$$

The problem is that $p^* D$ is not a sum of the divisors E'_{ij}.

To study the situation, choose indices j, k and points $y_1 \in E_{1j}$ and $y_2 \in E_{2k}$ such that $\sum_j E_{ij}$ is smooth at y_i. Let v_1 (resp. v_2) be a local defining equation of E_{1j} (resp. E_{2k}). By suitable choice of the v_i we may assume that locally near y

$$p_1^* f_1 = v_1^{b_{1j}}, \quad \text{and} \quad p_2^* f_2 = v_2^{b_{2k}}.$$

In a neighborhood of $y \in Y$ there are two exceptional divisors $E'_{1j} = (v_1 = 0)$ and $E'_{2k} = (v_2 = 0)$, and

$$p^* f = v_1^{b_{1j}} + v_2^{b_{2k}}, \quad \text{locally near } y.$$

Set $F_{jk} = (v_1^{b_{1j}} + v_2^{b_{2k}} = 0)$. If $(X, \Delta + cD)$ is lc, then by (3.10) we obtain that

$$(8.21.1) \qquad (Y, -a_{1j} E'_{1j} - a_{2k} E'_{2k} + c F_{jk}) \quad \text{is lc near } y.$$

(8.11.3) shows that (8.21.1) is equivalent to

$$c - \frac{a_{1j}}{b_{1j}} - \frac{a_{2k}}{b_{2k}} \le \frac{1}{b_{1j}} + \frac{1}{b_{2k}}, \quad \text{that is,} \quad c \le \frac{a_{1j} + 1}{b_{1j}} + \frac{a_{2k} + 1}{b_{2k}}.$$

By (8.5),

$$c_{x_1}(X_1, \Delta_1, D_1) = \min_j \frac{a_{1j} + 1}{b_{1j}} \quad \text{and} \quad c_{x_2}(X_2, \Delta_2, D_2) = \min_k \frac{a_{2k} + 1}{b_{2k}}.$$

Choose j and k such that they achieve the minima. Then we obtain that

$$c_x(X,\Delta,D) \le c_{x_1}(X_1,\Delta_1,D_1) + c_{x_2}(X_2,\Delta_2,D_2).$$

We have not proved equality, since the above procedure controls only those divisors F of $K(X)$ such that $\operatorname{Center}_Y(F) \supset E'_{1j} \cap E'_{2k}$ for some j,k. There are two ways to go to the general case.

First, we can do a similar computation at any point of Y. I found this somewhat cumbersome.

Second, we could try to show that the above computation accounts for all divisors F of $K(X)$, if we vary Y_1 and Y_2. Indeed, for suitable choice of Y_i we may assume that the image of F on Y_i contains a divisor E_{1j} (resp. E_{2k}) (3.17). This means that $\operatorname{Center}_Y(F) \supset E'_{1j} \cap E'_{2k}$, as required. □

8.21.2 Exercise. Use (8.21) to show that every element of $\mathcal{T}_n - \{1\}$ is an accumulation point of $\mathcal{T}_{n+1}$.

It is possible that the set of accumulation points of $\mathcal{T}_n$ is precisely $\mathcal{T}_{n-1}$. In the toric case this was proved by [Borisov95].

9. The Log Canonical Threshold and the Complex Singular Index

In this section we compare the lc-threshold and the complex singular index of an isolated singularity. The notion of complex singular index was introduced by Arnold, using the asymptotic behaviour of certain integrals over vanishing cycles. See [AGV85, II.Chap.13] for the motivation and for basic results.

The classical case is the following:

9.1 Definition. Let $f : (0,\mathbb{C}^{n+1}) \to (0,\mathbb{C})$ be a holomorphic function in the neighborhood of the origin. Assume that f has an isolated critical point at the origin. Set $D := (f=0)$. Let $B \subset \mathbb{C}^{n+1}$ be a small ball around the origin and $\Delta \subset \mathbb{C}$ an even smaller disc around the origin. Set $X = B \cap f^{-1}(\Delta)$. From now on, we restrict f to $f : X \to \Delta$.

By [Milnor68], the only interesting homology of $X_t := f^{-1}(t)$ for $t \neq 0$ is in dimension n. The corresponding cycles are called the *vanishing cycles.*

If σ is a section of $\omega_{X/\Delta}$ then σ restricts to a holomorphic n-form on each X_t. Thus if $\delta(t)$ is an n-cycle in X_t, then we can form the integral

$$\int_{\delta(t)} \sigma, \tag{9.1.1}$$

which depends only on the homology class of $\delta(t)$.

Let $x_1,\dots,x_{n+1}$ be local coordinates on $\mathbb{C}^{n+1}$. A local generator of $\omega_{X/\Delta}$ can be written down explicitly. Up-to a sign, it is

$$\frac{dx_1 \wedge \cdots \wedge dx_{n+1}}{df} := \pm \frac{dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1}}{\partial f/\partial x_i}. \tag{9.1.2}$$

9.2 General case. More generally, we can consider an arbitrary (normal) complex space X and any morphism $f : X \to \Delta$ such that f is smooth over Δ^* and

$X - f^{-1}(0) \to \Delta^*$ is a locally trivial topological fiber bundle over Δ^*. (The latter can usually be assumed by shrinking Δ.)

Let σ be a section of $\omega_{X/\Delta}$. σ restricts to a section of ω_{X_t} for each t. If X_t is smooth, then $\int_{\delta(t)} \sigma$ makes sense as above.

The following basic theorem describes the asymptotic behaviour of the integrals $\int_{\delta(t)} \sigma$ for small values of t. It can be approached from many different points of view. See, for instance, [AGV85, II.10.2] for a discussion and several references.

9.3 Theorem. *Notation and assumptions as above. Let $t \mapsto \delta(t) \in H_n(X_t, \mathbb{Q})$ be a continuous (multiple valued) section. There is an asymptotic expansion (as $t \to 0$)*

$$\int_{\delta(t)} \sigma = \sum_{\alpha \in \mathbb{Q}, k \in \mathbb{N}} a(\sigma, \delta, \alpha, k) t^{\alpha} (\log t)^k,$$

where the $a(\sigma, \delta, \alpha, k)$ are constants. $\square$

9.3.1 Complement. *One can get rather precise information about the possible values of α and k. The following are some of these:*

(9.3.1.1) There is a lower bound for the values of α, depending only on $f : X \to \Delta$. (This also follows from (9.5).)

(9.3.1.2) The values of $\alpha \mod 1$ can be described in terms of the eigenvalues of the monodromy.

(9.3.1.3) There is an upper bound for the values of k depending on the size of the Jordan blocks of the monodromy.

(9.3.1.4) If X_0 is a normal crossing divisor, then $a(\sigma, \delta, \alpha, k) = 0$ for $\alpha < 0$.

9.4 Definition. Let X be a normal complex space and $f : X \to \Delta$ a morphism. Set $D = f^{-1}(0)$. Let $x \in D$ be a closed point and assume that f is smooth on $X - x$. Assume also that ω_X is locally free.

The *complex singular index*, denoted by $\beta_{\mathbb{C}}(f)$ or by $\beta_{\mathbb{C}}(X, D)$ is defined by the formula

$$\beta_{\mathbb{C}}(X, D) = \beta_{\mathbb{C}}(f) := 1 + \inf\{\alpha | \exists \sigma, \delta, k \text{ such that } a(\sigma, \delta, \alpha, k) \neq 0\}.$$

The definition gives $\beta_{\mathbb{C}}(f) = \infty$ if there are no vanishing cycles at all. If X is smooth this happens only when D is also smooth.

In most cases the asymptotic expansion involves negative powers of t, thus the complex singular index measures the maximum rate of divergence of the above integrals as $t \to 0$.

The terminology is taken from [Steenbrink85]. In [Varchenko82, p.477] this is called the complex singular exponent and in [AGV85,II.13.1.5] the complex oscillation index.

The following theorem relates the lc-threshold to the complex singular index. For X smooth it was proved by [Varchenko82, §4]. The proof also works in a more general setting. Further generalizations are pointed out in (9.7).

9.5 Theorem. [Varchenko82, §4] *Notation and assumptions as in (9.4). Then*

$$c_x(f) = \min\{1, \beta_{\mathbb{C}}(f)\}.$$

Proof. Let $\pi_n : \Delta_n \to \Delta$ be the morphism $t_n \mapsto t = t_n^n$, and consider the fiber product diagram

$$\begin{array}{ccc} X_n & \xrightarrow{\pi_n} & X \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f} \\ \Delta_n & \xrightarrow{\pi_n} & \Delta. \end{array} \tag{9.5.1}$$

We can identify the central fiber of f_n with the central fiber D of f. For suitable choice of n we may assume that $f_n : X_n \to \Delta_n$ has a semistable resolution. That is, there is a proper birational morphism $g_n : Y_n \to X_n$ such that Y_n is smooth and

$$g_n^* D = D_0 + \sum_{i>0} D_i$$

is a reduced divisor with normal crossings only, where D_0 is the birational transform of D.

Using (3.16) we obtain that

$$c_x(f_n) = 1 - n(1 - c_x(f)),$$

where we use the general definition of the lc-threshold (8.4). Write

$$K_{Y_n} = g_n^* K_{X_n} + \sum_{i>0} a_i D_i, \quad \text{hence also}$$
$$K_{Y_n/\Delta_n} = g_n^* K_{X_n/\Delta_n} + \sum_{i>0} a_i D_i.$$

Set $a_0 = 0$, $c = \min_{i\geq 0}\{a_i\}$ and note that $c_x(f_n) = c + 1$. The crucial formula is

$$K_{Y_n/\Delta_n} = g_n^*(K_{X_n/\Delta_n} + cD) + \sum_{i\geq 0}(a_i - c) D_i, \tag{9.5.2}$$

where $a_i - c \geq 0$ for every $i \geq 0$. Let σ be any section of $\mathcal{O}(K_{X/\Delta})$. $K_{X_n/\Delta_n} = \pi_n^* K_{X/\Delta}$, thus $\pi_n^*\sigma$ is a section of $\mathcal{O}(K_{X_n/\Delta_n})$. Therefore $t_n^{-c}\pi_n^*\sigma$ is a section of $\mathcal{O}(K_{X_n/\Delta_n} + cD)$, hence by (9.1) it corresponds to a holomorphic section σ_n' of $\mathcal{O}(K_{Y_n/\Delta_n})$.

Up to an n^{th}-root of unity, $t_n^{-c}\pi_n^*\sigma = \pi_n^*(t^{1-c_x(f)}\sigma)$, and therefore

$$t^{1-c_x(f)} \int_{\delta(t)} \sigma = \int_{\delta(t_n)} \sigma_n', \quad \text{for } t \neq 0, \text{ where } t_n = t^{1/n}.$$

σ_n' is a holomorphic section of $\mathcal{O}(K_{Y_n/\Delta_n})$, and so, by (9.3.1.4),

$$\int_{\delta(t_n)} \sigma_n' \tag{9.5.3}$$

grows at most logarithmically as $t_n \to 0$. This shows that $c_x(f) \le \beta_{\mathbb{C}}(f)$, hence also $c_x(f) \le \min\{1, \beta_{\mathbb{C}}(f)\}$ and equality holds if $c_x(f) = 1$.

In order to see the equality in the remaining cases, we have to find $\delta(t_n)$ such that the integral (9.5.3) grows as a nonzero constant times a power of $\log t_n$. Thus assume that $c_x(f) < 1$, which is equivalent to $c < 0$. Let $E := D_j,\ j > 0$ be an irreducible component such that $a_j = c$. Such a component exists since $c < 0$ and it is proper since D has isolated singularities. Set $E^0 = E - \cup_{i\neq j} D_i$; this is an open set of E.

σ_n' restricts to a holomorphic section of the dualizing sheaf of $\cup_{i\ge 0} D_i$, thus $\sigma_n'|E$ is a holomorphic n-form on E with at worst simple poles along $E - E^0$ (for top degree forms this is the same as having logarithmic poles). By [Deligne71], closed forms with logarithmic poles at infinity compute the cohomology of a smooth variety, thus there is an n-cycle $Z \subset E^0$ such that $\int_Z (\sigma_n'|E) \ne 0$.

f_n is a locally trivial fibration near E^0, thus Z can be extended to an n-cycle $\delta(t_n)$ for small values of t_n. (We even get a monodromy invariant cycle, but this is not important for now.) Let σ be a local generator of $\mathcal{O}(K_{X/\Delta})$. By construction,

$$\lim_{t\to 0} t^{1-c_x(f)} \int_{\delta(t)} \sigma = \lim_{t_n \to 0} \int_{\delta(t_n)} \sigma_n' = \int_Z (\sigma_n'|E) \ne 0,$$

thus the asymptotic expansion of

$$\int_{\delta(t)} \sigma$$

does contain a nonzero term $const \cdot t^{c_x(f)-1}$. $\square$

* * * * *

9.6 Generalizations. From the point of view of the lc-threshold, the assumptions that D has an isolated singularity and that f is smooth over Δ^* are rather restrictive. Some of these conditions can be weakened.

If D does not have an isolated singularity, then it is not clear what exactly happens. For instance, assume that $f(x, y)$ defines an isolated singularity. Viewed as a map $f_2 : \mathbb{C}^2 \to \mathbb{C}$ the 1-dimensional homology of the nearby fibers gives the vanishing cycles. If we view f as a morphism $f_3 : \mathbb{C}^3 \to \mathbb{C}$, then the central fiber has a nonisolated singularity. Furthermore, $f_3^{-1}(t) \cong f_2^{-1}(t) \times \mathbb{C}$, thus all the interesting homology is in 1-dimension and we cannot integrate a 2-form. I do not know how to overcome this problem, except in some special cases.

If f is not smooth over Δ^*, we can proceed as follows.

Let $p : X' \to X$ be a resolution of singularities and $f' : X' \to \Delta$ the induced morphism. Assume that f' is smooth over Δ^*. Let $\delta(t) \in H_n(X_t')$ be a continuous (multiple valued) section. Assume furthermore that X_t has canonical singularities for $t \ne 0$.

Under these assumptions, the integral $\int_{\delta(t)} p^*\sigma$ makes sense and it behaves like the integral (9.1.1).

The proof of (9.5) shows that the result also holds more generally:

9.7 Theorem. *Let X be a normal analytic space and $f : X \to \Delta$ a morphism. Assume that*

(9.7.1) ω_X is locally free;

(9.7.2) $f^{-1}(0)$ has rational singularities except at a single point $x \in f^{-1}(0)$;

(9.7.3) $f^{-1}(t)$ has rational singularities for $t \neq 0$.

Then

$$c_x(f) = \min\{1, \beta_{\mathbb{C}}(f)\}. \quad \square$$

The log canonical threshold is also related to the constants of quasiadjunction intruduced in [Libgober83] and further studied in [Loeser-Vaquié90].

9.8 Proposition. *Let $f(x_1, \dots, x_n)$ define a singularity at the origin. For every m let $\psi(m)$ be the smallest integer such that $y^{\psi(m)}$ is contained in the adjoint ideal of the hypersurface $X_m := (y^m = f)$. Then $\psi(m) = \llcorner m(c_0(f) + 1)\lrcorner$.*

Proof. $y^{1-m} dx_1 \wedge \cdots \wedge dx_n$ is a local generator of ω_{X_m}, thus $y^{\psi(m)}$ is contained in the adjoint ideal iff $y^{\psi(m)+1-m} dx_1 \wedge \cdots \wedge dx_n$ is L^2. Pushing down to $\mathbb{C}^n$, this is equivalent to $|f|^{(\psi(m)+1-m)/m}$ being L^2. This happens precisely when $(\psi(m)+1-m)/m > c_0(f)$, which is eqivalent to $\psi(m) = \llcorner m(c_0(f)+1)\lrcorner$. $\square$

10. The Log Canonical Threshold and the Bernstein-Sato Polynomial

The aim of this section is to compare the log canonical threshold of a function f to the Bernstein-Sato polynomial of f. The basic definitions are given bellow.

10.1 Theorem. [Bernstein71; Björk79] *Let $f = f(z_1, \dots, z_n)$ be a polynomial (resp. a convergent power series) and s a variable. There is a nonzero polynomial $b(s) \in \mathbb{C}[s]$ and a linear differential operator*

$$P = \sum_{I,j} f_{I,j} s^j \frac{\partial^I}{\partial z^I},$$

whose coefficients $f_{I,j}$ are polynomials (resp. convergent power series) such that

(10.1.1) $$b(s) f^s = P f^{s+1}. \quad \square$$

10.1.2 Remark. It is easiest to interpret (10.1.1) as a formal equality, where we do not assign any meaning to the powers f^s, just handle them as symbols with the usual roles of differentiation assumed. If the powers have a well defined meaning as functions (for instance, f is everywhere nonnegative on $\mathbb{R}^n$) then the formal equality becomes an actual equality of functions.

10.2 Definition. All the polynomials satisfying (10.1) form an ideal in $\mathbb{C}[s]$. The unique generator of this ideal with leading coefficient 1 is called the *Bernstein-Sato polynomial* of f. It is denoted by $b_f(s)$.

In singularity theory, many people use the defining equation $b(s)f^{s-1} = Pf^s$; this corresponds to the substitution $s := s+1$ in the polynomial b_f.

10.3 Remark. The polynomial b_f is a very interesting invariant of the singularity $(f = 0)$. It can be connected with with other types of invariants in many different ways, see, for instance, [Malgrange75; Loeser87] and the references there.

10.4 Definition. Setting $s = -1$, (10.1.1) becomes $b(-1)f^{-1} = \sum_j f_{0,j}$, which implies that $b(-1) = 0$. Thus $b_f(s) = (s+1)\tilde{b}_f(s)$. $\tilde{b}_f(s)$ is called the *reduced Bernstein-Sato polynomial* of f.

10.5 Examples. It is not easy to compute b_f and P in concrete examples.

(10.5.1) For quadratic forms the answer is rather obvious. Set $Q(z) = \sum z_i^2$, then

$$(s+1)(s+\frac{n+1}{2})Q(z)^s = \frac{1}{4}\left(\sum \frac{\partial^2}{\partial z_i^2}\right) Q(z)^{s+1}.$$

(10.5.2) Already the case of cusps is nontrivial:

$$(s+1)(s+\frac{5}{6})(s+\frac{7}{6})(x^2+y^3)^s = \left(\frac{1}{27}\frac{\partial^3}{\partial y^3} + \frac{y}{6}\frac{\partial^3}{\partial x^2 \partial y} + \frac{x}{8}\frac{\partial^3}{\partial x^3}\right)(x^2+y^3)^{s+1}.$$

(10.5.3) Assume that f defines an isolated singularity at the origin, and if we set $wt(z_i) = a_i$ then f is weighted homogeneous of degree 1. By [Yano78],

$$\prod_i \frac{t^{a_i} - t}{1 - t^{a_i}} = \sum_{\alpha \in \mathbb{Q}} q_\alpha t^\alpha \quad \text{is a finite sum, and} \quad \tilde{b}_f(s) = \prod_{\alpha : q_\alpha \neq 0} (s + \alpha).$$

(10.5.4) Let $f = \sum z_i^{m_i}$. One can easily compute using (10.5.3) that

$$\text{largest root of } \tilde{b}_f = -\sum \frac{1}{m_i}.$$

The following observation relates the roots of Bernstein-Sato polynomials to the lc-threshold of f:

10.6 Theorem. *Let $f = f(z_1, \dots, z_n)$ be a polynomial or a convergent power series. Then*

$$\textit{largest root of } b_f(s) = -(\textit{lc-threshold of } f)$$

Proof. By definition, we have $b(s)f^s = Pf^{s+1}$ and conjugating it we obtain $\bar{b}(s)\bar{f}^s = \bar{P}\bar{f}^{s+1}$. Two differential operators do not commute in general, but a holomorphic operator always commutes with an antiholomorphic one, and, moreover

$$Pf^{s+1} \cdot \bar{P}\bar{f}^{s+1} = (P\bar{P})|f^2|^{s+1}.$$

Let ϕ be any C^∞ function supported in a small neighborhood of the origin. Then we have the equality

$$|b(s)|^2 \int |f^2|^s \phi dm(z) = \int |f^2|^{s+1} (P\bar{P}) \phi dm(z),$$

where $dm(z)$ is the Lebesgue measure.

Let $c(f)$ be the lc-threshold of f. As long as $s > -c(f)$, both sides are well defined and finite. If ϕ is positive and nonzero at the origin, then

$$\lim_{s\to -c(f)^+} \int |f^2|^s \phi dm(z) = +\infty, \quad \text{and} \quad \int |f^2|^{-c(f)+1}(P\bar{P})\phi dm(z) < \infty.$$

This shows that $-c(f)$ is a root of b_f.

Assume that $t > -c(f)$ is a root of b_f. We obtain that

$$\int |f^2|^{t+1}(P\bar{P})\phi dm(z) = 0 \quad \text{for every } \phi.$$

This is a rather rare accident, but cannot be excluded without knowing something about P.

The actual proof that $-c(f)$ is the largest root is unfortunately rather complicated. It follows from the next result of [Lichtin89, Thm. 5], which in turn is a modification of the arguments in [Kashiwara76]:

10.7 Theorem. *Set $D = (f = 0) \subset \mathbb{C}^n$ and let $\pi : Y \to \mathbb{C}^n$ be a log resolution with exceptional divisors $D_i : i > 0$. Set $D_0 := \pi_*^{-1} D$. Write*

$$K_Y = \sum_{i>0} a_i D_i, \quad \textit{and} \quad \pi^* D = \sum_{i\geq 0} d_i D_i.$$

Then every root of b_f is of the form

$$-\frac{a_i + e}{d_i} \quad \textit{for some } i \geq 0 \textit{ and } e \in \mathbb{N}. \qquad \square$$

By (8.5) we know that $c(f) = \min_i\{(a_i + 1)/d_i\}$, which shows that b_f does not have any root bigger than $-c(f)$. $\square$

10.8 Remark. If $f = 0$ defines a rational singularity, then the largest root of b_f is the trivial root -1 and the lc-threshold is 1, as it should be.

In this case it is natural to try to connect the largest nontrivial root of b_f with some geometric data coming from the resolution. The following may appear a rather natural candidate:

$$-(\text{largest root of } \tilde{b}_f) \stackrel{?}{=} \inf_{i:\ \pi(D_i)\subset \operatorname{Sing} D} \frac{a_i + 1}{d_i}.$$

Unfortunately, the right hand side depends on the resolution chosen.

More generally, it would be of interest to understand which exceptional divisors give roots of b_f in (10.7).

11. Rational and Canonical Singularities

The aim of this section to prove that canonical singularities are rational. This result was proved by [Elkik81; Flenner81]. The essential part of these proofs was generalized in [Fujita85] and treated systematically in [KaMaMa87, 1-3]. The treatment given here uses duality theory only for CM schemes, and this simplification makes the proofs even a little shorter.

11.1 Theorem. *Let X be a normal variety over a field of characteristic zero.*

(11.1.1) Assume that ω_X is locally free. Then X has rational singularities iff X has canonical singularities.

(11.1.2) Assume that (X,D) is a klt pair. Then X has rational singularities.

11.1.3 Remark. If ω_X is not locally free, then rational and klt are no longer equivalent. Most rational singularities are not klt and not even log canonical. For instance, a normal surface singularity is klt iff it is a quotient singularity (3.6), but there are many rational surface singularities which are not quotient.

At the end (11.15) we present a result about deformations of rational singularities. This is a generalization of [Elkik78].

* * * * *

As the first step of the proof, recall the Leray spectral sequence for local cohomology and some of its immediate consequences:

11.2 Theorem. *Let $f: Y \to X$ be a proper morphism, $x \in X$ a closed point, $F = f^{-1}(x)$ and G a sheaf on Y.*

(11.2.1) There is a Leray spectral sequence $E_2^{ij} = H^i_x(X, R^jf_*G) \Rightarrow H^{i+j}_F(Y,G)$.

(11.2.2) The spectral sequence gives an injection $H^1_x(X, f_*G) \hookrightarrow H^1_F(Y,G)$.

(11.2.3) If $R^if_*G = 0$ for $i > 0$ then $H^j_F(Y,G) = H^j_x(X, f_*G)$ for every j.

(11.2.4) If $\operatorname{Supp} R^if_*G \subset \{x\}$ for $i \ge 0$ then $H^j_F(Y,G) = R^jf_*G$ for every j.

(11.2.5) If $\operatorname{Supp} R^if_*G \subset \{x\}$ for $1 \le i < k$ and $H^i_F(Y,G) = 0$ for $i \le k$ then $R^jf_*G = H^{j+1}_x(X, f_*G)$ for $j = 1, \dots, k-1$.

Proof. It is clear that $H^0_x(X, f_*G) = H^0_F(Y,G)$. This gives a spectral sequence between the derived functors. The construction is the same as for the ordinary Leray spectral sequence (see e.g. [Griffiths-Harris78, p.462]).

Looking at the beginning of the spectral sequence gives (11.2.2). Under the assumptions (11.2.3) or (11.2.4) the spectral sequence degenerates at the E_2 term since all the nonzero E_2^{ij} are in one row or column.

Finally assume (11.2.5). Then the only nonzero E_2^{ij} for $0 \le i,j \le k$ are those with $ij = 0$. Thus for every $j < k$ there is only one possible nonzero differential $d: R^jf_*G \to H^{j+1}_x(X, f_*G)$ which must be an isomorphism since $H^i_F(Y,G) = 0$ for $i \le k$. $\square$

11.3 Definition. Let X be a scheme of pure dimension n and G a sheaf on X. We say that G is *CM* (which is an abbreviation for *Cohen–Macaulay*) if it satisfies the following equivalent conditions (cf. [Hartshorne77, Exercise III.3.4]):

(11.3.1) for every point $x \in X$, $\operatorname{depth}_x G = \operatorname{codim}(x, X)$,

(11.3.2) $H^i_x(X,G) = 0$ for every $x \in X$ and $i < \operatorname{codim}(x,X)$.

We say that X is CM if $\mathcal{O}_X$ is CM.

Basic properties of CM sheaves are recalled in the next lemma.

11.4 Lemma. *(11.4.1) Let X be a regular scheme and G a coherent sheaf. Then G is CM iff it is locally free.*

(11.4.2) Let $f : X \to Y$ be a finite morphism of schemes of pure dimension n and G a coherent sheaf on X. Then G is CM iff f_*G is CM.

Proof. The first part is proved in [Matsumura86, 19.1]. The second assertion follows from (11.2.3). □

11.5 Proposition. *Let X be an S_2 scheme of pure dimension n and assume that ω_X exists. Then $\mathcal{O}_X$ is CM iff ω_X is CM.*

More generally, if G is an S_2 sheaf then G is CM iff $\operatorname{Hom}(G, \omega_X)$ is CM.

Proof. We may clearly suppose that X is affine. Assume first that there is a finite morphism $f : X \to Z$ onto a regular scheme Z of dimension n.

Then $f_*\omega_X = \operatorname{Hom}(f_*\mathcal{O}_X, \omega_Z)$ and ω_Z is a line bundle. Since X is S_2, $f_*\mathcal{O}_X$ is reflexive hence $f_*\omega_X$ and $f_*\mathcal{O}_X$ are duals (up to a twist by a line bundle). Thus ω_X is CM iff $f_*\omega_X$ is locally free iff $f_*\mathcal{O}_X$ is locally free iff $\mathcal{O}_X$ is CM.

The general case is proved the same way since $f_* \operatorname{Hom}(G, \omega_X) = \operatorname{Hom}(f_*G, \omega_Z)$ (cf. [Hartshorne77, Exercise III.6.10]).

By Noether normalization f always exists if X is of finite type. f also exists if X is the spectrum of a complete local ring. The general case can be reduced to the latter by noting that a module over a local ring is CM iff its completion is CM over the completion of the local ring (this is a very special case of [Matsumura86, 23.3]). □

The next proposition is a collection of some duality statements. They are all special cases of the general duality theorem, but they can also be derived from ordinary duality easily.

11.6 Proposition. *Let $f : Y \to X$ be a proper morphism, $x \in X$ a closed point, $F = f^{-1}(x)$ and G a locally free sheaf on Y. Assume that Y is CM of pure dimension n.*

(11.6.1) If $\operatorname{Supp} R^if_*G \subset \{x\}$ then $R^if_*G \overset{\text{dual}}{\sim} H^{n-i}_F(\omega_Y \otimes G^{-1})$.

(11.6.2) If $\operatorname{Supp} R^if_*G \subset \{x\}$ for every $i \geq 0$ then $R^jf_*G \overset{\text{dual}}{\sim} R^{n-j}f_*(\omega_Y \otimes G^{-1})$ for every $j \geq 0$.

(11.6.3) Assume that $R^if_*(\omega_Y \otimes G^{-1}) = 0$ for $i > 0$. If $\operatorname{Supp} R^{n-i}f_*G = \{x\}$ for some i then

$$H^i_x(X, f_*(\omega_Y \otimes G^{-1})) \overset{\text{dual}}{\sim} R^{n-i}f_*G.$$

Proof. By duality (on Y) we obtain that $\operatorname{Ext}^{n-i}(\mathcal{O}_{mF}, \omega_Y \otimes G^{-1})$ is dual to $H^i(\mathcal{O}_{mF} \otimes G)$ for every $m \geq 1$. The inverse limit of the $H^i(\mathcal{O}_{mF} \otimes G)$ is the completion of R^if_*G at x which is finite dimensional by assumption. $H^{n-i}_F(Y, \omega_Y \otimes G^{-1})$ is the direct limit of the groups $\operatorname{Ext}^{n-i}(\mathcal{O}_{mF}, \omega_Y \otimes G^{-1})$, and is therefore finite dimensional by the above duality.

Thus for $m \gg 1$ we obtain that

$$H^{n-i}_F(Y, \omega_Y \otimes G^{-1}) = \operatorname{Ext}^{n-i}(\mathcal{O}_{mF}, \omega_Y \otimes G^{-1}) \overset{\text{dual}}{\sim} H^i(\mathcal{O}_{mF} \otimes G) = R^if_*G.$$

In order to show (11.6.2) assume for simplicity that ω_Y is locally free. (This is the only case that we use later.) Then $R^{n-j}f_*(\omega_Y \otimes G^{-1})$ is dual to $H^j_F(\omega_Y \otimes (\omega_Y \otimes G^{-1})^{-1}) = H^j_F(G)$ and $H^j_F(G) = R^jf_*G$ by (11.2.4).

Finally consider (11.6.3). By (11.2.3) and (11.6.1) we see that

$$H^i_x(X, f_*(\omega_Y \otimes G^{-1})) \cong H^i_F(Y, \omega_Y \otimes G^{-1}) \overset{\text{dual}}{\sim} R^{n-i} f_* G. \quad \square$$

The next two applications use these duality results to get information about the depth of direct image sheaves.

11.7 Corollary. *(cf.* [Fujita85]*) Let $f : Y^n \to X^k$ be a proper morphism between pure dimensional schemes, Y CM. Assume that every irreducible component of Y dominates an irreducible component of X. Let G be a locally free sheaf on Y such that $R^i f_*(\omega_Y \otimes G^{-1}) = 0$ for $i > n - k$.*

Then f_*G is S_2.

Proof. Pick $x \in X$ such that $j = \dim x \leq k-2$. We need to prove that $\operatorname{depth}_x f_*G \geq 2$. By localization we are reduced to the case when $f : Y^{n-j} \to X^{k-j}$ is proper and $x \in X$ is closed.

By (11.2.2) there is an injection $H^1_x(X, f_*G) \hookrightarrow H^1_F(Y, G)$ and by (11.6.1) $H^1_F(Y, G) \overset{\text{dual}}{\sim} R^{n-j-1} f_*(\omega_Y \otimes G^{-1})$. Since $n - j - 1 \geq n - (k-2) - 1 > n - k$, the latter group is zero by assumption. $\square$

The following result is a refined version of (11.7). It is stated in the dual form, since we use it mostly that way.

11.8 Corollary. *Let $f : Y^n \to X^k$ be a proper morphism of pure dimensional schemes, Y CM. Assume that every irreducible component of Y dominates an irreducible component of X. Let G be a locally free sheaf on Y such that $R^i f_*(\omega_Y \otimes G^{-1}) = 0$ for $i > 0$. The following are equivalent:*

(11.8.1) $R^i f_* G = 0$ for every $i > n - k$, and

(11.8.2) $f_*(\omega_Y \otimes G^{-1})$ is a CM sheaf.

Proof. There is nothing to prove if $k = 0$. The assumptions are stable under localization in X, thus by induction on k we may assume that there is a closed point $x \in X$ such that $\operatorname{Supp} R^i f_* G \subset \{x\}$ for every $i > n - k$ and $f_*(\omega_Y \otimes G^{-1})$ is a CM sheaf on $X - x$.

Then $f_*(\omega_Y \otimes G^{-1})$ is a CM sheaf iff $H^i_x(X, f_*(\omega_Y \otimes G^{-1})) = 0$ for $i < k$ which, by (11.6.3), is equivalent to $R^i f_* G = 0$ for every $i > n - k$. $\square$

11.9 Corollary. [KKMS73, p.50] *Let X be a normal scheme and $f : Y \to X$ a resolution of singularities. Assume that $R^i f_* \omega_Y = 0$ for $i > 0$. The following conditions are equivalent:*

(11.9.1) $R^i f_* \mathcal{O}_Y = 0$ for $i > 0$.

(11.9.2) $f_* \omega_Y = \omega_X$ and ω_X is a CM sheaf.

(11.9.3) $f_* \omega_Y = \omega_X$ and X is CM.

Proof. (11.9.2) $\Leftrightarrow$ (11.9.3) was established in (11.5). $f_*\omega_Y$ is a subsheaf of ω_X and they are equal in codimension one. Thus (11.9.2) is equivalent to the condition: $f_*\omega_Y$ is a CM sheaf. (11.8) for $G = \mathcal{O}_Y$ shows that the latter is equivalent to (11.9.1). $\square$

11.10 Definition. Let X be an excellent scheme over a field of characteristic zero. We say that X has *rational singularities* if it satisfies the equivalent conditions of (11.9). (By (2.17.6) $R^i f_*\omega_Y = 0$ for every $i > 0$ and for every resolution of singularities $f : Y \to X$ if X is over a field of characteristic zero.)

11.11 Exercise. Let X be a reduced and pure dimensional scheme, $f : Y \to X$ a resolution of singularities and $g : Z \to X$ a proper birational morphism, Z normal. Show that there are natural inclusions

$$f_*\omega_Y \subset f_*\omega_Z \subset \omega_X.$$

In particular, the conditions (11.9.2) and (11.9.3) are independent of the choice of $f : Y \to X$.

The following result, due to [Fujita85; KaMaMa87, 1-3], is the main technical result of the section. We give a simpler proof in a slightly more general form:

11.12 Theorem. *Let $f : Y^n \to X^k$ be a proper morphism of pure dimensional schemes, Y CM. Assume that every irreducible component of Y dominates an irreducible component of X. Let L_1, L_2 be line bundles on Y and E an effective Cartier divisor on Y. Assume that*
(11.12.1) $\operatorname{codim}(f(E), X) \geq 2$,
(11.12.2) $\omega_Y \cong L_1 \otimes L_2 \otimes \mathcal{O}_Y(E)$, *and*
(11.12.3) $R^i f_* L_j(E) = 0$ *for* $i > 0$ *and* $j = 1, 2$.
Then $R^i f_* L_j = 0$ *for* $i > 0$ *and* $j = 1, 2$.

Proof. The assumptions are stable under localization at a point of X. Thus by induction on $\dim X$ we may assume that there is a closed point $x \in X$ such that $\operatorname{Supp} R^i f_* L_j \subset \{x\}$ for $i > 0$ and $j = 1, 2$.

The main part of the proof is to establish two different dualities between the sheaves $R^i f_* L_j$. Note that $\omega_Y \otimes L_j^{-1} \cong L_{3-j}(E)$.

By (11.6.1) $R^i f_* L_j$ is dual to $H_F^{n-i}(Y, L_{3-j}(E))$ and using (11.2.3) we obtain that $H_F^{n-i}(Y, L_{3-j}(E)) = H_x^{n-i}(X, f_* L_{3-j}(E))$.

$f_* L_{3-j}$ is S_2 by (11.7), hence $f_* L_{3-j} = f_* L_{3-j}(E)$ and so $H_x^{n-i}(X, f_* L_{3-j}(E)) = H_x^{n-i}(X, f_* L_{3-j})$ for every i, j. $H_x^{n-i}(X, f_* L_{3-j}) = 0$ for $i = n, n-1$ since $f_* L_{3-j}$ is S_2, and this shows that

$$R^n f_* L_j = R^{n-1} f_* L_j = 0. \tag{11.12.4}$$

By (11.6.1) $H_F^{n-i}(Y, L_{3-j})$ is dual to $R^i f_* L_j(E) = 0$ for $i \geq 1$. Therefore by (11.2.5) $H_x^{n-i}(X, f_* L_{3-j}) = R^{n-i-1} f_* L_{3-j}$ for $1 \leq i \leq n-2$.

Putting all these together we obtain that

$$R^i f_* L_j \overset{\text{dual}}{\sim} R^{n-i-1} f_* L_{3-j} \quad \text{for } 1 \leq i \leq n-2. \tag{11.12.5}$$

On the other hand, look at the exact sequence

$$0 \to L_j \to L_j(E) \to L_j(E)|E \to 0.$$

By assumption $R^i f_* L_j(E) = 0$ for $i > 0$ and we proved that $f_* L_j = f_* L_j(E)$. Thus $R^i f_*(L_j(E)|E) = R^{i+1} f_* L_j$ for $i \geq 0$. In particular, $\operatorname{Supp} R^i f_*(L_j(E)|E) \subset \{x\}$ for $i \geq 0$.

By adjunction $\omega_E \otimes (L_1(E)|E)^{-1} \cong L_2(E)|E$. Thus by (11.6.2) $R^i f_*(L_j(E)|E) \overset{\text{dual}}{\sim} R^{n-1-i} f_*(L_{3-j}(E)|E)$. This gives that

$$R^i f_* L_j \overset{\text{dual}}{\sim} R^{n-i+1} f_* L_{3-j} \quad \text{for } 1 \le i \le n. \tag{11.12.6}$$

Put (11.12.5–6) together to conclude that

$$R^i f_* L_j \cong R^{i-2} f_* L_j \quad \text{for } 3 \le i \le n. \tag{11.12.7}$$

Starting with the vanishing (11.12.4) this completes the proof by descending induction on i. $\square$

The simplest application of this vanishing is the first part of (11.1):

11.13 Corollary. *Let X be an excellent normal scheme over a field of characteristic zero. Assume that ω_X exists and is locally free. Then X has rational singularities iff X has canonical singularities.*

Proof. Let $f : Y \to X$ be a resolution of singularities. Assume that X has rational singularities. Then $f_*\omega_Y = \omega_X$, hence there is a natural map $f^*\omega_X \to \omega_Y$. This shows that $\omega_Y = f^*\omega_X(E)$ for some effective divisor $E \subset Y$.

Conversely, assume that X has canonical singularities, that is $\omega_Y \cong f^*\omega_X \otimes \mathcal{O}_Y(E)$ for some effective divisor E. Apply (11.12) with $L_1 \cong \mathcal{O}_Y$ and $L_2 \cong f^*\omega_X$.

Then $L_2(E) \cong \omega_Y$, hence $R^i f_* L_2(E) = 0$ for $i > 0$ by (2.17.6). By the projection formula, $R^i f_* L_1(E) = \omega_X^{-1} \otimes R^i f_* L_2(E) = 0$. Thus by (11.12) we see that $R^i f_* \mathcal{O}_Y = R^i f_* L_1 = 0$ for $i > 0$. $\square$

If ω_X is not locally free, then we show that every klt singularity is rational. The sharpest technical result proved in [Fujita85; KaMaMa87, 1-3] asserts that if (X, D) is dlt then X has rational singularities. We have not defined dlt (cf. [Kollár et al.92, 2.13]), but the proof requires only small changes.

11.14 Corollary. *Let (X, D) be a klt pair over a field of characteristic zero. Then X has rational singularities.*

Proof. Let $f : Y \to X$ be a log resolution. Write

$$K_Y \equiv f^*(K_X + D) + F, \quad \text{where } \ulcorner F \urcorner \text{ is effective.}$$

Set $\Delta = \ulcorner F \urcorner - F$; this is an effective normal crossing divisor such that $\llcorner \Delta \lrcorner = 0$. Set

$$L_1 = \mathcal{O}_Y, \quad E = \ulcorner F \urcorner \quad \text{and} \quad L_2 = \mathcal{O}(K_Y - E).$$

Then

$$L_1(E) \equiv K_Y + \Delta - f^*(K_X + D), \quad \text{and} \quad L_2(E) \equiv K_Y.$$

$-f^*(K_X + D)$ is f-nef (even f-numerically trivial), thus (2.17.3) applies and we get that $R^i f_* L_j(E) = 0$ for $i > 0$ and $j = 1, 2$. Thus by (11.12) we see that $R^i f_* \mathcal{O}_Y = R^i f_* L_1 = 0$ for $i > 0$. $\square$

[Elkik78] proved that a small deformation of a rational singularity is again rational. The following is a slight variation of her arguments, which also says something about deformations of certain nonnormal schemes.

11.15 Theorem. *Let X be an excellent normal scheme over a field of characteristic zero and $h : X \to T$ a flat morphism to the spectrum of a DVR with closed point $0 \in T$ and local parameter $t \in \mathcal{O}_T$. Set $X_0 := h^{-1}(0)$. Let $\pi : \bar{X}_0 \to X_0$ be the normalization. Assume that π is an isomorphism in codimension one and that $\bar{X}_0$ has rational singularities.*

Then $\bar{X}_0 \cong X_0$ and X has rational singularities.

Proof. Take a resolution of singularities $f : X' \to X$ such that $X_0' + E := (f \circ h)^{-1}(0)$ is a divisor with normal crossings only where $f : X_0' \to \bar{X}_0 \to X_0$ is a resolution of singularities.

We denote h^*t and f^*h^*t again by t. On X' we have an exact sequence

$$0 \to \omega_{X'} \xrightarrow{t} \omega_{X'} \to \omega_{X_0'+E} \to 0. \tag{11.15.1}$$

On X we get an exact sequence

$$0 \to \omega_X \xrightarrow{t} \omega_X \to \tilde{\omega}_{X_0} \to 0, \tag{11.15.2}$$

where $\tilde{\omega}_{X_0} \subset \omega_{X_0}$ (see e.g. [Reid94, 2.13]). For any pure dimensional scheme Z, ω_Z is S_2 (see e.g. [Reid94, 2.12]), which in particular implies that if $\pi : \bar{Z} \to Z$ is finite and an isomorphism in codimension one then $\pi_*\omega_{\bar{Z}} \cong \omega_Z$. Since $\bar{X}_0$ has rational singularities, this implies that $\omega_{X_0} = f_*\omega_{X_0'}$

We have a natural injection $f_*\omega_{X'} \hookrightarrow \omega_X$. This gives the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & f_*\omega_{X'} & \xrightarrow{t} & f_*\omega_{X'} & \longrightarrow & f_*\omega_{X_0'+E} & \longrightarrow & R^1 f_*\omega_{X'} \\
 & & \Big\downarrow & & \Big\downarrow & & \uparrow & & \\
(11.15.3) & & & & & & \omega_{X_0} = f_*\omega_{X_0'} & & \\
 & & & & & & \uparrow & & \\
0 & \longrightarrow & \omega_X & \xrightarrow{t} & \omega_X & \longrightarrow & \tilde{\omega}_{X_0} & \longrightarrow & 0
\end{array}$$

$R^1 f_*\omega_{X'} = 0$ by (2.17.6), thus $f_*\omega_{X'} \to f_*\omega_{X_0'+E}$ is surjective. The vertical maps

$$\tilde{\omega}_{X_0} \to \omega_{X_0} = f_*\omega_{X_0'} \to f_*\omega_{X_0'+E}$$

are injections. By the commutativity of (11.15.3) we obtain that they are both isomorphisms:

$$\tilde{\omega}_{X_0} = f_*\omega_{X_0'} = f_*\omega_{X_0'+E}. \tag{11.15.4}$$

This implies that

$$f_*\omega_{X'} \to \omega_X \to \omega_X / t\omega_X$$

is surjective, hence by the Nakayama lemma $f_*\omega_{X'} \hookrightarrow \omega_X$ is an isomorphism. Also by (11.15.4), $\omega_X/t\omega_X \cong f_*\omega_{X_0'}$. By (11.4.2) we know that $f_*\omega_{X_0'}$ is a CM-sheaf, hence ω_X is also a CM-sheaf. By (11.9) we conclude that X has rational singularities.

Therefore X is CM and so is X_0. This means that $X_0 = \bar{X}_0$. $\square$

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