Skinner–Venkatesh learning seminar on Iwasawa theory 6 March 2025

Talk #3: Class numbers of p-power cyclotomic fields, mod p

Kenz Kallal

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1 Introduction

The following fact is a fairly straightforward consequence of class field theory:

Theorem 1.1 (Iwasawa, 1956)**.** *Let p be any rational prime, and $n \geq 1$ be a positive integer. The class number of $\mathbf{Q}(\zeta_{p^n})$ is divisible by p if and only if the class number of $\mathbf{Q}(\zeta_p)$ is divisible by p.*

Proof. During my talk, I explained Iwasawa's proof of this fact, which appears in the 1.5-page paper [Iwa1956]. That paper is in English and is very readable, so I will allow the interested reader to refer to it. $\square$

While Iwasawa's proof of Theorem 1.1 used class field theory as the main input, surprisingly, it turns out that Weber [Web1886] was able to prove Theorem 1.1 for the case $p = 2$ in 1886 (decades before the full statements of class field theory were even known, and even before the existence of the Hilbert class field had been conjectured). The paper [Web1886] also contains a slightly flawed proof of the Kronecker–Weber theorem, which I explained in a talk in a previous instance of the Skinner–Venkatesh seminar. Weber's proof of Theorem 1.1 in the $p = 2$ case seems fully correct to me, and it uses completely different tools: namely, the analytic class number formula and explicit comparison of L-values to regulators. Since I did not explain the full detail of Weber's proof during my talk, and since some of the audience were surprised that this kind of method could be made to work, I have written in this document a detailed account of Weber's proof. I depart slightly from a literal interpretation of the original text in places: most significantly, I use the language of Stark regulators (which technically did not exist in 1886) and some group cohomology to avoid having to pass to maximal real subfields as Weber does.

To recap, the goal of these notes is to prove the following result:

Theorem 1.2 (Weber, 1886)**.** *For all positive integers $n \geq 1$, the class number of $\mathbf{Q}(\zeta_{2^n})$ is odd.*

I have also written a version of the proof which is essentially a direct translation of [Web1886, part II] with a few small details filled in (https://web.math.princeton.edu/~kk2703/kallal_weber.pdf). In contrast, these notes are intended to make it slightly easier to understand the new ideas in that proof by modernizing some notation and skipping/citing steps that are now routine so that it is easier for the reader to find the new ideas, but it's probably better to give both options as I may have failed in making Weber's excellent paper more understandable.

2 Strategy of proof and setup of computations

Let $n \geq 2$, $K_n = \mathbf{Q}(\zeta_{2^n})$ and $h_n := |\mathrm{Cl}(K_n)|$. Then (since the absolute value of the discriminant of K_n is $2^{2^{n-1}(n-1)}$ [e.g. by using [Neu1999, Ch. III, Proposition 2.4 and Theorem 2.9] and the minimal polynomial $X^{2^{n-1}} + 1$ for ζ_{2^n}, or just directly applying [Neu1999, Ch. I, Lemma 10.1 and Proposition 10.2]] and it contains 2^n roots of unity [e.g. because if there were any more then it would create ramification at a prime other than 2]) the analytic class number formula (see e.g. [Neu1999, Ch. VII, Corollary 5.11]) works out to

Proposition 2.1 (Analytic class number formula for K_n)**.** *In this situation, we have*

$$\mathrm{res}_{s=1}\zeta_{K_n}(s) = \frac{(2\pi)^{2^{n-2}}\mathrm{Reg}(K_n)h_n}{2^n 2^{2^{n-2}(n-1)}} = \pi^{2^{n-2}} 2^{-n-(n-2)2^{n-2}}\mathrm{Reg}(K_n)h_n.$$

Weber's strategy was to compute the left hand side of Proposition 2.1, finding that up to a multipicative factor of an odd integer, it is of the form $\pi^{a_1}2^{a_2}\mathrm{Reg}(K_n)$ for some explicit integers a_1, a_2 (of course since π is transcendental over $\mathbf{Q}$ we can reverse engineer from the fact that $h_n \in \mathbf{Z}$ that he must have proved $a_1 = \pi^{2^{n-2}}$ and from Theorem 1.2 that $a_2 = 2^{-n-(n-2)2^{n-2}}$).

By [Neu1999, Proposition 5.12] and the fact that 2 is totally ramified in K_n (the prime above it is $(1-\zeta_{2^n})$ which has norm 2 — see e.g. [Neu1999, Lemma 10.1]), the left hand side of Proposition 2.1 further decomposes as

$$\mathrm{res}_{s=1}\zeta_{K_n}(s) = \prod_{\chi \in \widehat{(\mathbf{Z}/2^n\mathbf{Z})^*} - \{1\}} L(1,\chi), \tag{2.1}$$

where $L(1,\chi)$ is (for $\chi \neq 1$) the convergent Dirichlet L-series

$$L(1,\chi) := \sum_{\substack{n \geq 0 \\ n \text{ odd}}} \frac{\chi(n)}{n} < \infty.$$

Recall from [Elk2019, `l1x.pdf`] that for $\chi \neq 1$,

$$L(1,\chi) = \frac{1}{\tau(\overline{\chi})} \sum_{a \in (\mathbf{Z}/2^n)^\times} \overline{\chi}(a) \sum_{n \geq 1} \frac{1}{n} \exp\left(2\pi i n \frac{a}{q}\right) = \frac{1}{\tau(\overline{\chi})} \sum_{a \in (\mathbf{Z}/2^n)^\times} \overline{\chi}(a) \log(1 - \zeta_{2^n}^a). \tag{2.2}$$

Here the notation $\tau(\chi)$ means the Gauss sum

$$\tau(\chi) := \sum_{a \in (\mathbf{Z}/2^n)^\times} \chi(a) \exp\left(2\pi i \frac{a}{q}\right) = \sum_{a \in (\mathbf{Z}/2^n)^\times} \chi(a)\zeta_{2^n}^a.$$

The various elements $1 - \zeta_{2^n}^a$ are NOT units in K_n — they are Galois conjugate generators of the prime above 2. However, their pairwise quotients are units, as are the quantities $(1-\zeta_{2^n}^a)/(1-\zeta_{2^n}^{-a}) = -\zeta_{2^n}^a$. As a result, depending on whether $\chi(-1) = \pm 1$ (i.e. on whether χ is even or odd), our expression for $\prod L(1,\chi)$ will involve logs of very specific units in K_n (to be made more explicit than this). Weber's strategy was to study the explicit arithmetic of those systems of Galois conjugate units sitting inside $\mathcal{O}_{K_n}^\times$ to relate this product to the regulator of K_n.

We now make all of this more precise by setting up Weber's computation in more explicit detail.

2.1 Handling the Gauss sums

First we deal with the Gauss sums, whose computation we know will depend on the conductor of χ — so our first step must be to divide the computation by conductor. By (2.1) and (2.2), the left hand side of Proposition 2.1 is equal to

$$\prod_{\chi \in \widehat{(\mathbf{Z}/2^n)^\times} - \{1\}} L(1,\chi) = \prod_{\chi \in \widehat{(\mathbf{Z}/2^n)^\times} - \{1\}} \left(\tau(\overline{\chi})^{-1} \sum_{a \in (\mathbf{Z}/2^n)^\times} \overline{\chi}(a) \log(1 - \zeta_{2^n}^a) \right). \tag{2.3}$$

Now we pair up χ with $\overline{\chi}$ and use the standard identity $\tau(\chi)\tau(\overline{\chi}) = \chi(-1)q$, where χ has conductor q (which can be seen via the Poisson summation formula as in [Elk2019, `lsx.pdf`, (17)] or by direct computation). There is just one small wrinkle in this plan[1], which is that there are three nontrivial real characters of $(\mathbf{Z}/2^n)^\times = \langle -1 \rangle \times \langle 5 \rangle$, and these cannot be paired up since they satisfy $\chi = \overline{\chi}$. Those characters are defined as follows:

Definition 2.2. The nontrivial characters of conductor 4 and 8 are:

(1) $\chi_4((-1)^a 5^b) := (-1)^a$, the unique nontrivial character of conductor 4.

(2) $\chi_8^-((-1)^a 5^b) := (-1)^{a+b}$, the unique odd character of conductor 8.

(3) $\chi_8^+((-1)^a 5^b) := (-1)^b$, the unique even character of conductor 8.

The corresponding special values $L(1,\chi)$ are known (and they can be explicitly computed from the formula (2.2) because all the sums are finite and explicit for a given explicit χ), so we just keep them separate from the rest of the computation:

Lemma 2.3. *We have the following explicit special values.*

(1) $L(1,\chi_4) = \frac{\pi}{4}$.

(2) $L(1,\chi_8^-) = \frac{\pi}{2\sqrt{2}}$.

(3) $L(1,\chi_8^+) = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)$.

Carrying out our plan to pair up the remaining χ's with $\overline{\chi}$ and to use the identity $\tau(\chi)\tau(\overline{\chi}) = \chi(-1)\mathrm{cond}(\chi)$, we deduce (using the fact that $L(s,\chi)$ is the same regardless of whether χ is considered as a character mod

[1] Technically this is not a real wrinkle: after partitioning the χ according to the conductor 2^N, we can still compute all of the absolute values $|\tau(\chi)| = 2^{N/2}$ which are all that matter to us (we only care about 2-adic valuations so everything can be computed up to sign). Down the line, we will have no choice but to separate out the case of χ_4 (we will need to consider the element $5^{2^{N-3}} \in (\mathbf{Z}/2^N)^\times$ to carry out the general arguments), but there really would have been nothing wrong with making the prevailing hypothesis in all of these notes $N \geq 3$ instead of $N \geq 4$.

2^n or a character modulo the conductor of χ) from (2.3) and Lemma 2.3 that

$$
\begin{aligned}
\prod_{\chi \in \widehat{(\mathbf{Z}/2^n)^\times} - \{1\}} L(1,\chi) &= L(1,\chi_4)L(1,\chi_8^-)L(1,\chi_8^+) \prod_{N=4}^{n} \prod_{\substack{\chi \in \widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive}}} L(1,\chi) \\
&= \frac{\pi^2}{16} \log(\sqrt{2}+1) \prod_{N=4}^{n} \prod_{\substack{\chi \in \widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive}}} L(1,\chi) \\
&= \frac{\pi^2}{16} \log(\sqrt{2}+1) \prod_{N=4}^{n} \left(\left(\prod_{\substack{\chi \in \widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive}}} \tau(\overline{\chi}^{-1}) \right) \left(\prod_{\substack{\chi \in \widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive}}} \sum_{a \in (\mathbf{Z}/2^N)^\times} \overline{\chi}(a) \log(1-\zeta_{2^N}^a) \right) \right) \\
&= \frac{\pi^2}{16} \log(\sqrt{2}+1) \prod_{N=4}^{n} \left(\left((-1)^{2^{N-4}} 2^{-N2^{N-3}} \right) \left(\prod_{\substack{\chi \in \widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive}}} \sum_{a \in (\mathbf{Z}/2^N)^\times} \overline{\chi}(a) \log(1-\zeta_{2^N}^a) \right) \right) \\
&= -\pi^2 \log(\sqrt{2}+1) 2^{-(n-1)2^{n-2}} \prod_{N=4}^{n} \left(\prod_{\substack{\chi \in \widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive}}} \sum_{a \in (\mathbf{Z}/2^N)^\times} \overline{\chi}(a) \log(1-\zeta_{2^N}^a) \right)
\end{aligned}
\tag{2.4}
$$

where the second-to-last line is using the fact that $(\mathbf{Z}/2^N)^\times$ has $\varphi(2^{N-2}) = 2^{N-3}$ primitive odd characters and the same number of primitive even characters. This concludes our handling of the Gauss sums.

2.2 The Galois action on units and on archimedean places, and two important special elements of the Galois group

What remains is to compute the product from $N = 4$ to n of

$$
\prod_{\substack{\chi \in \widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive}}} \sum_{a \in (\mathbf{Z}/2^N)^\times} \overline{\chi}(a) \log(1-\zeta_{2^N}^a) = \left[\prod_{\substack{\chi \in \widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive and even}}} \cdot \prod_{\substack{\chi \in \widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive and odd}}} \right] \left(\sum_{a \in (\mathbf{Z}/2^N)^\times} \overline{\chi}(a) \log(1-\zeta_{2^N}^a) \right)
$$

From here, most of the content (with the exception of what will be done in Section 5.2, which requires a third special element 5^{N-4} when $N \geq 4$) comes from the following. The group $(\mathbf{Z}/2^N)^\times$ decomposes as $\langle -1 \rangle \times \langle 5 \rangle \cong (\mathbf{Z}/2) \times (\mathbf{Z}/2^{N-2})$. For all $a \in (\mathbf{Z}/2^N)^\times$, let $\sigma_a \in \mathrm{Gal}(K_N/\mathbf{Q})$ be the corresponding automorphism, $\sigma_a(\zeta_{2^N}) = \zeta_{2^N}^a$. This decomposition corresponds via $(\mathbf{Z}/2^N)^\times \cong \mathrm{Gal}(\mathbf{Q}(\zeta_{2^N})/\mathbf{Q})$ to both short short exact sequences

$$
\begin{tikzcd}
1 \arrow[r] & \mathrm{Gal}(\mathbf{Q}(\zeta_{2^N}/\mathbf{Q}(\zeta_{2^N})^+) \arrow[r] \arrow[d, equal] & \mathrm{Gal}(\mathbf{Q}(\zeta_{2^N})/\mathbf{Q}) \arrow[r] \arrow[d, equal] & \mathrm{Gal}(\mathbf{Q}(\zeta_{2^N})^+/\mathbf{Q}) \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & \langle -1 \rangle \arrow[r] & \langle -1 \rangle \times \langle 5 \rangle \arrow[r] & \langle 5 \rangle \arrow[r] & 0
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
1 \arrow[r] & \mathrm{Gal}(\mathbf{Q}(\zeta_{2^N})/\mathbf{Q}(i)) \arrow[r] \arrow[d, equal] & \mathrm{Gal}(\mathbf{Q}(\zeta_{2^N})/\mathbf{Q}) \arrow[r] \arrow[d, equal] & \mathrm{Gal}(\mathbf{Q}(i)/\mathbf{Q}) \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & \langle 5 \rangle \arrow[r] & \langle -1 \rangle \times \langle 5 \rangle \arrow[r] & \langle -1 \rangle \arrow[r] & 0
\end{tikzcd}
$$

where the point is that the automorphism σ_{-1} is just complex conjugation (which also induces the nontrivial automorphism of $\mathbf{Q}(i)/\mathbf{Q}$), and the automorphism σ_5 has fixed field $\mathbf{Q}(i)$ (whose totally real part is $\mathbf{Q}$). This can be summarized in the compositum diagram

$$
\begin{tikzcd}
 & \mathbf{Q}(\zeta_{2^N}) \arrow[dl, dash, "\langle 5 \rangle"] \arrow[dr, dash, "\langle -1 \rangle"] & \\
\mathbf{Q}(i) \arrow[dr, dash] & & \mathbf{Q}(\zeta_{2^N})^+ = \mathbf{Q}(\zeta_{2^N} + \zeta_{2^N}^{-1}) \arrow[dl, dash] \\
 & \mathbf{Q} &
\end{tikzcd}
$$

the left hand side of which fits inside a big tower

$$\mathbf{Q} \subset \mathbf{Q}(i) \subset \mathbf{Q}(\zeta_8) \subset \cdots \subset \mathbf{Q}(\zeta_{2^{N-1}}) \subset \mathbf{Q}(\zeta_{2^N}).$$

In fact, for the purpose of carrying out inductive arguments down this tower, we should consider the compositum diagram that involves the last part of this tower instead of the first part:

$$
\begin{tikzcd}
 & \mathbf{Q}(\zeta_{2^N}) \arrow[dl, dash, "\langle 5^{2^{N-3}} \rangle"] \arrow[dr, dash, "\langle -1 \rangle"] & \\
\mathbf{Q}(\zeta_{2^{N-1}}) \arrow[dr, dash] & & \mathbf{Q}(\zeta_{2^N})^+ \arrow[dl, dash] \\
 & \mathbf{Q} &
\end{tikzcd}
$$

where the point is that 5 has order 2^{N-2} in $(\mathbf{Z}/2^N)^\times$, so $5^{2^{N-3}} \equiv 1 \mod 2^{N-1}$ and it is of order 2 in $(\mathbf{Z}/2^N)^\times$, so $\sigma_{5^{2^{N-3}}}$ is the nontrivial automorphism of $\mathrm{Gal}(K_N/K_{N-1})$, namely $\sigma_{5^{2^{N-3}}}(\zeta_{2^N}) = \zeta_{2^N}^{5^{2^{N-3}}} -\zeta_{2^N}$.

Since we will have to deal with regulators coming from systems of units that are Galois conjugate to each other (as pointed out vaguely after (2.2)), it will be useful to understand the places of $\mathbf{Q}(\zeta_{2^N}) = K_N$ in terms of our explicit understanding of the group $(\mathbf{Z}/2^N)^\times$.

Definition 2.4. Denote by $\iota : \mathbf{Q}(\zeta_{2^N}) = K_N \to \mathbf{C}$ the complex embedding coming from $\zeta_{2^N} \mapsto \exp(2\pi i/2^N)$, i.e., the one already implicit in identifying the complex number $L(1,\chi)$ with a formula involving ζ_{2^N} in (2.2).

The field K_N is totally complex with 2^{N-2} complex places (up to complex conjugation – i.e. we have already quotiented out by σ_{-1}). It is of degree $2^{N-2} = |\langle 5 \rangle|$ over $\mathbf{Q}(i) = K_2$, which has a single complex place. The Galois group $\langle 5 \rangle = \mathrm{Gal}(K_N/\mathbf{Q}(i))$ therefore acts transitively on the archimedean places of K_N; and since there are the same number of those places as there are elements of $\langle 5 \rangle$, we have

Lemma 2.5. *The 2^{N-2} complex places of K_N are in bijection with $\langle 5 \rangle = \mathrm{Gal}(K_N/\mathbf{Q}(i))$ via*

$$a = 5^\beta \mapsto \iota \circ \sigma_a = \iota \circ \sigma_{5^\beta}.$$

Furthermore, K_N/K_{N-1} is quadratic, so each (complex) archimedean place v of K_{N-1} comes with (complex) places v_1 and v_2 of K_N lying over it. By the same transitivity argument, the nontrivial element $\sigma_{5^{N-3}} \in \mathrm{Gal}(K_N/K_{N-1})$ has the effect of switching v_1 and v_2 for all v, and we have

Lemma/Definition 2.6. (1) The archimedean places v of K_{N-1} are in bijection with $\langle 5\rangle/\langle 5^{2^{N-3}}\rangle = \mathrm{Gal}(K_{N-1}/\mathbf{Q}(i))$ via $a \mapsto \iota\circ\sigma_a$;

(2) For the system of representatives $\{5^\beta : \beta = 0,\dots,2^{N-3}-1\}$ for $\langle 5\rangle/\langle 5^{2^{N-3}}\rangle$ (which we take because we might as well), the embeddings $\iota\circ\sigma_{5^\beta}$ form a list of archimedean (complex) places of K_N with the property of that each pair $\{v_1, v_2\}$ of places of K_N over a given place v of K_{N-1} contains exactly one of the places $\iota\circ\sigma_{5^\beta}, \beta = 0,\dots,2^{N-3}$. What is logically equivalent to this choice of representatives $\{\beta\}$, for all archimedean places v of K_{N-1}, we choose to always define the place v_1 of K_N to be the one of the form $\iota\circ\sigma_{5^\beta}$ (for $\beta\in\{0,\dots,2^{N-3}-1\}$ uniquely determined by v), and v_2 to be the one that is not of this form.

The two elements σ_{-1} (which we quotient out by to get $\mathrm{Gal}(K_N/\mathbf{Q}(i))$) and $\sigma_{5^{N-3}}$ (which we quotient out by to get $\mathrm{Gal}(K_{N-1}/\mathbf{Q}(i))$) are then of importance for understanding $\mathrm{Reg}(K_N)$ in terms of $\mathrm{Reg}(K_{N-1})$, so it makes sense to use (quotients of $\mathrm{Gal}(K_N/\mathbf{Q})$ by) them in order to relate our product $\prod_\chi L(1,\chi)$ directly to $\mathrm{Reg}(K_n)$. From now on, we always keep the following notation:

Definition 2.7 (τ)**.** We always refer to $\sigma_{5^{2^{N-3}}}$ as τ.

The above discussion was about the complex places of K_N and K_{N-1}. To understand regulators, we also need to discuss the same actions of these two groups of order 2, namely $\langle -1\rangle \cong \langle\sigma_{-1}\rangle = \mathrm{Gal}(K_N/K_N^+)$ and $\langle 5^{2^{N-3}}\rangle \cong \langle\tau\rangle = \mathrm{Gal}(K_N/K_{N-1})$, on the units in K_N. Ideally we would use the pairs of idempotents $(1\pm[\sigma_{-1}])/2, (1\pm[\tau])/2$, to split all the unit groups in sight into a direct sum of isotypic components for $\sigma,\tau = 1$ and $\sigma,\tau=-1$. But this is not literally possible, since the required idempotents have denominators. For example, the inclusion

$$(\mathcal{O}_{K_N}^\times/\mu)^{\tau=1} \oplus (\mathcal{O}_{K_N}^\times/\mu)^{\tau=-1} \subset \mathcal{O}_{K_N}^\times/\mu$$

is almost never an equality. At the very least, though, those idempotents still map to their corresponding isotypic components for the usual formal reasons, so we still have the following fact:

Lemma 2.8. *Let $x \in K_N^\times$ such that $v_\mathfrak{p}(x) = 0$ for all $\mathfrak{p}$ except for the one lying over 2. Then, $(1-[\sigma_{-1}])x \in \mathcal{O}_{K_N}^\times$ is a root of unity, and*

$$(1-[\tau])(1-[\sigma_{-1}])x = \pm 1.$$

Proof. The element $(1-[\sigma_{-1}])x = x/\sigma_{-1}(x)$ is in $\mathcal{O}_{K_N}^\times$, since $\sigma_{-1}(\mathfrak{p}) = \mathfrak{p}$ for $\mathfrak{p}|2$ (as there is only one such prime), so $v_\mathfrak{p}$ of this quotient is 0. More specifically

$$(1-[\sigma_{-1}])x \in (\mathcal{O}_{K_N}^\times)^{\sigma_{-1}=-1}.$$

Since every unit in $\mathcal{O}_{K_N}^\times$ is a root of unity times a real unit (i.e. a unit u with the property that $\sigma_{-1}(u) = u$), it follows that $(1-[\sigma_{-1}])x =: u$ is a 2^M-th root of unity for some $M \le N$, hence $(1-[\tau])u = \pm 1$, since $\tau(\zeta_{2^N}) = -\zeta_{2^N}$. ☐

We will in the next few paragraphs (as soon as we introduce the analogous lemma for the even part) that Lemma 2.8 is relevant to the odd part of $\prod_\chi L(1,\chi)$, as we will find units of the form $(1-[\tau])(1-[\sigma_{-1}])x$ in our formula for that product.

As for the even terms, the relevant statement will be:

Lemma 2.9. *Let $x \in K_N^\times$ such that $v_\mathfrak{p}(x) = 0$ for all $\mathfrak{p}$ except for the one lying over 2. Then $(1+[\sigma_{-1}])x = x\sigma_{-1}(x) \in K_N^+$, and*

$$(1-[\tau])(1+[\sigma_{-1}])x \in (\mathcal{O}_{K_N^+}^\times)^{\tau=-1}.$$

In fact, we already have

$$(1-[\tau])x \in (\mathcal{O}_{K_N}^\times)^{\tau=-1}.$$

Proof. We have $(1+[\sigma_{-1}])x \in (K_N^\times)^{\sigma_{-1}=1} = (K_N^+)^\times$. Applying $1-[\tau]$ to this gives a unit by the same reasoning used in Lemma 2.8. ☐

We are about to see in the next paragraph why the elements considered Lemma 2.9 are relevant to the even part of $\prod_\chi L(1,\chi)$, but for now: such elements $(1-[\tau])(1+[\sigma_{-1}])x$, can therefore plausibly account for the "$\tau = -1$ part of $\mathrm{Reg}(K_N)$." In Section 4, we will make this into a precise notion, define it by the notation $\mathrm{Reg}(K_N/K_{N-1})^-$, and analyse the (finite) index of the inclusion

$$(\mathcal{O}_{K_N}^\times/\mu)^{\tau=1} \oplus (\mathcal{O}_{K_N}^\times/\mu)^{\tau=-1} \subset \mathcal{O}_{K_N}^\times/\mu$$

to relate $\mathrm{Reg}(K_N)$ to $\mathrm{Reg}(K_{N-1})\mathrm{Reg}(K_N/K_{N-1})^-$, so that $\prod_N \prod_\chi L(1,\chi)$ will be related to

$$\mathrm{Reg}(K_4/K_3)^- \cdots \mathrm{Reg}(K_n/K_{n-1})^- = \mathrm{Reg}(K_n)/\mathrm{Reg}(K_3).$$

In the next paragraphs, we will find units of the form $(1-[\tau])(1+[\sigma_{-1}])x$ in a formula for the even part of $\prod_\chi L(1,\chi)$, and in Section 5 we will relate that formula to $\mathrm{Reg}(K_N/K_{N-1})^-$.

Let us bring the automorphisms σ_{-1} and τ to bear on the odd part of the product we obtained in (2.4). The relevance of Lemma 2.8 will be clear from this discussion. For a given $N \geq 4$, we take $\langle 5\rangle \subset (\mathbf{Z}/2^N)^\times$ to be the quotient by $\langle -1\rangle$ of $(\mathbf{Z}/2^N)^\times$, and we rearrange the product over odd χ of conductor 2^N according to orbits under the action of $-1 \in (\mathbf{Z}/2^N)^\times$:

$$\begin{aligned}
\prod_{\substack{\chi\in\widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive odd}}} \sum_{a\in(\mathbf{Z}/2^N)^\times} \overline{\chi}(a)\log(1-\zeta_{2^N}^a) &= \prod_{\substack{\chi\in\widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive odd}}} \sum_{a\in\langle 5\rangle} \left(\overline{\chi}(a)\log(1-\zeta_{2^N}^a) + \overline{\chi}(-a)\log(1-\zeta_{2^N}^{-a})\right) \\
&= \prod_{\substack{\chi\in\widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive odd}}} \sum_{a\in\langle 5\rangle} \overline{\chi}(a)(\log(1-\zeta_{2^N}^a) - \log(1-\zeta_{2^N}^{-a})) \\
&= \prod_{\substack{\chi\in\widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive odd}}} \sum_{a\in\langle 5\rangle} \left(\overline{\chi}(a)\log\frac{1-\zeta_{2N}^a}{1-\zeta_{2^N}^{-a}}\right) \\
&= \prod_{\substack{\chi\in\widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive odd}}} \sum_{a\in\langle 5\rangle} \overline{\chi}(a)\log(-\iota(\zeta_{2^N}^a)).
\end{aligned} \tag{2.5}$$

where the second line uses the fact that χ is odd, and the last line is an explicit version of the part of Lemma 2.8 that says $(1-[\sigma_{-1}])x$ is a root of unity, applied to $x = 1-\zeta_{2^N}^a$. In the calculation above, we are always using the same branch of the logarithm, and passing differences into the log can be checked to always be okay because of where the arguments of the inputs are. The product in (2.5) only involves the logs of the very specific units $-\zeta_{2^N}^a$. In particular, it

(1) only involves units in the kernel of the Log map, so we expect it to have nothing to do with the regulator of K_n, and

(2) involves the terms $\log(-\zeta_{2^N}^a) = \log(-e^{2\pi i a/2^N}) = \pi i(2a/2^N - 1)$ which contribute some factors of π.

Having used σ_{-1}, we now turn to τ, where we use the main part of Lemma 2.8 and the set of representatives $a = 5^\beta, \beta = 0, \ldots, 2^{N-3}-1$ for $\langle 5\rangle/\langle 5^{2^{N-3}}\rangle$ to see that (2.5) continues to

$$
\begin{aligned}
\prod_{\substack{\chi\in\widehat{(\mathbf{Z}/2^N)^\times}\\ \chi \text{ primitive odd}}} \sum_{a\in(\mathbf{Z}/2^N)^\times} \overline{\chi}(a)\log(1-\zeta_{2^N}^a) &= \prod_{\substack{\chi\in\widehat{(\mathbf{Z}/2^N)^\times}\\ \chi \text{ primitive odd}}} \sum_{a\in\langle 5\rangle} \overline{\chi}(a)\log(-\zeta_{2N}^a)\\
&= \prod_{\substack{\chi\in\widehat{(\mathbf{Z}/2^N)^\times}\\ \chi \text{ primitive odd}}} \sum_{\beta=0}^{2^{N-3}-1} \overline{\chi}(5^\beta)(\log(-\zeta_{2N}^a)-\log(-\tau(\zeta_{2^N}^a)))\\
&= \prod_{\substack{\chi\in\widehat{(\mathbf{Z}/2^N)^\times}\\ \chi \text{ primitive odd}}} \sum_{\beta=0}^{2^{N-3}-1} \overline{\chi}(5^\beta)(\log(-\zeta_{2N}^a)-\log(\zeta_{2^N}^a))\\
&= \prod_{\substack{\chi\in\widehat{(\mathbf{Z}/2^N)^\times}\\ \chi \text{ primitive odd}}} \sum_{\beta=0}^{2^{N-3}-1} \overline{\chi}(5^\beta)(\epsilon_\beta\pi i)\\
&= (\pi i)^{2^{N-3}} \prod_{\substack{\chi\in\widehat{(\mathbf{Z}/2^N)^\times}\\ \chi \text{ primitive odd}}} \sum_{\beta=0}^{2^{N-3}-1} \overline{\chi}(5^\beta)\epsilon_\beta
\end{aligned}
\tag{2.6}
$$

where the point is that (because of our explicit computation that shows that we are in the -1 case of Lemma 2.8 applied to $x = 1-\zeta_{2^N}^a$) since we chose the specific branch of log with $\log(e^{2\pi i\theta}) \in [0, 2\pi i)$, the quantities ϵ_β are all ± 1 depending on β (exercise for the reader: what is the condition on β equivalent to having $\varepsilon_\beta = 1$? In fact, we really only need to know that the ϵ_β are odd integers, and to prove this we don't need to worry about the branch cut — it is just a direct consequence of the fact that we are in the -1 case of Lemma 2.8 [if we were in the 1 case, then the ϵ_β would be even]), and since χ is primitive, we have $\chi(5^{2^{N-3}}) = -1$. This is a product of certain linear combinations with ± 1 coefficients of certain 2^{N-2}-th roots of unity.

We state the exact outcome of this computation (including the exact powers of π and 2 that appear) in Proposition 2.10, and we carry it out in full detail in Section 3.

On the other hand, where again we are careful when passing products and quotients through our chosen branch of log, and we take $\beta = 0, \dots, 2^{N-3}-1$ as a choice of representatives for $(\mathbf{Z}/2^{N-2})/\langle 2^{N-3}\rangle$ the even part of the product from (2.4) is

$$
\begin{aligned}
\prod_{\substack{\chi\in\widehat{(\mathbf{Z}/2^N)^\times}\\ \chi \text{ primitive even}}} \sum_{a\in(\mathbf{Z}/2^N)^\times} \overline{\chi}(a)\log(1-\zeta_{2^N}^a) &= \prod_{\substack{\chi\in\widehat{(\mathbf{Z}/2^N)^\times}\\ \chi \text{ primitive odd}}} \sum_{a\in\langle 5\rangle\subset(\mathbf{Z}/2^N)^\times} \overline{\chi}(a)\log(1-\zeta_{2^N}^a)+\overline{\chi}(-a)\log(1-\zeta_{2^N}^{-a})\\
&= \prod_{\substack{\chi\in\widehat{(\mathbf{Z}/2^N)^\times}\\ \chi \text{ primitive even}}} \sum_{a\in\langle 5\rangle\subset(\mathbf{Z}/2^N)^\times} \overline{\chi}(a)\log\left((1-\zeta_{2^N}^a)(1-\zeta_{2^N}^{-a})\right)\\
&= \prod_{\substack{\chi\in\widehat{(\mathbf{Z}/2^N)^\times}\\ \chi \text{ primitive even}}} \sum_{\substack{\{a,5^{2^{N-3}}a\}\in\langle 5\rangle/\langle 5^{2^{N-3}}\rangle\\ a=5^\beta,0\le\beta<2^{N-3}}} \overline{\chi}(a)\log\frac{(1-\zeta_{2^N}^a)(1-\zeta_{2^N}^{-a})}{(1-\zeta_{2^N}^{5^{2^{N-3}}a})(1-\zeta_{2^N}^{-5^{2^{N-3}}a})}\\
&= \prod_{\substack{\chi\in\widehat{(\mathbf{Z}/2^N)^\times}\\ \chi \text{ primitive even}}} \sum_{\substack{\{a,5^{2^{N-3}}a\}\in\langle 5\rangle/\langle 5^{2^{N-3}}\rangle\\ a=5^\beta,0\le\beta<2^{N-3}}} \overline{\chi}(a)\log\frac{(1-\zeta_{2^N}^a)(1-\zeta_{2^N}^{-a})}{(1+\zeta_{2^N}^a)(1+\zeta_{2^N}^{-a})}
\end{aligned}
\tag{2.7}
$$

where the first line is by combining the terms a and $-a$ using $\langle 5\rangle$ as the quotient of $(\mathbf{Z}/2^N)^\times$ by $\langle -1\rangle$ in the same way as in the even case. After noticing that $(1-\zeta^a)(1-\zeta^{-a})$ is still not a unit (because we took the product instead of the quotient since the χ here are even instead of odd), the third line further breaks down the sum into classes (pairs) $a \mod 5^{2^{N-3}}$ (which works because χ is primitive mod 2^N so $\chi(5^{2^{N-3}}) = -1$ or else χ would factor through the quotient by $\langle 5^{2^{N-3}} \rangle$, and $\zeta_{2^N}^{5^{2^{n-3}}a} = \sigma_{5^{2^{N-3}}}(\zeta_{2^N}) = -\zeta_{2^N}$, since $\sigma_{5^{2^{n-3}}}$ is the nontrivial automorphism of K_N/K_{N-1}). This is an explicit incarnation with $x = 1 - \zeta_{2^N}^a$ of Lemma 2.9: in particular the quantity

$$(1+[\sigma_{-1}])(1-[\tau])x = \frac{(1-\zeta_{2^N}^a)(1-\zeta_{2^N}^{-a})}{(1+\zeta_{2^N}^a)(1+\zeta_{2^N}^{-a})}$$

is a real unit in $\mathcal{O}_{K_N}^\times$. In particular, remembering that all of this is an abuse of notation for applying ι and then taking logs, the sums we get in (2.7) involve logs of squared norms (according to the embedding ι) of Galois conjugate units

$$\log \iota \left(\frac{1-e^{2\pi i a/2^N}}{1+e^{2\pi i a/2^N}} \right) \bar{\iota} \left(\frac{1-e^{2\pi i a/2^N}}{1+e^{2\pi i a/2^N}} \right) = \log \left\| \frac{1-\zeta_{2^N}^a}{1+\zeta_{2^N}^a} \right\|_\iota = \log \left\| \sigma_a \left(\frac{1-\zeta_{2^N}}{1+\zeta_{2^N}} \right) \right\|_\iota = \log \left\| \frac{1-\zeta_{2^N}}{1+\zeta_{2^N}} \right\|_{\iota\circ\sigma_a} \tag{2.8}$$

where the notation $\|-\|_\iota$ is used in the way it is for defining the Log map (it is the squared absolute value instead of just the absolute value, since ι is complex), and a ranges over a set of representatives for $\langle 5 \rangle / \langle 5^{2^{N-3}} \rangle \cong \mathrm{Gal}(K_{N-1})/\mathbf{Q}(i)$ so that $\iota \circ \sigma_a$ ranges over the archimedean places of K_{N-1} or the places v_1 of K_N (according to Lemma-Definition 2.6). In particular,

(1) we do not expect these terms to contribute any factors of π, and

(2) we expect the even terms to be the only part of the left hand side of Proposition 2.1 accounting for the regulator of K_n on the right hand side.

Adding more detail to (2), the things we are taking squared norms under ι of in the even part of (2.4), as computed in (2.7), are the specific 2^{N-3} Galois conjugate units

$$(1-[\tau])x_a = \frac{1-\zeta_{2^N}^a}{1+\zeta_{2^N}^a}$$

which are in $(\mathcal{O}_{K_N}^\times)^{\tau=-1}$ by Lemma 2.9. Weber did the following:

(1) related the quantity we found at the end of (2.7) to $\mathrm{Reg}(K_N/K_{N-1})^-$ by showing it was equal to a certain determinant involving the quantities $\log\|(1+[\tau])x_a\|_\iota$ (which we do in Proposition 5.1) and that

(2) the submodule of $(\mathcal{O}_{K_N}^\times/\mu)^{\tau=-1}$ that the various units $(1-[\tau])x$ generate has odd index (which we do in Section 5.2 — the fact that this works is the most surprising part), so that the even part of $\prod_{N=4}^n \prod_\chi L(1,\chi)$ is related to $\prod_{N=4}^n \mathrm{Reg}(K_N/K_{N-1})^-$; then

(3) as mentioned above, he understood the index of $(\mathcal{O}_{K_{N-1}}^\times/\mu) \oplus (\mathcal{O}_{K_N}^\times)^{\tau=-1} \subset \mathcal{O}_{K_N}^\times/\mu$ well enough to relate $\mathrm{Reg}(K_N)$ to $\mathrm{Reg}(K_{N-1})\mathrm{Reg}(K_N)^-$ and therefore $\prod_{N=4}^n \mathrm{Reg}(K_N/K_{N-1})^-$ to $\mathrm{Reg}(K_n)/\mathrm{Reg}(K_3)$. This is carried out by us in Section 4 using more modern language (but the content is exactly the same as [Web1886, §8]).

2.3 The results of Weber's computations

We still have to work out the 2-adic valuation of the remaining algebraic number that will appear in the continuation of the calculation (2.6) in the odd case, and we still need to carry out Weber's strategy briefly outlined above for relating the quantity we arrived at in (2.7) to $\mathrm{Reg}(K_n)/\mathrm{Reg}(K_3)$.

Here are the exact statements we will prove:

Proposition 2.10 (The computation for odd characters)**.** *For $N \geq 4$, we have*

$$\prod_{\substack{\chi \in \widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive odd}}} \sum_{a \in (\mathbf{Z}/2^N)^\times} \overline{\chi}(a) \log(1 - \zeta_{2^N}^a) = \pi^{2^{N-3}} 2^{2^{N-3}-1} (\textit{some odd integer}).$$

Proof. Postponed for Section 3, where this will be accomplished by direct computation, continuing along the lines of (2.6). □

Proposition 2.11 (The computation for even characters)**.** *We have*

$$\prod_{N=4}^{n} \prod_{\substack{\chi \in \widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive even}}} \sum_{a \in (\mathbf{Z}/2^n)^\times} \overline{\chi}(a) \log(1 - \zeta_{2^n}^a) = \frac{2^{-1-\Sigma}}{\log(\sqrt{2}+1)} \mathrm{Reg}(K_n) (\textit{some odd integer})$$

where χ runs over the nontrivial even characters of $(\mathbf{Z}/2^n)^\times$, and Σ denotes some nonnegative integer.

Proof. Postponed for Section 5, where we will follow the outline given at the end of Section 2.2. For this outline to make sense, we first need to properly define $\mathrm{Reg}(K_N/K_{N-1})^-$, which we do in Section 4. □

Just to be on the same page regarding what was done in this section, we make the following explicit:

Proof of Theorem 1.2 assuming Proposition 2.10 and Proposition 2.11. The analytic class number formula (Proposition 2.1) says that

$$\prod_{\chi \neq 1} L(1, \chi) = \pi^{2^{n-2}} 2^{-n-(n-2)2^{n-2}} \mathrm{Reg}(K_n) h_n. \tag{2.9}$$

In (2.4), we saw that the left hand side decomposes as

$$\prod_{\chi \neq 1} L(1,\chi) = -\pi^2 \log(\sqrt{2}+1) 2^{-(n-1)2^{n-2}} \left(\prod_{N=4}^{n} \text{the quantity from Proposition 2.10} \right) \cdot (\text{the quantity from Proposition 2.11}),$$

and then Proposition 2.10 and Proposition 2.11 imply that

$$\begin{aligned} \prod_{\chi \neq 1} L(1,\chi) &= -\pi^2 \log(\sqrt{2}+1) 2^{-(n-1)2^{n-2}} \pi^{\sum_{N=4}^{n} 2^{N-3}} 2^{\sum_{N=4}^{n} (2^{N-3}-1)} \frac{2^{-1-\Sigma}}{\log(\sqrt{2}+1)} \mathrm{Reg}(K_n) \cdot (\text{some odd integer}) \\ &= \pi^{2^{n-2}} 2^{-n-(n-2)2^{n-2}-\Sigma} \mathrm{Reg}(K_n) \cdot (\text{some odd integer}). \end{aligned}$$

Comparing this to the right hand side of (2.9), we conclude that $\Sigma = 0$ (as $h_n \in \mathbf{Z}$), which is a strengthening of Proposition 2.11, and that h_n is odd. □

The remainder of these notes are hence devoted to proving Proposition 2.10 and Proposition 2.11.

3 Odd characters

In this section we will prove Proposition 2.10 by continuing the computations of (2.6). Let $N \geq 4$. Recall that (2.6) says

$$\begin{aligned}
\prod_{\substack{\chi \in \widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive odd}}} \sum_{a \in (\mathbf{Z}/2^N)^\times} \overline{\chi}(a) \log(1-\zeta_{2^N}^a) &= (\pi i)^{2^{N-3}} \prod_{\substack{\chi \in \widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive odd}}} \sum_{\beta=0}^{2^{N-3}-1} \overline{\chi}(5^\beta)\epsilon_\beta \\
&= \pm \pi^{2^{N-3}} \prod_{\substack{\chi \in \widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive odd}}} \sum_{\beta=0}^{2^{N-3}-1} \overline{\chi}(5^\beta)\epsilon_\beta \\
&= \pm \pi^{2^{N-3}} \prod_{x \in (\mathbf{Z}/2^{N-2})^\times} \sum_{\beta=0}^{2^{N-3}-1} \epsilon_\beta \zeta_{2^{N-2}}^{-x\beta} \\
&= \pm \pi^{2^{N-3}} \mathrm{N}_{\mathbf{Q}}^{\mathbf{Q}(\zeta_{2^{N-2}})} \left(\sum_{\beta=0}^{2^{N-3}-1} \epsilon_\beta \zeta_{2^{N-2}}^{-\beta} \right)
\end{aligned} \tag{3.1}$$

as the primitive odd characters χ are in bijection with the primitive 2^{N-2}-th roots of unity $\zeta_{2^{N-2}}^x$ ($x \in (\mathbf{Z}/2^{N-2})^\times$), via $\chi \mapsto \chi(5)$. Since $\zeta_{2^{N-2}}^{2^{N-3}} = -1$, we have

$$\frac{1}{2}(1-\zeta_{2^{N-2}}^{-1}) \sum_{\beta=0}^{2^{N-3}-1} \epsilon_\beta \zeta_{2^{N-2}}^{-\beta} = \frac{\epsilon_0 + \epsilon_{2^{N-3}-1}}{2} + \sum_{k=1}^{2^{N-3}-1} \frac{\epsilon_k - \epsilon_{k-1}}{2} \zeta_{2^{N-2}}^{-k}, \tag{3.2}$$

which is congruent to $\epsilon_{2^{N-3}-1} = \pm 1$ modulo the prime $(1-\zeta_{2^{N-2}}^{-1})$ above 2, since after subtracting $\epsilon_{2^{N-3}-1}$ we get (viewing $\epsilon_{-1} := \epsilon_{2^{N-3}-1}$)

$$\sum_{k=0}^{2^{N-3}-1} \frac{\epsilon_k - \epsilon_{k-1}}{2} \zeta_{2^{N-2}}^{-k} = (1-\zeta_{2^{N-2}}^{-1}) \sum_{k=0}^{2^{N-3}-2} \frac{\epsilon_k - \epsilon_{2^{N-3}-1}}{2} \zeta_{2^{N-2}}^{-k} \in (1-\zeta_{2^{N-2}}^{-1})\mathbf{Z}[\zeta_{2^{N-2}}],$$

from which we deduce that

$$\mathrm{N}_{\mathbf{Q}}^{\mathbf{Q}(\zeta_{2^{N-2}})} \left(\frac{1}{2}(1-\zeta_{2^{N-2}}^{-1}) \sum_{\beta=0}^{2^{N-3}-1} \epsilon_\beta \zeta_{2^{N-2}}^{-\beta} \right) \text{ is odd,}$$

as the quantity inside the norm is in $\mathbf{Z}[\zeta_{2^{N-2}}]$ by (3.2) and is not in the prime of $\mathbf{Q}(\zeta_{2^{N-2}})$ lying over 2. Since

$$\mathrm{N}_{\mathbf{Q}}^{\mathbf{Q}(\zeta_{2^{N-2}})}(2) = 2^{2^{N-3}} \text{ and } \mathrm{N}_{\mathbf{Q}}^{\mathbf{Q}(\zeta_{2^{N-2}})}(1-\zeta_{2^{N-2}}^{-1}) = 2$$

(2 is totally ramified), we deduce from this and from (3.1) that

$$\begin{aligned}
\prod_{\substack{\chi \in \widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive odd}}} \sum_{a \in (\mathbf{Z}/2^N)^\times} \overline{\chi}(a) \log(1-\zeta_{2^N}^a) &= \pm \pi^{2^{N-3}} \mathrm{N}_{\mathbf{Q}}^{\mathbf{Q}(\zeta_{2^{N-2}})} \left(\sum_{\beta=0}^{2^{N-3}-1} \epsilon_\beta \zeta_{2^{N-2}}^{-\beta} \right) \\
&= \pm \pi^{2^{N-3}} 2^{2^{N-3}-1} \mathrm{N}_{\mathbf{Q}}^{\mathbf{Q}(\zeta_{2^{N-2}})} \left(\frac{1}{2}(1-\zeta_{2^{N-2}}) \sum_{\beta=0}^{2^{N-3}-1} \epsilon_\beta \zeta_{2^{N-2}}^{-\beta} \right) \\
&= \pi^{2^{N-3}} 2^{2^{N-3}-1} (\text{an odd integer})
\end{aligned}$$

which concludes the proof of Proposition 2.10.

4 Preliminaries on units and Stark regulators

I will now recall some stadard material about Stark regulators from [Tat1984]. That book is hard to find online but this specific material is also found in Samit Dasgupta's undergraduate thesis. The purpose of this is to apply it to the situation of K_N/K_{N-1} in order to follow the outline from the end of Section 2 in relating $\mathrm{Reg}(K_N)$ to the product of $\mathrm{Reg}(K_{N-1})$ with $\mathrm{Reg}(K_N/K_{N-1})^-$. The first subsection will deal with the generalities required to define $\mathrm{Reg}(K_N/K_{N-1})^-$ and to set up this relation, and the second subsection will carry out this step of Weber's argument in the relevant case.

4.1 Generalities on Stark regulators

For a number field K, let Y_K be the free $\mathbf{Z}$-module of rank $r+s$ generated by the symbols $[v]$, where $[v]$ runs over the archimedean places of K. Let $X_K \subset Y_K$ be the free $\mathbf{Z}$-module of rank $r+s-1$ equal to the set of elements $\sum_v a_v[v]$ with the property that $\sum_v a_v = 0$. For all places v_0 of K, $\{v - v_0 : v \neq v_0\}$ is a $\mathbf{Z}$-basis for X_K of size $r+s-1$.

Dirichlet's unit theorem says that the quotient $\mathcal{O}_K^\times/\mu_K$ is a free abelian group of rank $r+s-1$, and provides an $\mathbf{R}$-linear isomorphism

$$\mathrm{Log}_K : \mathbf{R} \otimes (\mathcal{O}_K^\times/\mu_K) \xrightarrow{\sim} \mathbf{R} \otimes X_K$$

given on the pure tensors $1 \otimes [\varepsilon]$ by

$$\mathrm{Log}_K(1 \otimes \varepsilon) = \sum_v \log\|\varepsilon\|_v[v],$$

where $\|x\|$ means $|x|_v^2$ when v is complex and it means $|x|_v$ when v is real. Note that this morphism does NOT take $\mathcal{O}_K^\times/\mu_K$ to X_K: it only exists after tensoring with $\mathbf{R}$, since the logs of unit are in general transcendental quantities.

If K/k is a Galois extension of number fields, then both $\mathcal{O}_K^\times/\mu_K$ and X_K admit an action of $\mathrm{Gal}(K/k)$ and are therefore $\mathbf{Z}[\mathrm{Gal}(K/k)]$-modules which are free of rank $r+s-1$ as $\mathbf{Z}$-modules. The isomorphism Log_K is $\mathrm{Gal}(K/k)$-equivariant, so we manifestly have

$$\mathbf{C} \otimes (\mathcal{O}_K^\times/\mu_K) \cong \mathbf{C} \otimes X_K$$

as finite-dimensional complex representations of the finite group $\mathrm{Gal}(K/k)$. Since $\mathbf{Q}$ has characteristic zero, a finite-dimensinal $\mathbf{Q}[\mathrm{Gal}(K/k)]$-module is determined by its character, so this implies that there is an abstract (but no specific choice of) isomorphism

$$f : \mathbf{Q} \otimes X_K \xrightarrow{\sim} \mathbf{Q} \otimes (\mathcal{O}_K^\times/\mu_K)$$

of $\mathbf{Q}[\mathrm{Gal}(K/k)]$-modules, which has nothing to do with Log_K (since that map is not usually defined over $\mathbf{Q}$).

Definition 4.1. Let $f : \mathbf{Q} \otimes X_K \to \mathbf{Q} \otimes (\mathcal{O}_K^\times/\mu_K)$ be *any* isomorphism of $\mathbf{Q}[\mathrm{Gal}(K/k)]$-modules, and let V be *any* finite-dimensional complex representation of $\mathrm{Gal}(K/k)$, with character χ. The maps induced by f under $(-) \otimes_{\mathbf{Q}} \mathbf{C}$ which we abusively call $f : \mathbf{C} \otimes X_K \to \mathbf{C} \otimes (\mathcal{O}_K^\times/\mu_K)$ and $\mathrm{Log}_K : \mathbf{C} \otimes (\mathcal{O}_K^\times/\mu_K) \to \mathbf{C} \otimes X_K$ induce maps that we call

$$f_\chi : \mathrm{Hom}_{\mathrm{Gal}(K/k)}(V^\vee, \mathbf{C} \otimes X_K) \to \mathrm{Hom}_{\mathrm{Gal}(K/k)}(V^\vee, \mathbf{C} \otimes (\mathcal{O}_K^\times/\mu_K))$$

and

$$\mathrm{Log}_{K,\chi} : \mathrm{Hom}_{\mathrm{Gal}(K/k)}(V^\vee, \mathbf{C} \otimes (\mathcal{O}_K^\times/\mu_K)) \to \mathrm{Hom}_{\mathrm{Gal}(K/k)}(V^\vee, \mathbf{C} \otimes X_K).$$

These two maps compose to give a $\mathrm{Gal}(K/k)$-equivariant automorphism of $\mathbf{C} \otimes X_K$, and the quantity

$$R(\chi, f) := \det(\mathrm{Log}_{K,\chi} \circ f_\chi)$$

is called the *Stark regulator* of χ with respect to f.

The definition of the Stark regulator is not vacuous, because we showed right before it that such isomorphisms f exist. In fact, by multiplication by a large enough integer, f can even be taken to come from an *integral* datum of an injective $\mathbf{Z}[\mathrm{Gal}(K/k)]$-module morphism $f : X_K \to \mathcal{O}_K^\times/\mu_K$. We will have no need to worry about the presence of the dual on top of V in the definition, because the only representations we will care about are the two irreps of $\mathbf{Z}/2$.

The point of the Stark regulator is that it captures the part of the regulator that comes from the units in a certain isotypic component of the action of $\mathrm{Gal}(K/k)$. Indeed, if f is obtained by base change from an injective map of $\mathbf{Z}[\mathrm{Gal}(K/k)]$-modules $f : X_K \to \mathcal{O}_K^\times/\mu_K$, then we have the following examples:

Example 4.1. (1) If $K = k$ and V is the trivial representation of the trivial group, then f_χ and $\mathrm{Log}_{K,\chi}$ are identified with f, Log_K. Choosing an arbitrary infinite place v_0 of K, using the $\mathbf{Z}$-basis $\{[v] - [v_0] : v \neq v_0\}$ of X_K, and setting $\varepsilon_v := f([v] - [v_0])$ for $v \neq v_0$, the $r + s - 1$ units ε_v are (by the assumption that f is an isomorphism after $(-) \otimes \mathbf{Q}$) a basis of a finite-index free abelian subgroup of $\mathcal{O}_K^\times/\mu_K$. Since

$$(\mathrm{Log}_K \circ f)([v] - [v_0]) = \mathrm{Log}_K(\varepsilon_v) = \sum_w \log\|\varepsilon_v\|_w [w] = \sum_{w \neq v_0} \log\|\varepsilon_v\|_w ([w] - [v_0]),$$

the Stark regulator is then

$$R(f, \chi) = \det(\mathrm{Log}_K \circ f) = \det\left(\log\|\varepsilon_v\|_w\right)_{v,w \neq v_0} = \pm[\mathcal{O}_K^\times/\mu_K : f(X_K)]\mathrm{Reg}(K).$$

(2) Let K/k satisfy[2] $(\mathcal{O}_K^\times/\mu)^{\mathrm{Gal}(K/k)} = \mathcal{O}_k^\times/\mu$, and let V be the trivial representation of $\mathrm{Gal}(K/k)$. We can still let $\varepsilon_v = f([v] - [v_0])$, $v \neq v_0$ be the values of f on this specific basis of X_K. The equivariant map f also induces an injection of free abelian groups of the same rank $f : X_K^{\mathrm{Gal}(K/k)} \to (\mathcal{O}_K^\times/\mu_K)^{\mathrm{Gal}(K/k)}$, which necessarily has finite index. Moreover, $X_K^{\mathrm{Gal}(K/k)}$ is canonically identified with X_k regardless of whether we take $\mathbf{Z}$ or $\mathbf{C}$ coefficients (take the common value of a_w, $w|v$, to be the a_v of the corresponding element of X_k), from which we have $\mathbf{C} \otimes X_K^{\mathrm{Gal}(K/k)} = (\mathbf{C} \otimes X_K)^{\mathrm{Gal}(K/k)}$.

The induced maps we want to understand are therefore

$$f_\chi : \mathbf{C} \otimes X_k \to (\mathbf{C} \otimes (\mathcal{O}_K^\times/\mu_K))^{\mathrm{Gal}(K/k)}$$

and

$$\mathrm{Log}_{K,\chi} : (\mathbf{C} \otimes (\mathcal{O}_K^\times/\mu_K))^{\mathrm{Gal}(K/k)} \to \mathbf{C} \otimes X_k.$$

Choose a fixed archimedean place v_0 of k, so that $\{[v] - [v_0]\}$ is a $\mathbf{Z}$-basis of X_k. Without caring about the values of f on X_K outside of $X_K^{\mathrm{Gal}(K/k)} = X_k$, we set $\varepsilon_v = f([v] - [v_0])$ for all archimedean places v of k. This is an element of $(\mathcal{O}_K^\times/\mu_K)^{\mathrm{Gal}(K/k)} = \mathcal{O}_k^\times/\mu_k$, so we further have

$$\mathrm{Log}_{K,\chi}(\varepsilon_v) = \sum_w \log\|\varepsilon_v\|_w [w] \in (\mathbf{C} \otimes X_K)^{\mathrm{Gal}(K/k)},$$

where this last is taken to $\mathbf{C} \otimes X_k$ by observing that the $\log\|\varepsilon_v\|_w$ don't change when you change w lying over the same archimedean place of k, so really $[v] - [v_0]$, for $v \neq v_0$ a place of k, is taken by $\mathrm{Log}_{K,\chi} \circ f_\chi$ to

$$\sum_{w \neq v_0} \log\|\varepsilon_v\|_{w'} [w],$$

where for all places w of k, w' is taken to be an arbitrary place of K with $w'|w$. Since (and this is where we are actually using it) $\varepsilon_v \in \mathcal{O}_k^\times/\mu_k$, we really have $\|\varepsilon_v\|_{w'} = \|\varepsilon_v\|_w$, at least if we assume that

[2]Weber gets to ignore this condition because he passes to the maximal totally real subfield, where this is obvious. But it is almost equally obvious in our setting where $K/k = K_N/K_{N-1}$, because $H^1(\mathrm{Gal}(K_N/K_{N-1}), \mu) = 1$ for $N \geq 3$ (easy to check explicitly: it comes down to the fact that $\zeta_{2^N}\tau(\zeta_{2^N}) = -\zeta_{2^{N-1}} \neq 1$ if $N \neq 2$, so this H^1 is at worst the group $\{\pm 1\}$, but it is actually trivial because $\zeta_{2^N}/\tau(\zeta_{2^N}) = -1$.)

K and k are both totally complex or both totally real (if w' were complex and w were real then there would be a factor of 2 to insert), so

$$(\mathrm{Log}_{K,\chi} \circ f_\chi)([v] - [v_0]) = \sum_{w \neq v_0} \log\|\varepsilon_v\|_w [w],$$

and, taking determinants, we have

$$R(\chi, f) = \pm[\mathcal{O}_k^\times/\mu_k : f(X_k)]\mathrm{Reg}(k).$$

(3) Suppose that K/k is quadratic and that they are both totally complex. Let $\eta_- : \mathrm{Gal}(K/k) = \langle\tau\rangle \cong \mathbf{Z}/2 \to \{\pm 1\}$ be the nontrivial 1-dimensional character. This is a convenient case to be in because it is defined over $\mathbf{Z}$, and consequently the $\mathbf{Z}$-modules $\mathrm{Hom}_{\mathrm{Gal}(K/k)}(\eta_-, X_K), \mathrm{Hom}_{\mathrm{Gal}(K/k)}(\eta_-, \mathcal{O}_K^\times/\mu_K)$ are well-defined. The former is the same as the η_--isotypic component $X_K^{\tau=-1} \subset X_K$, and the latter is the same as the η_--isotypic component of $\mathcal{O}_K^\times/\mu_K$, namely $(\mathcal{O}_K^\times/\mu_K)^{\tau=-1}$. We also have

$$(\mathbf{C} \otimes X_K)^{\tau=-1} = \mathbf{C} \otimes X_K^{\tau=-1}, \quad \mathbf{C} \otimes (\mathcal{O}_K^\times/\mu_K)^{\tau=-1} = (\mathbf{C} \otimes (\mathcal{O}_K^\times/\mu_K))^{\tau=-1}.$$

The injection $f : X_K \to \mathcal{O}_K^\times/\mu_K$ is assumed to be $\mathrm{Gal}(K/k)$-equivariant and hence induces an injection

$$f_{\eta_-} : X_K^{\tau=-1} \to (\mathcal{O}_K^\times/\mu_K)^{\tau=-1}.$$

of free abelian groups of the same rank.

Since $\mathrm{Gal}(K/k) \cong \mathbf{Z}/2$, there are two complex places of K above each (complex) place of k, and τ acts by switching them (because it must act transitively). It follows that $X_K^{\tau=-1}$ has a natural basis given by $\{[v_1] - [v_2] : v \text{ archimedean place of } k\}$, where v_1, v_2 are the two places of K above v (a choice of which one is v_1 and which one is v_2 is made in order to choose this basis). So for each place v of k, we let $\varepsilon_v = f([v_1] - [v_2]) \in (\mathcal{O}_K^\times/\mu_K)^{\tau=-1}$, so that

$$(\mathrm{Log}_{K,\eta_-} \circ f)([v_1] - [v_2]) = \sum_w \log\|\varepsilon_v\|_w [w] = \sum_w \log\|\varepsilon_v\|_{w_1}([w_1] - [w_2]),$$

where in the second expression the w's go over the places of K and in the third one they go over the places of k. In particular,

$$R(\eta_-, f) = \det(\mathrm{Log}_{K,\eta_-} \circ f) = \det(\log\|\varepsilon_v\|_{w_1}).$$

Since $(\mathcal{O}_K^\times/\mu_K)^{\tau=-1}$ is a free abelian group of rank equal to the number of complex places of k (i.e. the rank of $X_K^{\tau=-1}$), it has a basis $\{\mathcal{E}_v : v \text{ archimedean places of } k\}$, and we have

$$R(\eta_-, f) = \pm[(\mathcal{O}_K^\times/\mu_K)^{\tau=-1} : f(X_K^{\tau=-1})]|\det(\log\|\mathcal{E}_v\|_w)|.$$

Let us make explicit the definition of "minus part of the regulator" implicit in the third example:

Definition 4.2. Let K/k be a quadratic extension of totally complex fields with Galois group $G = \langle\tau\rangle$. For a (complex) archimedean place v of k, denote by v_1, v_2 the two (complex) places of K lying over it. The abelian group $(\mathcal{O}_K^\times/\mu_K)^{\tau=-1}$ is a free $\mathbf{Z}$-module, and since it is isomorphic after $- \otimes \mathbf{Q}$ to $(X_K)^{\tau=-1}$, it is of rank equal to $[k : \mathbf{Q}]/2 = \#\{(\text{complex}) \text{ archimedean places of } k\} =: r$. For any basis $\mathcal{E}_1, \ldots, \mathcal{E}_r$ of $(\mathcal{O}_K^\times/\mu_K)^{\tau=-1}$, we define the minus-part of the regulator to be

$$\mathrm{Reg}(K/k)^- := |\det((\log\|\mathcal{E}_i\|_{v_1})_{i,v})|$$

where the rows are parametrized by the choice of $1 \leq i \leq r$ and the columns are parametrized by the choice of place v of k (this matrix is square by the remarks above, and this definition obviously does not depend on the choice of basis). Since we took absolute values, this doesn't depend on the choice of basis $\mathcal{E}_i$, nor on the choice of which of the places above v is called v_1 (changing that choice would just negate a whole column of the matrix and thus would only change the sign of the determinant).

4.2 Application to the 2-power cyclotomic tower

Now we apply this theory to our situation in order to relate $\mathrm{Reg}(K_N)$ to $\mathrm{Reg}(K_{N-1})\mathrm{Reg}(K_N/K_{N-1})^-$. Let $N \geq 3$, and let $k = \mathbf{Q}(\zeta_{2^{N-1}}) = K_{N-1}$ and $K = K_N = \mathbf{Q}(\zeta_{2^N}) = k(\sqrt{\zeta_{2^{N-1}}})$. Then K/k is quadratic with $\mathrm{Gal}(K/k) = \{1, \tau\}$ (where now τ is the element $\sigma_{5^{2^{N-3}}}$ defined in Section 2), and it satisfies $H^1(\mathrm{Gal}(K/k), \mu_K) = 1$ (the norm of $\zeta_{2^{N-1}}^x$ is $\zeta_{2^{N-1}}^{2x}$, which is 1 if and only if $\zeta_{2^{N-1}}^x = \pm 1$; and the norm of ζ_{2^N} is $\zeta_{2^N}\tau(\zeta_{2^N}) = -\zeta_{2^N}^2 = -\zeta_{2^{N-1}}$, which is not 1 because $\zeta_{2^{N-1}} \neq -1$ because $N \geq 3$ — so the norm 1 elements of μ_K are just $\{\pm 1\}$, and we note that $-1 = \zeta_{2^N}/\tau(\zeta_{2^N})$), so that the hypothesis $(\mathcal{O}_K^\times/\mu)^{\mathrm{Gal}(K/k)} = \mathcal{O}_k^\times/\mu$ to Example 4.1(2) is satisfied. Since the trivial character and η_- are a complete list of characters of $\mathrm{Gal}(K_N/K_{N-1})$, we have the decompositions

$$\mathbf{C} \otimes X_K = (\mathbf{C} \otimes X_K)^{\mathrm{Gal}(K/k)} \oplus (\mathbf{C} \otimes X_K)^{\tau=-1} = (\mathbf{C} \otimes X_k) \oplus (\mathbf{C} \otimes X_K^{\tau=-1})$$

and

$$\mathbf{C} \otimes (\mathcal{O}_K^\times/\mu_K) = (\mathbf{C} \otimes (\mathcal{O}_K^\times/\mu_K))^{\mathrm{Gal}(K/k)} \oplus (\mathbf{C} \otimes (\mathcal{O}_K^\times/\mu_K))^{\tau=-1} = (\mathbf{C} \otimes (\mathcal{O}_k^\times/\mu_k)) \oplus (\mathbf{C} \otimes (\mathcal{O}_K^\times/\mu_K)^{\tau=-1})$$

where η_- is as before the nontrivial character of $\mathrm{Gal}(K/k)$ sending τ to -1. There is an important thing to keep track of here, which is that this decomposition does NOT apply on the nose integrally, i.e., it is NOT true that

$$X_K = X_k \oplus X_K^{\tau=-1},$$

nor that

$$\mathcal{O}_K^\times/\mu_K = \mathcal{O}_k^\times/\mu_k \oplus (\mathcal{O}_K^\times/\mu_K)^{\tau=-1}.$$

The problem, as already discussed in Section 2, is that the idempotents $([1]-[\tau])/2, ([1]+\tau)/2$ in the group algebra that you would use to make this decomposition have denominators. So the index will be a power of 2, which we need to keep track of since our results are sensitive to powers of 2. In our particular situation, we have $\mathrm{rk}_{\mathbf{Z}}(X_K) = \mathrm{rk}_{\mathbf{Z}}(\mathcal{O}_K^\times/\mu_K) = 2^{N-2}-1$, $\mathrm{rk}_{\mathbf{Z}}(X_k) = \mathrm{rk}_{\mathbf{Z}}(\mathcal{O}_k^\times/\mu_k) = 2^{N-3}-1$, and (either by separate calculation or by subtraction), $\mathrm{rk}_{\mathbf{Z}}(X_K^\chi) = \mathrm{rk}_{\mathbf{Z}}((\mathcal{O}_K^\times/\mu_K)^\chi) = 2^{N-3}$.

Lemma 4.3. *The index of $X_k \oplus X_K^{\tau=-1} \subset X_K$ is exactly*

$$[X_K : X_k \oplus X_K^{\tau=-1}] = 2^{2^{N-3}-1}.$$

Proof. I will carry this out by explicit Smith normal form computation, using specific bases of all the modules involved. For all places $v|\infty$ of k, let v_1, v_2 be the two places of K lying over v. Since the group $\{1,\tau\}$ must act transitively on $\{v_1, v_2\}$, we know that τ acts on the places of K by switching v_1, v_2 for all v of k. Let us fix arbitrarily a certain place $v^0|\infty$ of k.

The free $\mathbf{Z}$-module X_K has a basis given by

$$\{[v_1^0]-[v_2^0]\} \sqcup \{[v_1]-[v_1^0] : v \neq v^0\} \sqcup \{[v_2]-[v_1^0] : v \neq v^0\}.$$

As a sanity check, note that K has 2^{N-2} (complex) places and k has 2^{N-3}, so we expect this basis to have size $2^{N-2}-1$, which is true because $1 + (2^{N-3}-1) + (2^{N-3}-1) = 2^{N-2}-1$.

The free $\mathbf{Z}$-module $X_k = X_K^{\mathrm{Gal}(K/k)}$ has a basis given by $\{[v_1]+[v_2]-[v_1^0]-[v_2^0] : v \neq v^0\}$, which has size $2^{N-3}-1$ as expected. Writing these in terms of the basis of X_K we just discussed, we have

$$[v_1]+[v_2]-[v_1^0]-[v_2^0] = ([v_1^0]-[v_2^0]) + ([v_1]-[v_1^0]) + ([v_2]-[v_1^0]).$$

Finally, the free $\mathbf{Z}$-module $X_K^{\tau=-1}$ has a basis given by

$$\{[v_1^0]-[v_2^0]\} \sqcup \{[v_1]-[v_2] : v \neq v^0\},$$

where the point of separating $v \neq v^0$ from v^0 is that $[v_1^0]-[v_2^0]$ is already in our chosen basis of X_K, while the decomposition of the remaining ones is

$$[v_1]-[v_2] = ([v_1]-[v_1^0]) - ([v_2]-[v_1^0]).$$

Figure 1: The matrix of $X_k \oplus X_K^{\tau=-1} \to X_K$, from the proof of Lemma 4.3. The labels of the columns remind us of which basis element of X_k or $X_K^{\tau=-1}$ that column represents the image of, and the labels of the rows remind us of the basis of X_K that those images are being represented in.

$$
\begin{array}{ccc}
 & & \begin{array}{cc} \overbrace{\hphantom{1\ 1\ 1\ \cdots\ 1\ 1\ 1}}^{\mathrm{rk}_{\mathbf{Z}}(X_k) = 2^{N-3}-1} & \overbrace{\hphantom{1\ 0\ 0\ 0\ \cdots\ 0\ 0\ 0}}^{\mathrm{rk}_{\mathbf{Z}}(X_K^{\tau=-1}) = 2^{N-3}} \end{array} \\
\mathrm{rk}_{\mathbf{Z}}(X_K) = 2^{N-2}-1 \left\{ \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right. &
\begin{array}{r} [v_1^0]-[v_2^0] \\ \\ [v_1]-[v_1^0], \\ [v_2]-[v_1^0] \\ (v \neq v^0) \\ \\ \end{array} &
\begin{pmatrix}
1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1
\end{pmatrix} \\
 & & \begin{array}{ccc} \begin{array}{c} [v_1]+[v_2]-[v_1^0]-[v_2^0] \\ (v \neq v^0) \end{array} & [v_1^0]-[v_2^0] & \begin{array}{c} [v_1]-[v_2] \\ (v \neq v^0) \end{array} \end{array}
\end{array}
$$

In other words, with this choice of bases, the matrix of the injection $X_k \oplus X_K^{\tau=-1} \to X_K$ is as pictured in Figure 1.

To compute the index of the submodule $X_k \oplus X_K^\chi \subset X_K$ can be done by computing the Smith normal form of this matrix. That computation is just many copies of the following 3×3 Smith normal form computation:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

which is the $N = 4$ case. One concludes from this that the elementary divisors of $X_K/(X_k \oplus X_K^{\tau=-1})$ are just $2^{N-3} - 1$ copies of 2, hence that the index is $2^{N-3} - 1$, as required. Note that the fact that no power of 2 greater than 2^1 can appear in the elementary divisors was clear at the outset, because (again thanks to the fact that the denominators in the relevant idempotents are just 2) $2X_K \subset X_k \oplus X_K^\chi \subset X_K$. ☐

In fact, the index appearing in Lemma 4.3 is the "most degenerate" possible case, as shown by the following general fact:

Lemma 4.4. *Let $G = \mathbf{Z}/2 = \langle\tau\rangle$, and let M be a $\mathbf{Z}[G]$-module which is free of finite rank as a $\mathbf{Z}$-module. Then*

$$[M : M^{\tau=1} \oplus M^{\tau=-1}] \leq 2^{\min(\mathrm{rk}_{\mathbf{Z}}(M^{\tau=1}, M^{\tau=-1}))}.$$

Proof. We consider the short exact sequence of $\mathbf{Z}[G]$-modules

$$0 \to M^{\tau=1} \oplus M^{\tau=-1} \to M \to Q \to 0,$$

and take the long exact sequence in cohomology, which starts out as

$$0 \to M^{\tau=1} \xrightarrow{\sim} M^{\tau=1} \to Q^{\tau=1} \to H^1(G, M^{\tau=1} \oplus M^{\tau=-1}).$$

We simplify the terms one at a time starting at $Q^{\tau=1}$. The point here is to clear the denominators in the idempotents we would like to use: $x + \tau(x) \in M^{\tau=1}$ and $x - \tau(x) \in M^{\tau=-1}$ are still true, so (only actually using the second one), we have $\overline{x} - \tau(\overline{x}) = 0$, where $\overline{x}$ denotes the class in Q represented by $x \in M$. In other words, $Q^{\tau=1} = Q$. So far, the content of the long exact sequence is that Q injects into $H^1(G, M^{\tau=1} \oplus M^{\tau=-1})$. This is where the upper bound on the index will come from.

Since $G = \langle\tau\rangle = \mathbf{Z}/2$, this H^1 is identified with

$$H^1(G, M^{\tau=1} \oplus M^{\tau=-1}) = \frac{\{m_1 + m_{-1} \in M^{\tau=1} \oplus M^{\tau=-1} : (m_1 + m_{-1}) + \tau(m_1 + m_{-1}) = 0\}}{\{(m_1 + m_{-1}) - \tau(m_1 - m_{-1}) : m_1 + m_{-1} \in M^{\tau=1} \oplus M^{\tau=-1}\}}.$$

By definition of $M^{\tau=\pm 1}$, the numerator simplifies to $\{m_1 + m_{-1} : 2m_1 = 0\} = M^{\tau=-1}$, and the denominator simplifies to $\{2m_{-1}\} = 2M^{\tau=-1}$. In other words,

$$H^1(G, M^{\tau=1} \oplus M^{\tau=-1}) = \frac{M^{\tau=-1}}{2M^{\tau=-1}} \cong \mathbf{F}_2^{\mathrm{rk}_{\mathbf{Z}}(M^{\tau=-1})},$$

so the injection of Q into this provides the upper bound $|Q| \leq 2^{\mathrm{rk}_{\mathbf{Z}}(M^{\tau=-1})}$.

The bound $|Q| \leq 2^{\mathrm{rk}_{\mathbf{Z}}(M^{\tau=1})}$ which was also claimed follows *a fortiori* from this, by considering the alternative action of G on M given by $\tau_{\mathrm{new}}(m) = -\tau_{\mathrm{original}}(m)$. ☐

Corollary 4.5. *Let $N \geq 3$. If $K = \mathbf{Q}(\zeta_{2^N})$ and $k = \mathbf{Q}(\zeta_{2^{N-1}})$ and τ denotes the nontrivial element of* $\mathrm{Gal}(K/k)$*, then*

$$[(\mathcal{O}_K/\mu_K) : (\mathcal{O}_k/\mu_k) \oplus (\mathcal{O}_k/\mu_k)^{\tau=-1}] \leq 2^{2^{N-3}-1}$$

and is a power of 2.

Let $\sigma_N \geq 0$ be the nonnegative integer such that this index is equal to $2^{2^{N-3}-1-\sigma_N}$.

We can put this together to obtain the following result that is relevant to our situation, where K/k is K_N/K_{N-1}. Let $f : X_K \to (\mathcal{O}_K^\times/\mu_K)$ be a $\mathrm{Gal}(K/k)$-equivariant injection. By restriction, we obtain two injections

$$f_1 : X_k \to \mathcal{O}_k^\times/\mu_k$$

and

$$f_{\eta_-} : X_K^\chi \to (\mathcal{O}_K^\times/\mu_K)^{\tau=-1}.$$

On one hand, we have $\mathbf{C}$-vector space isomorphisms

$$\mathbf{C} \otimes X_K \xrightarrow{f} \mathbf{C} \otimes (\mathcal{O}_K^\times/\mu_K) \xrightarrow{\mathrm{Log}_K} \mathbf{C} \otimes X_K,$$

for which (by Example 4.1(1)) we have

$$\det(\mathrm{Log}_K \circ f) \pm [\mathcal{O}_K^\times/\mu_K : f(X_K)]\mathrm{Reg}(K). \tag{4.1}$$

On the other hand, by decomposing both f and Log_K into $\mathrm{Gal}(K/k)$-isotypic components,

$$(\mathbf{C}\otimes X_k)\oplus(\mathbf{C}\otimes X_K^{\tau=-1}) \xrightarrow{f=f_1\oplus f_{\eta_-}} \mathbf{C}\otimes(\mathcal{O}_k^\times/\mu_k)\oplus\mathbf{C}\otimes(\mathcal{O}_K^\times/\mu_K)^{\tau=-1} \xrightarrow{\mathrm{Log}_K=\mathrm{Log}_{K,1}\oplus\mathrm{Log}_{K,\eta_-}} (\mathbf{C}\otimes X_k)\oplus(\mathbf{C}\otimes X_K^{\tau=-1}),$$

and we obtain

$$\begin{aligned}\det(\mathrm{Log}_K \circ f) &= \det(\mathrm{Log}_{K,1} \circ f_1)\cdot\det(\mathrm{Log}_{K,\eta_-} \circ f_{\eta_-}) \\ &= R(1,f)R(\chi,f) \\ &= \pm[(\mathcal{O}_k^\times/\mu_k) : f(X_k)][(\mathcal{O}_K^\times/\mu_k)^{\tau=-1} : f(X_K^{\tau=-1})]\mathrm{Reg}(k)\mathrm{Reg}(K/k)^- \\ &= \pm[(\mathcal{O}_k^\times/\mu_k) \oplus (\mathcal{O}_K^\times/\mu_k)^{\tau=-1} : f(X_k \oplus X_K^{\tau=-1})]\mathrm{Reg}(k)\mathrm{Reg}(K/k)^-.\end{aligned} \tag{4.2}$$

by Example 4.1((2) and (3)).

Combining (4.1) and (4.2) with Lemma 4.3 and Corollary 4.5, we end up with

$$\begin{aligned}\frac{\mathrm{Reg}(K)}{\mathrm{Reg}(k)\mathrm{Reg}(K/k)^-} &= \frac{[(\mathcal{O}_k^\times/\mu_k)\oplus(\mathcal{O}_K^\times/\mu_k)^\chi : f(X_k \oplus X_K^\chi)]}{[\mathcal{O}_K^\times/\mu_K : f(X_K)]} \\ &= \frac{[X_K : X_k \oplus X_K^\chi]}{[\mathcal{O}_K^\times/\mu_K : \mathcal{O}_k^\times/\mu_k \oplus (\mathcal{O}_K^\times/\mu_K)^\chi]} \\ &= 2^{\sigma_N},\end{aligned}$$

from which we have the main result of this section:

Proposition 4.6. *Let $N \geq 3$. Then there exists a nonnegative integer σ_N such that*

$$\mathrm{Reg}(K_N) = 2^{\sigma_N}\mathrm{Reg}(K_{N-1})\mathrm{Reg}(K_N/K_{N-1})^-.$$

Weber gets here through much more explicit (and probably more understable) means. Writing it this way gave me some practice with Stark regulators but it is up to the reader to decide whether this gives any useful clarifying perspective on what is already done pretty clearly in [Web1886, §8].

5 Even characters

In this section, we will prove Proposition 2.11, thus completing the proof of Theorem 1.2. Recall from (2.7) and (2.8) that the quantity (which, up to some Gauss sum factors, is the product of values $L(1,\chi)$ where χ

is even of conductor 2^N with $4 \le N \le n$) we are meant to compute (the left hand side of Proposition 2.11) is

$$\begin{aligned}
\prod_{N=4}^{n} \prod_{\substack{\chi \in \widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive even}}} \sum_{a \in (\mathbf{Z}/2^n)^\times} \overline{\chi}(a) \log(1-\zeta_{2^n}^a) &= \prod_{N=4}^{n} \prod_{\substack{\chi \in \widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive even}}} \sum_{\beta=0}^{2^{N-3}-1} \overline{\chi}(a) \log \frac{(1-\zeta_{2^N}^a)(1-\zeta_{2^N}^{-a})}{(1+\zeta_{2^N}^a)(1+\zeta_{2^N}^{a})} \\
&= \prod_{N=4}^{n} \prod_{\substack{\chi \in \widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive even}}} \sum_{\beta=0}^{2^{N-3}-1} \overline{\chi}(a) \log \left\| \sigma_a \left(\frac{1-\zeta_{2^N}}{1+\zeta_{2^N}} \right) \right\|_\iota \qquad (5.1) \\
&= \prod_{N=4}^{n} \prod_{\substack{\chi \in \widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive even}}} \sum_{\beta=0}^{2^{N-3}-1} \overline{\chi}(a) \log \left\| \frac{1-\zeta_{2^N}}{1+\zeta_{2^N}} \right\|_{\iota \circ \sigma_a}
\end{aligned}$$

where $a = 5^\beta \in \langle 5 \rangle \subset (\mathbf{Z}/2^N)^\times$ runs over our chosen set of representatives for $\langle 5 \rangle / \langle 5^{2^{N-3}} \rangle$. We will use the notation of Section 2.2 to talk about the Galois action on units and on archimedean places of K_N, K_{N-1}, recall especially Lemma-Definition 2.6. To briefly recall, the $\iota \circ \sigma_a$ as β ranges from 0 to $2^{N-3}-1$ are a choice of places v_1 of K_N lying over each of the 2^{N-3} archimedean place v of K_{N-1} (and we *always* take v_1 to mean this particular choice), and the elements of $K_N^\times$ given by

$$\varepsilon_\beta = \sigma_a \left(\frac{1-\zeta_{2^N}}{1+\zeta_{2^N}} \right) = \frac{1-\zeta_{2^N}^{5^\beta}}{1+\zeta_{2^N}^{5^\beta}}, \quad \beta = 0, \ldots, 2^{N-3}-1,$$

are a system of 2^{N-3} mutually $\mathrm{Gal}(K_N/\mathbf{Q}(i))$-conjugate elements of $(\mathcal{O}_{K_N}^\times)^{\tau=-1}$, by Lemma 2.9. There is nothing stopping us from considering the full Galois orbit of ε_0 under $\mathrm{Gal}(K_N/\mathbf{Q}(i))$, which is given by the

$$\varepsilon_\beta := \frac{1-\zeta_{2^N}^{5^\beta}}{1+\zeta_{2^N}^{5^\beta}} = \sigma_{5^\beta}(\varepsilon_0)$$

for all $\beta \in \mathbf{Z}/2^{N-2} \cong \langle 5 \rangle$. The full set of ε_β are still in $(\mathcal{O}_{K_N}^\times)^{\tau=-1}$ by Lemma 2.9; it's just that only one of these per class mod $5^{2^{N-3}}$ contribute to the sum above, and (precisely because $\varepsilon_\beta \in (\mathcal{O}_{K_N}^\times)^{\tau=-1}$), we have $\varepsilon_{\beta+2^{N-3}} = \tau(\varepsilon_\beta) = \varepsilon_\beta^{-1}$, so this construction produces at most 2^{N-3} independent elements of $(\mathcal{O}_{K_N}^\times/\mu)^{\tau=-1}$. In any event we know from our detailed computations in Section 4 that $(\mathcal{O}_{K_N}^\times/\mu)^{\tau=-1}$ is a free abelian group of rank 2^{N-3}, so $\mathrm{Reg}(K_N/K_{N-1})^{\tau=-1}$ will be a determinant involving exactly 2^{N-3} elements of $(\mathcal{O}_{K_N}^\times/\mu)^{\tau=-1}$. The ε_β, $0 \le \beta < 2^{N-3}$ are typically not a basis of $(\mathcal{O}_{K_N}^\times/\mu)^{\tau=-1}$, but it is of the right size to be a basis of a finite index subgroup — we will prove that this is the case.

More generally, for all $\beta, \beta' \in \mathbf{Z}/2^{N-2} \cong \langle 5 \rangle$, we have

$$\log \|\varepsilon_{\beta+\beta'}\|_\iota = \log \|\sigma_{\beta'}(\varepsilon_\beta)\|_\iota = \log \|\varepsilon_\beta\|_{\iota \circ \sigma_{5^\beta}}$$

which means that all the entries of $(\log\|\varepsilon_\beta\|_{v_1})_{\beta,v}$ where the coordinate v ranges over all 2^{N-3} archimedean places of K_{N-1} and the coordinate β ranges over $0, \ldots, 2^{N-3}-1$ (the same determinant that appeared in Example 4.1(3) for a set of units that forms a basis of $(\mathcal{O}_{K_N}^\times/\mu)^{\tau=-1}$, which the ε_β do not) appear in (5.1), though a direct relationship between the expression in (5.1) to any $\det(\log\|\varepsilon_\beta\|_{v_1})_{\beta,v}$ is not yet obvious. The proof of that relationship will be carried out in Section 5.1. Then, in Section 5.2, we will relate that determinant to $\mathrm{Reg}(K_N/K_{N-1})^-$ by proving that the subgroup of $(\mathcal{O}_{K_N}^\times/\mu)^{\tau=-1}$ spanned by the ε_β is of finite odd index, and in Section 5.3 we will put this together inductively using Proposition 4.6 to conclude the proof of Proposition 2.11.

5.1 From L-values to the regulator of some special units in the minus part of the unit group

The primitive even characters χ of $(\mathbf{Z}/2^N)^\times$ are in bijection with the primitive 2^{N-2}-th roots of unity $\zeta_{2^{N-2}}^x$, $x \in (\mathbf{Z}/2^{N-2})^\times$, so (continuing from (5.1)) we have

$$\prod_{N=4}^{n} \prod_{\substack{\chi \in \widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive even}}} \sum_{\beta=0}^{2^{N-3}-1} \overline{\chi}(5^\beta) \log\|\sigma_{5^\beta}(\varepsilon_0)\|_\iota = \prod_{N=4}^{n} \prod_{x \in (\mathbf{Z}/2^{N-2})^\times} \sum_{\beta=0}^{2^{N-3}-1} \zeta_{2^{N-2}}^{x\beta} \log\|\sigma_{5^\beta}(\varepsilon_0)\|_\iota.$$

In Section 3, a similar inside product appeared, and we wrote it as a norm from $\mathbf{Q}(\zeta_{2^{N-2}})$. We cannot do that in this case because the coefficients $\log\|\sigma_{5^\beta}(\varepsilon_0)\|_\iota$ are typically transcendental quantities. The point of this section is to relate that inside product to the determinant ananalous to the one defining $\mathrm{Reg}(K_N/K_{N-1})^-$ (see Example 4.1(3)) defined with respect to the $\{\varepsilon_\beta\}$ instead of with respect to a basis of $(\mathcal{O}_{K_N}^\times/\mu)^{\tau=-1}$, using a similar-in-spirit technique:

Proposition 5.1. *For all $N \geq 3$,*

$$\prod_{x \in (\mathbf{Z}/2^{N-2})^\times} \sum_{\beta=0}^{2^{N-3}-1} \zeta_{2^{N-2}}^{x\beta} \log\|\sigma_{5^\beta}(\varepsilon_0)\|_\iota = \pm \det(\log\|\varepsilon_\beta\|_{v_1})_{\beta,v}$$

where β runs over the set $\{0, \ldots, 2^{N-3}-1\}$ and v runs over the 2^{N-3} complex places of K_{N-1}.

Proof. Parametrizing the rows and columns by $5^\beta, 5^{\beta'} \in \mathrm{Gal}(K_{N-1}/\mathbf{Q}(i)) \cong \langle 5 \rangle / \langle 5^{2^{N-3}} \rangle$ so that $v_1 = \iota \circ \sigma_{5^{\beta'}}$ and $\varepsilon_\beta = \sigma_{5^\beta}(\varepsilon_0)$, we get

$$\begin{aligned} M &:= (\log\|\varepsilon_\beta\|_{v_1})_{\beta,v} \\ &= (\log\|\sigma_{5^\beta}(\varepsilon_0)\|_{\iota \circ \sigma_{5^{\beta'}}})_{\beta,\beta'} \\ &= (\log\|\varepsilon_{\beta+\beta'})\|_\iota)_{\beta,\beta'} \end{aligned}$$

$$= \begin{pmatrix} \log\|\varepsilon_0\|_\iota & \log\|\varepsilon_1\|_\iota & \log\|\varepsilon_2\|_\iota & \cdots & \log\|\varepsilon_{2^{N-3}-3}\|_\iota & \log\|\varepsilon_{2^{N-3}-2}\|_\iota & \log\|\varepsilon_{2^{N-3}-1}\|_\iota \\ \log\|\varepsilon_1\|_\iota & \log\|\varepsilon_2\|_\iota & \log\|\varepsilon_3\|_\iota & \cdots & \log\|\varepsilon_{2^{N-3}-2}\|_\iota & \log\|\varepsilon_{2^{N-3}-1}\|_\iota & \log\|\varepsilon_{2^{N-3}}\|_\iota \\ \log\|\varepsilon_2\|_\iota & \log\|\varepsilon_3\|_\iota & \log\|\varepsilon_4\|_\iota & \cdots & \log\|\varepsilon_{2^{N-3}-1}\|_\iota & \log\|\varepsilon_{2^{N-3}}\|_\iota & \log\|\varepsilon_{2^{N-3}+1}\|_\iota \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \log\|\varepsilon_{2^{N-3}-3}\|_\iota & \log\|\varepsilon_{2^{N-3}-2}\|_\iota & \log\|\varepsilon_{2^{N-3}-1}\|_\iota & \cdots & \log\|\varepsilon_{2^{N-2}-6}\|_\iota & \log\|\varepsilon_{2^{N-2}-5}\|_\iota & \log\|\varepsilon_{2^{N-2}-4}\|_\iota \\ \log\|\varepsilon_{2^{N-3}-2}\|_\iota & \log\|\varepsilon_{2^{N-3}-1}\|_\iota & \log\|\varepsilon_{2^{N-3}}\|_\iota & \cdots & \log\|\varepsilon_{2^{N-2}-5}\|_\iota & \log\|\varepsilon_{2^{N-2}-4}\|_\iota & \log\|\varepsilon_{2^{N-2}-3}\|_\iota \\ \log\|\varepsilon_{2^{N-3}-1}\|_\iota & \log\|\varepsilon_{2^{N-3}}\|_\iota & \log\|\varepsilon_{2^{N-3}+1}\|_\iota & \cdots & \log\|\varepsilon_{2^{N-2}-4}\|_\iota & \log\|\varepsilon_{2^{N-2}-3}\|_\iota & \log\|\varepsilon_{2^{N-2}-2}\|_\iota \end{pmatrix}$$

$$= \begin{pmatrix} \log\|\varepsilon_0\|_\iota & \log\|\varepsilon_1\|_\iota & \log\|\varepsilon_2\|_\iota & \cdots & \log\|\varepsilon_{2^{N-3}-3}\|_\iota & \log\|\varepsilon_{2^{N-3}-2}\|_\iota & \log\|\varepsilon_{2^{N-3}-1}\|_\iota \\ \log\|\varepsilon_1\|_\iota & \log\|\varepsilon_2\|_\iota & \log\|\varepsilon_3\|_\iota & \cdots & \log\|\varepsilon_{2^{N-3}-2}\|_\iota & \log\|\varepsilon_{2^{N-3}-1}\|_\iota & -\log\|\varepsilon_0\|_\iota \\ \log\|\varepsilon_2\|_\iota & \log\|\varepsilon_3\|_\iota & \log\|\varepsilon_4\|_\iota & \cdots & \log\|\varepsilon_{2^{N-3}-1}\|_\iota & -\log\|\varepsilon_0\|_\iota & -\log\|\varepsilon_1\|_\iota \\ \log\|\varepsilon_3\|_\iota & \log\|\varepsilon_4\|_\iota & \log\|\varepsilon_5\|_\iota & \cdots & -\log\|\varepsilon_0\|_\iota & -\log\|\varepsilon_1\|_\iota & -\log\|\varepsilon_2\|_\iota \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \log\|\varepsilon_{2^{N-3}-3}\|_\iota & \log\|\varepsilon_{2^{N-3}-2}\|_\iota & \log\|\varepsilon_{2^{N-3}-1}\|_\iota & \cdots & -\log\|\varepsilon_{2^{N-3}-6}\|_\iota & -\log\|\varepsilon_{2^{N-3}-5}\|_\iota & -\log\|\varepsilon_{2^{N-3}-4}\|_\iota \\ \log\|\varepsilon_{2^{N-3}-2}\|_\iota & \log\|\varepsilon_{2^{N-3}-1}\|_\iota & -\log\|\varepsilon_0\|_\iota & \cdots & -\log\|\varepsilon_{2^{N-3}-5}\|_\iota & -\log\|\varepsilon_{2^{N-3}-4}\|_\iota & -\log\|\varepsilon_{2^{N-3}-3}\|_\iota \\ \log\|\varepsilon_{2^{N-3}-1}\|_\iota & -\log\|\varepsilon_0\|_\iota & -\log\|\varepsilon_1\|_\iota & \cdots & -\log\|\varepsilon_{2^{N-3}-4}\|_\iota & -\log\|\varepsilon_{2^{N-3}-3}\|_\iota & -\log\|\varepsilon_{2^{N-3}-2}\|_\iota \end{pmatrix}$$

This is the matrix whose determinant we are after, and it involves the same terms $\|\varepsilon_\beta\|_\iota$, $\beta = 0, \ldots, 2^{N-3}-1$, that appear in the left hand side of the statement of this proposition.

For all $x \in (\mathbf{Z}/2^{N-2})^\times$, consider the quantity

$$r(x) := \sum_{\beta=0}^{2^{N-3}-1} \zeta_{2^{N-2}}^{x\beta} \log\|\sigma_{5^\beta}(\varepsilon_0)\|_\iota.$$

We want to show that $\prod_{x\in(\mathbf{Z}/2^{N-2})^\times} r(x) = \pm\det M$, and we will do this by showing that the $r(x)$ are precisely the eigenvalues (with algebraic multiplicity) of a diagonalizable matrix that is the same as M except with some rows swapped or negated.

Indeed, since $\zeta_{2^{N-2}}^{2^{N-3}} = -1$, which we also used in Section 3 to further simplify, we have (with $\|-\|$ always understood to mean $\|-\|_\iota$)

$$\begin{aligned}
r(x) &= \log\|\varepsilon_0\| + \zeta_{2^{N-2}}^{x}\log\|\varepsilon_1\| + \cdots + \zeta_{2^{N-2}}^{(2^{N-3}-3)x}\log\|\varepsilon_{2^{N-3}-3}\| + \zeta_{2^{N-2}}^{(2^{N-3}-2)x}\log\|\varepsilon_{2^{N-3}-2}\| + \zeta_{2^{N-2}}^{(2^{N-3}-1)x}\log\|\varepsilon_{2^{N-3}-1}\| \\
\zeta_{2^{N-2}}^{x} r(x) &= \zeta_{2^{N-2}}^{x}\log\|\varepsilon_0\| + \zeta_{2^{N-2}}^{2x}\log\|\varepsilon_1\| + \cdots + \zeta_{2^{N-2}}^{(2^{N-3}-2)x}\log\|\varepsilon_{2^{N-3}-3}\| + \zeta_{2^{N-2}}^{(2^{N-3}-1)x}\log\|\varepsilon_{2^{N-3}-2}\| + \zeta_{2^{N-2}}^{2^{N-3}x}\log\|\varepsilon_{2^{N-3}-1}\| \\
&= -\log\|\varepsilon_{2^{N-3}-1}\| + \zeta_{2^{N-2}}^{x}\log\|\varepsilon_0\| + \zeta_{2^{N-2}}^{2x}\log\|\varepsilon_1\| + \cdots + \zeta_{2^{N-2}}^{(2^{N-3}-2)x}\log\|\varepsilon_{2^{N-3}-3}\| + \zeta_{2^{N-2}}^{(2^{N-3}-1)x}\log\|\varepsilon_{2^{N-3}-2}\| \\
\zeta_{2^{N-2}}^{2x} r(x) &= \zeta_{2^{N-2}}^{2x}\log\|\varepsilon_0\| + \zeta_{2^{N-2}}^{3x}\log\|\varepsilon_1\| + \cdots + \zeta_{2^{N-2}}^{(2^{N-3}-1)x}\log\|\varepsilon_{2^{N-3}-3}\| + \zeta_{2^{N-2}}^{2^{N-3}x}\log\|\varepsilon_{2^{N-3}-2}\| + \zeta_{2^{N-2}}^{(2^{N-3}+1)x}\log\|\varepsilon_{2^{N-3}-1}\| \\
&= -\log\|\varepsilon_{2^{N-3}-2}\| - \zeta_{2^{N-2}}^{x}\log\|\varepsilon_{2^{N-3}-1}\| + \zeta_{2^{N-2}}^{2x}\log\|\varepsilon_0\| + \zeta_{2^{N-2}}^{3x}\log\|\varepsilon_1\| + \cdots + \zeta_{2^{N-2}}^{(2^{N-3}-1)x}\log\|\varepsilon_{2^{N-3}-3}\| \\
\vdots &= \vdots \\
\zeta_{2^{N-2}}^{(2^{N-3}-3)x} r(x) &= \zeta_{2^{N-2}}^{(2^{N-3}-3)x}\log\|\varepsilon_0\| + \cdots + \zeta_{2^{N-2}}^{(2^{N-2}-6)x}\log\|\varepsilon_{2^{N-3}-3}\| + \zeta_{2^{N-2}}^{(2^{N-2}-5)x}\log\|\varepsilon_{2^{N-3}-2}\| + \zeta_{2^{N-2}}^{(2^{N-2}-4)x}\log\|\varepsilon_{2^{N-3}-1}\| \\
&= -\log\|\varepsilon_3\| - \cdots - \zeta_{2^{N-2}}^{(2^{N-3}-4)x}\log\|\varepsilon_{2^{N-3}-1}\| + \zeta_{2^{N-2}}^{(2^{N-3}-3)x}\log\|\varepsilon_0\| + \zeta_{2^{N-2}}^{(2^{N-3}-2)x}\log\|\varepsilon_1\| + \zeta_{2^{N-2}}^{(2^{N-3}-1)x}\log\|\varepsilon_2\| \\
\zeta_{2^{N-2}}^{(2^{N-3}-2)x} r(x) &= \zeta_{2^{N-2}}^{(2^{N-3}-2)x}\log\|\varepsilon_0\| + \cdots + \zeta_{2^{N-2}}^{(2^{N-2}-5)x}\log\|\varepsilon_{2^{N-3}-3}\| + \zeta_{2^{N-2}}^{(2^{N-2}-4)x}\log\|\varepsilon_{2^{N-3}-2}\| + \zeta_{2^{N-2}}^{(2^{N-2}-3)x}\log\|\varepsilon_{2^{N-3}-1}\| \\
&= -\log\|\varepsilon_2\| - \cdots - \zeta_{2^{N-2}}^{(2^{N-3}-3)x}\log\|\varepsilon_{2^{N-3}-1}\| + \zeta_{2^{N-2}}^{(2^{N-3}-2)x}\log\|\varepsilon_0\| + \zeta_{2^{N-2}}^{(2^{N-3}-1)x}\log\|\varepsilon_1\| \\
\zeta_{2^{N-2}}^{(2^{N-3}-1)x} r(x) &= \zeta_{2^{N-2}}^{(2^{N-3}-1)x}\log\|\varepsilon_0\| + \cdots + \zeta_{2^{N-2}}^{(2^{N-2}-4)x}\log\|\varepsilon_{2^{N-3}-3}\| + \zeta_{2^{N-2}}^{(2^{N-2}-3)x}\log\|\varepsilon_{2^{N-3}-2}\| + \zeta_{2^{N-2}}^{(2^{N-2}-2)x}\log\|\varepsilon_{2^{N-3}-1}\| \\
&= -\log\|\varepsilon_1\| - \cdots - \zeta_{2^{N-2}}^{(2^{N-3}-2)x}\log\|\varepsilon_{2^{N-3}-1}\| + \zeta_{2^{N-2}}^{(2^{N-3}-1)x}\log\|\varepsilon_0\|
\end{aligned}$$

In other words, for all $x \in (\mathbf{Z}/2^{N-2})^\times$, the vector

$$v_x = \begin{pmatrix} 1 \\ \zeta_{2^{N-2}}^{x} \\ \vdots \\ \zeta_{2^{N-2}}^{(2^{N-3}-1)x} \end{pmatrix} \in \mathbf{C}^{2^{N-3}}$$

is an eigenvector for the matrix

$$\begin{pmatrix}
\log\|\varepsilon_0\| & \log\|\varepsilon_1\| & \log\|\varepsilon_2\| & \cdots & \log\|\varepsilon_{2^{N-3}-3}\| & \log\|\varepsilon_{2^{N-3}-2}\| & \log\|\varepsilon_{2^{N-3}-1}\| \\
-\log\|\varepsilon_{2^{N-3}-1}\| & \log\|\varepsilon_0\| & \log\|\varepsilon_1\| & \cdots & \log\|\varepsilon_{2^{N-3}-4}\| & \log\|\varepsilon_{2^{N-3}-3}\| & \log\|\varepsilon_{2^{N-3}-2}\| \\
-\log\|\varepsilon_{2^{N-3}-2}\| & -\log\|\varepsilon_{2^{N-3}-1}\| & \log\|\varepsilon_0\| & \cdots & \log\|\varepsilon_{2^{N-3}-5}\| & \log\|\varepsilon_{2^{N-3}-4}\| & \log\|\varepsilon_{2^{N-3}-3}\| \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-\log\|\varepsilon_2\| & -\log\|\varepsilon_3\| & -\log\|\varepsilon_4\| & \cdots & -\log\|\varepsilon_{2^{N-3}-1}\| & \log\|\varepsilon_0\| & \log\|\varepsilon_1\| \\
-\log\|\varepsilon_1\| & -\log\|\varepsilon_2\| & -\log\|\varepsilon_3\| & \cdots & -\log\|\varepsilon_{2^{N-3}-2}\| & -\log\|\varepsilon_{2^{N-3}-1}\| & \log\|\varepsilon_0\|
\end{pmatrix}$$

of eigenvalue $r(x)$. If you permute the set of rows excluding the first one by reversing the order, and then negate that same collection of rows, this is exactly the same as M, so its determinant is $\pm\det(M)$. By the standard Vandermonde determinant calculation, the set $\{v_x : x \in (\mathbf{Z}/2^{N-2})^\times\}$ is a basis of $\mathbf{C}^{2^{N-3}}$, which means that in fact we have diagonalized this matrix, and thus proven that

$$\prod_x r(x) = \pm\det(M),$$

as desired. $\square$

5.2 From the special units to the full minus part of the unit group

As a consequence of Proposition 5.1, $\det(\log\|\varepsilon_\beta\|_{v_1})_{\beta,v} \neq 0$ (combining (5.1), (2.4), (2.1), and Proposition 2.1, it implies that $\det(\log\|\varepsilon_\beta\|_{v_1})_{\beta,v}$ can be multiplied by some factor to get h_N, which is not zero because it is the class number of a number field). Recall that $\mathrm{Reg}(K_N/K_{N-1})^-$ was defined by $\det(\log\|\mathcal{E}_\beta\|_{v_1})_{\beta,v}$ for any basis $\{\mathcal{E}_\beta\}$ of $(\mathcal{O}_{K_N}^\times/\mu)^{\tau=-1}$, so this already implies:

Corollary 5.2. *The subgroup of $(\mathcal{O}_{K_N}^\times/\mu)^{\tau=-1}$ spanned by the ε_β is of finite index. In particular, the ε_β are a $\mathbf{Z}$-basis of that subgroup, since there are 2^{N-3} of them.*

More specifically, letting

$$[(\mathcal{O}_{K_N}^\times/\mu)^{\tau=-1} : \oplus_{\beta=0}^{2^{N-3}-1}\mathbf{Z}\cdot\varepsilon_\beta] =: d,$$

we have

$$\det(\log\|\varepsilon_\beta\|_{v_1})_{\beta,v} = \pm d\,\mathrm{Reg}(K_N/K_{N-1})^-.$$

It is therefore mandatory for carrying out Weber's strategy to compute the 2-adic valuation of d, which is what we do next:

Proposition 5.3. *The positive integer d is odd.*

Corollary 5.4. $\det(\log\|\varepsilon_\beta\|_{v_1})_{\beta,v} = (\textit{an odd integer})\mathrm{Reg}(K_N/K_{N-1})^-$.

Proof of Proposition 5.3. First, let us explicitly decompose ε_β into a root of unity times a real unit by rationalizing the denominator: if $a = 5^\beta \mod 2^N$, we have

$$\begin{aligned}
\varepsilon_\beta &= \frac{1-\zeta_{2^N}^a}{1+\zeta_{2^N}^a} \\
&= \frac{(1-\zeta_{2^N}^a)(1+\zeta_{2^N}^{-a})}{(1+\zeta_{2^N}^a)(1+\zeta_{2^N}^{-a})} \\
&= \frac{\zeta_{2^N}^{-a}-\zeta_{2^N}^a}{2+\zeta_{2^N}^a+\zeta_{2^N}^{-a}} \\
&= \zeta_{2^N}^{-2^{N-2}a}\frac{\zeta_{2^N}^{(2^{N-2}-1)a}-\zeta_{2^N}^{(2^{N-2}+1)a}}{2+\zeta_{2^N}^a+\zeta_{2^N}^{-a}} \\
&= \zeta_{2^N}^{-2^{N-2}a}\frac{\zeta_{2^N}^{(2^{N-2}-1)a}+\zeta_{2^N}^{(-2^{N-2}+1)a}}{2+\zeta_{2^N}^a+\zeta_{2^N}^{-a}}
\end{aligned}$$

where in the last line we are using $\zeta_{2^N}^{2^{N-1}} = -1$ to get a real[3] number in the numerator. We can conveniently simplify this remaining real unit to get a trigonometric function, which will be convenient for the purposes of our index computation:

$$\begin{aligned}
\frac{\zeta_{2^N}^{(2^{N-2}-1)a}+\zeta_{2^N}^{(-2^{N-2}+1)a}}{2+\zeta_{2^N}^a+\zeta_{2^N}^{-a}} &= \frac{2\cos\left(2\pi\frac{(2^{N-2}-1)a}{2^N}\right)}{2+2\cos\left(2\pi\frac{a}{2^N}\right)} \\
&= \frac{\cos\left(-2\pi\frac{a}{2^N}+\frac{\pi a}{2}\right)}{1+\cos\left(2\pi\frac{a}{2^N}\right)} \\
&= \frac{\sin\left(2\pi\frac{a}{2^N}\right)}{1+\cos\left(2\pi\frac{a}{2^N}\right)} \\
&= \frac{\sin\left(2\pi\frac{a}{2^N}\right)}{\cos^2\left(\frac{\pi a}{2^N}\right)},
\end{aligned}$$

[3] There is no difference between "real" and "totally real" here because K_N is CM.

where the second to last line is using the fact that $a = 5^\beta \equiv 1 \mod 4$, and the last line is the double-angle formula for cosine.

In fact, this is the same as the more pleasant $\tan(\pi a/2^N)$, but the current presentation will ultimately be more convenient.

Since we only care about the index of the span as a submodule of $(\mathcal{O}_{K_N}^\times/\mu)^{\tau=-1}$, for the purposes of proving this result, we can replace ε_β with the quantity which we call by the same name

$$\varepsilon_\beta := \frac{\sin\left(2\pi\frac{a}{2^N}\right)}{\cos^2\left(\frac{\pi a}{2^N}\right)} \in (\mathcal{O}_{K_N^+}^\times/\mu)^{\tau=-1}.$$

In fact, we will prove that the set $\{\varepsilon_\beta : \beta = 0, \dots, 2^{N-3}-1\} \subset (K_N^+)^\times$ has the following property:

(*) For all elements $x \in (K_N^+)^\times$ of the form $x = \prod_{\beta=0}^{2^{N-3}-1} \varepsilon_\beta^{e_\beta}$, if all of the embeddings $(\iota \circ \gamma)(x) \in \mathbf{R}^\times$, $\gamma \in \mathrm{Gal}(K_N/\mathbf{Q})$, have the same sign, then all the e_β are even.

The property (*) already implies that d is odd. Indeed, suppose that we have proved property (*) for $\{\varepsilon_\beta\}$, and assume for the sake of contradiction that $d = 2d'$. If $\mathcal{E} \in (\mathcal{O}_{K_N^+}^\times/\{\pm 1\})^{\tau=-1} \subset (\mathcal{O}_{K_N}^\times/\mu)^{\tau=-1}$, then there are integers e_β such that

$$\mathcal{E}^d = \pm \prod_{\beta=0}^{2^{N-3}-1} \varepsilon_\beta^{e_\beta}.$$

Since $\mathcal{E}^d = (\mathcal{E}^{d'})^2$ is the square of a real unit in K_N^+, all of its embeddings (which are real *a priori* because we have arranged to work only with real units) are positive, hence of the same sign. So the same is true of the product $\prod_\beta \varepsilon_\beta^{e_\beta}$, and property (*) tells us that the e_β's are all even. This means that d could have been replaced with $d' = d/2$, which means d was not the index in the first place — hence the index is odd.

It remains to prove the property (*) for the specific set $\{\varepsilon_\beta\}$. The numerator and denominator of

$$\varepsilon_\beta = \frac{\sin\left(2\pi\frac{a}{2^N}\right)}{\cos^2\left(\frac{\pi a}{2^N}\right)}$$

are elements of K_N^+. Moreover, the denominator is even the square of an element of K_N^+, so all of its embeddings into $\mathbf{R} \subset \mathbf{C}$ are positive. Therefore, it suffices to prove property (*) for the alternative set of elements

$$\delta_\beta := \sin\left(2\pi\frac{a}{2^N}\right) = \frac{\zeta_{2^N}^a - \zeta_{2^N}^{-a}}{2} \in K_N^+,$$

where we are reminded that $a := 5^\beta$. It is no longer the case (nor is it necessary for what we need) that we are dealing with units or even algebraic integers in K_N, and the action of τ needs to be refigured as well. For all $\sigma_{a'} \in \mathrm{Gal}(K_N/\mathbf{Q}(i))$, $a' = 5^{\beta'} \in \langle 5 \rangle$ we have

$$\sigma_{a'}(\delta_\beta) = \delta_{\beta+\beta'},$$

just like for the ε_β's, i.e., the δ_β are just the Galois conjugates by $\sigma_{5^\beta} \in \mathrm{Gal}(K_N/\mathbf{Q})$ of δ_0. Once again, it is legal to consider the entire orbit of δ_0 under all of $\mathrm{Gal}(K_N/\mathbf{Q}(i)) = \mathrm{Gal}(K_N^+/\mathbf{Q}) = \langle 5 \rangle$ instead of only the representatives $\beta = 0, \dots, 2^{N-3}-1$ for the quotient by $\langle 5^{2^{N-3}} \rangle$. But if we consider the action of τ, we see that

$$\tau(\delta_\beta) = \sigma_{5^{2^{N-3}}}(\delta_\beta) = \delta_{\beta+2^{N-3}} = \sin\left(2\pi\frac{a5^{2^{N-3}}}{2^N}\right) = -\delta_\beta, \tag{5.2}$$

since $5^{2^{N-3}} \equiv 1 \mod 2^{N-1}$ and $a := 5^\beta \equiv 1 \mod 4$. This is completely different from how τ acts on ε_β, which is by multiplicative inverse instead of additive inverse.

Finally, we will find it useful to fold $\langle 5\rangle/\langle 5^{2^{N-3}}\rangle$ in half again, by considering the action of $5^{2^{N-4}}$. Note that $5^{2^{N-4}} \equiv 1 + 2^{N-2} + 2^{N-1} \mod 2^N$, so

$$\delta_{\beta+2^{N-4}} = \sin\left(2\pi \frac{a5^{2^{N-4}}}{2^N}\right) = \sin\left(2\pi \frac{a}{2^N} + \frac{3\pi a}{2}\right) = -\cos\left(2\pi \frac{a}{2^N}\right)$$

and thus

$$\delta_\beta \delta_{\beta+2^{N-4}} = -\sin\left(2\pi \frac{a}{2^N}\right)\cos\left(2\pi \frac{a}{2^N}\right) = -\frac{1}{2}\sin\left(2\pi \frac{a}{2^{N-1}}\right) \tag{5.3}$$

by the double angle formula. Importantly, the right hand side of (5.3) is exactly the same as δ_β except considered with respect to K_{N-1} instead of K_N, which we will now use to prove (*) for the δ_β's by induction on N. We take $N = 3$ as the base case. Then $K_N = \mathbf{Q}(\zeta_8)$ and $K_N^+ = \mathbf{Q}(\cos(\pi/4)) = \mathbf{Q}(\sqrt{2})$. We have $2^{N-3} = 1$, so there is only one δ_β to consider, which is

$$\delta_0 = \sin\frac{\pi}{4} = \sqrt{2}/2.$$

If $x = \delta_0^e$, $e \in \mathbf{Z}$, then the two embeddings of x in $\mathbf{R} \subset \mathbf{C}$ according to the two genuinely different embeddings of K_3 into $\mathbf{C}$ are $(\pm\sqrt{2}/2)^e$. These have the same sign if and only if e is even, thus establishing the base case. Now assume that property (*) is established for the δ_β's defined with respect to the field K_{N-1}, and by induction it suffices to prove it for those defined with respect to K_N.

To make the notation clear, we add N as a parameter, setting

$$\delta_{\beta,N} := \sin\left(2\pi \frac{a}{2^N}\right),$$

so that (5.3) becomes the more usable

$$\delta_{\beta,N}\delta_{\beta+2^{N-4},N} = -\frac{1}{2}\delta_{\beta,N-1}. \tag{5.4}$$

Suppose that all the embeddings of

$$x := \prod_{\beta=0}^{2^{N-3}-1} \delta_\beta^{e_\beta}$$

have the same sign. We intend to prove that the e_β's are all even. Since $\sigma_{5^{2^{N-4}}}(\delta_\beta) = \delta_{\beta+2^{N-4}}$, the element

$$y := \sigma_{5^{2^{N-4}}}(x) = \prod_{\beta=0}^{2^{N-3}-1} \delta_{\beta+2^{N-4}}^{e_\beta}$$

must also have the property that all of its embeddings have the same sign. (in fact the embeddings of y are exactly the same as a set to those of x since y is defined as a Galois conjugate of x, which just permutes the embeddings). Since the infinite places of K_N are by definition ring homomorphisms, $xy \in K_N^+$ also has the property that all of its embeddings have the same sign. Using (5.4) followed by (5.2) (applied to $N-1$ instead of N), that element is equal to

$$\begin{aligned} xy &= \left(\prod_{\beta=0}^{2^{N-3}-1} \delta_\beta^{e_\beta}\right)\left(\prod_{\beta=0}^{2^{N-3}-1} \delta_{\beta+2^{N-4}}^{e_\beta}\right) \\ &= \prod_{\beta=0}^{2^{N-3}-1} (\delta_{\beta,N}\delta_{\beta+2^{N-4},N})^{e_\beta} \\ &= \pm(\text{some power of } 2) \prod_{\beta=0}^{2^{N-3}-1} \delta_{\beta,N-1}^{e_\beta} \end{aligned}$$

$$= \pm(\text{some power of } 2) \prod_{\beta=0}^{2^{N-4}-1} \delta_{\beta,N-1}^{e_\beta + e_{\beta+2^{N-4}}}$$

where the ± 1 on the last line might be different from the ± 1 on the line before, because (5.2) for $N-1$ says that that $\delta_{\beta+2^{N-4},N-1} = -\delta_{\beta,N-1}$. Since (*) has been assumed, as an inductive hypothesis, for the $\delta_{\beta,N-1} \in K_{N-1}$, it follows that

$$e_\beta \equiv e_{\beta+2^{N-4}} \mod 2$$

for all $\beta = 0, \dots, 2^{N-4}-1$.

Using this congruence along with (5.4), we obtain

$$\begin{aligned}
\prod_{\beta=0}^{2^{N-4}-1} \delta_{\beta,N-1}^{e_\beta} &= \pm(\text{some power of } 2) \prod_{\beta=0}^{2^{N-4}-1} \delta_{\beta,N}^{e_\beta}\delta_{\beta+2^{N-4},N}^{e_\beta} \\
&= \left(\prod_{\beta=0}^{2^{N-4}-1} \delta_{\beta,N}^{e_\beta}\right)\left(\prod_{\beta=2^{N-4}}^{2^{N-3}-1} \delta_{\beta,N}^{e_{\beta-2^{N-4}}}\right) \\
&= \left(\prod_{\beta=0}^{2^{N-4}-1} \delta_{\beta,N}^{e_\beta}\right)\left(\prod_{\beta=2^{N-4}}^{2^{N-3}-1} \delta_{\beta,N}^{e_\beta}\right)\alpha^2 \\
&= x\alpha^2
\end{aligned}$$

for some $\alpha \in (K_N^+)^\times$. The congruence $e_\beta \equiv e_{\beta+2^{N-4}}$ is being used in the second to last line to deduce the existence of α. Since the embeddings of α^2 are all positive (they are squares of nonzero real numbers since the embeddings are ring homomorphisms), it follows that

$$z := \prod_{\beta=0}^{2^{N-4}-1} \delta_{\beta,N-1}^{e_\beta}$$

has the property that all of its embeddings have the same sign (since x was assumed to have this property and we just saw that multiplication by α^2 doesn't affect whether that property holds).

Making use one last time of the inductive hypothesis that (*) holds for the $\delta_{\beta,N-1} \in K_{N-1}^+$, we conclude that the e_β, $\beta = 0, \dots, 2^{N-4}-1$, are all even. We already saw that $e_\beta \equiv e_{\beta+2^{N-4}} \mod 2$, so this is already enough to deduce that all of the e_β are even, and thus that (*) holds for $\delta_{\beta,N} \in K_N$, as desired. $\square$

5.3 Inductively recovering the regulator from those of the minus parts in the tower

Now we put everything together to deduce Proposition 2.11. The left hand side of Proposition 2.11 is

$$\begin{aligned}
\prod_{N=4}^{n} \prod_{\substack{\chi \in \widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive even}}} \sum_{a \in (\mathbf{Z}/2^n)^\times} \overline{\chi}(a) \log(1-\zeta_{2^n}^a) &\overset{(2.7)}{=} \prod_{N=4}^{n} \prod_{\substack{\chi \in \widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive even}}} \sum_{\{a, 5^{2^{N-3}}a\} \in \langle 5\rangle/\langle 5^{2^{N-3}}\rangle} \overline{\chi}(a) \log \frac{(1-\zeta_{2^N}^a)(1-\zeta_{2^N}^{-a})}{(1+\zeta_{2^N}^a)(1+\zeta_{2^N}^a)} \\
&\overset{(2.8)}{=} \prod_{N=4}^{n} \prod_{\substack{\chi \in \widehat{(\mathbf{Z}/2^N)^\times} \\ \chi \text{ primitive even}}} \sum_{\beta=0}^{2^{N-3}-1} \overline{\chi}(5^\beta) \log\|\sigma_{5^\beta}(\varepsilon_0)\|_\iota \\
&= \prod_{N=4}^{n} \prod_{x \in (\mathbf{Z}/2^{N-2})^\times} \sum_{\beta=0}^{2^{N-3}-1} \zeta_{2^{N-2}}^{x\beta} \log\|\sigma_{5^\beta}(\varepsilon_0)\|_\iota
\end{aligned}$$

$$
\begin{aligned}
\text{(Proposition 5.1)} \quad &= \pm \prod_{N=4}^{n} \det(\log\|\varepsilon_\beta\|_{v_1})_{\beta,v} \\
\text{(Corollary 5.4)} \quad &= \pm \prod_{N=4}^{n} (\text{some odd integer depending on } N)\mathrm{Reg}(K_N/K_{N-1})^- \\
&= (\text{some odd integer}) \prod_{N=4}^{n} \mathrm{Reg}(K_N/K_{N-1})^- \\
\text{(Proposition 4.6)} \quad &= (\text{some odd integer}) 2^{\sum_{N=4}^{n} -\sigma_N} \frac{\mathrm{Reg}(K_n)}{\mathrm{Reg}(K_3)} \\
&= (\text{some odd integer}) 2^{-\Sigma} \frac{\mathrm{Reg}(K_n)}{\mathrm{Reg}(K_3)},
\end{aligned}
$$

where the σ_N and hence $\Sigma := \sum_{N=4}^{n} \sigma_N$ is known to be nonnegative. Moreover, $K_3 = \mathbf{Q}(\zeta_8)$, which has unit group of rank 1, soits regulator is twice the regulator of its maximal real subfield $\mathbf{Q}(\sqrt{2})$, which is $\log(\sqrt{2}+1)$ by the basic theory of Pell equation. We conclude that the left hand side of Proposition 2.11 is

$$
(\text{some odd integer}) \frac{2^{-1-\Sigma}}{\log(\sqrt{2}+1)} \mathrm{Reg}(K_n),
$$

which is exactly the right hand side in the claim of Proposition 2.11, which we have therefore proved.

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