Theorie der abel'schen zahlkörper
(Theory of abelian number fields)

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(document translated and lightly modified by Kenz Kallal)

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Contents

Introduction

I have used Google Gemini to translate the second of the three parts of Weber's paper *Theorie der abel'schen zahlkörkper*, Acta Mathematica, vol. 8, p 193-263 (1886) into English in this more readable format. Other than occasionally fixing incorrect formulas that seem mostly due to OCR errors, I have found that Gemini is extremely accurate and saves me a lot of useless hassle with my limited German proficiency. For the convenience of the reader, I have added at most a few sentences of details in the places where Weber makes leaps that were not immediately obvious to me. Other than this and some very limited modernizing of what is now standard notation, I have kept the document much the same as Weber's original publication, even when our modern language would allow us to skip large sections of it.

Section I of Weber's paper mostly deals with standard material about the basic theory of cyclotomic fields — I felt there was no need to translate this. Section II (the only section that is presently translated by me) is entirely about Weber's proof of the fact that the cyclotomic field $\mathbf{Q}(\zeta_{2^m})$ has odd class number. Having been published in 1886, this was carried out without anything close to the content of class field theory being known: instead, Weber uses the analytic class number formula, together with detailed computations of regulators corresponding to the two characters of $\mathrm{Gal}(\mathbf{Q}(\zeta_{2^m})/\mathbf{Q}(\zeta_{2^{m-1}}))$. These computations should be viewed as a precursor to some ideas in Iwasawa theory from the second half of the 20th century. I presented a very brief sketch of this proof during a talk I gave in the spring 2025 Skinner–Venkatesh number theory learning seminar on the historical development of Iwasawa theory, where I also presented Iwasawa's generalization (*A note on class numbers of algebraic number fields*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, vol. 20, p. 257-258, 1956) of this fact to arbitrary p-power cyclotomic extensions, the proof of which is a short exercise in applying class field theory:

Theorem (Iwasawa 1956)**.** *Let p be a rational prime. The class number of $\mathbf{Q}(\zeta_p)$ is divisible by p if and only if the class number of $\mathbf{Q}(\zeta_{p^m})$ is divisible by p.*

This document contains the full detail of Weber's proof of the $p = 2$ case, translated essentially directly into English.

Section III of Weber's paper was his original slightly flawed proof of the Kronecker–Weber theorem. I have not provided a translation yet (though I expect to do so in the future given that I also presented that proof in a previous instance of the Skinner–Venkatesh seminar).

II. On the Number of Ideal Classes and the Units in Cyclotomic Fields Whose Order is a Power of 2.

In his fundamental works on the ideal prime factors of complex numbers, Kummer determined the number of ideal classes in cyclotomic fields of prime order.

In the present work, an investigation into the number of ideal classes will be carried out following the same principles, for the simplest case in which the order is a composite number: namely when it is a power of 2. This is necessary for later application to the theory of abelian fields, but can also claim some independent interest.

1 First expression for the number of ideal classes.

Let λ be a positive integer greater than 2, and we set

$$m = 2^{\lambda} \quad \nu = 2^{\lambda-2} \quad \mu = 2^{\lambda-3} \quad \varphi(m) = 2^{\lambda-1}. \tag{1.1}$$

Let r be a primitive m^{th} root of unity, i.e., a root of the irreducible equation

$$x^{2^{\lambda-1}} + 1 = 0 \tag{1.2}$$

and Ω_m or Ω_λ or simply Ω the complete cyclotomic field of order m.

If we denote by $\mathfrak{a}$ all ideals of the field Ω, and by $N(\mathfrak{a})$ their norms, then the determination of the number h of ideal classes of the field Ω depends on the determination of the limit

$$\lim_{s=1}(s-1)\sum \frac{1}{N(\mathfrak{a})^s} = gh, \tag{1.3}$$

where g is a numerical factor, whose definition and determination will be discussed further below.

The sum occurring in (1.3) can first be transformed into an infinite product, which extends over all prime ideals $\mathfrak{p}$ of the field Ω:

$$\sum \frac{1}{N(\mathfrak{a})^s} = \prod \frac{1}{1 - N(\mathfrak{p})^{-s}}. \tag{1.4}$$

Now among the ideals $\mathfrak{p}$ is the principal ideal $(1-r)$, whose norm is 2. Furthermore, if p is an odd prime number which is of order 2^k in $(\mathbf{Z}/m)^\times$ (here $(\mathbf{Z}/m)^\times = (\mathbf{Z}/2^\lambda)^\times$ is isomorphic to $(\mathbf{Z}/2) \times (\mathbf{Z}/2^{\lambda-2})$, so $0 \le k \le \lambda - 2$), then (p) splits into $2^{\lambda-k-1}$ distinct prime ideals of inertial degree 2^k, all of which have the same norm p^{2^k}. Therefore,

$$\sum \frac{1}{N(\mathfrak{a})^s} = \frac{1}{1-2^{-s}} \prod \frac{1}{(1-p^{-s2^k})^{2^{\lambda-k-1}}}, \tag{1.5}$$

where the product $\prod$ extends over all *odd* prime numbers p.

We now consider the group $(\mathbf{Z}/m)^\times$ and its characters χ. If n is odd of order 2^k in $(\mathbf{Z}/m)^\times$, then the powers of n form a subgroup of size 2^k, and among the characters χ there are exactly $2^{\lambda-k-1}$ that satisfy the condition

$$\chi(n) = 1.$$

Fix $n \in (\mathbf{Z}/m)^\times$ of order 2^k. The characters χ mod m are partitioned into 2^k sets $\{\chi : \chi(n) = \zeta\}$, where ζ ranges over the 2^k-th roots of unity. Moreover, since all $\chi(n)$ (because $n^{2^k} \equiv 1 \pmod{m}$) are 2^k-th roots of unity, it follows that:

Among the $2^{\lambda-1}$ values $\chi(n)$ as χ ranges over the characters of $(\mathbf{Z}/m)^\times$, every 2^k-th root of unity appears, and each exactly $2^{\lambda-k-1}$ times. We can also write this result, if x signifies a formal variable, as:

$$(1-x^{2^k})^{2^{\lambda-k-1}} = \prod_{\chi}[1-\chi(n)x], \tag{1.6}$$

where the product extends over all characters χ of $(\mathbf{Z}/m)^\times$, and n denotes an arbitrary element of $(\mathbf{Z}/m)^\times$ of order 2^k. If we set $x = p^{-s}$ and $n = p$, then (1.5) takes the form:

$$\sum \frac{1}{N(\mathfrak{a})^s} = \frac{1}{1-2^{-s}} \prod_{p\neq 2} \prod_{\chi \in \widehat{(\mathbf{Z}/m)^\times}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}. \tag{1.7}$$

If one expands, similarly to (1.5), the individual factors on the right side of (1.7) into powers of p^{-s} and then combines them by multiplication, it follows as before that

$$\sum \frac{1}{N(\mathfrak{a})^s} = \frac{1}{1-2^{-s}} \prod_{\chi \in \widehat{(\mathbf{Z}/m)^\times}} \sum_n \frac{\chi(n)}{n^s}, \tag{1.8}$$

where now the sum on the right is over all odd numbers n and the product over all characters χ. If this is substituted into (1.3), the limit can be taken; namely,

$$\lim_{s=1} \frac{s-1}{1-2^{-s}} \sum \frac{1}{n^s} = 1 \tag{1.9}$$

$$\lim_{s=1} \sum \frac{\chi(n)}{n^s} = \sum_n \frac{\chi(n)}{n} = L(1,\chi), \tag{1.10}$$

if in (1.10) the trivial character is excluded and on the right the infinite series are arranged according to the magnitude of the numbers n. Accordingly, from (1.3), *if the trivial character is now excluded* from the product $\prod_\chi$, we obtain

$$gh = \prod_{\chi\neq 1} \sum_{n \text{ odd}} \frac{\chi(n)}{n} = \prod_{\chi \neq 1} L(1,\chi). \tag{1.11}$$

2 Continuation.

Recall the decomposition $(\mathbf{Z}/m)^\times = \langle -1\rangle \times \langle 5\rangle \cong (\mathbf{Z}/2) \times (\mathbf{Z}/2^{\lambda-2})$. In particular, for all odd n, there are unique indices $\alpha \in (\mathbf{Z}/2), \beta \in (\mathbf{Z}/2^{\lambda-2})$ such that

$$(-1)^\alpha 5^\beta \equiv n (\mathrm{mod}\ 2^\lambda), \tag{2.1}$$

so the characters $\chi(n)$ are obtained as follows. Let α, β be the indices of n, and let $\epsilon = \pm 1$, and θ be *any* $2^{\lambda-2}$-th root of unity. Then in (1.11) χ runs over all functions of the form

$$\chi(n) = \epsilon^\alpha \theta^\beta \tag{2.2}$$

with the sole exception of $\epsilon = +1$, $\theta = +1$ (which corresponds to the trivial character, which is excluded). Here the data of n is equivalent to the data of (α, β) and the data of χ is equivalent to the data of (ϵ, θ). The sums appearing in (1.11) then fall into different classes, depending on the values of θ, ϵ. Namely, if θ runs through all *primitive* $2^{\lambda-2}$-th roots of unity, we set

$$\begin{aligned} P_\lambda &= \prod_\theta \sum_n \frac{\chi(n)}{n}, \\ Q_\lambda &= \prod_\theta \sum_n \frac{\chi(n)}{n}; \end{aligned} \tag{2.3}$$

for $\epsilon = +1$ and $\epsilon = -1$ (i.e. $\chi(-1) = 1, -1$) respectively. Since the indices for modulus 2^k are congruent to those for modulus 2^λ if $k < \lambda$, all factors occurring in (1.11) are obtained by substituting $3, 4, \ldots, \lambda$ for λ in P_λ and $2, 3, 4, \ldots, \lambda$ in Q_λ. Therefore,

$$gh = Q_2 P_3 Q_3 P_4 Q_4 \cdots P_\lambda Q_\lambda. \tag{2.4}$$

For Q_2 one directly obtains the value

$$Q_2 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4} \tag{2.5}$$

and we can therefore assume $\lambda \geq 3$ for further calculation of P_λ, Q_λ. To determine the sums P_λ, Q_λ, we set

$$\sum_t \chi(t) x^t = f(x), \tag{2.6}$$

where t *runs through all odd positive integers smaller than* m, so that, since

$$\chi(t + 2^{\lambda-1}) = -\chi(t)$$

(as χ is primitive), $f(x)$ vanishes as soon as a non-primitive m^{th} root of unity or zero is set for x (in fact those are all the roots of the polynomial f). Then we obtain

$$\sum_n \frac{\chi(n)}{n} = \int_0^1 \frac{f(x)dx}{x(1-x^m)} = -\frac{1}{m} \sum_t f(r^t) \log(1 - r^{-t}), \tag{2.7}$$

where the logarithms are to be taken such that their imaginary parts lie in the interval $\pm \frac{\pi}{2} i$. In (2.7), the first equality is by integrating one term at a time and using geometric series, and the second equality is by partial fraction decomposition[1]. The value of f at the primitive m-th root of unity r is the Gauss sum

$$f(r) = \sum_t \epsilon^a \theta^\beta r^t =: (\epsilon, \theta, r) = \tau(\chi) \tag{2.8}$$

where α, β are the indices of t. Note that

$$f(r^t) = \epsilon^{-\alpha} \theta^{-\beta} f(r). \tag{2.9}$$

[1]Nowadays (2.7) would be proved more concisely using the power series expansion of log and Fourier analysis on finite abelian groups. At least this it the first time I am seeing this particular justification for (2.7). Presumably these are ultimately the same thing under the hood.

And consequently

$$\sum \frac{\chi(n)}{n} = -\frac{1}{m}(\epsilon,\theta,r)\sum_t \epsilon^{-\alpha}\theta^{-\beta}\log(1-r^{-t}). \tag{2.10}$$

This formula also holds for $\lambda = 2$ and gives as above

$$Q_2 = \frac{\pi}{4}. \tag{2.11}$$

Furthermore, from (2.10), for $\lambda = 3$ we obtain

$$Q_3 = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \cdots = \frac{\pi}{2\sqrt{2}} \tag{2.12}$$

$$P_3 = 1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \frac{1}{11} - \cdots = \frac{1}{2\sqrt{2}}\log\frac{\sqrt{2}+1}{\sqrt{2}-1} = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1). \tag{2.13}$$

If $\lambda > 3$, the factors in the products P_λ, Q_λ always appear in pairs, where two factors corresponding to the values θ, θ^{-1} determining χ form a pair. In the calculation of products of these pairs, one must use the standard Gauss sum formula[2]

$$(\epsilon,\theta,r)(\epsilon,\theta^{-1},r) = \tau(\chi)\tau(\overline{\chi}) = \epsilon m. \tag{2.14}$$

3 Determination of the factors Q_λ.

The calculation of P_λ and Q_λ, $(\lambda \geq 4)$ proceeds differently. We begin with Q_λ, and for simplification we set

$$\varphi(\theta) = \frac{-m}{2\pi i}\sum_t (-1)^\alpha \theta^{-\beta}\log(1-r^{-t}), \tag{3.1}$$

so that according to (2.3), (2.10), (2.14):

$$Q_\lambda = \frac{(2\pi)^\mu}{m^{3\cdot 2^{\lambda-4}}}\prod_\theta \varphi(\theta). \tag{3.2}$$

[2] This formula, which is exactly as strong as knowing the magnitude, but not the sign, of the Gauss sum, is proven earlier in Weber's paper via the usual direct means: substituting $x = n/n'$, we have

$$(\epsilon,\theta,r)(\epsilon,\theta^{-1},r) = \sum_{n,n'}\chi(n)\chi((n')^{-1})r^{n+n'} = \sum_{x,n'}\chi(x)r^{n'x+n'}.$$

The sum with respect to $n' \in (\mathbf{Z}/m)^\times$, $\sum_{n'} r^{n'(x+1)}$, is a finite geometric series

$$r^{x+1} + r^{3(x+1)} + r^{5(x+1)} + \cdots + r^{(2^\lambda-1)(x+1)}$$

which is $r^{x+1}\frac{1-r^{m(x+1)}}{1-r^{2(x+1)}} = 0$ if $x+1$ is not divisible by $2^{\lambda-1}$. On the other hand if $x+1$ is divisible by $2^{\lambda-1}$ then this is just r^{x+1} times the sum of $2^{\lambda-1}$ copies of 1, i.e. it is $-2^{\lambda-1}$ if $x+1 = 2^{\lambda-1}$ (equivalent to x having indices $\alpha = 1, \beta = 2^{\lambda-3}$); and it is $2^{\lambda-1}$ if $x+1 = 2^\lambda$ (equivalent to x having indices $\alpha = 1$, $\beta = 0$). Summing over x, we obtain the desired result at once: the only two terms that contribute are $(\alpha,\beta) = (1, 2^{\lambda-3}), (1,0)$ which yields $\epsilon(-\theta^\mu + 1)2^{\lambda-1} = \epsilon m$.

To calculate $\varphi(\theta)$, we note that by interchanging t with $m-t$, the index α of t changes to $\alpha+1$, while β remains unchanged, and therefore we also obtain for $\varphi(\theta)$

$$\varphi(\theta) = \frac{m}{2\pi i}\sum_t (-1)^\alpha \theta^{-\beta}\log(1-r^t) \tag{3.3}$$

and by adding both expressions (3.3) and (3.1),

$$\varphi(\theta) = \frac{m}{4\pi i}\sum_t (-1)^\alpha \theta^{-\beta}\log(-r^t), \tag{3.4}$$

where the now purely imaginary logarithm lies in the interval $\pm\pi i$. We can therefore set

$$\begin{aligned} -r^t &= e^{\pi i\left(\frac{2t}{m}-1\right)} \\ \log(-r^t) &= \pi i\left(\frac{2t}{m}-1\right), \end{aligned} \tag{3.5}$$

and t is then, as above, the *smallest positive remainder* of

$$(-1)^\alpha 5^\beta \pmod{m}.$$

Then, combining (3.5) and (3.4), since $\sum_t (-1)^\alpha\theta^{-\beta} = \sum_{t\in(\mathbf{Z}/m)^\times}\chi(t)$ vanishes (as χ is assumed to be nontrivial):

$$\varphi(\theta) = \frac{1}{2}\sum_t (-1)^\alpha \theta^{-\beta} t. \tag{3.6}$$

Two values of t that correspond to the same β, but to different values of α, sum to m, and since $\sum_t \theta^\beta$ also vanishes, it is

$$\varphi(\theta) = \sum_{\beta=0}^{\nu-1}\theta^{-\beta} t, \tag{3.7}$$

if t denotes the smallest positive remainder of $5^\beta (\bmod\ m)$. However, the expression (3.7) can be further simplified. Namely,

$$\varphi(\theta) = \sum_{\beta=0}^{\mu-1}\theta^{-\beta} t + \sum_{\beta=\mu}^{\nu-1}\theta^{-\beta} t. \tag{3.8}$$

In the last sum, substitute $\beta+\mu$ in place of β and denote the *smallest positive remainder* of $5^{\beta+\mu} (\bmod\ m)$ by t_1. Then it follows, since $\theta^\mu = -1$, that

$$\varphi(\theta) = \sum_{\beta=0}^{\mu-1}\theta^{-\beta}(t-t_1). \tag{3.9}$$

By definition,

$$t_1 - t \equiv 5^\beta(5^\mu - 1) (\bmod\ m), \tag{3.10}$$

thus $t_1 \equiv t$ modulo $2^{\lambda-1}$; but not modulo 2^λ. We therefore have

$$t = t_1 + \epsilon_\beta 2^{\lambda-1} \tag{3.11}$$

and $\epsilon_\beta = \pm 1$, depending on whether $t \geq 2^{\lambda-1}$, since both t and t_1 are positive and smaller than m. We thus obtain from (3.9)

$$\varphi(\theta^{-1}) = 2^{\lambda-1}(\epsilon_0 + \epsilon_1\theta + \cdots + \epsilon_{\mu-1}\theta^{\mu-1}) \tag{3.12}$$

and

$$(1-\theta)\varphi(\theta^{-1}) = m\left(\frac{\epsilon_0 + \epsilon_{\mu-1}}{2} + \frac{\epsilon_1 - \epsilon_0}{2}\theta + \cdots + \frac{\epsilon_{\mu-1} - \epsilon_{\mu-2}}{2}\theta^{\mu-1}\right). \tag{3.13}$$

The expression

$$\psi(\theta) = \frac{\epsilon_0 + \epsilon_{\mu-1}}{2} + \frac{\epsilon_1 - \epsilon_0}{2}\theta + \cdots + \frac{\epsilon_{\mu-1} - \epsilon_{\mu-2}}{2}\theta^{\mu-1} \tag{3.14}$$

is now an *algebraic integer* in the field $\Omega_{\lambda-2}$. Letting $\mathrm{N}_{\lambda-2}$ denote the norm from that field to $\mathbf{Q}$, and setting

$$\mathrm{N}_{\lambda-2}\psi(\theta) = a_\lambda, \quad \mathrm{N}_{\lambda-2}(1-\theta) = 2 \tag{3.15}$$

then according to (3.2),

$$Q_\lambda = \frac{\pi^\mu a_\lambda}{2\nu^{2^{\lambda-4}}} \tag{3.16}$$

(because taking the norm of $\psi(\theta)$ is the same as multiplying it over all choices of θ a primitive $2^{\lambda-2}$-th root of unity). However, if one replaces the $\epsilon_0 + \epsilon_{\mu-1}$ in (3.14) with $\epsilon_0 - \epsilon_{\mu-1}$, then the resulting expression would be in $(1-\theta)$, as

$$\begin{aligned}&\frac{\epsilon_0 - \epsilon_{\mu-1}}{2} + \frac{\epsilon_1 - \epsilon_0}{2}\theta + \cdots + \frac{\epsilon_{\mu-1} - \epsilon_{\mu-2}}{2}\theta^{\mu-1}\\ &= (1-\theta)\left(\frac{\epsilon_0 - \epsilon_{\mu-1}}{2} + \frac{\epsilon_1 - \epsilon_{\mu-1}}{2}\theta + \cdots + \frac{\epsilon_{\mu-2} - \epsilon_{\mu-1}}{2}\theta^{\mu-2}\right),\end{aligned}$$

from which we conclude that

$$\psi(\theta) \equiv \epsilon_{\mu-1} \equiv \pm 1 \pmod{(1-\theta)}. \tag{3.17}$$

Consequently $\psi(\theta)$ is not divisible by $(1-\theta)$; thus the norm of $\psi(\theta)$ is also not divisible by 2, from which the main result of this section follows:

Proposition 3.1. *The integer a_λ is odd.*

This number can be calculated relatively easily for the first cases from (3.14) and (3.15); one finds, for example,

$$a_4 = 1, \quad a_5 = 1, \quad a_6 = 17, \quad a_7 = 21121.$$

4 Determination of the factors P_λ.

To compute P_λ, we group four terms at a time in the sum on the right side of (2.10), which correspond to a pair of values β and $\beta+\mu$ as well as pairing up $\alpha = 0$ with $\alpha = 1$, as before. In other words we group the four terms $t = \pm 5^\beta, \pm 5^{\beta+\mu}$ at a time. Since $\theta^\mu = -1$ and θ can be interchanged with θ^{-1} in the product, we obtain, according to (2.3), (2.10), (2.14),

$$P_\lambda = \frac{1}{m^{2^{\lambda-4}}} \prod_\theta \sum_{\beta=0}^{\mu-1} \theta^\beta \log \frac{(1-r^n)(1-r^{-n})}{(1+r^n)(1+r^{-n})}, \tag{4.1}$$

where $n \equiv 5^\beta (\mathrm{mod}\ m)$. (Here n is written for t to indicate it is not sensitive to addition of multiples of m.)

The quotients $(1-r^n)/(1+r^n)$ appearing under the logarithm are *units* of the field Ω_λ, which, up to a root of unity, can be expressed in terms of the two-term periods[3] $r + r^{-1}$. Indeed, if one sets

$$r = e^{\frac{2\pi i}{m}} \tag{4.2}$$

then:

$$\frac{1-r^n}{1+r^n} = r^{-\nu n} \frac{r^{(1-\nu)n} + r^{-(1-\nu)n}}{2 + r^n + r^{-n}} = r^{-\nu n} (-1)^{\frac{n-1}{2}} \tan \frac{n\pi}{m}. \tag{4.3}$$

Here the first equality is making the denominator real by multiplying numerator and denominator by $1 + r^{-n}$, and also using the fact that n is odd and $r^{2\nu} = r^{2^{\lambda-1}} = -1$ (this particular product decomposition is chosen to be a root of unity times a real unit). The second equality follows from the fact that $r^\nu = i$ and the half-angle formula for tangent. And we now want to introduce the following notation: if n is determined from one of the two congruences

$$n \equiv \pm 5^\beta \pmod m \tag{4.4}$$

then let

$$\tau_\beta = r^{\nu n} \frac{1-r^n}{1+r^n} = (-1)^{\frac{n-1}{2}} \tan\left(\frac{n\pi}{m}\right) = \tan\left(5^\beta \frac{\pi}{m}\right). \tag{4.5}$$

The functions τ_β form a system of *real* units of the field Ω_λ which remain unchanged by the substitution $r \mapsto r^{-1}$ (complex conjugation), and which transform into $\tau_{\beta+\beta'}$ under the substitution $r \mapsto r^{n'}$ (the Galois automorphism corresponding to $n' \in (\mathbf{Z}/m)^\times$), if β' depends on n' in the same way as β depends on n. In particular, one also has the relations (since $5^\mu \equiv 1 + 2^{\lambda-1} \pmod m$ so $r \mapsto r^{5^\mu}$ is the same as negating r, which we already used in the proof of (4.1))

$$\tau_\beta \tau_{\beta+\mu} = -1. \tag{4.6}$$

[3]The standard general fact that the units in the ring of integers of $\mathbf{Q}(\zeta_m)$ are all of the form $\pm\zeta_m^x$ times a totally real unit is proved in the first part of Weber's paper, though this is evidently not used here, as the decomposition is made explicitly.

If we further set

$$\log \tau_\beta^2 = l_\beta, \quad l_{\beta+\mu} = -l_\beta \tag{4.7}$$

then it follows from this and (4.1) that

$$P_\lambda = m^{-2^{\lambda-4}} \prod_\theta \sum_{\beta=0}^{\mu-1} \theta^\beta l_\beta. \tag{4.8}$$

The product of the sums on the right side of this expression can be represented in the form of a determinant, which no longer contains θ; the simplest way to achieve this is probably as follows. Let

$$x = \sum_{\beta=0}^{\mu-1} \theta^\beta l_\beta \tag{4.9}$$

and form by multiplication with the powers of θ the equations

$$\begin{aligned} x &= l_0 + \theta l_1 + \cdots + \theta^{\mu-1} l_{\mu-1} \\ \theta x &= -l_{\mu-1} + \theta l_0 + \cdots + \theta^{\mu-1} l_{\mu-2} \\ &\vdots \\ \theta^{\mu-1} x &= -l_1 - \theta l_2 - \cdots + \theta^{\mu-1} l_0, \end{aligned} \tag{4.10}$$

and from by elimination of the powers of θ (i.e. using the fact that the vector of powers of θ is in the kernel of this matrix, which is the content of (4.10))

$$\begin{vmatrix} l_0 - x & l_1 & \dots & l_{\mu-1} \\ -l_{\mu-1} & l_0 - x & \dots & l_{\mu-2} \\ \dots & \dots & \dots & \dots \\ -l_1 & -l_2 & \dots & l_0 - x \end{vmatrix} = 0, \tag{4.11}$$

This is an equation of the μ-th degree with respect to x, in which the highest power of x has the coefficient 1, and whose roots are the factors of the product (4.8). One therefore obtains this product by setting $x = 0$ in the preceding determinant. Thus, if

$$\Lambda(\tau_0, \tau_1, \dots, \tau_{\mu-1}) = (-1)^{2^{\lambda-4}} \begin{vmatrix} l_0 & l_1 & \dots & l_{\mu-1} \\ l_1 & l_2 & \dots & -l_0 \\ l_2 & l_3 & \dots & -l_1 \\ \dots & \dots & \dots & \dots \\ l_{\mu-1} & -l_0 & \dots & -l_{\mu-2} \end{vmatrix}, \tag{4.12}$$

then it follows after a rearrangement of the rows (the order of the last $\mu-1$ rows is reversed, which has the effect of multiplying by $(-1)^{\frac{(\mu-1)(\mu-2)}{2}}$, and all of those $\mu - 1$ rows are negated, bringing the extra sign to $(-1)^{2^{\lambda-4}(\mu-1)} = (-1)^{2^{\lambda-4}}$) that the product of the different values of x in (4.9) over all θ is equal to the value Λ. Note that this is the product of some pairwise conjugate complex numbers (the θ are permuted without fixed points by the complex conjugation operation $\theta \mapsto \theta^{-1}$ as $\lambda \geq 4$), so it is a positive real number. From this then follows

$$P_\lambda = m^{-2^{\lambda-4}} \Lambda(\tau_0, \tau_1, \ldots, \tau_{\mu-1}). \tag{4.13}$$

5 On the real units of the field Ω_λ.

We now again assume $\lambda > 3$ and consider the *real units* of the field Ω_λ, i.e., those that are written in terms of the two-term periods $r + r^{-1}$, thus belonging to the field

$$\Omega_\lambda^+ = \mathbf{Q}(r + r^{-1})$$

since through these, multiplied by the powers of r, all units in Ω_λ are exhausted. Furthermore, if $\mathcal{E}(r)$ is a real unit in Ω_λ, let $l\mathcal{E}(r)$ be understood as the *logarithm* of $\mathcal{E}(r)^2$ (which is a positive real number) or twice the *real part of the logarithm of* $\mathcal{E}(r)$. We further introduce the notation, using the convention of (4.2),

$$r^{5^\beta} = r_\beta \tag{5.1}$$

from which follows:

$$r_{\beta+\mu} = -r_\beta; \tag{5.2}$$

By a *primitive unit* of the field Ω_λ, we understand such a real unit $\mathcal{E}(r)$ which satisfies the condition

$$\mathcal{E}(r)\mathcal{E}(-r) = \pm 1 \quad [\implies l\mathcal{E}(r) + l\mathcal{E}(-r) = 0] \tag{5.3}$$

and which is not $= \pm 1$, thus in any case *does not belong to the field* $\Omega_{\lambda-1}$. This is because $\mathrm{Gal}(\Omega_\lambda/\Omega_{\lambda-1}) = \langle r \mapsto -r \rangle \cong \mathbf{Z}/2$, so $\mathcal{E}(r) \in \Omega_{\lambda-1}$ if and only if $\mathcal{E}(r) = \mathcal{E}(-r)$, which implies that $\mathcal{E}(r)\mathcal{E}(-r) = \mathcal{E}(r)^2$, which is ± 1 if and only if $\mathcal{E}(r) = \pm 1$ (as $\mathcal{E}(r)$ is assumed to be real so it cannot be $\pm i$). In fact the "$\implies$" in (5.3) can be upgraded to "$\iff$" by Dirichlet's unit theorem and the fact that the only real roots of unity are ± 1.

A system of such units

$$\mathcal{E}_0(r), \mathcal{E}_1(r), \ldots, \mathcal{E}_{\mu-1}(r) \tag{5.4}$$

is called a *system of mutually independent primitive units* if condition (5.3) is satisfied for each of them, and if the determinant

$$\sum_{\sigma \in S_\mu} \mathrm{sgn}(\sigma) l\mathcal{E}_0(r_{\sigma(0)}) l\mathcal{E}_1(r_{\sigma(1)}) \cdots l\mathcal{E}_{\mu-1}(r_{\sigma(\mu-1)}) = L(\mathcal{E}_0, \mathcal{E}_1, ..., \mathcal{E}_{\mu-1}) \tag{5.5}$$

has a *non-zero value.*

The units $\tau_\beta(r)$ defined in (4.5) satisfy the condition

$$\tau_\beta(r_{\beta'}) = \tau_{\beta+\beta'}(r) \tag{5.6}$$

and because of (4.6)

$$\tau_\beta(r)\tau_\beta(-r) = -1, \quad \tau_\beta\tau_{\beta+\mu} = -1. \tag{5.7}$$

The determinant $L(\tau_0, \tau_1, \dots, \tau_{\mu-1})$, apart from the factor $(-1)^{2^{\lambda-4}}$, is identical to the determinant (4.12) by (5.2), (5.6), (5.7) and as a factor of the class number, does not vanish. The units

$$\tau_0, \tau_1, \dots, \tau_{\mu-1} \tag{5.8}$$

therefore form a system of independent primitive units, whereby the existence of such systems is proven. (For $\lambda = 3$, this follows directly from the consideration of $\tau_0(r)$).

If $\mathcal{E}(r)$ is an arbitrary *primitive* unit in Ω_λ, then, since the determinant (5.5) is non-zero, numbers $e_0, e_1, \dots, e_{\mu-1}$, which may be called the *exponents* of the unit $\mathcal{E}(r)$, can be determined such that for $s = 0, 1, \dots, \mu-1$,

$$l\mathcal{E}(r_s) = e_0 l\mathcal{E}_0(r_s) + e_1 l\mathcal{E}_1(r_s) + \cdots + e_{\mu-1} l\mathcal{E}_{\mu-1}(r_s) \tag{5.9}$$

and because of the relations (5.3), these formulas also hold when r_s is replaced by $-r_s$, i.e., *for all Galois conjugate values* r.

The exponents of a product of several primitive units are the sums of the corresponding exponents of the individual factors.

It can now be shown, just as in the general theory of units, that *the numbers* $e_0, e_1, \dots, e_{\mu-1}$ *are rational numbers, and that there exists a certain smallest integer* e*, independent of* $\mathcal{E}(r)$*, by which the numbers* $e_0, e_1, \dots, e_{\mu-1}$ *must be multiplied so that the products become integers*[4]. We give this proof here.

(i) There is only a *finite number of integers* ρ in Ω_λ which have the property that the absolute values of all conjugates of ρ remain below a finite limit. For if

$$\rho = a_0 + a_1 r + \cdots + a_{\frac{1}{2}m-1} r^{\frac{1}{2}m-1}$$

where the a_i are *rational integers* (which is the general form for ρ because $r^{m/2} = -1$), then:

$$\frac{1}{2} m a_0 = \mathrm{Tr}_{\mathbf{Q}}^{\Omega_\lambda} \rho$$

$$\frac{1}{2} m a_1 = \mathrm{Tr}_{\mathbf{Q}}^{\Omega_\lambda} (\rho r^{-1})$$

$$\cdots$$

$$\frac{1}{2} m a_{\frac{1}{2}m-1} = \mathrm{Tr}_{\mathbf{Q}}^{\Omega_\lambda} \left(\rho r^{-(\frac{1}{2}m-1)}\right)$$

[4] For us this is obvious: the $\mathbf{Z}$-span of the $(l\mathcal{E}_i(r_0), \dots, l\mathcal{E}_i(r_{\mu-1}))$ is a full-rank sublattice in the log lattice of the primitive units (suitably interpreted to only involve the first μ Galois conjugates), and this number e is just the index. At the very most all we need here is the base adaptée, the fact that the logs of the units are discrete, and the fact that a discrete subgroup of $\mathbf{R}^N$ is a lattice. I still include Weber's proof below, with the disclaimer that most of it is standard.

(because for odd powers n, $\mathrm{Tr}_{\mathbf{Q}}^{\Omega_\lambda}(r^n) = \mathrm{Tr}_{\mathbf{Q}}^{\Omega_{\lambda-1}}(r^n - r^n) = 0$; and more generally if $\frac{1}{2}m - 1 \geq n \geq 2$,

$$\mathrm{Tr}_{\mathbf{Q}}^{\Omega_\lambda}(r^n) = 2^{v_2(n)}\mathrm{Tr}_{\mathbf{Q}}^{\Omega_{\lambda-v_2(n)}}(r^n) = 2^{v_2(n)}\mathrm{Tr}_{\mathbf{Q}}^{\Omega_{\lambda-v_2(n)-1}}(r^n - r^n) = 0$$

because r^n is a primitive $2^{\lambda - v_2(n)}$-th root of unity — note that we are using the fact that $\lambda - v_2(n) \geq 2$) from which the correctness of the assertion is immediately apparent[5].

(ii) If we let $e_0, e_1, \cdots, e_{\mu-1}$ on the right-hand side of (5.9) run through the interval from 0 to 1, the absolute values of these expressions remain below certain finite limits, and thus so do the absolute values of the $\mathcal{E}(r_s)$ determined by them. From this, according to (i), it follows that the tuple of exponents $(e_0, e_1, \ldots, e_{\mu-1})$, as long as they are in $[0, 1]$, can only assume a *finite number of values*.

(iii) Let $m_i', m_i'', m_i''', \ldots$ be the unique rational integers such that the differences

$$e_i - m_i', \quad 2e_i - m_i'', \quad 3e_i - m_i''', \ldots \quad (i = 0, 1, \ldots, \mu - 1)$$

are in the interval $[0, 1)$. Each of the sequences

$$ke_0 - m_0^{(k)}, \quad ke_1 - m_1^{(k)}, \quad \ldots, \quad ke_{\mu-1} - m_{\mu-1}^{(k)} \tag{5.10}$$

then forms an admissible system of exponents in (5.9). From this, it follows that for a certain value $k = e$, which is in any case not greater than the number of admissible proper fractional exponent systems according to (ii), all members of the series (5.10) must vanish.[6]

But with this, the fact to be proved is established. We can also express it in the following way.

If $\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_{\mu-1}$ is a system of mutually independent primitive units, then there exists a smallest integer e, depending only on this basis, such that for *every*

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[5] This is just the discreteness of the $\mathcal{O}_K$ inside $K \otimes \mathbf{R}$, and of the log lattice that we need for all of our regulators, proved by ad-hoc means for the cyclotomic case.

[6] I didn't want to modify the language here too much, but here is a version with more precision: for all $k \geq 1$, if $\mathcal{E}$ has exponents $e_0, \ldots, e_{\mu-1} \in \mathbf{R}$, then $\mathfrak{E}_k := \mathcal{E}^k \prod_{i=0}^{\mu-1} \mathcal{E}_i^{-m_i^{(k)}}$ has exponents $ke_0 - m_0^{(k)}, \ldots, ke_{\mu-1} - m_{\mu-1}^{(k)} \in [0, 1)$. By (i)/(ii), the infinite sequence of units $\mathfrak{E}_k$ / of tuples $(ke_0 - m_0^{(k)}, \ldots, ke_{\mu-1} - m_{\mu-1}^{(k)})$ can only take on finitely many values. By pigeonhole, there are $k' < k$ such that $\mathfrak{E}_k = \mathfrak{E}_{k'}$, and hence

$$((k - k')e_0 - (m_0^{(k)} - m_0^{k'}), \ldots, (k - k')e_{\mu-1} - (m_{\mu-1}^{(k)} - m_{\mu-1}^{k'})) = 0.$$

This shows that the e_i are rational. Moreover, the bound $k - k'$ on the denominators is at most the bound from (ii), which (by (i)) only depends on the choice of $\mathcal{E}_0, \ldots, \mathcal{E}_{\mu-1}$ (that choice influences the bound on the absolute value in (i)). So the exponent e is indeed independent of $\mathcal{E}$, only depending on the choice of fundamental system of primitive units.

primitive unit $\mathcal{E}$, *integers* $e_0, e_1, \ldots, e_{\mu-1}$ (which now have a slightly different meaning than above) can be determined such that

$$\mathcal{E}^e = \pm \mathcal{E}_0^{e_0} \mathcal{E}_1^{e_1} \cdots \mathcal{E}_{\mu-1}^{e_{\mu-1}} \tag{5.11}$$

(this uses the fact that the kernel of log is precisely the 2^λ-th roots of unity, but the relevant units here are all real, so the relevant quotient of units must be ± 1). The exponents of the unit $\mathcal{E}$ are then

$$\frac{e_0}{e}, \frac{e_1}{e}, \ldots, \frac{e_{\mu-1}}{e}.$$

6 The units τ_β.

We now consider the system of independent primitive units $\tau_0, \tau_1, \ldots, \tau_{\mu-1}$ and first prove the following fact.

Proposition 6.1. *If the product*

$$\tau_0^{e_0} \tau_1^{e_1} \cdots \tau_{\mu-1}^{e_{\mu-1}} \tag{6.1}$$

has the same sign for all its conjugate values, then the (integer) exponents $e_0, e_1, \ldots, e_{\mu-1}$ *must all be even.*

Proof. To prove this fact, we first transform the expression. According to our definition (4.5), it was

$$\tau_\beta = \tan\left(5^\beta \frac{\pi}{m}\right) = \frac{1}{2} \frac{\sin\left(\frac{2\pi}{m} 5^\beta\right)}{\left[\cos\left(\frac{\pi}{m} 5^\beta\right)\right]^2} \tag{6.2}$$

and we therefore set

$$\sigma_\beta = \sin\left(\frac{2\pi}{m} 5^\beta\right). \tag{6.3}$$

It is then (by (5.6)) obviously only necessary to prove that $e_0, e_1, \ldots, e_{\mu-1}$ must be even numbers if

$$\sigma_\beta^{e_0} \sigma_{\beta+1}^{e_1} \cdots \sigma_{\beta+\mu-1}^{e_{\mu-1}} =: S_\beta \tag{6.4}$$

has the same sign for all values of β. The numbers σ_β satisfy the following relations:

$$\sigma_{\beta+\mu} = -\sigma_\beta \tag{6.5}$$

$$\sigma_{\beta+\frac{1}{2}\mu} = -\cos\left(\frac{2\pi}{m} 5^\beta\right) \tag{6.6}$$

(because $5^{2^{\lambda-4}} \equiv 1 + 2^{\lambda-2} + 2^{\lambda-1} \pmod{2^\lambda}$ — this can presumably be checked directly, though I used the 2-adic logarithm to stay organized), and thus

$$\sigma_\beta \sigma_{\beta+\frac{1}{2}\mu} = -\frac{1}{2} \sin\left(\frac{4\pi}{m} 5^\beta\right) = -\frac{1}{2} \sigma'_\beta \tag{6.7}$$

where σ' arises from σ by replacing λ with $\lambda - 1$, so that

$$\sigma'_{\beta+\mu/2} = -\sigma'_\beta. \tag{6.8}$$

The correctness of our assertion is now evident if $\lambda = 3$; because in this case $\mu = 1$ and

$$\sigma_0 = \sin\frac{\pi}{4} = \sqrt{\frac{1}{2}}$$
$$\sigma_1 = \sin\frac{5\pi}{4} = -\sqrt{\frac{1}{2}}.$$

We therefore assume its correctness has been proven when λ is replaced by $\lambda - 1$ and seek to derive it for λ itself. To this end, we first form the product according to (6.4), (6.7), (6.8):

$$S_\beta S_{\beta+\frac{1}{2}\mu} = (-\frac{1}{2})^\mu(-1)^{\frac{1}{2}\mu}\sigma'^{\,e_0+e_{\frac{1}{2}\mu}}_\beta \sigma'^{\,e_1+e_{\frac{1}{2}\mu+1}}_{\beta+1} \cdots \sigma'^{\,e_{\frac{1}{2}\mu-1}+e_{\mu-1}}_{\beta+\frac{1}{2}\mu-1}$$

from which, according to the assumption made (that the product $S_\beta S_{\beta+\frac{1}{2}\mu}$ must have a constant sign as β varies [which follows from the same being assumed of S_β and therefore of $S_{\beta+\frac{1}{2}\mu}$] which implies its exponents must be even by the induction hypothesis), it follows that:

$$e_0 \equiv e_{\frac{1}{2}\mu}, \quad e_1 \equiv e_{\frac{1}{2}\mu+1}, \quad \cdots, \quad e_{\frac{1}{2}\mu-1} \equiv e_{\mu-1} \pmod 2.$$

According to this, it follows that also

$$\sigma'^{\,e_0}_\beta \sigma'^{\,e_1}_{\beta+1} \cdots \sigma'^{\,e_{\frac{1}{2}\mu-1}}_{\beta+\frac{1}{2}\mu-1} =: S'_\beta \tag{6.9}$$

has the same sign for all values of β, because by (6.4) and (6.7), up to even-order powers of the σ_β's, S_β coincides with S'_β, hence the sign of S_β coincides with the sign of S'_β. But according to the induction hypothesis, it follows from this that

$$e_0 \equiv e_1 \equiv \cdots \equiv e_{\frac{1}{2}\mu-1} \equiv 0 \pmod 2$$

and this is the claim to be proved. □

From this, in connection with the results of the previous section, it follows:

Proposition 6.2. *There exists an* odd rational integer e *such that, if* $\mathcal{E}(r)$ *denotes an arbitrary primitive unit in* Ω_λ*, there exist integers* $e_0, e_1, \ldots, e_{\mu-1}$ *such that*

$$\mathcal{E}(r)^e = \pm\tau_0^{e_0}\tau_1^{e_1}\cdots\tau_{\mu-1}^{e_{\mu-1}}. \tag{6.10}$$

For the conclusion of the previous section proves the existence of such a number e in the first place; but if it were even, while the numbers $e_0, e_1, \ldots, e_{\mu-1}$ are not all even at the same time, this would contradict Proposition 6.1 (as all the conjugates of the left hand side would be the square of a real number and hence would be positive).

As a corollary, we have the following theorem:

Theorem A. *A primitive unit of the field* Ω_λ *which has the same sign as all its conjugates is, up to a factor of* ±1*, a square of a primitive unit.*

7 Fundamental systems of primitive units.

Let

$$\mathcal{E}_0(r), \mathcal{E}_1(r), ..., \mathcal{E}_{\mu-1}(r) \tag{7.1}$$

now be any system of mutually independent primitive units; if the quantities $l_0, l_1, \ldots, l_{\mu-1}$ have the same meaning as in (4.7), namely

$$l_\beta = l\tau_\beta \tag{7.2}$$

and if e is the odd integer whose existence was proven in Proposition 6.2, then there exist integers $e_{i,j}$ such that

$$\begin{aligned} el\mathcal{E}_i(r_0) &= e_{i,0}l_0 + e_{i,1}l_1 + \cdots + e_{i,\mu-1}l_{\mu-1} \\ el\mathcal{E}_i(r_1) &= e_{i,0}l_1 + e_{i,1}l_2 + \cdots + e_{i,\mu-1}(-l_0) \\ &\cdots \\ el\mathcal{E}_i(r_{\mu-1}) &= e_{i,0}l_{\mu-1} + e_{i,1}(-l_0) + \cdots + e_{i,\mu-1}(-l_{\mu-2}), \end{aligned} \tag{7.3}$$

from which for $\lambda > 3$, using the notation (4.12), it follows that

$$(-1)^{2^{\lambda-4}} \det(l\mathcal{E}_i(r_j)) = \frac{\det(e_{i,j})}{e^\mu}\Lambda(\tau_0, \tau_1, \ldots, \tau_{\mu-1}). \tag{7.4}$$

Since the number e and likewise the determinant Λ have a value independent of the system of (7.1), it follows from this that one can choose this system such that the determinant

$$(-1)^{2^{\lambda-4}} \det(l\mathcal{E}_i(r_j)) \tag{7.5}$$

obtains the smallest possible positive value L_λ, and such a system shall be called a *fundamental system of primitive units*.

If the system (7.1) is such a fundamental system, then every primitive unit $\mathcal{E}(r)$ can be represented in the manner[7]

$$\mathcal{E} = \mathcal{E}_0^{g_0}\mathcal{E}_1^{g_1}\cdots\mathcal{E}_{\mu-1}^{g_{\mu-1}} \tag{7.6}$$

such that the exponents $g_0, g_1, \ldots, g_{\mu-1}$ are *integers*. For if, e.g., g_0 is a non-integer rational number[8], then by repeated division by $\mathcal{E}_0$ there also exists a primitive unit $\mathcal{E}$ for which g_0 *is a rational number in* $(0, 1)$, and the system $\mathcal{E}, \mathcal{E}_1, \ldots, \mathcal{E}_{\mu-1}$ is likewise independent (its span in the log space contains $\mathcal{E}_0^{g_0}$ and is in fact the same as the span of $\mathcal{E}_0^{g_0}$ and $\mathcal{E}_1, \ldots, \mathcal{E}_{\mu-1}$). But (because of the parenthetical remark in the previous sentence) for the determinant

$$(-1)^{2^{\lambda-4}} \sum_{\sigma\in S_\mu} \operatorname{sgn}(\sigma) l\mathcal{E}(r_{\sigma(0)}) l\mathcal{E}_1(r_{\sigma(1)}) \cdots l\mathcal{E}_{\mu-1}(r_{\sigma(\mu-1)}))$$

[7] I think Weber forgot a $\pm$ here.

[8] Here Weber is clearly just abusing notation briefly in order to take rational powers: $\mathcal{E}, \mathcal{E}_i$ in this paragraph should be taken to refer to their logs in $\mathbf{R}^\mu$.

the value $g_0 L_\lambda$ is obtained, which, contrary to the assumption, is smaller than L_λ. Accordingly, there are integers g_{ij} such that

$$l\tau_\beta(r_{\beta'}) = l_{\beta+\beta'} = g_{0,\beta} l\mathcal{E}_0(r_{\beta'}) + g_{1,\beta} l\mathcal{E}_1(r_{\beta'}) + \cdots + g_{\mu-1,\beta} l\mathcal{E}_{\mu-1}(r_{\beta'}) \quad (7.7)$$

from which, according to (7.3), it follows that

$$\det(g_{i,j}) \det(e_{i,j}) = e^\mu \quad (7.8)$$

and consequently

Proposition 7.1. *The number*

$$\det(g_{i,j}) =: b_\lambda \quad (7.9)$$

is a positive odd integer.

The calculation of the number b_λ encounters the known difficulties that always arise in the theory of units. Only for the case $\lambda = 3$ does it easily follow from the theory of Pell's equation that the unit $\tau_0 = \sqrt{2} - 1$ is itself a fundamental unit, since *all* units of the field Ω_3 can be represented in the form $r^{n_0}(\sqrt{2} - 1)^{n_1}$ (where r is an 8-th root of unity) with integer exponents n_0, n_1. We therefore have

$$L_3 = 2\log(\sqrt{2} + 1) \quad (7.10)$$

the minimal value of $l\mathcal{E}(r)$. One obtains from (7.4), (7.8) and (7.9)

$$\Lambda(\tau_0, \tau_1, \ldots, \tau_{\mu-1}) = b_\lambda L_\lambda$$

and from (2.13) and (4.13)

$$P_3 = \frac{L_3}{2\sqrt{2}}, \quad P_\lambda = m^{-2^{\lambda-4}} b_\lambda L_\lambda. \quad (7.11)$$

8 The fundamental units of the field Ω_λ.

It is now a matter of determining a complete system of fundamental units in Ω_λ, i.e. a system of $\nu - 1$ (not necessarily primitive) units

$$\delta_1(r), \delta_2(r), \ldots, \delta_{\nu-1}(r) \quad (8.1)$$

which can be assumed to be real, such that all units of the field Ω_λ can be represented as

$$r^{n_0} \delta_1^{n_1} \delta_2^{n_2} \cdots \delta_{\nu-1}^{n_{\nu-1}} \quad (8.2)$$

with integer exponents $n_0, n_1, \ldots, n_{\nu-1}$. Of particular importance here is the absolute value

$$L(\delta_1, \delta_2, \ldots, \delta_{\nu-1}) \quad (8.3)$$

of the determinant formed from the $(\nu-1)^2$ quantities

$$\log\delta_i(r_\beta)\delta_i(r_\beta^{-1}) = l\delta_i(r_\beta) \quad \begin{pmatrix}\beta = 0,1,...,\nu-2\\ i=1,2,...,\nu-1\end{pmatrix} \tag{8.4}$$

(The notation L should also be used in the same sense for any system of $\nu-1$ independent units in Ω, even if not fundamental)

If $\lambda = 3$, then $\delta_1 = \tau_0$ and $L(\delta_1)$ is equal to L_3. In the general case, we denote as above

$$\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_{\mu-1} \tag{8.5}$$

a *fundamental system of primitive units* in Ω_λ and by

$$\Delta_1, \Delta_2, \dots, \Delta_{\mu-1} \tag{8.6}$$

a complete fundamental system of the field $\Omega_{\lambda-1}$. The $\nu-1$ units

$$\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_{\mu-1}, \quad \Delta_1, \Delta_2, \dots, \Delta_{\mu-1} \tag{8.7}$$

together form a system of independent units in Ω_λ, and the determinant

$$L(\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_{\mu-1}, \Delta_1, \Delta_2, \dots, \Delta_{\mu-1}) \tag{8.8}$$

as a result of the properties of the units $\mathcal{E}$ and Δ

$$l\mathcal{E}_i(r) = -l\mathcal{E}_i(-r); \quad l\Delta_i(r) = l\Delta_i(-r) \tag{8.9}$$

is equal to

$$2^{\mu-1}L_\lambda L(\Delta_1, \Delta_2, ..., \Delta_{\mu-1}). \tag{8.10}$$

This is because

$$\begin{aligned}
L(\cdots) &= \begin{vmatrix}
l\mathcal{E}_0(r_0) & \cdots & l\mathcal{E}_0(r_{\mu-1}) & l\mathcal{E}_0(r_\mu) & \cdots & l\mathcal{E}_0(r_{\nu-2})\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
l\mathcal{E}_{\mu-1}(r_0) & \cdots & l\mathcal{E}_{\mu-1}(r_{\mu-1}) & l\mathcal{E}_{\mu-1}(r_\mu) & \cdots & l\mathcal{E}_{\mu-1}(r_{\nu-2})\\
l\Delta_1(r_0) & \cdots & l\Delta_1(r_{\mu-1}) & l\Delta_1(r_\mu) & \cdots & l\Delta_1(r_{\nu-2})\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
l\Delta_{\mu-1}(r_0) & \cdots & l\Delta_{\mu-1}(r_{\mu-1}) & l\Delta_{\mu-1}(r_\mu) & \cdots & l\Delta_{\mu-1}(r_{\nu-2})
\end{vmatrix}\\
&= \begin{vmatrix}
l\mathcal{E}_0(r_0) & \cdots & l\mathcal{E}_0(r_{\mu-1}) & 0 & \cdots & 0\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
l\mathcal{E}_{\mu-1}(r_0) & \cdots & l\mathcal{E}_{\mu-1}(r_{\mu-1}) & 0 & \cdots & 0\\
l\Delta_1(r_0) & \cdots & l\Delta_1(r_{\mu-1}) & 2l\Delta_1(r_0) & \cdots & 2l\Delta_1(r_{\mu-1})\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
l\Delta_{\mu-1}(r_0) & \cdots & l\Delta_{\mu-1}(r_{\mu-1}) & 2l\Delta_{\mu-1}(r_0) & \cdots & 2l\Delta_{\mu-1}(r_{\mu-1})
\end{vmatrix}\\
&= 2^{\mu-1}L(\mathcal{E}_0, \dots, \mathcal{E}_{\mu-1})L(\Delta_1, \dots, \Delta_{\mu-1})\\
&= 2^{\mu-1}L_\lambda L(\Delta_1, \dots, \Delta_{\mu-1})
\end{aligned}$$

(where in the second equality we have subtracted the first $\mu-1$ columns from the last $\mu-1$ columns, using the identities (8.9) and (5.2)).

It is now therefore still necessary to determine the relationship between the system (8.7) and (8.1). For this purpose, let $m_{i,i'}, M_{i,i'}$ be rational numbers such that

$$\begin{aligned} 2l\delta_i(r) &= m_{0,i}l\mathcal{E}_0(r) + m_{1,i}l\mathcal{E}_1(r) + \cdots + m_{\mu-1,i}l\mathcal{E}_{\mu-1}(r) \\ &\quad + M_{1,i}l\Delta_1(r) + M_{2,i}l\Delta_2(r) + \cdots + M_{\mu-1,i}l\Delta_{\mu-1}(r), \end{aligned} \tag{8.11}$$

and obtain according to (8.9):

$$\begin{aligned} l(\delta_i(r)\delta_i(-r)) &= M_{1,i}l\Delta_1(r) + M_{2,i}l\Delta_2(r) + \cdots + M_{\mu-1,i}l\Delta_{\mu-1}(r) \\ l(\delta_i(r)\delta_i(-r)^{-1}) &= m_{0,i}l\mathcal{E}_0(r) + m_{1,i}l\mathcal{E}_1(r) + \cdots + m_{\mu-1,i}l\mathcal{E}_{\mu-1}(r). \end{aligned} \tag{8.12}$$

Since now $\delta_i(r)\delta_i(-r)$ is a unit of the field $\Omega_{\lambda-1}$, and $\Delta_1,\ \Delta_2, \ldots, \Delta_{\mu-1}$ is a fundamental system of this field, and furthermore $\delta_i(r)\delta_i(-r)^{-1}$ is a *primitive* unit in Ω_λ, and $\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_{\mu-1}$ is a fundamental system of primitive units, it follows from these formulas that the $M_{i,i'}, m_{i,i'}$ are *integers*. If we denote by M the absolute value of the determinant of the numbers $m_{i,i'}, M_{i,i'}$, it follows from (8.11) and (8.10) that

$$\begin{aligned} L(\delta_1, \delta_2, \ldots, \delta_{\nu-1}) &= M2^{1-\nu}L(\mathcal{E}_0, \ldots, \mathcal{E}_{\mu-1}, \Delta_1, \ldots, \Delta_{\mu-1}) \\ &= M2^{-\mu}L_\lambda L(\Delta_1, \Delta_2, ..., \Delta_{\mu-1}). \end{aligned} \tag{8.13}$$

However, from the assumption that $\delta_1, \delta_2, \ldots, \delta_{\nu-1}$ form a fundamental system of units of the field Ω_λ, we have the existence of rational integers $n_{i,i'}, N_{i,i'}$ which satisfy the equations

$$\begin{aligned} l\mathcal{E}_i(r) &= n_{1,i}l\delta_1(r) + n_{2,i}l\delta_2(r) + \cdots + n_{\nu-1,i}l\delta_{\nu-1}(r), \quad (i = 0, 1, \ldots, \mu-1) \\ l\Delta_x(r) &= N_{1,x}l\delta_1(r) + N_{2,x}l\delta_2(r) + \cdots + N_{\nu-1,x}\delta_{\nu-1}(r), \quad (x = 1, 2, \ldots, \mu-1) \end{aligned} \tag{8.14}$$

and if we denote the absolute value of the determinant of $n_{i,i'}, N_{i,i'}$ by N, then it follows from (8.11) and (8.14) that

$$MN = 2^{\nu-1} \tag{8.15}$$

from which it follows that both M and N are powers of 2.

It can be shown that M is divisible by 2^μ. Namely, one can determine a system of $\nu-1 = 2\mu-1$ rational integers $x_1, x_2, \ldots, x_{\nu-1}$, *without a common divisor* such that they satisfy the $\mu-1 < \nu-1$ equations

$$\sum_{i=1}^{\nu-1} M_{s,i}x_i = 0 \quad (s = 1, 2, \ldots, \mu-1) \tag{8.16}$$

since then the product

$$\delta_1^{x_1}\delta_2^{x_2}\cdots\delta_{\nu-1}^{x_{\nu-1}}$$

as a consequence of (8.11) has the property

$$2l(\delta_1^{x_1}\delta_2^{x_2}\cdots\delta_{\nu-1}^{x_{\nu-1}}(r)) = \sum_{s=0}^{\mu-1}\left(\sum_{i=1}^{\nu-1} m_{s,i}x_i\right) l\mathcal{E}_s(r),$$

and is therefore *a primitive unit* in Ω_λ; and since $\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_{\mu-1}$ forms a fundamental system of primitive units, it follows that

$$\sum_{i=1}^{\nu-1} m_{s,i}x_i \equiv 0 \pmod 2, \quad (s = 0, 1, \ldots, \mu-1) \tag{8.17}$$

from which it follows that the determinant M is divisible by 2 (because the vector of x_i's is in the kernel of the relevant matrix modulo 2).

Since $x_1, x_2, \ldots, x_{\nu-1}$ are not all even, let us assume x_1 is odd, and now determine a second system of quantities $x'_2, x'_3, \ldots, x'_{\nu-1}$ from the equations

$$\sum_{i=2}^{\nu-1} M_{s,i}x'_i = 0 \quad (s = 1, 2, \ldots, \mu-1)$$

from which similarly the divisibility of M by 2^2 follows (setting $x'_1 = 0$ and assuming WLOG that x'_2 is odd, the two vectors $(x_1, \ldots, x_{\nu-1})$ and $(x'_1, \ldots, x'_{\nu-1})$ are linearly independent mod 2 and are elements of the kernel of the relevant matrix modulo 2 — this is enough because by the base adaptée and (8.11) there is a choice of bases such that this matrix is diagonal with only 1's and 2's on the diagonal, and producing linearly independent elements in the kernel modulo 2 is the same as increasing the lower bound on $v_2(M)$).

And so one can continue as long as the number of unknowns still exceeds the number of equations, i.e., until

$$\sum_{i=\mu}^{\nu-1} M_{s,i}x_i^{(\mu-1)} = 0 \quad (s = 1, 2, \ldots, \mu-1) \tag{8.18}$$

from which the divisibility of M by 2^μ follows. We can therefore according to (8.15) set

$$M = 2^{\mu+\sigma} \quad N = 2^{\nu-\sigma-1} \tag{8.19}$$

wherein σ is a non-negative integer. It will emerge in the following paragraph as a corollary that σ actually is equal to zero, so that M is not divisible by a *higher* power of 2 than the μ^{th}. However, in order not to interrupt the chain of inferences that continues from this consideration and leads to an important result, this theorem shall be assumed for the time being. Assume again in (8.18) that $x_\mu^{(\mu-1)}$ is odd, and determine coprime integers $x_i^{(\mu)}$ satisfying the equations

$$\sum_{i=\mu+1}^{\nu-1} M_{s,i}x_i^{(\mu)} = 0 \quad (s = 2, 3, \ldots, \mu-1). \tag{8.20}$$

Then it must necessarily be the case that

$$\sum_{i=\mu+1}^{\nu-1} M_{1,i} x_i^{(\mu)} \equiv 1 \pmod 2 \tag{8.21}$$

because if we assume the contrary, i.e., this sum is even, say $= 2\xi$, it follows from (8.11) that

$$\delta_{\mu+1}^{x_{\mu+1}^{(\mu)}} \delta_{\mu+2}^{x_{\mu+2}^{(\mu)}} \cdots \delta_{\nu-1}^{x_{\nu-1}^{(\mu)}} \Delta_1^{-\xi}$$

is a primitive unit in Ω_λ, from which, as above ($s = 0, 1, \ldots, \mu - 1$)

$$\sum_{i=\mu+1}^{\nu-1} m_{s,i} x_i^{(\mu)} \equiv 0 \pmod 2$$

follows; thus, M would be divisible by $2^{\mu+1}$, contrary to the assumption. Therefore, if one multiplies the equations (8.11) for $i = \mu+1, \mu+2, \ldots, \nu-1$ respectively by $x_i^{(\mu)}$ and forms the sum, one obtains (using (8.14)) an equation of the form

$$l\Delta_1(r) = \sum_{i=0}^{\mu-1} A_{i,1} l\mathcal{E}_i(r) + 2\sum_{i=0}^{\nu-1} a_{i,1} l\delta_i(r) \tag{8.22}$$

where $A_{i,1}, a_{i,1}$ are integers. In place of Δ_1, $\Delta_2, \Delta_3, \ldots, \Delta_{\mu-1}$ can equally well appear in this formula. This result can be summarized in the following remarkable *theorem*:

Theorem B. *Every unit of the field $\Omega_{\lambda-1}$ can be represented as the product of a primitive unit and the square of a unit of the field Ω_λ.*

This theorem, apart from an application to be made later, has the theoretical interest that it teaches that in the field Ω_λ there exist units that are more fundamental (to use this comparison according to Kummer's precedent) than the system of $\mathcal{E}, \Delta$.

However, for now, we drop the to-be-proved assumption that $\sigma = 0$, and thus obtain from (8.13) and (8.19)

$$L(\delta_1, \delta_2, \ldots, \delta_{\nu-1}) = 2^\sigma L_\lambda L(\Delta_1, \Delta_2, \ldots, \Delta_{\mu-1}).$$

The same consideration can now be repeated with respect to $L(\Delta_1, \Delta_2, \ldots, \Delta_{\mu-1})$, so that one finally obtains:

$$L(\delta_1, \delta_2, \ldots, \delta_{\nu-1}) = 2^{\sum \sigma} L_\lambda L_{\lambda-1} \cdots L_3 \tag{8.23}$$

where $\sum \sigma$ is a sum composed of non-negative integers.

9 The class number.

It only remains for us to substitute the results found into the formula (2.4), namely

$$gh = Q_2Q_3Q_4\cdots Q_\lambda \cdot P_3P_4\cdots P_\lambda \tag{9.1}$$

and to determine the value of g. But according to[9] (2.11), (2.12), and (3.16),

$$Q_2 = \frac{\pi}{4}, \quad Q_3 = \frac{\pi}{2\sqrt{2}}; \quad Q_\lambda = \frac{\pi^\mu a_\lambda}{2\cdot 2^{(\lambda-2)2^{\lambda-4}}}, \tag{9.2}$$

and from this one finds (using $2\cdot 1 + 3\cdot 2^1 + \cdots + (\lambda-2)2^{\lambda-4} = (\lambda-3)2^{\lambda-3}$)

$$Q_2Q_3Q_4\cdots Q_\lambda = \frac{\pi^\nu a_4a_5\cdots a_\lambda}{\sqrt{2}\,2^\lambda\,2^{(\lambda-3)2^{\lambda-3}}} \tag{9.3}$$

Likewise according to (7.11),

$$P_3 = \frac{L_3}{2\sqrt{2}}, \qquad P_\lambda = \frac{b_\lambda L_\lambda}{2^{\lambda 2^{\lambda-4}}}$$

from which we have

$$P_3P_4\cdots P_\lambda = \sqrt{2}\frac{L_3L_4\cdots L_\lambda b_4b_5\cdots b_\lambda}{2^{(\lambda-1)2^{\lambda-3}}}; \tag{9.4}$$

and consequently

$$gh = \nu^{-\nu}2^{-\lambda}L_3L_4\cdots L_\lambda \pi^\nu a_4b_4a_5b_5\cdots a_\lambda b_\lambda. \tag{9.5}$$

For g one has according to the analytic class number formula

$$g = \frac{E(2\pi)^\nu}{\sqrt{D}}, \tag{9.6}$$

wherein D is the discriminant of the field Ω_λ, and (since the number of roots of unity contained in Ω_λ is 2^λ)

$$E = 2^{-\lambda}L(\delta_1, \delta_2, \ldots, \delta_{\nu-1}). \tag{9.7}$$

The discriminant D of our field, however, can be computed as follows: if one lets

$$f(t) = t^{2\nu} + 1 \in \mathbf{Z}[t],$$

then

$$D = \mathrm{N}(f'(r)) = (2\nu)^{2\nu}. \tag{9.8}$$

[9] The formula

$$\tan\frac{\pi}{8} = \frac{\tan\frac{\pi}{16}}{\tan\frac{5\pi}{16}}\left(\frac{\cos\frac{\pi}{16}}{\tan\frac{5\pi}{16}}\right)^2$$

serves as an example for $\lambda = 4$.

If one finally substitutes the value for $L(\delta_1, \delta_2, \ldots, \delta_{\nu-1})$ from (8.23), it follows that

$$g = 2^{-\lambda}\nu^{-\nu}\pi^{\nu}2^{\sum\sigma}L_3L_4\cdots L_\lambda \tag{9.9}$$

and consequently

$$h = 2^{-\sum\sigma}a_4b_4a_5b_5\cdots a_\lambda b_\lambda. \tag{9.10}$$

Since now the $a_4, b_4, \ldots, a_\lambda, b_\lambda$ are, as proven above (see Proposition 3.1 and Proposition 7.1), *odd integers*, and the class number h is an integer, it follows that $\sum\sigma$ cannot be positive, and two conclusions can be drawn from (9.10):

The sum $\sum\sigma$, consisting of non-negative terms, and therefore each of its summands, vanishes, whereby the assumption made in the proof of Theorem B is justified.

And finally

Theorem C. *The class number of Ω_λ is an odd number.*